Existence of Equilibrium in a Dynamic Supply Chain Game with Vertical Coordination, Horizontal Competition, and Complementary Goods

Fourth Revised Version

Bertrand Crettez* Naila Hayek† Guiomar Martín-Herrán‡

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Abstract

We consider supply chain competition and vertical coordination in a linear-quadratic differential game setting. In this setting, supply chains produce complementary goods and each of them includes a single manufacturer and a single retailer who coordinate their decisions through a revenue-sharing contract with a wholesale price and a fixed sales revenue share. We study a multiple leader-follower Stackelberg game where the manufacturers are the leaders and the retailers are the followers. Competition occurs at both levels of the supply chains. Retailers play Nash and compete in price; manufacturers also play Nash but they compete in choosing their production capacities by exploiting the equilibrium price decisions made by the retailers. We show that open-loop Nash equilibria exist when the manufacturers only receive a wholesale price (there are no longer exploiting the equilibrium price decision made by the retailers, however). When the manufacturers receive both a wholesale-price and a share of the retailers' sales revenues, equilibria generally no longer exist. The non-existence of an equilibrium stems from the fact that the manufacturers' instant profits are discontinuous functions of their production capacities. This discontinuity leads to a major technical difficulty in that one cannot apply standard optimal control approaches to study the equilibria of the dynamic game. Our results illustrate the possibility that competition between supply chains might not be sustainable when they sell complementary products and rely on a revenue-sharing agreement.

Key Words : Game Theory, Cournot Model of Complements, Open-loop Nash equilibrium, Supply chain, Generalized Stackelberg Equilibrium.

Declaration of interest: none.[1]

*Université Paris-Panthéon-Assas, CRED, EA 7321, 21 Rue Valette - 75005 Paris, France, bertrand.crettez@u-paris2.fr.

†Université Paris-Panthéon-Assas, CRED, EA 7321, 21 Rue Valette - 75005 Paris, France, naila.hayek@u-paris2.fr.

‡IMUVA, Universidad de Valladolid, Avd. Valle de Esgueva, 6 47011 Valladolid, Spain, guiomar.martin-herran@uva.es

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1 Introduction

This paper studies supply chain competition and coordination in a linear-quadratic differential game setting. In this setting, two supply chains produce complementary goods (in the sense of Cournot (1838), Chapter 9): that is, consumers need one unit of both products provided by the supply chains.[2] Each supply chain is made up of a single supplier or manufacturer and a single seller or retailer who coordinate their decisions through a revenue-sharing contract including a wholesale price and a fixed sales revenue share. We study a multiple leader-follower Stackelberg game where the manufacturers are the leaders and the retailers are the followers. Competition between supply chains occurs at both levels of the chains. Retailers play Nash and compete in price; manufacturers also play Nash but they compete in choosing their production capacities. Moreover, as leaders of a Stackelberg game in their respective supply chain, both manufacturers exploit the equilibrium price decisions made by the retailers; that is, they choose their production capacity by taking into account the dependence of the Nash equilibrium prices decided by the retailers upon the production levels.

Production capacities for the manufacturers are determined by the manufacturers' investment decisions and decay rate.[3] Due to the cumulative nature of production capacities, a dynamic game setup is in order. In this context, we focus on the existence of equilibria and our research questions are as follows. How are prices of Cournot complements determined given the production capacities? How are investments determined in a dynamic setting where the manufacturers (leaders) exploit the equilibrium price decisions made by the retailers (followers) and face quadratic adjustment costs? What are the implications of a revenue-sharing agreement that includes a wholesale price and a fixed sales revenue share on the chain members' strategies?

To the best of our knowledge, De Giovanni (2021), which builds on El Ouardighi et al. (2016), is the closest paper to ours that has considered dynamic competition between supply chains.[4] Introducing competition between supply chains in a dynamic framework undoubtedly enriches the modeling, but at the cost of making it more difficult to characterize

on previous versions of this work. An initial version of the paper circulated under the title: Capacity games in Cournot's duopoly model of complements.

[2]Think of a computer which is made up of hardware and an operating system (software).

[3]Manufacturers' investments are reversible.

[4]Dynamic competition between supply chains is also considered in El Ouardighi et al. (2021). But they focus on pollution and do not consider a fixed sales revenue share.

equilibrium strategies. As in De Giovanni (2021), we focus on the case where manufacturers rely on open-loop strategies, which means that investments in production capacities are functions of time and the initial value of these capacities. Some papers studying competition within a supply chain also use open-loop strategies (see, e.g., El Ouardighi et al. (2010, 2013)).[5]

Yet, it is generally the case that open-loop Stackelberg equilibria are time-inconsistent, implying that given the choice, the manufacturers (the leaders) would prefer to re-optimize at an intermediate instant of time rather than commit to their announced investment strategies. In our setting, however, the (open loop) generalized Stackelberg equilibrium would actually be time-consistent, because the retailers' (the followers) best responses at each instant of time depend only on the production capacities at this time.[6] Therefore, the future actions of the leaders do not influence the followers' decisions, implying that the (open loop) generalized Stackelberg equilibrium is time-consistent. Bearing these considerations in mind, we now briefly summarize our key results.

We first show that there are multiple Nash price equilibria for the game played by the retailers. The multiplicity of price equilibria stems from the fact that the retailers sell complementary products. Focusing on the symmetric price equilibrium is not compelling, we believe, as it associates a symmetric outcome to a setting where the retailers can resort to different production capacities. So which asymmetric equilibrium should one pick (beside the case where capacities are equal)? We propose to single out the price equilibrium that gives the highest price to the retailer connected with the manufacturer having the lowest capacity (and whose production is therefore the rarest).

We also find that the Nash equilibria with open-loop strategies played by the manufacturers generally do not exist in our setting, unless they only receive a price transfer, but in that case there are de facto no longer Stackelberg leaders as their payoffs are disconnected from the retailers' sales revenues. Thus, there is generally no equilibrium for our multiple leader-follower Stackelberg game. The main reason for the non-existence of Nash equilibrium for the game played by the manufacturers is as follows. In the literature, firms accumulate capital and play a Cournot quantity game. Here, firms accumulate capacities and make price decisions (moreover, the goods are complements). But a key difference between the

[5]We discuss the relevance of open-loop strategies in Section 6.

[6]In the language of Xie (1997), the retailers' decisions are not controllable by the decisions of the manufacturers regarding their future capacities. We thank a referee for drawing our attention to this reference.

literature and our setting is that here the firms that make decisions on prices are different from those that make decisions on production capacities (and production). The latter are assumed to be the leaders (of their respective supply chain) and the former are the followers. Furthermore, the leaders exploit the *equilibrium* prices decided by the followers (the retailers). It turns out that for these equilibrium prices, the producers' objectives are *not* always continuous (these objectives are always continuous when the producers only receive a wholesale price, however). Because of this discontinuity, it is almost always profitable for the producers to deviate from any given investment path.[7] A consequence of our results is that the static Cournot Nash equilibrium (with capacity constraints) is no longer equal to the steady-state open-loop equilibrium, as is often the case in the differential games literature (see, *e.g.*, Reynolds (1987) or Dockner (1992)). This is because the steady-state open-loop equilibrium does not generally exist.

This paper also includes the following methodological contributions. Firstly, while the existence of multiple equilibria in Operations Research is not uncommon and occurs in very different settings,[8] we are not aware of a contribution where there is hierarchical play in a dynamic setting, and where the leaders face multiple equilibria (i.e., prices for the complements) for the Nash game played by the followers.

Secondly, we also tackle a technical difficulty. Indeed, as was indicated above, the manufacturers' instant profits may be discontinuous functions of their production capacities. This discontinuity implies that one cannot apply standard optimal control approaches to study the equilibria of the dynamic game played by these manufacturers. We show how to overcome the difficulty, and the arguments used to study the equilibria can be adapted to different alternative settings. Besides, the discontinuity of the manufacturers' profit functions leads to the non-existence of equilibria for the game played by the manufacturers. Proving this non-existence is far from trivial. Even when the manufacturers' payoffs are continuous, proving the existence of equilibria is not trivial either. This is due to the fact that the manufacturers produce complements, and thus must pay attention to the minimum value of their capacities. More precisely, we show that when equilibria exist they satisfy the property that overcapacity disappears in finite time. But a key technical difficulty is to

[7] If we impose that the equilibrium prices are the same, then the manufacturers' objectives are continuous. But as we have argued above, this case is not compelling.

[8] See, *e.g.*,Caulkins et al. (2013), Chua et al. (2016), Egerer et al. (2022), Oliveira et al. (2013), Löschenbrand (2020), Yang and Anderson (2014). Cachon and Netessine (2006) discuss the implications of multiple Nash equilibria in game theory models of operations and supply chains.

find the date at which overcapacity no longer exists. We also show how to overcome this difficulty.

We now present some managerial insights that can be drawn from our results.

The aim of Operations Research is to help make better decisions. But sometimes decision-makers must make simultaneous decisions, and studying this kind of decision-making is the scope of game theory, which belongs to the Operations Research toolbox. Without coordination, game theory predicts that the outcome of firms' interactions is a Nash equilibrium. In such an equilibrium, decisions are mutually best responses. Finding mutual best responses, however, is not always possible. Therefore, decision-makers must look for something else.[9]

In this respect, we stress the fact that for certain sectors selling complementary goods, some form of cooperation between the leaders (the upstream firms) or the retailers (the downstream firms) of the different supply chains may be necessary. For instance, it would be better for a manufacturer to create its own distribution channel or integrate with the existing retailer (or else coordinate its production with other manufacturers of complements).[10]

It is difficult, however, to check empirically that cooperation is necessary. But some features of markets for complements suggest that cooperation is particularly fruitful. Consider for instance the relationships between car and tire makers (*e.g.*, Renault and Michelin), who can be thought of as the manufacturers, as well as car dealerships and tire stores, who can be seen as the retailers. To the best of our knowledge, new cars are equipped with tires. Thus, tire makers sell directly to car makers. But they also sell to tire stores as well as to some car dealerships. Moreover, some franchised dealerships are linked to specific car makers. While the actual relationships between those manufacturers and retailers do not satisfy all of our assumptions[11], what we observe is not a Nash equilibrium for the game akin to the one that we have studied. Cooperation and integration seem prevalent. A similar conclusion can be

[9]The non-existence of equilibrium is admittedly rare in Operations Research, but does occur from time to time. See, e.g., Yang and Anderson, (2014). They study a two-period game where firms make capacity choices in the first period and production choices in the second period. They show that this game may have no (pure-strategy) Nash equilibrium for certain initial values of their production capacities.

[10]This assessment is somewhat similar to that obtained in Grim et al. (2019). In this paper, the authors study a competitive gaz market game with non convexities. While the equilibrium may exist, it may not have desirable outcomes from a welfare perspective. The authors "propose a design where the market solution corresponds to a welfare maximum and vice versa." To put it another way, production must be reorganized.

[11]That is because, in our framework we disregard competition between car makers or tire makers, assuming some form of monopolistic competition.

obtained for toothbrush and toothpaste makers (or razor and blade makers) because both products often seem to be sold by each retailer (*e.g.*, supermarkets).

Also consider the interaction between logistic supply chains. Suppose, for instance, that the two retailers are actually freight forwarders[12] and the two manufacturers are two carriers. Both carriers send chips along lanes that are complementary for some clients (one lane could be Asia-US and the other US-Europe). According to our model, stable mutually best responses are unlikely. What we observe is more or less in line with our conclusion. First of all, there has been a wave of mergers/acquisitions among global carriers and a rise in the number of alliances. In these alliances, carriers share vessels in order to decrease their unit cost and get broader service coverage.[13] Alliances especially make sense in complementary regions (Mitsuhashi and Greve, 2009 and Ghorbani *et al.*, 2022). Furthermore, carriers also cooperate with other parts of the maritime transport chain, including ports but also freight forwarders.[14] By increasing their market power, carriers obtain better contracts with the freight forwarders. Finally, some carriers have started to operate as freight forwarders (Merk *et al.*, 2018, p. 68). The fact that carriers merge, or make alliances, and try to integrate or to compete with freight forwarders indicate that competition (as described in our model) can be flawed and that firms adapt to this issue by cooperating (or coordinating in the case of integration).[15]

Finally, consider the case where the retailers are airlines and the manufacturers are airports Assume there is an interline market where the airlines serve the same hub but only from one airport (in this setting, a supply-chain includes one airport and one airline). That is, in order to make an interline trip, a passenger starting her travel from one airport must use the airline that serves the line between this airport and the hub, and then uses the other line that connects the hub to the second airport (think about a trip from Madrid to Montréal

[12]A freight forwarder is a a company who, for a fee, organizes shipments for the shipper by liaising with carriers.

[13]Notice that competition in line shipping has been limited at least since 1875, particularly with conferences, which are legal agreements between carriers for setting common freight and regulating capacity. Conferences are now illegal in the US and in the European Union, however. Interestingly, vessel sharing consortia are not illegal.

[14]In this connection, Yin and Kim (2012) view the relationship between freight forwarders and carriers as a game, in which carriers are rule makers whilst freight forwarders are followers. See also Wang *et al.* (2020).

[15]Our setting can be used to study other aspects of supply chains. For instance, one retailer could be a freightforwarder and the second one a steamship agent, and one maker could be a carrier and the second one a terminal operating company. Notice that the largest carriers are now major terminal operators (see, https://unctad.org/publication/review-maritime-transport-2022, chapter 6, page 3).

with Paris as a hub).[16] Each airport levies an airport charge paid by the airline to which it is associated. In turn, airlines set fares paid by the consumers. Like in the preceding example, what we observe does not seem to be consistent with a Nash equilibrium. First of all, airports charges have to comply with rules set by authorities. In particular, they must comply with article 15 of the 1944 Chicago Convention according to which there shall be no discrimination between users, particularly from different countries. Moreover, airlines often cooperate. Cooperation allows them to reduce the fares in the interline markets. In doing this, they take into account the fact that their price affects negatively the number of passengers of the other airline because both airlines provide complementary services. Cooperation is often achieved through alliances (instead of mergers, which can be prohibited). Nowadays, there are three major global alliance groups, Star Alliance, One World and Sky Team.

The remainder of the paper unfolds as follows. In the next section, we briefly summarize the relevant literatures. Our model is presented in Section 3. Section 4 studies the existence of open-loop equilibrium when the manufacturers only receive a transfer or wholesale price. Section 5 addresses the existence issue when the manufacturers receive a transfer price *and* a share of the sales revenues. Section 6 discusses some limitations of our study and Section 7 offers some concluding remarks. Almost all the proofs are relegated to the Supplementary Material.

2 Literature Review

This paper is at the crossroad of two strands of literature. Firstly, this work contributes to the Cournot model of complements literature. Secondly, it contributes to the literature on differential games in dynamic marketing channels and supply chain management. In what follows, we briefly review these two strands of literature in turn.

2.1 Cournot model of complements

In the Cournot model of complements, consumers have a downward-sloping demand for a final product which is made out of n different components, each of which is being produced by a monopoly supplier. These n components are perfect complements in the sense that

[16] This setting includes some modelling ideas presented in Benoot et al. (2013) and Brueckner et al. (2020).

one unit of the final good requires one unit of each of the complementary goods. Therefore, when consumers demand one unit of the final good, they actually demand one unit of each of the complementary goods and, as a consequence, the price of this final good is the sum of the prices of the complement goods. Amir and Gama (2019) present a general approach of the Cournot model of complements. They tackle the existence and uniqueness of an equilibrium and the effects of an increase in the number of complements.[17]

Price competition between the monopoly suppliers differs from a Bertrand competition between producers of similar products. First of all, in contrast to Bertrand competition, when a monopoly supplier reduces its price, the demand for the products of the other firms does *not* decrease. To wit, since the products are complementary, if the demand for the product whose price is reduced increases, then the demand for all the products must increase as well. Therefore, whereas in a Bertrand competition setting firms are prone to reduce their prices, in a Cournot complement setting, firms find reducing their prices less worthwhile. To put it another way, each firm welcomes (as opposed to fears) the decrease in the prices of the other ones as it is always more profitable to let other firms reduce their prices than to reduce one's price.

Another difference between price competition between the monopoly suppliers and Bertrand competition is that integrating the n monopoly suppliers into a super-monopoly would be Pareto-improving (prices would be lower, and profits higher). This property is a direct consequence of what was stated above. When they cooperate, the monopoly suppliers are better off by decreasing the prices of all their products. By doing so, they stimulate the demand for the final good and therefore the demand for all the complementary goods (which brings about a rise in their profits). This is in contrast with what occurs when products are not complements. In that case, cooperating firms increase their profits by reducing their production, which leads to an increase in their prices. These increases in prices prevent consumers from substituting one product for another.

To save space, and since this paper is only concerned with chain-to-chain competition and dynamics we shall not review the literature studying complementary goods from a static viewpoint.

[17]The existence of the equilibrium is also addressed in Babaioff et al. (2017) who focused on a discretized version of the model in which demand changes only finitely many times. Linnemer (2022) provides both a historical and an analytical analysis of the use of Cournot duopoly model of complements.

The dynamic analysis of the Cournot model of complements probably begins with Dobson (1992) who used a two-stage game in which firms first choose capacities and then prices. He obtains the result that "there will be multiple perfect equilibria, since it is optimal to match capacity choices up to (and including) the level that corresponds to output from the firms setting Cournot prices." We extend his setting to the continuous-time case. We also show that at the price decision stage, when capacities are given, there are multiple price equilibria (and for some of those there would be no equilibrium in the quantity setting stage).

Yalcin et al. (2013) also provides a two-stage analysis of what they called value-capture and value-creation problems. In their model, the demands of complementary goods depend on their prices *and* their qualities. Quality choices are made before the price decisions. Improving the quality of one complement enhances the demand for the others. But quality improvement is costly and there is a risk of quality underprovision for all products (this is the value-creation problem). They show that relying on a royalty fee (for rewarding the quality choice made by the first firm) does not solve the problem (allowing more competition is more efficient).

Avenali et al. (2013) considers systems of complementary products to discuss how a bundling firm uses mixed bundling (that is, the individual components that make up the bundle are also available for purchase individually) to affect its competitors' product quality.[18] They found that bundling results in low quality products and may be socially harmful if there is competition in the complementary market.[19]

Casadesus-Masanell and Yoffie (2007) provide a continuous-time analysis of the dynamics of competition between two complementary firms, the complementors, dubbed as Intel and Microsoft, which both make a final product, the PC. The value of the final product depends on how well the components work together. This, in turn, depends on the firms' investment in complementary R&D. A priori, both firms would want to cooperate and make the final product as valuable as possible. They show that when the two firms have different decision horizons, natural conflicts occur over pricing and the size of initial investments in complementary R&D. This is because, one firm pays attention to the installed base while the other is only concerned with new customers.

[18] They use a two-stage game. In the first stage, one firm chooses whether to sell its products independently or in a bundle. In the second stage, the other firm chooses whether or not to invest in product quality.

[19] Mantovani and Ruiz-Aliseda (2015) tackle partial cooperation (firms coordinate their investment decisions first, and then compete on prices).

Another contribution to the dynamic analysis of the Cournot model of complements that differs from ours is Laussel and Van Long (2012). They analyze a model where a monopolistic downstream firm (an assembler) looks for the best way to separate from its upstream subsidiaries across time.

This paper proposes a differential game analysis of competition between supply chains that sell Cournot complements and where firms face capacity constraints and adjustment costs in choosing their production capacities.[20]

2.2 Differential Games in the dynamic supply chain management literature

Differential game models have been used to study different problems arising in dynamic supply chain management literature (see, for example, Jørgensen and Zaccour (2004) and He et al. (2007) for early surveys).

Most of the dynamic analyses of supply chains suppose that the supply chain is made up of a single supplier or manufacturer and a single seller or retailer. An important issue is to determine how coordination between the supply chain's members can improve their payoffs. For instance, coordination can be useful to alleviate the double marginalization problem faced by these members - that is, the fact that the successive markups of the members raise the price paid by consumers of the final product, which results in a reduction in all firms' demands and profits (see Cachon (2003) for a comprehensive overview of the coordination mechanisms available for supply chain management). In this regard, the supply chain's members often have asymmetric roles which affects the way their decisions are coordinated. Much of the literature considers the supplier to be the leader of a Stackelberg game, and the seller to be the follower.

This paper contributes to the literature studying revenue-sharing contracts as a way to partially or completely coordinate decisions within a supply chain.[21] The main kinds of revenue-sharing contracts comprise the wholesale-price contract with an added revenue-sharing mechanism and the consignment contract with revenue sharing. Revenue-sharing

[20] Casadesus-Masanell and Yoffie (2007) focused on dynamic price competition. Investment is realized at the initial date.

[21] Cachon and Lariviere (2005) is a seminal paper on revenue-sharing contracts. See Bart et al. (2019) for recent studies of the literature on revenue-sharing contracts in a supply chain.

agreements generally eliminate the double marginalization problem (because each stakeholder maximizes a portion of the *same* payoff). The consequences for firms in supply chains with asymmetric players using consignment contracts with a revenue-sharing agreement are analyzed in Buratto et al. (2019), De Giovanni et al. (2019), and Liu et al. (2016). The case where players are symmetric has been notably analyzed by El Ouardighi et al. (2008, 2016, 2021).

To further the study of coordination in a supply chain, it is also important to pay attention to competition. Competition can occur at one level or at any level of the supply chain and/or between supply chains. This paper specifically addresses chain-to-chain competition with (possibly) asymmetric players who coordinate their decisions through a revenue-sharing scheme including a wholesale-price and a fixed sales revenues share. It thus builds on El Ouardighi et al. (2016) who considers a fixed-share setting with a wholesale price. It is furthermore related to De Giovanni (2021), which also builds on El Ouardighi et al. (ibid). In contrast to the latter paper and the present one, De Giovanni (ibid) considers either a wholesale price or a fixed-share setting. But like our contribution and in contrast with El Ouardighi et al. (ibid), he pays attention to competition between supply chains.[22] Here, however, we focus on Cournot complements and we pay attention to asymetric players withing each supply chain. De Giovanni finds that when both supply chains coordinate through a fixed-share contract, both are economically worse off with respect to the case in which both use a wholesale-price contract. He concludes that supply chain coordination stiffens competition but that competition destroys the advantages created by coordination even when using an effective compensation scheme such as a revenue-sharing contract. Our conclusion is in a sense more negative than De Giovanni's because we argue that competition may not be viable with such a contract.

3 Model

Our problem is a multiple leader-follower Stackelberg game (Sherali 1984, Julien, 2017) also called a generalized Stackelberg competition model (Sinha et al. 2014). We consider

[22] El Ouardighi et al. (2021) investigate "the impact of both horizontal and vertical competition, on the one hand, and strategy types (commitment-based versus contingent-based equilibrium strategies), on the other hand, on the pollution accumulated by two supply chains over time." They also investigate the determination of optimal transfer prices. We do not pay attention to this issue (nor do we consider pollution), but we consider a model with complements, an infinite horizon decision, and a revenue-sharing agreement that includes a fixed share of sales revenues.

a double supply chain. Each supply chain includes a retailer and a supplier. Each of the two manufacturers acts as a traditional Stackelberg leader for the retailers (which are the follower firms), but as a competitor firm with respect to the other manufacturer. A generalized Stackelberg equilibrium for this game is: a pair of price decisions for the retailers which is a Nash equilibrium given the investment choices made by the manufacturers; a pair of investment decisions for the manufacturers which is a Nash equilibrium when the manufacturers take into account the effect of their decisions on the price Nash equilibrium. Thus, the model consists of two Nash games encompassed by a Stackelberg competition model.

The products sold by the two retailers are complementary and there is horizontal competition between the supply chains. The two retailers compete in price while the manufacturers compete in production capacity. In each supply chain, there is a hierarchical relationship where the retailer is the follower and the manufacturer the leader. All decision-makers play open-loop strategies (that is, strategies that only depend on the initial capacities and time). Retailers make their decisions given the paths of production capacities resulting from the investment choices made by the manufacturers. In turn, manufacturers make their investment choices by exploiting the *equilibrium* price decisions made by the retailers. We first consider the retailers' choices before paying attention to the manufacturers' decisions.

3.1 The retailers' choices

Let $(K_i(t), K_j(t))$ be a given path of production capacities for manufacturers i and j. Assume that the production of each manufacturer is equal to its production capacity.[23] Each retailer i solves the following problem:

$$\begin{aligned}
&\max_{p_i(\cdot)} \int_0^{\infty} \left(\phi_i p_i(t) - w_i\right) \left(\frac{a - p_i(t) - p_j(t)}{b}\right) e^{-rt}\, dt \\
&0 \leq \frac{a - p_i(t) - p_j(t)}{b} \leq \min\{K_i(t), K_j(t)\}, \quad \forall t \\
&w_i \leq \phi_i p_i(t), \quad \forall t.
\end{aligned}$$

In the problem above, $p_i(t)$ is the price of the product sold by retailer i at date t and produced by manufacturer i. Moreover, $\frac{a-p_i(t)-p_j(t)}{b}$ is the demand for the good by the consumers at date t. The parameters a and b are positive parameters. The parameter a

[23] We shall make this assumption more precise later on.

denotes the choke price while b measures price sensitivity. To obtain one unit of this good, consumers need to buy one unit of each complement. This is why the total quantity sold to the consumers cannot be higher than the minimum production level (hence the inequality constraint). Thus, $p_i(t)\left(\frac{a-p_i(t)-p_j(t)}{b}\right)$ is the gross sales revenue obtained by retailer i. We let ϕ_i denote the share of this revenue that the retailer kept. The other part is received by manufacturer i. The parameter w_i is the wholesale price paid by the retailer to the manufacturer. We assume that both ϕ_i and w_i are fixed throughout the analysis.

The issue of the dynamic interactions of the retailers is given by an open-loop Nash equilibrium. A glance at this problem shows that since the retailers take the paths $(K_i(\cdot), K_j(\cdot))$ as given, they essentially face a static problem.[24] Therefore, all retailers' choices are strongly time-consistent.

Consider a given date t and assume without loss of generality that $K_i(t) < K_j(t)$. Then the production capacity of manufacturer i limits what can be sold to consumers; that is, there is a *coupled constraint* that is taken into account by all the chains' members and which reads:

$$\frac{a - \big(p_i(t) + p_j(t)\big)}{b} \leq K_i(t). \tag{1}$$

For the sake of notational simplicity, let us drop the time index t. A Nash equilibrium with the coupled constraint (1) is a pair of non-negative prices (p_i, p_j) such that each retailer h $(h = i, j)$ solves:

$$\max_{p_h} \quad (\phi_h p_h - w_h)\left(\frac{a - (p_h + p_{-h})}{b}\right) \tag{2}$$

$$\text{s.t.:} \quad \frac{a - (p_h + p_{-h})}{b} \leq K_i, \tag{3}$$

$$w_h \leq \phi_h p_h \tag{4}$$

where p_{-h} is the price of retailer h's rival.

Set $\hat{p}_i = p_i - \frac{w_i}{\phi_i}$ and $\hat{a} = a - \frac{w_i}{\phi_i} - \frac{w_j}{\phi_j}$, and suppose that $\hat{a} > 0$.

Proposition 1

[24] As a referee noted, they can also be considered myopic players.

1. *Assume that* $\frac{\hat{a}}{3b} \leq K_i$. *Then the only Nash equilibrium with coupled constraint is:*

$$\hat{p}_i = \hat{p}_j = \frac{\hat{a}}{3}. \tag{5}$$

2. *Assume that* $K_i < \frac{\hat{a}}{3b}$. *Then we have three kinds of equilibria:*

   (a) $\hat{p}_i = bK_i$, $\hat{p}_j = \hat{a} - 2bK_i$.

   (b) $\hat{p}_i = \hat{a} - 2bK_i$, $\hat{p}_j = bK_i$.

   (c) *All couples* $(\hat{p}_i, \hat{p}_j)$ *such that* $bK_i < \min\{\hat{p}_i, \hat{p}_j\}$, $\hat{p}_i + \hat{p}_j = \hat{a} - bK_i$.

Proof. See Supplementary material. ■

This proposition shows that there are multiple (actually a continuum) of price equilibria (provided that production capacities are not too high[25]: $K_i < \frac{\hat{a}}{3b}$). The reason for this multiplicity is because production capacities are relatively low so that it is profitable to produce at full capacity. Indeed, consider a pair of prices such that the final demand equals the minimum capacity (and such that each price is not too low as in, e.g., case 2.c). In that case, no retailer is better off by lowering its price. Indeed, either a retailer already sells all the available production, or it cannot sell more, because its product is a complementary good. Likewise, no retailer is better off by increasing its price. This is because, as prices are relatively high (recall that the demand for the final good is not too high by assumption), if a retailer raises its price, its sales revenue will decrease. To wrap up, each retailer would like to sell at a higher price, provided that the other retailer decreases its price, so that the final demand is unchanged.

Now, what would be a reasonable equilibrium price? It is reasonable to assume that if the manufacturers have the same capacities (and thus produce the same quantities since capacities are fully used), then the retailers should choose the same prices, that is $p_i = p_j = \frac{\hat{a}-bK_i}{2}$. Yet, where capacities are different, there is no reason why the prices chosen by the retailers should be equal. A symmetric price equilibrium is not compelling as it associates a symmetric outcome to a setting where the retailers can resort to different production capacities. So which asymmetric equilibrium should one pick ? Among the different equilibria, we propose to single out the equilibrium price that gives the highest price to the retailer connected with the manufacturer having the lowest capacity (and whose production is therefore the rarest).

[25] As was mentioned in the introduction, multiple equilibria are not uncommon in the studies of supply-chains (see, e.g., Cachon and Netessine, 2006).

This equilibrium corresponds to case 2.b.[26] Our rationale to single out the equilibrium above is as follows. Since manufacturer i has the lowest capacity, it actually controls the production of the good (in the sense that the production of that good cannot be higher than K_i). This gives manufacturer i a certain power (it is easier for this manufacturer to choose a high price).

Notice that this equilibrium also arises if the retailer with the lowest production capacity chooses its price first, and the other retailer chooses its own price after having observed the choice of the former. This is actually the Edgeworth equilibrium for the Cournot Complements oligopoly.[27]

Our choice of the price equilibrium corresponds to a certain form of a Stackelberg equilibrium arising in the study of optimistic or pessimistic Stackelberg equilibria (von Stengel and Zamir, 2010). To the best of our knowledge, in this literature, there is only one leader and one or more followers. The optimistic (respectively pessimistic) Stackelberg equilibrium gives the highest equilibrium (respectively lowest) payoffs. However, in our setting, there are two leaders. Our choice of the price equilibrium would correspond to an optimistic equilibrium from the viewpoint of the manufacturer with the lowest capacity and a pessimistic equilibrium from the viewpoint of the manufacturer with the highest capacity.

Now let us pay attention to manufacturer i's payoff which reads

$$\left(w_i + (1-\phi_i)p_i\right)\left(\frac{a - p_i - p_j}{b}\right) = \left(\frac{w_i}{\phi_i} + (1-\phi_i)\hat{p}_i\right)\left(\frac{\hat{a} - \hat{p}_i - \hat{p}_j}{b}\right). \tag{6}$$

The next Proposition gives the equilibrium values of this payoff.

Proposition 2 *The equilibrium value of manufacturer i's profit $R^i(K_i, K_j)$ and the Nash equilibria of the static price game are as follows:*

1. *If $K_i < \frac{\hat{a}}{3b}$, then $K_i < K_j$, $R^i(K_i, K_j) = (1-\phi_i)(\hat{a} - 2bK_i)K_i + \frac{w_i}{\phi_i}K_i$, $\hat{p}_i = \hat{a} - 2bK_i$, $\hat{p}_j = bK_i$.*

2. *If $K_j < \frac{\hat{a}}{3b}$, $K_j < K_i$ then $R^i(K_i, K_j) = (1-\phi_i)bK_j^2 + \frac{w_i}{\phi_i}K_j$, $\hat{p}_i = bK_j$, $\hat{p}_j = \hat{a} - 2bK_j$.*

[26] Indeed, we have $bK_i \leq \hat{a} - 2bK_i \iff K_i \leq \frac{\hat{a}}{3b}$, which is true by assumption.

[27] See Linnemer (2022) for the comparison between Cournot and Edgeworth solutions. Linnemer explains that for Edgeworth, the equilibrium is indeterminate as each firm would like to post its price first. Notice that in Linnemer's presentation, production capacities are considered unbounded.

3. If $K_i = K_j = K < \frac{\hat{a}}{3b}$*, then* $R^i(K,K) = (1-\phi_i)\frac{(\hat{a}-bK)}{2}K + \frac{w_i}{\phi_i}K,\ \hat{p}_i = \hat{p}_j = \frac{\hat{a}-bK}{2}$.

4. If $\frac{\hat{a}}{3b} \leq \min\{K_i, K_j\}$*, then* $R^i(K_i,K_j) = (1-\phi_i)\frac{\hat{a}^2}{9b} + \frac{w_i}{\phi_i}\frac{\hat{a}}{3b},\ \hat{p}_i = \hat{p}_j = \frac{\hat{a}}{3}$.

Observe that there is a discontinuity in the manufacturer's payoff. This discontinuity will be instrumental in the proof of the non-existence of a Nash equilibrium for the game played by the manufacturers.

3.2 Manufacturers' decisions

We now study the manufacturers' decisions. Because each manufacturer is the leader of its supply chain, we assume that it can take advantage of the dependence of the equilibrium prices decided by the retailers on its production capacity. Therefore, upon contemplating a change in their decisions, the manufacturers are aware that the retailers will adapt their price decisions to the new value of the capacities.

For simplicity, we suppose that there is zero production cost.[28] However, the manufacturers face adjustment costs, *à la* Hanig (1986) and Reynolds (1987), when they modify their capacity.[29] The dynamic problem faced by manufacturer i is then as follows:

$$\max_{I_i(\cdot)} \quad \int_0^\infty e^{-rt}\left[R^i(K_i(t),K_j(t)) - C(I_i(t))\right]dt$$

$$\text{s.t.:} \quad \dot{K}_i(t) = I_i(t) - \delta K_i(t),\ K_i(0) = K_{i0},\ K_{i0} \leq \frac{\hat{a}}{3b},\ i = 1,2,$$

where

$$C(I_i) = \alpha I_i + \beta\frac{I_i^2}{2},\ \alpha > 0,\ \beta > 0.$$

In this problem, manufacturer i maximizes the flow of discounted instant profits with respect to investment I_i. Instant profits at any date t are the difference between the manufacturer's revenues $R^i(K_i(t),K_j(t))$ and instantaneous investment costs $C(I_i(t))$. These costs are convex increasing in the investment rate and are identical for both manufacturers. The production capacity depreciates at the instantaneous rate δ, $\delta > 0$. The discount factor is

[28] As there is no production cost per se here, a cost-plus contract is not applicable.

[29] In contrast to, *e.g.*, El Ouardighi and Erickson (2015), capacity constraints are binding. That is, firms cannot satisfy all possible values of the final demand even by bearing extra-costs when producing above their capacity level (for instance by hiring temporary workers).

also the same for both manufacturers and is equal to r, with r strictly positive. Furthermore, we shall assume that $\alpha < \frac{\hat{a}}{r+\delta}$. This assumption guarantees that there is a positive steady-state capacity *if* there is an equilibrium. This assumption means that the (net) choke price ($\hat{a}$) is large enough to finance small investment expenditures. Otherwise, of course, there is no point in building long-term capacity.

As in Reynolds (1987), we assume that manufacturers can disinvest (capacities, however, are always non-negative). Our problem bears some similarity with that considered by Reynolds, with the major exception that the revenue function is not (always) continuous.

Now recall that a generalized Stackelberg equilibrium for the competition between the two supply chains where the manufacturers are the leaders and the retailers are the followers of their respective supply chain is:

- A pair of retailers' price strategies that is a Nash equilibrium and which is given by Proposition 1 (1 and 2c).

- A pair of open-loop strategies that is an open-loop equilibrium for the game played by the manufacturers, when the manufacturers rely on the Nash equilibrium price decisions made by the retailers.

As we have already studied the Nash price equilibrium, to study the existence of our generalized Stackelberg equilibrium, we now have to consider the open-loop Nash equilibrium for the game played by the manufacturers.

4 Existence of Open-Loop Equilibrium with a Wholesale Price, without a Sales Revenue Share

In this section, we study the open-loop Nash equilibrium for the game played by the manufacturers under the assumption that there is no sales revenue sharing agreement ($\phi_h = 1$, $h = i, j$). This implies that manufacturers' gross revenue is given by $w_h \left(\frac{a-p_i-p_j}{b}\right)$ and that they can no longer benefit from influencing the retailers's selling decisions. Stackelberg and Nash equilibria will therefore coincide. Recalling that $\hat{a} = a - w_i - w_j$, then relying on

Proposition 2, we get that the gross revenue $R^i(K_i, K_j)$ of manufacturer i reads

$$R^i(K_i, K_j) = \begin{cases} w_i K_i, \text{ if } K_i \leq \min\{\frac{\hat{a}}{3b}, K_j\} \\ w_i K_j, \text{ if } K_j < \min\{\frac{\hat{a}}{3b}, K_i\} \\ w_i \frac{\hat{a}}{3b}, \text{ if } \min\{K_i, K_j\} \geq \frac{\hat{a}}{3b}. \end{cases} \tag{7}$$

We denote by K_h^* firm h's capacity when both firms use equilibrium strategies (I_i^*, I_j^*). To find the open-loop Nash equilibria of the manufacturers' game, we shall rely on the following two lemmata.

Lemma 1 *There is no Nash open-loop equilibrium where* $K_i^*(t) < K_j^*(t)$ *for all* t.

Proof. See Supplementary material. ■

Lemma 2 *There is no open-loop Nash equilibrium in which* $K_i^*(\underline{t}) = K_j^*(\underline{t})$, $K_i^*(\bar{t}) = K_j^*(\bar{t})$, $\underline{t} < \bar{t}$, *and* $K_i^*(t) < K_j^*(t)$ *for all* t *in* $(\underline{t}, \bar{t})$.

Proof. See Supplementary material. ■

The intuition of the first Lemma is as follows. Suppose that firm i's capacity is always lower than firm j's. Then firm j's gross revenue would only depend on that of firm i (since goods are complements, one can only sell a quantity equal to the lowest capacity). But then firm j's would find it profitable to get rid of its overcapacity (by selling a part of its capacity). It is clear, however, that sooner or later, firm j's capacity will meet firm i's.

As for the second Lemma, it is not optimal for firm j to maintain temporary overcapacity because it brings about no additional gross revenues, and because it is not profitable to build capacity today only to get rid of it in the future.

It turns out that the open-loop Nash equilibrium differs depending on whether firm i is more profitable than firm j, that is whether $w_i > w_j$, or $w_i \leq w_j$. We consider each case in turn.

4.1 Case where $w_i < w_j$

Because $K_{i0} < K_{j0}$, we conjecture that $K_i^*(t) \leq K_j^*(t)$ for all t. Relying on this idea, we also conjecture that if there is an open-loop equilibrium, then firm i's investment policy is a solution to the following problem.

$$\max_{I_i(\cdot)} \int_0^\infty e^{-rt} \left[w_i K_i(t) - C(I_i(t)) \right] dt \tag{8}$$

$$\text{s.t.: } \dot{K}_i(t) = I_i(t) - \delta K_i(t), K_i(0) = K_{i0}. \tag{9}$$

Lemma 3 *The solution to the problem above is given by*

$$I_i^*(t) = \delta K_{i\infty} \tag{10}$$

$$K_i^*(t) = K_{i\infty} + (K_{i0} - K_{i\infty})e^{-\delta t} \tag{11}$$

$$K_{i\infty} = \frac{w_i - (r+\delta)\alpha}{\beta\delta(r+\delta)}. \tag{12}$$

Proof. See Supplementary material. ■

The above Lemma implies that there is a dominant investment strategy for firm i. The optimal value of firm i's investment is constant because its payoff is linear in its capacity and the investment cost is convex.

Now, relying on Lemmata 1 and 2 we conjecture that firm j's capacity will meet firm i's in finite time. The following lemma asserts that when the firms' capacities met at a certain date, the best policy for firm j is to mimic firm i's decisions from that date on.

Lemma 4 *Assume that there is a date* $\underline{t}$ *at which* $K_j(\underline{t}) = K_i^*(\underline{t})$ *and consider the problem*

$$\max_{I_j(\cdot)} \int_{\underline{t}}^\infty e^{-rt} \left[w_j K_j(t) - C(I_j(t)) \right] dt \tag{13}$$

$$\textit{s.t.: } \dot{K}_j(t) = I_j(t) - \delta K_j(t), \tag{14}$$

$$K_j(t) \leq K_i^*(t) \quad \textit{for all } t. \tag{15}$$

Then the solution to this problem is $I_j(t) = I_i^*(t)$ *for all* $t \geq \underline{t}$.

Proof. See Supplementary material. ■

To grasp the intuition of the above result, recall that firm j benefits from a higher wholesale-price. As it is more profitable than firm i, it would like to increase its capacity. But this is not possible as long as what can be sold is bounded by firm i's capacity, which must be lower than firm j's since this firm is less profitable than firm j.

To find an equilibrium strategy for firm j, we must look for the "optimal" date at which its capacity meets firm i's. Then taking into account the assumption that $K_i(0) < K_j(0)$, the fact that as long as $K_i^*(t) < K_j(t)$ firm j's gross income only depends on firm i's capacity, as well as Lemma 4, to find the date at which manufacturer's capacities meet we must solve the following problem.

$$\sup_{\underline{t}, I_j(t)} \left\{ \int_0^{\underline{t}} e^{-rt} \left(-\alpha I_j(t) - \frac{\beta}{2}(I_j(t))^2 \right) dt + \int_{\underline{t}}^{\infty} e^{-rt} \left(-\alpha I_i^*(t) - \frac{\beta}{2}(I_i^*(t))^2 \right) dt \right\} \quad (16)$$

$$\text{s.t.:} \quad K_i^*(t) \leq K_j(t), \quad (17)$$

$$\dot{K}_j(t) = I_j(t) - \delta K_j(t), \; K_j(0) = K_{j0}, \quad (18)$$

$$K_j(\underline{t}) = K_i^*(\underline{t}), \quad (19)$$

where $K_i^*(t) = (K_{i0} - K_{i\infty})e^{-\delta t} + K_{i\infty}$.

We shall solve this problem is three steps. In the first step, we consider that $\underline{t}$ is fixed and we shall look for the optimal path of firm j's investment under the assumption that at date $\underline{t}$ firms' capacities become equal. To obtain the optimal date at which capacities meet we intuitively rely on the idea that at such date firms' investments must also be equal. To wit, if one differentiate firm j's payoff with respect to $\underline{t}$, we get

$$e^{-r\underline{t}} \left(-\alpha I_j(\underline{t}) - \frac{\beta}{2}(I_j(\underline{t}))^2 + \alpha I_i^*(\underline{t}) + \frac{\beta}{2}(I_i^*(\underline{t}))^2 \right).$$

Thus, as long as investments are not equal, it is worthwhile to change the date at which firms' capacities meet.[30] We will show that there is a date at which firms' investments and capacities are equal, and this date will turn out to be optimal.

To proceed, let us first assume that $\underline{t}$ is given. The following Lemma gives the optimal investment and capacity of firm j.

Lemma 5 *Let $\underline{t}$ be given and let $\hat{I}_j(t, \underline{t})$ and $\hat{K}_j(t, \underline{t})$ solve problem (16)-(19) when $\underline{t}$ is given. Furthermore suppose that $K_i^*(t) < \hat{K}_j(t, \underline{t})$, for all $t < \underline{t}$ where $K_i^*(t)$ is as in Lemma*

[30] Of course, since firm j's objective is not concave in $\underline{t}$ the condition that investments be equal is only necessary, but is not sufficient. But one can show that firm j's objective is non-decreasing in $\underline{t}$. Hence, the condition that investment be equal is also sufficient.

3. Then we have for all $t < \underline{t}$

$$\hat{I}_j(t,\underline{t}) = \frac{D_1 e^{(r+\delta)t} - \alpha}{\beta}, \tag{20}$$

$$\hat{K}_j(t,\underline{t}) = D_2 e^{-\delta t} + \frac{D_1}{\beta(r+2\delta)} e^{(r+\delta)t} - \frac{\alpha}{\beta\delta}, \tag{21}$$

where

$$D_1 = \beta(r+2\delta)\frac{\left(K_{j0} + \frac{\alpha}{\beta\delta}\right)e^{-\delta\underline{t}} - \frac{\alpha}{\beta\delta} - K_i^*(\underline{t})}{e^{-\delta\underline{t}} - e^{(r+\delta)\underline{t}}}, \tag{22}$$

$$D_2 = \frac{K_i^*(\underline{t}) + \frac{\alpha}{\beta\delta} - \left(K_{j0} + \frac{\alpha}{\beta\delta}\right)e^{(r+\delta)\underline{t}}}{e^{-\delta\underline{t}} - e^{(r+\delta)\underline{t}}}, \tag{23}$$

and

$$\hat{I}_j(t,\underline{t}) = I_i^*(t), \tag{24}$$

$$\hat{K}_j(t,\underline{t}) = K_i^*(t), \tag{25}$$

for all $t \geq \underline{t}$.

Proof. See Supplementary material. ■

The next Lemma ensures that there exists a date $\underline{t}$ at which firms' optimal values of investment and capacity are equal (and firm j's capacity is higher than firm i's before that date).

Lemma 6 *There exists a date $\tau > 0$ such that the solution to problem (16)-(19) with $\underline{t}$ fixed and where $\underline{t} = \tau$, satisfies the following conditions*

$$\hat{I}_j(\tau,\tau) = I_i^*(\tau) \tag{26}$$

$$K_i^*(t) < \hat{K}_j(t,\tau),\ t < \tau. \tag{27}$$

We can rely on the preceding results to show that there exists an open-loop Nash equilibrium for the game played by the manufacturers.

Theorem 1 *There exists a date $\underline{t}$ such that the following manufacturers' strategies are an open-loop Nash equilibrium for the game played by the manufacturers:*

• *Firm i*

For all t,

$$I_i^*(t) = \delta K_{i\infty}, \tag{28}$$

$$K_i^*(t) = K_{i\infty} + (K_{i0} - K_{i\infty})e^{-\delta t}, \tag{29}$$

$$K_{i\infty} = \frac{w_i - (r+\delta)\alpha}{\beta\delta(r+\delta)}. \tag{30}$$

• *Firm j*

For all $t \leq \underline{t}$,

$$\hat{I}_j^*(t) \quad = \quad \frac{D_1 e^{(r+\delta)t} - \alpha}{\beta}, \tag{31}$$

$$\hat{K}_j^*(t) \quad = \quad D_2 e^{-\delta t} + \frac{D_1}{\beta(r+2\delta)} e^{(r+\delta)t} - \frac{\alpha}{\beta\delta}, \tag{32}$$

where

$$D_1 \quad = \quad \beta(r+2\delta) \frac{\left(K_{j0} + \frac{\alpha}{\beta\delta}\right) e^{-\delta \underline{t}} - \frac{\alpha}{\beta\delta} - K_i^*(\underline{t})}{e^{-\delta \underline{t}} - e^{(r+\delta)\underline{t}}}, \tag{33}$$

$$D_2 \quad = \quad \frac{K_i^*(\underline{t}) + \frac{\alpha}{\beta\delta} - \left(K_{j0} + \frac{\alpha}{\beta\delta}\right) e^{(r+\delta)\underline{t}}}{e^{-\delta \underline{t}} - e^{(r+\delta)\underline{t}}}, \tag{34}$$

For all $t \geq \underline{t}$,

$$I_j^*(t) = I_i^*(t), \tag{35}$$

$$K_j^*(t) = K_i^*(t). \tag{36}$$

Moreover, these strategies together with those of the retailers displayed in Proposition 1 are an open-loop Nash equilibrium for the game played by the retailers and the manufacturers.

Proof. Set $\underline{t} = \tau$, where τ is given by Lemma 6. Assume that firms choose $I_i = I_i^*$ as in Lemma 3 and I_j as in Lemma 5 for $t \leq \tau$, and $I_j^* = I_i^*$ for $\tau \leq t$. Then these decisions are an open-loop Nash equilibrium (see the Supplementary material for more details). Moreover, we can check that these decisions together with those of the retailers displayed in Proposition 1 are an open-loop Nash equilibrium for the game played by the retailers and the manufacturers. ■

Notice that the open-loop Nash equilibrium played by the retailers and the manufacturers coincides with a (degenerate) generalized Stackelberg equilibrium since the manufacturers cannot exploit the pricing decisions of the retailers.

4.2 Case where $w_j < w_i$

We now study the case where the firm with the lowest initial capacity is also the firm which receives the highest transfer price. We shall look again for an equilibrium in which firms' capacities meet in finite time. Contrary to the case where firm j receives the highest transfer price, however, no firm has a dominant strategy from date 0 on. This implies that both firms must determine simultaneously the date at which their capacities meet. Once these capacities meet, as firm j's payoff is lower than firm i's, we conjecture that firm i will align its investment decisions on firm j's. Indeed, because firm i receives a higher transfer price than firm j, it will seek to build a higher capacity than firm j. But as firms i and j produce complements, firm i must follow firm j's decisions.

Specifically, to find an open-loop Nash equilibrium we shall look for decisions $(I_i^*(.), \underline{t}_i^*, I_j^*(.), \underline{t}_j^*)$ such that $(I_i^*(\cdot), \underline{t}_i^*)$ solves the following problem[31]

$$\max_{I_i, \underline{t}_i} \int_0^{\underline{t}_i} \big(w_i K_i - C(I_i)\big) e^{-rt} dt + \int_{\underline{t}_i}^{\infty} \big(w_i K_j^* - C(I_i)\big) e^{-rt} dt \tag{37}$$

subject to

$$\dot{K}_i = I_i - \delta K_i, \; K_i(0) = K_{i0}, \tag{38}$$

$$K_i(t) \leq K_j^*(t) \text{ for } t < \underline{t}_i, \tag{39}$$

$$K_i(\underline{t}_i) = K_j^*(\underline{t}_i), \tag{40}$$

and $(I_j^*, \underline{t}_j^*)$ solves the following problem

$$\max_{I_j, \underline{t}_j} \int_0^{\underline{t}_j} \big(w_j K_i^* - C(I_j)\big) e^{-rt} dt + \int_{\underline{t}_j}^{\infty} \big(w_j K_j - C(I_j)\big) e^{-rt} dt \tag{41}$$

[31] Notice that the choice of $\underline{t}$ does not appear in the definition of our open-loop equilibrium. However, we know that from date $\underline{t}_i$ on, firm i's payoff will depend on firm j's capacity j, so what really matters is knowing the value of this date in equilibrium.

subject to

$$\dot{K}_j = I_j - \delta K_j,\ K_j(0) = K_{j0}, \tag{42}$$

$$K_i^*(t) \leq K_j(t),\ t < \underline{t}_j \tag{43}$$

$$K_i^*(\underline{t}_j) = K_j(\underline{t}_j) \tag{44}$$

and $\underline{t}_i^* = \underline{t}_j^*$.

When capacities meet, we conjecture that firm j's decisions will be similar to firm i's decisions given in Lemma 3, that is

$$I_j(t) = \delta K_{j\infty}, \tag{45}$$

$$K_j(t) = K_{j\infty} + (K_j(\underline{t}_j) - K_{j\infty})e^{-\delta(t-\underline{t}_j)}, \tag{46}$$

$$K_{j\infty} = \frac{w_j - (r+\delta)\alpha}{\beta\delta(r+\delta)}. \tag{47}$$

Moreover, for $\underline{t}_j$ given, we can rely on Lemma 5 to find the decisions made by firm j that solve the following problem

$$\max_{I_j(\cdot)} \int_0^{\underline{t}_j} \big(w_j K_i^*(t) - C(I_j(t))\big)e^{-rt}dt \tag{48}$$

$$\text{s.t.: } \dot{K}_j = I_j(t) - \delta K_j(t),\ K_j(0) = K_{j0}, \tag{49}$$

$$K_j(\underline{t}_j) = K_i^*(\underline{t}_j), \tag{50}$$

$$K_i^*(t) < K_j(t). \tag{51}$$

As for firm i, we can show that its decisions before the date at which capacities meet are as follows

$$K_i(t) = K_{i\infty} + D_{1i}e^{-\delta t} + D_{2i}e^{(r+\delta)t},$$

where

$$K_{i\infty} = \frac{w_i - \alpha(r+\delta)}{\beta\delta(\delta+r)}, \tag{52}$$

$$D_{1i} = \frac{(K_{i0} - K_{i\infty})e^{(r+\delta)\underline{t}_i} - (K_j^*(\underline{t}_i) - K_{i\infty})}{e^{(r+\delta)\underline{t}_i} - e^{-\delta\underline{t}_i}}, \tag{53}$$

$$D_{2i} = \frac{K_j^*(\underline{t}_i) - K_{i\infty} - (K_{i0} - K_{i\infty})e^{-\delta\underline{t}_i}}{e^{(r+\delta)\underline{t}_i} - e^{-\delta\underline{t}_i}}. \tag{54}$$

In equilibrium $\underline{t}_i = \underline{t}_j = \underline{t}$, and we can also rely on Lemma 4 to conjecture that

$$I_i^*(t) = I_j^*(t) = \delta K_{j\infty}, \tag{55}$$

$$K_i^*(t) = K_j^*(t),\ \forall t \geq \underline{t}. \tag{56}$$

To find the date at which capacities meet we rely on the two above equations as well as the further conjecture that both I_i^* and I_j^* are continuous at $\underline{t}$.[32] We can show that these conditions are satisfied for firm i only if $K_j^*(\underline{t})$ takes a specific value. The same conclusion is obtained for firm j. The optimal date is found by equating $K_i^*(\underline{t})$ and $K_j^*(\underline{t})$. We have the following result.[33]

Theorem 2 *There exists a date $\underline{t}$ such that the following manufacturers' strategies are an open-loop Nash equilibrium for the game played by the manufacturers:*

- *Firm i*

For all $t \leq \underline{t}$

$$I_i^*(t) = \delta K_{i\infty} + (r + 2\delta)e^{(r+\delta)t}, \tag{57}$$

$$K_i^*(t) = K_{i\infty} + D_{1i}e^{-\delta t} + D_{2i}e^{(r+\delta)t}, \tag{58}$$

[32] Otherwise, as we have observed in the previous subsection, the date at which capacities meet is not optimal.

[33] To save space we have not followed the detailed approach used in the previous subsection. The Supplementary material, however, contains a complete proof of the result.

where

$$K_{i\infty} = \frac{w_i - \alpha(r+\delta)}{\beta\delta(r+\delta)}, \tag{59}$$

$$D_{1i} = \frac{(K_{i0} - K_{i\infty})e^{(r+\delta)\underline{t}} - (K_j^*(\underline{t}) - K_{i\infty})}{e^{(r+\delta)\underline{t}} - e^{-\delta\underline{t}}}, \tag{60}$$

$$D_{2i} = \frac{K_j^*(\underline{t}) - K_{i\infty} - (K_{i0} - K_{i\infty})e^{-\delta\underline{t}}}{e^{(r+\delta)\underline{t}} - e^{-\delta\underline{t}}}. \tag{61}$$

For all $t \geq \underline{t}$,

$$I_i^*(t) = \delta K_{j\infty} \tag{62}$$

$$K_i^*(t) = K_j^*(t). \tag{63}$$

- *Firm* j

For all $t \leq \underline{t}$,

$$I_j^*(t) = \frac{D_{1j}e^{(r+\delta)t} - \alpha}{\beta}, \tag{64}$$

$$K_j^*(t) = \frac{D_{1j}}{\beta(r+2\delta)}e^{(r+\delta)t} + D_{2j}e^{-\delta t} - \frac{\alpha}{\beta\delta}, \tag{65}$$

where

$$D_{1j} = \beta(r+2\delta)\frac{(K_{j0} + \frac{\alpha}{\beta\delta})e^{-\delta\underline{t}} - \frac{\alpha}{\beta\delta} - K_j^*(\underline{t})}{e^{-\delta\underline{t}} - e^{(r+\delta)\underline{t}}}, \tag{66}$$

$$D_{2j} = \frac{\frac{\alpha}{\beta\delta} + K_j^*(\underline{t}) - (K_{j0} + \frac{\alpha}{\beta\delta})e^{(r+\delta)\underline{t}}}{e^{-\delta\underline{t}} - e^{(r+\delta)\underline{t}}}, \tag{67}$$

$$K_j^*(\underline{t}) = \left[\frac{\delta(K_{j\infty} - K_{i\infty})}{(r+\delta)}e^{-(r+\delta)\underline{t}}\right]\left(e^{(r+\delta)\underline{t}} - e^{-\delta\underline{t}}\right) + K_{i\infty} + (K_{i_0} - K_{i\infty})e^{-\delta\underline{t}}. \tag{68}$$

For all $t \geq \underline{t}$

$$I_j^*(t) = \delta K_{j\infty}, \tag{69}$$

$$K_j^*(t) = K_{j\infty} + (K_j(\underline{t}) - K_{j\infty})e^{-\delta(t-\underline{t})}, \tag{70}$$

$$K_{j\infty} = \frac{w_j - \alpha(r+\delta)}{\beta\delta(r+\delta)}. \tag{71}$$

Moreover, these strategies together with those of the retailers displayed in Proposition 1 are an open-loop Nash equilibrium for the game played by the retailers and the manufacturers.

Proof. See Supplementary material. ■

Notice that for the same reasons stated in the comments of Theorem 1, that open-loop Nash equilibrium coincides with a (degenerate) generalized Stackelberg equilibrium.

5 Non-existence of Open-Loop Equilibrium with a Wholesale Price and a Sales Revenue Share

We now address the existence of an open-loop generalized Stackelberg equilibrium when each supply chain resorts to a revenue-sharing agreement including a wholesale price and a sales revenue share. Our major result is as follows.

Theorem 3 *Suppose that there is a firm h such that $K_{h0} \neq \frac{\hat{a}}{3b}$, or else that there is a firm h for which*

$$(1-\phi_h)\frac{\hat{a}}{3b} > \frac{w_h}{\phi_h} \tag{72}$$

Then there is no open-loop Nash equilibrium for the game played by the manufacturers and thus, there is no generalized Stackelberg equilibrium.

Notice that the inequality above ensures that there is no equilibrium where firms both start with and maintain a capacity equal to $\hat{a}/3b$. To put it differently, if we do not make that assumption, we cannot rule out that there is this kind of equilibrium. Such an equilibrium, however, is very peculiar.

The non-existence of an open-loop Nash equilibrium is in contrast with previous results in the literature of capital accumulation games. In one of the first versions of these games, Spence (1979) studied open-loop Nash equilibria, assuming that there is no discounting, linear investment cost and that firms sell the same product. Fershtman and Muller (1984) showed the existence of an open-loop Nash equilibrium when investment takes nonnegative values and capital depreciates at a positive (firm specific) rate. While their model is more general than ours in some respects, they assume that instantaneous profits are continuous in their capital stocks (*i.e.*, production capacities), a property that does not hold in general in our setting. The discontinuity of the instantaneous profits in the production capacities plays a relevant role in the non-existence of equilibria. Theorem 3 also is in contrast with

Hanig (1986) and Reynolds (1987) where both open-loop and feedback Nash equilibria exist (although under some mild conditions).[34] Furthermore, several researchers, *e.g.*, Reynolds (1987) or Dockner (1992), have shown that the static Cournot Nash equilibrium coincides with the steady state of the open-loop equilibrium (see also Lambertini (2018)). This is no longer true in our setting: while there is an equilibrium in the static Cournot model of complements (with capacity constraints), there is generally no steady-state open-loop equilibrium.

The non-existence of an open-loop Nash equilibrium will be deduced from other results which are of independent interest. The two first of these results (Propositions 3 and 4) assert that the manufacturers' capacities cannot be equal in equilibrium. The other Propositions (Propositions 5 and 6) deal with the cases where the manufacturers' capacities are always ordered in the same way, and where the capacity paths double-cross at least once.

Proposition 3 *Assume that the initial production capacities of the two manufacturers are identical and such that* $K_{i0} = K_{j0} < \frac{\hat{a}}{3b}$*. Then there is no symmetric open-loop Nash equilibrium such that* $K_h(t) > 0$ *on an interval* $[\underline{t}, \bar{t}]$*, with* $\underline{t} < \bar{t}$*, and* $K_h(t) < \frac{\hat{a}}{3b}$*,* $h = i, j$*.*

Proof. See Supplementary material. ■

The intuition of the result is as follows. By slightly decreasing investment expenditures over a non-negligible time interval, a manufacturer obviously decreases its production capacity. Since this policy departs from a candidate symmetric equilibrium and the production capacity depreciates at a rate δ, this means that this manufacturer's capacity will always be lower than its rival's. Manufacturer i's profit will be enhanced by a higher price but will also be negatively affected by a lower volume of production. However, because of the discontinuity in the revenue function, it turns out that for a small decrease in investment (that also increases instant profits), the positive effect more than compensates the negative one and thus brings about a profitable deviation.

The next Proposition addresses the case where manufacturers' capacities are always equal to $\hat{a}/(3b)$.

[34] Non-existence of equilibrium occurs in Operations Research. See, e.g., Yang and Anderson (2014). Yet, they rely on a static setting, and equilibria exist in several non-trivial cases. Here, we use a dynamic setting and generally there is no equilibrium. Specifically, an open-loop Nash equilibrium may exist when the capacity of both firms is initially equal to $\hat{a}/3b$ and is maintained at this value along the infinite time horizon of the game.

Proposition 4 *Suppose that there is a firm h such that*

$$(1-\phi_h)\frac{\hat{a}}{3b} > \frac{w_h}{\phi_h}. \tag{73}$$

There is no open-loop Nash equilibrium such that the manufacturers' capacities satisfy $K_h(t) = \frac{\hat{a}}{3b}$, for all t, and $h = i, j$.

Proof. See Supplementary material. ■

The intuition of the result is as follows. Choosing to always maintain the same relatively large capacity does not pay in equilibrium since a manufacturer can increase instant profits by diminishing its production capacity. This increase in instant profits stems from an increase in the manufacturer's revenues (the increase in the sale price compensates for the decrease in the quantity sold), and from a decrease in investment expenditures.

Proposition 5 *Assume that the manufacturers' initial capacities are such that $K_{i0} < K_{j0}$, $K_{j0} \leq \frac{a}{3b}$. Then there is no open-loop Nash equilibrium such that $K_i^*(t) < K_j^*(t)$ for all t.*

Proof. See Supplementary material. ■

The gist of the Proposition is that because manufacturer j's objective does not depend upon its capacity, the best decision is to downsize this capacity as much as possible. Selling its capacity is indeed the only way for manufacturer j to increase its instant profit. But by so doing, manufacturer j's capacity is soon lower than manufacturer i's, and this is inconsistent with the assumption that manufacturer j's capacity is always the largest.

Proposition 6 *There is no open-loop Nash equilibrium such that the manufacturers' capacities satisfy $K_i^*(\underline{t}) = K_j^*(\underline{t})$, $K_i^*(\bar{t}) = K_j^*(\bar{t})$, and $K_j^*(t) < K_i^*(t)$ on $]\underline{t}, \bar{t}[$.*

Proof. See Supplementary material. ■

The intuition behind Proposition 6 is that if the manufacturers' capacities are equal at two different dates, and ordered in the same manner between these dates, the manufacturer having the largest capacity can always decrease it and therefore saves on investment expenditures. To put it another way, it does not pay for a manufacturer to have a larger capacity than the other manufacturer because the extra capacity is costly to build and maintain and because it does not yield higher revenues.

We are now in position to prove Theorem 3.

Proof of Theorem 3. From Propositions 3, 4 and 5, manufacturers' capacities cannot always be the same, nor always be ordered in the same manner. Therefore, if there is an open-loop Nash equilibrium for the game played by the manufacturers, there must be at least two crossings similar to those considered in Proposition 6. But this very Proposition rules out such a case, and thus there is no generalized Stackelberg equilibrium. ■

6 Discussion

In this section, we will consider in turn three limitations of our results with regard to the use of open-loop strategies and the assumptions of investment reversibility, homogenous adjustment cost functions, zero price-adjustment costs, and flexible choices respectively.

• Open-loop strategies

We have restricted ourselves to considering open-loop strategies. The use of open-loop strategies by manufacturers, however, implies a commitment for these decision-makers because, in the event that production capacities are not equal to their expected values, firms should not notice. From our point of view, this is not too strong an assumption in a setting with no uncertainty.[35] Furthermore, open-loop strategies also pertain to cases where there is no information across time on production capacities except at the initial date. In this regard, it is unclear why firms would be aware of their production capacities every time.[36]

Moreover, in our setup the manufacturers' instant profits are discontinuous functions of their production capacities, and to the best of our knowledge, there is no general theory that allows us to study feedback Nash equilibria when the players' payoff functions are discontinuous functions of the state variables.[37] For all these reasons, we believe that open-

[35] Surely, assuming away uncertainty is a clear limitation of our approach.

[36] This remark appears, *e.g.*, in El Ouardighi and Erickson (2015), page 1282. For instance, the manufacturers may know the initial sizes of their plants, but not the continuous increments in their production capacities.

[37] Most of the time, feedback Nash equilibria in differential games are studied in a linear-state or a linear-quadratic setting. The class of linear-state or linear-quadratic differential allows the analytical characterization of feedback Nash equilibria. When the formulation of the dynamic games leads to nonlinear-quadratic differential games, numerical algorithms and methods are needed to find the feedback Nash equilibria. Yet, in this numerical approach, see e.g., De Frutos and Martín-Herrán (2015, 2018, 2019), Jaakkola and Ploeg (2019), El Ouardighi et al. (2020), the functions defining the players' payoff are continuous functions of the state and control variables. Studying feedback Nash equilibria when the players' payoff functions are discontinuous functions of the state variables would imply some technical and methodological developments outside the scope of this paper.

loop strategies, are not unreasonable in our setting.

• Reversible investment

We have assumed that manufacturers can disinvest and decrease their production capacities. Disinvesting is not always possible, however, especially if manufacturers' capacity includes non-material assets, that is, intangible capital accumulating from R&D or advertising. Where investment is irreversible, open-loop Nash equilibrium may exist even with a wholesale price and a sales revenue share, albeit in the particular case where initial capacities are equal. Indeed, when the manufacturers have the same initial capacities and produce complements, increasing capacity is not profitable because this will not result in increased sales, only in a rise in costs. Given the discontinuity of the instant profit function, the manufacturers would actually be better off by slightly decreasing their capacities in order to raise their revenues (see the argument used in the proof of Proposition 3). Yet, this is impossible as investment is irreversible. Thus, the best decision for each manufacturer is to let its capacity depreciate. Therefore, in this equilibrium, manufacturers' capacities are equal at each date and their common value goes to zero. This equilibrium outcome is peculiar because manufacturers obtain negligible profits in the long run.

• Homogenous adjustment cost functions

To simplifiy the analysis we have assumed that the manufacturers face homogenous cost functions. Assuming instead different cost functions would only make more cumbersome the determination of the date at which capacities meet when there is an equilibrium (from that date on, investments would no be the same from one firm to another). But it also make harder to prove that an equilibrium does not exist as the proof of Proposition 6 would have to be adapted.

• Zero price-adjustment costs

In our approach, retailers face zero price-adjustment costs. As we have seen above, this property is key for our non-existence result. Indeed, manufacturers can exploit the discontinuity in their profit function only if their capacity choices can affect the prices set by the retailers. Where price-adjustment costs are non nil (as in, e.g., Cellini and Lambertini (2007)), and include a fixed part, it may no longer be in the interest of retailers to adjust their prices in relation to the changes in the manufacturers' capacities. As a consequence, the argument used in the proof of Proposition 3 no longer applies.[38]

• Flexible Choices

We have assumed that retailers and manufacturers' choices are flexible. But decision-makers often face lumpy choices. For instance, retailers may actually use a finite set of prices, or manufacturers face lumpy capacity and lumpy investment/disinvestment decisions (see, e.g. Besanko et al. (2010) or Oliveira and Costa (2018)).[39] As a result, the arguments used in the proof of Proposition 3 are no longer relevant. Thus, where choices are lumpy, open-loop equilibria may exist. Of course, when choices are lumpy, the strategy sets are different from those considered in this paper and a thorough new analysis is in order.

7 Conclusion

The main result of this paper is that competition between supply chains may not be sustainable when they sell complementary products and when a revenue-sharing contract including a wholesale price and a fixed sales revenue share is used by asymmetric firms.

We have substantiated the claim above by showing that there generally is no open-loop equilibrium for the game played by the manufacturers and no generalized Stackelberg equilibrium either. These results ensue because the manufacturers' revenues are discontinuous functions of their production capacities, and this discontinuity in turn stems from the fact that manufacturers produce complementary goods. Our results may be considered as an additional instance where some manufacturers need to cooperate in order to prevent chaotic outcomes that competition is likely to produce in some specific markets (see, *e.g.*, Telser (1994), (1987), (2017) and McWilliams (1990)).

There are at least three other avenues for future research. Firstly, it would be interesting to study the existence of a generalized Stackelberg equilibrium when the retailers are the leaders, and the manufacturer the followers. Secondly, it would be worthwhile to investigate other dynamic settings - for example, ones where manufacturers face production or price

[38] We thank Utsave Sadana for this remark.

[39] Notice that the lumpiness of investment decisions may be unrelated to the existence of fixed costs and only results from technical constraints. For example, we can only use an integer number of machine-tools.

adjustment costs, or where capacity adjustment costs depend both on capacity and capacity investment. Thirdly, it would also be interesting to study cooperative solutions (either between the retailers and/or between the manufacturers), since this is a likely outcome when equilibrium does not exist.

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References

Amir, R. and, Gama, A. (2019). On Cournot's Theory of Oligopoly with Perfect Complements, Working paper.

Avenali, A., D'Annunzio A., and Reverberi, P. (2013). Bundling, Competition and Quality Investment: A Welfare Analysis. *Review of Industrial Organization* 43(3), 221-241.

Babaioff, M., Blumrosen, L., and Nisan, N. (2017). Selling Complementary Goods: Dynamics, Efficiency and Revenue. arXiv preprint arXiv:1706.00219.

Bart, N., Chernonog T., and Avinadav T. (2019). Revenue Sharing Contracts in a Supply Chain: A Literature Review. *IFAC PapersOnLIne* 52-13, 1578-1583.

Benoot, W., Brueckner J.K, and Proost S. (2013). Intercontinental-Airport Regulation. *Transportation Research Part B: Methodological* 52, 56-72.

Besanko, D., Doraszelski, U., Lu, L. X., and Satterthwaite, M. (2010). Lumpy Capacity Investment and Disinvestment Dynamics. *Operations Research* 58(4-part-2), 1178-1193.

Brueckner, J. K., and Flores-Fillol R. (2020). Market Structure and Quality Determination for Complementary Products: Alliances and Service Quality in the Airline Industry. *International Journal of Industrial Organization* 68, 102557.

Buratto, A., Cesaretto R., and De Giovanni P. (2019). Consignment Contracts with Cooperative Programs and Price discount Mechanisms in a Dynamic Supply Chain. *International Journal of Production Economics* 218, 72-82.

Cachon, G.P. (2003). Supply Chain Coordination with Contracts. In: Graves S., de Kok, T. (eds.) Handbooks in Operations Research and Management Science: Supply Chain Management. Elsevier, Amsterdam.

Cachon, G. P., and Netessine, S. (2006). Game Theory in Supply Chain Analysis. *INFORMS TutORials in Operations Research.* Published online: 14 Oct 2014; 200-233. https://doi.org/10.1287/educ.1063.0023

Cachon, G.P., and Lariviere M.A. (2005). Supply Chain Coordination with Revenue-Sharing Contracts: Strengths and Limitations. *Management Science* 51(1), 30-44.

Casadesus-Masanell, R., and Yoffie, D. B. (2007). Wintel: Cooperation and Conflict. *Management Science* 53(4), 584-598.

Caulkins, J. P., Feichtinger, G., Grass, D., Hartl, R. F., Kort, P. M., Novak, A. J., and Seidl, A. (2013). Leading Bureaucracies to the Tipping Point: An Alternative Model of Multiple Stable Equilibrium Levels of Corruption. *European Journal of Operational Research* 225(3), 541-546.

Cellini, R. and Lambertini L. (2007). A Differential Oligopoly Game with Differentiated Goods and Sticky Prices. *European Journal of Operational Research* 176(2), 1131-1144.

Chua, G. A., Lim, W. S., and Yeo, W. M. (2016). Market Structure and the Value of Overselling under Stochastic Demands. *European Journal of Operational Research*, 252(3), 900-909.

Colombo, L., and Labrecciosa, P. (2018). Differential Games in Industrial Organization. In *Handbook of Dynamic Game Theory* (pp. 779-825). Springer.

Cournot, L.A (1838). *Recherches sur les Principes Mathématiques de la Théorie des Richesses*. Hachette, Paris.

De Frutos J., and Martín-Herrán G. (2015). Does Flexibility Facilitate Sustainability of Cooperation over Time? A Case Study from Environmental Economics. *Journal of Optimization Theory and Applications* 165(2), 657–677.

De Frutos J., and Martín-Herrán G. (2018). Selection of a Markov Perfect Nash Equilibrium in a Class of Differential Games. *Dynamic Games and Applications* 8, 620–636.

De Frutos J., and Martín-Herrán G. (2019). Spatial Effects and Strategic Behavior in a Multiregional Transboundary Pollution Game. *Journal of Environmental Economics and Management* 97, 182–207.

De Giovanni, P. (2021). A Dynamic Supply Chain Game with Vertical Coordination and Horizontal Competition. *International Transactions in Operational Research* 28(6), 3117-3146.

De Giovanni P., Karray S., and Martín-Herrán G. (2019). Vendor Management Inventory with Consignment Contracts and the Benefits of Cooperative Advertising. *European Journal of Operational Research* 272, 465-480.

Dobson, P.W. (1992). Mutually Related Firms and the Rationality of Producing Nothing. *Managerial and Decision Economics* 13(6), 485-491.

Dobson, P.W., and Chakraborty, R. (2020). Strategic Incentives for Complementary Producers to Innovate for Efficiency and Support Sustainability. *International Journal of Production Economics* 219, 431-439.

Dockner, E. J. (1992). A Dynamic Theory of Conjectural Variations. *The Journal of Industrial Economics* 40, 377-395.

Dockner E.J., Jørgensen, S., Long, N.V., and Sorger, G. (2000). *Differential Games in Economics and Management Science*. Cambridge University Press. Cambridge.

Egerer, J., Grimm, V., Grübel, J., and Zöttl, G. (2022). Long-Run Market Equilibria in Coupled Energy Sectors: A Study of Uniqueness. *European Journal of Operational Research* 303(3), 1335-1354.

El Ouardighi F., Jørgensen S., and Pasin F. (2008). A Dynamic Model of Operations and Marketing Management in a Supply Chain. *International Game Theory Review* 10(4), 373-397.

El Ouardighi, F., and Bowon K. (2010). Supply Quality Management with Wholesale Price and Revenue Sharing Contracts under Horizontal Competition. *European Journal of Operational Research* 206(2), 329-340.

El Ouardighi, F., Jørgensen, S., and Pasin F. (2013). A Dynamic Game with Monopolist Manufacturer and Price-Competing Duopolist Retailers. *OR spectrum* 35(4), 1059-1084.

El Ouardighi, F., and Erickson, G (2015). Production Capacity Buildup and Double Marginalization Mitigation in a Dynamic Supply Chain. *Journal of the Operational Research Society* 66(8), 1281-1296.

El Ouardighi F., Kogan K., Gnecco G., and Sanguinetti M. (2020). Transboundary Pollution Control and Environmental Absorption Efficiency Management. *Annal of Operations Research* 287,653–681.

El Ouardighi, F., Jeongeun S., and Bowon, K. (2021). Pollution Accumulation and Abatement Policies in Two Supply Chains Under Vertical and Horizontal Competition and Strategy Types. *Omega* 98, 102-108.

Engwerda, J.C. (2005). *LQ Dynamic Optimisation and Differential Games*, Johen Wiley & Sons.

Fershtman, C., and Muller, E. (1984). Capital Accumulation Games of Infinite Duration. *Journal of Economic Theory* 33(2), 322-339.

Fleming, W., and Rishel, R. (1975). *Deterministic and Stochastic Optimal Control*. Springer-Verlag.

Fudenberg, D., and Tirole, J. (1983). Capital as a Commitment: Strategic Investment to Deter Mobility. *Journal of Economic Theory* 31(2), 227-250.

Fudenberg, D., and Tirole, J. (1986). *Dynamic Models of Oligopoly*. Harwood. London.

Fudenberg, D., and Tirole, J. (1991). *Game Theory*. MIT press.

Ghorbani, M., Acciaro, M., Transchel, S., and Cariou, P. (2022). Strategic alliances in container shipping: A review of the literature and future research agenda. *Maritime Economics & Logistics*, 24(2), 439-465.

Giri R.N., Mondal S.K., and Maiti M. (2020). Bundle Pricing Strategies for Two Complementary Products with Different Channel Powers *Annals of Operations Research* 287, 701–725.

Grimm, V., Grübel, J., Schewe, L., Schmidt, M., and Zöttl, G. (2019). Non-convex equilibrium models for gas market analysis: Failure of standard techniques and alternative modeling approaches. *European Journal of Operational Research*, Volume 273, Issue 3, 1097-1108.

Hamed, H., Ansaripoor A.H., and De Giovanni P. (2020). Closed-Loop Supply Chains with Complementary Products. *International Journal of Production Economics* 229. Art. No. 107757.

Hanig, M. (1986). *Differential Gaming Models of Oligopoly*, Phd Dissertation, MIT.

Hartl, R.F. (1987). A Simple Proof of the Monotonicity of the State Trajectories in Autonomous Control Problems. *Journal of Economic Theory* 41(1), 211-215.

He, X., Prasad, A., Sethi S.P., and Gutierrez, G.J. (2007). A Survey of Stackelberg Differential Game Models in Supply and Marketing Channels. *Journal of System Sciences and System Engineering* 16, 385-413.

Jaakkola N., and Van der Ploeg F. (2019). Non-Cooperative and Cooperative Climate Policies with Anticipated Breakthrough Technology. *Journal of Environmental Economics and Management* 97, 42–66.

Julien, L. A. (2017). On Noncooperative Oligopoly Equilibrium in the Multiple Leader–Follower Game. *European Journal of Operational Research* 256(2), 650-662.

Jørgensen, S., and Zaccour, G. (2004). *Differential Games in Marketing.* Springer, New York.

Kogan, K., and Tapiero, C.T. (2007). *Supply Chain Games: Operations Management and Risk Valuation.* Vol. 113. New York: Springer.

Lambertini, L. (2018). *Differential Games in Industrial Economics.* Cambridge University Press.

Laussel, D., and Van Long, N. (2012). Vertical Disintegration: a Dynamic Markovian Approach. *Journal of Economics & Management Strategy* 21(3), 745–771.

Linnemer, L. (2022). Doubling Back on Double Marginalization. *Review of Industrial Organization* 61, 1-19.

Liu, G., Sethi, S.P., and Zhang, J. (2016). Myopic vs. Far-Sighted Behaviours in a Revenue-Sharing Supply Chain with Reference Quality Effects. *International Journal of Production Research* 54(5), 1334-1357.

Löschenbrand, Markus (2020). Finding multiple Nash equilibria via machine learning-supported Gröbner bases. *European Journal of Operational Research.* 284.3 : 1178-1189.

Mantovani, A., and Ruiz-Aliseda, F. (2015). Equilibrium Innovation Ecosystems: the Dark Side of Collaborating with Complementors. *Management Science* 62(2), 534-549.

Martín-Herrán, G., Taboubi, S., and Zaccour, G. (2005). A Time-Consistent Open-Loop Stackelberg Equilibrium of Shelf-Space Allocation. *Automatica* 41(6), 971-982.

Merk, O., Kirstein, L., and Salamitov, F. (2018). The impact of alliances in container shipping. International Transport Forum

Mitsuhashi H and Greve HR (2009) A Matching Theory of Alliance Formation and Organizational Success: Complementarity and Compatibility *The Academy of Management Journal*, Vol. 52, No. 5 (Oct., 2009), pp. 975-995

McWilliams, A. (1990). Rethinking Horizontal Market Restrictions: in Defense of Cooperation in Empty Core Markets. *Quarterly Review of Economics and Business* 30, 3–14.

Oliveira, F. S., Ruiz, C. and Conejo, A.J. (2013). Contract Design and Supply Chain Coordination in the Electricity Industry. *European Journal of Operational Research* 227(3), 527-537.

Oliveira, F. S. and Costa M. L.G. (2018). Capacity expansion under uncertainty in an oligopoly using indirect reinforcement-learning. *European Journal of Operational Research,* 267(3), 1039-1050.

Reynolds, S. (1987). Capacity Investment, Preemption and Commitment in an Infinite Horizon Model. *International Economic Review* 28, 69–88.

Sherali, H. D. (1984). A Multiple Leader Stackelberg Model and Analysis. *Operations Research*, 32(2), 390–404.

Sim, J.E., El Ouardighi, F., and Kim, B. (2019). Economic and Environmental Impact of Vertical and Horizontal Competition and Integration. *Naval Research Logistics Quarterly* 66(2), 133-153.

Sinha, A., Malo, P., Frantsev, A., and Deb, K. Finding optimal strategies in a multi-period multi-leader–follower Stackelberg game using an evolutionary algorithm, (2014,) *Computers & Operations Research,* 41, 374-385.

Spence, A.M. (1979). Investment Strategy and Growth in a New Market. *Bell Journal of Economics* 10(1), 1-19.

Telser, L.G. (1987). *A Theory of Efficient Cooperation and Competition.* Cambridge University Press.

Telser L.G. (1994). The Usefulness of Core Theory in Economics. *The Journal of Economic Perspectives* 8, Spring, 151-164.

Telser, L.G. (2017)[1972]. *Competition, Collusion and Game Theory*, Routledge.

von Stengel, B., Zamir, S. (2010). Leadership games with convex strategy sets. *Games and Economic Behavior*. 69(2), 446–457.

Xie D. (1997). On time inconsistency: a technical issue in Stackelberg differential games. *Journal of Economic Theory*. 76(2), p. 412-430.

Wang J., Liu J and Zhang X (2020), Service purchasing and market-entry problems in a shipping supply chain, *Transportation Research Part E: Logistics and Transportation Review*, Volume 136, 101895.

Yin, M. and Kim, K. H. (2012). Quantity discount pricing for container transportation services by shipping lines. *Computers and Industrial Engineering*, 63(1), 313-322.

Wei, J., Zhao J, and Hou X. (2019). Integration Strategies of Two Supply Chains with Complementary Products. *International Journal of Production Research* 57(7): 1972-1989.

Yalcin T., Ofek, E., and Koenigsberg (2013). Complementary Goods: Creating, Capturing and Competing for Value. *Marketing Science* 32(4), 554-569.

Yang, S. J. S., and Anderson, E. J. (2014). Competition Through Capacity Investment under Asymmetric Existing Capacities and Costs. *European Journal of Operational Research*, 237(1), 217-230.

Supplementary Material

Existence of Equilibrium in a Dynamic Supply Chain Game with Vertical Coordination, Horizontal Competition, and Complementary Goods

Second Revised Version

Bertrand Crettez, Naila Hayek, Guiomar Martín-Herrán

1. Proofs

Proof of Proposition 1. Consider the Lagrangean associated with problem (2)-(3), that is:

$$L(\hat{p}_h, \hat{p}_{-h}, \lambda_h) = \hat{p}_h \left(\frac{\hat{a} - (\hat{p}_h + \hat{p}_{-h})}{b} \right) + \lambda_h \left(K_i - \frac{\hat{a} - (\hat{p}_h + \hat{p}_{-h})}{b} \right). \tag{74}$$

Necessary and sufficient conditions for an optimal solution are as follows.

$$\frac{\partial L}{\partial p_h} = \frac{1}{b} \left(\hat{a} - 2\hat{p}_h - \hat{p}_{-h} + \lambda_h \right) = 0, \tag{75}$$

$$\lambda_h \left(K_i - \frac{\hat{a} - (\hat{p}_h + \hat{p}_{-h})}{b} \right) = 0, \tag{76}$$

$$\lambda_h \geq 0. \tag{77}$$

We consider four cases in turn.

1. $\lambda_i = \lambda_j = 0$. Therefore, in that case

$$\frac{\hat{a} - (\hat{p}_i + \hat{p}_j)}{b} \leq K_i. \tag{78}$$

Moreover, from

$$\hat{a} - 2\hat{p}_i - \hat{p}_j = 0, \tag{79}$$

$$\hat{a} - 2\hat{p}_j - \hat{p}_i = 0, \tag{80}$$

we get

$$\hat{p}_i = \hat{p}_j = \frac{\hat{a}}{3}. \tag{81}$$

Such an outcome is possible only if

$$\frac{\hat{a}}{3b} \leq K_i. \tag{82}$$

2. $\lambda_i = 0$, $\lambda_j > 0$.

   It follows that

$$\frac{\hat{a} - (\hat{p}_i + \hat{p}_j)}{b} = K_i. \tag{83}$$

   We thus get:

$$\hat{p}_i = bK_i, \tag{84}$$

$$\hat{p}_j = \hat{a} - 2bK_i. \tag{85}$$

   This case only arises when $K_i < \frac{\hat{a}}{3b}$.

3. $\lambda_i > 0$, $\lambda_j = 0$. By symmetry with the preceding case, we get:

$$\frac{\hat{a} - (\hat{p}_i + \hat{p}_j)}{b} = K_i. \tag{86}$$

   We thus get:

$$\hat{p}_i = \hat{a} - 2bK_i, \tag{87}$$

$$\hat{p}_j = bK_i. \tag{88}$$

   This case only arises when $K_i < \frac{\hat{a}}{3b}$.

4. $\lambda_i > 0$, $\lambda_j > 0$.

   We still have:

$$\frac{\hat{a} - (\hat{p}_i + \hat{p}_j)}{b} = K_i. \tag{89}$$

From the first-order conditions, we get:

$$bK_i < \hat{p}_i, \tag{90}$$

$$bK_i < \hat{p}_j, \tag{91}$$

$$K_i < \frac{\hat{a}}{3b}. \tag{92}$$

Notice that there is a symmetric equilibrium where:

$$\hat{p}_i = \hat{p}_j = \frac{\hat{a} - bK_i}{2}. \tag{93}$$

■

Existence

Proof of Lemma 1

Proof. Consider manufacturer j's problem. Under our assumption that $K_i^*(t) < K_j^*(t)$ for all t, it follows from (7) that manufacturer j's equilibrium payoff reads:

$$\int_0^\infty e^{-rt}\left[w_j K_i(t) - C(I_j^*(t))\right] dt.$$

This is because the sale proceeds only depend on manufacturer i's production capacity. Therefore, the equilibrium investment decision $I_j^*(\cdot)$ solves the following problem:

$$\begin{aligned} \max_{I_j(\cdot)} \quad & \int_0^\infty e^{-rt}\left[w_j K_i^*(t) - C(I_j(t))\right] dt \\ \text{s.t.:} \quad \dot{K}_j(t) &= I_j(t) - \delta K_j(t), \quad K_j(0) = K_{j0}, \\ K_i^*(t) &< K_j(t), \ \forall t. \end{aligned}$$

Neglecting the sale proceeds, we can write the Hamiltonian associated with the problem above as follows

$$-e^{-rt}\left(\alpha I_j(t) + \frac{\beta}{2}(I_j(t))^2)\right) + \lambda(t)\left(I_j(t) - \delta K_j(t)\right),$$

where λ denotes the costate variable associated with the state variable K_j. The first-order conditions are given by:

$$
\begin{aligned}
\lambda(t) &= (\alpha + \beta I_j(t))e^{-rt}, \\
\dot{\lambda}(t) &= \delta\lambda(t), \\
\dot{K}_j(t) &= I_j(t) - \delta K_j(t), \quad K_j(0) = K_{j0}.
\end{aligned} \tag{94}
$$

Because manufacturer j's objective is strictly concave, the solution $I_j(t)$ is unique and differentiable.[40] Thus, upon differentiating Equation (94) and using the two other first-order conditions we obtain:

$$
\begin{aligned}
\dot{\lambda}(t) &= -re^{-rt}\left(\alpha + \beta\big(\dot{K}_j(t) + \delta K_j(t)\big)\right) + \beta e^{-rt}\left(\ddot{K}_j(t) + \delta\dot{K}_j(t)\right) \\
&= \delta\lambda(t) = \delta\left(\alpha + \beta\big(\dot{K}_j(t) + \delta K_j(t)\big)\right)e^{-rt}.
\end{aligned}
$$

Rearranging, we arrive at the following differential equation:

$$
\ddot{K}_j(t) - r\dot{K}_j(t) - K_j(t)\delta(r+\delta) = \alpha\frac{(r+\delta)}{\beta}.
$$

The general solution of the equation above is given by:

$$
K_j(t) = \left(K_{j0} + \frac{\alpha}{\beta\delta}\right)e^{-\delta t} - \frac{\alpha}{\beta\delta}, \tag{95}
$$

where $-\delta$ is the negative root of the following characteristic equation:

$$
s^2 - rs - (r+\delta)\delta = 0.
$$

That is:

$$
\frac{r - \sqrt{r^2 + 4(r+\delta)\delta}}{2} = -\delta.
$$

It is clear, however, that $K_j(t)$ goes to a negative value, which contradicts the assumption that $K_j^*(t) > K_i^*(t)$ for all t and the Proposition follows. ■

Proof of Lemma 2

Proof. The proof is by way of a contradiction. Assume that I_j^* is such that $K_i^*(\underline{t}) =$

[40]See, *e.g.*, Corollary 6.1, page 77 in Fleming and Rishel (1975).

$K_j^*(t)$, $K_i^*(\bar{t}) = K_j^*(\bar{t})$, $\underline{t} < \bar{t}$, and $K_i^*(t) < K_j^*(t)$ for all t in $(\underline{t}, \bar{t})$. From the definition of $R^h(K_h, K_{-h})$, firm j's revenues is equal to $w_j K_i^*(t)$ for all t in $[\underline{t}, \bar{t}]$. Since firm j's profit depends on firm i's capacity, but firm j's capacity differs from firm i's, it must be that the discounted cost of investment of firm j is strictly lower than that of firm i. That is,

$$\int_{\underline{t}}^{\bar{t}} e^{-rt} C(I_j^*(t)) dt < \int_{\underline{t}}^{\bar{t}} e^{-rt} C(I_i^*(t)) dt. \tag{96}$$

Moreover, by assumption for all t in $]\underline{t}, \bar{t}[$, we have

$$K_i^*(t) < K_j^*(t). \tag{97}$$

This implies that for all t in $]\underline{t}, \bar{t}[$ we have $e^{-rt} w_i K_i^*(t) < e^{-rt} w_i K_j^*(t)$. But then using the two inequalities above it holds that

$$\int_{\underline{t}}^{\bar{t}} e^{-rt} w_i K_i^*(t) dt - \int_{\underline{t}}^{\bar{t}} e^{rt} C(I_i^*(t)) dt < \int_{\underline{t}}^{\bar{t}} e^{-rt} w_i K_j^*(t) dt - \int_{\underline{t}}^{\bar{t}} e^{-rt} C(I_j^*(t)) dt \tag{98}$$

which proves that $I_i^*(\cdot)$ is not an optimal policy for firm i. ■

Case where $w_i < w_j$

Proof of Lemma 3

Proof. Let us introduce the following Hamiltonian

$$H(I_i, K_i, \lambda) = e^{-rt} \left(w_i K_i(t) - \alpha I_i(t) - \frac{\beta}{2} I_i(t)^2 \right) + \lambda(t) \left[I_i(t) - \delta K_i(t) \right]. \tag{99}$$

The necessary (and sufficient) conditions are as follows

$$\lambda(t) = e^{-rt}(\alpha + \beta I_i(t)), \tag{100}$$

$$\dot{\lambda}(t) = -e^{-rt} w_i + \delta \lambda(t), \tag{101}$$

$$\dot{K}_i(t) = I_i(t) - \delta K_i(t), \tag{102}$$

$$K_i(0) = K_{i0}. \tag{103}$$

Differentiating the first condition we get

$$\dot{\lambda}(t) = -r e^{-rt} \big(\alpha + \beta I_i(t) \big) + e^{-rt} \beta \left(\ddot{K}_i(t) + \delta \dot{K}_i(t) \right). \tag{104}$$

Using the first-order necessary condition in the equation above and rearranging we get

$$\ddot{K}_i(t) - r\dot{K}_i(t) - (r+\delta)\delta K_i(t) + \frac{w_i - (r+\delta)\alpha}{\beta} = 0. \tag{105}$$

From there we can check that Lemma 3 holds. ■

Proof of Lemma 4

Proof. Consider the following version of the Hamiltonian

$$L_j\big(I_j(t), K_j(t), \lambda_j(t), \mu_j(t)\big) = e^{-rt}\left(w_j K_j(t) - \alpha I_j(t) - \frac{\beta}{2} I_j(t)^2\right) + \lambda_j(t)\big(I_j(t) - \delta K_j(t)\big) + \mu_j(t)\big(K_i^*(t) - K_j(t)\big).$$

Now given functions $\lambda_j(t)$ and $\mu_j(t)$, consider the following conditions.

$$\lambda_j(t) = e^{-rt}\left(\alpha + \beta I_j(t)\right), \tag{106}$$

$$\dot{\lambda}_j(t) = -w_j e^{-rt} + \delta\lambda_j(t) + \mu_j(t), \tag{107}$$

$$\mu_j(t) \geq 0, \tag{108}$$

$$\mu_j(t)(K_i^*(t) - K_j(t)) = 0. \tag{109}$$

Using the necessary condition satisfied by the optimal solution to firm i's problem, set for all $t \geq \underline{t}$

$$\mu_j(t) = e^{-rt}(w_j - w_i) > 0, \tag{110}$$

$$\lambda_j(t) = \lambda(t), \tag{111}$$

$$I_j^*(t) = I_i^*(t), \tag{112}$$

$$K_j^*(t) = K_i^*(t). \tag{113}$$

Observe that with these definitions $(I_j^*(\cdot), K_j^*(\cdot))$ solve the conditions (106)-(109) above. Observe also that I_j^* maximizes the Hamiltonian (because it is a concave function of $I_j(t)$). Now we shall show that this policy is the solution to firm j's problem above.

For all $T > \underline{t}$ and $K_j \leq K_i^*$ we have

$$\begin{aligned}
\int_{\underline{t}}^{T} e^{-rt}\big(w_j K_j^*(t) - C(I_j^*(t)) - w_j K_j(t) + C(I_j(t))\big)\,dt &= \int_{\underline{t}}^{T} e^{-rt}\big(w_j K_j^*(t) - C(I_j^*(t)) \pm \lambda_j(t)\big(I_j^*(t) - \delta K_j^*(t)\big) \\
&\quad - w_j K_j(t) + C(I_j(t)) \pm \lambda_j(t)\big(I_j(t) - \delta K_j(t)\big)\big)\,dt \\
&\geq \int_{\underline{t}}^{T} \big\{e^{-rt}\big(w_j K_j^*(t) - C(I_j^*(t))\big) + \lambda_j(t)\big(I_j^*(t) - \delta K_j^*(t)\big) \\
&\quad + \mu_j(t)\big(K_i^*(t) - K_j^*(t)\big) \\
&\quad - e^{-rt}\left(w_j K_j(t) + C(I_j(t))\right) - \lambda_j(t)\big(I_j(t) - \delta K_j(t)\big) \\
&\quad - \mu_j(t)\big(K_i^*(t) - K_j(t)\big)\big\}\,dt \\
&\quad + \int_{\underline{t}}^{T} \big(\lambda_j(t)\big(I_j(t) - \delta K_j(t)\big) - \lambda_j(t)\big(I_j^*(t) - \delta K_j^*(t)\big)\big)\,dt.
\end{aligned}$$

Using the concavity of the Hamiltonian, we get:

$$\begin{aligned}
\int_{\underline{t}}^{T} e^{-rt}\big(w_j K_j^*(t) - C(I_j^*(t)) - w_j K_j(t) + C(I_j(t))\big)\,dt &\geq \int_{\underline{t}}^{T} \big[\big(e^{-rt} w_j - \lambda_j(t)\delta - \mu_j(t)\big)(K_j^*(t) - K_j(t)) + \\
&\quad \big(-e^{-rt} C'(I_j^*(t)) + \lambda_j(t)\big)(I_j^*(t) - I_j(t))\big]\,dt \\
&\quad + \int_{\underline{t}}^{T} \big(\lambda_j(t)\big(I_j(t) - \delta K_j(t)\big) - \lambda_j(t)\big(I_j^*(t) - \delta K_j^*(t)\big)\big)\,dt.
\end{aligned}$$

Using the definition of $\dot{\lambda}_j(t)$ in equation (107) and the fact that $I_j^*(t)$ maximizes the Hamiltonian, the inequality above reduces to

$$\begin{aligned}
\int_{\underline{t}}^{T} e^{-rt}\big(w_j K_j^*(t) - C(I_j^*(t)) - w_j K_j(t) + C(I_j(t))\big)\,dt &\geq \int_{\underline{t}}^{T} \frac{d}{dt}\big(\lambda_j(t)(K_j(t) - K_j^*(t)\big)\,dt \\
&= \lambda_j(T) K_j(T) - \lambda_j(T) K_j^*(T). \qquad (114)
\end{aligned}$$

Now

$$\lambda_j(T) K_j^*(T) = \lambda_i^*(T) K_i^*(T) \qquad (115)$$

and we have

$$\lim_{T\to\infty} \lambda_i^*(T) K_i^*(T) = \lim_{T\to\infty} e^{-rT}(\alpha + \beta I_i^*(T)) K_i^*(T) = 0 \qquad (116)$$

since $I_i^*(t)$ is constant and $K_i^*(t)$ goes to $K_{i\infty}$. Thus, we have

$$\liminf_{T\to\infty} \int_{\underline{t}}^{T} e^{-rt} \left(w_j K_j^*(t) - C(I_j^*(t)) - w_j K_j(t) + C(I_j(t))\right) dt \geq \liminf_{T\to\infty} \ \lambda_j(T) K_j(T) \geq 0.$$

This also implies that

$$\int_{\underline{t}}^{\infty} e^{-rt} \left(w_j K_j^*(t) - C(I_j^*(t))\right) dt \geq \limsup_{T\to\infty} \int_{\underline{t}}^{T} e^{-rt} \left(w_j K_j(t) - C(I_j(t))\right) dt.$$

■

Proof of Lemma 5

Proof. Since $\hat{I}_j(t,\underline{t})$ and $\hat{K}_j(t,\underline{t})$ are a solution to the problem on $[0,\infty)$, its restriction to $[0,\underline{t})$ is a solution to the problem on $[0,\underline{t})$. By assumption, $K_i^*(t) < \hat{K}_j(t)$ for all $t \in [0,\underline{t})$. Then the Hamiltonian associated with the problem restricted to $[0,\underline{t})$ is given by

$$H_j = e^{-rt}\left(-\alpha I_j - \frac{\beta}{2}(I_j)^2\right) + \lambda_j(I_j - \delta K_j). \tag{117}$$

The first-order conditions are as follows:

$$\dot{\lambda}_j = \lambda_j \delta,$$
$$e^{-rt}(-\alpha - \beta \hat{I}_j) + \lambda_j = 0,$$

with $\hat{K}_j(\underline{t}) = K_i^*(\underline{t})$. We have

$$\lambda_j(t) = D_1 e^{\delta t}, \quad \hat{I}_j(t) = \frac{D_1 e^{(r+\delta)t} - \alpha}{\beta}.$$

Solving the differential equation

$$\dot{\hat{K}}_j(t) = \frac{D_1 e^{(r+\delta)t} - \alpha}{\beta} - \delta \hat{K}_j(t),$$

we have

$$\hat{K}_j(t,\underline{t}) = D_2 e^{-\delta t} + \frac{D_1}{\beta(r+2\delta)} e^{(r+\delta)t} - \frac{\alpha}{\beta\delta},$$

where constants D_1 and D_2 satisfy the following boundary conditions:

$$\begin{aligned} K_{j0} &= D_2 + \frac{D_1}{\beta(r+2\delta)} - \frac{\alpha}{\beta\delta}, \\ \hat{K}_j(\underline{t}, \underline{t}) &= D_2 e^{-\delta \underline{t}} + \frac{D_1}{\beta(r+2\delta)} e^{(r+\delta)\underline{t}} - \frac{\alpha}{\beta\delta} = K_i^*(\underline{t}). \end{aligned}$$

After a few algebra, we obtain

$$\begin{aligned} D_1 &= \beta(r+2\delta) \frac{\left(K_{j0} + \frac{\alpha}{\beta\delta}\right) e^{-\delta \underline{t}} - \frac{\alpha}{\beta\delta} - K_i^*(\underline{t})}{e^{-\delta \underline{t}} - e^{(r+\delta)\underline{t}}}, \\ D_2 &= \frac{K_i^*(\underline{t}) + \frac{\alpha}{\beta\delta} - \left(K_{j0} + \frac{\alpha}{\beta\delta}\right) e^{(r+\delta)\underline{t}}}{e^{-\delta \underline{t}} - e^{(r+\delta)\underline{t}}}. \end{aligned}$$

Finally, it is clear from Lemmata 1, 2 and 4 that $\hat{I}_j(t, \underline{t}) = I_i^*(t)$ and $\hat{K}_j(t, \underline{t}) = K_i^*(t)$ for $\underline{t} \leq t$. ■

Proof of Lemma 6

Proof. Set $\underline{t} = \tau$. Consider the pair I_i^* and $\hat{I}_j(t, \tau)$ obtained in the previous lemmata. From Lemma 5, if $K_i^*(t) < \hat{K}_j(t, \tau)$, $t < \tau$, we know that the first-order conditions hold

$$\hat{I}_j(t, \tau) = \frac{D_1 e^{(r+\delta)t} - \alpha}{\beta}, \tag{118}$$

$$\hat{K}_j(t, \tau) = D_2 e^{-\delta t} + \frac{D_1}{\beta(r+2\delta)} e^{(r+\delta)t} - \frac{\alpha}{\beta\delta}, \tag{119}$$

where D_1 and D_2 are such that

$$\hat{K}_j(0, \tau) = K_{j0},$$
$$\hat{K}_j(\tau, \tau) = K_i^*(\tau).$$

The equality $\hat{K}_j(0, \tau) = K_{j0}$ implies that

$$K_{j0} = D_2 + \frac{D_1}{\beta(r+2\delta)} - \frac{\alpha}{\beta\delta}. \tag{120}$$

Now the condition $\hat{I}_j(\tau, \tau) = I_i^*(\tau)$ implies in turn that

$$D_1 = [\alpha + \beta I_i^*(\tau)] e^{-(r+\delta)\tau}. \tag{121}$$

Further, the condition $\hat{K}_j(\tau,\tau) = K_i^*(\tau)$ reads

$$
\begin{aligned}
D_2 e^{-\delta\tau} + \frac{D_1}{\beta(r+2\delta)} e^{(r+\delta)\tau} - \frac{\alpha}{\beta\delta} &= K_i^*(\tau) \\
\iff \left[K_{j0} - \frac{D_1}{\beta(r+2\delta)} + \frac{\alpha}{\beta\delta}\right] e^{-\delta\tau} + \frac{[\alpha + \beta I_i^*(\tau)]}{\beta(r+2\delta)} e^{-(r+\delta)\tau} e^{(r+\delta)\tau} &= K_i^*(\tau) + \frac{\alpha}{\beta\delta} \\
\iff \left[K_{j0} - \frac{[\alpha + \beta I_i^*(\tau)] e^{-(r+\delta)\tau}}{\beta(r+2\delta)} + \frac{\alpha}{\beta\delta}\right] e^{-\delta\tau} + \frac{\alpha + \beta I_i^*(\tau)}{\beta(r+2\delta)} &= K_i^*(\tau) + \frac{\alpha}{\beta\delta} \\
\iff \left(K_{j0} + \frac{\alpha}{\beta\delta}\right) e^{-\delta\tau} + \frac{\alpha + \beta I_i^*(\tau)}{\beta(r+2\delta)} (1 - e^{-(r+2\delta)\tau}) &= K_i^*(\tau) + \frac{\alpha}{\beta\delta}. \quad (122)
\end{aligned}
$$

To see that there exists a real number τ that satisfies the equation above, observe that upon setting $\tau = 0$ in the two sides of the equation above we obtain

$$K_{j0} > K_{i0}. \quad (123)$$

On the other hand, let τ be large enough so that $(K_{j0} + \frac{\alpha}{\beta\delta})e^{-\delta\tau} + \frac{\alpha+\beta I_i^*(\tau)}{\beta(r+2\delta)}(1 - e^{-(r+2\delta)\tau}) \sim \frac{\alpha+\beta\delta K_{i\infty}}{\beta(r+2\delta)}$. From equation (122) we then obtain

$$\frac{\alpha + \beta\delta K_{i\infty}}{\beta(r+2\delta)} < K_{i\infty} + \frac{\alpha}{\beta\delta}. \quad (124)$$

Since both sides of the equation (122) are continuous, there exists a smallest date τ such that the equation is satisfied.

Now let us show that for this very value of τ we have: $K_i^*(t) < \hat{K}_j(t,\tau)$ for all $t \in [0,\tau)$ so that the process is admissible.

Assume by way of a contradiction that there exists $t < \tau$ such that $\hat{K}_j(t,\tau) \leq K_i^*(t)$. Since $K_{i0} < K_{j0}$, this implies that there is a date $t' \leq t$ such that $\hat{K}_j(t',\tau) = K_i^*(t')$. We consider two cases in turn.

- $K_{i\infty} < K_{i0}$

This implies that $\dot{K}_i(t) < 0$. In particular, we have $\dot{K}_i(\underline{t}) < 0$. Thus, $K_{i\infty} < K_i(\underline{t}) < K_{i0} < K_{j0}$. But $I_j(\underline{t},\underline{t}) = I_i(\underline{t}) = \delta K_{i\infty} > 0$. As $I_j(t,\underline{t}) = (D_1 e^{(r+\delta)t} - \alpha)/\beta$, this implies that $D_1 > 0$. Moreover, we must have $D_2 > 0$. Otherwise, as $K_j(t,\underline{t}) = (D_1/(\beta(r+2\delta)))e^{(r+\delta)t} + D_2 e^{-\delta t}$, K_j would be increasing and it would be impossible to satisfy the condition $K_i(\underline{t}) = K_j(\underline{t},\underline{t})$.

Assume that there is a date $t' < \underline{t}$ such that $K_i(t') = K_j(t', \underline{t})$ (and consider the first such date if there are more than one). Necessarily, $\dot{K}_j(t', \underline{t}) < \dot{K}_i(t')$, which implies that $I_j(t', \underline{t}) < \delta K_{i\infty}$. Since I_j is monotonic and $I_j(\underline{t}, \underline{t}) = \delta K_{i\infty}$, this implies that $I_j(t, \underline{t}) < \delta K_{i\infty}$ for all $t \in [t', \underline{t}[$. But then we have

$$K_i(\underline{t}) = \int_{t'}^{\underline{t}} I_i e^{-\delta t} dt + e^{-(\underline{t}-t')\delta} K_i(t') > K_j(\underline{t}, \underline{t}) = \int_{t'}^{\underline{t}} I_j e^{-\delta t} dt + e^{-(\underline{t}-t')\delta} K_i(t') \qquad (125)$$

which is impossible by assumption.

- $K_{i0} < K_{i\infty}$

In that case $\dot{K}_i(t) > 0$. Notice that we also have $\dot{K}_i(\underline{t}) > 0$, which implies that $K_i'(\underline{t}) > 0$. We can show as above that $D_1 > 0$.

Suppose that $K_{j0} > K_{i\infty}$. We must then have $D_2 > 0$, otherwise, K_j would be increasing. Moreover, it must be that $K_j'(0, \underline{t}) < 0$ (this is because, K_j is convex, and would be always increasing otherwise). Since $\dot{K}_j(\underline{t}, \underline{t}) > 0$, K_j has a local minimum between 0 and $\underline{t}$. Suppose that $K_j(t', \underline{t}) = K_i(t')$ for some date $t' \in]0, \underline{t}]$. Then using the same argument as above we can show that we run into a contradiction. So, we must have $K_i(t) < K_j(t, \underline{t})$ for all $t \in [0, \underline{t}[$.

Alternatively, suppose that $K_{j0} < K_{i\infty}$. The same argument as above applies (but it is possible that $\dot{K}_j(0, \underline{t}) > 0$).

■

Proof of Theorem 1

Proof. It is clear that by construction firm i's decision maximizes its objective (it is actually a *dominant* strategy). Consider firm j's problem.

Set $\underline{t} = \tau$, where τ is obtained from Lemma 6. Assume that firms choose $I_i = I_i^*$ and I_j as in Lemma 5 for $t \leq \tau$, and $I_j = I_i^*$ for $\tau \leq t$. Now, consider any alternative path satisfying the following conditions

$$\dot{K}_j(t) = I_j(t) - \delta K_j(t), \; K_j(0) = K_{j0},$$
$$K_i^*(t) \leq K_j(t).$$

Moreover, set $\lambda_j^*(t) = \lambda_j(t)$ where λ_j is the value obtained in Lemma 5 for $t < \underline{t} = \tau$ and $\lambda_j^*(t) = \lambda_i^*(t)$ for $t \geq \tau$.

Now, for any $T > \tau$, as $K_i^*(t) \leq K_j(t)$ and because firm j *can only sell a quantiy* K_i^* as the goods are complementary, we have

$$\int_0^T e^{-rt}\Big(w_j K_i^*(t) - C\big(I_j^*(t)\big)\Big)dt - \int_0^T e^{-rt}\Big(w_j K_i^*(t) - C\big(I_j(t)\big)\Big)dt$$

$$= \int_0^T \Big(e^{-rt} w_j K_i^*(t) - e^{-rt} C\big(I_j^*(t)\big) + \lambda_j^*(t)(I_j^*(t) - \delta K_j^*(t))\Big)dt$$

$$- \int_0^T \Big(e^{-rt} w_j K_i^*(t) - e^{-rt} C\big(I_j(t)\big) + \lambda_j^*(t)(I_j(t) - \delta K_j(t))\Big)dt$$

$$+ \int_0^T \lambda_j^*(t)\big(\dot{K}_j(t) - \dot{K}_j^*(t)\big)dt$$

Observe that $e^{-rt}\Big(w_j K_i^*(t) - C\big(I_j(t)\big)\Big) + \lambda_j^*(t)(I_j(t) - \delta K_j(t))$ is concave with respect to $(I_j(t), K_j(t))$. Thus it holds that

$$e^{-rt}\Big(w_j K_i^*(t)) - C\big(I_j^*(t)\big)\Big) + \lambda_j^*(t)\big(I_j^*(t) - \delta K_j^*(t)\big) - e^{-rt}\Big(w_j K_i^*(t) - C\big(I_j(t)\big)\Big) - \lambda_j^*(t)\big(I_j(t) - \delta K_j(t)\big)$$

$$\geq \delta \lambda_j^*(t)\big(K_j(t) - K_j^*(t)\big) - \Big(-e^{-rt} C'\big(I_j^*(t)\big) + \lambda_j^*(t)\Big)(I_j(t) - I_j^*(t)).$$

It follows that

$$\int_0^T e^{-rt}\Big(w_j K_i^*(t) - C\big(I_j^*(t)\big)\Big)dt - \int_0^T e^{-rt}\Big(w_j K_i^*(t) - C\big(I_j(t)\big)\Big)dt \geq$$

$$\int_0^T \Big(\delta \lambda_j^*(t)\big(K_j(t) - K_j^*(t)\big) - \big(-e^{-rt} C'\big(I_j^*(t)\big) + \lambda_j^*(t)\big)\big(I_j(t) - I_j^*(t)\big)\Big)dt$$

$$+ \int_0^T \lambda_j^*(t)\big(\dot{K}_j(t) - \dot{K}_j^*(t)\big)dt.$$

By definition of I_j^* and λ_j^*

$$\lambda_j^*(t) = e^{-rt} C'(I_j(t)), \ \forall t < \tau$$

$$\lambda_j^*(t) = \lambda_i^*(t) = e^{-rt} C'(I_j^*(t)) = e^{-rt} C'(I_i^*(t)), \ \forall t, \ \tau \leq t$$

$$\dot{\lambda}_j^*(t) = \dot{\lambda}_j(t) = \delta \lambda_j(t), \ \forall t < \tau$$

$$\dot{\lambda}_j^*(t) = \dot{\lambda}_i^*(t) = \delta \lambda_j^*(t) \ \forall t, \ \tau \leq t.$$

Then, the preceding inequality boils down to

$$\int_0^T e^{-rt}\Big(w_jK_i^*(t) - C\big(I_j^*(t)\big)\Big)dt - \int_0^T e^{-rt}\Big(w_jK_i^*(t) - C\big(I_j(t)\big)\Big)dt \geq$$
$$\int_0^\tau \Big(\dot{\lambda}_j^*(t)\big(K_j(t) - K_j^*(t)\big) + \lambda_j^*(t)\big(\dot{K}_j(t) - \dot{K}_j^*(t)\big)\Big)\,dt + \int_\tau^T \Big(\dot{\lambda}_i^*(t)\big(K_j(t) - K_i^*(t)\big) + \lambda_i^*(t)\big(\dot{K}_j(t) - \dot{K}_i^*(t)\big)\Big)\,dt$$
$$= \int_0^\tau \frac{d}{dt}\big(\lambda_j^*(t)(K_j(t) - K_j^*(t)\big)dt + \int_\tau^T \frac{d}{dt}\big(\lambda_i^*(t)(K_j(t) - K_i^*(t)\big)dt$$
$$= \lambda_j^*(\tau)\big((K_j(\tau) - K_i^*(\tau)\big) + \lambda_i^*(T)\big(K_j(T) - K_i^*(T)\big) - \lambda_i^*(\tau)\big(K_j(\tau) - K_i^*(\tau)\big)$$
$$= \lambda_i^*(T)K_j(T) - \lambda_i^*(T)K_i^*(T)$$

where we have used the fact that $I_j^*(\tau) = I_i^*(\tau)$ and thus the equality

$$\lambda_j^*(\tau) = e^{-rt}C'\big(I_j^*(\tau)\big) = e^{-rt}C'\big(I_i^*(\tau)\big)\lambda_i^*(\tau). \tag{126}$$

Since $\lim_{T\to+\infty} \lambda_i^*(T) = 0$, we get

$$\int_0^T e^{-rt}\left(w_jK_i^*(t) - C\big(I_j^*(t)\big)\right)dt - \int_0^T e^{-rt}\left(w_jK_i^*(t) - C\big(I_j(t)\big)\right)dt$$
$$\geq \liminf_{T\to\infty}\big(\lambda_i^*(T)K_j(T) - \lambda_i^*(T)K_i^*(T)\big) = \liminf_{T\to\infty} \lambda_i^*(T)K_j(T) \geq 0.$$

This proves the Theorem. ■

Case where $w_j < w_i$

Proof of Theorem 2

Proof.

• First step

Firm i's problem

We conjecture that once capacities meet, firm i's decisions will be similar to that of firm j. So what matters is the determination of firm i's decisions before the date at which capacities meet as well as this date itself.

To proceed, recall that firm i's problem is

$$\max_{I_i,\underline{t}_i} \int_0^{\underline{t}_i} \big(w_iK_i - C(I_i)\big)e^{-rt}dt + \int_{\underline{t}_i}^\infty \big(w_iK_j^* - C(I_i)\big)e^{-rt}dt \tag{127}$$

where

$$\dot{K}_i = I_i - \delta K_i,\ K_i(0) = K_{i0},$$
$$K_j^*(t) \le K_i(t),$$
$$K_i(\underline{t}_i) = K_j^*(\underline{t}_i).$$

Assume that $\underline{t}_i$ is given and consider the optimal decisions for firm i over the interval $[0, \underline{t}]$. To find these optimal decisions, let us introduce the following Hamiltonian

$$H(I_i, K_i, \lambda_i) = e^{-rt}\big(w_i K_i - \alpha I_i - \frac{\beta_i}{2} I_i^2\big) + \lambda_i\big(I_i - \delta K_i\big).$$

In our candidate equilibrium, the following first-order conditions must be satisfied

$$\lambda_i = e^{-rt}(\alpha + \beta I_i),$$
$$\dot{\lambda}_i = -e^{-rt} w_i + \delta\lambda_i.$$

We thus deduce that

$$I_i = \frac{\lambda_i e^{rt} - \alpha}{\beta}.$$

Using $\dot{K}_i = I_i - \delta K_i$, we get

$$\ddot{K}_i = \dot{I}_i - \delta\dot{K}_i = \frac{e^{rt}}{\beta}\big(\dot{\lambda}_i + r\lambda_i\big) - \delta\dot{K}_i.$$

After a few algebra, we obtain

$$\ddot{K}_i(t) - r\dot{K}_i(t) - \delta(r+\delta)K_i(t) + \frac{w_i - \alpha(r+\delta)}{\beta} = 0.$$

The solution to the above equation is

$$K_i(t) = K_{i\infty} + D_{1i}e^{-\delta t} + D_{2i}e^{(r+\delta)t},$$

where

$$K_{i\infty} = \frac{w_i - \alpha(r+\delta)}{\beta\delta(\delta+r)}, \tag{128}$$

$$D_{1i} = \frac{(K_{i0} - K_{i\infty})e^{(r+\delta)\underline{t}_i} - (K_j^*(\underline{t}_i) - K_{i\infty})}{e^{(r+\delta)\underline{t}_i} - e^{-\delta\underline{t}_i}}, \tag{129}$$

$$D_{2i} = \frac{K_j^*(\underline{t}_i) - K_{i\infty} - (K_{i0} - K_{i\infty})e^{-\delta\underline{t}_i}}{e^{(r+\delta)\underline{t}_i} - e^{-\delta\underline{t}_i}}, \tag{130}$$

where we have used the conditions $K_i(0) = K_{i0}$ and $K_i(\underline{t_i}) = K_j^*(\underline{t_i})$.

We conjecture that the optimal value of $\underline{t}_i$ must be such that $I_i(\underline{t}_i) = I_j(\underline{t}_i)$. But as $I_i(\underline{t}_i) = \dot{K}_i(\underline{t}_i) + \delta K_i(\underline{t}_i)$, using the expressions of $K_i(.)$ we get

$$I_i(\underline{t}_i) = (r+2\delta)e^{(r+\delta)\underline{t}_i}D_{2i} + \delta K_{i\infty}, \tag{131}$$

or

$$D_{2i} = \frac{I_j(\underline{t}_i) - \delta K_{i\infty}}{r+2\delta}e^{-(r+\delta)\underline{t}_i}. \tag{132}$$

Using equations (130) and (132), we obtain

$$K_j^*(\underline{t}_i) = \frac{I_j(\underline{t}_i) - \delta K_{i\infty}}{r+2\delta}e^{-(r+\delta)\underline{t}_i}\left(e^{(r+\delta)\underline{t}_i} - e^{-\delta\underline{t}_i}\right) + K_{i\infty} + (K_{i0} - K_{i\infty})e^{-\delta\underline{t}_i}. \tag{133}$$

Firm j's problem

First consider the decisions made by firm j from a date $\underline{t}_j$ on, where the capacities meet. The solution to firm j's problem is similar to that given in Lemma 3, i.e., we obtain

$$I_j(t) = \delta K_{j\infty}, \tag{134}$$

$$K_j(t) = K_{j\infty} + (K_j(\underline{t}_j) - K_{j\infty})e^{-\delta(t-\underline{t}_j)}, \tag{135}$$

$$K_{j\infty} = \frac{w_j - (r+\delta)\alpha}{\beta\delta(r+\delta)}. \tag{136}$$

Now, for $\underline{t}_j$ given, consider the first part of firm j's problem, i.e.,

$$\max_{I_j(\cdot)} \int_0^{\underline{t}_j} \left(w_j K_i^*(t) - C(I_j(t))\right)e^{-rt}dt \tag{137}$$

$$\text{s.t.: } \dot{K}_j = I_j(t) - \delta K_j(t),\ K_j(0) = K_{j0}, \tag{138}$$

$$K_j(\underline{t}_j) = K_i^*(\underline{t}_j) \tag{139}$$

$$K_i^*(t) < K_j(t). \tag{140}$$

We can rely on Lemma 5 to solve that problem. Let $\hat{I}_j(t,\underline{t}_j)$ and $\hat{K}_j(t,\underline{t}_j)$ be the solution of the problem when $\underline{t}_j$ is given. Then we have for all $t < \underline{t}$

$$\hat{I}_j(t,\underline{t}_j) = \frac{D_{1j}e^{(r+\delta)t} - \alpha}{\beta}, \tag{141}$$

$$\hat{K}_j(t,\underline{t}_j) = D_{2j}e^{-\delta t} + \frac{D_{1j}}{\beta(r+2\delta)}e^{(r+\delta)t} - \frac{\alpha}{\beta\delta}, \tag{142}$$

where

$$D_{1j} = \beta(r+2\delta)\frac{\left(K_{j0} + \frac{\alpha}{\beta\delta}\right)e^{-\delta\underline{t}_j} - \frac{\alpha}{\beta\delta} - K_i^*(\underline{t}_j)}{e^{-\delta\underline{t}_j} - e^{(r+\delta)\underline{t}_j}}, \tag{143}$$

$$D_{2j} = \frac{K_i^*(\underline{t}_j) + \frac{\alpha}{\beta\delta} - \left(K_{j0} + \frac{\alpha}{\beta\delta}\right)e^{(r+\delta)\underline{t}_j}}{e^{-\delta\underline{t}_j} - e^{(r+\delta)\underline{t}_j}}. \tag{144}$$

We conjecture again that in equilibrium the following condition must be satisfied (otherwise the date $\underline{t}_j$ would not be optimal)

$$\hat{I}_j(\underline{t}_j,\underline{t}_j) = I_j(\underline{t}_j) = \delta K_{j\infty}. \tag{145}$$

Thus using (141) we obtain

$$D_{1j} = (\beta\delta K_{j\infty} + \alpha)e^{-(r+\delta)\underline{t}_j}. \tag{146}$$

Using the definition of $\hat{K}_j$, we must also have

$$D_{2j} = K_{j0} - \frac{D_{1j}}{\beta(r+2\delta)} + \frac{\alpha}{\beta\delta}. \tag{147}$$

Thus

$$\hat{K}_j(t,\underline{t}_j) = \left[K_{j0} - \frac{(\beta\delta K_{j\infty} + \alpha)}{\beta(r+2\delta)}e^{-(r+\delta)\underline{t}_j} + \frac{\alpha}{\beta\delta}\right]e^{-rt} + \frac{(\beta\delta K_{j\infty} + \alpha)}{\beta(r+2\delta)}e^{-(r+\delta)\underline{t}_j}e^{(r+\delta)t} - \frac{\alpha}{\beta\delta}. \tag{148}$$

Notice, however, that using (143) and (146) we have

$$(\beta\delta K_{j\infty}+\alpha)e^{-(r+\delta)\underline{t}_j}=\beta(r+2\delta)\frac{\left(K_{j0}+\frac{\alpha}{\beta\delta}\right)e^{-\delta\underline{t}_j}-\frac{\alpha}{\beta\delta}-K_i^*(\underline{t}_j)}{e^{-\delta\underline{t}_j}-e^{(r+\delta)\underline{t}_j}}. \tag{149}$$

Therefore we deduce that

$$K_i^*(\underline{t}_j)=-\frac{\alpha}{\beta\delta}+(K_{j0}+\frac{\alpha}{\beta\delta})e^{-\delta\underline{t}_j}+\left(e^{(r+\delta)\underline{t}_j}-e^{-\delta\underline{t}_j}\right)\frac{(\beta\delta K_{j\infty}+\alpha)}{\beta(r+2\delta)}e^{-(r+\delta)\underline{t}_j}. \tag{150}$$

• Second step

We now look for an equilibrium, that is, decisions that are consistent. More precisely, we look for a value $\underline{t}$, such that

$$\underline{t}=\underline{t}_i=\underline{t}_j. \tag{151}$$

Moreover, in equilibrium we must check that $K_i^*(\underline{t})=K_j^*(\underline{t})$. Thus, using (133) and (150), $\underline{t}$ must solve the following equation

$$\begin{aligned}-\frac{\alpha}{\beta\delta}+(K_{j0}+\frac{\alpha}{\beta\delta})e^{-\delta\underline{t}_j}+\left(e^{(r+\delta)\underline{t}_j}-e^{-\delta\underline{t}_j}\right)\frac{(\beta\delta K_{j\infty}+\alpha)}{\beta(r+2\delta)}e^{-(r+\delta)\underline{t}_j}=\\ \frac{\delta K_{j\infty}-\delta K_{i\infty}}{r+2\delta}e^{-(r+\delta)\underline{t}}\left(e^{(r+\delta)\underline{t}}-e^{-\delta\underline{t}}\right)+K_{i\infty}+(K_{i0}-K_{i\infty})e^{-\delta\underline{t}}.\end{aligned} \tag{152}$$

To show that there exists a value of $\underline{t}$ that solves the above equation, let us introduce the function $\varphi:\mathbb{R}_+\to\mathbb{R}$ defined by

$$\begin{aligned}\varphi(\underline{t})=-\frac{\alpha}{\beta\delta}+(K_{j0}+\frac{\alpha}{\beta\delta})e^{-\delta\underline{t}_j}+\left(e^{(r+\delta)\underline{t}_j}-e^{-\delta\underline{t}_j}\right)\frac{(\beta\delta K_{j\infty}+\alpha)}{\beta(r+2\delta)}e^{-(r+\delta)\underline{t}_j}\\ -\left(\frac{\delta K_{j\infty}-\delta K_{i\infty}}{r+2\delta}e^{-(r+\delta)\underline{t}}\left(e^{(r+\delta)\underline{t}}-e^{-\delta\underline{t}}\right)+K_{i\infty}+(K_{i0}-K_{i\infty})e^{-\delta\underline{t}}\right).\end{aligned} \tag{153}$$

Observe that

$$\lim_{\underline{t}\to 0+}\varphi(\underline{t})=\left(K_{j0}+\frac{\alpha}{\beta\delta}\right)-\frac{\alpha}{\beta\delta}-K_{i\infty}-(K_{i0}-K_{i\infty})=K_{j0}-K_{i0}>0. \tag{154}$$

Moreover,

$$\lim_{\underline{t}\to\infty}\varphi(\underline{t}) = -\frac{\alpha}{\beta\delta}+\frac{\delta\beta K_{j\infty}+\alpha}{\beta(r+2\delta)}-\delta\frac{K_{j\infty}-K_{i\infty}}{r+2\delta}-K_{i\infty} \quad (155)$$

$$= -\frac{\alpha}{\beta\delta}+\frac{\alpha}{\beta(r+2\delta)}+\frac{\delta K_{i\infty}}{r+2\delta}-K_{i\infty} \quad (156)$$

$$= \frac{\alpha}{\beta}\left(\frac{1}{r+2\delta}-\frac{1}{\delta}\right)+K_{i\infty}\left(-1+\frac{\delta}{r+2\delta}\right) \quad (157)$$

$$= -\frac{r+\delta}{r+2\delta}\left(\frac{\alpha}{\beta\delta}+K_{i\infty}\right)<0. \quad (158)$$

By continuity, there thus exists $\underline{t}$ such that $\varphi(\underline{t})=0$.

- Third step

We must now check that the inequality $K_i^*(t)<K_j^*(t)$ holds for all t in $[0,\underline{t}]$.

To show this, recall that for all $t\leq\underline{t}$ we have found that

$$K_i^*(t)=K_{i\infty}+D_{1i}e^{-\delta t}+D_{i2}e^{(r+\delta)t}, \quad (159)$$

$$K_j^*(t)=-\frac{\alpha}{\beta\delta}+\frac{D_{j1}}{\beta(r+2\delta)}e^{(r+\delta)t}+D_{j2}e^{-\delta t}. \quad (160)$$

From (146) we can see that $D_{j1}>0$.

Moreover from (132), we have

$$D_{i2}=\frac{I_j(\underline{t})-\delta K_{i\infty}}{r+2\delta}e^{-(r+\delta)\underline{t}}=\frac{\delta K_{j\infty}-\delta K_{i\infty}}{r+2\delta}e^{-(r+\delta)\underline{t}}<0.$$

Notice that we cannot have both $D_{i1}>0$ and $D_{j2}<0$. Otherwise, we would have that K_i is always decreasing, whereas K_j is always increasing. Thus, it would be impossible to satisfy the condition $K_i(\underline{t})=K_j(\underline{t})$.

We next consider three cases in turn.

- $D_{j2}<0$

Then $D_{i1}<0$ and K_i is concave and K_j is increasing. Since $K_j'(0)>0\Rightarrow K_j'(\underline{t})>0$ (and then $K_{j\infty}>K_{j0}>K_{i0}$). It is then impossible that $K_i'(0)<0$ (K_i would be always decreasing, whereas K_j would be always increasing). Thus $K_i'(0)>0$ and $K_i(\cdot)$ is increasing.

Assume that there is a date t' at which $K_j(t') = K_i(t')$. Then it must be that $K_i'(t') > K_j'(t')$, and thus that $I_i(t') > I_j(t')$ since $\dot{K}_h = I_h - \delta K_h$ $(h = i, j)$.

Moreover,

$$\dot{I}_i(t) = (r + 2\delta)(r + \delta)D_{i2}e^{(r+\delta)t}. \tag{161}$$

We can see that $I_i' < 0$. This implies that I_i is decreasing and goes to $\delta K_{j\infty}$ while I_j is increasing and goes to $\delta K_{j\infty}$. But then, we can see that it is impossible that $K_i(\underline{t}) = K_j(\underline{t})$ (see the inequality (125)).

- $D_{i1} > 0$

Then $D_{j2} > 0$. Moreover, K_i is decreasing and K_j is convex. It is impossible that $K_j'(0) > 0$ (otherwise, we would have K_i decreasing and K_j increasing). It thus must be that $K_j'(0) < 0$. It is also impossible that $K_j'(\underline{t}) > 0$ (since $K_j'(\underline{t}) = K_i'(\underline{t}) < 0$ as K_i is decreasing). So K_j is also decreasing $(K_{j\infty} < K_{i0} < K_{j0})$.

Suppose that there is a date t' such that $K_i(t') = K_j(t')$. Then we have $I_i(t') > I_j(t')$ (because it must be that $K_i'(t') > K_j'(t')$). Recall that

$$\dot{I}_i(t) = (r + 2\delta)(r + \delta)D_{i2}e^{(r+\delta)t}. \tag{162}$$

As $\dot{I}_i < 0$ and relying on an argument used above, we conclude that $K_i(\underline{t}) > K_j(\underline{t})$ which is impossible.

- $D_{i1} < 0$, $D_{j2} > 0$

We can check that K_i is concave and K_j is convex.

Suppose that $K_j'(\underline{t}) > 0$ $(K_{j\infty} > K_{j0} > K_{i0})$. Then $K_i'(\underline{t}) > 0$ which implies that $K_i'(0) > 0$. Suppose also that $K_j'(0) > 0$. Then there is no date t' in $]0, \underline{t}[$ such that $K_i(t') = K_j(t')$ because as K_i is concave increasing and K_j is convex increasing it would be impossible to satisfy the condition $K_i(\underline{t}) = K_j(\underline{t})$.

So suppose that $K_j'(0) < 0$. Hence K_j has a global minimum at a date included in $]0, \underline{t}[$. It is, however, impossible that $K_j(t) = K_i(t)$ for a date t such that $K_j'(t) > 0$. Thus, if

$K_j(t) = K_i(t)$ it must be at a date t' such that $K_j'(t') < 0$, and $I_i(t') > I_j(t')$. Because $I_i' < 0$, as I_j is increasing and both investment rates are equal at date $\underline{t}$, we cannot have $K_i(\underline{t}) = K_j(\underline{t})$.

• Fourth step

In the previous steps we have found candidate equilibrium decisions by solving necessary conditions. We now check that these conditions are also sufficient.

Sufficient conditions for firm i

We distinguish two cases, depending on whether firm i's alternative investment policy I_i is such that its capacity meets firm j's before or after $\underline{t}$.

Remark 1 *Notice that from Lemmata 2 and 4 we know that whenever firm i's capacity meets firm j's it is optimal to choose firm j's investment policy. It is therefore sufficient to only consider alternative policies that satisfy this property.*

— Firm i's capacity meets firm j's at date $\underline{t}' < \underline{t}$.

Let us compare firm i's payoffs (and recall that $K_{i0} < K_{j0}$). We have

$$\int_0^{\underline{t}'} \left\{e^{-rt}\big(w_i K_i^* - C(I_i^*)\big) + \lambda_i\big(I_i^* - \delta K_i^*\big)\right\} dt + \int_{\underline{t}'}^{\underline{t}} \left\{e^{-rt}\big(w_i K_i^* - C(I_i^*)\big) + \lambda_i\big(I_i^* - \delta K_i^*\big)\right\} dt$$

$$- \int_0^{\underline{t}'} \underbrace{\left\{e^{-rt}\big(w_i K_i - C(I_i)\big) + \lambda_i\big(I_i - \delta K_i\big)\right\}}_{\text{as } K_i(t) < K_j^*(t), \forall t < \underline{t}} dt - \underbrace{\int_{\underline{t}'}^{\underline{t}} \left\{e^{-rt}\big(w_i K_j^* - C(I_j^*)\big) + \lambda_i\big(I_j^* - \delta K_j^*\big)\right\} dt}_{\text{See remark 1}}$$

$$+ \int_0^{\underline{t}} \lambda_i\big(\dot{K}_i - \dot{K}_i^*\big) dt$$

$$\underbrace{\geq}_{\text{by concavity of the Hamiltonian}} \int_0^{\underline{t}'} \left\{ (e^{-rt} w_i - \delta\lambda_i)(K_i^* - K_i) + \underbrace{\big(\lambda_i - e^{-rt} C'(I_i^*)\big)}_{=0}(I_i^* - I_i) \right\} dt$$

$$+ \int_{\underline{t}'}^{\underline{t}} (e^{-rt} w_i - \delta\lambda_i)(K_i^* - K_j^*) dt$$

$$+ \int_{\underline{t}'}^{\underline{t}} \underbrace{\big(\lambda_i - e^{-rt} C'(I_i^*)\big)}_{=0}(I_i^* - I_i) dt + \int_0^{\underline{t}} \lambda_i\big(\dot{K}_i - \dot{K}_i^*\big) dt$$

$$\underbrace{=}_{\text{as } \dot{\lambda}_i = -e^{-rt} w_i + \delta\lambda_i} \int_0^{\underline{t}} \dot{\lambda}_i\big(K_i - K_i^*\big) dt + \int_0^{\underline{t}} \lambda_i\big(\dot{K}_i - \dot{K}_i^*\big) dt$$

$$= \lambda_i(\underline{t})(K_i(\underline{t}) - K_i^*(\underline{t})) - \lambda_i(\underline{t}')(K_i(\underline{t}') - K_i^*(\underline{t})) + \lambda_i(\underline{t}')(K_i(\underline{t}') - K_i^*(\underline{t})) - \lambda_i(0)(K_{i0} - K_{i0})$$

$$= 0 \quad (163)$$

where the last line stems from the fact that $K_i(\underline{t}) = K_j^*(\underline{t}) = K_i^*(\underline{t})$.

— Firm i's capacity meets firm j's at date $\underline{t}' > \underline{t}$.

$$\begin{aligned}
&\int_0^{\underline{t}} \left\{e^{-rt}\big(w_i K_i^* - C(I_i^*)\big) + \lambda_i\big(I_i^* - \delta K_i^*\big)\right\} dt + \int_{\underline{t}}^{\underline{t}'} \left\{e^{-rt}\big(w_i K_j^* - C(I_j^*)\big) + \lambda_i\big(I_j^* - \delta K_j^*\big)\right\} dt \\
&\quad - \int_0^{\underline{t}'} \underbrace{\left\{e^{-rt}\big(w_i K_i - C(I_i)\big) + \lambda_i\big(I_i - \delta K_i\big)\right\}}_{\text{as } K_i(t)<K_j^*(t),\ \forall t<\underline{t}} dt + \int_0^{\underline{t}'} \lambda_i\big(\dot{K}_i(t) - \dot{K}_i^*(t)\big) dt \\
&\underbrace{\geq}_{\text{by concavity of the Hamiltonian}} \int_0^{\underline{t}'} \left\{ (e^{-rt} w_i - \delta\lambda_i)(K_i^* - K_i) + \underbrace{\big(\lambda_i - e^{-rt} C'(I_i^*)\big)}_{=0}(I_i^* - I_i) \right\} dt \\
&\quad + \int_0^{\underline{t}'} \lambda_i\big(\dot{K}_i - \dot{K}_i^*\big) dt \\
&= \underbrace{\int_0^{\underline{t}} \dot{\lambda}_i\big(K_i - K_i^*\big) dt}_{\text{as } \dot{\lambda}_i = -e^{-rt} w_i + \delta\lambda_i} + \underbrace{\int_{\underline{t}}^{\underline{t}'} \big(e^{-rt}(w_i - w_j) - \dot{\lambda}_j\big)\big(K_i^* - K_i\big) dt}_{\text{as } \delta\lambda_i = \delta\lambda_j = w_j e^{-rt} + \dot{\lambda}_j} + \int_0^{\underline{t}'} \lambda_i\big(\dot{K}_i - \dot{K}_i^*\big) dt \\
&\underbrace{\geq}_{\text{as } K_i<K_i^* \forall t<\underline{t}'} \lambda_i(\underline{t}')(K_i(\underline{t}') - K_i^*(\underline{t}')) - \lambda_i(\underline{t})(K_i(\underline{t}) - K_i^*(\underline{t})) + \lambda_i(\underline{t})(K_i(\underline{t}) - K_i^*(\underline{t})) - \lambda_i(0)(K_{i0} - K_{i0}) \\
&= 0 \qquad (164)
\end{aligned}$$

where the last line stems from the fact that $K_i(\underline{t}') = K_j^*(\underline{t}') = K_i^*(\underline{t}')$.

Sufficient conditions for firm j

Remark 2 *Observe that from Lemma 2, whenever firm j's capacity meets firm i's, it will never choose an investment policy such that its capacity becomes higher than firm i's in the future.*

Again, we consider two cases.

— Firm j's capacity meets firm i's at date $\underline{t}' < \underline{t}$.

Let us compare firm j's payoffs (and recall that $K_{i0} < K_{j0}$). Let $T > \underline{t}$. We have

$$\int_0^{\underline{t}'} \left\{e^{-rt}\big(w_j K_i^* - C(I_j^*)\big) + \lambda_j\big(I_j^* - \delta K_j^*\big)\right\} dt + \int_{\underline{t}'}^{\underline{t}} \left\{e^{-rt}\big(w_j K_i^* - C(I_j^*)\big) + \lambda_j\big(I_j^* - \delta K_j^*\big)\right\} dt$$
$$+ \int_{\underline{t}}^{T} \left\{e^{-rt}\big(w_j K_j^* - C(I_j^*)\big) + \lambda_j\big(I_j^* - \delta K_j^*\big)\right\} dt$$
$$- \int_0^{\underline{t}'} \underbrace{\left\{e^{-rt}\big(w_j K_i^* - C(I_j)\big) + \lambda_j\big(I_j - \delta K_j\big)\right\}}_{\text{as } K_j(t) > K_i^*(t), \forall t < \underline{t}'} dt - \underbrace{\int_{\underline{t}'}^{T} \left\{e^{-rt}\big(w_j K_j - C(I_j)\big) + \lambda_j\big(I_j - \delta K_j\big)\right\} dt}_{\text{See remark 2 and Lemma 3}}$$
$$+ \int_0^T \lambda_j\big(\dot{K}_j - \dot{K}_j^*\big) dt$$
$$\underbrace{\geq}_{\text{by concavity of the Hamiltonian}} \int_0^{\underline{t}'} \left\{ \delta\lambda_j(K_j - K_j^*) + \underbrace{\big(\lambda_j - e^{-rt}C'(I_j^*)\big)}_{=0}(I_j^* - I_j) \right\} dt$$
$$+ \int_{\underline{t}'}^{T} (e^{-rt} w_j - \delta\lambda_j)(K_j^* - K_j) dt$$
$$+ \int_{\underline{t}'}^{T} \underbrace{\big(\lambda_j - e^{-rt}C'(I_j^*)\big)}_{=0}(I_j^* - I_j) dt + \int_0^T \lambda_j\big(\dot{K}_j - \dot{K}_j^*\big) dt$$
$$\underbrace{\geq}_{\text{as } K_j^* \geq K_j \forall t > \underline{t}'} \underbrace{\int_0^T \dot{\lambda}_j\big(K_j - K_j^*\big) dt}_{\text{as } \dot{\lambda}_j = \delta\lambda_j \forall t < \underline{t} \text{ and } \dot{\lambda}_j = -e^{-rt} w_j + \delta\lambda_j \forall t > \underline{t}} + \int_0^T \lambda_j\big(\dot{K}_j - \dot{K}_j^*\big) dt$$
$$= \lambda_j(T)(K_j(T) - K_j^*(T)) - \lambda_j(0)(K_{j0} - K_{j0}) \quad (165)$$

We obtain the sufficient condition by taking the limsup when T goes to ∞ (as K_j^* is bounded).

— Firm j's capacity meets firm i's at date $\underline{t}' > \underline{t}$.

Observe that whenever firm j's capacity meets firm i's then, firm j's best choice is to choose firm i's investment policy (which in turn is that of firm j in equilibrium). Based on this remark, we get

$$\int_0^{\underline{t}} \left\{e^{-rt}\big(w_j K_i^* - C(I_j^*)\big) + \lambda_j\big(I_j^* - \delta K_j^*\big)\right\} dt + \int_{\underline{t}}^{\underline{t}'} \left\{e^{-rt}\big(w_j K_j^* - C(I_j^*)\big) + \lambda_j\big(I_j^* - \delta K_j^*\big)\right\} dt$$

$$- \int_0^{\underline{t}'} \underbrace{\left\{e^{-rt}\big(w_j K_j - C(I_j)\big) + \lambda_j\big(I_j - \delta K_j\big)\right\}}_{\text{as } K_i^*(t) < K_j(t), \forall t < \underline{t}} dt + \int_0^{\underline{t}'} \lambda_j\big(\dot{K}_j - \dot{K}_j^*\big)$$

$$\underbrace{\geq}_{\text{by concavity of the Hamiltonian}} \int_0^{\underline{t}'} \left\{ (e^{-rt} w_j - \delta\lambda_j)(K_j^* - K_j) + \underbrace{\big(\lambda_j - e^{-rt} C'(I_j^*)\big)}_{=0} (I_j^* - I_j) \right\} dt$$

$$+ \int_0^{\underline{t}'} \lambda_j\big(\dot{K}_j - \dot{K}_j^*\big) dt$$

$$= \underbrace{\int_0^{\underline{t}'} \dot{\lambda}_j\big(K_j - K_j^*\big) dt}_{\text{as } \dot{\lambda}_j = -e^{-rt} w_j + \delta\lambda_j} + \int_0^{\underline{t}'} \lambda_j\big(\dot{K}_j - \dot{K}_j^*\big) dt$$

$$= \lambda_j(\underline{t}')(K_j(\underline{t}') - K_j^*(\underline{t}')) - \lambda_j(\underline{t})(K_j(\underline{t}) - K_j^*(\underline{t})) + \lambda_j(\underline{t})(K_j(\underline{t}) - K_j^*(\underline{t})) - \lambda_j(0)(K_{j0} - K_{j0})$$

$$= 0 \quad (166)$$

where the last line stems from the fact that $K_j(\underline{t}') = K_j^*(\underline{t}') = K_i^*(\underline{t}')$.

■

Non-existence

Proof of Proposition 3.

By way of contradiction, suppose that there is an open-loop equilibrium as described in the statement of the Proposition. Denote by I_h^* the equilibrium decision of manufacturer h, $h = i, j$, and by K_h^* the corresponding equilibrium time path for the production capacity. Without loss of generality, we can assume that $\underline{t} = 0$.[41] By assumption, the equilibrium time path for manufacturer j's capacity K_j^* satisfies $K_j^*(t) > 0$ for all $t \in [0, \bar{t}]$. Consider the following deviation strategy for manufacturer i. Let $\epsilon > 0$ and $t_\epsilon > 0$ with $t_\epsilon < \bar{t}$ such that for all t in $[0, t_\epsilon]$, $I_i^\epsilon(t) = I_i^*(t) - \epsilon = I_j^*(t) - \epsilon$ (since by assumption $I_i^* = I_j^*$), and $I_i^\epsilon(t) = I_j^*(t)$ elsewhere. Thus $K_i^\epsilon(t) = \int_0^t e^{\delta(s-t)}(I_j^*(s) - \epsilon)ds + e^{-\delta t} K_{j0} > 0$.[42] Recall that

[41] That is because, in a symmetric equilibrium we would have $K_i^*(\underline{t}) = K_j^*(\underline{t})$ and we can focus on the dynamics from date $\underline{t}$ on.

[42] The fact that $K_j^*(t) > 0$ on $[0, t_\epsilon]$ allows us to choose such an ϵ.

for $t \geq t_\epsilon$, $\dot{K}_i^\epsilon(t) = I_j^*(t) - \delta K_i^\epsilon(t)$. Then for $t \leq t_\epsilon$ we have

$$\begin{aligned} K_i^\epsilon(t) &= \int_0^t e^{\delta(s-t)} I_i^\epsilon(s)ds + e^{-\delta t}K_{j0} = \int_0^t e^{\delta(s-t)} I_j^*(s)ds - \int_0^t \epsilon e^{\delta(s-t)}ds + e^{-\delta t}K_{j0} \\ &= K_j^*(t) - \epsilon \frac{e^{-\delta t}}{\delta}[e^{\delta t} - 1]. \end{aligned}$$

And for $t \geq t_\epsilon$ it holds that

$$\begin{aligned} K_i^\epsilon(t) &= \int_0^t e^{\delta(s-t)} I_i^\epsilon(s)ds + e^{-\delta t}K_{j0} = \int_0^t e^{\delta(s-t)} I_j^*(s)ds - \int_0^{t_\epsilon} \epsilon e^{\delta(s-t)}ds + e^{-\delta t}K_{j0} \\ &= K_j^*(t) - \epsilon \frac{e^{-\delta t}}{\delta}[e^{\delta t_\epsilon} - 1]. \end{aligned}$$

Now, since $K_i^\epsilon(t) < K_j^*(t)$ for all t, it follows that manufacturer i's revenue is written $(1-\phi_i)(\hat{a} - 2bK_i^\epsilon(t))K_i^\epsilon(t) + \frac{w_i}{\phi_i}K_i^\epsilon(t)$ instead of $(1-\phi_i)\frac{(\hat{a}-bK_j^*(t))}{2}K_j^*(t) + \frac{w_i}{\phi_i}K_i^*(t)$ when $I_i^*(t) = I_j^*(t)$ for all t. Let us now compare the profit obtained by manufacturer i when $\epsilon > 0$ with its value when $\epsilon = 0$ (in that last case, both manufacturers use the same strategy). Deviating from manufacturer j's strategy is profitable whenever

$$\begin{aligned} &\int_0^\infty \left\{ (1-\phi_i)\left((\hat{a} - 2bK_i^\epsilon(t))K_i^\epsilon(t) - \frac{\hat{a} - bK_j^*(t)}{2}K_j^*(t) \right) + \frac{w_i}{\phi_i}(K_i^\epsilon(t) - K_i^*) \right\} e^{-rt}dt \\ &\qquad - \int_0^{t_\epsilon} \left(\alpha I_i^\epsilon(t) + \frac{\beta}{2}(I_i^\epsilon(t))^2 - \alpha I_i^*(t) - \frac{\beta}{2}(I_i^*(t))^2 \right) e^{-rt}dt > 0 \end{aligned}$$

for some $\epsilon > 0$.

Let us then compute $\hat{a}K_i^\epsilon(t) - 2b(K_i^\epsilon(t))^2$. For $t \leq t_\epsilon$, we have:

$$\begin{aligned} \hat{a}K_i^\epsilon(t) - 2b(K_i^\epsilon(t))^2 &= \hat{a}K_j^*(t) - \hat{a}\epsilon\frac{e^{-\delta t}}{\delta}[e^{\delta t} - 1] \\ &- 2b\left[(K_j^*(t))^2 - 2\epsilon K_j^*(t)\frac{e^{-\delta t}}{\delta}[e^{\delta t} - 1] + \epsilon^2 \frac{e^{-2\delta t}}{\delta^2}[e^{\delta t} - 1]^2 \right] \end{aligned}$$

and for $t \geq t_\epsilon$ we get

$$\begin{aligned} \hat{a}K_i^\epsilon(t) - 2b(K_i^\epsilon(t))^2 &= \hat{a}K_j^*(t) - \hat{a}\epsilon\frac{e^{-\delta t}}{\delta}[e^{\delta t_\epsilon} - 1] \\ &- 2b\left[(K_j^*(t))^2 - 2\epsilon K_j^*(t)\frac{e^{-\delta t}}{\delta}[e^{t_\epsilon} - 1] + \epsilon^2 \frac{e^{-2\delta t}}{\delta^2}[e^{\delta t_\epsilon} - 1]^2 \right]. \end{aligned}$$

In addition, for $t \leq t_\epsilon$ we have

$$\frac{w_i}{\phi_i}\big(K_i^\epsilon(t) - K_i^*(t)\big) = \frac{w_i}{\phi_i}\left(-\epsilon\frac{e^{-\delta t}}{\delta}[e^{\delta t} - 1].\right)$$

and for $t > t_\epsilon$ we have

$$\frac{w_i}{\phi_i}\big(K_i^\epsilon(t) - K_i^*(t)\big) = \frac{w_i}{\phi_i}\left(-\epsilon\frac{e^{-\delta t}}{\delta}[e^{\delta t_\epsilon} - 1]\right).$$

For $t \leq t_\epsilon$, we then get

$$\begin{aligned}(1-\phi_i)\left(\hat{a}K_i^\epsilon(t) - 2b(K_i^\epsilon(t))^2 - \frac{\hat{a}-bK_j^*(t)}{2}K_j^*(t))\right) + \frac{w_i}{\phi_i}\big(K_i^\epsilon(t) - K_i^*(t)\big) = \\ (1-\phi_i)\left(\frac{\hat{a}}{2}K_j^*(t) - \frac{3b}{2}(K_j^*(t))^2 - \hat{a}\epsilon\frac{e^{-\delta t}}{\delta}[e^{\delta t}-1]\right. \\ \left. +4b\epsilon K_j^*(t)\frac{e^{-\delta t}}{\delta}[e^{\delta t}-1] - 2b\epsilon^2\frac{e^{-2\delta t}}{\delta^2}[e^{\delta t}-1]^2\right) + \frac{w_i}{\phi_i}\left(-\epsilon\frac{e^{-\delta t}}{\delta}[e^{\delta t}-1]\right),\end{aligned}$$

whereas for $t \geq t_\epsilon$ we have

$$\begin{aligned}(1-\phi_i)\left(\hat{a}K_i^\epsilon(t) - 2b(K_i^\epsilon(t))^2 - \frac{\hat{a}-bK_j^*(t)}{2}K_j^*(t))\right) + \frac{w_i}{\phi_i}\big(K_i^\epsilon(t) - K_i^*(t)\big) = \\ (1-\phi_i)\left(\frac{\hat{a}}{2}K_j^*(t) - \frac{3b}{2}(K_j^*(t))^2 - \hat{a}\epsilon\frac{e^{-\delta t}}{\delta}[e^{\delta t_\epsilon}-1]\right. \\ \left. +4b\epsilon K_j^*(t)\frac{e^{-\delta t}}{\delta}[e^{\delta t_\epsilon}-1] - 2b\epsilon^2\frac{e^{-2\delta t}}{\delta^2}[e^{\delta t_\epsilon}-1]^2\right) + \frac{w_i}{\phi_i}\left(-\epsilon\frac{e^{-\delta t}}{\delta}[e^{\delta t_\epsilon}-1]\right).\end{aligned}$$

Now, observe that for all t, $t \leq t_\epsilon$, it holds that

$$\begin{aligned}\alpha I_i^\epsilon(t) + \frac{\beta}{2}(I_i^\epsilon(t))^2 - \left(\alpha I_j^*(t) + \frac{\beta}{2}(I_j^*(t))^2\right) &= \alpha(I_j^*(t)-\epsilon) + \frac{\beta}{2}(I_j^*(t)-\epsilon)^2 - \left(\alpha I_j^*(t) + \frac{\beta}{2}(I_j^*(t))^2\right) \\ &= -\alpha\epsilon + \frac{\beta}{2}\left(-2\epsilon I_j^*(t) + \epsilon^2\right).\end{aligned}$$

Using the above results, we obtain

$$\begin{aligned}\int_0^\infty \left\{(1-\phi_i)\left((\hat{a} - 2bK_i^\epsilon(t))K_i^\epsilon(t) - \frac{\hat{a}-bK_j^*(t)}{2}K_j^*(t)\right) + \frac{w_i}{\phi_i}(K_i^\epsilon(t) - K_i^*)\right\}e^{-rt}dt \\ -\int_0^{t_\epsilon}\left(\alpha I_i^\epsilon(t) + \frac{\beta}{2}(I_i^\epsilon(t))^2 - \alpha I_i^*(t) - \frac{\beta}{2}(I_i^*(t))^2\right)e^{-rt}dt\end{aligned}$$

$$= \int_0^{\infty} e^{-rt}(1-\phi_i)\left(\frac{\hat{a}}{2}K_j^*(t) - \frac{3b}{2}(K_j^*(t))^2\right)dt$$
$$-\epsilon(1-\phi_i)\int_0^{t_\epsilon} e^{-rt}\left((\hat{a} + \frac{w_i}{(1-\phi_i)\phi_i})\frac{e^{-\delta t}}{\delta}[e^{\delta t}-1] - 4bK_j^*(t)\frac{e^{-\delta t}}{\delta}[e^{\delta t}-1] + 2b\epsilon\frac{e^{-2\delta t}}{\delta^2}[e^{\delta t}-1]^2]\right)dt$$
$$-\epsilon(1-\phi_i)\int_{t_\epsilon}^{\infty} e^{-rt}\left((\hat{a} + \frac{w_i}{(1-\phi_i)\phi_i})\frac{e^{-\delta t}}{\delta}[e^{\delta t_\epsilon}-1] - 4bK_j^*(t)\frac{e^{-\delta t}}{\delta}[e^{\delta t_\epsilon}-1] + 2b\epsilon\frac{e^{-2\delta t}}{\delta^2}[e^{\delta t_\epsilon}-1]^2]\right)dt$$
$$+\epsilon\int_0^{t_\epsilon} e^{-rt}\left(\alpha + \beta I_j^*(t) - \frac{\beta}{2}\epsilon\right)dt. \tag{167}$$

Now since $K_j^*(t) < \hat{a}/(3b)$ by assumption, we have

$$\int_0^{\infty}\left(\frac{\hat{a}}{2}K_j^*(t) - \frac{3b}{2}(K_j^*(t))^2\right)e^{-rt}dt > 0.$$

Moreover, we can always choose ϵ small enough so that the expression (167) is positive. Therefore, we have found a profitable deviation. The result follows. ■

Proof of Proposition 4

Proof. Suppose by way of a contradiction that such an equilibrium exists. Assume that the inequality in the statement of the Proposition is satisfied for firm i. From Proposition 2, we see that along the candidate equilibrium path, manufacturer i's revenues are equal to $(1-\phi_i)\frac{\hat{a}^2}{9b} + \frac{w_i}{\phi_i}\frac{\hat{a}}{3b}$.

For any $K < \frac{\hat{a}}{3b}$, the payoff of manufacturer i is given (up to the investment expenditures) by the function $\zeta(K) : (1-\phi_i)(\hat{a} - 2bK)K + \frac{w_i}{\phi_i}K$. Notice that $\zeta'(\frac{\hat{a}}{4b}) = \frac{w_i}{\phi_i} > 0$ and under assumption (73) $\zeta'(\frac{\hat{a}}{3b}) = -(1-\phi_i)\frac{\hat{a}}{3b} + \frac{w_i}{\phi_i} < 0$. Since ζ' is continuous there exists a value $\tilde{K} \in]\frac{\hat{a}}{4b}, \frac{\hat{a}}{3b}[$, such that $\zeta'(\tilde{K}) = 0$. Since ζ is concave, it attains its maximum value at $\tilde{K}$.

Consider the following deviation strategy for manufacturer i. Set $I_i(t) = 0$, as long as $K_i(t)$ is not equal to $\tilde{K}$, and $I_i(t) = \delta\tilde{K}$ afterwards. Let t' be such that $K_i(t') = K_{i0}e^{-\delta t'} = \frac{\hat{a}}{3b}e^{-\delta t'} = \tilde{K}$. So for $t \in [0, t']$, $K_i(t) = \frac{\hat{a}}{3b}e^{-\delta t}$ and for $t > t'$, $K_i(t) = \tilde{K}$. Under our assumption it holds that for all t, manufacturer i's revenues is higher to the candidate equilibrium income (since $\zeta'(K) < 0$ if $K > \tilde{K}$).

Moreover, the investment expenditures of manufacturer i are also always lower than the equilibrium ones since they are nil until the date at which $K_i(t) = \tilde{K}$ and equal to $\delta\tilde{K}$ afterwards, which is lower than $\delta\frac{\hat{a}}{3b}$. The Proposition follows. ■

Proof of Proposition 5

Proof. The proof is essentially that of Lemma 1. The instantaneous profit of firm j is no longer $w_j K_i^*(t) - C(I_j)$ but $(1-\phi_j)b(K_i^*(t))^2 + \frac{w_j}{\phi_j} K_i^*(t) - C(I_j(t))$, however. Yet the proof does not depend on this difference. ■

Proof of Proposition 6

Proof.

Assume by way of a contradiction that the manufacturers' capacities satisfy $K_i^*(\underline{t}) = K_j^*(\underline{t})$, $K_i^*(\bar{t}) = K_j^*(\bar{t})$, and $K_j^*(t) < K_i^*(t)$ on $]\underline{t}, \bar{t}[$. Then from Proposition 2, Firm i's income can be written $(1-\phi_i)b\big(K_j^*(t)\big)^2 + \frac{w_i}{\phi_i} K_j^*(t) - C(I_i^*(t))$. Since firm i's income only depends on firm j's capacity, and firm i's capacity is different from firm j's, this implies that

$$\int_{\underline{t}}^{\bar{t}} e^{-rt} C(I_i^*(t))dt \leq \int_{\underline{t}}^{\bar{t}} e^{-rt} C(I_j^*(t))dt. \tag{168}$$

But it must be that $\int_{\underline{t}}^{\bar{t}} e^{-rt} C(I_i^*(t))dt < \int_{\underline{t}}^{\bar{t}} e^{-rt} C(I_j^*(t))dt$. Otherwise, choose $\lambda \in]0,1[$ and set $I_i^\lambda(t) \equiv \lambda I_i^*(t) + (1-\lambda) I_j^*(t)$. Since the cost function $C(\cdot)$ is strictly convex, it holds that

$$\int_{\underline{t}}^{\bar{t}} e^{-rt} C\big(I_i^\lambda(t)\big)dt < \lambda \int_{\underline{t}}^{\bar{t}} e^{-rt} C(I_i^*(t))dt + (1-\lambda) \int_{\underline{t}}^{\bar{t}} e^{-rt} C(I_j^*(t))dt = \int_{\underline{t}}^{\bar{t}} e^{-rt} C(I_i^*(t))dt. \tag{169}$$

But then, firm i would be better off by choosing investment decision I_i^λ because it is less costly than I_i^* and its income would be unchanged as $K_i^\lambda = \lambda K_i^*(t) + (1-\lambda) K_j^*(t) > K_j^*(t)$ for all t in $]\underline{t}, \bar{t}[$ (according to Proposition 2, firm i income with I_j^λ would only depend on K_j^*). Therefore I_i^* cannot be an equilibrium decision. So it must be that $\int_{\underline{t}}^{\bar{t}} e^{-rt} C(I_i^*(t))dt < \int_{\underline{t}}^{\bar{t}} e^{-rt} C(I_j^*(t))dt$. Now, since $K_j^*(t) < K_i^*(t)$ for all $t \in]\underline{t}, \bar{t}[$, if firm j makes the same investment decision I_i^* as firm i, it can always only use a share $Z_j = K_j^*(t) < K_i^*(t)$ of capacity $K_i^*(t)$, and thus obtain the income associated with K_j^*. However, since inequality (168) is actually a strict inequality, firm j would be better off by choosing firm i's investment policy I_i^* (such policy would yield the same revenues while reducing the investment costs). Thus I_i^* cannot be an equilibrium decision and we obtain a contradiction. ■