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Gradient Vector Fields Do Not Generate Twister Dynamics

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Thom's Gradient Conjecture states that a solution γ of an analytic gradient vector field X approaching to a singularity P of X has a tangent at P. A stronger version asserts that γ does not meet an analytic hypersurface an infinite number of times (it is non-oscillating). We prove, in dimension 3, that if γ is "infinitely near" an analytic curve Γ not composed of singularities of X, then γ is non-oscillating and, moreover, it does not spiral around Γ in a precise sense.
Key Words: trajectories of vector fields; gradient conjecture; oscillation; spiraling.

INTRODUCTION

Let $X = \nabla_g f$ be the gradient of an analytic function in $\mathbb{R}^n$ with respect to an analytic Riemannian metric g. In [3], S. Łojasiewicz proved that if γ is a solution of X which remains bounded in a relatively compact set, then γ accumulates at a single point P. Afterwards, R. Thom conjectured (cf. [4]) that γ *has a tangent at* P. That is to say, the transformation of γ after the blowing-up of $\mathbb{R}^n$ at P accumulates at a single point in the exceptional divisor. This result is proved in a recent manuscript of Kurdyka, Mostowski and Parusinski [2].

In a more general way, R. Moussu has proposed the "strong gradient conjecture": solutions of analytic gradient vector fields are non-oscillating. This means that γ meets each analytic hypersurface not containing it only a finite number of times.

In [1], Cano *et al.* they study the relation between nonoscillation and the existence of "all iterated tangents," that is, the existence of a tangent for the transform of γ after any number of point blow-ups. In an ambient space

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of dimension 3, nonoscillation for γ is equivalent to the following conditions:

(a) The solution γ has all iterated tangents, and

(b) There is no "spiraling axis" Γ for γ.

The aim of this paper is to prove property (b) for analytic gradient vector fields when Γ is not a union of stationary points. These axes are called "twister axes," and they give rise to a twister dynamics locally around them, as shown in [1]. To be precise, our main result is the following

THEOREM 5.1. *Let g be an analytic metric on a three-dimensional analytic manifold M. Given an analytic function f on M, the gradient vector field $\nabla_g f$ has no non-degenerate twister axes.*

This paper is structured as follows: in Section 1 we present the notions of spiraling, oscillation, iterated tangents, and twister axis. In Section 2, we establish a necessary condition for a smooth invariant axis Γ not to be a spiraling axis. This condition is given in terms of the linear term of the "normal" component of X with respect to Γ. Then we prove (Section 3) that spiraling axes are preserved by ramifications. This leads in a natural way to the study, made in Section 4, of a generalization of gradient vector fields: those arising from a bilinear form obtained as the ramification of a Riemannian metric. We prove that if $X = \nabla_g f$ is an ordinary gradient in M, Γ is a semi-branch and $\rho: M' \to M$ is a ramification with $\rho^{-1}(\Gamma)$ smooth, then this new branch cannot be a spiraling axis for the generalized gradient $X' = \nabla_{\rho^* g} \rho^* f$. The essence of this proof is the appearance of the Hessian of $\rho^* f$ in the normal component of X with respect to Γ. The symmetry inherent to the Hessian is the obstruction to the existence of spiraling. Theorem 5.1 is deduced in the last section as a consequence of all these results.

1. PRELIMINARIES

Let X be an analytic vector field on a three-dimensional real analytic manifold M. Let γ be a trajectory of X whose ω-limit set $\omega(\gamma)$ is a single point P. By an *analytic semi-branch at* P, we mean the image of $(0, \varepsilon)$ by a non-constant analytic map $\sigma: (-\varepsilon, \varepsilon) \to M$ with $\sigma(0) = P$. The semi-branch Γ is *smooth* (or *non-singular*) at P if $\sigma(-\varepsilon, \varepsilon)$ is a non-singular analytic curve. Suppose Γ is a semi-branch at P and let $\pi: \widetilde{M} \to M$ be a sequence of blow-ups of points such that $\pi^{-1}(\Gamma)$ is a smooth semi-branch at a point $\widetilde{P} \in \pi^{-1}(P)$. Fix a local coordinate system (x, y, z) at $\widetilde{P}$ such that

$\pi^{-1}(\Gamma)=\{x=y=0, z>0\}$. Let $\tilde{\gamma}=\pi^{-1}(\gamma)$. The definition of spiraling axis for X is introduced in [1]. We give here an equivalent property:

DEFINITION 1.1. The curve γ *spirals around* Γ (or equivalently, Γ is a *spiraling axis for* γ) if $\tilde{\gamma}$ admits a parametrization

$$\tilde{\gamma}(t)=(\rho(t)\cos(\varphi(t)), \rho(t)\sin(\varphi(t)), z(t)), \qquad t\in(0,\infty),$$

where ρ, φ, z are analytic in t, $\lim(\rho(t)/z(t)^n)=0$ for all $n\geqslant 0$, and $\lim_{t\to\infty}(\varphi(t))=\infty$.

Any spiraling axis is invariant for X (see [1]) and the property is preserved by blowing up P. Moreover, if Γ is non-singular, γ spirals around it and $\eta\colon M'\to M$ is the blowing-up of M at P with center Γ; then $\eta^{-1}(\gamma)$ is the whole projective line $\eta^{-1}(P)$.

DEFINITION 1.2. The curve γ is said to *have all the iterated tangents* if for any sequence of point blow-ups

$$M_n \xrightarrow{\pi_{n-1}} M_{n-1} \xrightarrow{\pi_{n-2}} \cdots \xrightarrow{\pi_1} M_1 \xrightarrow{\pi_0} M_0=M$$

with π_0 centered at P, the ω-limit set of $\pi_{n-1}^{-1}(\gamma)$ is a single point. If this is the case, then one defines the sequence $TI(\gamma)=\{P_n\}$ recursively: $P_0=P$, $P_{i+1}=\omega(\pi_i^{-1}(\gamma_i))$, where π_i is the blowing-up of M_i with center P_i and γ_i is the pull-back of γ by $\pi_{i-1}\circ\cdots\circ\pi_0$.

Notice that if Γ is a spiraling axis for γ, then γ has all the iterated tangents and $TI(\gamma)$ is exactly the sequence of infinitely near points of Γ. We remark also that γ has all the iterated tangents and $TI(\gamma)$ is the sequence of infinitely near points of an analytic branch Γ if and only if for any semi-analytic open set $V\supset\Gamma$, one has $\gamma(t)\in V$ for $t\gg 0$.

DEFINITION 1.3. The trajectory γ is said to *oscillate* at P if there is an analytic surface $f=0$ such that $\gamma\not\subset(f=0)$ but the set $(f=0)\cap|\gamma|$ is infinite.

The main result from [1] which we are going to use is the following

PROPOSITION [1]. *If the trajectory* γ *has all the iterated tangents at* P *and oscillates, then there is an analytic semi-branch* Γ *which is a spiraling axis for* γ.

DEFINITION 1.4. The semi-branch Γ is a *twister axis* for X if there is a positively invariant neighbourhood U of Γ such that for every $Q\in U-\Gamma$, the trajectory of X passing through Q spirals around Γ.

When Γ is non-degenerate (which means that it is not composed of singularities of X), then Γ is a spiraling axis if and only if it is a twister axis [1]. Thus, in order to know if a non-degenerate semi-branch is a twister axis, one need only test the existence of *one* solution spiraling around it.

Let $\pi: \widetilde{M} \to M$ be the blowing-up of M with center P. Denote by $\widetilde{\Gamma}$, $\tilde{\gamma}$, and $\widetilde{X}$ the transforms of Γ, γ, and X by π. We know (see [1]) that Γ is a spiraling axis for γ, if and only if $\widetilde{\Gamma}$ is a spiraling axis for $\tilde{\gamma}$.

2. SUFFICIENT CONDITIONS FOR NON-SPIRALING

Let X, M be as above. In this section, Γ will denote a *non-singular* analytic semi-branch at a point P, invariant and non-degenerate for X. Let $r \geqslant 0$ be the *order of X along Γ*, that is, the algebraic order at the origin of the restriction of X to the analytic curve in M containing Γ. Consider a system of coordinates (x, y, z) in a neighbourhood of P such that $\Gamma \equiv \{x = y = 0,\ z > 0\}$. In these coordinates, $r = \nu_z(X(z))$. Write X as follows,

$$X = \sum_{i=0}^{\infty} L_i z^i + v(x, y, z) \frac{\partial}{\partial z} + \widetilde{X},$$

where the L_i are linear vector fields in the variables (x, y) and

$$\widetilde{X} = a(x, y, z) \frac{\partial}{\partial x} + b(x, y, z) \frac{\partial}{\partial y},$$

with $\nu_{x,y}(a, b) \geqslant 2$. Let $k = \min\{i : L_i \text{ is not a radial vector field}\}$. This number k is independent of the chosen coordinate system. As Γ is non-degenerate, then $v(0, 0, z) = z^r u(z)$, with $\alpha = u(0) \neq 0$.

PROPOSITION 2.1. *If X and Γ satisfy one of the following conditions*:

(0) $\alpha > 0$,

(1) $\alpha < 0$ *and* $k \geqslant r$,

(2) $\alpha < 0$, $k \leqslant r - 1$, *and L_k has two different real eigenvalues*,

then Γ is not a twister axis for X.

Proof. Each of those conditions is stable under blowing-up of P. After making $2r$ blow-ups we can write X as follows, up to multiplication by an analytic unit

$$X = \sum_{i=0}^{r-1} L_i z^i + z^r \left(\alpha \frac{\partial}{\partial z} + Z \right),$$

where L_i are linear vector fields in (x, y), and Z is a vector field with $Z(z) \equiv 0$ and $\nu_{x,y}(Z(x), Z(y)) \geqslant 1$. If condition (0) holds, then Γ is not positively invariant. Assume that either (1) or (2) holds. Let γ be a trajectory of X such that $\omega(\gamma) = P$ and suppose, in order to obtain a contradiction, that γ spirals around Γ. By the expression of X above, there is a parametrization $\gamma(z) = (x(z), y(z), z)$ for $z > 0$ (not necessarily analytic at $z = 0$). Let η be the blowing-up of M along the z-axis, with local equations $x = x'$, $y = x'y'$, $z = z'$. The curve $\gamma' = \eta^{-1}(\gamma)$ admits a parametrization $\gamma'(z') = (x'(z'), y'(z'), z')$. If condition (1) holds, then we have

$$\frac{d}{dz'}(y'(z')) = A(x'(z'), y'(z'), z'),$$

where A is an analytic function. As γ' accumulates along the whole $\eta^{-1}(0)$, there is a z'_0 such that $1 > z'_0 > 0$ and $|y'(z'_0)| < 1$. Then $|\frac{d}{dz'}(y'(z'))|$ is bounded for $0 < z' < z'_0$, which implies that $|y'(z')|$ is also bounded near $z' = 0$, contradicting the accumulation of γ' along $\eta^{-1}(0)$. If condition (2) holds, then the transform of X by η is

$$X = \lambda(z')\, x' \frac{\partial}{\partial x'} + \mu z'^k y' \frac{\partial}{\partial y'} + z'^{k+1} Y,$$

where $k \in \mathbb{Z}_+$, λ is a polynomial in z' of degree at most k, μ is a non-zero constant, and Y is analytic. This gives

$$\frac{d}{dz'}(y'(z')) = \frac{1}{\alpha z'^{r-k}}(\mu y'(z') + z' B(x'(z'), y'(z'), z')),$$

B being an analytic function. From this equation we infer that near the points $P_1 = (0, 1, 0)$, $P_2 = (0, -1, 0)$, $\frac{d}{dz'}(y'(z'))$ has opposite signs for z' positive. This contradicts again the accumulation of $y'(z)$ along the whole real line. ∎

3. RAMIFICATIONS

Let Γ be a (not necessarily smooth) semi-branch at a point $P \in M$. Take local analytic coordinates (x, y, z) at P such that $\Gamma \subset \{z > 0\}$. We consider the ramification-rectification morphism

$$\rho \colon M' \to M$$

$$(x', y', t) \mapsto (x, y, z) = (x' + \alpha(t), y' + \beta(t), t^q),$$

where $(\alpha(t), \beta(t), t^q)$ is a Puiseux parametrization of Γ for $t>0$. Denote $\Gamma' = \rho^{-1}(\Gamma)$. We shall assume that ρ is an algebraic morphism, which can be accomplished after an analytic coordinate change in M (see [5]).

Given an analytic vector field X in M, let us consider the analytic vector field X' in M' such that $\rho_\star(X') = zX$. Notice that Γ is a spiraling axis for X if and only if it is so for zX.

PROPOSITION 3.1. *Suppose Γ is a spiraling axis for X. Then Γ' is also a spiraling axis for X'.*

Proof. Let γ be a trajectory of X spiraling around Γ and take $\gamma' = \rho^{-1}(\gamma)$, which is a trajectory of X'. We may suppose that $|\gamma| \subset \{z>0\}$ and $|\gamma'| \subset \{t>0\}$. As ρ is a homeomorphism between $\{t \geqslant 0\}$ and $\{z \geqslant 0\}$, the set $\omega(\gamma) = \rho^{-1}(P)$ consists of a single point. Let us prove that γ' spirals around Γ'. It is clearly oscillating, since γ is so. Thus, we shall finish if we prove that γ' has all the iterated tangents and $TI(\gamma')$ is the set of infinitely near points of Γ'. With the system of coordinates we are using, it suffices to prove that for any $k = 1, 2, ..., \gamma'(t)$ is in the open cone

$$C'_k = \{x'^2 + y'^2 < t^{2k}\}$$

for $t \gg 0$. Since these cones are algebraic, they project by ρ into semi-analytic sets C_k containing Γ in their interior. Thus, $\gamma(t)$ is in C_k for $t \gg 0$ because $TI(\gamma)$ coincides with the sequence of infinitely near points of Γ. So, for any $k = 1, 2, ..., \gamma'(t) \in C'_k$ for $t \gg 0$, which implies that $TI(\gamma')$ exists and is the sequence of infinitely near points of Γ', which completes the proof. ∎

Note that the converse is also true.

4. RAMIFIED GRADIENTS

Let g be an analytic symmetric bilinear form on M. Given a point P, denote by $\mathcal{M}_P$ the field of germs of meromorphic functions at P, that is, the field of quotients of the ring $\mathcal{O}_P$ of germs of analytic functions at P. Let Θ_P be the $\mathcal{O}_P$-module of germs of analytic vector fields. The bilinear operator $g_P: \Theta_P \times \Theta_P \to \mathcal{O}_P$ induces a symmetric bilinear form $\tilde{g}: \tilde{\Theta}_P \times \tilde{\Theta}_P \to \mathcal{M}_P$, with $\tilde{\Theta}_P = \Theta_P \otimes \mathcal{M}_P$. If $\tilde{g}$ is non-degenerate, we shall call $\Psi_{\tilde{g}}$ to the natural isomorphism

$$\Psi_{\tilde{g}}: \tilde{\Theta}_P \to \tilde{\Theta}^\star_P = \mathrm{Hom}(\tilde{\Theta}_P, \mathcal{M}_P)$$

induced by $\tilde{g}$. Taking into account that $\tilde{\Theta}_P^\star$ is the $\mathcal{M}_P$-vector space of germs of meromorphic 1-forms at P, we give the following

DEFINITION 4.1. For $f \in \mathcal{M}_P$, the *generalized gradient* of f with respect to g is the meromorphic vector field

$$\nabla_g f = \Psi_{\tilde{g}}^{-1}(df).$$

Consider a non-singular divisor $D \subset M$ containing P. Let g be as before and let $q \geqslant 0$ be a non-negative integer,

DEFINITION 4.2. The form g is a *metric of q-ramified type relative to D* if there is a local coordinate system (x, y, z) at P with $D = \{z = 0\}$ such that

1. The restriction $g_{\mathcal{N}}$ of $\tilde{g}$ to the $\mathcal{O}_P$-module $\mathcal{N}$ generated by the meromorphic vector fields $\{\partial/\partial x, \partial/\partial y$ and $z^{-q}\partial/\partial z\}$ is an $\mathcal{O}_P$-bilinear form, that is, $g_{\mathcal{N}}(\mathcal{N} \times \mathcal{N}) \subset \mathcal{O}_P$.
2. The specialization $g_{\mathcal{N}}(P) = g_{\mathcal{N}} \otimes \mathbb{R}$ defines a positive definite bilinear form on the three-dimensional real vector space $\mathcal{N} \otimes \mathbb{R}$.

In these conditions, we shall say that the coordinate system (x, y, z) is *appropriate* for g.

Notice that a coordinate change of the form

$$x' \mapsto \varphi(x, y) + z^{q+1}\varphi_1, \qquad y' \mapsto \psi(x, y) + z^{q+1}\psi_1, \qquad z' \mapsto zu(x, y, z)$$

respects the lattice $\mathcal{N} \subset A\tilde{\Theta}_P$ and gives another appropriate system. If the matrix of $g_{\mathcal{N}}(P)$ is the identity for the base of $\mathcal{N} \otimes \mathbb{R}$ corresponding to $\{\partial/\partial x, \partial/\partial y, z^{-q}\partial/\partial z\}$, we say that the coordinate system (x, y, z) is *normal* for g. Making coordinate changes as above, we can always get a normal system; we remark that the curve $x = y = 0$ need not be preserved under these changes.

PROPOSITION 4.1. *Let g be a metric of q-ramified type relative to a divisor D at P. Consider an appropriate and normal system of coordinates (x, y, z) and a germ of analytic function $f \in \mathcal{O}_P$. The meromorphic vector field $X = z^{2q}\nabla_g f$ is in fact analytic. If the curve $Y = \{x = y = 0\}$ is invariant and non-degenerate for X, then the branch $\Gamma = Y \cap \{z > 0\}$ is not a twister axis for X.*

Proof. Let A be the matrix of $g_{\mathcal{N}}$ in the basis $\{\partial/\partial x, \partial/\partial y, z^{-q}\partial/\partial z\}$. Notice that A is invertible and, in fact, the coefficients of A^{-1} belong to $\mathcal{O}_P$, since $A(0)$ is the identity matrix. Put

$$G=(G^{ij})=\begin{pmatrix} z^q & & \\ & z^q & \\ & & 1\end{pmatrix} A^{-1}\begin{pmatrix} z^q & & \\ & z^q & \\ & & 1\end{pmatrix}$$

The vector field $X=z^{2q}\nabla_g f=a\,\frac{\partial}{\partial x}+b\,\frac{\partial}{\partial y}+c\,\frac{\partial}{\partial z}$ is analytic, for

$$(a \quad b \quad c)=(f_x \quad f_y \quad f_z)\,G^t,$$

where subindices indicate partial derivation. From now on, we shall write $\bar{h}$ to indicate the restriction of an analytic function h in M to the curve Y. The following bounds follow from the fact that $A^{-1}(0)$ is the identity matrix

$$\begin{aligned} &v_z(\overline{G^{ii}})=2q,\ v_z(\overline{G^{ii}_x}, \overline{G^{ii}_y})\geqslant 2q\\ &v_z(\overline{G^{12}})\geqslant 2q+1,\ v_z(\overline{G^{12}_x}, \overline{G^{12}_y})\geqslant 2q \\ &v_z(\overline{G^{i3}})\geqslant q+1,\ v_z(\overline{G^{i3}_x}, \overline{G^{i3}_y})\geqslant q,\ v_z(\overline{G^{33}})=0,\end{aligned} \tag{1}$$

where $i=1, 2$. Let us show that X and Γ satisfy one of the conditions of Proposition 2.1. Let r be the order of X along the curve Y and let k be the least integer such that L_k is not radial in the expression

$$X=\sum_{i=0}^{\infty} L_i z^i+v(x, y, z)\,\frac{\partial}{\partial z}+\tilde{X}$$

used in Section 2. If $k\geqslant r$, then condition (0) or (1) holds. Thus, assume $k<r$. First, we note that the invariant r is equal to $v_z(\bar{f}_z)$. This follows from the fact that $X|_Y=z^{2q}\nabla_{g|_Y}\bar{f}$ and that the matrix of $g|_Y$ is exactly $(\overline{G^{33}})$ in the basis $\{z^{-q}\partial/\partial z\}$.

In the expression of X above, one can write the linear part normal to Γ:

$$\sum_{i=0}^{\infty} L_i z^i=(x \quad y)(N+H)\begin{pmatrix}\partial/\partial x\\ \partial/\partial y\end{pmatrix},$$

where N is the 2×2 matrix

$$\begin{pmatrix} \overline{G^{11}_x}\bar{f}_x+\overline{G^{12}_x}\bar{f}_y+\overline{G^{13}_x}\bar{f}_z+\overline{G^{13}}\bar{f}_{xz} & \overline{G^{12}_x}\bar{f}_x+\overline{G^{22}_x}\bar{f}_y+\overline{G^{23}_x}\bar{f}_z+\overline{G^{23}}\bar{f}_{xz}\\ \overline{G^{11}_y}\bar{f}_x+\overline{G^{12}_y}\bar{f}_y+\overline{G^{13}_y}\bar{f}_z+\overline{G^{13}}\bar{f}_{yz} & \overline{G^{12}_y}\bar{f}_x+\overline{G^{22}_y}\bar{f}_y+\overline{G^{23}_y}\bar{f}_z+\overline{G^{23}}\bar{f}_{yz}\end{pmatrix}$$

and H is obtained from the Hessian of f as follows:

$$H = \begin{pmatrix} \overline{G^{11}} & \overline{G^{12}} \\ \overline{G^{12}} & \overline{G^{22}} \end{pmatrix} \begin{pmatrix} \bar{f}_{xx} & \bar{f}_{xy} \\ \bar{f}_{xy} & \bar{f}_{yy} \end{pmatrix}.$$

Notice that this Hessian H can be written $H = z^{2q}(\sum A_i z^i)(\sum B_i z^i)$ where A_0 is the identity and A_i, B_i are symmetric matrices. Then, the first non-radial term appearing in the power series expansion of H is symmetric and so it has two different real eigenvalues. Thus, we shall finish if we prove that $v_z(N) \geqslant r$, for we are assuming $k < r$, from which we infer that the first non-radial term in the linear part of X normal to Γ comes, in fact, from H.

To see that $v_z(N) \geqslant r$, take $l = \min\{v_z(\bar{f}_x), v_z(\bar{f}_y), v_z(\bar{f}_z)\}$. From the bound (1), one sees that $v_z(N) \geqslant q + l$. There are two possibilities: if $l = v_z(\bar{f}_z) = r$ then $v_z(N) \geqslant r$. If $l < r$ assume, by symmetry, that $l = v_z(\bar{f}_x)$. As Y is an invariant curve for X, one must have

$$\bar{a} = \overline{G^{11} f_x} + \overline{G^{12} f_y} + \overline{G^{13} f_z} = 0.$$

Since $v_z(\overline{G^{12} f_y}) > v_z(\overline{G^{11} f_x})$ then $v_z(\overline{G^{11} f_x}) = v_z(\overline{G^{13} f_z})$, whence $2q + l = v_z(\overline{G^{13}}) + r \geqslant q + 1 + r$, from where $v_z(N) = q + l > r$ and we are done. ∎

5. GRADIENTS DO NOT GENERATE TWISTER AXES

THEOREM 5.1. *Let g be an analytic metric on a three-dimensional analytic manifold M. Given an analytic function f on M, the gradient vector field $\nabla_g f$ has no non-degenerate twister axes.*

Proof. Suppose, on the contrary, that Γ is a twister axis for X at P. Let (x, y, z) be a coordinate system appropriate for g and normal at P. Moreover, we can take (x, y, z) such that Γ is tangent to $x = y = 0$ and is contained in $z > 0$. Let $\rho: M' \to M$ be an algebraic ramification-rectification morphism as in Section 3

$$\rho(x', y', z') = \left(x' + \sigma(z'),\ y' + \tau(z'),\ \frac{1}{q+1} z'^{q+1} \right)$$

such that $\Gamma' = \rho^{-1}(\Gamma) = \{x' = y' = 0, z' > 0\}$. Let $g' = \rho^\star g$ be the transformed bilinear form. A computation shows that g' is a metric of q-ramified type relative to $\{z' = 0\}$ and (x', y', z') is a coordinate system appropriate and normal for g'. Let $X' = \nabla_{g'}(\rho^\star f)$. By Proposition 4.1, Γ' cannot be a twister axis for $z^{2q} X'$, in contradiction with Proposition 3.1, since $\rho_\star X' = X$. ∎

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