CRITICAL TRANSITIONS FOR ASYMPTOTICALLY CONCAVE OR D-CONCAVE NONAUTONOMOUS DIFFERENTIAL EQUATIONS WITH APPLICATIONS IN ECOLOGY

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Abstract. The occurrence of tracking or tipping situations for a transition equation $x' = f(t, x, \Gamma(t, x))$ with asymptotic limits $x' = f(t, x, \Gamma_\pm(t, x))$ is analyzed. The approaching condition is just $\lim_{t\to\pm\infty}(\Gamma(t, x) - \Gamma_\pm(t, x)) = 0$ uniformly on compact real sets, and so there is no restriction to the dependence on time of the asymptotic equations. The hypotheses assume concavity in x either of the maps $x \mapsto f(t, x, \Gamma_\pm(t, x))$ or of their derivatives with respect to the state variable (d-concavity), but not of $x \mapsto f(t, x, \Gamma(t, x))$ nor of its derivative. The analysis provides a powerful tool to analyze the occurrence of critical transitions for one-parametric families $x' = f(t, x, \Gamma^c(t, x))$. The new approach significatively widens the field of application of the results, since the evolution law of the transition equation can be essentially different from those of the limit equations. Among these applications, some scalar population dynamics models subject to non trivial predation and migration patterns are analyzed, both theoretically and numerically.

Some key points in the proofs are: to understand the transition equation as part of an orbit in its hull which approaches the α-limit and ω-limit sets; to observe that these sets concentrate all the ergodic measures; and to prove that in order to describe the dynamical possibilities of the equation it is sufficient that the concavity or d-concavity conditions hold for a complete measure subset of the equations of the hull.

1. Introduction

Tipping points or *critical transitions* are significant nonlinear phenomena that occur in complex systems subject to smooth changes of the external conditions. Roughly speaking, they are sudden and often irreversible changes in the state of the system caused by small changes in the external input. During the last years, they have frequently appeared in the literature as an explanation of abrupt changes in climate [2, 9, 27, 42], ecology [4, 40, 41, 46], biology [23, 32] or finances [31, 49], among other scientific areas of great interest. For this reason, critical transitions have become an important topic of multidisciplinary research.

A branch of the mathematical formulation of this problem focuses on one-parametric ordinary differential equations (ODEs). The parameter is replaced by a map (a *parameter shift*) with constant asymptotic limits as $t \to \pm\infty$, and the

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two *limit equations*, given by these constant limits of the parameter, are frequently assumed to have the same type of global dynamics. The initial ODE is understood as a transition between the past equation and the future equation. Sometimes, the dynamics of this transition equation reproduces those of the asymptotic limits and approaches them as time decreases and increases. This situation is usually referred to as *tracking*, and the remaining situations as *tipping*. In general, tracking means the survival over time of a local pullback attractor which represents the desirable state of the system, and hence tipping may mean a catastrophe. If the parameter shift, itself, depends on a parameter, a *critical value* of the parameter, resulting in a *critical transition*, occurs if there is tracking to its left and tipping to its right, or viceversa.

This is the approach of the reference work [8], as well as of [3, 26, 36, 47, 48], among many other papers. In the recent works [17, 19, 28, 29, 30, 38], the limit equations are allowed to be nonautonomous, and the law of the (scalar) ODE is assumed coercive and either concave or with concave derivative (*d-concave*) with respect to the state variable. The analysis of ODEs for which the evolution law satisfies certain coercive and concavity conditions is a classical subject in the theory of (autonomous or nonautonomous) dynamical systems: on the one hand, these properties imply a structure in the space of solutions that simplifies the study; and, on the other hand, the number of mathematical models that respond to laws of this type is high.

However, many interesting basic models do not fit such conditions, as the next populations dynamics cases will show. The starting point are the classical nonautonomous equations $x' = -r(t)\,x\,(1-x/K(t))$ and $x' = -r(t)\,x\,(S(t)-x)\,(1-x/K(t))$, with r, K and S continuous and r and K positively bounded from below, which model the evolution of a single population without or with Allee effect, and which are respectively given by a concave map and a d-concave map. That is, the maps sending x to $-r(t)\,x\,(1-x/K(t))$ (or to $-r(t)\,x\,(S(t)-x)\,(1-x/K(t))$) are concave (or d-concave) for all $t \in \mathbb{R}$. Let us focus on the concave case, introducing a continuous net emigration rate per unit of time $\phi(t) < 0$ and a predation term $\Delta_d(t,x) := -d\,\Gamma(t)\,x^2/(b(t)+x^2)$ which is a modification of a Holling type III functional response term: the continuous map $\Gamma \geq 0$ vanishes outside an interval $[-t_0, t_0]$ (responding to predators attacking for a finite period of time), $d > 0$ determines the intensity of predation, and the continuous map $b > 0$ corresponds to the average time between attacks. Taken together, we obtain the model

$$x' = r(t)\,x\left(1 - \frac{x}{K(t)}\right) + \phi(t) - d\,\Gamma(t)\,\frac{x^2}{b(t)+x^2}\,,$$

whose law, in general, does not provide a concave map in x for all $t \in \mathbb{R}$. The framework we present in this paper allows the analysis of this model. This is done in Section 4.2, where we establish conditions that guarantee, among other properties, the existence of a single critical transition as d increases: there exists a value $d_0 > 0$ of the intensity before which an initially healthy population survives, and after which it is doomed to extinction.

Since $\lim_{t\to\pm\infty} \Delta_d(t,x) = 0$ uniformly in x, the previous example introduces some "autonomous ingredients" of the asymptotic equations. In Section 6.2, we add a predation term to the equation $x' = -r(t)\,x\,(S(t)-x)\,(1-x/K(t))$ similar to the previous one, but now without assuming $\lim_{t\to\infty} \Delta_d(t) = 0$: the predation term also appears in the future equation, and it depends on t and x. Again, we study the occurrence of critical transitions as the involved parameters change. These two examples are chosen to show that the theoretical analysis performed in the paper considerably widens the field of application of the results.

Weaker hypotheses and main results. In order to briefly describe the main results of the paper and their hypotheses (which, as already mentioned, are less restrictive than in previous approaches), we need to explain the framework of the analysis: the classical hull construction allows us to embed one single nonautonomous ODE in a family which provides a skewproduct flow, to which techniques coming from topological dynamics and ergodic theory can be applied. The conclusions obtained for the flow are then particularized to the original equation. In our opinion, the skewproduct formalism provides the most suitable framework to understand the occurrence of critical transitions in nonautonomous models, and pointing out this fact is one of the main contributions of this paper.

So, we work with a nonautonomous scalar ODE

$$x' = g(t,x)\,, \tag{1.1}$$

given by a regular enough map g for which the hull Ω_g, defined as the closure of the set of time shifts $g_s(t,x) := g(t+s,x)$ in the compact-open topology of $C(\mathbb{R}\times\mathbb{R},\mathbb{R})$, is compact. Equation (1.1) is hence embedded in the family

$$x' = \omega(t,x)\,, \qquad \omega\in\Omega_g\,, \tag{1.2}$$

which defines a skewproduct flow on $\Omega_g\times\mathbb{R}$. The set Ω_g is composed by three (possibly nondisjoint) sets: $\{g_s \mid s\in\mathbb{R}\}$ and the corresponding α-limit and ω-limit sets, Ω_g^α and Ω_g^ω; and, as a consequence, the ergodic measures on the hull are concentrated in $\Omega_g^\alpha\cup\Omega_g^\omega$. This point is key in our approach: it turns out that assuming conditions on concavity with respect to x of ω (in the so-called *concave case*) or of its derivative ω_x (in the *d-concave case*) for all $\omega\in\Omega_g^\alpha\cup\Omega_g^\omega$ suffices to determine the maximum number of hyperbolic solutions for (1.1): two in the concave case, and three in the d-concave case. In addition, if they exist, and if an extra coercivity assumption on all the involved equations holds and hence the set of bounded solutions is bounded (if nonempty), then the hyperbolic solutions yield a very fixed type of global dynamics. These assertions are proved in the concave and d-concave cases in the main Theorems 3.6 and 5.6, respectively, as consequences of general results on skewproducts, with interest by their own, that we previously prove. We point out that no concavity condition is required on g or on g_x, which is one of the two most significative advantages with respect to previous approaches.

In order to formulate the necessary conditions on the α-limit and ω-limit sets in the language of processes rather than in the language of skewproducts, and thus be able to apply our results to particular examples, we assume the existence of two regular enough functions $g_\pm$ with $\lim_{t\to\pm\infty}(g(t,x)-g_\pm(t,x))=0$ uniformly on compact sets of $\mathbb{R}$, and which are concave and coercive with respect to x. These conditions are inherited by all the elements of the corresponding hulls $\Omega_{g_\pm}$, and the asymptotic approach yields $\Omega_g^\alpha=\Omega_{g_-}^\alpha$ and $\Omega_g^\omega=\Omega_{g_+}^\omega$. So, we are in the suitable framework of the previous paragraph. In order to describe scenarios as rich in dynamical possibilities as possible, we assume that the *limit equations* $x'=g_\pm(t,x)$ have the maximal number of hyperbolic solutions. And then we describe all the dynamical cases for (1.1) (which are three) and explain the strong connection among critical transition and nonautonomous saddle-node bifurcation. The classification is provided by the main Theorems 4.7 and 6.4 in the concave and d-concave cases, respectively. In both cases, all the scenarios are persistent under small perturbations excepting one of them. A critical transition will occur when the dynamics jumps, as the external input varies, from one stable scenario to another one, which (as Theorems 4.9 and 6.6 show) means crossing the nonstable scenario and can be understood as a nonautonomous saddle-node bifurcation phenomenon.

The words *asymptotically concave or d-concave ODEs*, appearing in the title of this work, refer to equations $x' = g(t,x)$ satisfying the conditions described in the previous paragraph. Once again, observe that no concavity condition is required on g or on g_x, but in the (not necessarily univocally determined) maps $g_\pm$.

When trying to build realistic mathematical models, it is common to replace a parameter that appears in a first and simple approximation to the law of evolution of a given system, say $x' = f(t,x,\gamma)$, by a map that may depend on time, state, and new parameters. Our approach allows us to deal with

$$x' = f(t,x,\Gamma(t,x)) \tag{1.3}$$

assuming the existence of two maps $\Gamma_\pm$ such that $\lim_{t\to\pm\infty}(\Gamma(t,x) - \Gamma_\pm(t,x)) = 0$ uniformly on compact sets of $\mathbb{R}$. Of course, we must assume conditions on f, Γ and $\Gamma_\pm$ guaranteing the previous hypotheses on $g(t,x) := f(t,x,\Gamma(t,x))$ and $g_\pm(t,x) := f(t,x,\Gamma_\pm(t,x))$. The second fundamental advantage arises here: in previous approaches, Γ is just a map of t (and often the unique time-dependent part of the evolution law) with constant asymptotic limits; but, in this new formulation, also the asymptotic part $\Gamma_\pm$ of the past and future equations $x' = f(t,x,\Gamma_\pm(t,x))$ may depend on t and x. In order to analyze the occurrence of critical transitions, we let the transition map to depend on a parameter c which moves, getting $x' = f(t,x,\Gamma^c(t,x))$. It is reasonable to assume that the asymptotic equations do not depend on the parameter c, whose variations represent different ways to approach the past and the future. Theorems 4.15, 4.17 and 4.18 (in the concave case) and 6.10, 6.11 and 6.12 (in the d-concave setting) describe several scenarios of occurrence and/or absence of critical transitions, focusing on rate-induced, phase-induced and size-induced tipping points.

As already pointed out, the lack of general requirements on concavity of f or f_x with respect to x combined with the possible dependence of $\Gamma^c_\pm$ on t and x significatively increases the number of possible applications of our results. We complete the paper by combining theoretical and numerical techniques to analyze the examples mentioned above in Sections 4.2 and 6.2.

Paper structure. Let us describe with some more detail the contents of the paper. Section 2 summarizes several basic concepts and results used throughout the paper: we characterize concavity and d-concavity in terms of divided differences, describe the skewproduct construction from a scalar nonautonomous ODE, and recall some properties of hyperbolicity and Lyapunov exponents. The remaining sections present the analysis in the concave case (Sections 3 and 4) and the d-concave case (Sections 5 and 6). Sections 3 and 5 deal with general properties of skewproduct flows, which include those arising from families of the type (1.2). Section 3 deals with the concave case. Under some assumptions which are weaker than the strict concavity of the equations of a set with full measure for any ergodic measure, we prove the existence of at most two hyperbolic solutions for each one of the equations, characterize this existence by the occurrence of two uniformly separated bounded solutions, and describe the global dynamics in this situation. Section 5 contains the analogous results for the d-concave case: now there are at most three hyperbolic solutions for each equation, which happens if and only if there are three uniformly separated bounded solutions and forces a certain type of global dynamics.

At the beginning of Section 4, with g and $g_\pm$ as above described, we assume the strict concavity on x of the maps $g_\pm(t,x)$ and the existence of the maximum number of hyperbolic solutions for $x' = g_\pm(t,x)$: two, which form an attractor-repeller pair. Then, there are three possibilities for (1.1): Case A, when it also has an attractor-repeller pair which connects with that of the past as time decreases and with that of the future as time increases (i.e., when there is tracking); Case C, when it has no bounded solutions; and Case B, when it has exactly one bounded solution, which is nonhyperbolic. In Section 6, we assume the strict concavity on x of the derivatives $(g_\pm)_x(t,x)$ and the existence of the maximum number of hyperbolic solutions for $x' = g_\pm(t,x)$: three. Again, the dynamical possibilities for (1.1) are three: Case A, if it also has three hyperbolic solutions which connect with those of the past as time decreases and with those of the future as time increases (tracking); Case C, if it has two hyperbolic solutions, which approach each other as time increases; or Case B, if it has just one hyperbolic solution. In both cases, a typical critical transition occurs when a small variation on g causes the dynamics to move from Case A to Case C. We also establish nonrestrictive conditions guaranteing the persistence under small perturbations of these two cases, and show that a critical transition means that the (highly nonpersistent) Case B occurs and can be understood as a nonautonomous saddle-node bifurcation phenomenon (see [6, 19, 30, 33]).

Sections 4.1 and 6.1 are centered in equations of the type (1.3), and hence the corresponding general hypotheses are given for f, Γ^c, and the maps $\Gamma^c_\pm$ before mentioned. In scenarios suitable for raising the question of the occurrence of rate-induced, phase-induced and size-induced critical transitions, we add extra conditions of f and Γ ensuring the existence and/or absence of these types of tipping points. These results, as well as the techniques used in their proofs, are the key points to analyze some population dynamics models including those mentioned at the beginning of this Introduction, which is the goal of Sections 4.2 and 6.2: the survival of a given population subject to emigration and predation depends on several factors, as the speed of arrival of the predators, the quantity of them, the moment at which they arrive, or the length of their periods of permanence in the preys' habitat. We point out once again that the purpose of the models that we consider is showing the applicability of this new approach to the analysis of critical transitions due to a fairly general parametric variation in a nonautonomous dynamical system.

2. Some preliminary results

The following subsections collect basic concepts and some general results needed throughout the paper. We also provide some suitable references for further information on those results which we do not prove here.

2.1. **Concave and d-concave real functions, and divided differences.**

Recall that a map $h \in C(\mathbb{R},\mathbb{R})$ is *concave* if $h(\alpha x_1 + (1-\alpha)x_2) \ge \alpha h(x_1) + (1-\alpha)h(x_2)$ for all $x_1, x_2 \in \mathbb{R}$ and $\alpha \in [0,1]$ (which ensures that h' is nonincreasing if $h \in C^1(\mathbb{R},\mathbb{R})$). And we say that $h \in C^1(\mathbb{R},\mathbb{R})$ is *d-concave* if h' is concave. The simplest examples of concave and d-concave maps are $h(x) := -x^2$ and $h(x) := -x^3$, respectively. Our next result explains that both properties can be characterized in terms of the *divided differences of first* and *second order* of h, defined as

$$h[x_1, x_2] := \frac{h(x_2) - h(x_1)}{x_2 - x_1} \quad \text{and} \quad h[x_1, x_2, x_3] := \frac{h[x_2, x_3] - h[x_1, x_2]}{x_3 - x_1}\,.$$

Both of them are invariant under any permutation of their nodes.

Proposition 2.1. (i) *$h \in C(\mathbb{R}, \mathbb{R})$ is concave if and only if $h[x_0, x_1] \geq h[x_0, x_2]$ whenever $x_1 < x_2$ and $x_0 \neq x_i$ for $i \in \{1, 2\}$.*
(ii) *$h \in C^1(\mathbb{R}, \mathbb{R})$ is d-concave if and only if $h[x_1, x_0, x_2] \geq h[x_1, x_0, x_3]$ whenever $x_1 < x_2 < x_3$ and $x_0 \neq x_i$ for $i \in \{1, 2, 3\}$.*

Proof. (i) Sufficiency is proved by taking $x_0 := \alpha x_1 + (1 - \alpha)\, x_2$ for $\alpha \in (0, 1)$. To check necessity, we rewrite the intermediate node as a convex combination of the other two, write the parameter (always in $(0, 1)$) in terms of the nodes, and apply the definition of concavity. (ii) See [45, Lemma 2.1 and remark after it]. □

Let us take $x_1 < x_2 < x_3$ and $x_0 \neq x_i$ for $i \in \{1, 2, 3\}$, and define

$$\begin{aligned} a(x_0, x_1, x_2) &:= h[x_0, x_1] - h[x_0, x_2]\,, \\ b(x_0, x_1, x_2, x_3) &:= h[x_1, x_0, x_2] - h[x_1, x_0, x_3]\,. \end{aligned}$$

Clearly, $\lim_{x_0 \to x_i} h[x_0, x_i] = h'(x_i)$ for all $x_i \in \mathbb{R}$ if $h \in C^1(\mathbb{R}, \mathbb{R})$. Hence, there exist $\lim_{x_0 \to x_i} h[x_1, x_0, x_j]$ for $i \in \{1, 2, 3\}$ and $j \in \{2, 3\}$. We call

$$\begin{aligned} a_i(x_1, x_2) &:= \lim_{x_0 \to x_i} a(x_0, x_1, x_2)\,, \\ b_j(x_1, x_2, x_3) &:= \lim_{x_0 \to x_j} b(x_0, x_1, x_2, x_3) \end{aligned} \tag{2.1}$$

for $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$. Our next result establishes an equivalence between the sign of a_i (resp. b_i) and the decreasing properties of h' (resp. h'').

Proposition 2.2. (i) *Let $h \in C^1(\mathbb{R}, \mathbb{R})$ be concave and $x_1 < x_2$. Then, for $i \in \{1, 2\}$, $a_i(x_1, x_2) \geq 0$, and $a_i(x_1, x_2) > 0$ if and only if $h'(x_1) > h'(x_2)$.*
(ii) *Let $h \in C^2(\mathbb{R}, \mathbb{R})$ be d-concave and $x_1 < x_2 < x_3$. Then, for $i \in \{1, 2, 3\}$, $b_i(x_1, x_2, x_3) \geq 0$, and $b_i(x_1, x_2, x_3) > 0$ if and only if $h''(x_1) > h''(x_3)$.*

Proof. (i) The first assertion follows from Proposition 2.1(i). For the second one, we take $i = 1$ and write

$$a_1(x_1, x_2) = h'(x_1) - h[x_1, x_2] = \int_0^1 \big(h'(x_1) - h'(sx_1 + (1 - s)x_2)\big)\, ds\,. \tag{2.2}$$

Since h is C^1 and concave, the integrand is continuous on s, nonnegative for all $s \in [0, 1]$ and nonincreasing with respect to s. Hence, the integral is strictly positive if and only if $h'(x_1) > h'(x_2)$. An analogous argument proves the assertion for $i = 2$.

(ii) The first assertion follows from Proposition 2.1(ii). For the second one, we work in the case $i=2$. It is easy to check that

$$\begin{aligned} b_2(x_1,x_2,x_3) &= \frac{1}{x_2-x_1}\left(h'(x_2) - \frac{x_3-x_2}{x_3-x_1}\,h[x_1,x_2] - \frac{x_2-x_1}{x_3-x_1}\,h[x_2,x_3]\right) \\ &= \frac{1}{x_2-x_1}\left(\frac{x_3-x_2}{x_3-x_1}\int_0^1 \big(h'(x_2)-h'(sx_1+(1-s)x_2)\big)\,ds \right.\\ &\qquad\qquad \left. +\frac{x_2-x_1}{x_3-x_1}\int_0^1 \big(h'(x_2)-h'(sx_3+(1-s)x_2)\big)\,ds\right) \\ &= \frac{x_3-x_2}{x_3-x_1}\int_0^1\int_0^1 s\big(h''(sx_1+(1-s)x_2+ts(x_2-x_1)) \\ &\qquad\qquad - h''(sx_3+(1-s)x_2-ts(x_3-x_2))\big)\,dt\,ds\,. \end{aligned}$$

Since h is C^2 and d-concave, the integrand is continuous on s,t and nonnegative for all $t\in[0,1]$. If $h''(x_1)>h''(x_3)$, then the integrand is strictly positive at $(t,s)=(0,1)$, and hence $b_2(x_1,x_2,x_3)>0$. Conversely, if $h''(x_1)=h''(x_3)$, then h'' is constant on $[x_1,x_3]$, and hence the integrand is identically zero. We proceed analogously with b_1 and b_3. □

2.2. **Skew-product flows.** Throughout the paper, the basic concepts of flows, orbits, invariant sets, ergodic measures, α-limit sets and ω-limit sets will be used. Their well-known definitions and some basic properties can be found, e.g., in [10].

Let Ω be a compact metric space and $\sigma\colon\mathbb{R}\times\Omega\to\Omega$, $(t,\omega)\mapsto\sigma(t,\omega)=:\omega{\cdot}t$ a global continuous flow on Ω. Throughout the paper, $C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$ represents the set of continuous functions $\mathfrak{h}\colon\Omega\times\mathbb{R}\to\mathbb{R}$ for which the derivative $\mathfrak{h}_x$ with respect to the second variable exists and is continuous, and $C^{0,2}(\Omega\times\mathbb{R},\mathbb{R})$ is the subset of $C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$ of maps $\mathfrak{h}$ for which the second derivative $\mathfrak{h}_{xx}$ exists and is continuous. Given $\mathfrak{h}\in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$, we consider the family of scalar nonautonomous differential equations

$$x'=\mathfrak{h}(\omega{\cdot}t,x)\,,\quad \omega\in\Omega\,. \tag{2.3}$$

For each $\omega\in\Omega$, $(2.3)_\omega$ is the particular equation of the family. We use similar notation throughout the paper to refer to elements of the hull or parameters. For each $\omega\in\Omega$ and $x\in\mathbb{R}$, the map $t\mapsto v(t,\omega,x)$ is the maximal solution of $(2.3)_\omega$ with $v(0,\omega,x)=x$, and $(\alpha_{\omega,x},\beta_{\omega,x})$ is its interval of definition, with $-\infty\le\alpha_{\omega,x}<0<\beta_{\omega,x}\le\infty$. Throughout the paper, any solution will be assumed to be maximal. By uniqueness of solutions, $v(t+s,\omega,x)=v(t,\omega{\cdot}s,v(s,\omega,x))$ when the right-hand term is defined. Hence, if $\mathcal{V}:=\bigcup_{(\omega,x)\in\Omega\times\mathbb{R}}((\alpha_{\omega,x},\beta_{\omega,x})\times\{(\omega,x)\})$, then

$$\tau\colon\mathcal{V}\subseteq\mathbb{R}\times\Omega\times\mathbb{R}\to\Omega\times\mathbb{R}\,,\ \ (t,\omega,x)\mapsto(\omega{\cdot}t,v(t,\omega,x)) \tag{2.4}$$

defines a (possibly local) continuous flow on $\Omega\times\mathbb{R}$, of *skewproduct* type. As we will see in Section 2.3, families of this type appear in a natural way when we construct the hull of a single equation.

A τ-*equilibrium* is a map $\mathfrak{b}\colon\Omega\to\mathbb{R}$ whose graph is τ-invariant (i.e., with $v(t,\omega,\mathfrak{b}(\omega))=\mathfrak{b}(\omega{\cdot}t)$ for all $\omega\in\Omega$ and $t\in\mathbb{R}$). If it is continuous, then its compact graph, which we represent by $\{\mathfrak{b}\}$, is a τ-*copy of the base* or τ-*copy of* Ω. A τ-copy of the base $\{\mathfrak{b}\}$ is *hyperbolic attractive* if there exists $\delta>0$, $\gamma>0$ and $k\ge1$ such that, if $|\mathfrak{b}(\omega)-x|<\delta$ for any $\omega\in\Omega$, then $v(t,\omega,x)$ is defined for all $t\ge0$ and $|\mathfrak{b}(\omega{\cdot}t)-v(t,\omega,x)|\le k\,e^{-\gamma\,t}\,|\mathfrak{b}(\omega)-x|$ for $t\ge0$. Replacing $t\ge0$ by $t\le0$

and $-\gamma$ by γ provides the definition of *repulsive hyperbolic* τ-copy of the base. We will usually call $\tilde{\mathfrak{b}} := \mathfrak{b}$ if $\{\mathfrak{b}\}$ is a hyperbolic τ-copy of the base, and we will often omit the prefix τ.

Given a bounded τ-invariant set $\mathcal{B} \subset \Omega \times \mathbb{R}$ projecting onto Ω, the maps $\omega \mapsto \inf\{x \in \mathbb{R} \,|\, (\omega, x) \in \mathcal{B}\}$ and $\omega \mapsto \sup\{x \in \mathbb{R} \,|\, (\omega, x) \in \mathcal{B}\}$ define τ-equilibria. We will refer to these maps as the *lower* and *upper equilibria of* $\mathcal{B}$. Observe that, if $\mathcal{B}$ is compact, then they are lower and upper semicontinuous, respectively, and hence m-measurable for all $m \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$, where $\mathfrak{M}_{\text{erg}}(\Omega, \sigma)$ is the (nonempty) set of σ-ergodic measures on Ω.

We complete this part with two more definitions. If there exists a compact τ-invariant set $\mathcal{A} \subset \Omega \times \mathbb{R}$ such that $\lim_{t\to\infty} \text{dist}(\mathcal{C}{\cdot}t, \mathcal{A}) = 0$ for every bounded set $\mathcal{C}$, where $\mathcal{C}{\cdot}t = \{(\omega{\cdot}t, v(t, \omega, x)) \,|\, (\omega, x) \in \mathcal{C}\}$ and

$$\text{dist}(\mathcal{C}_1, \mathcal{C}_2) = \sup_{(\omega_1,x_1)\in\mathcal{C}_1} \left(\inf_{(\omega_2,x_2)\in\mathcal{C}_2} \big(\text{dist}_{\Omega\times\mathbb{R}}((\omega_1, x_1), (\omega_2, x_2))\big) \right),$$

then $\mathcal{A}$ is the *global attractor for* τ. And given two compact subsets $\mathcal{K}_1$ and $\mathcal{K}_2$ of $\Omega \times \mathbb{R}$, we say that they are *ordered* with $\mathcal{K}_1 < \mathcal{K}_2$ if $x_1 < x_2$ whenever there exists $\omega \in \Omega$ such that $(\omega, x_1) \in \mathcal{K}_1$ and $(\omega, x_2) \in \mathcal{K}_2$.

2.3. **Admissible processes and their hull extensions.** Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set. We say that a continuous map $h\colon \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ is *admissible* and write $h \in C^{0,0}(\mathbb{R} \times \mathcal{U}, \mathbb{R})$ if the restriction of h to $\mathbb{R} \times \mathcal{J}$ is bounded and uniformly continuous for any compact set $\mathcal{J} \subset \mathcal{U}$. In most of the cases, we will work with $\mathcal{U} = \mathbb{R}$. We say that $h\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1*-admissible* (resp. C^2*-admissible*) and write $h \in C^{0,1}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ (resp. $h \in C^{0,2}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$) if there exists its derivative h_x with respect to the second variable and it is admissible (resp. there exist h_x and h_{xx} and they are admissible).

Given $h \in C^{0,1}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, we represent by $x_h(t, s, x)$ the maximal solution of

$$x' = h(t, x) \tag{2.5}$$

with $x_h(s, s, x) = x$. By uniqueness of solutions, $x_h(t, s, x_h(s, r, x)) = x_h(t, r, x)$. Often, the map $(t, s, x) \mapsto x_h(t, s, x)$ is called a *process*.

We say that two solutions $b_1(t)$ and $b_2(t)$ of (2.5) are *uniformly separated* if they are bounded and $\inf_{t\in\mathbb{R}} |b_1(t)-b_2(t)| > 0$. A bounded solution $\tilde{b}(t)$ of (2.5) is *hyperbolic attractive* (resp. *hyperbolic repulsive*) if there exist $k \geq 1$ and $\gamma > 0$ such that $\exp\left(\int_s^t h_x(r, \tilde{b}(r))\, dr\right) \leq k e^{-\gamma(t-s)}$ whenever $t \geq s$ (resp. $\exp\left(\int_s^t h_x(r, \tilde{b}(r))\, dr\right) \leq k e^{\gamma(t-s)}$ whenever $t \leq s$); and, in both cases, (k, γ) is a *dichotomy constant pair of* $\tilde{b}$. For the reader's convenience, we state the next fundamental result. A partial proof, strongly based on [13, Lecture 3], can be found in [19, Theorem 2.2]; and a proof of the last assertion, strongly based on [22, Theorem III.2.4], can be found in [16, Theorem 3.2.3] (just use the admissibility of h_x instead of the existence and boundedness of the second derivative). We denote $\|h\|_{1,\rho} := \sup_{(t,x)\in\mathbb{R}\times[-\rho,\rho]} |h(t, x)| + \sup_{(t,x)\in\mathbb{R}\times[-\rho,\rho]} |h_x(t, x)|$ and $\|b\|_\infty := \sup_{t\in\mathbb{R}} |b(t)|$.

Theorem 2.3. *Let h be C^1-admissible, let $\tilde{b}_h$ be an attractive (resp. repulsive) hyperbolic solution of (2.5) with dichotomy constant pair (k_0, γ_0), and take $\rho > \|\tilde{b}_h\|_\infty$. Then, for every $\gamma \in (0, \gamma_0)$ and $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ and $\rho_\varepsilon > 0$ such that, if g is C^1-admissible and $\|h - g\|_{1,\rho} < \delta_\varepsilon$, then*

(i) *there exists an attractive (resp. repulsive) hyperbolic solution* $\tilde{b}_g$ *of* $x' = g(t,x)$ *with dichotomy constant pair* (k_0,γ) *which satisfies* $\|\tilde{b}_h - \tilde{b}_g\|_\infty < \varepsilon$*;*

(ii) *if* $|\tilde{b}_g(t_0) - x_0| \le \rho_\varepsilon$*, then* $|\tilde{b}_g(t) - x_g(t,t_0,x_0)| \le k_0\, e^{-\gamma(t-t_0)}|\tilde{b}_g(t_0) - x_0|$ *for all* $t \ge t_0$ *(resp.* $|\tilde{b}_g(t) - x_g(t,t_0,x_0)| \le k_0\, e^{\gamma(t-t_0)}|\tilde{b}_g(t_0) - x_0|$ *for all* $t \le t_0$*).*

A solution $\bar{b}\colon (-\infty,\beta) \to \mathbb{R}$ of (2.5) is *locally pullback attractive* if there exists $s_0 < \beta$ and $\delta > 0$ such that, if $s \le s_0$, then $x_h(t,s,\bar{b}(s) \pm \delta)$ exists for $t \in [s,s_0]$, and

$$\lim_{s\to-\infty} |\bar{b}(t) - x_h(t,s,\bar{b}(s) \pm \delta)| = 0 \qquad \text{for all } t \le s_0\,.$$

Analogously, a solution $\bar{b}\colon (\alpha,\infty) \to \mathbb{R}$ of (2.5) is said to be *locally pullback repulsive* if and only if there exist $s_0 > \alpha$ and $\delta > 0$ such that, if $s \ge s_0$ and $|x - \bar{b}(s)| < \delta$, then $x_h(t,s,\bar{b}(s) \pm \delta)$ exists for $t \in [s_0,s]$ and

$$\lim_{s\to\infty} |\bar{b}(t) - x_h(t,s,\bar{b}(s) \pm \delta)| = 0 \qquad \text{for all } t \ge s_0\,.$$

Let us describe the already mentioned *hull construction*. Given an admissible function $h\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, we define $h{\cdot}t(s,x) := h(t+s,x)$. The *hull* Ω_h of h is the closure of the set $\{h{\cdot}t \mid t \in \mathbb{R}\}$ on the set $C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ provided with the compact-open topology. The set Ω_h is a compact metric space, the time-shift map $\sigma_h\colon \mathbb{R}\times\Omega_h \to \Omega_h, (t,\omega) \mapsto \omega{\cdot}t$ defines a global continuous flow, and the map $\mathfrak{h}$ given by $\mathfrak{h}(\omega,x) = \omega(0,x)$ is continuous on $\Omega_h\times\mathbb{R}$. In addition, if h is C^1-admissible then $\Omega_h \subset C^{0,1}(\mathbb{R}\times\mathbb{R},\mathbb{R})$, and the continuous map $\mathfrak{h}_x(\omega,x) := \omega_x(0,x)$ is the derivative of $\mathfrak{h}$ with respect to x; and, if h is C^2-admissible then $\Omega_h \subset C^{0,2}(\mathbb{R}\times\mathbb{R},\mathbb{R})$, and the continuous map $\mathfrak{h}_{xx}(\omega,x) := \omega_{xx}(0,x)$ is the second derivative of $\mathfrak{h}$ with respect to x. The proof of these properties can be found in [44, Theorem I.3.1] and [43, Theorem IV.3]. Note that (Ω_h,σ_h) is a *transitive flow*, i.e., there exists a dense σ_h-orbit: that of the point $h \in \Omega_h$. More precisely, if Ω_h^{α} and Ω_h^{ω} are the α-limit set and ω-limit set of the element $h \in \Omega_h$, then

Lemma 2.4. $\Omega_h = \Omega_h^{\alpha} \cup \{h{\cdot}t \mid t \in \mathbb{R}\} \cup \Omega_h^{\omega}$.

Proof. We can write any $\omega \in \Omega_h$ as $\omega = \lim_{n\to\infty} h{\cdot}t_n$ in the compact-open topology for a suitable sequence (t_n). If a subsequence (t_k) has limit $-\infty$ or $+\infty$, then ω belongs to Ω_h^{α} or Ω_h^{ω}. Otherwise, there exists a subsequence (t_k) with limit $t_0 \in \mathbb{R}$, and it is easy to check that $\omega = h{\cdot}t_0$. □

The map h is *recurrent* if (Ω_h,σ_h) is a *minimal flow*, i.e., if every σ_h-orbit is dense in Ω_h.

Assume that h is (at least) C^1-admissible, and let us call τ_h the skewproduct flow defined on $\Omega_h\times\mathbb{R}$ by the family of equations (2.3) corresponding to the constructed function $\mathfrak{h}$. Note that this family includes (2.5): it is given by the element $\omega = h \in \Omega_h$, and, if $\tau_h(t,\omega,x) = (\omega{\cdot}t, v_h(t,\omega,x))$, then $x_h(t,s,x) = v_h(t-s,h{\cdot}s,x)$. This is the *skewproduct flow induced by* h *on its hull*. The next basic result will be used in Sections 4 and 6.

Proposition 2.5. *Let* h *be* C^1*-admissible and let* Ω_h *be its hull. If* $x' = h(t,x)$ *has a bounded solution* b *(resp.* n *uniformly separated solutions* $b_1 < b_2 < \ldots < b_n$*), then* $x' = \omega(t,x)$ *has a bounded solution (resp.* n *uniformly separated solutions) for all* $\omega \in \Omega_h$*.*

Proof. We write $\omega = \lim_{n\to\infty} h{\cdot}t_n$ in the compact-open topology for a sequence (t_n). Let x_0 be the limit of a suitable subsequence $(b(t_k))$ of $(b(t_n))$. Then, the solution $v_h(t,\omega,x_0)$ of $x' = \omega(t,x)$ (with value x_0 at 0) is bounded, since $v_h(t,\omega,x_0) = \lim_{k\to\infty} b(t+t_k)$. The same argument proves the other assertion. □

In what follows, we will consider both processes and skewproduct flows. Observe that $(t,x)\mapsto \mathfrak{h}(\omega{\cdot}t,x)$ is C^1-admissible for all $\omega\in\Omega$ if $\mathfrak{h}\in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$.

Proposition 2.6. (i) *Let $\tilde b(t)$ be an attractive (resp. repulsive) hyperbolic solution of (2.5) for $h\in C^{0,1}(\mathbb{R}\times\mathbb{R},\mathbb{R})$. Then, $\inf_{t<t_0}|\tilde b(t)-\bar x(t)|>0$ (resp. $\inf_{t>t_0}|\tilde b(t)-\bar x(t)|>0$) for any $t_0\in\mathbb{R}$ and any solution $\bar x(t)\neq\tilde b(t)$ defined on $(-\infty,t_0]$ (resp on $[t_0,\infty)$).*

(ii) *Let the family (2.3) be given by $\mathfrak{h}\in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$, and assume that the α-limit set (resp. ω-limit set) of $(\bar\omega,b_0)$ is an attractive (resp. repulsive) hyperbolic copy of the α-limit set $\Omega^{\alpha}_{\bar\omega}$ (resp. ω-limit set $\Omega^{\omega}_{\bar\omega}$) of $\bar\omega$, say $\{\tilde{\mathfrak{b}}\}$. Then, $\{\tilde{\mathfrak{b}}\}$ does not intersect the α-limit set (resp. ω-limit set) of any $(\bar\omega,x)$ with $x\neq b_0$ and bounded backward semiorbit (resp. bounded forward semiorbit).*

Proof. (i) We reason in the attractive case, assuming for contradiction the existence of $(t_n)\downarrow-\infty$ such that $\lim_{n\to\infty}|\tilde b(t_n)-\bar x(t_n)|=0$. According to the First Approximation Theorem (see [22, Theorem III.2.4] and [19, Proposition 2.1]), the attractive hyperbolicity of $\tilde b$ provides $k\ge1$ and $\gamma>0$ such that, for large enough n,

$$|\tilde b(t_0)-\bar x(t_0)| = |x_h(t_0,t_n,\tilde b(t_n)) - x_h(t_0,t_n,\bar x(t_n))| \le k\,e^{-\gamma(t_0-t_n)}|\tilde b(t_n)-\bar x(t_n)|\,.$$

The contradiction follows, since the last term tends to 0 as $n\to\infty$.

(ii) We reason in the attractive case. Assume the existence of the α-limit set $\mathcal{K}$ of a point $(\bar\omega,x)$ with $x\neq b_0$, and, for contradiction, the existence of $(\omega,\tilde{\mathfrak{b}}(\omega))\in\mathcal{K}$. We write $(\omega,\tilde{\mathfrak{b}}(\omega)) = \lim_{n\to\infty}(\bar\omega{\cdot}t_n, v(t_n,\bar\omega,x))$ for a suitable sequence $(t_n)\downarrow-\infty$, assume without restriction the existence of $\lim_{n\to\infty}v(t_n,\bar\omega,b_0)$, observe that this limit is also $\tilde{\mathfrak{b}}(\omega)$, and note that this contradicts (i). □

Proposition 2.7. *Let $\mathfrak{h}\in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$, and let $\tilde{\mathfrak{b}}\colon\Omega\to\mathbb{R}$ determine an attractive (resp. repulsive) copy of the base for (2.3). For any $\omega\in\Omega$, the function $\tilde b_\omega$ defined by $\tilde b_\omega(t):=\tilde{\mathfrak{b}}(\omega{\cdot}t)$ is an attractive (resp. repulsive) hyperbolic solution of $(2.3)_\omega$.*

Proof. Let us reason in the attractive case, fixing $\omega\in\Omega$. Let us define $\omega^*(t,x):=\mathfrak{h}(\omega{\cdot}t,x)$ and v as in (2.4). Then, the solution $x_\omega(t,s,x)$ of $x'=\omega^*(t,x)$ (i.e., of $(2.3)_\omega$), coincides with $v(t-s,\omega{\cdot}s,x)$, and $\tilde b_\omega(t)=v(t-s,\omega{\cdot}s,\tilde b_\omega(s))$. Hence, the hyperbolicity of $\tilde{\mathfrak{b}}$ ensures that, if $|x-\tilde b_\omega(s)|\le\delta$ for an $s\in\mathbb{R}$, then $x_\omega(t,s,x)$ exists for all $t\ge s$ and it satisfies $|x_\omega(t,s,x)-\tilde b_\omega(t)|\le ke^{-\gamma(t-s)}|x-\tilde b_\omega(s)|$. Therefore, $(\partial/\partial x)x_\omega(t,s,x)|_{x=\tilde b_\omega(s)} = \lim_{\varepsilon\to0}\big(x_\omega(t,s,\tilde b_\omega(s)+\varepsilon)-x_\omega(t,s,\tilde b_\omega(s))\big)/\varepsilon \le ke^{-\gamma(t-s)}$. This derivative solves the variational equation $z'=(\omega^*)_x(t,\tilde b_\omega(t))\,z$ and has value 1 at $t=s$, from where the assertion follows. □

2.4. **Lyapunov exponents.** Let $\mathcal{K}\subset\Omega\times\mathbb{R}$ be τ-invariant compact set projecting onto Ω, and let $\mathfrak{M}_{\rm inv}(\mathcal{K},\tau)$ and $\mathfrak{M}_{\rm erg}(\mathcal{K},\tau)$ be the (nonempty) sets of the τ-invariant and τ-ergodic measures on $\mathcal{K}$. A value $\gamma\in\mathbb{R}$ is a *Lyapunov exponent of* $\mathcal{K}$ if there exists $(\omega,x)\in\mathcal{K}$ such that $\gamma=\lim_{t\to\pm\infty}(1/t)\int_0^t\mathfrak{h}_x(\tau(r,\omega,x))\,dr$. In this case, there exists $\nu\in\mathfrak{M}_{\rm inv}(\mathcal{K},\tau)$ such that $\gamma=\int_{\mathcal{K}}\mathfrak{h}_x(\omega,x)\,d\nu$: this fact can be deduced from Riesz Representation Theorem and Kryloff-Bogoliuboff's Theorem. In addition, Birkhoff's Ergodic Theorem ensures that $\gamma(\mathcal{K},\nu) := \int_{\mathcal{K}} \mathfrak{h}_x(\omega,x)\,d\nu$ is a Lyapunov exponent of $\mathcal{K}$ for each $\nu \in \mathfrak{M}_{\text{erg}}(\mathcal{K},\tau)$. Since the ergodic measures are the extremal points in the set of invariant measures, the upper and lower Lypaunov exponents of $\mathcal{K}$ are $\gamma(\mathcal{K},\nu^u)$ and $\gamma(\mathcal{K},\nu^l)$ for suitable measures $\nu^u,\nu^l \in \mathfrak{M}_{\text{erg}}(\mathcal{K},\tau)$. According to [20, Theorem 4.1] and [7, Theorem 1.8.4], if $\nu \in \mathfrak{M}_{\text{erg}}(\mathcal{K},\tau)$ projects onto $m \in \mathfrak{M}_{\text{erg}}(\Omega,\sigma)$, then there exists an m-measurable τ-equilibrium $\mathfrak{b}\colon \Omega \to \mathbb{R}$ such that $\gamma(\mathcal{K},\nu) = \int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}(\omega))\,dm$. Therefore, there exists $m^l, m^u \in \mathfrak{M}_{\text{erg}}(\Omega,\sigma)$, an m^l-measurable equilibrium $\mathfrak{b}^s\colon \Omega \to \mathbb{R}$ and an m^u-measurable equilibrium $\mathfrak{b}^u\colon \Omega \to \mathbb{R}$ such that the lower and upper Lyapunov exponents of $\mathcal{K}$ are given by $\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}^l(\omega))\,dm^l$ and $\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}^l(\omega))\,dm^l$, respectively. Finally, if $m \in \mathfrak{M}_{\text{erg}}(\Omega,\sigma)$ and $\mathfrak{b}\colon \Omega \to \mathbb{R}$ is an m-measurable τ-equilibrium with graph in $\mathcal{K}$, then $\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}(\omega))\,dm$ is one of the Lyapunov exponents of $\mathcal{K}$.

Theorem 2.8. *Let $\mathcal{K} \subset \Omega \times \mathbb{R}$ be a τ-invariant compact set projecting onto Ω. Assume that its upper and lower equilibria coincide (at least) on a point of each minimal subset $\mathcal{M} \subseteq \Omega$. Then, all the Lyapunov exponents of $\mathcal{K}$ are strictly negative (resp. positive) if and only if $\mathcal{K}$ is an attractive (resp. repulsive) hyperbolic copy of the base.*

In addition, if either $\mathcal{K}$ (and hence Ω) is minimal or its upper and lower equilibria coincide on a τ-invariant subset $\Omega_0 \subseteq \Omega$ with $m(\Omega_0) = 1$ for all $m \in \mathfrak{M}_{\text{erg}}(\Omega,\sigma)$, then the condition on its upper and lower equilibria holds.

Proof. We reason in the attractive case. Let $\mathfrak{l}$ and $\mathfrak{u}$ be the lower and upper equilibria of $\mathcal{K}$. We take $(\omega,\mathfrak{l}(\omega)) \in \mathcal{K}$, a point ω_0 in a minimal subset of the $\boldsymbol{\alpha}$-limit set of ω with $\mathfrak{l}(\omega_0) = \mathfrak{u}(\omega_0)$, and a sequence $(t_n) \downarrow -\infty$ with $(\omega_0,\mathfrak{l}(\omega_0)) = \lim_{n\to\infty} \tau(t_n,\omega,\mathfrak{l}(\omega))$ and such that there exists $(\omega_0,x_0) := \lim_{n\to\infty} \tau(t_n,\omega,\mathfrak{u}(\omega))$. Then, $\mathfrak{l}(\omega_0) \le x_0 \le \mathfrak{u}(\omega_0) = \mathfrak{l}(\omega_0)$; i.e., $x_0 = \mathfrak{l}(\omega_0)$. This property allows us to repeat the proof [10, Proposition 2.8] in the attractive case: just replace the points (ω_1,x_1) and (ω_1,x_2) of that proof by $(\omega,\mathfrak{l}(\omega))$ and $(\omega,\mathfrak{u}(\omega))$. For the repulsive case, we work with the $\boldsymbol{\omega}$-limit set.

The last assertion is proved in [10, Section 2.4] in the minimal case. In the other one, it follows from the existence of a measure $m \in \mathfrak{M}_{\text{erg}}(\Omega,\sigma)$ concentrated on each minimal set. □

3. The concave and nonlinear case

The main purpose in this section is to extend previous results on families of equations on which hypotheses on concavity were globally assumed (as in [5, 34, 28, 30, 17]) to a significantly less restrictive setting, on which the concavity hypotheses are assumed just in measure.

Let (Ω,σ) be a global continuous real flow on a compact metric space, and let us consider the family of scalar ordinary differential equations

$$x' = \mathfrak{h}(\omega{\cdot}t, x) \tag{3.1}$$

for $\omega \in \Omega$, where $\mathfrak{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies (all or part of) the next conditions:

c1 $\mathfrak{h} \in C^{0,1}(\Omega \times \mathbb{R}, \mathbb{R})$,
c2 $\limsup_{x\to\pm\infty} \mathfrak{h}(\omega,x) < 0$ uniformly on Ω,
c3 $m(\{\omega \in \Omega \,|\, x \mapsto \mathfrak{h}(\omega,x) \text{ is concave}\}) = 1$ for all $m \in \mathfrak{M}_{\text{erg}}(\Omega,\sigma)$,

c4 $m(\{\omega \in \Omega \mid x \mapsto \mathfrak{h}_x(\omega, x)$ is strictly decreasing on $\mathcal{J}\}) > 0$ for all compact interval $\mathcal{J} \subset \mathbb{R}$ and all $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$.

Recall that $\mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ is the (nonempty) set of σ-ergodic measures on Ω. To simplify language, under these conditions we will say that (3.1) is a *family of concave ordinary differential equations*, although the concavity of the map $x \mapsto \mathfrak{h}(\omega, x)$ is not required for all $\omega \in \Omega$. Note also that the words do not make reference to the coercive character of the equation, although it is required.

Remark 3.1. Let $\Omega_0 \subset \Omega$ be a nonempty compact σ-invariant subset. Then, any $m_0 \in \mathfrak{M}_{\mathrm{erg}}(\Omega_0, \sigma)$ can be extended to $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ by $m(\mathcal{U}) = m_0(\mathcal{U} \cap \Omega_0)$. So, if $\mathfrak{h}$ satisfies **cj** for $\mathbf{j} \in \{1, 2, 3, 4\}$, also the restriction $\mathfrak{h} \colon \Omega_0 \times \mathbb{R} \to \mathbb{R}$ satisfies **cj**.

Let τ be the skewproduct flow defined by (2.4), with $\tau(t, \omega, x) = (\omega{\cdot}t, v(t, \omega, x))$. Before working under the coercive property **c2**, we want to explain some consequences of the concavity assumptions **c3** and **c4**, fundamental in what follows. They are based on the next fundamental proposition, that establishes conditions under which two different bounded ordered m-measurable τ-equilibria give rise to two Lyapunov exponents with different signs on two compact τ-invariant sets (or only one) containing their graphs (see Section 2.4). Its proof is based on that of [18, Theorem 4.1].

Proposition 3.2. *Let* $\mathfrak{h} \colon \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfy* **c1***, let us fix* $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$*, and let* $\mathfrak{b}_1, \mathfrak{b}_2 \colon \Omega \to \mathbb{R}$ *be bounded* m*-measurable* τ*-equilibria with* $\mathfrak{b}_1(\omega) < \mathfrak{b}_2(\omega)$ *for* m*-a.e.* $\omega \in \Omega$*. Assume that* $m(\{\omega \in \Omega \mid x \mapsto \mathfrak{h}(\omega, x)$ *is concave*$\}) = 1$ *and* $m(\{\omega \in \Omega \mid \mathfrak{h}_x(\omega, \mathfrak{b}_1(\omega)) > \mathfrak{h}_x(\omega, \mathfrak{b}_2(\omega))\}) > 0$*. Then,*

$$\int_\Omega \mathfrak{h}_x(\omega, \mathfrak{b}_1(\omega))\, dm > 0 \quad \text{and} \quad \int_\Omega \mathfrak{h}_x(\omega, \mathfrak{b}_2(\omega))\, dm < 0\,.$$

In particular, there are at most two bounded m*-measurable* τ*-equilibria which are strictly ordered* m*-a.e.*

Proof. We call $\Omega^c := \{\omega \in \Omega \mid x \mapsto \mathfrak{h}(\omega, x)$ is concave$\}$, which satisfies $m(\Omega^c) = 1$, and $\Omega_0 := \{\omega \in \Omega \mid \mathfrak{b}_1(\omega) < \mathfrak{b}_2(\omega)\}$, which is σ-invariant (since $\mathfrak{b}_i(\omega{\cdot}t) = v(t, \omega, \mathfrak{b}_i(\omega))$ for $i \in \{1, 2\}$) and with $m(\Omega_0) = 1$. For each $\omega \in \Omega^c$, we represent by $a_i(\omega, x_1, x_2)$ the expression $a_i(x_1, x_2)$ of (2.1) associated to the concave map $x \mapsto \mathfrak{h}(\omega, x)$ and observe that $(x_1, x_2) \mapsto a_i(\omega, x_1, x_2)$ is continuous on $\mathbb{R}^2$ for every $\omega \in \Omega^c$: see (2.2). For $i \in \{1, 2\}$, we define $a_i^* \colon \Omega \to \mathbb{R}$ by $a_i^*(\omega) := a_i(\omega, \mathfrak{b}_1(\omega), \mathfrak{b}_2(\omega))$ if $\omega \in \Omega^c \cap \Omega_0$ and $a_i^*(\omega) := 0$ otherwise, and observe that a_i^* is m-measurable and that $a_i^* \ge 0$ (see Proposition 2.2(i)). Let us take $i = 1$ and write

$$\mathfrak{h}_x(\omega{\cdot}t, \mathfrak{b}_1(\omega{\cdot}t)) - a_1^*(\omega{\cdot}t) = \frac{\mathfrak{h}(\omega{\cdot}t, \mathfrak{b}_2(\omega{\cdot}t)) - \mathfrak{h}(\omega, \mathfrak{b}_1(\omega{\cdot}t))}{\mathfrak{b}_2(\omega{\cdot}t) - \mathfrak{b}_1(\omega{\cdot}t)} = \frac{\mathfrak{b}_2'(\omega{\cdot}t) - \mathfrak{b}_1'(\omega{\cdot}t)}{\mathfrak{b}_2(\omega{\cdot}t) - \mathfrak{b}_1(\omega{\cdot}t)}$$

for $\omega \in \Omega_0$, where $\mathfrak{b}_i'(\omega{\cdot}t)$ is the derivative of $t \mapsto \mathfrak{b}_i(\omega{\cdot}t)$. This yields

$$\frac{1}{t}\int_0^t \mathfrak{h}_x(\omega{\cdot}s, \mathfrak{b}_1(\omega{\cdot}s))\, ds = \frac{1}{t}\int_0^t a_1^*(\omega{\cdot}s)\, ds + \frac{1}{t}\log\left(\frac{\mathfrak{b}_2(\omega{\cdot}t) - \mathfrak{b}_1(\omega{\cdot}t))}{\mathfrak{b}_2(\omega) - \mathfrak{b}_1(\omega)}\right). \quad (3.2)$$

Lusin's Theorem provides a compact subset $\Delta \subset \Omega^c$ with $m(\Delta) > 0$ such that $\mathfrak{b}_1|_\Delta, \mathfrak{b}_2|_\Delta \colon \Delta \to \mathbb{R}$ are continuous. Since $\mathfrak{h}_x(\cdot, \mathfrak{b}_1(\cdot))$ is bounded and $a_1^*(\cdot)$ is nonnegative, Birkhoff's Ergodic Theorem (see [14, Theorem 1 in Section 1.2] and

[25, Proposition 1.4]) ensures the existence of a σ-invariant subset $\Omega_0^*\subseteq\Omega_0$ with $m(\Omega_0^*)=1$ such that, for every $\omega\in\Omega_0^*$,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathfrak{h}_x\big(\omega{\cdot}s,\mathfrak{b}_1(\omega{\cdot}s)\big)\,ds=\int_\Omega \mathfrak{h}_x\big(\omega,\mathfrak{b}_1(\omega)\big)\,dm\in\mathbb{R}\,,$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t a_1^*(\omega{\cdot}s)\,ds=\int_\Omega a_1^*(\omega)\,dm\in[0,\infty]\,,$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \chi_\Delta(\omega{\cdot}s)\,ds=m(\Delta)>0\,.$$

In particular, if $\omega\in\Omega_0^*$, there exists a sequence $(t_n)\uparrow\infty$ such that $\omega{\cdot}t_n\in\Delta$. Hence, the sequence $\{\log((\mathfrak{b}_2(\omega{\cdot}t_n)-\mathfrak{b}_1(\omega{\cdot}t_n))/(\mathfrak{b}_2(\omega)-\mathfrak{b}_1(\omega))\}_{n\in\mathbb{N}}$ is bounded. We write (3.2) for $t=t_n$ and take limit as $n\to\infty$ to get $\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}_1(\omega))\,dm=\int_\Omega a_1^*(\omega)\,dm$. So, $\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}_1(\omega))\,dm>0$ follows from $\int_\Omega a_1^*(\omega)\,dm>0$. To prove this last inequality, we deduce from Proposition 2.2(i) that $a_1^*(\omega)>0$ if and only if $\mathfrak{h}_x(\omega,\mathfrak{b}_1(\omega))>\mathfrak{h}_x(\omega,\mathfrak{b}_2(\omega))$, and hence use the last hypothesis on $\mathfrak{h}_x$ to get $m(\{\omega\in\Omega^c\colon\, a_1^*(\omega)>0\})>0$. An analogous argument proves that $\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}_2(\omega))\,dm<0$. The last assertion is an easy consequence of the previous ones. □

Theorem 3.3. *Let $\mathfrak{h}$ satisfy* **c1**, **c3** *and* **c4**. *Then, there exist two disjoint and ordered τ-invariant compact sets $\mathcal{K}_1<\mathcal{K}_2$ projecting onto Ω if and only if there exist two different hyperbolic copies of the base $\{\tilde{\mathfrak{r}}\}$ and $\{\tilde{\mathfrak{a}}\}$ with $\tilde{\mathfrak{r}}<\tilde{\mathfrak{a}}$. In this case, $\mathcal{K}_1=\{\tilde{\mathfrak{r}}\}$ and it is repulsive; $\mathcal{K}_2=\{\tilde{\mathfrak{a}}\}$ and it is attractive; and $\mathcal{B}:=\{(\omega,x)\in\Omega\times\mathbb{R}\,|\,\tilde{\mathfrak{r}}(\omega)\le x\le\tilde{\mathfrak{a}}(\omega)\}$ is the set of globally bounded orbits. In particular, there are at most two disjoint and ordered τ-invariant compact sets projecting onto Ω.*

Proof. Sufficiency is obvious. To check necessity, we observe that $\mathfrak{h}$ satisfies the conditions of Proposition 3.2 for any $m\in\mathfrak{M}_{\text{erg}}(\Omega,\sigma)$ and any pair of bounded ordered m-measurable equilibria. This result ensures that all the Lyapunov exponents of $\mathcal{K}_1$ are positive and all the Lyapunov exponents of $\mathcal{K}_2$ are negative: the lower one of $\mathcal{K}_1$ is given by $\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}(\omega))\,dm$ for $m\in\mathfrak{M}_{\text{erg}}(\Omega,\sigma)$ and an m-measurable τ-equilibrium $\mathfrak{b}$ which is strictly smaller than the (m-measurable) lower τ-equilibrium of $\mathcal{K}_2$, so that Proposition 3.2 ensures that it is positive; and the other property is proved similarly. The last assertion in Proposition 3.2 ensures that the upper and lower equilibria of $\mathcal{K}_i$ coincide on a τ-invariant set with $m(\Omega_0)=1$ for all $m\in\mathfrak{M}_{\text{erg}}(\Omega,\sigma)$ for $i\in\{1,2\}$, and hence Theorem 2.8 ensures that $\mathcal{K}_1$ is a repulsive hyperbolic copy of Ω and $\mathcal{K}_2$ is an attractive hyperbolic copy of Ω. Observe that this fact precludes the existence of more than two disjoint and ordered τ-invariant compact sets projecting onto Ω.

Let us write $\mathcal{K}_1=\{\tilde{\mathfrak{r}}\}$ and $\mathcal{K}_2=\{\tilde{\mathfrak{a}}\}$ for continuous maps $\tilde{\mathfrak{r}},\tilde{\mathfrak{a}}\colon\Omega\to\mathbb{R}$. Clearly, $\bigcup_{\omega\in\Omega}(\{\omega\}\times[\tilde{\mathfrak{r}}(\omega),\tilde{\mathfrak{a}}(\omega)])\subseteq\mathcal{B}$. To prove the converse inclusion, we assume for contradiction the existence of (ω_0,x_0) with $x_0>\tilde{\mathfrak{a}}(\omega_0)$ and with globally defined and bounded τ-orbit. Then, the α-limit set $\mathcal{K}$ of this orbit exists and is a τ-invariant compact set projecting onto a compact set $\Omega_\mathcal{K}\subseteq\Omega$. Since $\{\tilde{\mathfrak{a}}\}$ is attractive, Proposition 2.6(ii) restricted to $\Omega_\mathcal{K}$ (see Remark 3.1) ensures that $\mathcal{K}>\mathcal{K}_2|_{\Omega_\mathcal{K}}>\mathcal{K}_1|_{\Omega_\mathcal{K}}$, which contradicts the last assertion of the previous paragraph. A similar argument working with ω-limit sets shows that $x_0\ge\tilde{\mathfrak{r}}(\omega_0)$ for all $(\omega_0,x_0)\in\mathcal{B}$. □

We say that *there exists an attractor-repeller pair* $(\tilde{\mathfrak{a}},\tilde{\mathfrak{r}})$ *of copies of the base* (or *of* Ω) *for* (3.1), or that $(\tilde{\mathfrak{a}},\tilde{\mathfrak{r}})$ *is an attractor-repeller pair of copies of the base for*

(3.1), if $\{\tilde{\mathfrak{a}}\}$ is an attractive hyperbolic copy of Ω and $\{\tilde{\mathfrak{r}}\}$ is a repulsive hyperbolic copy of Ω. So, Theorem 3.3 characterizes its existence under conditions **c1**, **c3** and **c4**, in which case, in addition, $\tilde{\mathfrak{r}} < \tilde{\mathfrak{a}}$.

Remark 3.4. Theorem 3.3 shows that, if there exists an attractor-repeller pair of copies of the base, then the set $\mathcal{B}$ of bounded τ-orbits is nonempty, bounded, and the pair is given by its upper and lower equilibria. Assume now that we previously know that $\mathcal{B}$ is nonempty and bounded, with $\mathcal{B} \subseteq \Omega \times \mathcal{J}_{\mathcal{B}}$ for a compact interval $\mathcal{J}_{\mathcal{B}} \subset \mathbb{R}$. Then all the conclusions of Theorem 3.3 apply if, for all $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$, $m(\{\omega \in \Omega \,|\, x \mapsto \mathfrak{h}(\omega, x) \text{ is concave}\}) = 1$ and $m(\{\omega \in \Omega \,|\, x \mapsto \mathfrak{h}_x(\omega, x) \text{ is strictly decreasing on } \mathcal{J}_{\mathcal{B}}\}) > 0$.

Let us now derive consequences of the coercivity property.

Proposition 3.5. *Let $\mathfrak{h}$ satisfy* **c1** *and* **c2***, and take $\delta > 0$ and $m_1 < m_2$ with $\mathfrak{h}(\omega, x) \le -\delta$ for all $\omega \in \Omega$ if $x \notin (m_1, m_2)$. Then,*

(i) $\liminf_{t\to(\alpha_{\omega,x})^+} v(t,\omega,x) > m_1$ *and* $\limsup_{t\to(\beta_{\omega,x})^-} v(t,\omega,x) < m_2$ *for any solution $v(t,\omega,x)$: any solution remains lower bounded as time decreases and upper bounded as time increases.*

(ii) *If $v(t,\omega,x)$ is bounded, then $v(t,\omega,x) \in [m_1, m_2]$ for all $t \in \mathbb{R}$: the set*
$$\mathcal{B} := \Big\{(\omega, x) \,|\, \sup_{t\in\mathbb{R}} |v(t,\omega,x)| < \infty\Big\}$$
is either empty or contained in $\Omega \times [m_1, m_2]$.

(iii) *If $\mathcal{B}$ is nonempty, then the projection Ω^b of $\mathcal{B}$ onto Ω is a σ-invariant compact set.*

(iv) *For each $\omega \in \Omega^b$, let us write $\mathcal{B}_\omega := \{x \,|\, (\omega, x) \in \mathcal{B}\} = [\mathfrak{r}(\omega), \mathfrak{a}(\omega)]$. Then, the maps $\mathfrak{r}, \mathfrak{a} \colon \Omega^b \to [m_1, m_2]$ are lower and upper semicontinuous equilibria for the restriction of τ to $\Omega^b \times \mathbb{R}$.*

(v) *If, for a point $\omega \in \Omega$, there exists a bounded C^1 function $b \colon \mathbb{R} \to \mathbb{R}$ such that $b'(t) \le \mathfrak{h}(\omega{\cdot}t, b(t))$ for all $t \in \mathbb{R}$, then $\omega \in \Omega^b$, and $\mathfrak{r}(\omega{\cdot}t) \le b(t) \le \mathfrak{a}(\omega{\cdot}t)$ for all $t \in \mathbb{R}$. If $b'(t) < \mathfrak{h}(\omega{\cdot}t, b(t))$ for all $t \in \mathbb{R}$, then $\mathfrak{r}(\omega{\cdot}t) < b(t) < \mathfrak{a}(\omega{\cdot}t)$ for all $t \in \mathbb{R}$.*

(vi) *If $\omega \in \Omega^b$, then $v(t,\omega,x)$ is bounded from below if and only if $x \ge \mathfrak{r}(\omega)$, and from above if and only if $x \le \mathfrak{a}(\omega)$.*

(vii) *Assume that $\mathfrak{h}$ satisfies also* **c3** *and* **c4***, and that $(\tilde{\mathfrak{a}}, \tilde{\mathfrak{r}}) := (\mathfrak{a}, \mathfrak{r})$ is an attractor-repeller pair of copies of the base. Then, $\lim_{t\to\infty}(v(t,\omega,x) - \tilde{\mathfrak{a}}(\omega{\cdot}t)) = 0$ if and only if $x > \tilde{\mathfrak{r}}(\omega)$, $\lim_{t\to-\infty}(v(t,\omega,x) - \tilde{\mathfrak{r}}(\omega{\cdot}t)) = 0$ if and only if $x < \tilde{\mathfrak{a}}(\omega)$, and $t \mapsto \tilde{\mathfrak{r}}(\omega{\cdot}t), \tilde{\mathfrak{a}}(\omega{\cdot}t)$ define the two unique hyperbolic solutions of* $(3.1)_\omega$.

Proof. The existence of m_1 and m_2 is ensured by property **c2**. The proofs of (i) and (ii) are classical exercises on ODEs. It is easy to check that $\mathcal{B}$ is closed, and hence compact, and clearly it is τ-invariant. Assertions (iii) and (iv) follow from here. The properties stated in (v) follow from (i) and standard comparison arguments: see e.g. the proof of [30, Theorem 3.1(v)]. Easy contradiction arguments using (i) prove (vi).

To check the first assertion in (vii), we first take $x > \tilde{\mathfrak{r}}(\omega)$, assume for contradiction that the ω-limit set of (ω, x) is not contained in $\{\tilde{\mathfrak{a}}\}$, deduce from Theorem 3.3 (restricted to the projection of the ω-limit set: see Remark 3.4) that it intersects $\{\tilde{\mathfrak{r}}\}$, and observe that this contradicts Proposition 2.6(ii). Conversely, if

$\lim_{t\to\infty}(v(t,\omega,x)-\tilde{\mathfrak{a}}(\omega{\cdot}t))=0$, then (vi) and the τ-invariance of $\{\tilde{\mathfrak{r}}\}$ ensure that $x>\tilde{\mathfrak{r}}(\omega)$. The same arguments prove the second assertion in (vii), and the last one follows from Propositions 2.7 and 2.6. □

Note that the previous property (iv) ensures that $\mathfrak{r}$ and $\mathfrak{a}$ are m-measurable equilibria for all $m\in\mathfrak{M}_{\rm erg}(\Omega^b,\sigma)$. We will use this property when we apply Proposition 3.2 to these equilibria.

The last result in this section characterizes the existence of an attractor-repeller pair of copies of the base in terms of the existence of two uniformly separated hyperbolic solutions of a given equation when the base is constructed as the hull of that equation: see Section 2.3.

Theorem 3.6. *Let $\mathfrak{h}\colon\Omega\times\mathbb{R}\to\mathbb{R}$ satisfy* **c1**, **c2**, **c3** *and* **c4**. *Let us fix $\bar\omega\in\Omega$. Then, the following assertions are equivalent:*

- (a) *Equation $(3.1)_{\bar\omega}$ has two hyperbolic solutions.*
- (b) *Equation $(3.1)_{\bar\omega}$ has two uniformly separated hyperbolic solutions.*
- (c) *Equation $(3.1)_{\bar\omega}$ has two uniformly separated bounded solutions.*
- (d) *There exists an attractor-repeller pair $(\tilde{\mathfrak{a}},\tilde{\mathfrak{r}})$ of copies of the base for the restriction of the family (3.1) to the closure $\Omega_{\bar\omega}$ of $\{\bar\omega{\cdot}t\mid t\in\mathbb{R}\}$.*

In this case, $t\mapsto\tilde a(t):=\tilde{\mathfrak{a}}(\bar\omega{\cdot}t)$ and $t\mapsto\tilde r(t):=\tilde{\mathfrak{r}}(\bar\omega{\cdot}t)$ are the two unique uniformly separated solutions of $(3.1)_{\bar\omega}$, they are hyperbolic, and there are no more hyperbolic solutions. In addition, if $x_{\bar\omega}(t,s,x)$ is the solution of $(3.1)_{\bar\omega}$ with $x_{\bar\omega}(s,s,x)=x$, then: it is bounded if and only if $x\in[\tilde r(s),\tilde a(s)]$, $\lim_{t\to\infty}|x_{\bar\omega}(t,s,x)-\tilde a(t)|=0$ if and only if $x>\tilde r(s)$, and $\lim_{t\to-\infty}|x_{\bar\omega}(t,s,x)-\tilde r(t)|=0$ if and only if $x<\tilde a(s)$.

Proof. The statements after the equivalences follow from (d) and Proposition 3.5(vi) and (vii). We will check (b) ⇒ (c) ⇒ (d) ⇒ (a) ⇒ (b). Recall that the hypotheses on $\mathfrak{h}$ are also valid for its restriction to $\Omega_{\bar\omega}\times\mathbb{R}$: see Remark 3.1.

(b) ⇒ (c) ⇒ (d). Obviously, (b) implies (c). Now we assume (c) and observe that it ensures that the lower and upper bounded solutions, $r(t)$ and $a(t)$, are uniformly separated. We call $\delta:=\inf_{t\in\mathbb{R}}(a(t)-r(t))>0$. Let $\mathcal{K}_a$ be the closure of the τ-orbit of $(\bar\omega,a(0))$. It projects on $\Omega_{\bar\omega}$, and hence $\Omega_{\bar\omega}\subset\Omega^b$: there exist $\mathfrak{r}(\omega)$ and $\mathfrak{a}(\omega)$ for all $\omega\in\Omega_{\bar\omega}$. Let us check that $x_0\ge\mathfrak{r}(\omega_0)+\delta$ for all $(\omega_0,x_0)\in\mathcal{K}_a$. We write $(\omega_0,x_0)=\lim_{n\to\infty}(\bar\omega{\cdot}t_n,a(t_n))$ and assume without restriction the existence of $(\omega_0,x^0):=\lim_{n\to\infty}(\bar\omega{\cdot}t_n,r(t_n))$, which belongs to the (closed) set $\mathcal{B}$. If $x_0<\mathfrak{r}(\omega_0)+\delta$, then $x^0\le x_0-\delta<\mathfrak{r}(\omega_0)+\delta-\delta=\mathfrak{r}(\omega_0)$, impossible. Let us consider the restriction $\bar\tau$ of τ to $\Omega_{\bar\omega}\times\mathbb{R}$. Since any $\bar\tau$-equilibrium with graph in $\mathcal{K}_a$ is strictly above $\mathfrak{r}$, Proposition 3.2 shows that all the Lyapunov exponents of $\mathcal{K}_a$ are strictly negative, and that its upper and lower equilibria coincide on a σ-invariant set Ω_0 with $m_0(\Omega_0)=1$ for all $m_0\in\mathfrak{M}_{\rm erg}(\Omega_{\bar\omega},\sigma)$. Hence, Theorem 2.8, ensures that $\mathcal{K}_a$ is an attractive hyperbolic copy of $\Omega_{\bar\omega}$. This fact and the previous property ensure that $\mathcal{K}_a$ is strictly above the closure $\mathcal{K}_r$ of $\{(\omega,\mathfrak{r}(\omega))\mid\omega\in\Omega_{\bar\omega}\}$. Hence, Theorem 3.3 ensures that $\mathcal{K}_r$ is a repulsive hyperbolic copy of $\Omega_{\bar\omega}$: (d) holds.

(d) ⇒ (a) ⇒ (b). If (d) holds, then $t\mapsto\tilde{\mathfrak{a}}(\bar\omega{\cdot}t)$ and $t\mapsto\tilde{\mathfrak{r}}(\bar\omega{\cdot}t)$ are two hyperbolic solutions of $(3.1)_{\bar\omega}$ (see Proposition 2.7), which ensures (a). Let us assume (a), and let $\tilde x_1<\tilde x_2$ be the two hyperbolic solutions of $(3.1)_{\bar\omega}$. Let us first check that $\tilde x_1$ is repulsive, assuming for contradiction that it is attractive. We call $a(t):=\mathfrak{a}(\bar\omega{\cdot}t)$, with $\mathfrak{a}$ given by Proposition 3.5(iv). Proposition 2.6(i) ensures that $\delta:=\inf_{t\le 0}(a(t)-\tilde x_1(t))>0$. Let $\mathcal{M}_1$ and $\mathcal{M}_a$ be the $\boldsymbol{\alpha}$-limit sets of $(\bar\omega,\tilde x_1(0))$ and

$(\bar\omega, a(0))$, which project on the α-limit set $\Omega^\alpha_{\bar\omega} \subseteq \Omega_{\bar\omega}$ of $\bar\omega$. Repeating the argument of the previous paragraph, we check that $x_0 \le \mathfrak{a}(\omega_0) - \delta$ whenever $(\omega_0, x_0) \in \mathcal{M}_1$, and deduce that $\mathcal{M}_1$ is a repulsive copy of $\Omega^\alpha_{\bar\omega}$. Proposition 2.7 shows that any orbit in $\mathcal{M}_1$ corresponds to a repulsive hyperbolic solution. On the other hand, it is easy to check that any orbit in the α-limit set of the orbit of an attractive hyperbolic solution corresponds to an attractive hyperbolic solution. And these facts provide the sought-for contradiction, since a solution cannot be at the same time hyperbolic attractive and repulsive.

Hence, $\tilde x_1$ is repulsive. Proposition 2.6(i) provides $\delta > 0$ such that $\inf_{t\ge 0}(a(t) - \tilde x_1(t)) > \delta$. Let $\bar{\mathcal{M}}_1$ be the ω-limit set of $(\bar\omega, \tilde x_1(0))$, which projects on the ω-limit set $\Omega^\omega_{\bar\omega} \subseteq \Omega_{\bar\omega}$ of $\bar\omega$. Repeating again the arguments used to prove (c)$\Rightarrow$(d), we check that $x_0 \le \mathfrak{a}(\omega_0) - \delta$ whenever $(\omega_0, x_0) \in \bar{\mathcal{M}}_1$; and we deduce that $\bar{\mathcal{M}}_1$ is a repulsive copy of $\Omega^\omega_{\bar\omega}$. Hence, $\bar{\mathcal{M}}_1$ does not intersect the ω-limit set $\bar{\mathcal{M}}_2$ of $(\bar\omega, \tilde x_2(0))$: see Proposition 2.6(ii). So, we have $\bar{\mathcal{M}}_1 < \bar{\mathcal{M}}_2$. Theorem 5.3 applied to $\Omega^\omega_{\bar\omega} \times \mathbb{R}$ ensures that $\bar{\mathcal{M}}_2$ is an attractive hyperbolic copy of $\Omega^\omega_{\bar\omega}$, which according to Proposition 2.7 is only possible if $\tilde x_2$ is attractive. Proposition 2.6(i) ensures that the two solutions are uniformly separated. So, (b) holds. □

We will refer to the situation described by the equivalences of Theorem 3.6 as the *existence of an attractor-repeller pair of solutions of* $(3.1)_{\bar\omega}$.

4. Asymptotically concave transition equations

Let $g\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a C^1-admissible function. The hull construction described in Section 2.3 allows us to understand the σ-orbit of g, $\{g{\cdot}t \mid t \in \mathbb{R}\}$, which is dense in the hull Ω_g, as a connection between its α-limit set Ω^α_g and its ω-limit set Ω^ω_g. In fact, the hull Ω_g is the union of these three sets: see Lemma 2.4. Our goal in this section is to describe the dynamical possibilities of an "asymptotically concave" equation

$$x' = g(t, x) \tag{4.1}$$

under conditions which ensure that the families of equations defined over Ω^α_g (α-family) and Ω^ω_g (ω-family) satisfy the regularity, coercivity and strict concavity properties **c1**-**c4**, as well as the existence of attractor-repeller pairs of copies of the base for the α-family and the ω-family. Since the structures of these sets represent the past and future of g, we are understanding (4.1) as a transition between the α-limit and ω-limit families.

Proposition 2.5 precludes the existence of uniformly separated solutions of (4.1) unless all the equations of the α-family and the ω-family have uniformly separated solutions. Hence, to consider a transition scenario with interesting dynamical possibilities, it is reasonable to assume the existence of attractor-repeller pairs of solutions for all the (concave) equations of the limit families: see Theorem 3.6. We will achieve these properties by assuming the existence of strictly concave (in x) maps g_- and g_+ such that g and g_- (resp. g and g_+) form an asymptotic pair as $t \to -\infty$ (resp. as $t \to \infty$) in the common hull of g and g_- (resp. g and g_+). (The common hull of two admissible maps h_1 and h_2 is the compact metric space defined as the closure of $\{h_i{\cdot}t \mid i = 1, 2,\, t \in \mathbb{R}\}$ in the compact-open topology, and ω_1 and ω_2 form an asymptotic pair as $t \to \pm\infty$ if the distance from $\omega_1{\cdot}t$ to $\omega_2{\cdot}t$ tends to 0.) The required existence of these maps does not imply their uniqueness,

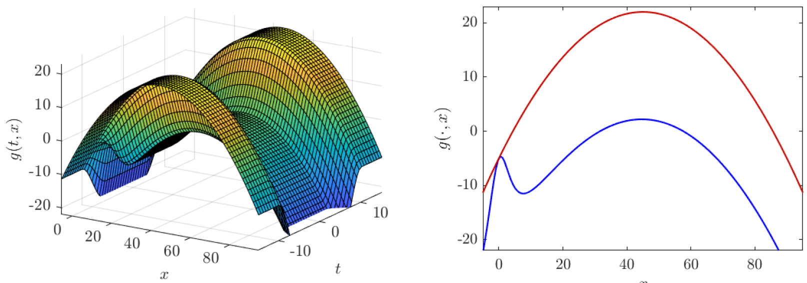

FIGURE 1. In the left panel, the surface $z = g(t,x)$ for $g(t,x) := 1.2\,x\,(1-x/90) - 5 - 20\,\Gamma(t)\,x^2/(20+x^2)$, and the plane $z = 0$. The map Γ is the unique C^1 cubic spline which takes value 1 on $[-5,5]$ and 0 outside $[-10,10]$: see Figure 2. It is easy to check that conditions **gc1**-**gc4** are satisfied by this map g and the maps $g_\pm(t,x) = g_\pm(x) := 1.2\,x\,(1-x/90)-5$, as well as the no concavity of the maps $x \mapsto g(t,x)$ for t close to 0: the right panel depicts the maps $x \mapsto g(0,x)$ (in red) and $x \mapsto g(t_0,x) = g_\pm(x)$ for $|t_0| \geq 10$ (in blue). It is clear that the limit equations $x' = g_\pm(x)$ have two hyperbolic constant solutions, given by the zeros of $g_\pm$. So, **gc5** also holds. The map g is a simplification from that giving rise to the model analyzed in Section 4.2, which has the same properties but a less clear graph.

but Lemma 4.2 below also shows that Ω_g^α and Ω_g^ω respectively coincide with $\Omega_{g_-}^\alpha$ and $\Omega_{g_+}^\omega$, which is a key point in our analysis.

So, we fix g and assume the existence of g_- and g_+ such that:

gc1 $g, g_-, g_+ \in C^{0,1}(\mathbb{R}\times\mathbb{R},\mathbb{R})$.
gc2 $\lim_{t\to\pm\infty}(g(t,x) - g_\pm(t,x)) = 0$ uniformly on each compact subset $\mathcal{J} \subset \mathbb{R}$.
gc3 $\limsup_{x\to\pm\infty} h(t,x) < 0$ uniformly on $\mathbb{R}$ for $h = g, g_-, g_+$.
gc4 $\inf_{t\in\mathbb{R}} \big((g_\pm)_x(t,x_1) - (g_\pm)_x(t,x_2)\big) > 0$ whenever $x_1 < x_2$.
gc5 Each one of the equations

$$x' = g_-(t,x) \quad \text{and} \quad x' = g_+(t,x) \tag{4.2}$$

has two hyperbolic solutions, $\tilde r_{g_-} < \tilde a_{g_-}$ and $\tilde r_{g_+} < \tilde a_{g_+}$.

Under these conditions we will say that (4.1) is an *(asymptotically) concave ordinary differential equation*. Observe that the concavity of the map $x \mapsto g(t,x)$ is not required for all $t \in \mathbb{R}$: see Figure 1. Note also that the coercive character of the equation is required without making explicit reference to it.

As Lemma 4.3 will prove, conditions **gc1**-**gc4** provide a setting satisfying the hypotheses of Section 3, that is, a family of concave ordinary differential equations (see Section 3).

Remarks 4.1. 1. Slightly abusing language, we will say that "g satisfies conditions **gc1**-**gc5**" if there exist g_- and g_+ such that all the listed conditions are satisfied.

2. To simplify the language, we will refer to (4.1) as a *transition equation* between the *past equation* and the *future equation*, which are the first one and the second one in (4.2). That the use of these words is accurate is partly justified by the previously mentioned equalities $\Omega_{g_-}^\alpha = \Omega_g^\alpha$ and $\Omega_{g_+}^\omega = \Omega_g^\omega$, which mean that the hyperbolic structures of the equations (4.2) condition that of (4.1) and viceversa; and it will be better justified by the main results of this section. But observe that the future of the dynamics of the nonautonomous equation $x' = g_-(t,x)$ is not necessarily related to its past (since $\Omega_{g_-}^\alpha$ can be different $\Omega_{g_-}^\omega$), and hence it can be not related

to the dynamics of $x' = g(t,x)$. And the same happens with the past dynamics of $x' = g_+(t,x)$ and $x' = g(t,x)$.

We will classify the dynamical scenarios for (4.1) and relate them to those of (4.2) under the above conditions, which include coercivity of all the involved equations (**gc3**) but strict concavity in x only of the limit ones (**gc4**). As said in the Introduction, several fundamental differences arise with respect to previous approaches, which, on the one hand, are restricted to maps $g(t,x) := f(t,x,\Gamma(t))$ and $g_\pm(t,x) := f(t,x,\gamma_\pm)$, where $\gamma_\pm := \lim_{t\to\pm\infty}\Gamma(t)$ are assumed to exist and be real; and, on the other hand, are analyzed under much more exigent concavity hypothesis. So, we will extend part of the results of [30], formulated for $x' = -(x-\Gamma(t))^2 + p(t)$ and with constant asymptotic limits of Γ, to a much more general setting. The problem is also analyzed in [28] for $x' = h(t, x-\Gamma(t))$ assuming (less restrictive) Carathéodory conditions on h and properties concerning its concavity with respect to the second variable and the asymptotic limits of Γ which are much stronger than those assumed here. Thus, the current formulation of our results considerably broadens their possibilities of application, as we will see in Section 4.2.

Theorem 4.7 provides the above mentioned classification of the dynamical possibilities for (4.1) when all the previous conditions hold. Its proof is strongly based on Theorem 3.6 and Proposition 4.6, and this last one also requires some previous work. Our first two results, fundamental for the subsequent application of Theorem 3.6, refer to the hull extensions (see Section 2.3). Recall that we represent by $x_h(t,s,x)$ the maximal solution of $x' = h(t,x)$ which satisfies $x_h(s,s,x) = x$: we will use this notation for h equal to g, g_-, g_+, and some other auxiliary admissible functions. In all these cases, the set Ω_h is the hull of h, and Ω_h^α and Ω_h^ω are the α-limit set and ω-limit set of the element $h \in \Omega_h$. Recall that $h{\cdot}t(s,x) := h(t+s,x)$, and that $\mathfrak{h}(\omega,x) := \omega(0,x)$ if $\omega \in \Omega_h$. We represent by $\mathfrak{g}$, $\mathfrak{g}_-$ and $\mathfrak{g}_+$ the extensions to the corresponding hulls of g, g_- and g_+. These auxiliary results do not need all the conditions **gc1**-**gc5**: we will specify the required ones.

Lemma 4.2. *Let g and $g_\pm$ satisfy* **gc1** *and* **gc2**. *Then, $\Omega_g^\alpha = \Omega_{g_-}^\alpha$ and $\Omega_g^\omega = \Omega_{g_+}^\omega$. Hence, $\Omega_g = \Omega_{g_-}^\alpha \cup \{g{\cdot}t \mid t\in\mathbb{R}\} \cup \Omega_{g_+}^\omega$.*

Proof. Given a sequence (t_n) with limit $\pm\infty$, it is easy to check that $\omega(t,x) = \lim_{n\to\infty} g(t+t_n,x)$ uniformly on the compact subsets of $\mathbb{R}\times\mathbb{R}$ if and only if $\omega(t,x) = \lim_{n\to\infty} g_\pm(t+t_n,x)$ uniformly on the compact subsets of $\mathbb{R}\times\mathbb{R}$. This proves the first equalities, which combined with Lemma 2.4 prove the last one. □

Lemma 4.3. *If $h \in C^{0,1}(\mathbb{R}\times\mathbb{R},\mathbb{R})$ then $\mathfrak{h}$ satisfies* **c1** *on Ω_h. If $h \in C^{0,1}(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $\limsup_{x\to\pm\infty} h(t,x) < 0$ uniformly on $\mathbb{R}$, then $\mathfrak{h}$ satisfies* **c2** *on Ω_h. And, if* **gc1**, **gc2** *and* **gc4** *hold, then $\mathfrak{g}$ and $\mathfrak{g}_\pm$ satisfy* **c3** *and* **c4** *on Ω_g and $\Omega_{g_\pm}$, respectively.*

Proof. As explained in Section 2.3, **c1** for $\mathfrak{h}$ follows from the C^1-admissibility of h. If, in addition, $\limsup_{x\to\pm\infty} h(t,x) < 0$ uniformly on $\mathbb{R}$, then there exists $\delta > 0$ and $\rho_\delta > 0$ such that $h(t,x) \le -\delta$ if $|x| \ge \rho_\delta$ and $t \in \mathbb{R}$. Since any $\omega \in \Omega_h$ satisfies $\omega(0,x) = \lim_{n\to\infty} h(t_n,x)$ for a sequence (t_n), we have $\mathfrak{h}(\omega,x) = \omega(0,x) \le -\delta$ if $|x| \ge \rho_\delta$: **c2** holds on Ω_h.

Now, we assume that **gc1**, **gc2** and **gc4** hold. To prove the last assertion, it is enough to reason with g, since g_- and g_+ satisfy the conditions assumed on g.

To check that $\mathfrak{g}$ satisfies **c3** and **c4** on Ω_g, we use the ideas of the proof of [19, Proposition 3.16]. Lemma 2.4 ensures that $\Omega_g = \Omega_g^{\alpha} \cup \{g{\cdot}t \mid t \in \mathbb{R}\} \cup \Omega_g^{\omega}$. In particular, given $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega_g, \sigma_g)$, $m(\Omega_g^{\alpha}) = 1$ or $m(\Omega_g^{\omega}) = 1$ (or both): this is trivial if g is independent of t or t-periodic (since $\Omega_g = \Omega_g^{\alpha} = \Omega_g^{\omega}$); and, in the remaining cases, $\{g{\cdot}t \mid t \in \mathbb{R}\} = \bigcup_{n\in\mathbb{Z}} \sigma_n(\{g{\cdot}t \mid t \in [0,1)\})$ (where $\sigma_n(\omega) = \omega{\cdot}n$) is a nonfinite union of disjoint sets. Therefore, $m(\sigma_n(\{g{\cdot}t \mid t \in [0,1)\})) = 0$ for all $n \in \mathbb{N}$, since this measure is independent of n. Hence, it suffices to check that $x \mapsto \mathfrak{g}_x(\omega, x)$ is strictly decreasing on $\mathbb{R}$ for all $\omega \in \Omega_g^{\alpha} \cup \Omega_g^{\omega}$: this ensures that $m(\{\omega \in \Omega_g \mid x \mapsto \mathfrak{g}_x(\omega, x) \text{ is strictly decreasing on } \mathbb{R}\}) = 1$ for all $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega_g, \sigma_g)$, which is stronger than **c3** and **c4**. We reason for $\omega \in \Omega_g^{\omega}$. According to Lemma 4.2, $\omega = \lim_{n\to\infty} g_+{\cdot}t_n$ (in the compact-open topology) for a sequence (t_n) with limit ∞. Then, ω_x is the limit of any subsequence of $((g_+)_x{\cdot}t_n)$ which uniformly converges on the compact subsets of $\mathbb{R}\times\mathbb{R}$, and hence $\omega_x = \lim_{n\to\infty}(g_+)_x{\cdot}t_n$ uniformly on the compact subsets of $\mathbb{R}\times\mathbb{R}$. We take $x_1 < x_2$, and apply **gc4** to get

$$\begin{aligned}\mathfrak{g}_x(\omega, x_1) - \mathfrak{g}_x(\omega, x_2) &= \omega_x(0, x_1) - \omega_x(0, x_2)\\ &= \lim_{n\to\infty}((g_+)_x(t_n, x_1) - (g_+)_x(t_n, x_2)) > 0\,,\end{aligned}$$

which completes the proof. □

Remark 4.4. Lemma 4.3 shows that $\mathfrak{g}_-$ satisfies **c1**, **c2**, **c3** and **c4** if g_- satisfies the conditions assumed on it on **gc1**, **gc3** and **gc4**. Hence, in this case, and according to Theorem 3.6, the property corresponding to g_- in condition **gc5** can be reformulated as: "the equation $x' = g_-(t,x)$ has an attractor-repeller pair of solutions $(\tilde{a}_{g_-}, \tilde{r}_{g_-})$", which determines its corresponding global dynamics: see Theorem 3.6. The same applies to g_+. We will use these facts without further reference.

The next result allows us to apply Theorem 2.3 in the proof of Proposition 4.6.

Lemma 4.5. *If g and $g_\pm$ satisfy* **gc1** *and* **gc2**, *then* $\lim_{t\to\pm\infty}(g_x(t,x) - (g_\pm)_x(t,x)) = 0$ *uniformly on each compact subset* $\mathcal{J} \subset \mathbb{R}$.

Proof. Let us reason for g_-, taking $(t_n) \downarrow -\infty$ and a compact subset $\mathcal{J} \subset \mathbb{R}$. Since $h_- := g - g_-$ is C^1-admissible, every subsequence (t_m) has a subsequence (t_k) such that there exists $d_-(x) := \lim_{k\to\infty}(h_-)_x(t_k, x)$ and is uniform on $\mathcal{J}$. We assume for contradiction that $d_- \not\equiv 0$ and, without restriction, that $d_-(x) \geq \varepsilon > 0$ for $x \in [x_1, x_2] \subseteq \mathcal{J}$. Then, $0 = \lim_{k\to\infty}(h_-(t_k, x_2) - h_-(t_k, x_1)) = \lim_{k\to\infty}\int_{x_1}^{x_2}(h_-)_x(t_k, s)\,ds = \int_{x_1}^{x_2} d_-(s)\,ds \geq \varepsilon(x_2 - x_1)$, which is impossible. □

Proposition 4.6. *Assume that g satisfies* **gc1-gc5**, *and let $(\tilde{a}_{g_\pm}, \tilde{r}_{g_\pm})$ be the attractor-repeller pairs of solutions of $x' = g_\pm(t,x)$ given by* **gc5**. *Then,*

(i) *there exists a unique solution a_g of (4.1) defined at least on a negative half-line and characterized by "$x > a_g(s)$ if and only if $x_g(t,s,x)$ is unbounded from above as time decreases". It satisfies $\lim_{t\to-\infty}(a_g(t) - \tilde{a}_{g_-}(t)) = 0$ and, if there exists $a_g(s)$, then $x < a_g(s)$ if and only if $\lim_{t\to-\infty}|x_g(t,s,x) - \tilde{r}_{g_-}(t)| = 0$. In addition, a_g is locally pullback attractive.*

(ii) *There exists a unique solution r_g of (4.1) defined at least on a positive half-line and characterized by "$x < r_g(s)$ if and only if $x_g(t,s,x)$ is unbounded from below as time increases". It satisfies $\lim_{t\to\infty}(r_g(t) - \tilde{r}_{g_+}(t)) = 0$ and, if there exists $r_g(s)$, then $x > r_g(s)$ if and only if $\lim_{t\to\infty}|x_g(t,s,x) - \tilde{a}_{g_+}(t)| = 0$. In addition, r_g is locally pullback repulsive.*

(iii) *There exists a bounded solution* $b\colon \mathbb{R} \to \mathbb{R}$ *of* (4.1) *if and only if* r_g *and* a_g *are globally defined, in which case* $r_g \le b \le a_g$.

(iv) *If* a_g *and* r_g *are bounded and different, then* $(\tilde a_g, \tilde r_g) := (a_g, r_g)$ *is an attractor-repeller pair of solutions for* (4.1). *In particular, this is the situation if: there exists* $s \in \mathbb{R}$ *such that* $a_g(s)$ *and* $r_g(s)$ *exist and satisfy* $r_g(s) < a_g(s)$; *or if* $a_g \ne r_g$ *and one of them is bounded.*

(v) *If* (4.1) *has no hyperbolic solutions, then it has at most one bounded solution* $a_g = r_g$.

Proof. If **gc1** and **gc3** hold, then there exist $m > 0$ and $\varepsilon > 0$ such that $g(t, \pm x) \le -\varepsilon$ and $g_\pm(t, \pm x) \le -\varepsilon$ for all $t \in \mathbb{R}$ if $x \ge m$. So, we can repeat the proofs of points (i) and (ii) of [30, Theorem 3.1], in order to proof that, if there are solutions which remain bounded as time decreases (resp. as time increases), then there exists the (possibly local) map $a_g\colon (-\infty, s_a) \to (-\infty, m]$, with $-\infty < s_a \le \infty$ (resp. $r_g\colon (s_r, \infty) \to [-m, \infty)$ with and $-\infty \le s_r < \infty$), characterized as in the first assertion of (i) (resp. of (ii)). The same arguments show that m is a bound for the absolute value of any bounded solution of $x' = g_\pm(t, x)$. Hence, $-m \le \tilde r_{g_\pm}(t) \le \tilde a_{g_\pm}(t) \le m$ for all $t \in \mathbb{R}$. We also point out that the characterizations of a_g and r_g combined with the existence of an upper bound for a_g and a lower bound for r_g prove (iii).

Now, we proceed as in the proof of [28, Theorem 3.4]. We detail it, since the scenario here is much more general, and some technical differences arise.

Let us take $\varepsilon > 0$. Since **gc1** holds, Theorem 2.3 provides $\delta_- = \delta_-(\varepsilon) > 0$ such that, if f is C^1-admissible and $\|g_- - f\|_{1,m} < \delta_-$, then $x' = f(t, x)$ has an attractor-repeller pair $(\tilde a_f, \tilde r_f)$ with $\|\tilde a_{g_-} - \tilde a_f\|_\infty \le \varepsilon$ and $\|\tilde r_{g_-} - \tilde r_f\|_\infty \le \varepsilon$. It also ensures the existence of a common dichotomy constant pair for all these hyperbolic solutions. We fix the same for both hyperbolic solutions: $(k_\varepsilon, \beta_\varepsilon)$.

We choose $t^- = t^-(\varepsilon) < 0$ such that $|g(t, x) - g_-(t, x)| < \delta_-/2$ and $|g_x(t, x) - (g_-)_x(t, x)| < \delta_-/2$ if $t \le t^-$ and $|x| \le m$ (see Lemma 4.5), and define $f_-(t, x)$ as $g(t, x)$ if $t < t^-$ and as $g_-(t, x) - g_-(t^-, x) + g(t^-, x)$ otherwise. It is easy to check that f_- is C^1-admissible and $\|g_- - f_-\|_{1,m} \le \delta_-$, and so $x' = f_-(t, x)$ has an attractor-repeller pair $(\tilde a_{f_-}, \tilde r_{f_-})$, with $\|\tilde a_{f_-} - \tilde a_{g_-}\|_\infty \le \varepsilon$ and $\|\tilde r_{f_-} - \tilde r_{g_-}\|_\infty \le \varepsilon$.

Let us now define $\hat a_{f_-}$ as the solution of $x' = g(t, x)$ with $\hat a_{f_-}(t^-) = \tilde a_{f_-}(t^-)$. We will check that $\hat a_{f_-} = a_g$. Since $\hat a_{f_-}(t) = \tilde a_{f_-}(t)$ for $t \le t^-$, it remains bounded as t decreases, which, as seen before, ensures that a_g exists and that $\hat a_{f_-} \le a_g$. To prove that $\hat a_{f_-} \ge a_g$, we take $x > \hat a_{f_-}(t^-)$ in order to check that $x_g(t, t^-, x)$ is unbounded as time decreases: Lemma 4.3 guarantees that the map $\mathfrak{f}_-$ defined on the hull of f_- satisfies the hypotheses of Proposition 3.5; hence, this result ensures that the solution $x_{f_-}(t, t^-, x)$ of $x' = f_-(t, x)$ is unbounded as time decreases; and the assertion follows from here and from $x_g(t, t^-, x) = x_{f_-}(t, t^-, x)$ for $t \le t^-$.

Note that we have proved that $\lim_{t\to-\infty}(a_g(t) - \tilde a_{g_-}(t)) = 0$. On the other hand, if $x < a_g(s)$, then there exists $t_0 < t^- = t^-(\varepsilon)$ such that $x_g(t_0, s, x) < a_g(t_0) = \tilde a_{f_-}(t_0)$. Since $x_g(t, s, x) = x_g(t, t_0, x_g(t_0, s, x))$ solves $x' = f_-(t, x)$ for $t \le t_0$, we conclude from Theorem 3.6 that $\lim_{t\to-\infty} |x_g(t, s, x) - \tilde r_{f_-}(t)| = 0$. Therefore, $|x_g(t, s, x) - \tilde r_{g_-}(t)| < 2\,\varepsilon$ if $t \le t^-(\varepsilon)$, which ensures that $\lim_{t\to-\infty} |x_g(t, s, x) - \tilde r_{g_-}(t)| = 0$. To check that a_g is locally pullback attractive (and so complete the proof of (i)), we observe that the attractive hyperbolicity of $\tilde a_{f_-}$ provides $\delta > 0$,

$k \geq 1$ and $\gamma > 0$ such that $|a_g(t) - x_g(t, s, a_g(s) \pm \delta)| = |\tilde{a}_{f_-}(t) - x_{f_-}(t, s, \tilde{a}_{f_-}(s) \pm \delta)| \leq k\,\delta\, e^{-\gamma(t-s)}$ if $s \leq t^-$ and $t \in [s, t^-]$ (see, e.g., [19, Proposition 2.1]).

Analogous arguments prove (ii). If a_g and r_g are bounded and different, then (iii) yields $r_g < a_g$, and hence the limiting properties established in (i) and (ii) prove that they are uniformly separated. Therefore, Theorem 3.6 proves that they form an attractor-repeller pair. The last assertions in (iv) follow from (i), (ii) and (iii), and the bounds $-m \leq r_g$ and $a_g \leq m$. Finally, (v) follows easily from (iv). □

Theorem 4.7. *Assume that g satisfies* **gc1**-**gc5***, let $(\tilde{a}_{g_\pm}, \tilde{r}_{g_\pm})$ be the attractor-repeller pairs of solutions of $x' = g_\pm(t, x)$ given by* **gc5***, and let a_g and r_g be the solutions of* (4.1) *provided by Proposition* 4.6*. Then, the dynamics of the transition equation* (4.1) *fits in one of the following dynamical scenarios:*

- Case A*: there exists an attractor-repeller pair of solutions $(\tilde{a}_g, \tilde{r}_g)$, with $\tilde{a}_g := a_g$ and $\tilde{r}_g := r_g$. In this case, $\lim_{t\to\pm\infty}(\tilde{r}_g(t) - \tilde{r}_{g_\pm}(t)) = 0$ and $\lim_{t\to\pm\infty}(\tilde{a}_g(t) - \tilde{a}_{g_\pm}(t)) = 0$.*
- Case B*: there are bounded solutions but no hyperbolic ones. In this case, $r_g = a_g$ is the unique bounded solution, and it is locally pullback attractive and repulsive.*
- Case C*: there are no bounded solutions.*

Proof. Assume that Case C does not hold; i.e., that there exists a bounded solution of (4.1). Proposition 4.6(iii) ensures that r_g and a_g are bounded. If they are different, point (iv) of Proposition 4.6 ensures that they form an attractor-repeller pair of solutions, and points (i) and (ii) yield the asymptotic behaviour described in Case A. If, on the contrary, $a_g = r_g$, then Proposition 4.6(iii) ensures that $a_g = r_g$ is the unique bounded solution. As stated in Theorem 2.3, its hyperbolicity would ensure its exponential asymptotic stability as time either increases or decreases, which contradicts either point (ii) or (i) of Proposition 4.6. Hence, Case B holds. □

Let us analyze part of the information provided by Proposition 4.6 and Theorem 4.7. In all the cases, the locally pullback attractive solution a_g of the transition equation "connects" with the attractive hyperbolic solution of the past equation as time decreases. The differences arise with its behavior in the future: in Case A, usually referred to as (*end-point*) *tracking*, $\tilde{a}_g := a_g$ also connects with the attractive hyperbolic solution of the future as time increases, while in Case C, of *tipping*, a_g is unbounded, and hence the connection is lost. In the extremely unstable Case B, a_g is still bounded but it connects with the repulsive hyperbolic solution of the future. The interested reader can in find [30, Figures 1-6] some drawings depicting the dynamical behavior in each one of these three cases. (There is a typo there: the graphs of Cases A and C are interchanged).

The next result provides a useful comparison criterion ensuring Case A.

Proposition 4.8. *Assume that g satisfies* **gc1**-**gc5***. If there exists a continuous map $h\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ such that $h(t, x) \leq g(t, x)$ for all $(t, x) \in \mathbb{R}\times\mathbb{R}$ and*

- *either $x' = h(t, x)$ has two different bounded solutions,*
- *or $x' = h(t, x)$ has a bounded solution which does not solve $x' = g(t, x)$,*

then (4.1) *is in* Case A*.*

Proof. In both cases, Proposition 3.5(v) ensures that (4.1) has two bounded solutions, so that Theorem 4.7 proves the assertion. □

As explained in the Introduction, a *critical transition* (or *tipping point*) occurs when a small variation on the external input of the equation causes a dramatic variation on the dynamics. We will focus on critical transitions associated to one-parametric families of equations which occur when the dynamics moves from Case A to Case C of Theorem 4.7 as the parameter crosses a particular *critical value*. Theorem 4.9 shows that, if the parametric variation is smooth enough, this transition means Case B for the critical value, and that these critical transitions can be understood as nonautonomous saddle-node bifurcations: they occur as a consequence of the collision of an attractive hyperbolic solution with a repulsive one as c varies. It also shows the persistence of Cases A and C.

Theorem 4.9. *Let $\mathcal{C} \subseteq \mathbb{R}$ be an open interval, and let $\bar{g}\colon \mathbb{R}\times\mathbb{R}\times\mathcal{C} \to \mathbb{R}$ be a map such that $g^c(t,x) := \bar{g}(t,x,c)$ satisfies* **gc1-gc5** *for all $c \in \mathcal{C}$. Let $\bar{g}_x$ be the partial derivative with respect to the second variable and assume that $\bar{g}$ and $\bar{g}_x$ are admissible on $\mathbb{R}\times\mathbb{R}\times\mathcal{C}$. Assume also that $\limsup_{x\to\pm\infty} \bar{g}(t,x,c) < 0$ uniformly on $\mathbb{R}\times\mathcal{J}$ for any compact interval $\mathcal{J} \subset \mathcal{C}$.*

(i) *Assume that there exist c_1, c_2 in $\mathcal{C}$ with $c_1 < c_2$ such that the dynamics of $x' = g^c(t,x)$ is in Case A for $c = c_1$ and not for $c = c_2$. If $c_0 := \inf\{c > c_1 \mid$ Case A does not hold$\}$, then $c_0 > c_1$. Let $(\tilde{a}_{g^c}, \tilde{r}_{g^c})$ be the attractor-repeller pair for $c \in [c_1, c_0)$. Then, the dynamics of $x' = g^{c_0}(t,x)$ is in Case B, and $\lim_{c\to c_0^-}(\tilde{a}_{g^c}(t) - \tilde{r}_{g^c}(t)) = 0$ for all $t \in \mathbb{R}$. The result is analogous if $c_1 > c_2$.*

(ii) *Assume that there exist c_3, c_4 in $\mathcal{C}$ with $c_3 < c_4$ such that the dynamics of $x' = g^c(t,x)$ is in Case C for $c = c_3$ and not for $c = c_4$. If $c_0 := \inf\{c > c_3 \mid$ Case C does not hold$\}$, then $c_0 > c_3$, and the dynamics of $x' = g^{c_0}(t,x)$ is in Case B. The result is analogous if $c_3 > c_4$.*

Proof. (i) The admissibility hypotheses ensure that, for $c \in \mathcal{C}$, $\rho > 0$ and $\delta > 0$ fixed, there exists $\varepsilon_0 > 0$ such that

$$\sup_{(t,x)\in\mathbb{R}\times[-\rho,\rho]} |g^c(t,x) - g^{c+\varepsilon}(t,x)| + \sup_{(t,x)\in\mathbb{R}\times[-\rho,\rho]} |g^c_x(t,x) - g^{c+\varepsilon}_x(t,x)| < \delta$$

if $|\varepsilon| \le \varepsilon_0$. Hence, Theorems 4.7 and 2.3 guarantee the persistence of Case A under small variations of c, which in turn ensures that $c_0 > c_1$ and that $x' = g^{c_0}(t,x)$ is not in Case A. (Note that the last condition on coercivity is not yet required.) On the other hand, the hypothesis on $\limsup_{x\to\pm\infty} \bar{g}(t,x,c)$ (which is stronger than "**gc3** for all c") ensures the existence of a constant $m > 0$ and $\delta > 0$ such that $g^c(t,x) \le -\delta$ if $t \in \mathbb{R}$, $|x| > m$, and $c \in \mathcal{C}$ is close enough to c_0. This fact allows us to reason as in the proof of Proposition 3.5(ii) in order to check that the lower and upper bounded solutions ($\tilde{r}_{g^c}$ and $\tilde{a}_{g^c}$) of $x' = g^c(t,x)$ are lower bounded by $-m$ and upper bounded by m for $c \in \mathcal{C}$ close enough to c_0. Hence, for a common sequence $(c_n) \uparrow c_0$, there exist $\bar{r}_0 := \lim_{n\to\infty} \tilde{r}_{g^{c_n}}(0)$ and $\bar{a}_0 := \lim_{n\to\infty} \tilde{a}_{g^{c_n}}(0)$. It is easy to deduce that the solutions of $x' = g^{c_0}(t,x)$ with values $\bar{r}_0$ and $\bar{a}_0$ at $t = 0$ are bounded. Hence we are in Case B, and both solutions coincide. It is easy to check that this unique bounded solution, b^{c_0}, satisfies $b^{c_0}(t) = \lim_{n\to\infty} \tilde{a}_{g^{c_n}}(t) = \lim_{n\to\infty} \tilde{r}_{g^{c_n}}(t)$ for all $t \in \mathbb{R}$, which combined with the uniqueness of b^{c_0} guarantees that $\lim_{c\to c_0^-}(\tilde{a}_{g^c}(t) - \tilde{r}_{g^c}(t)) = 0$ for all $t \in \mathbb{R}$, as asserted. It is clear that the argument can be repeated if $c_1 > c_2$.

(ii) We assume for contradiction the existence of $(c_n) \downarrow c_3$ such that there exists a bounded solution b^{c_n} for all n, and reason as before to conclude that $\bar{b}_3 :=$

$\lim_{n\to\infty} b^{c_n}(0)$ is finite and provides the value at 0 of a bounded solution for c_3, impossible. This shows the persistence of Case C under small variations in c, which in turn ensures that $c_0 > c_3$ and that $x' = g^{c_0}(t,x)$ is not in Case C. The previously proved persistence of Case A proves the last assertion. And the argument can be repeated if $c_3 > c_4$. □

We complete this part with a consequence of Proposition 4.8 which ensures the existence of at most a unique tipping point for certain parametric families:

Corollary 4.10. *Let $\mathcal{C} \subseteq \mathbb{R}$ be an open interval and let $\{g^c \mid c \in \mathcal{C}\}$ be a family of functions satisfying* **gc1**-**gc5**. *Assume that there exists $c_0 \in \mathcal{C}$ such that the dynamics of $x' = g^{c_0}(t,x)$ is in* Case B, *and such that, for all $c_-, c_+ \in \mathcal{C}$ with $c_- < c_0 < c_+$: $g^{c_-}(t,x) \le g^{c_0}(t,x) \le g^{c_+}(t,x)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}$; and there exist t_{c_-} and t_{c_+} such that the first and second inequality are strict for $t = t_{c_-}$ and $t = t_{c_+}$ (respectively) and all $x \in \mathbb{R}$. Then, $x' = g^c(t,x)$ is in* Case C *for $c \in \mathcal{C}$ with $c < c_0$ and in* Case A *for $c \in \mathcal{C}$ with $c > c_0$.*

Proof. The hypotheses ensure that, if $c_+ > c_0$, any bounded solution of $x' = g^{c_0}(t,x)$ does not solve $x' = g^{c_+}(t,x)$, and hence Proposition 4.8 shows that $x' = g^c(t,x)$ is in Case A for $c > c_0$ in $\mathcal{C}$. Analogously, if $c_- < c_0$, any bounded solution of $x' = g^{c_-}(t,x)$ does not solve $x' = g^{c_0}(t,x)$, and hence Proposition 4.8 also shows that $x' = g^{c_0}(t,x)$ would be in Case A (which is not true, by hypothesis) if $x' = g^c(t,x)$ were in Cases A or B for $c < c_0$ in $\mathcal{C}$. □

4.1. **Some scenarios of critical transitions in the concave case.** Let $\mathcal{I} \subseteq \mathbb{R}$ be an open interval, and let the functions $f\colon \mathbb{R}\times\mathbb{R}\times\mathcal{I} \to \mathbb{R}$ and $\Gamma, \Gamma_-, \Gamma_+ \colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfy

- **fc1** there exist the derivatives f_x and f_γ, and f, f_x and f_γ are admissible on $\mathbb{R}\times\mathbb{R}\times\mathcal{I}$.
- **fc2** Γ, Γ_- and Γ_+ take values in $[a,b] \subset \mathcal{I}$, are C^1-admissible, and $\lim_{t\to\pm\infty}(\Gamma(t,x) - \Gamma_\pm(t,x)) = 0$ uniformly on each compact subset $\mathcal{J} \subset \mathbb{R}$.
- **fc3** $\limsup_{x\to\pm\infty} f(t,x,\gamma) < 0$ uniformly in $(t,\gamma) \in \mathbb{R}\times\mathcal{J}$ for all compact interval $\mathcal{J} \subset \mathcal{I}$.
- **fc4** $\inf_{t\in\mathbb{R}} \big((\partial/\partial x) f(t,x,\Gamma_\pm(t,x))|_{x=x_1} - (\partial/\partial x) f(t,x,\Gamma_\pm(t,x))|_{x=x_2}\big) > 0$ whenever $x_1 < x_2$.
- **fc5** Each equation $x' = f(t,x,\Gamma_\pm(t,x))$ has two hyperbolic solutions $\tilde r_{\Gamma_\pm} < \tilde a_{\Gamma_\pm}$.

Observe that condition **fc2** allows us to understand the equations

$$x' = f(t,x,\Gamma_-(t,x)) \quad \text{and} \quad x' = f(t,x,\Gamma_+(t,x)) \tag{4.3}$$

as the "past" and "future" of

$$x' = f(t,x,\Gamma(t,x))\,. \tag{4.4}$$

We will say that the pair (f,Γ) satisfies **fc1**-**fc5** whenever there exist $\Gamma_\pm$ such all the listed properties hold. Note that, in this case, also the pairs $(f,\Gamma_\pm)$ satisfy **fc1**-**fc5**. We omit the almost immediate proof of the next result, which shows that the previous ones apply to the current setting.

Proposition 4.11. *Assume that (f,Γ) satisfies* **fc1**-**fc5**. *Then, the maps g, g_- and g_+ given by $g(t,x) := f(t,x,\Gamma(t,x))$, $g_-(t,x) := f(t,x,\Gamma_-(t,x))$ and $g_+(t,x) := f(t,x,\Gamma_+(t,x))$ satisfy the conditions* **gc1**-**gc5**. *Therefore, the dynamical possibilities for* (4.4) *are those described in Theorem* 4.7.

Remarks 4.12. 1. It is easy to check that the proof of Proposition 4.11 can be repeated in the next cases: if we remove the boundedness of Γ and $\Gamma_\pm$ from condition **fc2** but assume that $\mathcal{I} = \mathbb{R}$ and that the limit in **fc3** is uniform in $(t, \gamma) \in \mathbb{R} \times \mathbb{R}$; and if we remove the assumptions on the derivative f_γ of **fc1** but assume that Γ, and hence $\Gamma_\pm$, depend only on t. Hence, the conclusions of Proposition 4.6 and Theorem 4.7 also hold under these conditions.

2. As explained in Remark 4.4, Proposition 4.11 applied to the pairs $(f, \Gamma_\pm)$ allows us to reformulate condition **fc5** as: "each equation $x' = f(t, x, \Gamma_\pm(t, x))$ has an attractor-repeller pair of solutions", which determines its global dynamics.

In this section (as in Section 6.1), we analyze some mechanisms of occurrence (or lack) of tipping points for transition equations (4.4) due to small parametric variations in the transition function: we work with one-parametric families

$$x' = f(t, x, \Gamma^c(t, x)) \,. \tag{4.5}$$

Let us mention three of the large variety of physical mechanisms that may cause critical transitions:

- *Rate-induced critical transitions:* if $\Gamma^c(t, x) = \Gamma(ct, x)$ for a fixed Γ and any $c > 0$, then the parameter $c > 0$ determines the speed of the transition Γ^c. In order to have a past and a future independent of the rate, we require Γ_- and Γ_+ to be independent of t. So, a larger c means a significant distance from $\Gamma(ct, x)$ to $\Gamma_\pm(x)$ during a shorter period.
- *Phase-induced critical transitions:* if $\Gamma^c(t, x) = \Gamma(c + t, x)$, then the parameter $c \in \mathbb{R}$ represents the initial phase of the transition function. As before, we assume Γ_- and Γ_+ independent of t.
- *Size-induced critical transitions:* with $\Gamma_- \equiv 0$ and $\Gamma^c(t, x) := c\,\Gamma(t, x)$, different values of $c > 0$ mean different sizes of the transition function which "takes" $x' = f(t, x, 0)$ to $x' = f(t, x, c\,\Gamma_+(t, x))$.

The next result establishes conditions on a parametric family of maps $\{\Gamma^c\}$ and f which are enough to guarantee the persistence of Cases A and C, and to show that the occurrence of a critical transition means the occurrence of Case B and can be understood as a nonautonomous saddle-node bifurcation: see Theorem 4.9. We omit the (easy) proof.

Proposition 4.13. *Let $\mathcal{C} \subsetneq \mathbb{R}$ be an open interval, and let the maps $\{\Gamma^c \mid c \in \mathcal{C}\}$ be a family of functions such that all the pairs (f, Γ^c) satisfy* **fc1-fc5** *and such that $\mathbb{R} \times \mathbb{R} \times \mathcal{C} \to \mathbb{R}$, $(t, x, c) \mapsto \Gamma^c(t, x)$ is admissible. Assume also that, for any $c \in \mathcal{C}$, there exists $\delta_c > 0$ such that $\sup_{(t,x)\in\mathbb{R}\times\mathbb{R},\, |\varepsilon|\le\delta_c} |\Gamma^{c+\varepsilon}(t, x)| < \infty$. Then, the map $\bar{g}(t, x, c) := f(t, x, \Gamma^c(t, x))$ satisfies all the hypotheses of Theorem 4.9.*

Remark 4.14. Note that if, in the considered case of rate and phase variation, with $\Gamma_\pm$ independent of t, all the pairs (f, Γ^c) satisfy **fc1-fc5** if (f, Γ) does, with the same maps $\Gamma_\pm$. The same occurs in the size-variation case if we also assume $\Gamma_+ \equiv 0$. In addition, in the three considered cases, the admissibility and boundedness hypotheses of Proposition 4.13 also hold.

In the rest of this section, we will describe conditions ensuring the lack of rate-induced and phase-induced critical transitions (in Theorem 4.15), as well as the occurrence of size-induced critical transitions (in Theorem 4.17). These scenarios assume monotonicity of f with respect to γ. Theorem 4.15 establishes conditions on $f(t,x,\gamma_0)$ ensuring that $[\gamma_0,\infty)$ or $(-\infty,\gamma_0]$ is a *safety halfline*: if Γ takes values in it, then neither rate-induced tipping nor phase-induced tipping occurs.

Theorem 4.15. *Assume that (f,Γ) satisfy* **fc1-fc5**. *Assume also that $\gamma\mapsto f(t,x,\gamma)$ is nondecreasing (resp. nonincreasing) for all $(t,x)\in\mathbb{R}\times\mathbb{R}$ and that one of the following situations holds:*

(1) *there exists a constant $\gamma_0\le\Gamma$ (resp. $\gamma_0\ge\Gamma$) such that $x'=f(t,x,\gamma_0)$ has either two different bounded solutions or a bounded solution which does not solve (4.4);*
(2) *there exists a continuous map $\Delta\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ with $\Delta\le\Gamma$ (resp. $\Delta\ge\Gamma$) such that $x'=f(t,x,\Delta(t,x))$ has either two different bounded solutions, or a bounded solution which does not solve (4.4);*

Then, (4.4) is in Case A.

In particular, if (1) or (2) holds, and if we assume in addition that $\Gamma_\pm$ do not depend on t, then the equations $x'=f(t,x,\Gamma(ct,x))$ and $x'=f(t,x,\Gamma(t+c,x))$ are in Case A *for all $c>0$ and $c\in\mathbb{R}$, respectively: there are neither rate-induced nor phase-induced critical transitions.*

Proof. All the assertions follow easily from Propositions 4.11 and 4.8, and from Remark 4.14. □

Remark 4.16. In the increasing (resp. decreasing) scenario of Theorem 4.15, condition **fc5** ensures (2) if either $\Gamma_-(t,x)\le\Gamma(t,x)$ or $\Gamma_+(t,x)\le\Gamma(t,x)$ (resp. either $\Gamma_-(t,x)\ge\Gamma(t,x)$ or $\Gamma_+(t,x)\ge\Gamma(t,x)$) for all $(t,x)\in\mathbb{R}\times\mathbb{R}$: it suffices to take $\Delta=\Gamma_-$ or $\Delta=\Gamma_+$.

Our next result provides another scenario of lack of critical transitions or occurrence of exactly one. Now, the variation of the parameter precludes the transition map to remain always in the safety half-line, and hence the dynamics cannot be always (if ever) in Case A. The interval $\mathcal{I}$ of variation of the third argument of f must be $\mathbb{R}$.

Theorem 4.17. *Assume that $\mathcal{I}=\mathbb{R}$. Let $\Gamma\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\Gamma_0\colon\mathbb{R}\times\mathbb{R}\to[0,\infty)$ be globally bounded and C^1-admissible, and such that the pair $(f,\Gamma+d\,\Gamma_0)$ satisfies* **fc1-fc5** *for all $d\in\mathbb{R}$. Assume that $\Gamma_0(t_0,x)>0$ for all $x\in\mathbb{R}$ and a $t_0\in\mathbb{R}$. Assume also that $\gamma\mapsto f(t,x,\gamma)$ is strictly increasing on $\mathbb{R}$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}$, with $\lim_{\gamma\to-\infty}f(t,x,\gamma)=-\infty$ uniformly on compact sets of $\mathbb{R}\times\mathbb{R}$. Then, either*

$$x'=f(t,x,\Gamma(t,x)+d\,\Gamma_0(t,x)) \tag{4.6}$$

is in Case C *for all $d\in\mathbb{R}$, or there exists d_0 such that (4.6) is in* Case A *for $d>d_0$, in* Case B *for $d=d_0$ and in* Case C *for $d<d_0$.*

Proof. Let us assume for contradiction that $(4.6)_d$ is in Case A for all $d\in\mathbb{R}$, and fix $\bar d\in\mathbb{R}$. We take $\delta>0$ and $m_1,m_2\in\mathbb{R}$ such that $f(t,x,\Gamma(t,x)+\bar d\,\Gamma_0(t,x))\le-\delta$ for all $t\in\mathbb{R}$ if $x\notin(m_1,m_2)$. Since f is nondecreasing in γ and $\Gamma_0\ge0$, the map $d\mapsto f(t,x,\Gamma(t,x)+d\,\Gamma_0(t,x))$ is nondecreasing, and hence, for all $d\le\bar d$, $f(t,x,\Gamma(t,x)+d\,\Gamma_0(t,x))\le-\delta$ for all $t\in\mathbb{R}$ if $x\notin(m_1,m_2)$. As in the proof of Proposition 3.5(ii), we check that $m_1\le\tilde a_d\le m_2$ if $d\le\bar d$, where $\tilde a_d$ is the upper bounded solution of $(4.6)_d$. We look for $t_1<t_0<t_2$ and $k>0$ such that $\Gamma_0(t,x)>k$ if $t\in[t_1,t_2]$ and $x\in[m_1,m_2]$, and call $k_d:=\sup_{(t,x)\in[t_1,t_2]\times[m_1,m_2]}f(t,x,\Gamma(t,x)+d\,\Gamma_0(t,x))$ for $d\le\bar d$. Then, $k_d\le\sup_{(t,x)\in[t_1,t_2]\times[m_1,m_2]}f(t,x,\Gamma(t,x)+dk)$ if $d\le\min(0,\bar d)$, which combined with the hypothesis on $\lim_{\gamma\to-\infty} f(t,x,\gamma)$ ensures that $\lim_{d\to-\infty} k_d = -\infty$. Hence, $m_1 - m_2 \le \tilde a_d(t_2) - \tilde a_d(t_1) \le (t_2 - t_1)\, k_d$ for all $d \le \bar d$, which is impossible.

Now we assume that $(4.6)_d$ is not in Case C for all $d \in \mathbb{R}$. Theorem 4.9 ensures the existence of d_0 such that $(4.6)_{d_0}$ is in Case B, and Corollary 4.10, all whose hypotheses are satisfied, completes the proof. □

By reviewing the proof of Theorem 4.17, we observe that we have in fact proved the next result, which considerably weakens the conditions of the previous one (but with a statement quite harder to read, so we keep both of them).

Theorem 4.18. *Assume that $\mathcal{I} = \mathbb{R}$. Let $\Gamma\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\Gamma_0\colon \mathbb{R}\times\mathbb{R}\to[0,\infty)$ be globally bounded and C^1-admissible, and such that the pair $(f, \Gamma + d\,\Gamma_0)$ satisfies* **fc1**-**fc5** *for all $d\in\mathbb{R}$. Assume that there exists $\bar d \in \mathbb{R}$ such that*

$$x' = f(t, x, \Gamma(t,x) + d\,\Gamma_0(t,x)) \tag{4.7}$$

is in Cases A or B for $d = \bar d$. Let $\delta > 0$ and $m_1, m_2 \in \mathbb{R}$ satisfy $f(t,x,\Gamma(t,x) + \bar d\,\Gamma_0(t,x)) \le -\delta$ for all $t\in\mathbb{R}$ if $x \notin (m_1, m_2)$. Assume that there exists t_0 such that $\Gamma_0(t_0,x) > 0$ for all $x \in [m_1,m_2]$, that $\gamma \mapsto f(t,x,\gamma)$ is nondecreasing for all $(t,x)\in \mathbb{R}\times\mathbb{R}$ and strictly increasing for $(t,x)\in\mathbb{R}\times[m_1,m_2]$, with $\lim_{\gamma\to-\infty} f(t,x,\gamma) = -\infty$ uniformly on compact sets of $\mathbb{R}\times[m_1,m_2]$. Then, there exists $d_0 \le \bar d$ such that (4.7) is in Case A for $d > d_0$, in Case B for $d = d_0$ and in Case C for $d < d_0$.

Remark 4.19. Frequently, the limit maps providing condition **fc2** for all d are $\Gamma_\pm + d\,\Gamma_{0,\pm}$ for C^2-admissible maps $\Gamma_\pm$ and $\Gamma_{0,\pm} \ge 0$. If so, condition **fc5** for all $x' = f(t,x,\Gamma_\pm(t,x) + d\,\Gamma_{0,\pm}(t,x))$ is only possible if, for any $t_0\in\mathbb{R}$, each map $x \mapsto \Gamma_{0,\pm}(t_0,x)$ vanishes at least for an $x^\pm_{t_0} \in [m_1,m_2]$: otherwise, Theorem 4.18 precludes hyperbolic solutions if $-d$ is large enough. Also often, $\Gamma_\pm \equiv \Gamma$ and $\Gamma_{0,-} \equiv 0$, and so Theorems 4.17 and 4.18 study the occurrence of size-induced critical transitions: just define $f^*(t,x,d\,\Gamma_0(t,x)) := f(t,x,\Gamma(t,x) + d\,\Gamma_0(t,x))$.

4.2. **Numerical simulations in asymptotically concave models.** In this section, we consider a single-species population whose intrinsic dynamics is governed by a nonautonomous quadratic equation (see [17]) subject to two external phenomena: migration and predation. The inclusion of time-dependent intrinsic parameters and functions in the model naturally arises when considering the influence of external factors which vary over time, such as climatic or meteorological factors (see e.g. [39]). In this way, the evolution of the population size is governed by

$$x' = r(t)\,x\left(1 - \frac{x}{K(t)}\right) + \Delta(t,x)\,, \tag{4.8}$$

where $r(t)$ is the growth rate of the population at time t, $K(t)$ is closely related to the carrying capacity of the environment, and $\Delta(t,x)$ represents the migration and predation factors. We assume r and K to be positively bounded from below quasiperiodic functions, and Δ to be C^1-admissible: so, if we define $h(t,x,\delta) := r(t)\,x\,(1 - x/K(t)) + \delta$, then h satisfies **fc1** and **fc3**. In addition, we assume (h,Δ) to satisfy conditions **fc2**, **fc4** and **fc5** for certain maps $\Delta_\pm$ which do not play a role for the moment. The function r represents the intrinsic growth rate of the species, that is, the growth rate per individual in an ideal situation of unlimited resources; K does no longer represent, as in the autonomous case, the maximal population allowed by the resources if $\Delta \equiv 0$ (unless it is constant), but there exists positive hyperbolic attractive solution which describes this target population; and the function Δ quantifies the net contribution per unit of time of both external phenomena: migration and predation. In this framework, the locally pullback attractive solution a_h of (4.8) provided by Proposition 4.6 (which can be applied, as Proposition 4.11 ensures) describes the evolution of the target population during the transition. The dynamical possibilities described by Theorem 4.7 are: the stable Case A, that means the survival of the population a_h, which approaches the upper bounded solution of the future equation as time increases; the stable Case C, which means the extinction of that population due to predation and migration; and Case B, which is the highly unstable situation which separates the other two.

Before choosing a particular function Δ, we apply Theorem 4.15(1) to prove the existence of a safety halfline: a threshold $\delta_0 < 0$ such that, if $\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}} \Delta(t,x) \geq \delta_0$ and $\Delta \not\equiv \delta_0$, then (4.8) is in Case A. That is, if the combined effect of predation and migration never eliminates more than $|\delta_0|$ individuals per unit of time, then the target population persists. To apply Theorem 4.15, we use the information provided by [30, Theorem 3.6] (which remains unchanged except for the bound for λ^* when multiplying the leading term by r), according to which there exists $\delta_0 \in \mathbb{R}$ such that $x' = r(t)\, x\, (1-x/K(t))+\delta$ is in Case A if $\delta > \delta_0$, in Case B if $\delta = \delta_0$, and in Case C if $\delta < \delta_0$. In addition, since we can take $\varepsilon > 0$ such that $0 < r(t)\, \varepsilon\, (1-\varepsilon/K(t))$ for all $t \in \mathbb{R}$, Proposition 3.5(v) ensures that $x' = r(t)\, x\, (1 - x/K(t))$ has two distinct bounded solutions, and hence it is in Case A; that is, $\delta_0 < 0$. Observe that this choice of δ_0 is optimal for the application of Theorem 4.15(1), since it is the smallest value of δ for which $x' = r(t)\, x\, (1 - x/K(t)) + \delta$ has a bounded solution.

In the following examples, we will give explicit expressions to the function Δ, depending on several bifurcation parameters, and we will find critical transitions: changes from Cases A to Case C through Case B as one of those parameters changes. We will prove the uniqueness of almost all those critical transitions and find numerical evidence of nonuniqueness of the remaining one.

Example 4.20. We assume that the predation in (4.8) can be suitably modeled by a Holling type III functional response term $-\gamma\, x^2/(b(t) + x^2)$ (see [17, 19]), where γ represents the predator density and b depends on the average time between two attacks of a predator (which is related to the food processing time); and we also assume that the net migration rate per unit of time $\phi(t)$ is negative for all $t \in \mathbb{R}$. Therefore, the model is

$$x' = r(t)\, x \left(1 - \frac{x}{K(t)}\right) + \phi(t) - \gamma\, \frac{x^2}{b(t) + x^2}\,, \tag{4.9}$$

where $-\phi$ and b are positively bounded from below and quasiperiodic (as r and K).

Let us introduce a specific time-variation on the predator density γ: we assume that the habitat is initially free of predators, that a certain time a group of predators arrives in, and that all of them leave away after some time. A wide range of causes can give rise to this transitory phenomenon in the case of predatory birds: adverse winds, storms, orientation errors, changing attractiveness of the breeding colony, etc. (see [37]). A somehow related real-life example, with foxes as predators, can be found in the work [35], which describes the colonization of Punta de la Banya by the Audouin's gull: an increasing population began to severely decline from a certain time due to the arrival of foxes, whom later were removed; and then the gull population began to increase again.

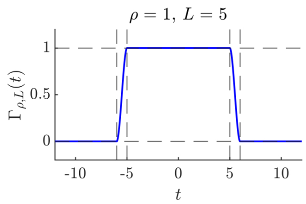

FIGURE 2. The C^1 transition map $\Gamma_{\rho,L}$ for $L = 5$ and several values of $\rho > 0$. This map is defined as the unique C^1 cubic spline which takes value 1 on $[-L, L]$ and 0 outside $[-L-\rho, L+\rho]$: if $Q(y) := 2y^3 - 3y^2 + 1$, then $\Gamma_{\rho,L}(t) := Q(-(t+L)/\rho)$ for $t \in [-L-\rho, -L]$ and $\Gamma_{\rho,L}(t) := Q((t-L)/\rho)$ for $t \in [L, L+\rho]$. This map is increasing on $[-L-\rho, -L]$ and decreasing on $[L, L+\rho]$, and hence $\Gamma_{\rho,L}(\cdot)$ is nondecreasing with respect to L and with respect to ρ.

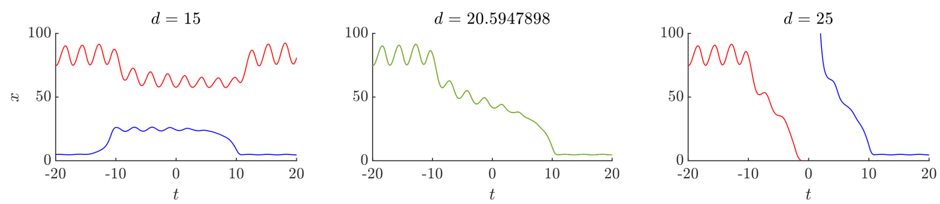

FIGURE 3. Numerical depiction of the existence of a unique size-tipping point for $(4.10)_d$ for $\rho = 1$, $L = 10$ and $p = 0$. The central panel depicts the dynamics just before the tipping point $d(1, 10, 0)$ (see Table 1): the two hyperbolic solutions are so close within the representation window that any of them is a good approximation (green) to the unique (nonhyperbolic) bounded solution of CASE B. The left panel depicts CASE A (persistence), which is the dynamics for any $d < d(1, 10, 0)$: the attractive hyperbolic solution which represents the behavior of the population in red, and the repulsive one in blue. The right panel depicts CASE C (extinction), which is the dynamics for any $d > d(1, 10, 0)$: the locally pullback attractive solution which represents the behavior of the population in red, and the locally pullback repulsive in blue. (They are given by Proposition 4.6.)

To model the effect of this type of phenomena in a simple way, we use a multiple of a C^1 approximation to the characteristic function of $[-L, L]$. More precisely, we substitute the predator density parameter γ in (4.9) by the compactly supported four-parametric transition function $t \mapsto d\,\Gamma_{\rho,L}(t-p)$, where $d \geq 0$ and $\Gamma_{\rho,L}$ is the unique C^1 cubic spline which takes the value 1 on $[-L, L]$ and 0 outside $[-L - \rho, L+\rho]$. Its asymptotic limits are 0 for any choices of ρ and L, and so, the past and future equations coincide: the predation term disappears. Figure 2 depicts $\Gamma_{\rho,L}$ for $L = 5$ and some values of ρ. Altogether, we get the four-parameter model

$$x' = r(t)\,x\left(1 - \frac{x}{K(t)}\right) + \phi(t) - d\,\Gamma_{\rho,L}(t-p)\,\frac{x^2}{b(t)+x^2}\,, \tag{4.10}$$

where $d \geq 0$ is proportional to the size of the group of predators, $\rho > 0$ is inversely related to the average speed at which the predators arrive and leave, $2L \geq 0$ is the period of time during which the action of the predators is most intense, and $p \in \mathbb{R}$ fixes the arrival and departure times $p - L - \rho$ and $p + L + \rho$ of the predators.

Let us define $f(t, x, \gamma) := r(t)\,x\big(1 - x/K(t)\big) + \phi(t) - \gamma\,x^2/(b(t) + x^2)$. It is easy to check that $(f, d\,\Gamma_{\rho,L}(\cdot - p))$ satisfies **fc1-fc4** with $\Gamma_\pm \equiv 0$ independently of the choice of the parameters. We emphasize that $f(t, x, d\,\Gamma_{\rho,L}(t-p))$ is not a concave function if $d > 4\,\max_{t\in\mathbb{R}} r(t)\,b(t)/K(t)$, what is easy to check: we are dealing with an asymptotically concave equation which is not concave. In addition, we assume that r, K and ϕ are chosen in such a way that $x' = f(t,x,0)$ has two (bounded) hyperbolic solutions: **fc5** is also fulfilled. Observe also that we can take $m_1 < m_2$ as in Proposition 3.5 with $m_1 > 0$: we take $m_1 \in (0, \inf_{t\in\mathbb{R}}(-\phi(t)/r(t)))$ and check that $f(t,x,d\,\Gamma_{\rho,L}(t-p)) \le f(t,x,0) \le r(t)\,m_1 + \phi(t) < -\delta$ for certain $\delta > 0$ if $x \le m_1$. Hence, all the bounded solutions of (4.10) are strictly positive (and hence they are biologically meaningful) for all $d \ge 0$, $\rho > 0$, $L \ge 0$ and $p \in \mathbb{R}$. Besides, $\gamma \mapsto f(t,x,\gamma)$ is nonincreasing for all $(t,x) \in \mathbb{R}\times\mathbb{R}$ and strictly decreasing for $(t,x) \in \mathbb{R}\times(0,\infty) \supset \mathbb{R}\times[m_1,m_2]$, and $\lim_{\gamma\to\infty} f(t,x,\gamma) = -\infty$ uniformly on $\mathbb{R}\times[m_1,m_2]$. Recall also that we have chosen the coefficients to ensure that $x' = f(t,x,0)$ is in Case A. So, if we fix $(\rho,L,p) \in (0,\infty)\times[0,\infty)\times\mathbb{R}$ and define $g(t,x,\gamma) := f(t,x,-\gamma)$, then the pairs $(g, d\,\Gamma_{\rho,L}(\cdot - p))$ satisfy all the hypotheses of Theorem 4.18 (with $\Gamma(t) := 0$, $\Gamma_0(t) := \Gamma_{\rho,L}(t-p)$ and $\bar{d} = 0$). Therefore, there exists a unique tipping point $d(\rho,L,p) > 0$: for $0 \le d < d(\rho,L,p)$ (as for $d < 0$), the dynamics of $(4.10)_d$ fits Case A, and it fits Cases B and C for $d = d(\rho,L,p)$ and for $d > d(\rho,L,p)$, respectively. In addition, $d(\rho,L,p)$ varies continuously with respect to each parameter, as Theorem 4.9 shows.

This critical transition is depicted in Figure 3 for $\rho = 1$, $L = 10$ and $p = 0$, with the next choices: $r(t) := 1 + 0.2\sin^2(t)$, $K(t) := 90 + 20\sin(\sqrt{5}\,t)$, $\phi(t) := -5$, and $b(t) := 20 + \cos(t)$. Numerical evidences show that the corresponding equation $x' = f(t,x,0)$ has two (strictly positive) hyperbolic solutions, as our analysis requires. For these choices, the right size of (4.10) is not concave in x if $d > 1.44$.

In Table 1, we numerically approximate the unique bifurcation points $d(1,L,p)$ for some pairs (L,p) and the previous choices. The bifurcation points have been approximated through bisection methods.

$d(1,L,p)$	$p = 0$	$p = 2$	$p = 5$
$L = 1$	40.2455300	42.2034404	41.9617506
$L = 5$	23.0532048	22.9017928	22.8172667
$L = 10$	20.5947898	20.5342198	20.4768856
$L = 15$	19.9425668	19.9151819	19.8875532
$L = 20$	19.6805426	19.6731947	19.6649049

Table 1. Numerical approximations up to seven decimal places to the bifurcation point $d(1,L,p)$ of $(4.10)_d$. The displayed number is a value of d for which $(4.10)_d$ is in Case A and $(4.10)_{d+1e-7}$ is in Case C. The Matlab2023a `ode45` algorithm has been used with `AbsTol` and `RelTol` equal to `1e-12`. The final integration has been carried out over the interval [−`1e4`, `1e4`].

Let us briefly extract some conclusions from the data of Table 1. In this three-parametric model (we have fixed $\rho = 1$), we may find changes from Case A to Case C by varying any of the three parameters d, L or p (and ρ, although we work with a fixed value for simplicity). In fact, since $\Gamma_{\rho,L}(\cdot)$ is nondecreasing and nonconstant with respect to L (and also with respect to ρ), Corollary 4.10 (applied to the maps $g(t,x,d\,\Gamma_{\rho,L}(t-p))$) shows that there exists at most a critical value L_0 (or ρ_0) if the rest of the parameters are fixed, and that Case A holds to its left. Figure 4 depicts the occurrence of this critical transition as L increases for a fixed value of d (which is our previous approximation to $d(1,L,0)$). (In fact, it can be proved that this tipping point indeed exists if d is large enough and ρ is small enough, but we

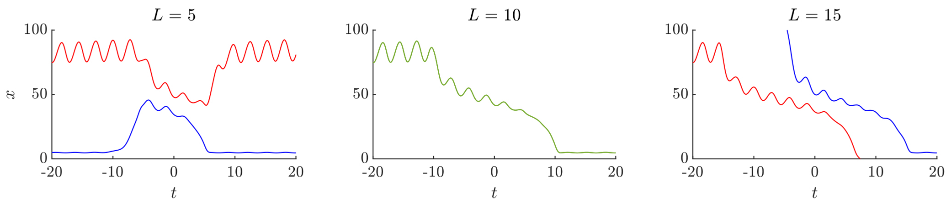

FIGURE 4. Numerical depiction of a time-of-action-tipping point for $(4.10)_L$ for $d = 20.5947898$, $\rho = 1$ and $p = 0$. The central panel approximates the dynamics at the tipping point $d(1, 10, 0)$ (see Table 1 and Figure 3): CASE B. The left panel depicts CASE A (persistence) and the right panel depicts CASE C (extinction). Recall that the predators act during $t \in [-L - \rho, L + \rho]$.

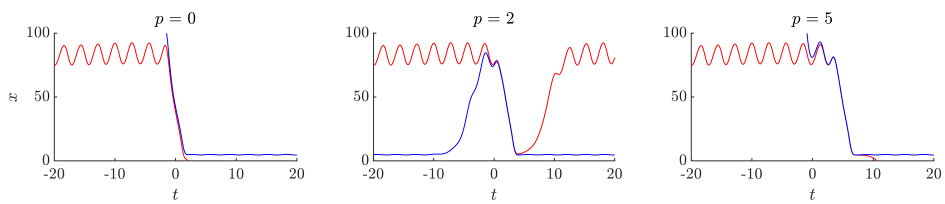

FIGURE 5. Numerical depiction of the existence of at least two critical values of the phase p: for $L = 1$ and $d = 42$, we find CASE C (extinction) for $p = 0$ and $p = 5$, and CASE A (persistence) for $p = 2$. See Figure 3 for the meaning of the different elements.

omit this nontrivial analysis.) On the other hand, the lack of monotononicity in the first row of Table 1 shows the possible lack of uniqueness of the tipping point as p varies: for $L = 1$ and $d = 42$, we are in CASE C for $p = 0$ and $p = 5$ (in those cases $d(1, 1, p) < 42$), and in CASE A for $p = 2$ (because $d(1, 1, 2) > 42$): there are at least two critical transitions, i.e., two CASES B. This fact is depicted in Figure 5. This lack of uniqueness of phase-induced critical transitions was already observed in [29, Section 6].

Summing up, we find a unique size-induced critical transition as d varies (see Figure 3), a unique time-of-action-induced critical transition as L varies (see Figure 4) if d is large enough, and possibly several phase-induced critical transitions as p varies (see Figure 5), also for a large enough d.

Let us finally analyze the occurrence of rate-induced critical transitions, fixing d, L, ρ and taking $p = 0$. By writing the expression of $\Gamma_{\rho,L}$, we check that $\Gamma_{\rho,L}(ct) = \Gamma_{\rho/c,L/c}(t)$ for all $\rho, L, c > 0$ and $t \in \mathbb{R}$. This formula relates the study of rate-induced critical transitions to one induced by a simultaneous change of time-of-action L and the time spent in arrival ρ. It also shows that $c \mapsto \Gamma_{\rho,L}(ct)$ is a nonincreasing and nonconstant map. Therefore, Corollary 4.10 shows the uniqueness of the tipping point in case of existence, and that CASE A holds to its right: we are in cases of rate-induced tracking. (And, as in the case of variation of L, the existence of this unique tipping value c_0 for each fixed $\rho > 0$, $L > 0$, $p \in \mathbb{R}$ and large enough d can be indeed proved.)

Example 4.21. The analysis in Example 4.20 has been carried out considering an x-independent transition function $t \mapsto d\,\Gamma_{\rho,L}(t-p)$. Now we add another parameter shift $t \mapsto \Lambda(t - p)$ in the migration part of the driving law. The results presented in this paper can still be applied to an x-dependent transition function $(t, x) \mapsto$

$\Delta_{d,\rho,L,p}(t,x)$ (recalling the formulation of (4.8)) that encompasses the above two parameter shifts. The presented theory allows to understand separately a stationary part and a transient part of the law that generates the dynamics.

So, we consider a slightly more complicated emigration phenomena: due to the arrival to the habitat of the predator species, the attractiveness of the habitat is reduced, and, as a consequence, the emigration increases. To model this, we consider that r, K, b and ϕ satisfy the conditions assumed in Example 4.20, and replace the quasiperiodic emigration function ϕ by an asymptotically quasiperiodic emigration function $\phi(t) + (\psi(t) - \phi(t))\,\Lambda_{\rho,L}(t-p)$, where ψ is quasiperiodic with $\psi(t) \le \phi(t)$ for all $t \in \mathbb{R}$, and $\Lambda_{\rho,L}(t)$ is the unique C^1 cubic piecewise polynomial which takes values 0 for all $x \le -L-\rho$ and 1 for all $x \ge -L$. So, we work with

$$x' = r(t)\,x\left(1 - \frac{x}{K(t)}\right) + \phi(t) + (\psi(t) - \phi(t))\,\Lambda_{\rho,L}(t-p) - \frac{d\,\Gamma_{\rho,L}(t-p)\,x^2}{b(t)+x^2}\,. \quad (4.11)$$

If, as before, $h(t,x,\delta) := r(t)\,x\,(1 - x/K(t)) + \delta$, and $\Delta_{d,\rho,L,p}(t,x) := \phi(t) + (\psi(t) - \phi(t))\,\Lambda_{\rho,L}(t-p) - d\,\Gamma_{\rho,L}(t-p)\,x^2/(b(t)+x^2)$, then $(h, \Delta_{d,\rho,L,p})$ satisfies **fc1-fc4** with $\Delta_- := \phi$ and $\Delta_+ := \psi$. We also assume that $x' = h(t,x,\psi(t))$ has two hyperbolic solutions and deduce from Proposition 3.5(v) that this property also holds for $x' = h(t,x,\phi(t))$: **fc5** is also fulfilled. Proposition 3.5(v) also shows that $x' = h(t,x,\phi(t) + (\psi(t) - \phi(t))\,\Lambda_{\rho,L}(t-p))$ has two hyperbolic solutions for all $L \ge 0$ and $\rho > 0$. This fact allows us to repeat the arguments of Example 4.20 in order to prove the existence and uniqueness of the bifurcation point $d(\rho,L,p) > 0$, giving rise to a unique size-induced critical transition.

$d(1,L,p)$	$p=0$	$p=2$	$p=5$
$L=1$	34.1938684	36.9449750	35.7506039
$L=5$	18.8506812	18.6486286	18.6059557
$L=10$	16.4930418	16.4318568	16.3869395
$L=15$	15.8700118	15.8460071	15.8202938
$L=20$	15.6203137	15.6150934	15.6065018

TABLE 2. Numerical approximations up to seven decimal places to the bifurcation point $d(1,L,p)$ of $(4.11)_d$. The displayed number is a value of d for which $(4.11)_d$ is in CASE A and $(4.11)_{d+1e-7}$ is in CASE C. The Matlab2023a `ode45` algorithm has been used with `AbsTol` and `RelTol` equal to `1e-12`. The final integration has been carried out over the interval $[-1e4, 1e4]$.

We repeat the choices of r, K, ϕ and b of the first example, and take $\psi(t) := -9-\cos t$. Again, numerical evidences show that $x' = h(t,x,\psi(t))$ has two (positive) hyperbolic solutions, as required. Table 2 shows numerical approximations of the unique bifurcation point $d(1,L,p)$ for some pairs (L,p). The arguments of Example 4.20 also work to prove the existence of a unique bifurcation value $L_0 > 0$ if d is large enough, $\rho > 0$ is small enough, and $p \in \mathbb{R}$. Note that, for $L = 1$, we find the phenomenon of multiple phase bifurcation points already mentioned in Table 1 and shown in Figure 5. As expected, the bifurcation points $d(1,L,p)$ for (4.11) in Table 2 are lower than those for (4.10) in Table 1: an increased emigration means that fewer predators are needed to cause the population extinction. We point out that, in this example, the asymptotic limits of the transition map are time-dependent.

5. The d-concave and nonquadratic case

The results of this section partly extend those of [18] to a less restrictive setting. Let (Ω,σ) be defined as in Section 3. Now, we work with the family

$$x' = \mathfrak{h}(\omega{\cdot}t, x) \tag{5.1}$$

for $\omega\in\Omega$, and with the flow τ defined by (2.4), $\tau(t,\omega,x)=(\omega{\cdot}t, v(t,\omega,x))$, assuming that $\mathfrak{h}\colon\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies (all or part of) the next conditions:

d1 $\mathfrak{h}\in C^{0,2}(\Omega\times\mathbb{R},\mathbb{R})$,
d2 $\limsup_{x\to\pm\infty}(\pm\mathfrak{h}(\omega,x))<0$ uniformly on Ω,
d3 $m(\{\omega\in\Omega\,|\; x\mapsto\mathfrak{h}_x(\omega,x) \text{ is concave}\})=1$ for all $m\in\mathfrak{M}_{\rm erg}(\Omega,\sigma)$,
d4 $m(\{\omega\in\Omega\,|\; x\mapsto\mathfrak{h}_{xx}(\omega,x) \text{ is strictly decreasing on } \mathcal{J}\})>0$ for all compact interval $\mathcal{J}\subset\mathbb{R}$ and all $m\in\mathfrak{M}_{\rm erg}(\Omega,\sigma)$.

We will refer to the concavity of the derivative as *d-concavity* and, for the sake of simplicity, we will say that, under all these hypotheses, (5.1) is a (coercive) *family of d-concave ordinary differential equations.* Note that the concavity of all the maps $x\mapsto\mathfrak{h}(\omega,x)$ is not required.

Remark 5.1. As explained in Remark 3.1, if $\mathfrak{h}$ satisfies **dj** for $\mathbf{j}\in\{1,2,3,4\}$ and $\Omega_0\subset\Omega$ is a nonempty compact σ-invariant subset, then also the restriction $\mathfrak{h}\colon\Omega_0\times\mathbb{R}\to\mathbb{R}$ satisfies **dj**.

The following results establish properties analogous to those of Proposition 3.2 and Theorem 3.3, now for the new hypotheses on d-concavity.

Proposition 5.2. *Let $\mathfrak{h}\colon\Omega\times\mathbb{R}\to\mathbb{R}$ satisfy* **d1***, let us fix $m\in\mathfrak{M}_{\rm erg}(\Omega,\sigma)$, and let $\mathfrak{b}_1,\mathfrak{b}_2,\mathfrak{b}_3\colon\Omega\to\mathbb{R}$ be bounded m-measurable τ-equilibria with $\mathfrak{b}_1(\omega)<\mathfrak{b}_2(\omega)<\mathfrak{b}_3(\omega)$ for m-a.e. $\omega\in\Omega$. Assume that $m(\{\omega\in\Omega\,|\; x\mapsto\mathfrak{h}_x(\omega,x)$ is concave$\})=1$ and $m(\{\omega\in\Omega\,|\;\mathfrak{h}_{xx}(\omega,\mathfrak{b}_1(\omega))>\mathfrak{h}_{xx}(\omega,\mathfrak{b}_3(\omega))\})>0$. Then,*

$$\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}_2(\omega))\,dm>0 \quad\textit{and}\quad \int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}_i(\omega))\,dm<0 \;\;\textit{for}\;\; i=1,3\,.$$

In particular, there are at most three bounded m-measurable τ-equilibria which are strictly ordered m-a.e.

Proof. We call $\Omega^c:=\{\omega\in\Omega\,|\;x\mapsto\mathfrak{h}_x(\omega,x)$ is concave$\}$, which satisfies $m(\Omega^c)>0$, and $\Omega_0:=\{\omega\in\Omega\,|\;\mathfrak{b}_1(\omega)<\mathfrak{b}_2(\omega)<\mathfrak{b}_3(\omega)\}$, which is invariant and with $m(\Omega_0)=1$. For each $\omega\in\Omega^c$, we represent by $b_i(\omega,x_1,x_2,x_3)$ the expression $b_i(x_1,x_2,x_3)$ of (2.1) associated to the d-concave map $x\mapsto\mathfrak{h}(\omega,x)$. For $i\in\{1,2,3\}$, we define $b_i^*\colon\Omega\to\mathbb{R}$ by $b_i^*(\omega):=b_i(\omega,\mathfrak{b}_1(\omega),\mathfrak{b}_2(\omega),\mathfrak{b}_3(\omega))$ for $\omega\in\Omega_0\cap\Omega^c$ and $b_i^*(\omega):=0$ if $\omega\notin\Omega_0\cap\Omega^c$. The hypothesis on $\mathfrak{h}_{xx}$ and Proposition 2.2(ii) (see also its proof) ensure that b_i^* is m-measurable, with $b_i^*\ge 0$ and $m(\{\omega\in\Omega\,|\;b_i^*(\omega)>0\})>0$. These are the key properties to repeat the arguments of [18, Theorem 4.1] in order to check that, if $v_1:=1/(\mathfrak{b}_2-\mathfrak{b}_1)-1/(\mathfrak{b}_3-\mathfrak{b}_1)$ (which satisfies $v_1(\omega)>0$ for $\omega\in\Omega_0$), then $\int_\Omega\mathfrak{h}_x(\omega,\mathfrak{b}_1(\omega))\,dm=-\int_\Omega(b_1^*(\omega)/v_1(\omega))\,dm<0$. The argument is similar for $\mathfrak{b}_2$ and $\mathfrak{b}_3$, and the last assertion follows from the previous ones and a simple contradiction argument. □

Theorem 5.3. *Let $\mathfrak{h}$ satisfy* **d1**, **d3** *and* **d4**. *Then, there exist three disjoint and ordered τ-invariant compact sets $\mathcal{K}_1<\mathcal{K}_2<\mathcal{K}_3$ projecting onto Ω if and only if there exist three hyperbolic copies of the base $\{\tilde{\mathfrak{l}}\}$, $\{\tilde{\mathfrak{m}}\}$ and $\{\tilde{\mathfrak{u}}\}$ with $\tilde{\mathfrak{l}}<\tilde{\mathfrak{m}}<\tilde{\mathfrak{u}}$. In this case, $\mathcal{K}_1=\{\tilde{\mathfrak{l}}\}$ and $\mathcal{K}_3=\{\tilde{\mathfrak{u}}\}$ and they are attractive; $\mathcal{K}_2=\{\tilde{\mathfrak{m}}\}$*

and it is repulsive; and $\mathcal{B} := \{(\omega, x) \in \Omega \times \mathbb{R} \mid \tilde{\mathfrak{l}}(\omega) \le x \le \tilde{\mathfrak{u}}(\omega)\}$ *is the set of globally bounded orbits. In particular, there are at most three disjoint and ordered* τ*-invariant compact sets projecting onto* Ω.

Proof. Observe that $\mathfrak{h}$ satisfies the conditions of Proposition 5.2 for any $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ and any three bounded ordered m-measurable equilibria. This fact allows us to use the arguments of the proof of Theorem 3.3 to check all the assertions. □

Remark 5.4. An analogue of Remark 3.4 applies to Theorem 5.3.

Now, the coercivity property **d2** comes into play. Recall that a τ-invariant compact set $\mathcal{A} \subset \Omega \times \mathbb{R}$ is the *global attractor of* τ if it attracts every bounded set $\mathcal{C} \subset \Omega \times \mathbb{R}$; that is, if $\tau_t(\mathcal{C})$ is defined for any $t \ge 0$ and $\lim_{t\to\infty} \mathrm{dist}(\tau_t(\mathcal{C}), \mathcal{A}) = 0$, where $\mathrm{dist}(\mathcal{C}_1, \mathcal{C}_2)$ is the Hausdorff semidistance from $\mathcal{C}_1$ to $\mathcal{C}_2$ and $\tau_t(\mathcal{C}) := \{\tau(t, \omega, x) \mid (\omega, x) \in \mathcal{C}\}$.

Proposition 5.5. *Let* $\mathfrak{h} \in C^{0,1}(\Omega \times \mathbb{R}, \mathbb{R})$ *satisfy* **d2**, *and take* $\delta > 0$ *and* $m_1, m_2 \in \mathbb{R}$ *with* $\mathfrak{h}(\omega, x) \ge \delta$ *if* $x \le m_1$ *and* $\mathfrak{h}(\omega, x) \le -\delta$ *if* $x \ge m_2$ *for all* $\omega \in \Omega$. *Then,*

- (i) $v(t, \omega, x)$ *exists for* $(t, \omega, x) \in [0, \infty) \times \Omega \times \mathbb{R}$, *and* $m_1 \le \liminf_{t\to\infty} v(t, \omega, x) \le \limsup_{t\to\infty} v(t, \omega, x) \le m_2$*: any forward* τ*-semiorbit is bounded.*
- (ii) *There exists the global attractor for* τ, *it is of the form*
$$\mathcal{A} = \bigcup_{\omega\in\Omega} \left(\{\omega\} \times [\mathfrak{l}(\omega), \mathfrak{u}(\omega)]\right), \tag{5.2}$$
it is the union of all the globally defined and bounded τ*-orbits, and it is contained in* $\Omega \times [m_1, m_2]$.
- (iii) *The maps* $\mathfrak{l}$ *and* $\mathfrak{u}$ *are, respectively, lower and upper semicontinuous* τ*-equilibria.*
- (iv) *If, for a point* $\omega \in \Omega$, *there exists a bounded* C^1 *function* $b\colon \mathbb{R} \to \mathbb{R}$ *such that* $b'(t) \le \mathfrak{h}(\omega{\cdot}t, b(t))$ *(resp.* $b'(t) \ge \mathfrak{h}(\omega{\cdot}t, b(t))$*) for all* $t \in \mathbb{R}$, *then* $b(t) \le \mathfrak{u}(\omega{\cdot}t)$ *(resp.* $b(t) \ge \mathfrak{l}(\omega{\cdot}t)$*) for all* $t \in \mathbb{R}$. *If* $b'(t) < \mathfrak{h}(\omega{\cdot}t, b(t))$ *(resp.* $b'(t) > \mathfrak{h}(\omega{\cdot}t, b(t))$*) for all* $t \in \mathbb{R}$, *then* $b(t) < \mathfrak{u}(\omega{\cdot}t)$ *(resp.* $\mathfrak{l}(\omega{\cdot}t) < b(t)$*) for all* $t \in \mathbb{R}$.
- (v) $v(t, \omega, x)$ *is bounded from below if and only if* $x \ge \mathfrak{l}(\omega)$, *and from above if and only if* $x \le \mathfrak{u}(\omega)$.
- (vi) *Assume that* $\mathfrak{h}$ *satisfies also* **d1**, **d3** *and* **d4**, *and that* $\{\tilde{\mathfrak{l}}\}$, $\{\tilde{\mathfrak{m}}\}$ *and* $\{\tilde{\mathfrak{u}}\}$ *are three hyperbolic copies of the base with* $\tilde{\mathfrak{l}} < \tilde{\mathfrak{m}} < \tilde{\mathfrak{u}}$. *Then,* $\mathcal{A} = \bigcup_{\omega\in\Omega}(\{\omega\} \times [\tilde{\mathfrak{l}}(\omega), \tilde{\mathfrak{u}}(\omega)])$; $\{\tilde{\mathfrak{l}}\}$ *and* $\{\tilde{\mathfrak{u}}\}$ *are attractive and* $\{\mathfrak{m}\}$ *is repulsive;* $\lim_{t\to\infty}(v(t, \omega, x) - \tilde{\mathfrak{u}}(\omega{\cdot}t)) = 0$ *if and only if* $x > \tilde{\mathfrak{m}}(\omega)$; $\lim_{t\to\infty}(v(t, \omega, x) - \tilde{\mathfrak{l}}(\omega{\cdot}t)) = 0$ *if and only if* $x < \tilde{\mathfrak{m}}(\omega)$; *and* $\lim_{t\to-\infty}(v(t, \omega, x) - \tilde{\mathfrak{m}}(\omega{\cdot}t)) = 0$ *if and only if* $x \in (\tilde{\mathfrak{l}}(\omega), \tilde{\mathfrak{u}}(\omega))$.

Proof. The existence of m_1 and m_2 is ensured by **d2**. The properties stated in (i) are a nice exercise on ODEs. To prove (ii), we take $n_1 < m_1$ and $n_2 > m_2$ and check that $v(t, \omega, n_i) \in [m_1, m_2]$ for all $\omega \in \Omega$ and $i = 1, 2$ if $t \ge (1/\delta)\max(m_1 - n_1, n_2 - m_2)$. We deduce from this fact that $\lim_{t\to\infty} \mathrm{dist}(\tau_t(\mathcal{C}), \Omega \times [m_1, m_2]) = 0$ for every bounded set $\mathcal{C} \subset \Omega \times \mathbb{R}$; i.e., $\Omega \times [m_1, m_2]$ is a compact absorbing set. This property and [12, Theorem 2.2] prove the existence of the global attractor $\mathcal{A} \subseteq \Omega \times [m_1, m_2]$, and [11, Theorem 1.7] ensures the last assertion in (ii).

Assertion (iii) is a consequence of the compactness of $\mathcal{A}$; and the properties in (iv) follow from (i) and standard comparison results: see, e.g., the proof of [18, Theorem 5.1(iii)]. The assertions in (v) follow from (i) and (ii). In the conditions of (vi), Theorem 5.3 shows that $\mathcal{A} = \bigcup_{\omega\in\Omega}(\{\omega\}\times[\tilde{\mathfrak{l}}(\omega),\tilde{\mathfrak{u}}(\omega)])$, and that $\{\tilde{\mathfrak{l}}\}$ and $\{\tilde{\mathfrak{u}}\}$ are attractive and $\{\tilde{\mathfrak{m}}\}$ is repulsive. The remaining assertions are proved with the arguments used to check Proposition 3.5(vii), working with the ω-limit sets in the cases of $x > \tilde{\mathfrak{m}}(\omega)$ and $x < \tilde{\mathfrak{m}}(\omega)$, and with the α-limit set for $\tilde{\mathfrak{l}}(\omega) < x < \tilde{\mathfrak{u}}(\omega)$. □

It follows from the previous property (iii) that $\mathfrak{l}$ and $\mathfrak{u}$ are m-measurable equilibria for any $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$, which we will use without further reference. In the line of Theorem 3.6, our next result establishes equivalences regarding the existence of three uniformly separated hyperbolic solutions of a given equation in terms of the existence of three ordered hyperbolic copies of the corresponding hull.

Theorem 5.6. *Let $\mathfrak{h}\colon \Omega\times\mathbb{R}\to\mathbb{R}$ satisfy* **d1***,* **d2***,* **d3** *and* **d4***. Let us fix $\bar\omega\in\Omega$. Then, the following assertions are equivalent:*

(a) *Equation $(5.1)_{\bar\omega}$ has three hyperbolic solutions.*
(b) *Equation $(5.1)_{\bar\omega}$ has three uniformly separated hyperbolic solutions.*
(c) *Equation $(5.1)_{\bar\omega}$ has three uniformly separated bounded solutions.*
(d) *There exist three hyperbolic copies of the base for the restriction of the family (5.1) to the closure $\Omega_{\bar\omega}$ of $\{\bar\omega{\cdot}t\mid t\in\mathbb{R}\}$, given by $\tilde{\mathfrak{l}} < \tilde{\mathfrak{m}} < \tilde{\mathfrak{u}}$.*

In this case, $t\mapsto \tilde l(t) := \tilde{\mathfrak{l}}(\bar\omega{\cdot}t)$, $t\mapsto\tilde m(t) := \tilde{\mathfrak{m}}(\bar\omega{\cdot}t)$ and $t\mapsto\tilde u(t) := \tilde{\mathfrak{u}}(\bar\omega{\cdot}t)$ are the three unique uniformly separated solutions of $(5.1)_{\bar\omega}$, they are hyperbolic, and there are no more hyperbolic solutions. In addition, if $x_{\bar\omega}(t,s,x)$ is the solution of $(5.1)_{\bar\omega}$ with $x_{\bar\omega}(s,s,x) = x$, then: $\lim_{t\to\infty}(x_{\bar\omega}(t,s,x)-\tilde u(t)) = 0$ if and only if $x > \tilde m(s)$; $\lim_{t\to\infty}(x_{\bar\omega}(t,s,x)-\tilde l(t)) = 0$ if and only if $x < \tilde m(s)$; and $\lim_{t\to-\infty}(x_{\bar\omega}(t,s,x)-\tilde m(t)) = 0$ if and only if $x\in(\tilde l(s),\tilde u(s))$.

Proof. The proof of this result follows the line of that Theorem 3.6. The assertions after the equivalences follow from (d) and Proposition 5.5(v) and (vi). We will check (b) ⇒ (c) ⇒ (d) ⇒ (a) ⇒ (b). Recall that the hypotheses on $\mathfrak{h}$ are also valid for its restriction to $\Omega_{\bar\omega}\times\mathbb{R}$: see Remark 5.1.

(b) ⇒ (c) ⇒ (d). Obviously, (b) implies (c). Now we assume (c) and observe that there is no restriction in assuming that the three uniformly separated solutions are $l(t) := \mathfrak{l}(\bar\omega{\cdot}t)$, $u(t) := \mathfrak{u}(\bar\omega{\cdot}t)$, and $m(t)$ with $l(t) < m(t) < u(t)$. We will check that the closures $\mathcal{K}_l$, $\mathcal{K}_m$ and $\mathcal{K}_u$ of the corresponding τ-orbits are three different ordered compact sets projecting on $\Omega_{\bar\omega}$. Reasoning as in Theorem 3.6, we check that, for a $\delta > 0$, $x_0 \le \mathfrak{u}(\omega_0)-\delta$ and $x_0 \ge \mathfrak{l}(\omega_0)+\delta$ whenever $(\omega_0,x_0)\in\mathcal{K}_m$. Proposition 5.2 allows us to assert that all the Lyapunov exponents of $\mathcal{K}_m$ are positive, and that the upper and lower equilibria of $\mathcal{K}_m$ coincide on a set Ω_0 with $m_0(\Omega_0) = 1$ for all $m_0\in\mathfrak{M}_{\mathrm{erg}}(\Omega_{\bar\omega},\sigma)$. Theorem 2.8 ensures that $\mathcal{K}_m$ is a repulsive hyperbolic copy of the base $\Omega_{\bar\omega}$, say $\mathcal{K}_m = \{\tilde{\mathfrak{m}}\}$. Now, it is easy to deduce from the previous separation properties that $\mathcal{K}_l < \mathcal{K}_m < \mathcal{K}_u$, as asserted. Theorem 5.3 applied to $\Omega_{\bar\omega}\times\mathbb{R}$ shows that (d) holds.

(d) ⇒ (a) ⇒ (b). If (d) holds, then $t\mapsto\tilde{\mathfrak{l}}(\bar\omega{\cdot}t)$, $t\mapsto\tilde{\mathfrak{m}}(\bar\omega{\cdot}t)$ and $t\mapsto\tilde{\mathfrak{u}}(\bar\omega{\cdot}t)$ are three hyperbolic solutions of $(5.1)_{\bar\omega}$ (see Proposition 2.7), so (a) holds. Let us assume (a), and let $\tilde x_1 < \tilde x_2 < \tilde x_3$ be the three hyperbolic solutions of $(5.1)_{\bar\omega}$. Let us first eliminate the possibility that $\tilde x_2$ is attractive, assuming it for contradiction.

Let us call $l(t) := \mathfrak{l}(\bar\omega{\cdot}t)$ and $u(t) := \mathfrak{u}(\bar\omega{\cdot}t)$, where $\mathfrak{l}$ and $\mathfrak{u}$ are the lower and upper τ-equilibria: see (5.2). Proposition 2.6(i) yields $\delta > 0$ such that $\inf_{t\le 0}(u(t)-\tilde x_2(t)) > \delta$ and $\inf_{t\le 0}(\tilde x_2(t)-l(t)) > \delta$. Let $\mathcal{M}_l$, $\mathcal{M}_2$ and $\mathcal{M}_u$ be the α-limit sets of $(\bar\omega, l(0))$, $(\bar\omega, \tilde x_2(0))$ and $(\bar\omega, u(0))$, which project on the α-limit set $\Omega_- \subseteq \Omega_{\bar\omega}$ of $\bar\omega$. As in the previous paragraph, we check that $\mathfrak{l}(\omega_0)+\delta \le x_0 \le \mathfrak{u}(\omega_0)-\delta$ if $(\omega_0, x_0) \in \mathcal{M}_2$; and deduce that $\mathcal{M}_2$ is a repulsive copy of Ω_-. As explained in the proof of Theorem 3.6, this contradicts Proposition 2.7.

Hence, $\tilde x_2$ is repulsive. Proposition 2.6(i) yields $\delta > 0$ such that $\inf_{t\ge 0}(u(t) - \tilde x_2(t)) > \delta$ and $\inf_{t\ge 0}(\tilde x_2(t) - l(t)) > \delta$. Let $\bar{\mathcal{M}}_2$ be the ω-limit set of $(\bar\omega, \tilde x_2(0))$, which projects on the ω-limit set $\Omega_+ \subseteq \Omega_{\bar\omega}$ of $\bar\omega$. As in the proof of (c) $\Rightarrow$ (d), we check that $\mathfrak{l}(\omega_0) + \delta \le x_0 \le \mathfrak{u}(\omega_0) - \delta$ whenever $(\omega_0, x_0) \in \bar{\mathcal{M}}_2$; and we deduce that $\bar{\mathcal{M}}_2$ is a repulsive copy of Ω_+. Hence, $\bar{\mathcal{M}}_2$ does not intersect the ω-limit sets $\bar{\mathcal{M}}_1$ of $(\bar\omega, \tilde x_1(0))$ and $\bar{\mathcal{M}}_3$ of $(\bar\omega, \tilde x_3(0))$: see Proposition 2.6(ii). So, we have $\bar{\mathcal{M}}_1 < \bar{\mathcal{M}}_2 < \bar{\mathcal{M}}_3$. Theorem 5.3 applied to $\Omega_+ \times \mathbb{R}$ ensures that $\bar{\mathcal{M}}_1$ and $\bar{\mathcal{M}}_3$ are attractive hyperbolic copies of Ω_+, which according to Proposition 2.7 is only possible if $\tilde x_1$ and $\tilde x_3$ are attractive. Proposition 2.6(i) ensures that the three solutions are uniformly separated: (b) holds. ☐

In Section 6.1, we will analyze a situation precluding the occurrence of critical transitions. Some of the hypotheses will refer to the relative order of the three hyperbolic copies of the base of two "ordered" equations which are in the situation of Theorem 5.3. For the sake of completeness, Proposition 5.8 proves that there are just two relative positions in the case of a minimal base under the hypotheses so far assumed. In Section 6.3, we will check that the minimality of the base is indeed required for Proposition 5.8, whose proof is based on the next result.

Lemma 5.7. *Let $\mathfrak{h}_0, \mathfrak{h}_1 \colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy* **d1** *and* **d2***, with $\mathfrak{h}_0(\omega, x) \le \mathfrak{h}_1(\omega, x)$ for all $(\omega, x) \in \Omega \times \mathbb{R}$.*

(i) *Let $\mathfrak{l}_i$ (resp. $\mathfrak{u}_i$) be the lower (resp. upper) bounds of the global attractor (5.2) of $x' = \mathfrak{h}_i(\omega{\cdot}t, x)$ for $i = 0, 1$. Then, $\mathfrak{l}_0 \le \mathfrak{l}_1$ and $\mathfrak{u}_0 \le \mathfrak{u}_1$.*

(ii) *Assume also that $\mathfrak{h}_0$ and $\mathfrak{h}_1$ satisfy* **d3** *and* **d4***, and that there exists $\bar\omega \in \Omega$ such that $x' = \mathfrak{h}_i(\bar\omega{\cdot}t, x)$ has three hyperbolic solutions $\tilde l_i < \tilde m_i < \tilde u_i$ for $i = 0, 1$. Then, $\inf_{t\in\mathbb{R}}(\tilde m_0(t) - \tilde l_1(t)) > 0$ if and only if $\inf_{t\in\mathbb{R}}(\tilde u_0(t) - \tilde m_1(t)) > 0$, in which case $\tilde l_0 \le \tilde l_1 < \tilde m_1 \le \tilde m_0 < \tilde u_0 \le \tilde u_1$. If, in addition, $\mathfrak{h}_0(\bar\omega{\cdot}t, x) < \mathfrak{h}_1(\bar\omega{\cdot}t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$, then all the inequalities are strict.*

Proof. The proof repeats that of [19, Proposition 4.2], which is generalized by this result. We must just use Theorem 5.6 instead of [19, Theorem 3.3]. ☐

Proposition 5.8. *Assume that (Ω, σ) is minimal. Let $\mathfrak{h}_0, \mathfrak{h}_1 \colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy* **d1***,* **d2***,* **d3** *and* **d4***, and with $\mathfrak{h}_0(\omega, x) < \mathfrak{h}_1(\omega, x)$ for all $(\omega, x) \in \Omega \times \mathbb{R}$. Assume that the family $x' = \mathfrak{h}_i(\omega{\cdot}t, x)$ has three hyperbolic copies of the base $\tilde{\mathfrak{l}}_i < \tilde{\mathfrak{m}}_i < \tilde{\mathfrak{u}}_i$ for $i = 0, 1$. Then, one of the following orders holds:*

(1) $\tilde{\mathfrak{l}}_0 < \tilde{\mathfrak{l}}_1 < \tilde{\mathfrak{m}}_1 < \tilde{\mathfrak{m}}_0 < \tilde{\mathfrak{u}}_0 < \tilde{\mathfrak{u}}_1$,

(2) $\tilde{\mathfrak{l}}_0 < \tilde{\mathfrak{m}}_0 < \tilde{\mathfrak{u}}_0 < \tilde{\mathfrak{l}}_1 < \tilde{\mathfrak{m}}_1 < \tilde{\mathfrak{u}}_1$.

Proof. Let us assume (2) does not hold. Hence, there exists $\bar\omega \in \Omega$ with $\tilde{\mathfrak{l}}_1(\bar\omega) \le \tilde{\mathfrak{u}}_0(\bar\omega)$. A comparison argument ensures that $\tilde{\mathfrak{l}}_1(\bar\omega{\cdot}t) < \tilde{\mathfrak{u}}_0(\bar\omega{\cdot}t)$ for all $t < 0$, and hence the minimality of Ω ensures that $\tilde{\mathfrak{l}}_1 \le \tilde{\mathfrak{u}}_0$. If, in addition, $\tilde{\mathfrak{l}}_1(\omega_0) = \tilde{\mathfrak{u}}_0(\omega_0)$

for an $\omega_0 \in \Omega$, then a new comparison argument shows that $\tilde{\mathfrak{u}}_0(\omega_0{\cdot}t) < \tilde{\mathfrak{l}}_1(\omega_0{\cdot}t)$ for all $t > 0$, impossible. Therefore, $\tilde{\mathfrak{l}}_1 < \tilde{\mathfrak{u}}_0$. Now, for contradiction, we assume that neither (1) holds, and deduce from Lemma 5.7(ii) and the minimality of Ω the existence of $\bar\omega \in \Omega$ with $\tilde{\mathfrak{m}}_0(\bar\omega) \le \tilde{\mathfrak{l}}_1(\bar\omega)$. Hence, $\tilde{\mathfrak{m}}_0(\bar\omega{\cdot}t) < \tilde{\mathfrak{l}}_1(\bar\omega{\cdot}t)$ for all $t > 0$. We fix $t_0 > 0$ and call $\omega_0 := \bar\omega{\cdot}t_0$. Proposition 5.5(vi) yields $\lim_{t\to\infty}(\tilde{\mathfrak{u}}_0(\omega_0{\cdot}t) - v_0(t,\omega_0,\tilde{\mathfrak{l}}_1(\omega_0))) = 0$, where v_0 stands for the solutions of $x' = \mathfrak{h}_0(\omega{\cdot}t, x)$. Since $v_0(t,\omega_0,\tilde{\mathfrak{l}}_1(\omega_0)) < \tilde{\mathfrak{l}}_1(\omega_0{\cdot}t)$ for all $t > 0$, we deduce that $\limsup_{t\to\infty}(\tilde{\mathfrak{u}}_0(\omega_0{\cdot}t) - \tilde{\mathfrak{l}}_1(\omega_0{\cdot}t))) \le 0$, which combined with the minimality of Ω contradicts $\tilde{\mathfrak{l}}_1 < \tilde{\mathfrak{u}}_0$. □

We point out that, under the hypotheses of Proposition 5.8, part of the arguments of [18, Section 5] show that the situation (1) is equivalent to the absence of a bifurcation value $\lambda_0 \in [0,1]$ for the family $x' = \mathfrak{h}_0(\omega{\cdot}t,x) + \lambda(\mathfrak{h}_1(\omega{\cdot}t,x) - \mathfrak{h}_0(\omega{\cdot}t,x))$.

6. Asymptotically d-concave transition equations

Let $g\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a C^2-admissible function. As explained at the beginning of Section 4, we can understand

$$x' = g(t,x) \tag{6.1}$$

as a transition between the corresponding α-family and ω-family. Now, we will assume that these limit families satisfy conditions **d1-d4**, as well as the existence of three hyperbolic copies of the base for the α-family and the ω-family. These last conditions are those which provide a wider range of dynamical possibilities for (6.1) under conditions **d1-d4**: the maximum number of uniformly separated solutions for each equation of the α-family or the ω-family is three (see Theorem 5.6); hence, Proposition 2.5 precludes the existence of more than three uniformly separated solutions of (6.1); and, if there are three, then Theorem 5.6 yields three hyperbolic copies of the base for the α-family and the ω-family.

As in Section 4, we will achieve the required properties by assuming the existence of strictly d-concave (in x) maps g_- and g_+ forming asymptotic pairs with g. More precisely, we fix g and assume the existence of g_- and g_+ such that

gd1 $g, g_-, g_+ \in C^{0,2}(\mathbb{R}\times\mathbb{R},\mathbb{R})$.
gd2 $\lim_{t\to\pm\infty}(g(t,x) - g_\pm(t,x)) = 0$ uniformly on each compact subset $\mathcal{J} \subset \mathbb{R}$.
gd3 $\limsup_{x\to\pm\infty}(\pm h(t,x)) < 0$ uniformly on $\mathbb{R}$ for $h = g, g_-, g_+$.
gd4 $\inf_{t\in\mathbb{R}}\big((g_\pm)_{xx}(t,x_1) - (g_\pm)_{xx}(t,x_2)\big) > 0$ whenever $x_1 < x_2$.
gd5 Each one of the equations

$$x' = g_-(t,x) \quad \text{and} \quad x' = g_+(t,x) \tag{6.2}$$

has three hyperbolic solutions, $\tilde l_{g_-} < \tilde m_{g_-} < \tilde u_{g_-}$ and $\tilde l_{g_+} < \tilde m_{g_+} < \tilde u_{g_+}$.

If all these conditions hold, we say that (6.1) is a *(coercive and asymptotic) d-concave ordinary differential equation*. Note once again that the d-concavity of $x \mapsto g(t,x)$ is not required for all $t \in \mathbb{R}$. The maps $g(t,x) := -x(x-1)(x-2) - \Gamma(t)\,x^2/(1+x^2)$ with Γ continuous and $\lim_{t\to\pm\infty}\Gamma(t) = 0$ and $g_\pm(t,x) := -x(x-1)(x-2)$ satisfy these conditions, and if we choose Γ with $\Gamma(0) = 5$, then $x \mapsto g_x(0,x)$ is not concave.

As in Section 4 (see Remarks 4.1), we will say that "g satisfies conditions **gd1-gd5**" if there exist g_- and g_+ such that all the listed conditions are satisfied, and we will refer to the first and second equations in (6.2) as the *past* and *future equations* of the *transition equation* (6.1). Some of the results of this section extend part of those of [19] to a much more general setting: the setting and hypotheses of this section are considerably less restrictive than those leading to the analogous results in [19].

Our initial purpose is to classify the dynamical scenarios for the transition equation (6.1) when g satisfies **gd1-gd5**, which is achieved in Theorem 6.4. Its proof is based on some previous results. The notation established before Lemma 4.2 is used in what follows. The next result shows that conditions **gd1-gd4** provide a family of d-concave ordinary differential equations (see Section 5) via the hull construction. Its proof, which uses Lemma 4.2, is almost identical to that of Lemma 4.3. The only difference is that we must work with the second derivative in the last step of the proof.

Lemma 6.1. *If $h \in C^{0,2}(\mathbb{R}\times\mathbb{R},\mathbb{R})$ then $\mathfrak{h}$ satisfies* **d1** *on Ω_h. If $h \in C^{0,2}(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $\limsup_{x\to\pm\infty}(\pm h(t,x)) < 0$ uniformly on $\mathbb{R}$, then $\mathfrak{h}$ satisfies* **d2** *on Ω_h. And, if* **gd1**, **gd2** *and* **gd4** *hold, then $\mathfrak{g}$ and $\mathfrak{g}_\pm$ satisfy* **d3** *and* **d4** *on Ω_g and $\Omega_{g_\pm}$, respectively.*

Assume that $h \in C^{0,2}(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $\limsup_{x\to\pm\infty}(\pm h(t,x)) < 0$. Lemma 6.1 and Proposition 5.5 ensure the existence of the global attractor $\mathcal{A}_h = \bigcup_{\omega\in\Omega_h}\big(\{\omega\}\times[\mathfrak{l}_h(\omega),\mathfrak{u}_h(\omega)]\big)$ of the flow τ_h defined by $x' = \mathfrak{h}(\omega{\cdot}t, x)$ on $\Omega_h\times\mathbb{R}$. In particular, if $\omega_0 := h$, then the maps $l_h(t) := \mathfrak{l}_h(\omega_0{\cdot}t)$ and $u_h(t) := \mathfrak{u}_h(\omega_0{\cdot}t)$ define the lower and upper bounded solutions of $x' = h(t,x)$. In addition, the global pullback attractor of the induced process is $\{[l_h(s), u_h(s)] \mid s\in\mathbb{R}\}$ (see e.g. [11, Definition 1.12, Theorem 2.12 and Corollary 1.18], and the proof of Proposition 5.5(ii)). Recall that $x_h(t,s,x)$ satisfies $x' = h(t,x)$ and $x_h(s,s,x) = x$.

Proposition 6.3, key in the proof of Theorem 6.4, establishes the existence of three solutions which govern the dynamics of (6.1) if **gd1-gd5** hold: the two previously described solutions l_g and u_g, which are locally pullback attractive, and a locally pullback repulsive one, m_g. Its proof requires the next previous result:

Proposition 6.2. *Assume that (6.1) has three uniformly separated hyperbolic solutions $\tilde{l}_g < \tilde{m}_g < \tilde{u}_g$, with $\tilde{l}_g$ and $\tilde{u}_g$ attractive and $\tilde{m}_g$ repulsive.*

(i) *If g and g_+ satisfy all the conditions involving them in* **gd1-gd5**, *then $\lim_{t\to\infty}(x_g(t,s,x) - \tilde{u}_{g_+}(t)) = 0$ for $x > \tilde{m}_g(s)$ and $\lim_{t\to\infty}(x_g(t,s,x) - \tilde{l}_{g_+}(t)) = 0$ for $x < \tilde{m}_g(s)$.*

(ii) *If g and g_- satisfy all the conditions involving them in* **gd1-gd5**, *then $t \mapsto x_g(t,s,x)$ is bounded from above (resp. from below) as time decreases if and only if $x \le \tilde{u}_g(s)$ (resp. $x \ge \tilde{l}_g(s)$); and $\lim_{t\to-\infty}(x_g(t,s,x) - \tilde{m}_{g_-}(t)) = 0$ for $x \in (\tilde{l}_g(s), \tilde{u}_g(s))$.*

Proof. We proceed as in the proof of [19, Proposition 3.5]: Lemma 6.1 allows us to use the information of Theorem 5.6 instead of that of [19, Theorem 3.3], and Proposition 2.6 provides the necessary information on uniform separation of the α-limit and ω-limit sets of the points $(g, \tilde{l}_g(0))$, $(g, \tilde{m}_g(0))$ and $(g, \tilde{u}_g(0))$ for the corresponding skewproduct. □

Proposition 6.3. *Assume that g satisfies* **gd1-gd5**, *let $\tilde{l}_{g_\pm} < \tilde{m}_{g_\pm} < \tilde{u}_{g_\pm}$ be the hyperbolic solutions given by* **gd5**, *and let l_g and u_g be the lower and upper bounded solutions of (6.1). Then,*

(i) u_g *and* l_g *are the unique solutions of* (6.1) *satisfying* $\lim_{t\to-\infty}(u_g(t) - \tilde{u}_{g_-}(t)) = 0$ *and* $\lim_{t\to-\infty}(l_g(t) - \tilde{l}_{g_-}(t)) = 0$, *and they are locally pullback attractive.*

(ii) *There exists a unique solution* m_g *of* (6.1) *defined at least on a positive half-line and satisfying* $\lim_{t\to\infty}(m_g(t) - \tilde{m}_{g_+}(t)) = 0$, *and it is locally pullback repulsive.*

Moreover, for $s \in \mathbb{R}$ *in the interval of definition of* m_g, $\lim_{t\to\infty}(x_g(t,s,x) - \tilde{u}_{g_+}(t)) = 0$ *if and only if* $x > m_g(s)$, *and* $\lim_{t\to\infty}(x_g(t,s,x) - \tilde{l}_{g_+}(t)) = 0$ *if and only if* $x < m_g(s)$. *In addition, for any* $s \in \mathbb{R}$, $\lim_{t\to-\infty}(x_g(t,s,x) - \tilde{m}_{g_-}(t)) = 0$ *if and only if* $x \in (l_g(s), u_g(s))$.

Proof. The assertions reproduce those of [19, Theorem 3.7], formulated under more restrictive hypotheses. The proof basically repeats step by step that one, using the information provided by Proposition 6.2 to check the last assertions. The differences rely on the first steps, which we detail. We take $m > 0$ such that $\|b\|_\infty \le m$ for any bounded solution of $x' = g(t,x)$ and $x' = g_\pm(t,x)$ (see Proposition 5.5(ii)). Given $\varepsilon > 0$, Theorem 2.3 provides $\delta_\pm > 0$ such that, if $\|g_\pm - h_\pm\|_{1,m} < \delta_\pm$, then each one of the equations $x' = h_\pm(t,x)$ has three hyperbolic solutions, at a uniform distance from those of $x' = g_\pm(t,x)$ bounded by ε. We choose $t^0 = t^0(\varepsilon) > 0$ such that $|g(t,x) - g_\pm(t,x)| < \delta_\pm/2$ and $|g_x(t,x) - (g_\pm)_x(t,x)| < \delta_\pm/2$ if $\pm t \ge t^0$ and $|x| \le m$ (see Lemma 4.5), and define $f_\pm(t,x)$ as $g(t,x)$ if $\pm t > t^0$ and as $g_\pm(t,x) - g_\pm(\pm t^0, x) + g(\pm t^0, x)$ otherwise. The solutions of $x' = g(t,x)$ with values $\tilde{l}_{f_-}(-t^0)$, $\tilde{u}_{f_-}(-t^0)$ and $\tilde{m}_{f_+}(t^0)$ provide the solutions l_g, u_g and m_g of the statement, as we can proof by repeating the remaining arguments of [19]. □

We will denote l_g, m_g and u_g by $\tilde{l}_g$, $\tilde{m}_g$ and $\tilde{u}_g$ when they are hyperbolic. Now we will formulate the announced result concerning the dynamical possibilities for (6.1). Recall that two uniformly separated solutions are, by definition, bounded. Clearly, there exist (at least) two uniformly separated solutions if and only if l_g and u_g satisfy this property.

Theorem 6.4. *Assume that* g *satisfies* **gd1**-**gd5**, *let* $\tilde{l}_{g_\pm} < \tilde{m}_{g_\pm} < \tilde{u}_{g_\pm}$ *be the hyperbolic solutions given by* **gd5**, *and let* l_g, u_g *and* m_g *be the solutions of* (6.1) *provided by Proposition* 6.3. *Then, the dynamics of the transition equation* (6.1) *fits in one of the following dynamical scenarios:*

- Case A*: there exist exactly three hyperbolic solutions,* $\tilde{l}_g := l_g$ *and* $\tilde{u}_g := u_g$, *which are attractive, and* $\tilde{m}_g := m_g$, *which is repulsive. In addition, the unique solution uniformly separated from* $\tilde{l}_g$ *and* $\tilde{u}_g$ *is* $\tilde{m}_g$. *In this case,* $\tilde{l}_g < \tilde{m}_g < \tilde{u}_g$, $\lim_{t\to\pm\infty}(\tilde{l}_g(t)-\tilde{l}_{g_\pm}(t)) = 0$, $\lim_{t\to\pm\infty}(\tilde{m}_g(t)-\tilde{m}_{g_\pm}(t)) = 0$ *and* $\lim_{t\to\pm\infty}(\tilde{u}_g(t)-\tilde{u}_{g_\pm}(t)) = 0$.
- Case B*: there exists exactly one hyperbolic solution, which is attractive, and uniformly separated only from another solution, which is locally pullback attractive and repulsive. There are two possibilities:*
  - Case B1*:* $\tilde{u}_g = u_g$ *is hyperbolic attractive and uniformly separated of* $l_g = m_g$. *In this case,* $\lim_{t\to\infty}(\tilde{u}_g(t) - \tilde{u}_{g_+}(t)) = 0$ *and* $\lim_{t\to\infty}(l_g(t) - \tilde{m}_{g_+}(t)) = 0$.
  - Case B2*:* $\tilde{l}_g = l_g$ *is hyperbolic attractive and uniformly separated of* $m_g = u_g$. *In this case,* $\lim_{t\to\infty}(\tilde{l}_g(t) - \tilde{l}_{g_+}(t)) = 0$ *and* $\lim_{t\to\infty}(u_g(t) - \tilde{m}_{g_+}(t)) = 0$.

- Case C*: there are no uniformly separated solutions. In this case,* $\tilde{l}_g = l_g$ *and* $\tilde{u}_g = u_g$ *are the unique hyperbolic solutions, they are attractive, and the locally pullback repulsive solution* m_g *is unbounded. There are two possibilities:*
  - Case C1*:* $m_g < \tilde{l}_g$ *in its domain of definition. In this case,* $\lim_{t\to\infty}(\tilde{l}_g(t) - \tilde{u}_{g_+}(t)) = \lim_{t\to\infty}(\tilde{u}_g(t) - \tilde{u}_{g_+}(t)) = 0$.
  - Case C2*:* $m_g > \tilde{u}_g$ *in its domain of definition. In this case,* $\lim_{t\to\infty}(\tilde{l}_g(t) - \tilde{l}_{g_+}(t)) = 0 = \lim_{t\to\infty}(\tilde{u}_g(t) - \tilde{l}_{g_+}(t)) = 0$.

Proof. Proposition 6.3 allows us to repeat the proofs of [19, Theorems 3.9 and 3.10 and Corollary 3.11] under conditions **gd1-gd5**. The statement of this theorem follows from those ones and Proposition 6.3. □

Figures 2, 3 and 4 of [19] depict these five dynamical possibilities in the case of a map f which is asymptotically periodic with respect to t. In addition, they can be characterized in terms of the forward attraction properties of the global pullback attractor for (6.1): the proof of [19, Proposition 3.12] can be repeated in our framework.

In the line of Proposition 4.8, the next result establishes two conditions precluding some of the five cases, and which, together, guarantee Case A. The example depicted in Figure 6 shows that the hypotheses concerning the relative order of $\tilde{m}_{g_+}$ and the bounded solution b_i are not superfluous. So, the conditions are more exigent than those of the analogous result in the concave case, Proposition 4.8.

Proposition 6.5. *Assume that g satisfies* **gd1-gd5***. Then,*

(i) *if there exists $h_1 \colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that $h_1(t,x) \le g(t,x)$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}$, and $x' = h_1(t,x)$ has a bounded solution b_1 such that* $\liminf_{t\to\infty}(b_1(t) - \tilde{m}_{g_+}(t)) > 0$, *then $x' = g(t,x)$ is in* Case A, B1 *or* C1.

(ii) *If there exists $h_2 \colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that $h_2(t,x) \ge g(t,x)$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}$, and $x' = h_2(t,x)$ has a bounded solution b_2 such that* $\liminf_{t\to\infty}(\tilde{m}_{g_+}(t) - b_2(t)) > 0$, *then $x' = g(t,x)$ is in* Case A, B2 *or* C2.

Proof. Let us check (i): the second proof is analogous. Let u_g be the upper bounded solution for $x' = g(t,x)$ (see Proposition 5.5(ii)). Then, $b_1 \le u_g$ (see Proposition 5.5(iv)). Hence, $\liminf_{t\to\infty}(u_g(t) - \tilde{m}_{g_+}(t)) \ge \liminf_{t\to\infty}(b_1(t) - \tilde{m}_{g_+}(t)) > 0$, which according to Theorem 6.4 precludes Cases B2 and C2. □

As in the concave case, we will focus on critical transitions associated to one-parametric families of equations which occur when the dynamics moves from Case A to one of the Cases C of Theorem 6.4 as the parameter crosses a *critical value*. Theorem 6.6 shows the persistence of Cases A, C1 and C2 under small suitable parametric variations, as well as the occurrence of a saddle-node bifurcation phenomenon when Case A transits to one of the Cases B as the parameter varies.

Theorem 6.6. *Let $\mathcal{C} \subseteq \mathbb{R}$ be an open interval, and let $\bar{g} \colon \mathbb{R}\times\mathbb{R}\times\mathcal{C}\to\mathbb{R}$ be a map such that $g^c(t,x) := \bar{g}(t,x,c)$ satisfies* **gd1-gd5** *for all $c \in \mathcal{C}$. Let $\bar{g}_x$ be the partial derivative with respect to the second variable, and assume that $\bar{g}$ and $\bar{g}_x$ are admissible on $\mathbb{R}\times\mathbb{R}\times\mathcal{C}$. Assume also that* $\limsup_{x\to\pm\infty}(\pm\bar{g}(t,x,c)) < 0$ *uniformly on $\mathbb{R}\times\mathcal{J}$ for any compact interval $\mathcal{J}\subset\mathcal{C}$.*

(i) *Assume that there exist c_1, c_2 in $\mathcal{C}$ with $c_1 < c_2$ such that the dynamics of $x' = g^c(t,x)$ is in* Case A *for $c = c_1$ and not for $c = c_2$. If $c_0 := \inf\{c > c_1 \mid$* Case A *does not hold$\}$, then $c_0 > c_1$. Let $\tilde{l}_{g^c} < \tilde{m}_{g^c} < \tilde{u}_{g^c}$ be the three*

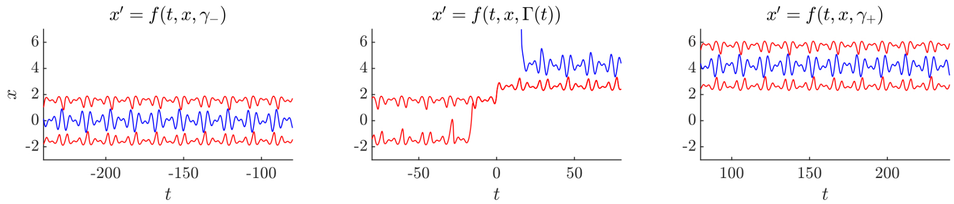

FIGURE 6. We define $\Gamma(t) = \arctan(5t)/\pi + 1/2 \in (0,1)$, $g(t,x) = -x^3 + \sin(t) + \sin(\sqrt{2}\,t) + (5/2)x + \Gamma(t)\,a(3x^2 - 3ax + a^2 - 5/2)$, and $g_-(t,x)$ and $g_+(t,x)$ by replacing $\Gamma(t)$ by 0 and 1 in $g(t,x)$, respectively. Then, **gd1-gd5** hold. The central panel shows that the dynamics of $x' = g(t,x)$ for $a = 4.2$ corresponds to CASE C2: we depict in red the two attractive hyperbolic solutions $\tilde{l}_g$ and $\tilde{u}_g$, and in blue the unbounded locally pullback repulsive solution m_g. It is easy to check that $g_-(t,x) \le g(t,x)$ for this choice of a. But, as checked below, any bounded solution of $x' = g_-(t,x)$ (which are bound by the red curves in the left panel) is below m_{g_+} (depicted in blue in the right panel), and hence neither the hypotheses nor the thesis of Proposition 6.5(i) are fulfilled. It is also easy to check that $g_-(t,x-a) = g_+(t,x)$ and that $\pm g_-(t,r) < 0$ for $\pm r > 2$. Hence, $-2 \le \tilde{l}_{g_-}(t) < \tilde{u}_{g_-}(t) \le 2$, and $2.2 \le \tilde{l}_{g_+}(t) < \tilde{u}_{g_+}(t) \le 6.2$, which implies the assertion. In addition, since $\liminf_{t\to-\infty}(\tilde{l}_{g_+}(t) - u_g(t)) > \lim_{t\to-\infty}(\tilde{u}_{g_-}(t) - u_g(t)) + 0.2 = 0.2$, we get $u_g(t) < \tilde{l}_{g_+}(t)$ for $t \le t_0$; and $u_g'(t) < \tilde{l}'_{g_+}(t)$ if $u_g(t) = \tilde{l}_{g_+}(t)$, from where we deduce that $u_g(t) < \tilde{l}_{g_+}(t)$ for all $t \in \mathbb{R}$. This is only possible in CASE C2.

hyperbolic solutions of $x' = g^c(t,x)$ for $c \in [c_1, c_0)$. Then, the dynamics of $x' = g^{c_0}(t,x)$ is either in CASE B1*, with $\lim_{c\to c_0^-}(\tilde{m}_{g^c}(t) - \tilde{l}_{g^c}(t)) = 0$ for all $t \in \mathbb{R}$, or in* CASE B2*, with $\lim_{c\to c_0^-}(\tilde{u}_{g^c}(t) - \tilde{m}_{g^c}(t)) = 0$ for all $t \in \mathbb{R}$. The results are analogous if $c_1 > c_2$.*

(ii) *Assume that there exist c_3, c_4 in $\mathcal{C}$ with $c_3 < c_4$ such that the dynamics of $x' = g^c(t,x)$ is in* CASE C1 *for $c = c_3$ and not for $c = c_4$. If $c_0 := \inf\{c > c_3 \mid$ CASE C1 *does not hold$\}$, then $c_0 > c_3$, and the dynamics of $x' = g^{c_0}(t,x)$ is in* CASE B1*. The results are analogous by replacing* C1 *and* B1 *by* C2 *and* B2*, and also if $c_3 > c_4$.*

Proof. As in the proof of Theorem 4.9, the admissibility hypotheses combined with Theorems 6.4 and 2.3 guarantee the persistence of CASE A under small variations of c. Let us check that also CASE C1 is persistent, assuming for contradiction that $x' = g^{c_3}(t,x)$ is in this case and the existence of a sequence (c_n) with limit c_3 such that $x' = g^{c_n}(t,x)$ is not CASE C1 for all $n \in \mathbb{N}$. (The same argument works for CASE C2.) Theorem 2.3 shows that $x' = g^{c_n}(t,x)$ has two different attractive hyperbolic solutions for large enough n, which must be $\tilde{l}_{g^{c_n}}$ and $\tilde{u}_{g^{c_n}}$ (see Theorem 6.4), and which satisfy $\lim_{n\to\infty}\left\|\tilde{l}_{g^{c_n}} - \tilde{l}_{g^{c_3}}\right\|_\infty = \lim_{n\to\infty}\left\|\tilde{u}_{g^{c_n}} - \tilde{u}_{g^{c_3}}\right\|_\infty = 0$. This precludes CASES B. Let ρ be the radio of the common domains of attraction also provided by Theorem 2.3, and let us take n_0 such that $\left\|\tilde{l}_{g^{c_n}} - \tilde{l}_{g^{c_3}}\right\|_\infty < \rho/3$ and $\left\|\tilde{u}_{g^{c_n}} - \tilde{u}_{g^{c_3}}\right\|_\infty < \rho/3$ for all $n \ge n_0$. If $n \ge n_0$, we deduce from $\lim_{t\to\infty}(\tilde{u}_{g^{c_3}}(t) - \tilde{l}_{g^{c_3}}(t)) = 0$ the existence of t_0 such that $|\tilde{u}_{g^{c_n}}(t_0) - \tilde{l}_{g^{c_n}}(t_0)| < \rho$, and hence that $\lim_{t\to\infty}(\tilde{u}_{g^{c_n}}(t) - \tilde{l}_{g^{c_n}}(t)) = 0$, which precludes CASE A. That is, $x' = g^{c_n}(t,x)$ is in CASE C2 for all $n \ge n_0$.

Let k be a common bound for the $\|\cdot\|_\infty$-norm of the bounded solutions of $x' = g^{c_3}(t,x)$ and $x' = g_+^{c_3}(t,x)$, and let $\varepsilon > 0$ be smaller than $\inf_{t\in\mathbb{R}}(\tilde{u}_{g_+^{c_3}}(t) - \tilde{m}_{g_+^{c_3}}(t))$ and $\inf_{t\in\mathbb{R}}(\tilde{m}_{g_+^{c_3}}(t) - \tilde{l}_{g_+^{c_3}}(t))$. Theorem 2.3 applied to $\varepsilon/4$ provides $\delta > 0$ such that, if f is C^1-admissible and $\left\|f - g_+^{c_3}\right\|_{1,k} < \delta$, then $x' = f(t,x)$ has three hyperbolic solutions at a $\|\cdot\|_\infty$-distance of those of $x' = g_+^{c_3}(t,x)$ less that $\varepsilon/4$, and hence with a separation between of at least $\varepsilon/2$. The admissibility of $\bar{g}$ and condition **gd2** applied to g^{c_3} and $g_+^{c_3}$ allow us to choose t_0 and n_0 large enough to get

$$\sup_{(t,x)\in[t_0,\infty)\times[-k,k]} |g^{c_n}(t,x) - g_+^{c_3}(t,x)| + \sup_{(t,x)\in[t_0,\infty)\times[-k,k]} |(g^{c_n})_x - (g_+^{c_3})_x(t,x)| < \delta$$

for all $n \geq n_0$: we just write $|g^{c_n} - g_+^{c_3}| \leq |g^{c_n} - g^{c_3}| + |g^{c_3} - g_+^{c_3}|$, do the same with the derivatives, and apply Lemma 4.5. Let us define $f_+^{c_n}(t,x)$ by truncating g^{c_n} at t_0, as in the proof of Proposition 6.3. Since $\left\|f_+^{c_n} - g_+^{c_3}\right\|_{1,k} < \delta$, $x' = g^{c_n}(t,x)$ has three (possibly locally defined) solutions, $b_1^{c_n} < b_2^{c_n} < b_3^{c_n}$, with $|b_i^{c_n}(t)| \leq k + \varepsilon/4$ and $b_{i+1}^{c_n}(t) - b_i^{c_n}(t) \geq \varepsilon/2$ for all $t \geq t_0$ and $n \geq n_0$. We define $\bar{b}_i^{c_3}(t) := \lim_{n\to\infty} b_i^{c_n}(t)$ for $i = 1, 2, 3$, and get three solutions of $x' = g^{c_3}(t,x)$ defined and uniformly separated by $\varepsilon/2$ on $[t_0,\infty)$. Since we are in Case C2, we have $b_2^{c_n}(t) \geq \tilde{u}_{g^{c_n}}(t)$ for all $t \in [t_0,\infty)$: there cannot be two different solutions separated on $[t_0,\infty)$ strictly below $\tilde{u}_{g^{c_n}}$. Hence, $\bar{b}_3^{c_3}(t) \geq \bar{b}_2^{c_3}(t) + \varepsilon/2 \geq \tilde{u}_{g^{c_3}}(t) + \varepsilon/2$ for all $t \in [t_0,\infty)$, which is not possible in Case C1. This is the sought-for contradiction.

Let us complete the proof of (i) with $c_1 < c_2$. The persistence of Cases A and C ensures that $c_0 > c_1$ and that $x' = g^{c_0}(t,x)$ is in one of the Cases B, say B1. Let us prove that $\lim_{c\to c_0^-}(\tilde{m}_{g^c}(t) - \tilde{l}_{g^c}(t)) = 0$ by checking that, given $(c_n) \uparrow c_0$, $\lim_{n\to\infty} \tilde{m}_{g^{c_n}}(t) = \lim_{n\to\infty} \tilde{l}_{g^{c_n}}(t) = m_{g^{c_0}}(t)$ for all $t \in \mathbb{R}$. The existence of these limits follows from the existence of a common bound for all the bounded solutions if n is large enough. A new application of last assertion of Theorem 2.3 applied to $\tilde{u}_{g^{c_0}}$ and its approximants $\tilde{u}_{g_{c_n}}$ shows that $\tilde{u}_{g_{c_n}}(t) - \tilde{l}_{g_{c_n}}(t) > \tilde{u}_{g_{c_n}}(t) - \tilde{m}_{g_{c_n}}(t) \geq \rho$ if n is large enough, with a common $\rho > 0$. And hence both limits are $m_{g^{c_0}}$, which is the unique bounded solution of $x' = g^{c_0}(t,x)$ uniformly separated from $\tilde{u}_{g^{c_0}} = \lim_{n\to\infty} \tilde{u}_{g^{c_n}}$ (see Theorem 6.4). The remaining situations are proved with similar arguments.

To complete the proof of (ii) if $c_3 < c_4$ and with $x' = g^{c_3}(t,x)$ in Case C2, we deduce from the proved persistence that $c_0 > c_3$ and that the dynamics of $x' = g^{c_0}(t,x)$ is in one of the Cases B. Let us assume for contradiction that it is in Case B1, so that $\tilde{u}_{g^{c_0}}$ is hyperbolic. We take $(c_n) \uparrow c_0$, with $x' = g^{c_n}(t,x)$ in Case C2, and get the sought-for contradiction by repeating the last paragraph of the proof of the persistence of Case C1: just replace c_3 by c_0. The remaining cases are proved with similar arguments. □

We complete this part with an analogue of Corollary 4.10 for the d-concave case.

Proposition 6.7. *Let $\mathcal{C} \subseteq \mathbb{R}$ be an open interval, and let $\{g^c \mid c \in \mathcal{C}\}$ be a family of functions satisfying* **gd1-gd5** *and such that, if $\bar{g}(t,x,c) := g^c(t,x)$, then $\bar{g}$ and $\bar{g}_x$ are admissible on $\mathbb{R}\times\mathbb{R}\times\mathcal{C}$. Assume that there exists $\bar{c} \in \mathcal{C}$ such that the dynamics of $x' = g^{\bar{c}}(t,x)$ is in* Case B1 *(resp.* Case B2*), and such that, for all $c_-, c_+ \in \mathcal{C}$ with $c_- < \bar{c} < c_+$: $g^{c_-}(t,x) \leq g^{\bar{c}}(t,x) \leq g^{c_+}(t,x)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}$; and there exist t_{c_-} and t_{c_+} such that the first and second inequalities are strict for $t = t_{c_-}$ and $t = t_{c_+}$ (respectively) and all $x \in \mathbb{R}$. Then, there exists $\rho > 0$ such that $x' = g^c(t,x)$ is in* Case A *(resp.* Case C2*) for $c \in (\bar{c}-\rho,\bar{c})$ and in* Case C1 *(resp.* Case A*) for $c \in (\bar{c},\bar{c}+\rho)$.*

Proof. Let l_c, m_c and u_c be the three solutions of $x' = g^c(t,x)$ given by Proposition 6.3. Let g_+^c be the globally bounded and C^2-admissible function associated to g^c

by **gd2** at $+\infty$, and let $\tilde l_{g_+^c} < \tilde m_{g_+^c} < \tilde u_{g_+^c}$ be the three hyperbolic solutions of $x' = g_+^c(t,x)$ provided by **gd5**. Let k be a common bound for the $\|\cdot\|_\infty$-norm of these three solutions. We take $2\varepsilon > 0$ smaller than $\inf_{t\in\mathbb{R}}(\tilde u_{g_+^{\bar c}}(t) - \tilde m_{g_+^{\bar c}}(t))$ and $\inf_{t\in\mathbb{R}}(\tilde m_{g_+^{\bar c}}(t) - \tilde l_{g_+^{\bar c}}(t))$. Then, Theorem 2.3 provides $\delta > 0$ such that if $\|g_+^{\bar c} - g\|_{1,k} < \delta$ for a C^1-admissible map g, then $x' = g(t,x)$ has three hyperbolic solutions at $\|\cdot\|_\infty$-distance of those of $x' = g_+^{\bar c}(t,x)$ less than ε.

The admissibility of $\bar g$ and $(\bar g)_x$ provide $\rho > 0$ such that $\|g^{\bar c} - g^c\|_{1,k} < \delta/3$ for all $c \in (\bar c - \rho, \bar c + \rho)$. Our hypotheses provide $t_c \ge 0$ with $\sup_{(t,x)\in[t_c,\infty)\times[-k,k]} |g^c(t,x) - g_+^c(t,x)| + \sup_{(t,x)\in[t_c,\infty)\times[-k,k]} |g_x^c(t,x) - (g_+^c)_x(t,x)| < \delta/3$: see **gd2** and Lemma 4.5. We take $c^* \in (\bar c - \rho, \bar c + \rho)$ and $t^0 := \max(t_{\bar c}, t_{c^*})$. For $h(t,x)$, we denote by $\hat h(t,x)$ the map given by $h(t,x)$ for $t \ge t^0$ and by $g_+^{\bar c}(t,x) - g_+^{\bar c}(t^0,x) + h(t^0,x)$ for $t < t^0$. In this way, we construct $\hat g_+^{c^*}$, $\hat g^{\bar c}$ and $\hat g^{c^*}$ from $g_+^{c^*}$, $g^{\bar c}$ and g^{c^*}, and note that they are C^1-admissible. Then: $\|\hat g^{\bar c} - \hat g^{c^*}\|_{1,k} < \delta/3$, since the difference is $\hat g^{\bar c}(t_0,x) - \hat g^{c^*}(t_0,x)$ for $t < t^0$ and $g^{\bar c}(t,x) - g^{c^*}(t,x)$ for $t \ge t^0$; and $\|g^{c^*} - g_+^{c^*}\|_{1,k} < \delta/3$ and $\|g_+^{\bar c} - \hat g^{\bar c}\|_{1,k} < \delta/3$ for analogous reasons. So, $\|g_+^{\bar c} - \hat g_+^{c^*}\|_{1,k} < \delta$, and hence $x' = \hat g_+^{c^*}(t,x)$ has three hyperbolic solutions at a distance less than ε of those of $x' = g_+^{\bar c}(t,x)$. In addition, since they solve $x' = g_+^{c^*}(t,x)$ on $[t^0,\infty)$, the middle one coincides with $\tilde m_{g_+^{c^*}}$ on $[t^0,\infty)$: $\tilde m_{g_+^{c^*}}$ is the unique solution of $x' = g_+^{c^*}(t,x)$ uniformly separated from two other solutions as t increases. Hence, $\tilde u_{g_+^{\bar c}}(t) \ge \tilde m_{g_+^{c^*}}(t) + \varepsilon$ and $\tilde l_{g_+^{\bar c}}(t) \le \tilde m_{g_+^{c^*}}(t) - \varepsilon$ for $t \ge t^0$.

Let us assume that $x' = g^c(t,x)$ is in Case B1 for $c = \bar c$, associate ρ to $\bar c$ as above, and check that $x' = g^c(t,x)$ is in Case C1 for any $c^* \in (\bar c, \bar c + \rho)$, which we fix. Since $g^{c^*}(t, l_{c^*}(t)) \ge g^{\bar c}(t, l_{c^*}(t))$ for all $t \in \mathbb{R}$, Proposition 5.5(iv) shows that $l_{\bar c} \le l_{c^*}$. These inequalities combined with $g^{c^*}(t_0, l_{c^*}(t_0)) > g^{\bar c}(t_0, l_{c^*}(t_0))$ yield $l_{\bar c}(t) < l_{c^*}(t)$ for all $t > t_0$, and hence $\lim_{t\to\infty}(x_{\bar c}(t, t_0+1, l_{c^*}(t_0+1)) - \tilde u_{g_+^{\bar c}}(t)) = 0$: see Proposition 6.3 and Theorem 6.4. A standard comparison argument shows that $x_{\bar c}(t, t_0+1, l_{c^*}(t_0+1)) \le l_{c^*}(t)$ for $t \ge t_0+1$, and hence $\liminf_{t\to\infty}(l_{c^*}(t) - \tilde u_{g_+^{\bar c}}(t)) \ge 0$. Thus, $\liminf_{t\to\infty}(l_{c^*}(t) - \tilde m_{g_+^{c^*}}(t)) \ge \varepsilon$, which means Case C1 for c^*: see Theorem 6.4. Similar comparison arguments show that Case A holds $c^* \in (\bar c - \rho, \bar c)$, as well as the stated properties if $x' = g^{\bar c}(t,x)$ is in Case B2. □

6.1. **Some scenarios of critical transitions in the d-concave case.** Let $\mathcal{I} \subseteq \mathbb{R}$ be an open interval, and let $f\colon \mathbb{R}\times\mathbb{R}\times\mathcal{I} \to \mathbb{R}$ and $\Gamma, \Gamma_-, \Gamma_+ \colon \mathbb{R}\times\mathbb{R} \to \mathcal{I}$ satisfy

- **fd1** there exist the derivatives f_x, f_{xx}, f_γ, $f_{\gamma\gamma}$, $f_{x\gamma}$ and $f_{\gamma x}$, and f, f_x, f_γ, f_{xx}, $f_{x\gamma}$, $f_{\gamma x}$ and $f_{\gamma\gamma}$ are admissible on $\mathbb{R}\times\mathbb{R}\times\mathcal{I}$.
- **fd2** Γ, Γ_- and Γ_+ take values in $[a,b] \subset \mathcal{I}$, are C^2-admissible, and $\lim_{t\to\pm\infty}(\Gamma(t,x) - \Gamma_\pm(t,x)) = 0$ uniformly on each compact subset $\mathcal{J} \subset \mathbb{R}$.
- **fd3** $\limsup_{x\to\pm\infty}(\pm f(t,x,\gamma)) < 0$ uniformly in $(t,\gamma) \in \mathbb{R}\times\mathcal{J}$ for all compact interval $\mathcal{J} \subset \mathcal{I}$.
- **fd4** $\inf_{t\in\mathbb{R}}\big((\partial^2/\partial x^2) f(t,x,\Gamma_\pm(t,x))|_{x=x_1} - (\partial^2/\partial x^2) f(t,x,\Gamma_\pm(t,x))|_{x=x_2}\big) > 0$ whenever $x_1 < x_2$.
- **fd5** Each equation $x' = f(t,x,\Gamma_\pm(t,x))$ has three hyperbolic solutions $\tilde l_{\Gamma_\pm} < \tilde m_{\Gamma_\pm} < \tilde u_{\Gamma_\pm}$.

With the same abuse of language as in Section 4.1, we will say that (f,Γ) satisfies **fd1**-**fd5** if there exist maps Γ_- and Γ_+ such that the previous conditions are satisfied, and refer to the equations

$$x' = f(t, x, \Gamma_-(t, x)) \quad \text{and} \quad x' = f(t, x, \Gamma_+(t, x)) \tag{6.3}$$

as the "past" and "future" of

$$x' = f(t, x, \Gamma(t, x)) \,. \tag{6.4}$$

We can easily prove the next result, analogous to Proposition 4.11:

Proposition 6.8. *Assume that* (f, Γ) *satisfies* **fd1-fd5**. *Then, the maps* g, g_-, g_+ *given by* $g(t, x) := f(t, x, \Gamma(t, x))$, $g_-(t, x) := f(t, x, \Gamma_-(t, x))$ *and* $g_+(t, x) := f(t, x, \Gamma_+(t, x))$ *satisfy the conditions* **gd1-gd5**. *Therefore, the dynamical possibilities for* (6.4) *are those described in Theorem* 6.4.

Remark 6.9. The conditions on (f, Γ) can be weakened, as in Remark 4.12.1.

In the line of Theorem 4.15, Theorem 6.10, based on Proposition 6.5, establishes conditions providing a *safety interval* $[\gamma_1, \gamma_2]$: if $\Gamma(\mathbb{R} \times \mathbb{R}) \subseteq [\gamma_1, \gamma_2]$, then neither rate-induced tipping nor phase-induced tipping takes place. As seen in its statement, this safety interval depends on the C^2-admissible function Γ_+ determining the future equation, which is an important difference with respect to the concave analogue, Theorem 4.15. And Theorems 6.11 and 6.12, based on Proposition 6.7, provide the d-concave analogues of Theorems 4.17 and 4.18: under hypotheses precluding the transition map Γ^d to take values in any fixed interval for all the values of the parameter, they show either the absence of critical transition or the occurrence of exactly two tipping points. Looking for clarity in their statements, we just analyze the situation precluding Γ^d to be always bounded from below.

Theorem 6.10. *Assume that* (f, Γ) *satisfies* **fd1-fd5**. *Assume also that* $\gamma \mapsto f(t, x, \gamma)$ *is nondecreasing for all* $(t, x) \in \mathbb{R} \times \mathbb{R}$, *and that there exist* $\gamma_1 \leq \gamma_2$ *such that:* $\Gamma(\mathbb{R} \times \mathbb{R}) \subseteq [\gamma_1, \gamma_2]$, $x' = f(t, x, \gamma_1)$ *has a bounded solution* b_1 *with* $\liminf_{t\to\infty}(b_1(t) - \tilde{m}_{\Gamma_+}(t)) > 0$*; and* $x' = f(t, x, \gamma_2)$ *has a bounded solution* b_2 *with* $\liminf_{t\to\infty}(\tilde{m}_{\Gamma_+}(t) - b_2(t)) > 0$. *Then,* (6.4) *is in* CASE A.

If, in addition, we assume that $\Gamma_\pm$ *do not depend on* t, *then the equations* $x' = f(t, x, \Gamma(ct, x))$ *and* $x' = f(t, x, \Gamma(t + c, x))$ *are in* CASE A *for all* $c > 0$ *and* $c \in \mathbb{R}$, *respectively: there is neither rate-induced tipping nor phase-induced tipping.*

Proof. Take $h_i(t, x) = f(t, x, \gamma_i)$ for $i = 1, 2$ and apply Proposition 6.5. □

Theorem 6.11. *Assume that* $\mathcal{I} = \mathbb{R}$. *Let* $\Gamma\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $\Gamma_0\colon \mathbb{R} \times \mathbb{R} \to [0, \infty)$ *be globally bounded and* C^2*-admissible, and such that the pair* $(f, \Gamma + d\,\Gamma_0)$ *satisfies* **fd1-fd5** *for all* $d \in \mathbb{R}$. *Assume that* $\Gamma_0(t_0, x) > 0$ *for all* $x \in \mathbb{R}$ *and a* $t_0 \in \mathbb{R}$. *Assume also that* $\gamma \mapsto f(t, x, \gamma)$ *is strictly increasing on* $\mathbb{R}$ *for all* $(t, x) \in \mathbb{R} \times \mathbb{R}$, *with* $\lim_{\gamma\to\pm\infty} f(t, x, \gamma) = \pm\infty$ *uniformly on compact sets of* $\mathbb{R} \times \mathbb{R}$. *Then,*

$$x' = f(t, x, \Gamma(t, x) + d\,\Gamma_0(t, x)) \tag{6.5}$$

is either in CASE C1 *for all* $d \in \mathbb{R}$, *in* CASE C2 *for all* $d \in \mathbb{R}$, *or there exist* $d_- < d_+$ *such that it is in* CASE C2 *for* $d < d_-$, *in* CASE B2 *for* $d = d_-$, *in* CASE A *for* $d \in (d_-, d_+)$, *in* CASE B1 *for* $d = d_+$, *and in* CASE C1 *for* $d > d_+$.

Proof. It is easy to check that the family of maps $g^d(t, x) := f(t, x, \Gamma(t, x) + d\,\Gamma_0(t, x))$ satisfies all the hypotheses of Proposition 6.7, with $\mathcal{C} = \mathbb{R}$ and $\bar{c}$ equal to any $\bar{d} \in \mathbb{R}$. We assume that $(6.5)_d$ is in CASE A for $d = \bar{d}$ and, for contradiction that $d_+ := \inf\{d > \bar{d} \,|\, \text{CASE A does not hold}\} = \infty$. Theorem 2.3 ensures that $\bar{d} < d_+$.

Let $\tilde{l}_d < \tilde{m}_d < \tilde{u}_d$ be the three hyperbolic solutions of $(6.5)_d$ for $d \in [\bar{d}, d_+)$. Proposition 5.5(iv) yields $\tilde{l}_{\bar{d}} \le \tilde{l}_d$ for all $d \in [\bar{d}, d_+)$. Let us prove that $\tilde{m}_d \le \tilde{m}_{\bar{d}}$ for all $d \in [\bar{d}, d_+)$. Clearly, it suffices to check it for $d \in [\bar{d}, \bar{d}+\rho)$ for a small $\rho > 0$, which we choose (applying Theorem 2.3) to ensure $\tilde{u}_{\bar{d}} > \tilde{m}_d + \varepsilon$ for an $\varepsilon > 0$ common for all $d \in [\bar{d}, \bar{d}+\rho)$. For contradiction, we take $x_0 \in (\tilde{m}_{\bar{d}}(0), \tilde{m}_d(0))$. Theorem 6.4 yields $\lim_{t\to\infty}(x_d(t,0,x_0) - \tilde{l}_d(t)) = 0$ and $\lim_{t\to\infty}(x_{\bar{d}}(t,0,x_0) - \tilde{u}_{\bar{d}}(t)) = 0$, and hence $x_d(t,0,x_0) \ge x_{\bar{d}}(t,0,x_0)$ for $t \ge 0$ yields $\limsup_{t\to\infty}(x_d(t,0,x_0) - \tilde{u}_{\bar{d}}(t)) \ge 0$. Therefore, $\limsup_{t\to\infty}(x_d(t,0,x_0) - \tilde{m}_d(t)) \ge \varepsilon$, impossible.

So, we can take $m_1 < m_2$ such that $m_1 \le \tilde{l}_{\bar{d}} \le \tilde{l}_d < \tilde{m}_d \le \tilde{m}_{\bar{d}} \le m_2$. An argument similar to that involving k_d in the proof of Theorem 4.17 provides the sought-for contradiction. Similarly, $d_- := \sup\{d < \bar{d} \mid$ CASE A does not hold$\} < \bar{d}$ is finite. It easy to deduce from Proposition 6.7 that the variation is the stated one: C2 for all $d < d_-$, B2 at d_-, B1 at d_+, and C1 for all $d > d_+$.

Since CASE A cannot occur for all d, there are equations either in CASE C1 or in CASE C2. Assume that $(6.5)_d$ is in CASE C2 for $d = \bar{d}$, but not for all d. Theorem 6.6 and Proposition 6.7 ensure the existence of $d_- > \bar{d}$ for which the dynamics is in CASE B2 and that CASE A holds for close values of $d > d_-$. So, we are in the situation of the previous paragraphs. The argument is analogous if $(6.5)_d$ is in CASE C1 for $d = \bar{d}$, but not for all d, and the proof is complete. □

By reviewing the previous proof, we observe that we have proved the next result:

Theorem 6.12. *Assume that $\mathcal{I} = \mathbb{R}$. Let $\Gamma\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $\Gamma_0\colon \mathbb{R}\times\mathbb{R} \to [0,\infty)$ be globally bounded and C^2-admissible, and such that the pair $(f, \Gamma + d\,\Gamma_0)$ satisfies* **fd1-fd5** *for all $d \in \mathbb{R}$. Assume that there exists $\bar{d} \in \mathbb{R}$ such that*

$$x' = f(t, x, \Gamma(t,x) + d\,\Gamma_0(t,x)) \tag{6.6}$$

is in CASE A for $d = \bar{d}$, and let $\tilde{l}_{\bar{d}} < \tilde{m}_{\bar{d}} < \tilde{u}_{\bar{d}}$ be its three hyperbolic solutions. Let $m_1 < m_2$ and $m_3 < m_4$ be such that $m_1 \le \tilde{l}_{\bar{d}}(t) < \tilde{m}_{\bar{d}}(t) \le m_2$ for all $t \in \mathbb{R}$ and $m_3 \le \tilde{m}_{\bar{d}}(t) < \tilde{u}_{\bar{d}}(t) \le m_4$ for all $t \in \mathbb{R}$.

(1) *Assume that there exists t_0 such that $\Gamma_0(t_0, x) > 0$ for all $x \in [m_1, m_2]$, that $\gamma \mapsto f(t,x,\gamma)$ is nondecreasing for all $(t,x) \in \mathbb{R}\times\mathbb{R}$ and strictly increasing for $(t,x) \in \mathbb{R}\times[m_1,m_2]$, with $\lim_{\gamma\to\infty} f(t,x,\gamma) = \infty$ uniformly on compact sets of $\mathbb{R}\times[m_1,m_2]$. Then, there exists $d_+ > \bar{d}$ such that $(6.6)_d$ is in CASE C1 for $d > d_+$, in CASE B1 for $d = d_+$, in CASE A for $d \in [\bar{d}, d_+)$.*
(2) *Assume that there exists t_0 such that $\Gamma_0(t_0, x) > 0$ for all $x \in [m_3, m_4]$, that $\gamma \mapsto f(t,x,\gamma)$ is nondecreasing for all $(t,x) \in \mathbb{R}\times\mathbb{R}$ and strictly increasing for $(t,x) \in \mathbb{R}\times[m_3,m_4]$, with $\lim_{\gamma\to-\infty} f(t,x,\gamma) = -\infty$ uniformly on compact sets of $\mathbb{R}\times[m_3,m_4]$. Then, there exists $d_- < \bar{d}$ such that $(6.6)_d$ is in CASE C2 for $d < d_-$, in CASE B2 for $d = d_-$, in CASE A for $d \in (d_-, \bar{d}]$.*

An analysis similar to that of Remark 4.19 applies to this d-concave case.

6.2. **Numerical simulations in asymptotically d-concave models.** In this section, we consider two different single-species population models whose internal dynamics are driven by nonautonomous cubic equations and which include predation phenomena. The intrinsic cubic dynamics is indebted to the Allee effect (see [15, 17, 19]), e.g., due to some breeding cooperative mechanism or to an easier mate

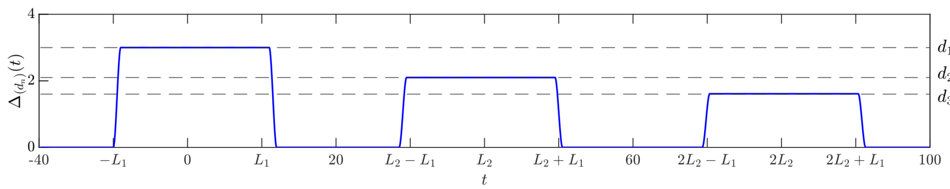

FIGURE 7. The transition map $\Delta_{(d_n)}$ defined in (6.9) for $\rho = 1$, $L_1 = 10$, $L_2 = 40$, $d_+ = 0.5$, $d = 2.5$, $d_n = d_+ + d/((n-1)/4+1)^2$, and $p_n = (n-1)L_2 + (-1)^{(n-1)}/n$ for all $n \in \mathbb{N}$.

finding. In both cases, the evolution of the population is modeled by

$$x' = r(t)\,x\left(1-\frac{x}{K(t)}\right)\frac{x-S(t)}{K(t)} + \Delta(t,x)\,, \tag{6.7}$$

where we assume r, K and S to be quasiperiodic functions with r and K positively bounded from below, and Δ to be C^2-admissible. So, analogously to Section 4.2, the map $h(t,x,\delta) := r(t)\,x\,(1-x/K(t))(x-S(t))/K(t)+\delta$ satisfies **fd1** and **fd3**. In addition, we will assume that (h,Δ) satisfies **fd2**, **fd4** and **fd5** for some maps $\Delta_\pm$. The meaning of r and K is the same as in Section 4.2, while S (on which we assume $K(t)+S(t) \geq 0$ for all $t \in \mathbb{R}$) stands for the force of the Allee effect, and Δ models the contribution of a single external effect: predation.

Each one of the two examples tries to emphasize some of the novel aspects of the theory presented in this paper: the possibility of non asymptotically constant transition functions in the first one, and the possibility of intrinsically x-dependent transition functions in the second one. As in Section 4.2, we find CASES A, B and C for different values of certain parameters, and we point to certain parametric variations as possible causes of tipping. Throughout the section, CASE A means the survival of the species, while CASE C2 means its extinction (and CASE B2 is the highly unstable situation which separates the other two).

Example 6.13. We begin by assuming that, in (6.7), the predation is modeled by a Holling type III functional response term $-\gamma\,x^2/(b(t)+x^2)$, where γ and $b>0$ have the same meaning as in Example 4.20:

$$x' = r(t)\,x\left(1-\frac{x}{K(t)}\right)\frac{x-S(t)}{K(t)} - \gamma\,\frac{x^2}{b(t)+x^2}\,. \tag{6.8}$$

Next, we assume that the population is attacked by a predator species which behaves as follows: the habitat is initially free of predators; at a certain time a group of predators arrives at the ecosystem, which they leave after some time; and this behavior repeats yearly. Such a pattern may correspond to the colonization of a new patch by a migratory species of predators, due to the reproductive, nutritional, breeding or wintering interest of the habitat. (See, e.g., [1] for a study on the evolution of some migration patterns of common swift, an insectivorous bird.)

Let L_2 be the length of the year. We assume that the n-th predation season occurs during the time $[p_n-L_1-\rho, p_n+L_1+\rho]$, with maximum number of predators during $[p_n-L_1, p_n+L_1]$: $\rho>0$ is the (short) time needed to reach and leave the patch. We assume $L_2 > 2(L_1+\rho)$, $p_{n+1}-p_n > 2(L_1+\rho)$ for all $n \in \mathbb{N}$, and $p_n-(n-1)\,L_2 \to 0$ as $n\to\infty$. The size of the n-th group of predators is determined by the constant $d_n \geq 0$, and we assume that the sequence (d_n) is bounded with limit d_+. The possible differences between p_n and $(n-1)L_2$ capture variations in

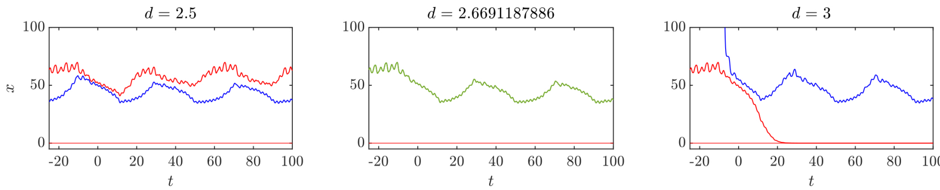

FIGURE 8. Numerical depiction of the existence of a unique size-tipping point for $(6.10)_d$. The central panel shows the dynamics for an accurate approximation to the tipping point d_0: the two upper hyperbolic solutions are so close within the representation window that are a good approximation (green) to the nonhyperbolic solution of CASE B2. The left panel depicts CASE A, which is the dynamics for any $d \in [0, d_0)$ and means survival: the attractive hyperbolic solutions in red, and the repulsive one in blue. The right panel depicts CASE C2, which is the dynamics for any $d > d_0$ and means extinction: the hyperbolic solutions in red, and the locally pullback repulsive solution in blue.

the start date of the yearly predation season, and the hypothesis $p_{n+1} - p_n \to L_2$ is made for the sake of simplicity: combined with the existence of d_+, it describes an asymptotically periodic phenomenon, which means that the behavior of the predators becomes as regular as possible over time. Other more complicated types of recurrence in the future equation may fit in the model. (See [21] for a study on the variation of arrival dates of common swift and barn swallow to the Iberian Peninsula.) The phenomenon of lack of predators in some occasional years can be described through null elements in the sequence (d_n). We use the map $\Gamma_{\rho,L}$ of Example 4.20 (see Figure 2) to model this behavior: the amount of predators at the ecosystem at time t is

$$\Delta_{(d_n)}(t) := \sum_{n=1}^{\infty} d_n \, \Gamma_{\rho,L_1}(t - p_n) \,, \tag{6.9}$$

which is a bounded continuous function due to the boundedness of (d_n) and to the disjointness of the intervals of predation. Figure 7 depicts $\Delta_{(d_n)}$ for $\rho = 1$, $L_1 = 10$, $L_2 = 40$ and certain sequences (d_n) and (p_n).

So, we study the transition equation

$$x' = r(t)\, x \left(1 - \frac{x}{K(t)}\right) \frac{x - S(t)}{K(t)} - \Delta_{(d_n)}(t)\, \frac{x^2}{b(t) + x^2} \,, \tag{6.10}$$

which represents the dynamics of the single-species population through the repeated passage (which tends to be periodic) of groups of predators starting at certain time $p_1 - L_1 - \rho$. We define $\Delta_- := 0$ and

$$\Delta_+(t) := \sum_{n=-\infty}^{\infty} d_+ \Gamma_{\rho,L_1}\big(t - (n-1)L_2\big) \,,$$

which is bounded, continuous, and L_2-periodic in time. Then, $\lim_{t\to-\infty}(\Delta_{(d_n)}(t) - \Delta_-(t)) = 0$, since $\Delta_{(d_n)}(t) = 0$ for all $t \le p_1 - L_1 - \rho$, and $\lim_{t\to\infty}(\Delta_{(d_n)}(t) - \Delta_+(t)) = 0$, since the uniform continuity of Γ_{ρ,L_1} on compact sets ensures that $\|\Gamma_{\rho,L_1}(t - p_n) - \Gamma_{\rho,L_1}(t - (n-1)L_2)\|_\infty \to 0$ as $n \to \infty$, and the separation of the supports of the terms of the series guaranteed by the conditions $L_2 > 2(L_1 + \rho)$ and $p_{n+1} - p_n > 2(L_1 + \rho)$ ensures that we can compare the series term-by-term. That is, (6.9) corresponds to a transition between these two limit functions, and

fd2 is fulfilled. It can be checked that the right-hand side of equation (6.10) is not d-concave if $\max_{t\in\mathbb{R}} \Delta_{(d_n)}(t) = \max_{n\in\mathbb{N}} d_n$ is large enough, while $r(t)\,x\,(1 - x/K(t))(x - S(t))/K(t) - \Delta_+(t)\,x^2/(b(t) + x^2)$ is d-concave if d_+ is not too large, in which case also **fd4** is fulfilled: see [19, Section 5.2].

Let us choose: $r(t) := 0.7+0.3\sin^2(t)$, $K(t) := 70+20\cos(\sqrt{5}\,t)$ and $S(t) = 20+30\cos^2(\sqrt{3}\,t)$ for the internal dynamics of the species, $b(t) := 200$ for the influence of the predation, and $L_1 = 10$, $L_2 = 40$, $d_+ = 0.3$, $d_n = d_+ + d/((n-1)/20+1)^2$ and $p_n = (n-1)L_2 + (-1)^{(n-1)}/n$ (for all $n \in \mathbb{N}$) for the shape of the transition function. The particular expression of d_n implies that the yearly density of predators d_n decreases to d_+. The decreasing attractiveness of the habitat can be indebted to different causes: learning of defensive mechanisms, overpopulation in the previous season, insufficient nesting or breeding space, etc. The constant d of the definition of d_n is a size bifurcation parameter in terms of which we will study the dynamical cases of (6.10). The choice of d_+ (below 0.32) guarantees **fd4**. We numerically check **fd5**, and hence **fd1**-**fd5** hold for all $d \geq 0$. In addition, the size of d_n for small n provides a not d-concave equation (6.10) if d is large enough (above 0.96).

Clearly, $\Delta_{(d_n)} = \tilde{\Delta}_+ + d\,\Delta_0$ for $\tilde{\Delta}_+(t) := \sum_{n=1}^{\infty} d_+\Gamma_{\rho,L_1}\big(t - p_n\big)$ and the continuous nonnegative map $\Delta_0(t) := \sum_{n=1}^{\infty}(1/((n-1)/20+1)^2)\Gamma_{\rho,L_1}(t-p_n)$, whose limits as $t \to \pm\infty$ are 0. We define $f(t,x,\gamma) := r(t)\,x\,(1-x/K(t))(x-S(t))/K(t) - \tilde{\Delta}_+(t)\,x^2/(b(t)+x^2) - \gamma\,x^2/(b(t)+x^2)$ and $g(t,x,\gamma) := f(t,x,-\gamma)$, and check that the pairs $(g, d\,\Delta_0)$ satisfy the hypotheses of Theorem 6.12(ii) (with $\Gamma(t) := 0$, $\Gamma_0(t) := \Delta_0(t)$, and $\bar{d} = 0$). To this end, we numerically check that $x' = g(t,x,0)$ has three hyperbolic copies of the base and that the lower one, attractive, is 0 (and hence $\tilde{m}_0$ is positively bounded from below). Hence, Theorem 6.11 ensures the existence of a unique size-induced tipping point $d_0 > 0$ for $x' = f(t,x,d\,\Delta_0(t))$ (i.e., for (6.10)): Case A holds for $0 \leq d < d_0$, and Case C2 holds for $d > d_0$. That is, an excessive increase in the number of predators visiting the habitat leads to the extinction of the species. The existence of this critical transition is depicted in Figure 8.

Example 6.14. Now, we consider that a flock of x animals described by (6.7) grazes in a patch which is initially free of predators. We assume that at time $t = 0$ a group of predators, which we suppose that have constant density d (due to the time scale in which we work) and whose predation mechanism is assumed to be suitably modeled by a Holling type III functional response term $-d\,x^2/(b(t)+x^2)$ reaches the patch. (See Example 4.20 for the meaning of d and b.) The function b is assumed to be quasiperiodic and positively bounded from below. At time $L > 0$, the threat is identified by the flock owner and s shepherds per unit of time are hired to protect the flock: there are $s\,(t-L)$ shepherds at time $t \geq L$, and each shepherd is assumed to be able to protect h heads of livestock. As soon as there are enough shepherds to protect the whole herd, i.e, when $x \leq h\,s\,(t-L)$, predators are not able to attack the flock. That is, predation occurs while $0 \leq t \leq L\,(cx+1)$, where $c = 1/hsL$. So, for $x \geq 0$, we can model the evolution of the flock by the equation

$$x' = r(t)\,x\left(1 - \frac{x}{K(t)}\right)\frac{x - S(t)}{K(t)} - d\,\Gamma_L\left(\frac{2\,t}{cx+1} - L\right)\frac{x^2}{b(t)+x^2}\,, \tag{6.11}$$

where we take $\Gamma_L := \Gamma_{\rho,L}$ for some small fixed $\rho > 0$, with $\Gamma_{\rho,L}$ defined in Example 4.20 (see Figure 2). So, the predation term practically vanishes when t is outside the interval $[0, L\,(cx+1)]$. By multiplying the Holling type III functional response

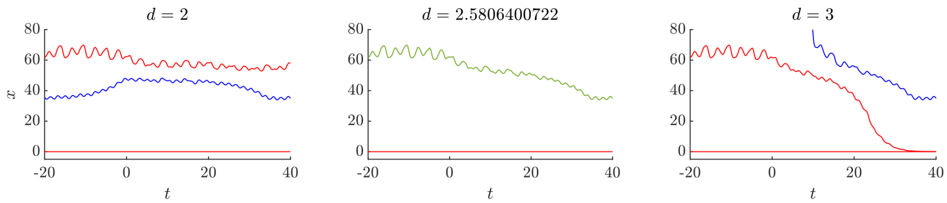

FIGURE 9. Numerical depiction of the existence of a unique size-tipping point for $(6.12)_d$ when L and c are fixed. In this example, $L = 20$ and $c = 0.02$. The central panel corresponds to the approximation for $d(20, 0.02)$ of Table 3: CASE B2. To its left and right, we find CASES A and C2. See Figure 8 to understand the color code.

term by Γ_L, it is implicitly assumed that the search for prey mechanism, i.e. the Holling type III interaction, is not affected by the presence of shepherds as long as there are not enough of them to protect the whole herd. (See the biological meaning of Holling functional response in [24].) This assumption, made for the sake of simplicity, can be understood as follows: if a shepherd has more sheep in his care than he can protect, then a predator, once it has located its prey, can wait a negligible amount of time on the timescale we are working with until the shepherd moves on to other sheep, far enough away to allow the predator to hunt the prey.

Since $\mathbb{R} \times [0, \infty)$ is an invariant set for the process given by (6.11) and only nonnegative solutions have biological meaning, we can replace the predation term by a globally defined one. To this end, we take a a globally defined C^2-map $k(x)$ which coincides with $1/(cx + 1)$ on $[0, \infty)$, and consider the equation

$$x' = r(t)\, x \left(1 - \frac{x}{K(t)}\right) \frac{x - S(t)}{K(t)} - d\, \Gamma_L(2\, t\, k(x) - L)\, \frac{x^2}{b(t) + x^2}\,, \tag{6.12}$$

Let $\Lambda_{L,c}(t, x) := \Gamma_L(2\, t\, k(x) - L)$. Then, for any choices of $d \geq 0$, $L > 0$ and $c > 0$, $d\, \Lambda_{L,c}$ is globally bounded, C^2-admissible on $\mathbb{R}\times\mathbb{R}$, and with $\lim_{t\to\pm\infty} d\, \Lambda_{L,c}(t, x) = 0$ uniformly on each compact set $\mathcal{J} \subset \mathbb{R}$. That is, $d\, \Lambda_{L,c}$ globally satisfies **fd2**, with $\Lambda_\pm = 0$. In addition, if f is the right-hand term of (6.8), then it is not difficult to check that **fd1**, **fd3** and **fd4** hold.

We choose $r(t) := 0.7 + 0.3 \sin^2(t)$, $K(t) := 70 + 20 \cos(\sqrt{5}\, t)$, $S(t) := 20 + 30 \cos^2(\sqrt{3}\, t)$, and $b(t) := 20 + \cos(t)$ to construct Table 3 and Figures 9, 10 and 11, and numerically check that **fd5** holds for these choices, being 0 the lower bounded solution of $x' = f(t, x, 0)$. That is $(f, d\, \Lambda_{L,c})$ satisfies **fd1**-**fd5** for all $d \geq 0$, $L > 0$ and $c > 0$, and hence the dynamics of (6.12) fits in one of the cases described by Theorem 6.4. Moreover, since 0 is the lowest bounded solution for the past and future equations, CASES B1 and C1 are preluded. In addition, if $g(t, x, \gamma) := f(t, x, -\gamma)$, then the pairs $(g, d\, \Lambda_{L,c})$ satisfy all the hypotheses of Theorem 6.12(2) (with $\Gamma(t) := 0$, $\Gamma_0(t) := \Lambda_{L,c}(t)$ and $\bar{d} = 0$). This result shows the existence of a unique tipping value $d(L, c) > 0$: $(6.12)_d$ is in CASE A for all $d \in [0, d(L, c))$, in CASE B2 for $d = d(L, c)$ and in CASE C2 for all $d > d(L, c)$. Figure 9 depicts the upper locally pullback attractive and the locally pullback repulsive solutions of the transition equation $(6.12)_d$ for d close to the bifurcation point, for some fixed L and c, and Table 3 shows numerical approximations to $d(L, c)$ for different $L, c > 0$.

Let the other parameters vary. The monotonicity of $L \mapsto \Gamma_L(2t/(cx + 1) - L)$ for any $(t, x) \in \mathbb{R} \times [0, \infty)$ yields the uniqueness of a possible tipping point L_0

$d(L,c)$	$c = 0.01$	$c = 0.02$	$c = 0.03$
$L = 2$	9.5918417988	7.8400146619	6.6406325271
$L = 10$	3.5156887400	3.1640725896	2.9522195572
$L = 20$	2.7559336044	2.5806400722	2.4622290038
$L = 30$	2.4757094854	2.3677420953	2.3132184604
$L = 40$	2.3543746813	2.2850546293	2.2459305139

TABLE 3. Numerical approximations up to ten places to the bifurcation points $d(L, c)$ of $(6.12)_d$. The displayed number is a value of d for which $(6.12)_d$ is in CASE A and such that $(6.12)_{d+1e-10}$ is in CASE C. The numerical integration has been done using Matlab2023a `ode45` algorithm with `AbsTol` and `RelTol` equal to `1e-12`. The final integration has been carried out over the interval $[-\mathtt{1e4}, \mathtt{1e4}]$.

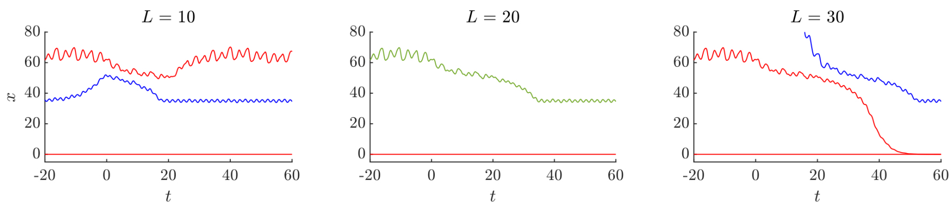

FIGURE 10. Numerical depiction of an L-induced tipping point: for $c = 0.02$ and $d = 2.5806400722$, we find CASE A for $L = 10$, CASE B2 for $L = 20$ and CASE C2 for $L = 30$.

for $(6.12)_L$ for d and c fixed. In fact, if $\tilde{u}_L$ and $\tilde{m}_L$ are the upper and middle hyperbolic solutions if $(6.12)_L$ is in CASE A and $L_1 < L_2$ provide this case, then Proposition 5.5(iv) shows that $\tilde{u}_{L_1} > \tilde{u}_{L_2}$, and a new comparison argument shows that $\tilde{m}_{L_1} \le \tilde{m}_{L_2}$. So, if CASE B2 (the unique possible one) occurs as $L \uparrow L_0$, then they collide, and CASE A cannot occur for $L > L_0$.

Analogously, the monotonicity of $c \mapsto \Gamma_L(2t/(cx+1) - L)$ for any $(t, x) \in \mathbb{R} \times [0, \infty)$ ensures the uniqueness of the bifurcation for $(6.12)_c$ for d and L fixed in the case of existence. The biological sense of the problem makes reasonable expecting no more than one critical transition as L or c varies: the decrease in L means an earlier detection of the problem and therefore the extinction of the hinders; and the decrease in c means an increase in the rate of recruitment of shepherds, i.e., a faster response to the problem that facilitates survival.

Figures 10 and 11 represent the behaviour of the locally pullback attractive and locally pullback repulsive solutions of the transition equation $(6.12)_L$ for fixed d and c, and $(6.12)_c$ for fixed d and L, respectively. As in the case of Figure 9, the left-hand panel corresponds to the survival of the species (CASE A), the right-hand panel corresponds to extinction (CASE C2), and the middle panel is an approximation to the intermediate unstable situation between them (CASE B2).

6.3. **Example on the necessity of minimality in Proposition 5.8.** In this section, we present an example which shows that the minimality of (Ω, σ) is indeed required in Proposition 5.8. We will construct a non minimal set Ω and a pair of functions $\mathfrak{h}_1, \mathfrak{h}_2 \colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying **d1-d4** with $\mathfrak{h}_1(\omega, x) > \mathfrak{h}_2(\omega, x)$ for all $(\omega, x) \in \Omega \times \mathbb{R}$, such that $x' = \mathfrak{h}_i(\omega{\cdot}t, x)$ has three hyperbolic copies of the base $\mathfrak{l}_i < \mathfrak{m}_i < \mathfrak{u}_i$ for $i = 1, 2$ which do not satisfy none of the two possible

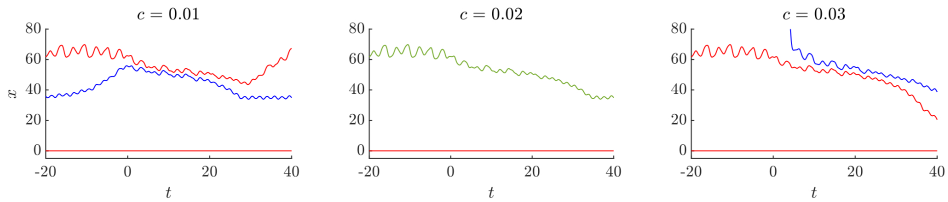

FIGURE 11. Numerical depiction of a c-induced tipping point: for $L = 20$ and $d = 2.5806400722$, we find CASE A for $c = 0.01$, CASE B2 for $c = 0.02$ and CASE C2 for $c = 0.03$.

orders described in Proposition 5.8. We make use of the transition framework of Section 6 to construct the example: $x' = \mathfrak{h}_i(\omega{\cdot}t, x)$ for $i = 1, 2$ will be transition equations, with Ω composed by a heteroclinic orbit connecting two singletons, which are minimal. The cornerstone of the example is the fact that we construct three hyperbolic copies of the base Ω contained in $\Omega \times \mathbb{R}$ projecting onto each one of the two minimal subsets of Ω, and the three copies of the base have the two distinct orders allowed by Proposition 5.8 over each minimal.

Let $\Gamma\colon \mathbb{R} \to (0, 1)$ be a continuous map with $\lim_{t\to\infty} \Gamma(t) = \Gamma_+ := 1$ and $\lim_{t\to-\infty} \Gamma(t) = \Gamma_- := 0$ (as $\Gamma(t) := \arctan(t)/\pi + 1/2$). We take $a \geq \sqrt{10}$ and

$$h_b(x, \alpha) := -x^3 + x + \alpha\,(3x^2a - 3xa^2 + a^3 - a) + \alpha\,(1 - \alpha)\,b\,,$$

for some $b \geq 0$ which will be properly fixed later on. Note that: $h_b(x, \alpha) = h_0(x, \alpha) + \alpha(1-\alpha)\,b$; $h_b(x, 0) = -x(x-1)(x+1)$; $h_b(x, 1) = -(x-a)(x-a-1)(x-a+1)$; and $3x^2a - 3xa^2 + a^3 - a \geq 0$ for all $x \in \mathbb{R}$ by the choice of a, so $\alpha \mapsto h_0(x, \alpha)$ is nondecreasing for all $x \in \mathbb{R}$. For each $b \geq 0$, we consider the equation

$$x' = h_b(x, \Gamma(t))\,. \tag{6.13}$$

It is easy to check that (h_b, Γ) satisfies **fd1-fd5** for any $b \geq 0$: the past equation $x' = h_b(x, 0)$ has three hyperbolic critical points -1, 0 and 1, and the future equation $x' = h_b(x, 1)$, which is a shift of the past one, has three hyperbolic critical points $a - 1$, a and $a + 1$. So, the dynamics of $(6.13)_b$ fits in one of the dynamical cases of Theorem 6.4.

We will check later the existence of $b_0 > 0$ such that $(6.13)_b$ is in CASE A for $b = b_0$. Let Ω be the hull of $(t, x) \mapsto h_{b_0}(x, \Gamma(t))$ (see Section 2.3), and let $\mathfrak{h}_1\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be given by $\mathfrak{h}_1(\omega, x) := \omega(0, x)$ for $(\omega, x) \in \Omega \times \mathbb{R}$, that is, the extension of h_{b_0} to Ω. Then, $\mathfrak{h}_1(\omega, x)$ is a cubic polynomial with -1 as leading coefficient for all $\omega \in \Omega$, and hence $\mathfrak{h}_1$ satisfies **d1-d4**. Note that Ω is the union of the (heteroclinic) σ-orbit $\{h_{b_0}(x, \Gamma(t + s)) \,|\, s \in \mathbb{R}\}$ and its α-limit and ω-limit sets, $\{h_{b_0}(x, 0)\}$ and $\{h_{b_0}(x, 1)\}$: see Lemma 2.4. Theorem 5.6 ensures that $x' = \mathfrak{h}_1(\omega{\cdot}t, x)$ has three hyperbolic copies of the base $\mathfrak{l}_1 < \mathfrak{m}_1 < \mathfrak{u}_1$. In particular, the restrictions of these three copies to the α-limit set $\{h_{b_0}(x, 0)\}$ are -1, 0 and 1, and to the ω-limit set $\{h_{b_0}(x, 1)\}$ are $a - 1$, a and $a + 1$.

Next, we define $\mathfrak{h}_2(\omega, x) := -x^3 + x - \varepsilon$ for $\varepsilon \in (0, 2/(3\sqrt{3}\,))$, which clearly satisfies **d1-d4** and $\mathfrak{h}_1(\omega, x) > \mathfrak{h}_2(\omega, x)$ for all $(\omega, x) \in \Omega \times \mathbb{R}$. It can be checked that $x' = \mathfrak{h}_2(\omega{\cdot}t, x)$ has three copies of the base: three constant equilibria. So, the order of $\mathfrak{l}_1$, $\mathfrak{m}_1$, $\mathfrak{u}_1$ and $\mathfrak{l}_2$, $\mathfrak{m}_2$, $\mathfrak{u}_2$ is $\mathfrak{l}_2 < -1 < 0 < \mathfrak{m}_2 < \mathfrak{u}_2 < 1$ (like in Proposition 5.8(i)) over the minimal set $\{h_{b_0}(x, 0)\} \subset \Omega$, and $\mathfrak{l}_2 < \mathfrak{m}_2 < \mathfrak{u}_2 < a - 1 < a < a + 1$ (like in Proposition 5.8(ii)) over the minimal set $\{h_{b_0}(x, 1)\} \subset \Omega$. Hence, the continuity of the copies of the base preclude any of the two possibilities of Proposition 5.8 to hold over the whole Ω.

It remains to check the existence of $b_0 > 0$ such that $(6.13)_{b_0}$ is in CASE A, for which it suffices to check $(6.13)_0$ is in CASE C2 and that there exists $b_1 > 0$ such that $(6.13)_{b_1}$ is in CASE C1: Theorem 6.6 precludes moving from CASE C2 to CASE C1 as b varies without crossing A.

We denote by l_b and u_b (resp. m_b) the locally pullback attractive (resp. repulsive) solutions of $(6.13)_b$ provided by Proposition 6.3, and recall that $\lim_{t\to-\infty} u_b(t) = 1$, $\lim_{t\to-\infty} l_b(t) = -1$, and $\lim_{t\to\infty} m_b(t) = a$. Since $\Gamma(t) \le 1$ for all $t \in \mathbb{R}$ and $\alpha \mapsto h_0(x,\alpha)$ is nondecreasing for all $x \in \mathbb{R}$, we have $h_0(a-1,\Gamma(t)) \le h_0(a-1,1) = 0$ for all $t \in \mathbb{R}$, so $\mathbb{R}\times(-\infty, a-1]$ is positively invariant for $(6.13)_0$. Since $\lim_{t\to-\infty} u_0(t) = 1 < a-1$, we have $u_0(t) \in (-\infty, a-1]$ for all $t \in \mathbb{R}$, and hence $\lim_{t\to\infty} u_0(t) = a-1$: the other possible future limits a and $a+1$ are uniformly separated from u_0. That is, $(6.13)_0$ is in CASE C2. To look for b_1, we first check that all the bounded solutions of $(6.13)_b$ take values in $[-1,\infty)$, since $h_b(x,0) < h_b(x,\Gamma(t))$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}$, and hence any $m_1 < -1$ satisfies the initial hypothesis of Proposition 5.5. Next, we take $t_0 > 0$ in the domain of definition of m_0 with $m_0(t) < a+1/2$ for all $t \ge t_0$ and assume for contradiction that $l_b(t) \le a+1/2$ for all $b > 0$ and $t \in [t_0, t_0+1]$. Let γ be a lower bound for $\Gamma(t)(1-\Gamma(t))$ for $t \in [t_0, t_0+1]$. Then $l_b(t_0+1) \ge -2 + \int_{t_0}^{t_0+1}(-(a+1/2)^3 + \gamma\, b)\, ds$ for all $b > 0$, which is impossible. We take b_0 and t_1 with $l_{b_0}(t_1) > a+1/2 > m_0(t_1)$. Proposition 6.3 ensures that $\lim_{t\to\infty}(x_0(t,t_1,l_{b_0}(t_1)) - (a+1)) = 0$, and a comparison argument yields $l_{b_0}(t) = x_{b_0}(t,t_1,l_{b_0}(t_1)) \ge x_0(t,t_1,l_{b_0}(t_1))$ for $t \ge t_1$. That is, $\liminf_{t\to\infty}(l_{b_0}(t) - (a+1)) \ge 0$, which may only happen in CASE C1 (see Theorem 6.4). This completes the proof.

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