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STRUCTURE OF NON-AUTONOMOUS ATTRACTORS FOR A CLASS OF DIFFUSIVELY COUPLED ODE

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Abstract

In this work we will study the structure of the skew-product attractor for a planar diffusively coupled ordinary differential equation, given by $\dot{x} = k(y-x) + x - \beta(t)x^3$ and $\dot{y} = k(x-y)+y-\beta(t)y^3$, $t \geq 0$. We identify the non-autonomous structures that completely describes the dynamics of this model giving a Morse decomposition for the skew-product attractor. The complexity of the isolated invariant sets in the global attractor of the associated skew-product semigroup is associated to the complexity of the attractor of the associated driving semigroup. In particular, if β is asymptotically almost periodic, the isolated invariant sets will be almost periodic hyperbolic global solutions of an associated globally defined problem.

Key words and phrases. skew-product semiflow, gradient system, pullback attractor, uniform attractor.

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1. **Introduction.** The qualitative analysis of autonomous and non-autonomous dynamical systems coming from differential equations is a powerful tool to give some information about the behaviour of many phenomena in several areas, such as Physics, Biology and Engineering ([20, 32, 30, 27, 15, 25, 16, 33]). The internal dynamics on the global attractor of such systems plays a fundamental role in the understanding of the models, as the system tends to mimic the dynamics ([3, 28]) and structures inside the attractor. Indeed, the asymptotic behaviour of a given system is fully described if we characterize the topological and geometrical structures of a global attractor, which has been intensively studied in the last decades in an infinite-dimensional framework in the case of autonomous equations ([20, 25, 24, 23, 22, 21]), non-autonomous ([5, 26, 31, 11, 1, 2]) or even random perturbations ([7, 9]). But there are no many papers studying the full structure of attractors for non-autonomous differential equations, more than making a perturbation of an autonomous system. Indeed, in the general case in which time dependent nonlinearities are not close to a fixed vector field, even the generalization of an hyperbolic stationary point becomes problematic, much more if we try to characterize the gradient structure of the associated attractors ([13, 8, 6]).

In this work, we will study the structure of the attractors (pullback, uniform and skew-product attractor) of the following non-autonomous planar diffusively coupled ordinary differential equation:

$$\begin{aligned} &\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = k \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x - \beta(t)x^3 \\ y - \beta(t)y^3 \end{pmatrix} = f\left(t, \begin{pmatrix} x \\ y \end{pmatrix}\right), \quad t > \tau \geq 0, \\ &\begin{pmatrix} x(\tau) \\ y(\tau) \end{pmatrix} = \begin{pmatrix} x_\tau \\ y_\tau \end{pmatrix}. \end{aligned} \tag{1}$$

where $\beta : \mathbb{R}^+ \to \mathbb{R}$ is a globally Lipschitz function such that $\beta(\mathbb{R}^+) \subset [\beta_0, \beta_1]$, $0 < \beta_0 < \beta_1 < \infty$.

Firstly, we will study the autonomous case, that is, when $\beta(\cdot)$ is a constant function, identifying the key objects of study so we can tackle the non-autonomous case. We will then show that (1) has an attractor whose dynamics can be decomposed, in some sense, between its gradient part and its recurrent part ([17, 29]) through the introduction of the natural concept of skew-product attractor ([15]), which allows us to identify structures in non-autonomous problems that are similar to the ones found on the autonomous ones.

We will associate a skew-product semigroup to (1) in the following way: consider $\beta : \mathbb{R}^+ \to [\beta_0, \beta_1]$ a globally Lipschitz function and the space $C(\mathbb{R}^+, [\beta_0, \beta_1])$ endowed with the metric ρ of the uniform convergence in compact subsets of $\mathbb{R}^+$.

If

$$\Sigma = \overline{\{\beta(t+\cdot) \in C(\mathbb{R}^+, [\beta_0, \beta_1]) : t \in \mathbb{R}^+\}}^{\rho},$$

define the driving semigroup $\Theta(t) : \Sigma \to \Sigma$ by $(\Theta(t)\sigma)(\cdot) = \sigma(t+\cdot)$, $\sigma \in \Sigma$.

Moreover, given $\left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right) \in \mathbb{R}^2$, define the cocycle $\mathbb{R} \times \Sigma \ni (t, \sigma) \to \mathcal{K}(t, \sigma) \left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right) \in \mathbb{R}^2$ as being the solution of the initial value problem (at initial time $s = 0$)

$$\begin{aligned} &\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = k \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x - \sigma(t)x^3 \\ y - \sigma(t)y^3 \end{pmatrix} \\ &\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \end{aligned} \tag{2}$$

By this approach, we can consider the associated skew-product semigroup $\Pi(\cdot)$ in $\mathbb{X} = \mathbb{R}^2 \times \Sigma$ given by

$$\Pi(t)(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \sigma) = (\mathcal{K}(t, \sigma) \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \Theta(t)\sigma).$$

If $\eta : \mathbb{R} \to \Sigma$ is a global solution for Θ, then $\eta(\tau)(0)$ denotes the function $\eta(\tau)$ evaluated at zero. We can define the evolution process $S_\eta(t, s) = \mathcal{K}(t - s, \eta(s))$ for every $t \geq s$ given by the solution of the initial value problem

$$\begin{aligned} &\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = k \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x - \eta(t)(0)x^3 \\ y - \eta(t)(0)y^3 \end{pmatrix} \\ &\begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \end{aligned} \tag{3}$$

We will show that, for each global solution $\eta : \mathbb{R} \to \Sigma$ of Θ, problem (3) has five hyperbolic bounded global solutions, denoted by $\xi_{0,\eta}(t) \equiv 0$ and $\xi^{\pm}_{i,\eta}(t)$, $i = 1, 2$. We will also show that the sets $\Xi_0 = \{(0, \eta(0)) : \eta \text{ is a bounded solution for } \Theta\}$ and $\Xi^{\pm}_i = \{(\xi^{\pm}_{i,\eta}(0), \eta(0)) : \eta \text{ is a bounded solution for } \Theta\}$, $i = 1, 2$, constitute a family of isolated invariants sets of $\Pi(\cdot)$. Lastly, we conclude that $\Pi(\cdot)$ is a gradient semigroup ([10, 11]) relatively to $\boldsymbol{\Xi} = \{\Xi_0, \Xi^{\pm}_i, i = 1, 2\}$ and that this gradient structure is stable under perturbation. Finally, assuming that β is asymptotically almost periodic, we prove that all non-autonomous equilibria are also almost periodic.

2. **Non-autonomous case: Pullback attractors.** Let $\mathscr{S}$ be the global attractor of Θ in Σ. If $\eta : \mathbb{R} \to \Sigma$ is a global solution of the driving semigroup Θ associated to (1), we can consider $\gamma : \mathbb{R} \to \mathbb{R}$ a globally Lipschitz function such that $\gamma(\mathbb{R}) \subset [\beta_0, \beta_1]$ with $\gamma(t) = \eta(t)(0)$ for all $t \in \mathbb{R}$.

Remark 2.1. *The global attractor $\mathscr{S}$ of Θ in Σ is the omega limit set of $\beta \in \Sigma$ and it is chain recurrent. We note that the driving semigroup Θ restricted to $\mathscr{S}$ does not need to be a flow, as backwards uniqueness may fail (see [5], Chapter 6).*

Now consider the initial value problem

$$\begin{aligned}&\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=k\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}x-\gamma(t)x^3\\ y-\gamma(t)y^3\end{pmatrix}=\left(f_\gamma\left(t,\begin{pmatrix}x\\ y\end{pmatrix}\right)\right),\quad t>\tau,\\ &\begin{pmatrix}x(\tau)\\ y(\tau)\end{pmatrix}=\begin{pmatrix}x_\tau\\ y_\tau\end{pmatrix}.\end{aligned}\tag{4}$$

Problem (4) (or (1)) is well-defined and its solutions are defined for all $t \geq \tau$ (or $t \geq 0$), once

$$\frac{d}{dt}\left\|\begin{pmatrix}x\\ y\end{pmatrix}\right\|^2=2f_\gamma(t,x)\cdot\begin{pmatrix}x\\ y\end{pmatrix}=2(-k(x-y)^2+x^2+y^2-\gamma(t)(x^4+y^4))<0$$

whenever $\left\|\begin{pmatrix}x\\ y\end{pmatrix}\right\|_{\mathbb{R}^2} \geq M$, for some $M > 0$.

Denote the solution of (4) (note that depends on γ) by $S(t,\tau)\left(\begin{smallmatrix}x_\tau\\ y_\tau\end{smallmatrix}\right)=\left(\begin{smallmatrix}x(t,\tau,x_\tau)\\ y(t,\tau,y_\tau)\end{smallmatrix}\right)$, $t \geq \tau$. Then $\{S(t,\tau) \in C(\mathbb{R}^2) : t \geq \tau\}$ is an evolution process, that is, a two-parameter family such that $S(t,t) = I_{\mathbb{R}^2}$ for every $t \in \mathbb{R}$, $S(t,\tau)S(\tau,s) = S(t,s)$, for every $t \geq \tau \geq s$, and $(t,\tau,\left(\begin{smallmatrix}x_\tau\\ y_\tau\end{smallmatrix}\right)) \to S(t,\tau)\left(\begin{smallmatrix}x_\tau\\ y_\tau\end{smallmatrix}\right)$ is continuous, for $t \geq \tau$ and $\left(\begin{smallmatrix}x_\tau\\ y_\tau\end{smallmatrix}\right) \in \mathbb{R}^2$.

Definition 2.2. *Let (X, d_X) be a metric space and $\{S(t,\tau) : t \geq \tau\} \subset C(X)$ be an evolution process in X. A family of compact sets $\{\mathcal{A}(t) \subset X : t \in \mathbb{R}\}$ is said to be the pullback attractor for $\{S(t,\tau) : t \geq \tau\}$ if it is invariant (that is, $S(t,s)\mathcal{A}(s) = \mathcal{A}(t)$ for every $t \geq s$), $\cup_{s\leq t}\mathcal{A}(s)$ is bounded for each $t \in \mathbb{R}$ and, for each $t \in \mathbb{R}$, $\mathcal{A}(t)$ pullback attracts bounded subsets of X at time t, that is, if $B \subset X$ is bounded, then*

$$d_H(S(t,s)B,\mathcal{A}(t)) \overset{s\to-\infty}{\to} 0,$$

where d_H is the Hausdorff semidistance between A and B given by

$$d_H(A,B)=\sup_{a\in A}\inf_{b\in B} d_X(a,b).$$

The following theorem gives conditions for an evolution process to have a pullback attractor ([26, 11]).

Theorem 2.3. *Let $\{S(t,\tau) \in C(X) : t \geq \tau\}$ be an evolution process in a metric space (X, d_X). If there is a compact subset $K \subset X$ that pullback attracts all bounded subsets of X at time t for every $t \in \mathbb{R}$, then $\{S(t,\tau) \in C(X) : t \geq \tau\}$ has a pullback attractor $\{\mathcal{A}(t) \subset X : t \in \mathbb{R}\}$ given by $\mathcal{A}(t) = \omega(K,t)$ for every $t \in \mathbb{R}$, with*

$$\omega(K,t)=\bigcap_{\tau\leq t}\overline{\bigcup_{s\leq\tau} S(t,s)K}.$$

Since $K = \overline{B}^{\mathbb{R}^2}_M(\binom{0}{0})$ (the closed ball of radius M around $\binom{0}{0}$ in $\mathbb{R}^2$) is a compact subset that pullback attracts bounded subsets of X at time t, for every $t \in \mathbb{R}$, under action of (4), it follows that $\{S(t,\tau) \in C(\mathbb{R}^2) : t \geq \tau\}$ has a pullback attractor $\{\mathcal{A}(t) \subset \mathbb{R}^2 : t \in \mathbb{R}\}$ given by $\mathcal{A}(t) = \omega(K,t)$ for every $t \in \mathbb{R}$.

2.1. **Autonomous case: Global attractor and its characterization.** Before we proceed, consider the case where $\gamma(\cdot)$ is a constant function in (4), denoted by γ . Thus, consider the initial value problem

$$\begin{aligned} \dot{z} &= f_\gamma(z), \\ z(0) &= z_0 \in \mathbb{R}^2 \end{aligned} \tag{5}$$

where $z = \begin{pmatrix} x \\ y \end{pmatrix}$, $z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ and $f_\gamma \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k(y-x) + x - \gamma x^3 \\ k(x-y) + y - \gamma y^3 \end{pmatrix}$.

First, note that, if $z(\cdot, z_0)$ is a solution for (5), then $\gamma^{\frac{1}{2}} z(\cdot)$ is a solution for

$$\begin{aligned} \dot{z} &= f_1, \\ z(0) &= \gamma^{\frac{1}{2}} z_0 \in \mathbb{R}^2 \end{aligned} \tag{6}$$

where $z = \begin{pmatrix} x \\ y \end{pmatrix}$, $z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ and $f_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k(y-x) + x - x^3 \\ k(x-y) + y - y^3 \end{pmatrix}$. Thus, there is no loss of generality if we consider $\gamma = 1$ in the following analysis.

The equilibria of (6) are the roots of

$$\begin{aligned} k(y-x) + x - x^3 &= 0 \\ k(x-y) + y - y^3 &= 0 \end{aligned} \tag{7}$$

From the general case, we know that given $z_0 \in \mathbb{R}^2$, (6) has a unique solution, which is defined in $[0,\infty)$. If $T(t)z_0 = z(t,z_0)$ for each $t \geq 0$, then $\{T(t) \in C(\mathbb{R}^2) : t \geq 0\}$ is a semigroup (a special case of evolution processes where $S_\gamma(t,s) = S_\gamma(t-s,0) = T(t-s)$ for every $t \geq s$) that has a global attractor $\mathcal{A}$, that is, a compact invariant set ($T(t)\mathcal{A} = \mathcal{A}$ for every $t \geq 0$) that attracts bounded subsets of $\mathbb{R}^2$ in the sense that if $B \subset \mathbb{R}^2$ is bounded, then $d_H(T(t)B, \mathcal{A}) \overset{t\to\infty}{\to} 0$ ([20, 30]).

Consider the rectangle $R_M = [-M, M]^2$, $M > 1$. Then, whenever z is on any of its faces, the vector field $f(z)$ points into the rectangle. It follows that a solution that starts in R_M never leaves R_M. Therefore, $\mathcal{A} \subset R_M$.

Clearly, $z_1^* = (1,1), z_2^* = (0,0)$ and $z_3^* = (-1,-1)$ are solutions of (7). If $k < \frac{1}{2}$, there are two other equilibria of (6) on the secundary diagonal $x = -y$. They are

$$z_4^* = (\sqrt{1-2k}, -\sqrt{1-2k}) \text{ and } z_5^* = (-\sqrt{1-2k}, \sqrt{1-2k}).$$

If $k > \frac{1}{3}$, there are no equilibria other than these five. In fact, from (7) we get

$$k^2(2k-1)x + [-2k^3 + 4k^2 - 3k + 1]x^3 - 3(k-1)^2x^5 - 3(k-1)x^7 - x^9 = 0,$$

whose non-trivial solutions satisfy (for $z = x^2$)

$$\begin{aligned} 0 &= p(z) \\ &= k^2(2k-1) + [-2k^3+4k^2-3k+1]z - 3(k-1)^2z^2 - 3(k-1)z^3 - z^4 \end{aligned} \tag{8}$$

Since the solutions of $x = y$, $x - x^3 = 0$ and $x = -y$, $(1-2k)x - x^3 = 0$ are roots of the above equation, we conclude that $q(z) := (1-z)((1-2k) - z)$ divides $p(z)$. Such division gives us the polynomial $r(z) = z^2 + (k-1)z - k^2$, whose roots are

$$z_{\pm} = \frac{1}{2}((1-k) \pm \sqrt{-3k^2 - 2k + 1}).$$

Observe that $-3k^2 - 2k + 1 < 0$ when $k \in (\frac{1}{3}, \frac{1}{2})$. Therefore, these roots do not contribute to the roots of (7).

It follows that the only roots are these found on the lines $x = y$ and $x = -y$, which are: $z_1^* = (1, 1)$, $z_2^* = (0, 0)$, $z_3^* = (-1, -1)$, $z_4^* = (\sqrt{1-2k}, -\sqrt{1-2k})$ and $z_5^* = (-\sqrt{1-2k}, \sqrt{1-2k})$.

Now, we will sketch the phase portrait of $\dot{z} = A_i z$, $1 \leq i \leq 5$, where $A_i = f'(z_i^*) \in M_{2\times 2}$, $1 \leq i \leq 5$.

For $z_1^* = \binom{1}{1}$ and $z_3^* = \binom{-1}{-1}$, we have $f'(z_1^*) = f'(z_3^*) = \begin{pmatrix} -(2+k) & k \\ k & -(2+k) \end{pmatrix}$ with eigenvalues $\lambda_+^1 = -2 < 0$ and $\lambda_-^1 = -2-2k < 0$ and corresponding eigenvectors $v_+ = \binom{1}{1}$ and $v_- = \binom{-1}{1}$.

For $z_4^* = \begin{pmatrix} \sqrt{1-2k} \\ -\sqrt{1-2k} \end{pmatrix}$ and $z_5^* = \begin{pmatrix} -\sqrt{1-2k} \\ \sqrt{1-2k} \end{pmatrix}$, we have $f'(z_3^*) = f'(z_4^*) = \begin{pmatrix} 5k-2 & k \\ k & 5k-2 \end{pmatrix}$ with eigenvalues $\lambda_+^k = 2(3k-1) > 0$ and $\lambda_-^k = 2(2k-1) < 0$ and corresponding eigenvectors $v_+ = \binom{1}{1}$ and $v_- = \binom{-1}{1}$.

Lastly, for $z_2^* = \binom{0}{0}$, we have $f'(z_2^*) = \begin{pmatrix} 1-k & k \\ k & 1-k \end{pmatrix}$ with eigenvalues $\lambda_+^0 = 1 > 0$ and $\lambda_-^0 = 1-2k > 0$ and corresponding eigenvectors $v_+ = \binom{1}{1}$ and $v_- = \binom{-1}{1}$.

From this analysis and the saddle point property, we have

- $z_1^* = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ and $z_3^* = \left(\begin{smallmatrix}-1\\-1\end{smallmatrix}\right)$ are asymptotically stable.

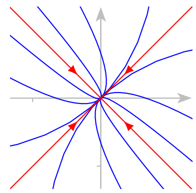

FIGURE 1. Linearization around $\pm(1,1)$.

- $z_4^* = \begin{pmatrix}\sqrt{1-2k}\\-\sqrt{1-2k}\end{pmatrix}$ and $z_5^* = \begin{pmatrix}-\sqrt{1-2k}\\\sqrt{1-2k}\end{pmatrix}$ are saddle points with one dimensional unstable and stable manifolds, then unstable.

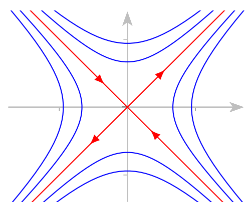

FIGURE 2. Linearization around $\pm(\sqrt{1-2k}, -\sqrt{1-2k})$

- $z_2^* = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$ is a unstable knot (two dimensional unstable manifold).

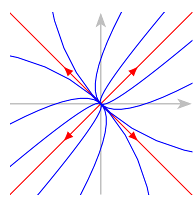

FIGURE 3. Linearization around $(0,0)$

Moreover, the semigroup $\{T(t) \in C(\mathbb{R}^2) : t \geq 0\}$ associated to (6) is gradient, that is, there exists an associated Lyapunov functional $V : \mathbb{R}^2 \to \mathbb{R}$, given by $V\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = \frac{k}{2}(x-y)^2 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{y^2}{2} + \frac{y^4}{4}$ which is such that $-\nabla V\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = f\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)$. Then, we can sketch its global attractor as the green part of Figure 4.

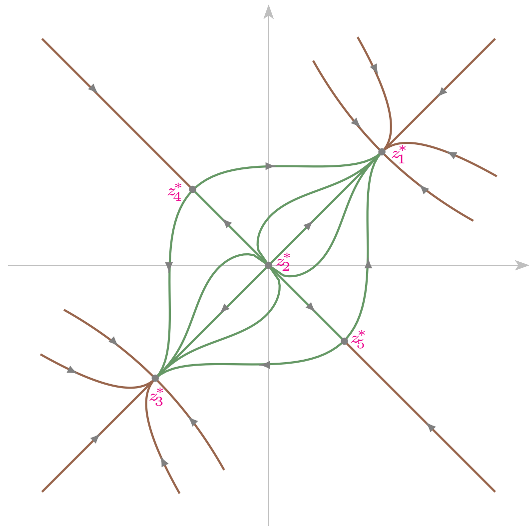

FIGURE 4. Phase portrait for $k \in \left(\frac{1}{3}, \frac{1}{2}\right)$.

If $k > \frac{1}{2}$, the only equilibria of (7) are the ones on the main diagonal. On the other hand, if $k \in [0, \frac{1}{3})$, then we have four additional equilibria, given by

$$
\begin{aligned}
z_6^* &= \left(-\tfrac{1}{\sqrt{2}}\sqrt{(1-k)-\sqrt{-3k^2-2k+1}},\ \tfrac{1}{\sqrt{2}}\sqrt{(1-k)+\sqrt{-3k^2-2k+1}}\right)\\
z_7^* &= \left(-\tfrac{1}{\sqrt{2}}\sqrt{(1-k)+\sqrt{-3k^2-2k+1}},\ \tfrac{1}{\sqrt{2}}\sqrt{(1-k)-\sqrt{-3k^2-2k+1}}\right)\\
z_8^* &= \left(\tfrac{1}{\sqrt{2}}\sqrt{(1-k)+\sqrt{-3k^2-2k+1}},\ -\tfrac{1}{\sqrt{2}}\sqrt{(1-k)-\sqrt{-3k^2-2k+1}}\right)\\
z_9^* &= \left(\tfrac{1}{\sqrt{2}}\sqrt{(1-k)-\sqrt{-3k^2-2k+1}},\ -\tfrac{1}{\sqrt{2}}\sqrt{(1-k)+\sqrt{-3k^2-2k+1}}\right)
\end{aligned}
$$

The equilibria $\{z_6^*, z_7^*, z_8^*, z_9^*\}$ are asymptotically stable when $k \in [0, \frac{1}{3})$. From now on, our interest will concentrate on the case $k \in (\frac{1}{3}, \frac{1}{2})$.

2.2. **Existence and hyperbolicity of the non-autonomous equilibria.** In Section 2.1 we have seen that the global attractor of the semigroup associated to (5) is completely described by five global bounded solutions (the equilibria) and that all other solutions in the attractor must converge (when $t \to \pm\infty$ to one of these equilibria, i.e, the global attractor is gradiente-like ([11, 12]). Furthermore we know exactly which is the structure of connections between equilibria (see Figure 4). The equilibria and the gradient structure play a fundamental role in the description of the dynamics of (5). Indeed, they induce a precise landscape in the phase space informing every initial data the way it has to follow until reaching one of the stable stationary solutions in which they asymptotically end. Thus, a gradient-like picture not only describes the asymptotic behaviour of the system, but also the metastability phenomena of solutions ([18]) before its stabilization in one of the invariant sets (stationary solutions in our case).

We now go back to the case where γ may be non-constant and, therefore, concentrate on problem (4). Our aim is to provide a description as in Figure 4 for the case when γ is no longer constant.

To start addressing this question we must find the solutions playing the role of equilibria in the description of the dynamics when $\gamma(\cdot)$ is time-dependent. These special solutions will be the hyperbolic global solutions which we will describe next. Note that

(i) The linear manifold $\ell_1 = \{(x,y) \in \mathbb{R}^2 : x = y\}$ is invariant for the evolution process associated to (4), that is, if $x_\tau = y_\tau$, then the solution

$$S(t,\tau)(x_\tau, y_\tau) = (x(t,\tau,x_\tau), y(t,\tau,y_\tau))$$

satisfies $x(t,\tau,x_\tau) = y(t,\tau,y_\tau))$, $t \geq \tau$.

In this case, both coordinates $x(t,\tau,x_\tau) = y(t,\tau,y_\tau)$ and satisfy

$$\dot{x} = x - \gamma(t)x^3, \quad t > \tau, \quad x(\tau) = x_\tau = y_\tau. \tag{9}$$

(ii) In the same way, the linear manifold $\ell_2 = \{(x,y) \in \mathbb{R}^2 : x = -y\}$ is invariant for the evolution process associated to (4).

In this case, both coordinates $x(t,\tau,x_\tau) = -y(t,\tau,y_\tau)$ satisfy

$$\dot{x} = (1-2k)x - \gamma(t)x^3, \quad t > \tau, \quad x(\tau) = x_\tau = -y_\tau = -y(\tau). \tag{10}$$

(iii) If $0 \leq k < \frac{1}{2}$, the linearization of (4) around the equilibrium solution $(x^*, y^*) = (0,0)$ has two positive eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1 - 2k$.

We will prove that, if $k < \frac{1}{2}$, there exist four bounded global solutions of (4) which remains away from $(0,0)$ for every $t \in \mathbb{R}$, two in ℓ_1 and two in ℓ_2.

For simplicity, consider the case in ℓ_2 and problem (10). The case in ℓ_1 is equivalent by making $k = 0$.

First, note that if $x_\tau \in I_k := \left[\left(\frac{1-2k}{\beta_1}\right)^{\frac{1}{2}}, \left(\frac{1-2k}{\beta_0}\right)^{\frac{1}{2}}\right]$, then $x(t,\tau,x_\tau) \in I_k$ for every $t \geq \tau$; i.e., the interval I_k is positively invariant for the evolution process $\{S(t,\tau) \in C(\mathbb{R}^2) : t \geq \tau\}$.

Now, consider $\{T_{\beta_i}(t) \in C(\mathbb{R}^2) : t \geq 0\}$, $i = 0,1$, the solution operator of the autonomous problem

$$\dot{x} = (1-2k)x - \beta_i x^3, \quad t > 0, \quad x(0) = x_0. \tag{11}$$

This problem is gradient and has three equilibria $x_0^* = 0, x_{1,i}^{\pm} = \pm\left(\frac{1-2k}{\beta_i}\right)^{\frac{1}{2}}$. If $\{S_\gamma(t,\tau) \in C(\mathbb{R}^2) : t \geq \tau\}$ is the evolution process associated to (10), we have

$$x_{1,1}^+ = T_{\beta_1}(t-s)x_{1,1}^+ \leq S_\gamma(t,s)x_{1,1}^+ \leq S_\gamma(t,s)x_{1,0}^+ \leq T_{\beta_0}(t-s)x_{1,0}^+ = x_{1,0}^+$$

and, if $s_1 \leq s_2 \leq t$, we have

$$\begin{aligned} x_{1,1}^+ \leq S_\gamma(t,s_1)x_{1,0}^+ &= S_\gamma(t,s_2)x_{1,0}^+ S_\gamma(s_2,s_1)x_{1,0}^+ \\ &\leq S_\gamma(t,s_2)x_{1,0}^+ \leq T_{\beta_0}(t-s)x_{1,0}^+ = x_{1,0}^+ \end{aligned}$$

and

$$x_{1,1}^+ \leq S_\gamma(t,s_2)x_{1,1}^+ = S_\gamma(t,s_2)T_{\beta_1}(s_2-s_1)x_{1,1}^+ \leq S_\gamma(t,s_2)S_\gamma(s_2,s_1)x_{1,1}^+$$
$$\leq S_\gamma(t,s_1)x_{1,1}^+ \leq S_\gamma(t,s_1)x_{1,0}^+ \leq T_{\beta_0}(t-s_1)x_{1,0}^+ = x_{1,0}^+$$

With this, we have that the limits

$$\eta_0(t) = \lim_{s\to-\infty} S_\gamma(t,s)x_{1,0}^+,\ t\in\mathbb{R}, \quad \text{and}$$
$$\eta_1(t) = \lim_{s\to-\infty} S_\gamma(t,s)x_{1,1}^+,\ t\in\mathbb{R}.$$

exist and correspond to bounded global solutions of (10) which lie in I_k. Also, $\eta_0(t)\geq\eta_1(t)$ for every $t\in\mathbb{R}$. If $\xi(t)=\eta_0(t)-\eta_1(t)\geq 0$, then

$$\left.\begin{array}{l}\dot{\eta_0}(t) = (1-2k)\eta_0(t)-\gamma(t)\eta_0(t)^3\\ \dot{\eta_1}(t) = (1-2k)\eta_1(t)-\gamma(t)\eta_1(t)^3\end{array}\right\} \dot{\xi}(t) = [1-2k-\gamma(t)(\eta_0(t)^2+\eta_0\eta_1(t)+\eta_1(t)^2)]\xi(t).$$

Let $q(t)=1-2k-\gamma(t)\eta_0(t)^2$. Then $\dot{\eta_0}(t)=q(t)\eta_0(t)$ and

$$\eta_0(t) = e^{\int_s^t q(\theta)d\theta}\eta_0(s).$$

Since $\eta_0(t)\in I_k$, for every $t\in\mathbb{R}$, we must have that $e^{\int_s^t q(\theta)d\theta}\in\left[\sqrt{\frac{\beta_0}{\beta_1}},\sqrt{\frac{\beta_1}{\beta_0}}\right]$, for every $t\geq s$.

Furthermore

$$\dot{\xi}(t) = q(t)\xi(t) - \gamma(t)[\eta_0(t)\eta_1(t)+\eta_1(t)^2\,]\,\xi(t)$$

with $p(t)=\gamma(t)[\eta_0(t)\eta_1(t)+\eta_1(t)^2]\geq 2\frac{\beta_0}{\beta_1}(1-2k)>0$ and

$$\begin{aligned}0\leq\xi(t) = e^{\int_s^t q(\theta)d\theta}\xi(s) - \int_s^t e^{\int_r^t q(\theta)d\theta}p(r)\xi(r)dr &\leq e^{\int_s^t q(\theta)d\theta}\xi(s)\\ &\leq e^{\int_s^t q(\theta)d\theta}\eta_0(s) = \eta_0(t)\end{aligned} \tag{12}$$

Hence, making $s\to-\infty$, we have that the integral

$$\int_{-\infty}^t e^{\int_r^t q(\theta)d\theta}p(r)\xi(r)dr$$

is convergent and, since $e^{\int_r^t q(\theta)d\theta}p(r)\geq 2\left(\frac{\beta_0}{\beta_1}\right)^{\frac{3}{2}}(1-2k)>0$, there is a sequence $s_n\to-\infty$ such that $\xi(s_n)\overset{n\to\infty}{\longrightarrow}0$. So, replacing s by s_n in (12) and making $n\to\infty$ we have that

$$0\leq\xi(t) = \int_{-\infty}^t e^{\int_r^t q(\theta)d\theta}p(r)\xi(r)dr\leq 0$$

and we conclude that $\eta_0(t)=\eta_1(t)$ for every $t\in\mathbb{R}$. This unique solution will be denoted by ξ_k, where $k\in(0,\frac{1}{2})$ is the parameter of (4).

This proves the existence of a unique bounded global solution of (10) that lies in I_k for every $t\in\mathbb{R}$.

This argument can be repeated, without changes, for the case $k=0$.

Our next result summarizes the results just proven for problem (4):

Theorem 2.4. *Let $0 \leq k < \frac{1}{2}$. Then* (10) *has exactly one global solution $\xi_k : \mathbb{R} \to \mathbb{R}^2$ which remains away from zero. As a consequence,* (4) *has exactly four solutions which remains away from zero and are given by*

$$\xi^*_{1,+}(t) = (\xi_0, \xi_0)(t),\ t \in \mathbb{R},$$
$$\xi^*_{1,-}(t) = (-\xi_0, -\xi_0)(t),\ t \in \mathbb{R},$$
$$\xi^*_{2,+}(t) = (\xi_k, -\xi_k)(t),\ t \in \mathbb{R},$$
$$\xi^*_{2,-}(t) = (-\xi_k, \xi_k)(t),\ t \in \mathbb{R}.$$

We call these four solutions bounded non-degenerate global solutions. We will denote $\xi^*_0(t) = (0, 0)$, for every $t \in \mathbb{R}$.

We will now show that every bounded global solution of (4) tends to ℓ_2 when $t \to -\infty$ and, except the bounded global solutions $\xi^*_{2,\pm}$ which remains in ℓ_2 for every $t \in \mathbb{R}$, tend to ℓ_1 when $t \to +\infty$. We will first linearize around these non-degenerated global solutions and prove that the linearizations possess exponential dichotomies.

Theorem 2.5. *Let $0 \leq k < \frac{1}{2}$. Then $\xi^*_0 = (0, 0)$, $\xi^*_{1,\pm}$ and $\xi^*_{2,\pm}$ are hyperbolic bounded global solutions, that is, a linearizations around each of these solutions have exponential dichotomy. These hyperbolic bounded global solutions will be called non-autonomous equilibria.*

Proof. We affirm that the linearization of the evolution process $\{S(t, \tau) \in C(\mathbb{R}^2) : t \geq \tau\}$ associated to (4) around ξ^*_0 has exponential dichotomy with constant projections $Q(t) = I$ and $(I - Q(t)) = 0$ for every $t \in \mathbb{R}$. Note that this linearizations is a autonomous dynamical system given by

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-k & k \\ k & 1-k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{13}$$

Along the invariant subspace ℓ_1, the problem reduces to $\dot{x}(t) = x$. It follows that $x(t) = e^t x(0)$ and ℓ_1 is an unstable direction. Now, along the invariant subspace ℓ_2, the problem reduces to $\dot{x}(t) = (1 - 2k)x$. It follows that $x(t) = e^{(1-2k)t}x(0)$ and ℓ_2 also is an unstable direction. Any other solution of the linear system is a linear combination of these two solutions that have exponential dichotomy; it follows that ξ^*_0 is hyperbolic.

Now, we will prove that the linearization of the evolution process $\{S(t, \tau) \in C(\mathbb{R}^2) : t \geq \tau\}$ associated to (4) around $\xi^*_{1,+}$ is a linear evolution process $\{L_{1,+}(t, \tau) \in C(\mathbb{R}^2) : t \geq \tau\}$ that has exponential dichotomy. This linearization is the evolution process associated to the linear ODE

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-k-3\gamma(t)\xi_0^2 & k \\ k & 1-k-3\gamma(t)\xi_0^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{14}$$

with $\xi_0 = \xi^*_{1,+}$.

Note that ℓ_1 is still an invariant manifold for (14) and that along ℓ_1, the system reduces to, for each coordinate, to

$$\dot{x}(t) = (1 - 3\gamma(t)\xi_0^2)x.$$

Recalling that $\dot{\xi_0}(t) = \xi_0(t) - \gamma(t)\xi_0(t)^3$, we have

$$3\frac{\dot{\xi_0}(t)}{\xi_0(t)} = 2 + (1 - 3\gamma(t)\xi_0(t)^2),$$

from which we obtain that

$$e^{\int_s^t (1-3\gamma(\theta)\xi_0(\theta)^2)d\theta} = e^{-2(t-s)} \left(\frac{\xi_0(t)}{\xi_0(s)}\right)^3$$

and, once $\xi_0(t) \in I_0 = \left[\beta_1^{-\frac{1}{2}}, \beta_0^{-\frac{1}{2}}\right]$ for every $t \in \mathbb{R}$, we must have that

$$\left(\frac{\beta_0}{\beta_1}\right)^{\frac{3}{2}} \leq \left(\frac{\xi_0(t)}{\xi_0(s)}\right)^3 \leq \left(\frac{\beta_1}{\beta_0}\right)^{\frac{3}{2}}$$

so that the solutions of (14) in ℓ_1 satisfy

$$\left\|\begin{pmatrix} x(t) \\ x(t) \end{pmatrix}\right\| \leq \left(\frac{\beta_1}{\beta_0}\right)^{\frac{3}{2}} e^{-2(t-s)} \left\|\begin{pmatrix} x(s) \\ x(s) \end{pmatrix}\right\|.$$

Therefore, if $I - P$ is the orthogonal projection in ℓ_1, we have

$$\|L_{1,+}(t,s)(I-P)\| \leq \left(\frac{\beta_1}{\beta_0}\right)^{\frac{3}{2}} e^{-2(t-s)},\ t \geq s. \tag{15}$$

Then, ℓ_1 is a stable direction.

Now we study what happens in the image of P, that is, in ℓ_2. Remember that it corresponds to $x = -y$ satisfying $\dot{x}(t) = (1 - 2k - 3\gamma(t)\xi_0^2)x$. Since $\dot{\xi_0}(t) = \xi_0(t) - \gamma(t)\xi_0(t)^3$, we have that

$$3\frac{\dot{\xi_0}(t)}{\xi_0(t)} = 2(1+k) + (1 - 2k - 3\gamma(t)\xi_0(t)^2),$$

from which we obtain that

$$e^{\int_s^t (1-2k-3\gamma(\theta)\xi_0(\theta)^2)d\theta} = e^{-2(1+k)(t-s)} \left(\frac{\xi_0(t)}{\xi_0(s)}\right)^3$$

Since $\xi_0(t) \in I_0 = \left[\beta_1^{-\frac{1}{2}}, \beta_0^{-\frac{1}{2}}\right]$ for every $t \in \mathbb{R}$, we have that

$$\left(\frac{\beta_0}{\beta_1}\right)^{\frac{3}{2}} \leq \left(\frac{\xi_0(t)}{\xi_0(s)}\right)^3 \leq \left(\frac{\beta_1}{\beta_0}\right)^{\frac{3}{2}}$$

and the solutions of (14) in ℓ_2 satisfy

$$\left\|\begin{pmatrix} x(t) \\ -x(t) \end{pmatrix}\right\| \leq \left(\frac{\beta_1}{\beta_0}\right)^{\frac{3}{2}} e^{-2(1+k)(t-s)} \left\|\begin{pmatrix} x(s) \\ -x(s) \end{pmatrix}\right\|$$

so

$$\|L_{1,+}(t,s)P\| \leq \left(\frac{\beta_1}{\beta_0}\right)^{\frac{3}{2}} e^{-2(1+k)(t-s)},\ t \geq s. \tag{16}$$

This indicates that ℓ_2 is a stable direction.

Combining the estimates in (15) and (16) together, we obtain that

$$\|L_{1,+}(t,s)\| \leq 2\left(\frac{\beta_1}{\beta_0}\right)^{\frac{3}{2}} e^{-2(t-s)},\ t \geq s.$$

The estimate for the linearization around $\xi^*_{1,-}$ is exactly the same with the same exponent.

Now, we will prove that the linearization of the evolution process $\{S(t,\tau) \in C(\mathbb{R}^2 : t \geq \tau\}$ associated to (4) around $\xi^*_{2,+}(t) = (\xi_k(t), -\xi_k(t))$, $t \in \mathbb{R}$, is a linear evolution process $\{L_{2,+}(t,\tau) \in C(\mathbb{R}^2) : t \geq \tau\}$ that has exponential dichotomy. This linearization is the linear evolution process associated to the linear ODE

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-k-3\gamma(t)\xi_k^2 & k \\ k & 1-k-3\gamma(t)\xi_k^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{17}$$

Note that ℓ_1 is still an invariant manifold to (17) and, along ℓ_1, the above system reduces to

$$\frac{d}{dt}\begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} 1-k-3\gamma(t)\xi_k^2 & k \\ k & 1-k-3\gamma(t)\xi_k^2 \end{pmatrix}\begin{pmatrix} x \\ x \end{pmatrix}$$

which corresponds to each coordinate satisfying $\dot{x}(t) = (1-3\gamma(t)\xi_k^2)x$. Remembering that $\dot{\xi_k}(t) = (1-2k)\xi_k(t) - \gamma(t)\xi_k(t)^3$, we have

$$3\frac{\dot{\xi_k}(t)}{\xi_k(t)} = 2 - 6k + (1-3\gamma(t)\xi_k(t)^2)$$

from which we obtain that

$$e^{\int_s^t (1-3\gamma(\theta)\xi_k(\theta)^2)d\theta} = e^{(6k-2)(t-s)}\left(\frac{\xi_k(t)}{\xi_k(s)}\right)^3$$

and since $\xi_k(t) \in I_k = \left[(1-2k)^{\frac{1}{2}}\beta_1^{-\frac{1}{2}}, (1-2k)^{\frac{1}{2}}\beta_0^{-\frac{1}{2}}\right]$ for every $t \in \mathbb{R}$, it follows that

$$\left(\frac{\beta_0}{\beta_1}\right)^{\frac{3}{2}} \leq \left(\frac{\xi_k(t)}{\xi_k(s)}\right)^3 \leq \left(\frac{\beta_1}{\beta_0}\right)^{\frac{3}{2}}$$

and the solutions of (17) along ℓ_1 satisfy

$$\left\|\begin{pmatrix} x(t) \\ x(t) \end{pmatrix}\right\| \geq \left(\frac{\beta_0}{\beta_1}\right)^{\frac{3}{2}} e^{(6k-2)(t-s)}\left\|\begin{pmatrix} x(s) \\ x(s) \end{pmatrix}\right\|$$

or, if Q is the orthogonal projection along ℓ_1 and $k \in (\frac{1}{3}, \frac{1}{2})$

$$\|L_{2,+}(s,t)Q\| \leq \left(\frac{\beta_1}{\beta_0}\right)^{\frac{3}{2}} e^{(6k-2)(s-t)},\ t \geq s.$$

Therefore, ℓ_1 is an unstable direction.

Let's see what happens in the image of $I-Q$, that is, along ℓ_2. This corresponds to $x=-y$ satisfying $\dot{x}(t)=(1-2k-3\gamma(t)\xi_k^2)x$. Since $\dot{\xi_k}(t)=\xi_k(t)-\gamma(t)\xi_k(t)^3$, we obtain that

$$3\frac{\dot{\xi_k}(t)}{\xi_k(t)}=2-4k+(1-2k-3\gamma(t)\xi_k(t)^2).$$

Hence we obtain that

$$e^{\int_s^t(1-2k-3\gamma(\theta)\xi_k(\theta)^2)d\theta}=e^{-2(1-2k)(t-s)}\left(\frac{\xi_k(t)}{\xi_k(s)}\right)^3$$

and, since $\xi_k(t)\in I_0=\left[\beta_1^{-\frac{1}{2}},\beta_0^{-\frac{1}{2}}\right]$ for every $t\in\mathbb{R}$, we must have

$$\left(\frac{\beta_0}{\beta_1}\right)^{\frac{3}{2}}\leq\left(\frac{\xi_k(t)}{\xi_k(s)}\right)^3\leq\left(\frac{\beta_1}{\beta_0}\right)^{\frac{3}{2}}$$

and the solutions of (17) along ℓ_2 satisfy

$$\left\|\begin{pmatrix}x(t)\\-x(t)\end{pmatrix}\right\|\leq\left(\frac{\beta_1}{\beta_0}\right)^{\frac{3}{2}}e^{-2(1-2k)(t-s)}\left\|\begin{pmatrix}x(s)\\-x(s)\end{pmatrix}\right\|,\ t\geq s,$$

That is,

$$\|L_{2,+}(t,s)(I-Q)\|\leq\left(\frac{\beta_1}{\beta_0}\right)^{\frac{3}{2}}e^{-2(1-2k)(t-s)},\ t\geq s.$$

and ℓ_2 is a stable direction.

This shows that the linearization of $\{S(t,s)\in C(\mathbb{R}^2):t\geq s\}$ around $\xi_{2,+}^*$ has exponential dichotomy with projections $Q(t)\equiv Q$, where Q is the orthogonal projection with range ℓ_1.

The estimate of the linearization around $\xi_{2,-}^*$ is exactly the same with the same exponent. □

Remark 2.6. *We note that, if $k\in(0,\frac{1}{3})$, the above computations will lead to $\xi_{2,\pm}^*$ being stable in the invariant manifolds ℓ_1 and ℓ_2. We expect that this situation will require the existence of 4 (four) other non-degenerated global solutions, each one possessing one dimensional stable and unstable manifolds. That is exactly what happens when $\gamma=const$ for which we can then explicitly compute the 4 additional equilibria that arise. In the non-autonomous case, we have not been able to find these additional 4 solutions. We will not be interested in that situation since it does not correspond to an attractor of the Chafee-Infante problem ([14]) which our ODE's are aimed to mimic (as in [13]).*

2.3. **Asymptotic symmetry of bounded global solutions.** Our next step is to show that solutions obtained in Section 2.2 play the same role of equilibria of (5).

The reasoning is divided in four parts, all similar, related to the quadrants determined by ℓ_1 and ℓ_2 in the plane. Observe that, once the lines ℓ_1 and ℓ_2 are invariant subspaces for (4), the quadrants determined by these lines are also invariants. First, we consider

$$Q_1=\{(x,y):x-y\geq0\text{ and }x+y\geq0\}$$

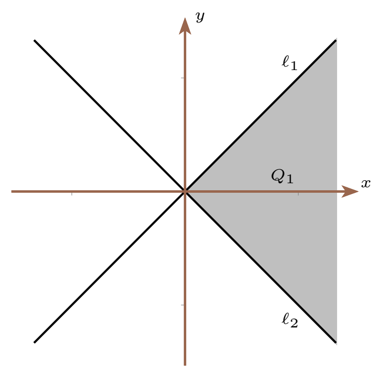

FIGURE 5. Region Q_1

Proposition 2.7. *If $(x(t), y(t))$ is a bounded global solution inside Q_1, then*

$$\|(x(t), y(t)) - \ell_1\| \overset{t\to\infty}{\longrightarrow} 0 \text{ and } \|(x(t), y(t)) - \ell_2\| \overset{t\to-\infty}{\longrightarrow} 0.$$

Proof. Consider the change of coordinate ($\frac{\pi}{4}$ counterclockwise rotation):

$$(x, y) \overset{\mathrm{T}}{\to} (\frac{x-y}{\sqrt{2}}, \frac{y+x}{\sqrt{2}}) = (z_1, z_2)$$

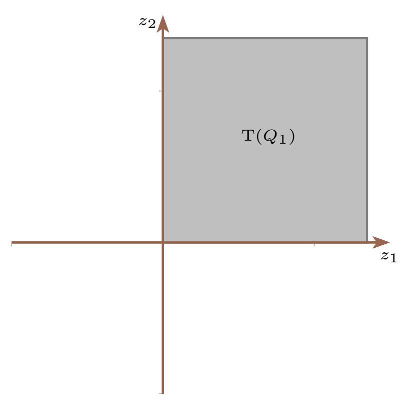

FIGURE 6. Region $\mathrm{T}(Q_1)$

Hence, inside Q_1, we have $x = \frac{\sqrt{2}}{2}(z_2 + z_1)$ and $y = \frac{\sqrt{2}}{2}(z_2 - z_1)$, where $z_1, z_2 \geq 0$. In the new coordinates, the flow (4) is given by

$$\begin{cases} \dot{z_1} = z_1(1-2k) - \frac{\gamma(t)}{4}((z_1+z_2)^3 - (z_2-z_1)^3) =: f_{z,1}(z_1, z_2) \\ \dot{z_2} = z_2 - \frac{\gamma(t)}{4}((z_1+z_2)^3 + (z_2-z_1)^3) =: f_{z,2}(z_1, z_2) \end{cases}$$

Consider now a curve $z_2 = rz_1^n$; its normal vector is given by $n_{(z_1,z_2)} = (-1, \frac{1}{nrz_1^{n-1}})$ when $z_1 \neq 0$.

Along this curve, we have the vector field

$$\begin{cases} f_{z,1}(z_1, rz_1^n) = z_1(1-2k) - \frac{\gamma(t)}{4}(2z_1^3 + 6r^2 z_1^{2n+1}) \\ f_{z,2}(z_1, rz_1^n) = rz_1^n - \frac{\gamma(t)}{4}(6rz_1^{2+n} + 2r^3 z_1^{3n}) \end{cases}$$

We show that there exists $n \in \mathbb{N}$ such that the field $(f_{z,1}(z_1, rz_1^n), f_{z,2}(z_1, rz_1^n))$ points into the interior of the domain bounded by the z_2-axis and the curve $z_2 = rz_1^n$ ($z_1 \geq 0$) for all $r > 0$. Observe that it will imply that if $(z_1(s), z_2(s))$ satisfy $z_2(s) = rz_1(s)^n$ and $t > s$, then $(z_1(t), z_2(t))$ satisfy $z_2(t) = r' z_1(t)^n$, with $r' > r$. Hence, the solutions will converge to the line $z_1 = 0$ (corresponding to the line $x = y$) when $t \to +\infty$ and to the line $z_2 = 0$ (corresponding to the line $x = -y$) when $t \to -\infty$.

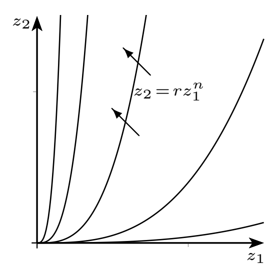

Figure 7. Contour lines

Hence, we will check under which conditions we have $(f_{z,1}, f_{z,2}) \cdot n_{(z_1,z_2)} > 0$ along the curve.

We have

$$\begin{aligned}(f_{z,1}, f_{z,2}) \cdot n_{(z_1,z_2)} &= \frac{z_1}{n} - \frac{\gamma(t)}{4n}(6z_1^3 + 2r^2 z^{(2n+1)}) - z_1(1-2k) + \frac{\gamma(t)}{4}(2z_1^3 + 6r^2 z_1^{(2n+1)}) \\ &= \frac{z_1(2kn - n + 1)}{n} + \frac{\gamma(t)(z_1^3(n-3) + r^2 z_1^{2n+1}(3n-1))}{2n} > 0 \end{aligned}$$

$$\iff 2(2kn - n + 1) + \gamma(t)(z_1^2(n-3) + r^2 z_1^{2n}(3n-1)) > 0$$

Observe that if $n = 3$, the last inequality becomes

$$4(3k-1) > -8\gamma(t) r^2 z_1^6,$$

which is true for every $r, z_1 \geq 0$, because $k > \frac{1}{3}$ and $\gamma(t) > 0$ for every $t \in \mathbb{R}$. Therefore, taking $n = 3$, we obtain the desired result.

□

This result will be refined when we treat the associated skew-product semigroup.

3. **Gradient structure of the skew-product semiflow.** In this section, we will take advantage of the skew-product semigroup associated to (1) and describe its attractor from the results in the previous sections.

Let $\beta : \mathbb{R}^+ \to [\beta_1, \beta_2]$ be a globally Lipschitz function and consider the space $C(\mathbb{R}^+, [\beta_1, \beta_2])$ endowed with the metric ρ of the uniform convergence in compact subsets of $\mathbb{R}^+$. If $\Sigma = \overline{\{\beta(t+\cdot) \in C(\mathbb{R}^+, [\beta_1, \beta_2]) : t \in \mathbb{R}^+\}}^{\rho}$, define the driving semigroup $\Theta(t) : \Sigma \to \Sigma$ by $(\Theta(t)\sigma)(\cdot) = \sigma(t+\cdot)$, $\sigma \in \Sigma$.

Moreover, given $\left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right) \in \mathbb{R}^2$, define the cocycle $\mathbb{R} \times \Sigma \ni (t, \sigma) \to \mathcal{K}(t, \sigma) \left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right) \in \mathbb{R}^2$ as being the solutions of the initial value problem (2). With this, we can consider the associated skew-product semigroup $\Pi(\cdot)$ in $\mathbb{X} = \mathbb{R}^2 \times \Sigma$ given by

$$\Pi(t)(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \sigma) = (\mathcal{K}(t, \sigma) \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \Theta(t)\sigma).$$

Given a compact $K \subset \mathbb{R}^+$, it is not difficult to see that $\{\beta(t+\cdot) : t \in \mathbb{R}^+\}$ is uniformly bounded and uniformly equicontinuous. Then, by Arzelá-Ascoli's Theorem, Σ is compact with the metric ρ of the uniform convergence in compact subsets. Hence, $\{\Theta(t) : t \geq 0\}$ has a global attractor $\mathscr{S}$ in Σ.

If $\eta : \mathbb{R} \to \Sigma$ is a global solution of Θ, then we can define the evolution process $S_\eta(t, s) = \mathcal{K}(t-s, \eta(s))$, for every $t \geq s$, being the solutions of

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{pmatrix} k(y-x) + x - \eta(s)(t-s)x^3 \\ k(x-y) + y - \eta(s)(t-s)y^3 \end{pmatrix} = f_\eta\left(t, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) \\ \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} &= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2 \end{aligned} \tag{18}$$

which have already been studied in the last session with $\gamma(t) = \eta(t)(0)$. We now describe the attractor of the semigroup $\Pi(\cdot)$. For this purpose, we will need a definition and a technical lemma ([5, 4]):

Definition 3.1. *Consider X a metric space. We say that a family of continuous semigroups $\{T_\eta(t) \in C(X) : t \geq 0\}_{\eta \in [0,1]}$ is* collectively asymptotically compact at *$\eta = 0$ if, given a sequence $\{\eta_k\}_{k\in\mathbb{N}}$ with $\eta_k \overset{k\to\infty}{\longrightarrow} 0$, a bounded sequence $\{x_k\}_{k\in\mathbb{N}}$ in X and a sequence $\{t_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^+$ with $t_k \overset{k\to\infty}{\longrightarrow} \infty$, then $\{T_{\eta_k}(t_k)x_k\}$ is relatively compact.*

Lemma 3.2. *Let $\{T_\eta(t) \in C(X) : t \in \mathbb{R}^+\}_{\eta\in[0,1]}$ be a family of semigroups that is continuous and collectively asymptotically compact at $\eta = 0$. Let $\eta_n \overset{n\to\infty}{\longrightarrow} 0$, $a_n < b_n < c_n$ such that $b_n - a_n, c_n - b_n \overset{n\to\infty}{\longrightarrow} \infty$, and set $\mathbb{J}_n = [a_n, c_n]$. Let $\xi_n : \mathbb{J}_n \to X$ be a solution of T_{η_n} and assume that $\overline{\bigcup_n \xi_n(\mathbb{J}_n)}$ is bounded. Then there exists a subsequence $(\eta_k)_{k\in\mathbb{N}}$ and a bounded global solution $\xi_0 : \mathbb{R} \to X$ of $\{T_0 \in C(X) : t \in \mathbb{T}^+\}$ such that $\xi_{n_k} \to \xi_0$ uniformly on compact subintervals of $\mathbb{R}$. In particular,*

$$\lim_{k\to\infty} \xi_{\eta_k}(s) \to \xi_0(s)$$

for every $s \in \mathbb{R}^+$.

If $\eta(t)$ is a global solution of Θ, let $\xi^{\pm}_{i,\eta}$, $i = 1, 2$, be the associated hyperbolic bounded global solutions of problem (18).

We will refine the results of Proposition 2.7. Define

$$\Xi_0 = \{((0,0), \eta(0)) : \eta \text{ is a bounded solution of } \Theta\}$$

and $\Xi^{\pm}_i = \{(\xi^{\pm}_{i,\eta}(t), \eta(t)) : \eta(\cdot) \text{ is a bounded solution of } \Theta(\cdot) \text{ and } t \in \mathbb{R}\}$, $i = 1, 2$. We know that $Q_1 \times \Sigma$ is invariant under the action of $\Pi(\cdot)$. Then

Proposition 3.3. *If* $(\xi_\eta(\cdot), \eta(\cdot)) : \mathbb{R} \to Q_1 \times \Sigma$ *is a bounded solution of* $\Pi(\cdot)$, *then* $d((\xi_\eta(t), \eta(t)), \Xi^+_1) \overset{t\to\infty}{\longrightarrow} 0$. *Moreover, either* $d((\xi_\eta(t), \eta(t)), \Xi_0) \overset{t\to-\infty}{\longrightarrow} 0$ *or* $d((\xi_\eta(t), \eta(t)), \Xi^+_2) \overset{t\to-\infty}{\longrightarrow} 0$.

Proof. Let $(\xi_\eta(t), \eta(t))$ be a bounded global solution in Q_1 for $\Pi(\cdot)$ and consider a sequence $t_n \to -\infty$. From Lemma 3.2, it follows that there exists a subsequence of $\{t_n\}_{n\in\mathbb{N}}$ (which we denote the same) and a bounded global solution $(\sigma(\cdot), \vartheta(\cdot)) : \mathbb{R} \to Q_1 \times \Sigma$ of $\Pi(\cdot)$ and, consequently, $\vartheta(\cdot) : \mathbb{R} \to \Sigma$ is a bounded global solution of $\Theta(\cdot)$ and $\sigma : \mathbb{R} \to Q_1$ is a bounded global solution of the evolution process $\{K(t-s, \vartheta(s)) : t \geq s\}$, given by $(\sigma(t), \vartheta(t)) = \lim_{n\to\infty}(\xi_\eta(t+t_n), \eta(t+t_n))$. If $(\xi_\eta(t), \eta(t)) \overset{t\to-\infty}{\longrightarrow} \Xi_0$, the result follows.

If not, $(\xi_\eta(t), \eta(t))$ stays away from a ball of radius δ and center at the origin. In fact, note that

$$\left(\begin{smallmatrix} \dot{x} \\ \dot{y} \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = (1-k)(x^2+y^2) + 2kxy - \eta(s)(t-s)(x^4+y^4).$$

Since $\eta(\cdot) \in \mathscr{S}$ is bounded, there is a $\delta > 0$ such that $(\dot{x}, \dot{y}) \cdot (x, y) > 0$ for every $t > s$, $t, s \in \mathbb{R}$ and $x < \delta$. Hence, the solution would converge backwards to Ξ_0 if it had entered the ball of radius δ centered at the origin.

Therefore, from the results in Section 2.3, $(\sigma(t), \vartheta(t))$ is a bounded global solution in $\ell_2 \times \mathscr{S}$ and $\sigma(\cdot)$ remains away from $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \in \mathbb{R}^2$. Then $(\sigma(t), \vartheta(t)) = (\xi^+_{2,\vartheta}, \vartheta)$ and $(\xi_\eta(t), \eta(t)) \overset{t\to-\infty}{\longrightarrow} \Xi^+_2 \ni (\xi^+_{2,\vartheta}, \vartheta)$.

Similarly, we obtain that $(\xi_\eta(t), \eta(t)) \overset{t\to\infty}{\longrightarrow} \Xi^+_1$. □

By symmetry, the solutions in the other quadrants enjoy the same property. Thereby, we obtain that the sets Ξ_0 and $\Xi^{\pm}_i$ are maximal invariants and conclude that $\Pi(\cdot)$ is a dynamically gradient semigroup relatively to the isolated invariants $\{\Xi_0, \Xi^{\pm}_i, i = 1, 2\}$ and, therefore, $\Pi(\cdot)$ is gradient. We can say more, as we will see in the next result.

Proposition 3.4. *There are global solutions* $(\zeta^{\pm}_{i,\eta}(\cdot), \eta(\cdot))$ *in* $\ell_i \times \mathscr{S}$ *such that*

$$d(\Xi_0, (\zeta^{\pm}_{i,\eta}(t), \eta(t))) \overset{t\to-\infty}{\longrightarrow} 0 \text{ and } d(\Xi^{\pm}_i, (\zeta^{\pm}_{i,\eta}(t), \eta(t))) \overset{t\to\infty}{\longrightarrow} 0$$

and a solution that connects $\Xi^+_2 (\Xi^-_2)$ *to* $\Xi^+_1 (\Xi^-_1)$.

Proof. We will prove the result in $\overline{Q_1}\times\mathscr{S}$. Let $\eta(t)$ be a global solution of Θ. Then ℓ_1 is the unstable manifold of the zero solution $\xi_{0,\eta}(t)\equiv\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$. Hence, there is a non-zero global solution $(\zeta_\eta(t),\eta(t))$ of $\Pi(\cdot)$ such that $d((\xi_{0,\eta}(t),\eta(t)),(\zeta_\eta(t),\eta(t)))\overset{t\to-\infty}{\longrightarrow}0$. By lemma 3.2, if $\{t_n\}_{n\in\mathbb{N}}$ is a sequence such that $t_n\to\infty$ and $(\zeta_\eta(t_n),\eta(t_n))\to(\sigma,\vartheta)$, we can construct a bounded global solution $(\sigma(t),\vartheta(t))$ given by $(\sigma(t),\vartheta(t))=\lim_{n\to\infty}(\zeta_\eta(t+t_n),\eta(t+t_n))$ and, once $(\sigma(t),\vartheta(t))$ remains away from Ξ_0, we have that $(\sigma(t),\vartheta(t))=(\xi^+_{1,\vartheta},\vartheta)\in\Xi^+_1$. The reasoning to find ζ^+_2 is entirely analogous.

Moreover, since each $\xi^+_{2,\vartheta}(\xi^-_{2,\vartheta})$ has a one-dimensional unstable manifold, there is a bounded global solution $(\sigma(t),\vartheta(t))$, such that $(\sigma(t),\vartheta(t))\overset{t\to-\infty}{\longrightarrow}(\xi^+_{2,\eta},\vartheta)((\xi^-_{2,\eta},\vartheta))$, that converges to $\Xi^+_1(\Xi^-_1)$ by the last proposition. □

Now we can illustrate the uniform attractor $\Pi_{\mathbb{R}^2}\mathbb{A}$, of the problem (1), as below, where $E^\pm_j=\Pi_{\mathbb{R}^2}\Xi^\pm_j$. As a result of this, any solution of (1) converge to one of the $E^\pm_j$, $j=0,1,2$. The solutions represented in Figure 8 are all global bounded solutions of (3) for all global solution $\eta:\mathbb{R}\to\mathscr{S}$ of $\Theta(\cdot)$.

This illustrates the fact that, when considering the problem (3) we must indeed consider all limiting vector fields associated to the one given in (3). In the autonomous case, the interpretation is the same, but the set of limiting vector fields is unitary.

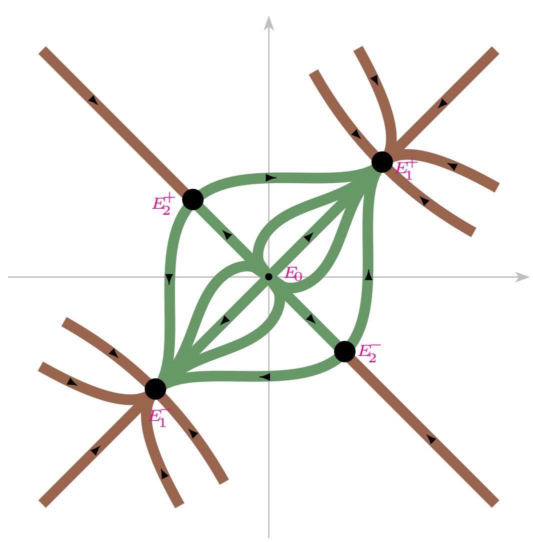

FIGURE 8. Representation of the uniform attractor for (1)

3.1. **Characterization of hyperbolic global solutions.** If $\vartheta:\mathbb{R}\to\Sigma$ is a global solution of Θ, then we can define the evolution process $S_\vartheta(t,s)=\mathcal{K}(t-s,\vartheta(s))$ for every $t\geq s$ as defined in (18), with η replaced by ϑ.

Now, if $\xi \in \{\xi_0, \xi_{1,\vartheta}^{\pm}, \xi_{2,\vartheta}^{\pm}\}$, let $\{L_{\vartheta,\xi}(t,s) : t \geq s\}$ be the solution operator for

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-k-3\vartheta(s)(t-s)\xi(t)^2 & k \\ k & 1-k-3\vartheta(s)(t-s)\xi(t)^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = A_{\vartheta,\xi}(t)\begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2 \tag{19}$$

We now give a characterization of bounded global solutions under the assumption that there exists a hyperbolic global solution.

Lemma 3.5. *If $\xi : \mathbb{R} \to \mathbb{R}^2$ is a hyperbolic bounded global solution of $S_\vartheta(\cdot,\cdot)$, $L_{\vartheta,\xi}(\cdot,\cdot)$ has an exponential dichotomy with projections $\{P(t) : t \in \mathbb{R}\}$, constant M, and exponent ω. If $\phi : \mathbb{R} \to \mathbb{R}^2$ is a bounded global solution of $S_\vartheta(\cdot,\cdot)$, then*

$$\phi(t) = \int_{-\infty}^{\infty} G_\xi(t,s)[f_\vartheta(s,\phi(s)) - A_{\vartheta,\xi}(s)\phi(s)]\,\mathrm{d}s,$$

where

$$G_\xi(t,s) = \begin{cases} L_{\vartheta,\xi}(t,s)(I - P(s)), & t \geq s \\ -L_{\vartheta,\xi}(t,s)P(s), & t \leq s. \end{cases} \tag{20}$$

Proof. First note that $\phi : \mathbb{R} \to \mathbb{R}^2$ is a solution of

$$\phi(t) = L_{\vartheta,\xi}(t,\tau)\phi(\tau) + \int_\tau^t L_{\vartheta,\xi}(t,s)[f_\vartheta(s,\phi(s)) - A_{\vartheta,\xi}(s)\phi(s)]\,\mathrm{d}s. \tag{21}$$

Recall that $L_{\vartheta,\xi}(\cdot,\cdot)$ has an exponential dichotomy. Applying $P(t)$ to (21)

$$P(t)\phi(t) = L_{\vartheta,\xi}(t,\tau)P(\tau)\phi(\tau) + \int_\tau^t L_{\vartheta,\xi}(t,s)P(s)[f_\vartheta(s,\phi(s)) - A_{\vartheta,\xi}(s)\phi(s)]\,\mathrm{d}s,$$

and consequently

$$L_{\vartheta,\xi}(\tau,t)P(t)\phi(t) = P(\tau)\phi(\tau) + \int_\tau^t L_{\vartheta,\xi}(\tau,s)P(s)[f_\vartheta(s,\phi(s)) - A_{\vartheta,\xi}(s)\phi(s)]\,\mathrm{d}s.$$

Letting $t \to \infty$, the left-hand side goes to zero and we obtain

$$P(t)\phi(t) = -\int_t^\infty L_{\vartheta,\xi}(t,s)P(s)[f_\vartheta(s,\phi(s)) - A_{\vartheta,\xi}(s)\phi(s)]\,\mathrm{d}s, \quad t \in \mathbb{R}.$$

Similarly,

$$\begin{aligned}(I - P(t))\phi(t) &= L_{\vartheta,\xi}(t,\tau)(I - P(\tau))\phi(\tau) \\ &\quad + \int_\tau^t L_{\vartheta,\xi}(t,s)(I - P(s))[f_\vartheta(s,\phi(s)) - A_{\vartheta,\xi}(s)\phi(s)]\,\mathrm{d}s,\end{aligned}$$

and sending $\tau \to -\infty$ produces

$$(I - P(t))\phi(t) = \int_{-\infty}^t L_{\vartheta,\xi}(t,s)(I - P(s))[f_\vartheta(s,\phi(s)) - A_{\vartheta,\xi}(s)\phi(s)]\,\mathrm{d}s.$$

By these two equalities we prove the lemma. □

Remark 3.6. *Since*

$$\rho(\epsilon) := \sup_{\substack{t\in\mathbb{R}\\ \|x\|\le\epsilon}} \frac{\|f_\vartheta(t,\xi(t)+x) - f_\vartheta(t,\xi(t)) - A_{\vartheta,\xi}(t)x\|_Y}{\|x\|_{\mathbb{R}^2}} \to 0 \quad as \quad \epsilon\to 0,$$

using Lemma 3.5, it is easy to see that $\xi:\mathbb{R}\to\mathbb{R}^2$ is isolated in the set $C_b(\mathbb{R},\mathbb{R}^2)$ of continuous and bounded functions from $\mathbb{R}$ into $\mathbb{R}^2$. Indeed, if $\phi:\mathbb{R}\to\mathbb{R}^2$ is a bounded solution of (18) satisfying $\sup_{t\in\mathbb{R}}\|\phi(t)-\xi(t)\|_{\mathbb{R}^2}\le\epsilon$, then

$$\sup_{t\in\mathbb{R}}\|\phi(t)-\xi(t)\|_{\mathbb{R}^2} \le 2M\rho(\epsilon)\omega^{-1}\sup_{t\in\mathbb{R}}\|\phi(t)-\xi(t)\|_{\mathbb{R}^2};$$

if ϵ is sufficiently small, it follows that $\phi(t)=\xi(t)$ for all $t\in\mathbb{R}$.

3.2. **Stability under perturbation.** We will now show that our problem is stable under perturbation. In [1, 10], we find conditions so gradient semigroups are stable under perturbation, as sumarized in the next theorem.

Definition 3.7. *Let $\{T(t)\in C(X): t\in\mathbb{R}^+\}$ be a semigroup possessing a global attractor $\mathcal{A}$ and a disjoint family of bounded isolated invariant sets $\boldsymbol{\Xi}=\{\Xi_1,\dots,\Xi_n\}$. We say that $T(\cdot)$ is* ***dynamically gradient*** *with respect to Ξ if we have the following properties*

- *G1) if $\xi:\mathbb{R}\to X$ is a bounded global solution of $T(\cdot)$, then there are $1\le i,j\le n$ such that*

$$\Xi_i \overset{t\to-\infty}{\longleftarrow} \xi(t) \overset{t\to\infty}{\longrightarrow} \Xi_j$$

- *G2) There is no homoclinic structure in $\boldsymbol{\Xi}$. That is, there is no subset $\{\Xi_{j_1},\dots,\Xi_{j_k}\}$ of $\boldsymbol{\Xi}$ and no set of bounded global solutions $\{\xi_i:\mathbb{R}\to X: i=1,\dots,k\}$ such that, setting $\Xi_{j_{k+1}}:=\Xi_{j_1}$, we have that, for every $1\le i\le k$, there exists $t_i\in\mathbb{T}^+$ such that $\xi(t_i)\notin\Xi_{j_i}\cup\Xi_{j_{i+1}}$ and $\Xi_{j_i}\overset{t\to-\infty}{\longleftarrow}\xi_i(t)\overset{t\to\infty}{\longrightarrow}\Xi_{j_{i+1}}$.*

Theorem 3.8. *Let $\{T_\nu(t)\in C(X): t\in\mathbb{R}^+\}_{\nu\in[0,1]}$ a family of semigroups that is continuous and collectively assymtoptically compact at $\nu=0$. Suppose that*

- *i) $T_\nu(\cdot)$ has a global attractor $\mathcal{A}_\nu$ for every $\nu\in[0,1]$ and $\cup_{\nu\in[0,1]}\mathcal{A}_\nu$ is bounded.*
- *ii) for each $\nu\in[0,1]$, $\mathcal{A}_\nu$ contains a finite family of isolated invariants $\boldsymbol{\Xi}_\nu=\{\Xi_{1,\nu},\cdots,\Xi_{n,\nu}\}$ such that $dist_H(\Xi_{i,\nu},\Xi_{i,0})\overset{\nu\to0}{\longrightarrow}0$ for each $1\le i\le n$.*
- *iii) There exists $\delta>0$ such that $\Xi_{i,\nu}$ is the maximal invariant in $\mathcal{O}_\delta(\Xi_{i,\nu})$ for every $1\le i\le n$ and $\nu\in[0,1]$.*
- *iv) $T_0(\cdot)$ is gradient with respect to $\boldsymbol{\Xi}_0$.*

Then there is $\nu_0\in(0,1]$ such that for every $\nu\in[0,\nu_0]$, $T_\nu(\cdot)$ is gradient.

Now, consider the skew product semigroup associated to (1) endowed with the structures described in the last section, that is, the driving semigroup $\Theta(\cdot)$ having a global attractor $\mathscr{S}$, the cocycle $\mathcal{K}(\cdot,\cdot)$ and the skew-product semigroup $\Pi(\cdot)$ having global attractor $\mathbb{A}$.

If the family $\{f_\nu(t,\cdot)\}_{\nu\in[0,1]}$ is a, uniformly in time, small C^1 perturbation in the second variable, of $f(t,\cdot)$ (given in (1)) consider the skew-product semigroups $\{\Pi_\nu\}_{\nu\in[0,1]}$ associated to (1), with f replaced by f_ν, having global attractors $\mathbb{A}_\nu$ uniformly bounded (denote by $\mathscr{S}_\nu$ the corresponding attractor for the driving semigroup $\Theta_\nu(\cdot)$). We will show next that the structure of the skew-product attractor given in Figure 8 remains the same for suitably small ν.

First note that, given a global solution $\eta_\nu : \mathbb{R} \to \mathscr{S}_\nu$ of the driving semigroup $\Theta_\nu(\cdot)$, there is an associated global solution $\eta : \mathbb{R} \to \mathscr{S}$ of the driving semigroup $\Theta(\cdot)$, and vice-versa. If $g(t,x) = \eta(t)(0,x)$ and $g_\nu(t,x) = \eta_\nu(t)(0,x)$ we have that g and g_ν are, uniformly in time, close in the C^1 topology with respect to the second variable.

Thus, if ξ_η is a hyperbolic bounded solution for the associated evolution process $S_\eta(t,s) = K(t-s,\eta(s))$, from the continuity of hyperbolic bounded global solutions under perturbation ([11, Lemma 8.3]), we obtain the existence of hyperbolic bounded global solutions ξ_{η_ν} for the evolution processes $S_{\eta_\nu}(t,s) = K(t-s,\eta_\nu(s))$ close to ξ_η. Then we can consider, for $\nu \in [0,1]$, the families $\boldsymbol{\Xi_\nu}$ given by the invariant sets $\Xi_{0,\nu} = \{(0,\eta_\nu(0)) : \eta_\nu \text{ is a bounded solution of } \Theta_\nu\}$ and

$$\Xi^{\pm}_{i,\nu} = \{(\xi^{\pm}_{i,\eta_\nu}(t),\eta_\nu(t)) : \eta_\nu \text{ is a bounded solution of } \Theta_\nu \text{ and } t\in\mathbb{R}\},\ i=1,2.$$

It follows from the above reasoning that the family $\boldsymbol{\Xi_\nu}$ behaves continuously when $\nu \to 0$.

In order to see that the invariant sets $\Xi^{\pm}_{i,\nu}$, $i=0,1,2$, belonging to $\boldsymbol{\Xi_\nu}$ are isolated invariant sets, we will show that there is a neighborhood of each $\Xi^{\pm}_{i,\nu}$, $i=0,1,2$, that does not contain any bounded global solution (other than those in $\Xi^{\pm}_{i,\nu}$). First, note that a bounded global solution of the skew-product corresponds to a bounded global solution $\xi_\nu(\cdot)$ of $S_{\eta_\nu}(\cdot,\cdot)$ for some fixed global solution $\eta_\nu(\cdot) : \mathbb{R} \to \mathscr{S}$ of the driving semigroup Θ_ν. Now, let η be a global solution $\eta : \mathbb{R} \to \mathscr{S}$ of the driving semigroup Θ that is a, uniformly in time, approximation of η_ν in the C^1 topology with respect to the second variable.

If $\xi^{\pm}_{i,\eta}$ is the hyperbolic global solution associated to η, as in Lemma 3.5 write ξ_ν in terms of the exponential dichotomy of the linearization around $\xi^{\pm}_{i,\eta}$

$$\xi_\nu(t) = \int_{-\infty}^{\infty} G_{\xi^{\pm}_{i,\eta}}(t,s)[g_\nu(s,\xi_\nu(s)) - A_{\eta,\xi^{\pm}_{i,\eta}}(s))\xi_\nu(s)]\,\mathrm{d}s,$$

where

$$G_{\xi^{\pm}_{i,\eta}}(t,s) = \begin{cases} L_{\eta,\xi^{\pm}_{i,\eta}}(t,s)(I-P(s)), & t\geq s \\ -L_{\eta,\xi^{\pm}_{i,\eta}}(t,s)P(s), & t\leq s. \end{cases} \tag{22}$$

It is easy to see that $\xi_\nu \to \xi_0$ as $\nu \to 0$, where ξ_0 is a global bounded solution of $S_\eta(\cdot,\cdot)$. Since ξ_0 remains in a neighborhood of $\xi^{\pm}_{i,\eta}$, by Remark 3.6, it must be equal to $\xi^{\pm}_{i,\eta}$. So ξ_ν also stays in a neighborhood of $\xi^{\pm}_{i,\eta}$ and consequently, by Remark 3.6, must be equal to $\xi^{\pm}_{i,\eta}$.

It follows that $\Xi^{\pm}_{i,\nu}$ are isolated invariant sets for $\Pi_\nu(\cdot)$ and from Theorem 3.8 $\Pi_\nu(\cdot)$ must be gradient relatively to $\boldsymbol{\Xi_\nu}$.

4. **Almost periodicity of the non-autonomous equilibria.** Recall the following characterization of almost periodic functions

Theorem 4.1 ([19, Page 341, Theorem 2.]). *Consider the metric d of the uniform convergence in the space $C_b(\mathbb{R},\mathbb{R})$ of the continuous and bounded functions from $\mathbb{R}$ into $\mathbb{R}$. Then, $\gamma \in C_b(\mathbb{R},\mathbb{R})$ is almost periodic if and only if the closure $H(\gamma)$ of $\{\gamma(t+\cdot) \in C_b(\mathbb{R},\mathbb{R}) : t \in \mathbb{R}\}$ with respect to d is compact.*

Definition 4.2. *Consider in $C([0,\infty),[\beta_1,\beta_2])$ the metric ρ of the uniform convergence in compact subsets of $\mathbb{R}^+$. We will say that $\beta \in C([0,\infty),[\beta_1,\beta_2])$ is asymptotically almost periodic if β is uniformly continuous and the global attractor $\mathscr{S}$ of $\{\Theta(t) : t \geq 0\}$ in $\mathcal{H}(\beta) = \overline{\{\Theta(t)\beta : t \geq 0\}}^{\rho}$ is such that*

$$\mathbb{S} = \{\gamma \in C(\mathbb{R},[\beta_1,\beta_2]) : \gamma(t) = \vartheta(t)(0), t \in \mathbb{R}, \vartheta : \mathbb{R} \to \mathscr{S} \text{ global solution of } \Theta\}$$

is compact in the topology of the uniform convergence in $\mathbb{R}$ of $C(\mathbb{R},[\beta_1,\beta_2])$.

Assuming that β is asymptotically almost periodic, we prove in this section that all non-autonomous equilibria (see Theorem 2.5) are also almost periodic.

If $\gamma_n = \vartheta_n(\cdot)(0)$ is a sequence in $\mathbb{S}$ which converges, uniformly in $\mathbb{R}$, to $\gamma \in \mathbb{S}$, we must show that the associated non-autonomous equilibria $\xi_n := \xi^{\iota}_{j,\vartheta_n}$, for fixed $(j,\iota) \in \{0,1,2\} \times \{+,-\}$, converge, uniformly in $\mathbb{R}$, to $\xi := \xi^{\iota}_{j,\vartheta}$.

From Lemma 3.5

$$\xi(t) = \int_{-\infty}^{\infty} G_\xi(t,s)[f_\vartheta(s,\xi(s)) - A_{\vartheta,\xi}(s)\xi(s)]\,\mathrm{d}s,$$

and

$$\xi_n(t) = \int_{-\infty}^{\infty} G_\xi(t,s)[f_{\vartheta_n}(s,\xi_n(s)) - A_{\vartheta,\xi}(s)\xi_n(s)]\,\mathrm{d}s,$$

where G_ξ is defined in (22). So, making $\phi_n(t) = \xi_n(t) - \xi(t)$, $t \in \mathbb{R}$,

$$\phi_n(t) = L_{\vartheta,\xi}(t,\tau)\phi_n(\tau) + \int_\tau^t L_{\vartheta,\xi}(t,s)\tilde{g}(s,(\phi_n(s)))\,\mathrm{d}s, \tag{23}$$

where $\tilde{g}(t,\phi) = f_{\vartheta_n}(t,\phi_n(t)+\xi(t)) - f_\vartheta(t,\xi(t)) - A(t)\phi_n(t)$.

Applying $I - P(t)$ to (23) and taking the limit as $\tau \to -\infty$ gives

$$(I - P(t))\phi_n(t) = \int_{-\infty}^t L_{\vartheta,\xi}(t,s)(I-P(s))\tilde{g}(s,(\phi_n(s)))\,\mathrm{d}s.$$

Applying the projection $P(t)$ to (23) yields, for $t \geq \tau$,

$$P(t)\phi_n(t) = L_{\vartheta,\xi}(t,\tau)P(\tau)\phi_n(\tau) + \int_\tau^t L_{\vartheta,\xi}(t,s)P(s)\tilde{g}(s,(\phi_n(s)))\,\mathrm{d}s,$$

and consequently

$$L_{\vartheta,\xi}(\tau,t)P(t)\phi_n(t) = P(\tau)\phi_n(\tau) + \int_\tau^t L_{\vartheta,\xi}(\tau,s)P(s)\tilde{g}(s,(\phi_n(s)))\,\mathrm{d}s.$$

Taking the limit as $t \to \infty$ we obtain

$$P(\tau)\phi_n(\tau) = -\int_\tau^\infty L_{\vartheta,\xi}(\tau,s)P(s)\tilde{g}(s,(\phi_n(s)))\,\mathrm{d}s.$$

Hence ϕ_n is the unique fixed point of $\mathcal{T}$

$$\begin{aligned}\mathcal{T}(\phi)(t) &= -\int_t^\infty L_{\vartheta,\xi}(t,s)P(s)\tilde{g}(s,(\phi_n(s)))\mathrm{d}s + \int_{-\infty}^t L_{\vartheta,\xi}(t,s)(I-P(s))\tilde{g}(s,(\phi_n(s)))\mathrm{d}s\\ &= \int_{-\infty}^\infty G_\xi(t,s)\tilde{g}(s,(\phi_n(s)))\,\mathrm{d}s\end{aligned}$$

in

$$B_\epsilon := \{\phi : \mathbb{R} \to X : \phi \text{ is continuous and } \sup_{t\in\mathbb{R}} \|\phi_n(t)\|_X \le \epsilon\}$$

for ϵ sufficiently small. To see that $\mathcal{T}$ indeed has a unique fixed point in B_ϵ note that:

Using the exponential dichotomy of $L_{\vartheta,\xi}(\cdot,\cdot)$

$$\begin{aligned}\|\mathcal{T}(\phi)(t)\|_{\mathbb{R}^2} &\le M\int_{-\infty}^\infty \mathrm{e}^{-\omega|t-s|}\|\tilde{g}(s,(\phi(s)))\|_{\mathbb{R}^2}\,\mathrm{d}s\\ &\le 2M\omega^{-1}\sup_{t\in\mathbb{R}}\|f_{\vartheta_n}(t,\xi(t)+\phi(t)) - f_\vartheta(t,\xi(t)+\phi(t))\|_{\mathbb{R}^2}\\ &\quad + 2M\omega^{-1}\sup_{\|x\|\le\epsilon}\sup_{t\in\mathbb{R}}\|f_\vartheta(t,\xi(t)+\phi(t)) - f_\vartheta(t,\xi(t)) - A(t)\phi(t)\|_{\mathbb{R}^2}\\ &\le \epsilon,\end{aligned}$$

Now, it is easy to see that $\mathcal{T}$ is a contraction for ϵ sufficiently small,

$$\|\mathcal{T}(\phi_1)(t) - \mathcal{T}(\phi_2)(t)\|_{\mathbb{R}^2} \le \frac{1}{2}\sup_{t\in\mathbb{R}}\|\phi_1(t)-\phi_2(t)\|_{\mathbb{R}^2}.$$

This guarantees that there is a unique global solution $\phi_n : \mathbb{R} \to X$ of (23) in B_ϵ for all n suitably large. Of course, $\xi_n(\cdot) = \phi_n(\cdot) + \xi(\cdot)$.

This shows that the set $\Xi_j^\pm = \{\xi_{j,\vartheta}^\pm : \vartheta : \mathbb{R} \to \mathscr{S} \text{ a global solution of } \Theta\}$ is compact with respect to the metric of the uniform convergence in $C(\mathbb{R},[\beta_1,\beta_2])$ and therefore almost-periodic.

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