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D-CONCAVE NONAUTONOMOUS DIFFERENTIAL EQUATIONS AND APPLICATIONS TO CRITICAL TRANSITIONS

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Wisdom has built herself a house,
she has erected her seven pillars,
she has slaughtered her beasts, prepared her wine,
she has laid her table.
She has despatched her maidservants
and proclaimed from the city's heights:
'Who is ignorant? Let him step this way.'
To the fool she says,
'Come and eat my bread,
drink the wine I have prepared!
Leave your folly and you will live,
walk in the ways of perception.'

Prov 9 : 1-6

Sapientia aedificavit sibi domvm, the beginning of Prov 9 : 1, is the motto of Universidad de Valladolid.

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Introduction

Dynamical systems are the mathematical framework used to describe how a phenomenon evolves over time according to certain laws or rules that govern its behavior. When the laws themselves are explicitly time-dependent, the system is said to be nonautonomous. The concept of nonautonomous dynamical systems encompasses a solid mathematical theory developed over the last century, which continues to be a very active area of research today. This theory requires its own set of tools and methods, different from those of autonomous dynamics, and includes areas of study as diverse as stability and attraction, bifurcation theory, control theory, dynamical complexity and chaos, or multiscale systems. While the results of nonautonomous dynamics occasionally mirror known scenarios observed in autonomous dynamics, in other instances, this field unveils novel dynamical scenarios inaccessible within the purview of autonomous models. These scenarios typically exhibit elements of high dynamical complexity.

An essential aspect of dynamical systems theory, and consequently of nonautonomous theory, lies in the modeling of real-world events. Endeavors in this direction seek to describe, predict, and/or control the behavior of such phenomena. The dynamical abundance achievable through nonautonomous models enables the description of real-world phenomena that autonomous dynamics alone cannot adequately capture. A topic in the applied sciences that has generated growing interest in recent years is the study of critical transitions [105], which are significant changes in the dynamics of a complex system that occur as a consequence of small variations in its inputs. This concept appears repeatedly in the literature in recent years in areas such as climate [4, 13, 68, 101, 108], ecology [5, 28, 79, 106, 107, 115], biology [48, 84] or finances [77, 112, 121]. The present PhD dissertation has two main objectives: to obtain new results on nonautonomous dynamics, paying special attention to nonautonomous bifurcation theory, and to apply the conclusions obtained to the study of critical transitions in ecology, among other fields of interest.

A commonly used approach to formalizing the theory of critical transitions was instigated by Ashwin et al. [12]. This approach usually involves a parametric family of ordinary differential equations as a starting point. Subsequently, the parameter is replaced by a function, referred to as a *parameter shift*, which has asymptotically constant limits γ_- and γ_+ as $t \to -\infty$ and $t \to \infty$ respectively. The system resulting from this substitution is called the *transition system* (or *transition equation* in the scalar case), and it is the one whose dynamics are supposed to describe the phenomenon to be modeled. The equations obtained by substituting the values of the parameter γ_- and γ_+ in the initial parametric family are called *past system* and *future system* (or *past equation* and *future equation* in the scalar case) respectively, and it is often assumed that both exhibit the same type of global dynamics. This approach is used, for instance, in Alkhayuon and Ashwin [3], Kiers and Jones [61], Kuehn and Longo [65], O'Keeffe and Wieczorek [89], Wieczorek and Xie [118], Wieczorek et al. [119] among others, where the past and future systems are autonomous. In the recent works of Longo et al. [71, 72, 73], the limit equations are inherently nonautonomous, and the law of the (scalar) differential equation is concave with respect to the state variable. In any case, the nonautonomous transition system can be understood as a connection between the past system, which is approached by the transition system as t decreases, and the future system, which is approached as t increases. It is thus apparent that this mathematical framework incorporates key elements from the theory of dynamical systems, and it is logical to anticipate that the advances in the theory of nonautonomous dynamical systems will facilitate progress in addressing various types of problems and refining the conclusions drawn from them.

Throughout the document, we will always restrict ourselves to the study of scalar equations. If h is regular enough, in order to analyze the nonautonomous scalar ordinary differential equation

$$x' = h(t,x)\,, \tag{1}$$

we consider the *hull* Ω_h of h, that is, the closure of the set of time shifts $h_t(s,x) = h(s+t,x)$ in the compact-open topology of $C(\mathbb{R}\times\mathbb{R},\mathbb{R})$, and study simultaneously the family of nonautonomous scalar ordinary differential equations

$$x' = \omega(t,x) \quad \text{for } \omega \in \Omega_h\,. \tag{2}$$

The properties of the function h that guarantee that Ω_h is compact and that the time-shift global flow $\sigma_h\colon \mathbb{R}\times\Omega_h \to \Omega_h$, $(t,\omega)\mapsto\omega_t$ is continuous will be collected in the definition of *admissible function*. In all cases, the flow (Ω_h,σ_h) is *transitive*, i.e., it has at least one dense orbit: that of h. Particular emphasis will be placed on scenarios wherein (Ω_h,σ_h) is *minimal*, meaning that every orbit is dense. Such circumstance entails a profound relationship between all the orbits in Ω_h. Considering simultaneously all the solutions of all the equations of the family (2) defines a *skewproduct flow*; i.e., a continuous and possibly local flow on the product space $\Omega_h\times\mathbb{R}$ whose action preserves the flow σ_h on the *base* Ω_h. This framework, known as the *skewproduct formalism*, enables the application of tools from topological dynamics and ergodic theory to explore the behavior of solutions of these equations. Within this framework, it becomes feasible to transfer information between different orbits and to identify collective properties, which are often easier to recognize than those that are shown by individual equations. These collective properties will sometimes be exhibited by all the equations of the hull, while at other times they will be exhibited only by an invariant set with complete measure or by a residual invariant set, i.e., by a large invariant set from the point of view of ergodic theory or topology, respectively. Frequently, the initial differential equation (1) belongs to these sets where some collective property is present. And, even in instances where this is not directly applicable, significant aspects of the behavior of the solutions of (1) may be inferred from these collective properties.

In this document, we focus on a specific type of nonlinearities. A continuous map $h\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $(t,x)\mapsto h(t,x)$ with continuous derivative $h_x\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ with respect to the state variable x is said to be *d-concave* if h_x is concave with respect to x. These kind of functions appear frequently in the mathematical modeling of problems in ecology [27], climate [15], optics and electronics [59], biology [82], and circuits [47] among many other fields of interest. It is worth noting, furthermore, that the d-concavity of h is transmitted to all equations (2) of the hull Ω_h.

Not only is d-concavity important in the analysis, but most of the main results also require a second hypothesis, namely the coercivity of h. This property is also transmitted to the hull and ensures the dissipativity of the skewproduct flow, thus guaranteeing the existence of a global attractor.

Bifurcation theory addresses relevant changes in the local or global dynamics generated by the solutions of a differential equation caused by small changes in the model parameters or coefficients. We remark that the shallow descriptions of critical transitions and bifurcations given so far closely resemble each other. In fact, it is noteworthy that bifurcation theory has played a fundamental role in the development of a mathematical theory of critical transitions since its inception. Autonomous bifurcation theory has been studied extensively in both finite and infinite dimension. Nevertheless, nonautonomous bifurcation theory, which has been subject to intensive investigation in recent years, presents new difficulties and challenges in both its formulation and development. The works of Alonso and Obaya [6], Anagnostopoulou and Jäger [8], Anagnostopoulu et al. [10, 9], Braaksma et al. [17], Fabbri and Johnson [39], Fuhrmann [42], Johnson and Mantellini [53], Johnson et al. [54], Jäger [51], Kloeden [62], Langa et al. [67], Núñez and Obaya [86, 87], Pötzsche [93, 94], Rasmussen [97, 98], Remo et al. [99] and the references therein, offer an overview of the present state of this theory, focusing on scalar ordinary differential equations. Since in general a nonautonomous differential equation does not have constant or periodic solutions, it is not even clear from which class of objects the bifurcation is to be studied. The skewproduct formulation gives a natural answer in this context: we will study the variation with the parameter of the global attractor and of the compact invariant sets that it contains; and, in the case that (Ω_h, σ) is minimal, we will study the variation with the parameter of the number and hyperbolic structure of minimal sets in $\Omega_h \times \mathbb{R}$. For instance, it is well known that a saddle-node bifurcation substantially changes the size of the global attractor, potentially leading to transitions from hyperbolic attractors to others that can be sensitive with respect to the initial conditions or even chaotic; and that, in the context of multi-scale dynamics, a double saddle-node bifurcation can give rise to the so-called relaxation oscillations and other solutions with irregular fluctuations of great importance in applications. This is just a sample of how the development of bifurcation theory is, in many cases, deeply intertwined with that of its applications.

This work offers a comprehensive analysis of the bifurcation theory concerning dissipative d-concave nonautonomous scalar ordinary differential equations with minimal base flow. This analysis represents a highly nontrivial extension of previously known results established for autonomous models. Specifically, as noted earlier, aspects of dynamical complexity that were absent in the autonomous realm may emerge. Furthermore, we wish to underscore our discovery of bifurcation patterns that are unattainable in the autonomous scenario: this is the case of the so-called *generalized pitchfork bifurcation*, to which considerable effort is devoted in this document and of which precise examples are constructed. In our results, the occurrence of this bifurcation stems from the interplay between a topological property and an ergodic property in the skewproduct base: minimality, a prevalent attribute in deterministic dynamics, and the presence of multiple ergodic measures, a widespread characteristic in random dynamical systems.

With a view to the application of bifurcation theory results in the modeling of critical transitions in ecological systems, and considering the longstanding tradition of employing d-concave scalar equations to represent populations exhibiting the Allee effect [27], we introduce a new dynamical formulation. Let $g\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a sufficiently regular, coercive and admissible map defining a dissipative transition equation

$$x' = g(t,x)\,. \tag{3}$$

We assume that there exist two sufficiently regular, coercive, admissible, and d-concave functions $g_\pm$ satisfying $\lim_{t\to\pm\infty}(g(t,x)-g_\pm(t,x))=0$ uniformly on compact sets of $\mathbb{R}$ and defining respectively a past equation $x'=g_-(t,x)$ and a future equation $x'=g_+(t,x)$. Coercivity and d-concavity properties are inherited by all the elements of the corresponding hulls. The asymptotic approach provides the relations $\Omega_g^\alpha = \Omega_{g_-}^\alpha$ and $\Omega_g^\omega = \Omega_{g_+}^\omega$ between the α-limit sets of g and g_- and the ω-limit sets of g and g_+. These relations serve as key elements in describing the dynamical possibilities of the transition equation (3) under the assumption that both the past and the future equations possess the maximum number of hyperbolic solutions allowed by the strict d-concavity, which as we previously prove, is three. It is worth noting that we do not impose d-concavity on the transition equation, i.e., on the function g itself, so the transition equation may be given by a law of evolution that may be significantly different, at least at very large time intervals, from the laws of evolution governing past and future systems. This fact significantly broadens the scope of applications.

The dissertation is organized into four chapters. A brief overview of the contents of each chapter serves to conclude this introduction.

Chapter 1 provides an introduction to the foundational concepts necessary for the subsequent discussion. A preliminary section elucidates the basic principles of both topological dynamics and ergodic dynamics. Following this, the chapter proceeds to introduce the essential elements of the two formulations to be utilized: the skewproduct formulation and the processes formulation. Finally, it concludes with the presentation of definitions and properties of spaces of continuous functions defined on a compact set.

In Chapter 2, we delineate the fundamental properties of families of d-concave nonautonomous scalar equations, employing the skewproduct formulation on a compact base Ω. That is, Ω is a compact metric space endowed with a continuous flow $\sigma\colon \mathbb{R}\times\Omega\to\Omega$, denoted by $\omega{\cdot}t=\sigma(t,\omega)$, and we consider families

$$x' = \mathfrak{h}(\omega{\cdot}t,x) \quad \text{for } \omega\in\Omega\,, \tag{4}$$

which induce a local continuous flow $\tau(t,\omega,x)=(\omega{\cdot}t, v(t,\omega,x))$ on $\Omega\times\mathbb{R}$. Here, $v(t,\omega,x)$ stands for the maximal solution of $(4)_\omega$ that satisfies $v(0,\omega,x)=x$, as the conditions on $\mathfrak{h}$ guarantee the existence and uniqueness of this solution. The family (2) is included in this framework by defining $\mathfrak{h}(\omega,x)=\omega(0,x)$ (and hence $\mathfrak{h}(\omega{\cdot}t,x)=\omega(t,x)$). Invariant and ergodic measures on Ω are of paramount importance in this discussion. It is assumed that $\mathfrak{h}\colon\Omega\times\mathbb{R}\to\mathbb{R}$ possesses adequate regularity, as well as the concavity of $x\mapsto\mathfrak{h}_x(\omega,x)$ over a complete measure subset of Ω for all ergodic measures, and the strict concavity of $x\mapsto\mathfrak{h}_x(\omega,x)$ over a positive measure subset of Ω for all the ergodic measures. Under these conditions, it is demonstrated that at most three ordered compact invariant sets may exist projecting over the entire base, and if three such compact sets exist, then they are hyperbolic copies of the base. Assuming in addition that $\mathfrak{h}$ satisfies a coercive property ensuring the dissipativity of $x' = \mathfrak{h}(\omega{\cdot}t, x)$, we delve into properties concerning the global attractor and those relative to the Lyapunov exponents of compact invariant sets. The findings presented in this chapter are primarily drawn from [34, 37].

The core of this dissertation is found in Chapters 3 (bifurcation) and 4 (critical transitions). In Chapter 3, we examine three different bifurcation problems for one-parameter families of nonautonomous scalar differential equations, employing the skewproduct formulation on a compact base. Most of the results are stated for the case of minimal base flow on Ω, so we assume this condition in what follows. Given a function $\mathfrak{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the conditions of the previous chapter, the bifurcation problems $x' = \mathfrak{h}(t, x) + \lambda$, $x' = \mathfrak{h}(t, x) + \lambda\, x$ and $x' = \mathfrak{h}(t, x) + \lambda\, x^2$ are studied, where in the last two problems it is also assumed that $\mathfrak{h}(\cdot, 0) \equiv 0$. In each problem, our aim is to ascertain the number of minimal sets in $\Omega \times \mathbb{R}$ for each parameter value, and from there, to infer properties of the global attractor. In the case of $x' = \mathfrak{h}(\omega{\cdot}t, x)+\lambda$, two of the possible bifurcation diagrams are described: the *double saddle node* bifurcation diagram and the *one minimal* bifurcation diagram, which are the nonautonomous analogs of the bifurcation diagrams of $x' = -x^3 + x + \lambda$ and of $x' = -x^3 + \lambda$, respectively, with the substantial difference that the dynamics at the bifurcation points may exhibit considerably greater complexity. Furthermore, it is proved that these are the only two possible diagrams if the flow on Ω is uniquely ergodic. In the case of $x' = \mathfrak{h}(\omega{\cdot}t, x) + \lambda\, x$ with $\mathfrak{h}(\cdot, 0) \equiv 0$, we encounter three potential bifurcation diagrams: the *classical pitchfork* bifurcation diagram, serving as the nonautonomous counterpart to the bifurcation diagram of $x' = -x^3+\lambda\, x$; the *local saddle-node and transcritical* bifurcation diagram, serving as the nonautonomous counterpart to the bifurcation diagram of $x' = -x^3 + 2x^2 + \lambda\, x$; and the *generalized pitchfork* bifurcation, for which there exists no nonautonomous equivalent. These three diagrams exhaust all the possibilities. Besides, three possible bifurcation diagrams are also described for $x' = \mathfrak{h}(\omega{\cdot}t, x) + \lambda\, x^2$ with $\mathfrak{h}(\cdot, 0) \equiv 0$: the *no bifurcation* diagram, which is the nonautonomous analog of the bifurcation diagram of $x' = -x^3 + x + \lambda\, x^2$; the *two saddle-node* bifurcation diagram, which is the nonautonomous analog of the bifurcation diagram of $x' = -x^3 - x + \lambda\, x^2$; and the *weak generalized transcritical* bifurcation diagram, which in some cases is the nonautonomous analog to the one of $x' = -x^3 + \lambda\, x^2$. These three diagrams also exhaust all the possibilities of this bifurcation problem. The findings concerning this last bifurcation problem offer supplementary insights into the immediately preceding problem. The theory presented in this chapter reproduces results from [34, 35].

In Chapter 4, we start by introducing models of single species populations subject to the Allee effect, whose evolution is described by equations defined by d-concave functions. We recall that these models have served as the impetus for investigating such equations. Additionally, a nonautonomous perspective on the two types of Allee effect, strong and weak, is presented. Subsequently, a mathematical framework for critical transitions in asymptotically d-concave equations, where all components are intrinsically nonautonomous, is formulated. The dynamical possibilities for a transition equation of the general form (3) are described, as well as the consequences of each one of them on the structures of the global attractor and the pullback attractor. Case A, where the transition equation possesses three distinct hyperbolic solutions, may signify the persistence of a species at risk of extinction or the continued control of a potential invasive species. This scenario is consistently desired in many practical applications. Cases C, where all bounded solutions converge to the same limit as time increases, may correspond to the catastrophic situations of extinction or invasion. Of course, there are contexts in which the desired situation may be one of the CASES C, such as the recovery of an endangered species. CASES B represent unstable situations that separate the two aforementioned cases. When rate-induced, phase-induced or size-induced tipping mechanisms are introduced, the theory illustrates how the transition from CASE A to one of the CASES C, namely the occurrence of a critical transition in the system under study, arises from a saddle-node bifurcation of hyperbolic solutions. Next, these results are translated into a specific formulation of the transition equation

$$x' = f(t, x, \Gamma(t, x)) \,,$$

which is more readily employed in numerous applications, where there exist two maps Γ_- and Γ_+ such that $\lim_{t\to\pm\infty}(\Gamma(t, x) - \Gamma_\pm(t, x)) = 0$ uniformly on compact subsets of $\mathbb{R}$ and such that the properties of strict d-concavity and maximum hyperbolic solutions apply to the past equation $x' = f(t, x, \Gamma_-(t, x))$ and future equation $x' = f(t, x, \Gamma_+(t, x))$, respectively. Numerical simulations and examples of critical transitions in various population models discussed earlier complete the chapter. This chapter jointly presents the contents of [36, 37].

Finally, we note that each chapter begins with a somewhat more detailed description of its structure and contents.

Chapter 1

Preliminaries

As explained in the Introduction, nonautonomous modeling of real-world phenomena requires specific techniques and tools, different from those used in autonomous modeling. This preliminary chapter encompasses the basics of time-dependent scalar ordinary differential equations and nonautonomous one-dimensional dynamical systems, as well as several results that will be needed in the following chapters. Some of them are formulated and proved specifically for this work. Many others are known to specialists in nonautonomous dynamics. Of these, we include those proofs for which it is not easy to find a suitable reference.

Section 1.1 summarizes some classical concepts and properties of topological dynamics and ergodic theory, paying special attention to the definition and main features of the Lyapunov exponents of a nonautonomous scalar linear ordinary differential equation. Its significant relationship with integrals with respect to ergodic measures is emphasized, and in the case of minimal base flow, with the dynamical spectrum. Section 1.2 considers a *base flow* on a compact metric space and a map, which evaluated on the orbits of the previously fixed base flow defines a family of nonautonomous scalar ordinary differential equations. The method for defining a *skewproduct flow* on the product space of the compact metric space and the real line, which projects onto the base flow, is explained. It also introduces the concepts of equilibria, semiequilibria and global upper and lower solutions, and describes the main properties of these dynamical objects, fundamental in our analysis. The notion of global attractor, and some characteristics of compact invariant subsets and minimal subsets for the flow, including hyperbolicity and its relation with the Lyapunov exponents of the variational equations, are also described in this section. In Section 1.3, it is explained how to obtain a skewproduct flow from a suitable single time-dependent equation, and the relation between this flow and the *process* defined by the solutions of the initial equation. Different notions of (local or global) attraction of the solutions, and their relation with the correspondig properties for the skewproduct flow, complete the section. Finally, the short Section 1.4 describes some features of sets of continuous functions on a compact metric space. In particular, the set of continuous maps with continuous primitive will play a remarkable role in the construction of some examples.

1.1 Some fundamental notions

Suitable references for this section are e.g. [26], [38], [54], [83], [103, 104] and [117].

1.1.1 Basics of topological dynamics

In the following section we introduce some basic concepts of topological dynamics that will be used repeatedly throughout the document.

Definition 1.1 (Continuous flow)**.** Let Θ be a metric space. A map $\phi\colon \mathcal{V} \subseteq \mathbb{R}\times\Theta \to \Theta$, $(t,\theta) \mapsto \phi(t,\theta)$ is said to be a *continuous flow* on Θ if

(i) $\mathcal{V} = \bigcup_{\theta\in\mathbb{R}} \mathcal{I}_\theta \times \{\theta\}$, where $\mathcal{I}_\theta$ is an open interval which contains 0 for all $\theta \in \Theta$,

(ii) $\phi(0,\theta) = \theta$ for all $\theta \in \Theta$,

(iii) if $(s,\theta), (t,\phi(s,\theta)) \in \mathcal{V}$, then $(t+s,\theta) \in \mathcal{V}$ and $\phi(t+s,\theta) = \phi(t,\, \phi(s,\theta))$,

(iv) $(t,\theta) \mapsto \phi(t,\theta)$ is continuous.

A continuous flow ϕ is said to be *global* if $\mathcal{V} = \mathbb{R} \times \Theta$.

If ϕ is a global continuous flow, then (iii) reads as $\phi(t+s,\theta) = \phi(t,\phi(s,\theta))$ for all $t,s \in \mathbb{R}$ and $\theta \in \Theta$. It will be usually said that the pair (Θ,ϕ) is a continuous flow on a metric space (or simply a continuous flow), meaning that ϕ is a continuous flow on the metric space Θ. We will frequently denote $\phi(t,\theta) = \theta{\cdot}t$, and therefore, given $\mathcal{C} \subseteq \Theta$, we will denote $\mathcal{C}{\cdot}t = \{\phi(t,\theta) \mid \theta \in \mathcal{C}\}$ whenever it is defined. Sometimes, to avoid risk of confusion, we will denote $\phi(t,\theta) = \phi_t(\theta)$ and $\phi_t(\mathcal{C}) = \mathcal{C}{\cdot}t$.

The following definitions refer to a continuous flow (Θ,ϕ).

Definition 1.2 (Orbit and semiorbits)**.** Let $\theta \in \Theta$. The set

(i) $\{\phi(t,\theta) \mid t \in \mathcal{I}_\theta\}$ is called the *ϕ-orbit of θ*,

(ii) $\{\phi(t,\theta) \mid t \in \mathcal{I}_\theta \cap [0,\infty)\}$ is called the *forward ϕ-semiorbit of θ*,

(iii) $\{\phi(t,\theta) \mid t \in \mathcal{I}_\theta \cap (-\infty,0]\}$ is called the *backward ϕ-semiorbit of θ*.

The ϕ-orbit of θ is said to be *globally defined* if $\mathbb{R} = \mathcal{I}_\theta$, the forward ϕ-semiorbit of θ is said to be *globally defined* if $[0,\infty) \subset \mathcal{I}_\theta$, and the backward ϕ-semiorbit of θ is said to be *globally defined* if $(-\infty,0] \subset \mathcal{I}_\theta$. The prefix ϕ will be sometimes omitted.

Definition 1.3 (Invariant set)**.** A set $\mathcal{C} \subseteq \Theta$ is said to be *ϕ-invariant* if the ϕ-orbit of θ is globally defined for all $\theta \in \mathcal{C}$ and $\phi_t(\mathcal{C}) = \mathcal{C}$ for all $t \in \mathbb{R}$.

Throughout the document, the subscript $n \in \mathbb{N}$ in sequences will often be omitted when it does not generate a risk of confusion.

Definition 1.4 (α-limit and ω-limit sets)**.** Let $\theta \in \Theta$ have globally defined forward (resp. backward) ϕ-orbit. The *ω-limit* (resp. *α-limit*) *set of θ*, or of its orbit, is the set of all the possible limits of sequences of the form $(\phi(t_n,\theta))$, where the sequence (t_n) has limit ∞ (resp. $-\infty$).

It is known that both the α-limit set and the ω-limit set of any $\theta \in \Theta$ are closed ϕ-invariant sets and that, if θ has relatively compact forward (resp. backward) ϕ-semiorbit, then the ω-limit set (resp. α-limit set) of θ is also compact and connected. See e.g. [46, Chapter I, Theorem 8.1].

Definition 1.5 (Asymptotic pair)**.** A pair of elements $\theta_1, \theta_2 \in \Theta$ is said to be an *asymptotic pair as* $t \to \infty$ (resp. *as* $t \to -\infty$) if the forward (resp. backward) semiorbit of both θ_1 and θ_2 is globally defined and

$$\lim_{t\to\infty} d_\Theta\big(\phi(t,\theta_1),\phi(t,\theta_2)\big) = 0 \qquad \Big(\text{resp. } \lim_{t\to-\infty} d_\Theta\big(\phi(t,\theta_1),\phi(t,\theta_2)\big) = 0\Big)\,,$$

where d_Θ stands for the distance on Θ.

Definition 1.6 (Minimal set)**.** A set $\mathcal{M} \subseteq \Theta$ is said to be ϕ-*minimal* if it is compact, ϕ-invariant and it does not contain any proper, compact and ϕ-invariant subset. The continuous flow (Θ, ϕ) is said to be *minimal* if Θ itself is ϕ-minimal.

It is not difficult to check that a compact set $\mathcal{M} \subseteq \Theta$ is ϕ-minimal if and only if the ϕ-orbit of every $\theta \in \mathcal{M}$ is dense in $\mathcal{M}$. In particular, a ϕ-minimal set coincides with the α-limit and ω-limit sets of any of its elements. The following important result will be used repeatedly: every compact ϕ-invariant set contains some ϕ-minimal set (see e.g. [83, Chapter V, Theorem 7.02]).

Definition 1.7 (Transitive set)**.** A compact set $\mathcal{C} \subseteq \Theta$ is said to be ϕ-*transitive* if there exists $\theta \in \mathcal{C}$ such that the ϕ-orbit of θ is dense in $\mathcal{C}$. The continuous flow (Θ, ϕ) is said to be *transitive* if Θ itself is compact and ϕ-transitive.

The following lemma gives some useful information about a minimal flow on a compact metric space. Note that $B_\Theta(\theta, \delta)$ stands for the open ball of center θ and radius δ in the metric space Θ.

Lemma 1.8. *Let (Θ, ϕ) be a minimal flow on a compact metric space Θ. Given an open set $\mathcal{U} \subseteq \Theta$, there exists $s_\mathcal{U} > 0$ such that, for all $\theta \in \Theta$, there exists $s_\theta \in (0, s_\mathcal{U}]$ such that $\phi(s_\theta, \theta) \in \mathcal{U}$.*

Proof. Given any $\theta \in \Theta$, since its ϕ-orbit is dense in Θ, there exists $t_\theta > 0$ such that $\phi(t_\theta, \theta) \in \mathcal{U}$. Therefore, there exists $\delta_\theta > 0$ such that $\phi(t_\theta, \bar\theta) \in \mathcal{U}$ for all $\bar\theta \in B_\Theta(\theta, \delta_\theta)$. Let $\theta_1, \theta_2, \ldots, \theta_n \in \Theta$ be such that the balls $B_\Theta(\theta_j, \delta_{\theta_j})$ for $0 \le j \le n$ cover Θ. Then, $s_\mathcal{U} = \max\{t_{\theta_j} \mid 0 \le j \le n\}$ is the looked-for amount. □

1.1.2 Basics of ergodic dynamics

In this section, we introduce some basic concepts and results of ergodic dynamics that will be used repeatedly throughout the document, that is, concepts referring to invariant and ergodic measures of a continuous flow (Θ, ϕ) on a locally compact metric space Θ. The measures considered in this document are normalized positive Borel measures. Unless otherwise indicated, given a positive Borel measure m, we will use its extension to the m-completion of the Borel σ-algebra, using the same symbol m to denote it. The sets of the m-completion are referred to as *m-measurable sets*. In what follows, as usual, the notation "m-a.e." means *almost everywhere with respect to the measure m*. A map $\mathfrak{b}\colon \Theta \to \mathbb{R}$ is said to be *m-measurable* if it is measurable with respect to the m-completion of the Borel sigma-algebra, and it is said to be simply *measurable* if it is measurable with respect to the Borel sigma-algebra. Recall that every normalized Borel measure m on a locally compact metric space is regular (see [41, Theorem 7.8]) and that, if Θ is compact, then the set of normalized Borel measures on Θ is compact in the topology of the weak* convergence of measures, defined by $\lim_{n\to\infty} m_n = m$ if $\lim_{n\to\infty} \int_\Theta \mathfrak{b}(\theta)\, dm_n = \int_\Theta \mathfrak{b}(\theta)\, dm$ for all continuous map $\mathfrak{b}\colon \Theta \to \mathbb{R}$ (see [117, Theorems 6.4 and 6.5]).

Definition 1.9 (Invariant and ergodic measures)**.** Let (Θ, ϕ) be a continuous global flow. A normalized Borel measure m on Θ is said to be

(i) *ϕ-invariant* if $m(\phi_t(\mathcal{B})) = m(\mathcal{B})$ for every $t \in \mathbb{R}$ and every m-measurable subset $\mathcal{B} \subseteq \Theta$,

(ii) *ϕ-ergodic* if it is ϕ-invariant and $m(\mathcal{B}) \in \{0, 1\}$ for every m-measurable ϕ-invariant set $\mathcal{B} \subseteq \Theta$.

The sets of normalized ϕ-invariant and ϕ-ergodic measures on Θ are represented by $\mathfrak{M}_{\rm inv}(\Theta, \phi)$ and $\mathfrak{M}_{\rm erg}(\Theta, \phi)$, respectively.

If Θ is a compact metric space, then the Kryloff-Bogoliuboff Theorem (see e.g. [83, Chapter VI, Theorem 9.05]) ensures that there always exists at least one ϕ-invariant measure. It is also known that $\mathfrak{M}_{\rm inv}(\Theta, \phi)$ is a compact and convex set whose set of extreme points is $\mathfrak{M}_{\rm erg}(\Theta, \phi)$ (a point of a convex set is called *extreme* if it does not lie on any open segment joining two points of the convex set), and that there always exists at least one ϕ-ergodic measure (see [117, Theorem 6.10] for the case of a transformation and [54, Theorem 1.9] for a flow). That is, both $\mathfrak{M}_{\rm inv}(\Theta, \phi)$ and $\mathfrak{M}_{\rm erg}(\Theta, \phi)$ are nonempty. The flow (Θ, ϕ) is said to be *uniquely ergodic* if $\mathfrak{M}_{\rm inv}(\Theta, \phi)$ reduces to just one element m, in which case m is ergodic; and it is said to be *finitely ergodic* if $\mathfrak{M}_{\rm erg}(\Theta, \phi)$ is a finite set. The *support* of $m \in \mathfrak{M}_{\rm inv}(\Theta, \phi)$, $\mathrm{Supp}(m)$, is the complement of the largest open set with zero measure, and it is a compact invariant set. If (Θ, ϕ) is minimal, then $\mathrm{Supp}(m) = \Theta$ for every $m \in \mathfrak{M}_{\rm inv}(\Theta, \phi)$ (see [54, Proposition 1.11(iii)]), and hence $m(\mathcal{U}) > 0$ for every open set $\mathcal{U} \subseteq \Theta$.

One of the main results of ergodic dynamics is Birkhoff's Ergodic Theorem. The next result, which does not require the compactness of Θ, summarizes part of its information (see [54, Theorem 1.3 and Proposition 1.4], and the references therein).

Theorem 1.10 (Birkhoff's Ergodic Theorem)**.** *Let (Θ, ϕ) be a global continuous flow, let $m \in \mathfrak{M}_{\rm erg}(\Theta, \phi)$, and let $\mathfrak{a} \in L^1(\Theta, m)$ (resp. let $\mathfrak{a}\colon \Theta \to [0, \infty)$ be an m-measurable map). Then, there exists a ϕ-invariant and m-measurable set $\Theta_{\mathfrak{a}} \subseteq \Theta$ with $m(\Theta_{\mathfrak{a}}) = 1$ such that, for all $\theta \in \Theta_{\mathfrak{a}}$, the limits*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathfrak{a}(\phi(t, \theta))\, dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^T \mathfrak{a}(\phi(t, \theta))\, dt = \lim_{T\to-\infty} \frac{-1}{T} \int_T^0 \mathfrak{a}(\phi(t, \theta))\, dt$$

exist, coincide, and take the value $\int_\Theta \mathfrak{a}(\theta)\, dm \in \mathbb{R}$ (resp. $\int_\Theta \mathfrak{a}(\theta)\, dm \in \mathbb{R} \cup \{\infty\}$).

1.1.3 Lyapunov exponents

This section deals with Lyapunov exponents of families of scalar linear differential equations. A detailed account on this topic for non necessarily scalar families can be found in [55] and [103]. Let (Θ, ϕ) be a continuous global flow on a compact metric space Θ and let $\mathfrak{a}\colon \Theta \to \mathbb{R}$ be a continuous map. Recall the notation $\theta{\cdot}t = \phi(t, \theta)$. We consider the family of scalar linear ordinary differential equations

$$z' = \mathfrak{a}(\theta{\cdot}t)\, z\,, \quad \theta \in \Theta\,. \tag{1.1}$$

Definition 1.11 (Lyapunov exponent)**.** A value $\gamma \in \mathbb{R}$ is a *Lyapunov exponent* of (1.1), or of the map $\mathfrak{a}$, if there exists $\theta \in \Theta$ such that

$$\gamma = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathfrak{a}(\theta{\cdot}s)\, ds = \lim_{t\to-\infty} \frac{1}{t} \int_0^t \mathfrak{a}(\theta{\cdot}s)\, ds\,.$$

The set of all the Lyapunov exponents of (1.1) will be denoted by $\mathrm{Lyap}(\mathfrak{a})$. The values $\inf \mathrm{Lyap}(\mathfrak{a})$ and $\sup \mathrm{Lyap}(\mathfrak{a})$ are called the *lower and upper Lyapunov exponents* of (1.1).

Note that $\inf \mathrm{Lyap}(\mathfrak{a})$ and $\sup \mathrm{Lyap}(\mathfrak{a})$ are finite since $\mathfrak{a}$ is bounded. The next result shows the relation between Lyapunov exponents and ergodic and invariant measures. In particular, $\inf \mathrm{Lyap}(\mathfrak{a})$ and $\sup \mathrm{Lyap}(\mathfrak{a})$ are indeed Lyapunov exponents of (1.1).

Proposition 1.12. *Let $\mathfrak{a}\colon \Theta \to \mathbb{R}$ be a continuous map. Then,*

(i) $\int_\Theta \mathfrak{a}(\theta)\, dm \in \mathrm{Lyap}(\mathfrak{a})$ *for all* $m \in \mathfrak{M}_{\mathrm{erg}}(\Theta,\phi)$.

(ii) *If* $\gamma \in \mathrm{Lyap}(\mathfrak{a})$, *then there exist* $m \in \mathfrak{M}_{\mathrm{inv}}(\Theta,\phi)$ *such that* $\gamma = \int_\Theta \mathfrak{a}(\theta)\, dm$.

(iii) *There exist* $m^l, m^u \in \mathfrak{M}_{\mathrm{erg}}(\Theta,\phi)$ *such that*

$$\inf \mathrm{Lyap}(\mathfrak{a}) = \int_\Theta \mathfrak{a}(\theta)\, dm^l \quad \textit{and} \quad \sup \mathrm{Lyap}(\mathfrak{a}) = \int_\Theta \mathfrak{a}(\theta)\, dm^u\,,$$

and $[\inf \mathrm{Lyap}(\mathfrak{a}), \sup \mathrm{Lyap}(\mathfrak{a})] = \{\int_\Theta \mathfrak{a}(\theta)\, dm \mid m \in \mathfrak{M}_{\mathrm{inv}}(\Theta,\phi)\}$.

(iv) *If* $\inf \mathrm{Lyap}(\mathfrak{a}) > 0$ *and* $\gamma \in (0, \inf \mathrm{Lyap}(\mathfrak{a}))$, *then there exists* $k \geq 1$ *such that*

$$\exp \int_0^t \mathfrak{a}(\theta{\cdot}s)\, ds \leq k\, e^{\gamma t} \quad \textit{for all } \theta \in \Theta \textit{ and } t \leq 0\,.$$

If $\sup \mathrm{Lyap}(\mathfrak{a}) < 0$ *and* $\gamma \in (0, -\sup \mathrm{Lyap}(\mathfrak{a}))$, *then there exists* $k \geq 1$ *such that*

$$\exp \int_0^t \mathfrak{a}(\theta{\cdot}s)\, ds \leq k\, e^{-\gamma t} \quad \textit{for all } \theta \in \Theta \textit{ and } t \geq 0\,.$$

Proof. (i) Given $m \in \mathfrak{M}_{\mathrm{erg}}(\Theta,\phi)$, Birkhoff's Ergodic Theorem 1.10 ensures that there exists $\theta_0 \in \Theta$ such that $\int_\Theta \mathfrak{a}(\theta)\, dm = \lim_{t\to\pm\infty} (1/t) \int_0^t \mathfrak{a}(\theta_0{\cdot}s)\, ds \in \mathrm{Lyap}(\mathfrak{a})$.

(ii) This proof is based on Kryloff–Bogoliuboff Theorem (see [83, Chapter VI, Theorem 9.05]). Let $\gamma \in \mathrm{Lyap}(\mathfrak{a})$ and let $\theta_0 \in \Theta$ be taken according to the definition of γ, that is, $\gamma = \lim_{t\to\pm\infty} (1/t) \int_0^t \mathfrak{a}(\theta_0{\cdot}s)\, ds$. Take $(t_n) \uparrow \infty$. According to Riesz Representation Theorem (see e.g. [117, Theorem 6.3]), $\int_\Theta \mathfrak{b}(\theta)\, dm_n = (1/t_n) \int_0^{t_n} \mathfrak{b}(\theta_0{\cdot}s)\, ds$ for all continuous map $\mathfrak{b}\colon \Theta \to \mathbb{R}$ defines a normalized Borel measure m_n on Θ. The compactness of the set of normalized Borel measures on the weak* topology ensures that there exists a normalized Borel measure m on Ω and a subsequence (m_k) of (m_n) such that $\int_\Theta \mathfrak{b}(\theta)\, dm = \lim_{k\to\infty} \int_\Theta \mathfrak{b}(\theta)\, dm_k = \lim_{k\to\infty} (1/t_k) \int_0^{t_k} \mathfrak{b}(\theta_0{\cdot}s)\, ds$ for all continuous map $\mathfrak{b}\colon \Theta \to \mathbb{R}$. Since, for any $t \in \mathbb{R}$,

$$\Big|\frac{1}{t_k} \int_0^t \mathfrak{b}(\theta_0{\cdot}s)\, ds - \frac{1}{t_k} \int_{t_k}^{t_k+t} \mathfrak{b}(\theta_0{\cdot}s)\, ds\Big| \leq \frac{2t}{t_k} \max_{\theta\in\Theta} |\mathfrak{b}(\theta)| \xrightarrow{k\to\infty} 0\,,$$

$\lim_{k\to\infty}(1/t_k)\int_0^{t_k}\mathfrak{b}(\theta_0{\cdot}s)\,ds = \lim_{k\to\infty}(1/t_k)\int_t^{t_k+t}\mathfrak{b}(\theta_0{\cdot}s)\,ds$ for any $t\in\mathbb{R}$, and hence $m\in\mathfrak{M}_{\rm inv}(\Theta,\phi)$ (see e.g. [54, Proposition 1.7]). Consequently, $\int_\Theta \mathfrak{a}(\theta)\,dm = \lim_{k\to\infty}(1/t_k)\int_0^{t_k}\mathfrak{a}(\theta_0{\cdot}s)\,ds=\gamma$.

(iii) Since $\mathfrak{M}_{\rm inv}(\Theta,\phi)\to\mathbb{R}$, $m\mapsto\int_\Theta\mathfrak{a}(\theta)\,dm$ is continuous in the weak* topology, for which $\mathfrak{M}_{\rm inv}(\Theta,\phi)$ is a compact set, there exist $m^l,m^u\in\mathfrak{M}_{\rm inv}(\Theta,\phi)$ such that $\int_\Theta\mathfrak{a}(\theta)\,dm^l\le\int_\Theta\mathfrak{a}(\theta)\,dm\le\int_\Theta\mathfrak{a}(\theta)\,dm^u$ for all $m\in\mathfrak{M}_{\rm inv}(\Theta,\phi)$. Since $\mathfrak{M}_{\rm erg}(\Theta,\phi)$ is the set of extreme points of $\mathfrak{M}_{\rm inv}(\Theta,\phi)$, [91, Choquet's Theorem, Lecture 3] ensures that there exists a normalized measure μ^l defined on the Borel sigma-algebra of $\mathfrak{M}_{\rm erg}(\Theta,\phi)$ such that,

$$\int_\Theta\mathfrak{a}(\theta)\,dm^l=\int_{\mathfrak{M}_{\rm erg}(\Theta,\phi)}\left(\int_\Theta\mathfrak{a}(\theta)\,dm\right)d\mu^l(m)\,,$$

so assuming that $\int_\Theta\mathfrak{a}(\theta)\,dm>\int_\Theta\mathfrak{a}(\theta)\,dm^l$ for all $m\in\mathfrak{M}_{\rm erg}(\Theta,\phi)$ leads us to a contradiction. This shows that m^l can be taken in $\mathfrak{M}_{\rm erg}(\Theta,\phi)$. The same argument works for m^u. Properties (i) and (ii) prove the $\supseteq$ part of the last assertion in (iii), since $\int_\Theta\mathfrak{a}(\theta)\,dm^l,\int_\Theta\mathfrak{a}(\theta)\,dm^u\in{\rm Lyap}(\mathfrak{a})\subseteq\{\int_\Theta\mathfrak{a}(\theta)\,dm\mid m\in\mathfrak{M}_{\rm inv}(\Theta,\phi)\}$; and the convexity of $\mathfrak{M}_{\rm inv}(\Theta,\phi)$ proves the equality.

(iv) We reason in the case $\sup{\rm Lyap}(\mathfrak{a})<0$, which combined with (iii) yields $\int_\Theta\mathfrak{a}(\theta)\,dm<0$ for all $m\in\mathfrak{M}_{\rm inv}(\Theta,\phi)$. Let us take $\gamma\in(0,-\sup{\rm Lyap}(\mathfrak{a}))$. Our goal is to check that there exist $t_0>0$ such that $(1/t)\int_0^t\mathfrak{a}(\theta{\cdot}s)\,ds\le-\gamma$ for all $t\ge t_0$ and $\theta\in\Theta$. For contradiction, assume that there exist $(t_n)\uparrow\infty$ and (θ_n) in Θ such that $(1/t_n)\int_0^{t_n}\mathfrak{a}(\theta_n{\cdot}s)\,ds>-\gamma$. Proceeding as in the proof of (ii), we define normalized Borel measures m_n on Θ satisfying $\int_\Theta\mathfrak{b}(\theta)\,dm_n=(1/t_n)\int_0^{t_n}\mathfrak{b}(\theta_n{\cdot}s)\,ds$ for all continuous map $\mathfrak{b}\colon\Theta\to\mathbb{R}$, and prove the existence of $m\in\mathfrak{M}_{\rm inv}(\Theta,\phi)$ such that $\int_\Theta\mathfrak{b}(\theta)\,dm=\lim_{k\to\infty}(1/t_k)\int_0^{t_k}\mathfrak{b}(\theta_k{\cdot}s)\,ds$ for all continuous map $\mathfrak{b}\colon\Theta\to\mathbb{R}$ for some subsequence (t_k) of (t_n). Hence, $\int_\Theta\mathfrak{a}(\theta)\,dm\ge-\gamma>\sup{\rm Lyap}(\mathfrak{a})$, which is not the case. This proves the assertion, which in turn yields $\exp\int_0^t\mathfrak{a}(\theta{\cdot}s)\,s\le k\,e^{-\gamma t}$ for all $t\ge0$ and $\theta\in\Theta$, where $k=\max\{\exp(\gamma\,t+\int_0^t\mathfrak{a}(\theta{\cdot}s)\,ds)\mid t\in[0,t_0]\,,\ \theta\in\Theta\}$. The proof is analogous in the other case. □

The previous theorem shows that there is a strong connection between the set of Lyapunov exponents of $\mathfrak{a}\colon\Theta\to\mathbb{R}$ and its dynamical spectrum:

Definition 1.13 (Dynamical spectrum)**.** Assume that (Θ,ϕ) is minimal. Given a continuous map $\mathfrak{a}\colon\Theta\to\mathbb{R}$, we define its *dynamical spectrum* as

$${\rm sp}(\mathfrak{a})=\left\{\int_\Theta\mathfrak{a}(\theta)\,dm\,\middle|\,m\in\mathfrak{M}_{\rm inv}(\Theta,\phi)\right\}.$$

We say that $\mathfrak{a}$ has *point spectrum* if ${\rm sp}(\mathfrak{a})$ reduces to a point, and *band spectrum* otherwise.

Remarks 1.14. 1. Proposition 1.12(iii) shows that

$${\rm sp}(\mathfrak{a})=[\inf{\rm Lyap}(\mathfrak{a}),\,\sup{\rm Lyap}(\mathfrak{a})]\,.$$

2. The results of [55] and [103, 104] ensure that, in this scalar case with minimal base Θ, the dynamical spectrum of $\mathfrak{a}\colon\Theta\to\mathbb{R}$ coincides with its Sacker and Sell spectrum, which is the set of values of λ such that none of the equations of the family $x'=(\mathfrak{a}(\theta{\cdot}t)-\lambda)\,x$ has an exponential dichotomy (see Definition 1.48 below).

1.2 Scalar skewproduct flows defined by families of nonautonomous differential equations

Let (Ω, σ) be a global continuous flow on a compact metric space. As in the previous sections, we will usually denote $\omega{\cdot}t = \sigma(t, \omega)$, and less frequently $\sigma_t(\omega) = \sigma(t, \omega)$.

Throughout the document, $C^{0,1}(\Omega\times\mathbb{R}, \mathbb{R})$ represents the set of continuous functions $\mathfrak{h}\colon \Omega\times\mathbb{R}\to\mathbb{R}$ for which the derivative $\mathfrak{h}_x$ with respect to the second variable exists and is continuous, and $C^{0,2}(\Omega\times\mathbb{R}, \mathbb{R})$ is the subset of $C^{0,1}(\Omega\times\mathbb{R}, \mathbb{R})$ of maps $\mathfrak{h}$ for which the second derivative $\mathfrak{h}_{xx}$ exists and is continuous. Analogously, $C^{0,3}(\Omega\times\mathbb{R}, \mathbb{R})$ is the subset of $C^{0,2}(\Omega\times\mathbb{R}, \mathbb{R})$ of maps $\mathfrak{h}$ for which the third derivative $\mathfrak{h}_{xxx}$ exists and is continuous.

We take $\mathfrak{h} \in C^{0,1}(\Omega\times\mathbb{R}, \mathbb{R})$ and consider the family of nonautonomous scalar differential equations

$$x' = \mathfrak{h}(\omega{\cdot}t, x)\,, \quad \omega\in\Omega\,. \tag{1.2}$$

Throughout the document, t stands for the independent variable, that is, $x' = dx/dt$. For each fixed $\omega\in\Omega$, a nonautonomous scalar ordinary differential equation is obtained from the family (1.2), which will be denoted by $(1.2)_\omega$. For each $\omega\in\Omega$ and $x\in\mathbb{R}$, let $\mathcal{I}_{\omega,x} = (\alpha_{\omega,x}, \beta_{\omega,x}) \to \mathbb{R}$, $t\mapsto v(t,\omega,x)$ denote the maximal solution of $(1.2)_\omega$ satisfying $v(0,\omega,x) = x$ with $-\infty\le\alpha_{\omega,x} < 0 < \beta_{\omega,x}\le\infty$. Let

$$\mathcal{V} = \bigcup_{(\omega,x)\in\Omega\times\mathbb{R}} \big((\alpha_{\omega,x}, \beta_{\omega,x})\times\{(\omega,x)\}\big)$$

be the domain of v, that is, $v\colon \mathcal{V}\subseteq\mathbb{R}\times\Omega\times\mathbb{R}\to\mathbb{R}$. The standard properties of existence, uniqueness and continuous dependence of the solutions of ordinary differential equations ensure that:

- *(cocycle property)* if $(s,\omega,x), (t,\omega{\cdot}s, v(s,\omega,x))\in\mathcal{V}$, then $(t+s,\omega,x)\in\mathcal{V}$ and
$$v(t+s,\omega,x) = v(t,\omega{\cdot}s, v(s,\omega,x))\,,$$
- $(t,\omega,x)\mapsto v(t,\omega,x)$ is continuous.

We will often refer to $v\colon\mathcal{V}\to\mathbb{R}$ as the *cocycle of solutions of* (1.2). A remarkable property of v, which follows from its continuity, is its *fiber monotonicity* (or simply *monotonicity*), that is, $v(t,\omega,x) < v(t,\omega,y)$ for all $x<y$, $\omega\in\Omega$, and $t\in\mathbb{R}$ for which both solutions are defined. This property will play a key role in many proofs.

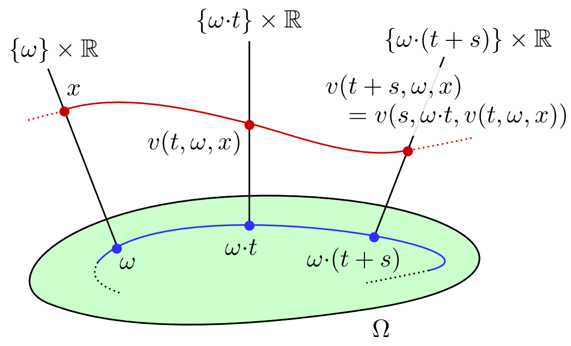


Figure 1.1: Sketch of a skewproduct flow.

Definition 1.15 (Scalar skewproduct flow associated to (1.2))**.** Let (Ω,σ) be a global continuous flow on a compact metric space and let v be the cocycle of solutions of (1.2). Then, the map

$$\tau\colon \mathcal{V}\subseteq\mathbb{R}\times\Omega\times\mathbb{R}\to\Omega\times\mathbb{R}\,,\quad (t,\omega,x)\mapsto(\omega{\cdot}t, v(t,\omega,x)) \tag{1.3}$$

is called the *scalar skewproduct flow* on $\Omega\times\mathbb{R}$ induced by the family (1.2).

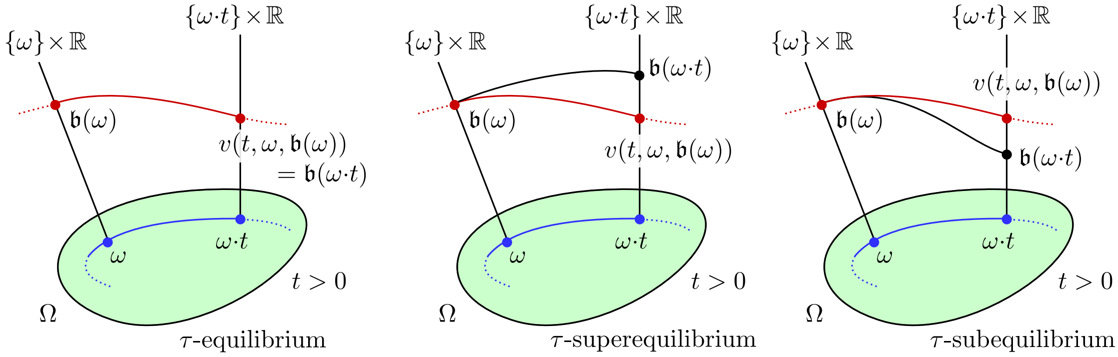


Figure 1.2: Sketch of a τ-equilibrium, a τ-superequilibrium and a τ-subequilibrium.

It is easy to deduce from the previous properties that τ is a continuous flow, local in general, and global if all the solutions of all the equations are globally defined. The space Ω and the flow σ will be called *base* and *base flow* of the skewproduct, respectively. Fugure 1.1 depicts a sketch of a skewproduct flow.

In the following subsections, some important elements, tools and properties of the skewproduct formalism will be described in some detail.

1.2.1 Equilibria, superequilibria and subequilibria

The main concept of this section, equilibrium, allows us to work simultaneously with solutions of all the equations of the family (1.2). As we will see later, equilibria will play a significant role in explaining properties which refer to the whole skewproduct.

Given a map $\mathfrak{b}\colon \Omega \to \mathbb{R}$, the *graph* of $\mathfrak{b}$ will be usually denoted by

$$\{\mathfrak{b}\} = \{(\omega, \mathfrak{b}(\omega)) \mid \omega \in \Omega\}\,.$$

Definition 1.16 (Equilibrium)**.** A map $\mathfrak{b}\colon \Omega \to \mathbb{R}$ such that $t \mapsto v(t,\omega,\mathfrak{b}(\omega))$ is defined for all $t \in \mathbb{R}$ and $\omega \in \Omega$ is said to be a τ-*equilibrium* if $\mathfrak{b}(\omega{\cdot}t) = v(t,\omega,\mathfrak{b}(\omega))$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$, that is, if the graph of $\mathfrak{b}$ is τ-invariant.

All the equilibria we will handle will be either continuous, semicontinuous (see Defintion 1.19 below) or m-measurable with respect to some $m \in \mathfrak{M}_{\text{inv}}(\Omega,\sigma)$, although this is not required in all the definitions and results of the present section.

Definition 1.17 (Copy of the base)**.** A subset of $\Omega \times \mathbb{R}$ is said to be a τ-*copy of the base* if it is the graph $\{\mathfrak{b}\}$ of a continuous τ-equilibrium $\mathfrak{b}\colon \Omega \to \mathbb{R}$.

Definition 1.18 (Semiequilibria)**.** A map $\mathfrak{b}\colon \Omega \to \mathbb{R}$ such that $t \mapsto v(t,\omega,\mathfrak{b}(\omega))$ is defined for all $t \geq 0$ and $\omega \in \Omega$ is said to be

(i) a τ-*subequilibrium* if $\mathfrak{b}(\omega{\cdot}t) \leq v(t,\omega,\mathfrak{b}(\omega))$ for all $\omega \in \Omega$ and $t \geq 0$,

(ii) a τ-*superequilibrium* if $\mathfrak{b}(\omega{\cdot}t) \geq v(t,\omega,\mathfrak{b}(\omega))$ for all $\omega \in \Omega$ and $t \geq 0$.

In both cases $\mathfrak{b}$ is said to be a τ-*semiquilibrium*. Figure 1.2 shows a local depiction of these dynamical objects.

The reference to the flow τ in all the previous definitions will be frequently omitted if there is no risk of confusion. For the sake of clarity, at other times, it will be said that $\mathfrak{b}\colon \Omega \to \mathbb{R}$ is an equilibrium or semiequilibrium for (1.2).

We say that an equilibrium (resp. semiequilibrium) $\mathfrak{b}\colon \Omega \to \mathbb{R}$ is a *semicontinuous equilibrium* (resp. *semiequilibrium*) if $\mathfrak{b}$ is a bounded semicontinuous map. For the convenience of the reader, we recall here some equivalent definitions of semicontinuity which will prove useful later.

Definition 1.19 (Semicontinuity)**.** A map $\mathfrak{b}\colon \Omega \to \mathbb{R}$ is *upper* (resp. *lower*) *semicontinuous* at $\omega_0 \in \Omega$ if for every $x > \mathfrak{b}(\omega_0)$ (resp. $x < \mathfrak{b}(\omega_0)$) there exists $\delta > 0$ such that $x > \mathfrak{b}(\omega)$ (resp. $x < \mathfrak{b}(\omega)$) for all $\omega \in B_\Omega(\omega_0, \delta)$. Equivalently, $\mathfrak{b}$ is upper (resp. lower) semicontinuous at $\omega_0 \in \Omega$ if and only if $\limsup_{\omega\to\omega_0} \mathfrak{b}(\omega) \le \mathfrak{b}(\omega_0)$ (resp. $\liminf_{\omega\to\omega_0} \mathfrak{b}(\omega) \ge \mathfrak{b}(\omega_0)$).

A map $\mathfrak{b}\colon \Omega \to \mathbb{R}$ is *upper* (resp. *lower*) *semicontinuous* if it is upper (resp. lower) semicontinuous at every $\omega \in \Omega$ or, equivalently, if $\{(\omega, x) \mid x \le \mathfrak{b}(\omega)\}$ (resp. $\{(\omega, x) \mid x \ge \mathfrak{b}(\omega)\}$) is closed.

In particular, a semicontinuous map is m-measurable for any measure m. The set of continuity points of a semicontinuous map $\mathfrak{b}\colon \Omega \to \mathbb{R}$ is a *residual set* $\Omega_0 \subseteq \Omega$ (see [23, Corollary 7.6]), that is, its complement $\Omega \setminus \Omega_0$ is a Baire first category set (i.e. $\Omega \setminus \Omega_0$ is a countable union of sets whose closure has empty interior). It can be easily checked that Ω_0 is σ-invariant if $\mathfrak{b}$ is a semicontinuous τ-equilibrium. Another property which will be repeatedly used, which can be directly deduced from the definitions, is that the limit of a nonincreasing (resp. nondecreasing) sequence of upper (resp. lower) semicontinuous maps is upper (resp. lower) semicontinuous.

The proof of the following proposition, which constructs a monotonic family of semiequilibria, can be found in [24, Proposition 3.4.1] and [85, Theorem 3.6].

Proposition 1.20. *Let $\mathfrak{b}\colon \Omega \to \mathbb{R}$ be a τ-superequilibrium (resp. τ-subequilibrium). Then,*

$$\mathfrak{b}_s\colon \Omega \to \mathbb{R}\,, \quad \omega \mapsto v(s, \omega{\cdot}(-s), \mathfrak{b}(\omega{\cdot}(-s))) \tag{1.4}$$

is a τ-superequilibrium (resp. τ-subequilibrium) for all $s \ge 0$, and $\mathfrak{b}_{s_2}(\omega) \le \mathfrak{b}_{s_1}(\omega) \le \mathfrak{b}(\omega)$ (resp. $\mathfrak{b}(\omega) \le \mathfrak{b}_{s_1}(\omega) \le \mathfrak{b}_{s_2}(\omega)$) for all $\omega \in \Omega$ and $0 \le s_1 \le s_2$. If, in addition, $\mathfrak{b}$ is upper (resp. lower) semicontinuous and $\{v(t, \omega, \mathfrak{b}(\omega)) \mid t \ge 0,\, \omega \in \Omega\}$ is bounded, then $\mathfrak{b}_\infty(\omega) = \lim_{s\to\infty} \mathfrak{b}_s(\omega)$ is an upper (resp. lower) semicontinuous τ-equilibrium.

Definition 1.21 (Strong semiequilibrium)**.** A τ-superequilibrium (resp. τ-subequilibrium) $\mathfrak{b}\colon \Omega \to \mathbb{R}$ is said to be *strong* if there exists a time $s_* > 0$ such that $\mathfrak{b}(\omega{\cdot}s_*) > v(s_*, \omega, \mathfrak{b}(\omega))$ (resp. $\mathfrak{b}(\omega{\cdot}s_*) < v(s_*, \omega, \mathfrak{b}(\omega))$) for all $\omega \in \Omega$.

In the previous definition, note that the nonincreasing (resp. nondecreasing) monotonicity of the family $\{\mathfrak{b}_s\}_{s\ge0}$ given by Proposition 1.20 ensures that, if $\mathfrak{b}$ is a strong τ-superequilibrium (resp. τ-subequilibrium), then $\mathfrak{b}(\omega{\cdot}s) > v(s, \omega, \mathfrak{b}(\omega))$ (resp. $\mathfrak{b}(\omega{\cdot}s) < v(s, \omega, \mathfrak{b}(\omega))$) for all $s \ge s_*$.

The next result, which corresponds to [85, Proposition 4.3], gives a useful uniform property of semicontinuous strong semiequilibria if (Ω, σ) is minimal.

Proposition 1.22. *Let (Ω, σ) be minimal and let $\mathfrak{b}\colon \Omega \to \mathbb{R}$ be a semicontinuous strong τ-superequilibrium (resp. τ-subequilibrium). Then, there exist $e_0 > 0$ and a time $s^* > 0$ such that $\mathfrak{b}(\omega) \ge \mathfrak{b}_{s^*}(\omega) + e_0$ (resp. $\mathfrak{b}(\omega) \le \mathfrak{b}_{s^*}(\omega) - e_0$) for every $\omega \in \Omega$, where $\mathfrak{b}_{s^*}$ is defined by (1.4).*

We highlight that the proof of the previous result in [85, Proposition 4.3] does not depend on the type of semicontinuity of $\mathfrak{b}$. Note that the nonincreasing (resp. nondecreasing) monotonicity of the family $\{\mathfrak{b}_s\}_{s\geq 0}$ given by Proposition 1.20 ensures that $\mathfrak{b}(\omega) \geq \mathfrak{b}_s(\omega) + e_0$ (resp. $\mathfrak{b}(\omega) \leq \mathfrak{b}_s(\omega) - e_0$) for every $\omega \in \Omega$ and all $s \geq s^*$.

The concepts of superequilibria and subequilibria are strongly related to those of global upper and lower solutions. A map $\mathfrak{b}\colon \Omega \to \mathbb{R}$ will be said to be C^1 *along the base orbits* if, for any $\omega \in \Omega$, the map $t \mapsto \mathfrak{b}_\omega(t) = \mathfrak{b}(\omega{\cdot}t)$ is C^1 on $\mathbb{R}$. In this case, we represent $\mathfrak{b}'(\omega) = \mathfrak{b}_\omega'(0)$. It is clear by definition that every equilibrium is C^1 along the base orbits.

Definition 1.23 (Global upper and lower solutions)**.** A map $\mathfrak{b}\colon \Omega \to \mathbb{R}$ which is C^1 along the base orbits is a *global upper* (resp. *lower*) *solution* for (1.2) if $\mathfrak{b}'(\omega) \geq \mathfrak{h}(\omega, \mathfrak{b}(\omega))$ (resp. $\mathfrak{b}'(\omega) \leq \mathfrak{h}(\omega, \mathfrak{b}(\omega))$) for every $\omega \in \Omega$, and it is *strict* if the inequalities are strict for every $\omega \in \Omega$.

Constant maps $\mathfrak{b}(\omega) = \rho \in \mathbb{R}$ will be often used as global upper (resp. lower) solutions. Note that ρ is a global upper (resp. lower) solution for (1.2) if $0 \geq \mathfrak{h}(\omega, \rho)$ (resp. $0 \leq \mathfrak{h}(\omega, \rho)$) for every $\omega \in \Omega$, and they are strict if they are so the inequalities.

The following proposition establishes relations between semiequilibria and upper and lower global solutions. Recall that a forward (resp. backward) τ-semiorbit of $(\omega, x) \in \Omega \times \mathbb{R}$ is globally defined if $[0, \infty) \subseteq \mathcal{I}_{\omega,x}$ (resp. $(-\infty, 0] \subseteq \mathcal{I}_{\omega,x}$). The arguments of the proof of the next proposition are what we will hereafter call *standard comparison arguments*, and from now on they will often be used without further explanation.

Proposition 1.24. *Let $\mathfrak{b}\colon \Omega \to \mathbb{R}$ be C^1 along the base orbits. Then,*

(i) *if the forward τ-semiorbit of $(\omega, \mathfrak{b}(\omega))$ is globally defined for all $\omega \in \Omega$ and $\mathfrak{b}$ is a strict global upper (resp. lower) solution for (1.2), then it is a strong τ-superequilibrium (resp. τ-subequilibrium).*

(ii) *If the forward τ-semiorbit of $(\omega, \mathfrak{b}(\omega))$ is globally defined for all $\omega \in \Omega$, then $\mathfrak{b}$ is a τ-superequilibrium (resp. subequilibrium) if and only if it is a global upper (resp. lower) solution for (1.2).*

Proof. We work only with global upper solutions and superequilibria. The proofs for the global lower solutions and subequilibria are analogous.

(i) Let $\mathfrak{b}$ be a strict global upper solution for (1.2). Let us check that, if there exist $t \in \mathbb{R}$ and $\omega \in \Omega$ such that $\mathfrak{b}(\omega{\cdot}t) = v(t, \omega, \mathfrak{b}(\omega))$, then there exists $\varepsilon > 0$ such that $\mathfrak{b}(\omega{\cdot}s) > v(s, \omega, \mathfrak{b}(\omega))$ for $s \in (t, t{+}\varepsilon]$ and $\mathfrak{b}(\omega{\cdot}s) < v(s, \omega, \mathfrak{b}(\omega))$ for $s \in [t{-}\varepsilon, t)$. The continuity of $s \mapsto \mathfrak{b}'(\omega{\cdot}s) - \mathfrak{h}(\omega{\cdot}s, v(s, \omega, \mathfrak{b}(\omega)))$ and its strictly positive sign for $s = t$ ensure that there exists $\varepsilon > 0$ such that $\mathfrak{b}'(\omega{\cdot}s) > \mathfrak{h}(\omega{\cdot}s, v(s, \omega, \mathfrak{b}(\omega)))$ for all $s \in [t - \varepsilon, t + \varepsilon]$. Hence, for all $s \in (t, t + \varepsilon]$,

$$\mathfrak{b}(\omega{\cdot}s) = \mathfrak{b}(\omega{\cdot}t) + \int_t^s \mathfrak{b}'(\omega{\cdot}r)\, dr > \mathfrak{b}(\omega{\cdot}t) + \int_t^s \mathfrak{h}(\omega{\cdot}r, v(r, \omega, \mathfrak{b}(\omega)))\, dr = v(s, \omega, \mathfrak{b}(\omega))\,,$$

and the sign of the inequality is reverted for $s \in [t - \varepsilon, t)$. Let us deduce that $\mathfrak{b}(\omega{\cdot}t) > v(t, \omega, \mathfrak{b}(\omega))$ for all $t > 0$ and all $\omega \in \Omega$. The previous property applied to $t = 0$ shows that $\mathcal{I} = \{s > 0 \text{ such that } \mathfrak{b}(\omega{\cdot}s) > v(s, \omega, \mathfrak{b}(\omega))\}$ is nonempty, and precludes the possibility of $\sup I \in \mathbb{R}$. This shows the assertion.

(ii) Assume that $\mathfrak{b}$ is a superequilibrium and, for contradiction, assume that there exists $\omega \in \Omega$ such that $\mathfrak{b}'(\omega) < \mathfrak{h}(\omega, \mathfrak{b}(\omega))$. Then, by continuity, there exists $t > 0$ such that $\mathfrak{b}'(\omega{\cdot}s) < \mathfrak{h}(\omega{\cdot}s, v(s, \omega, \mathfrak{b}(\omega)))$ for all $s \in [0, t]$. So,

$$\mathfrak{b}(\omega{\cdot}t) = \mathfrak{b}(\omega) + \int_0^t \mathfrak{b}'(\omega{\cdot}s)\, ds < \mathfrak{b}(\omega) + \int_0^t \mathfrak{h}(\omega{\cdot}s, v(s, \omega, \mathfrak{b}(\omega)))\, ds = v(t, \omega, \mathfrak{b}(\omega))\,,$$

which contradicts the fact that $\mathfrak{b}$ is a superequilibrium.

Now, assume that $\mathfrak{b}$ is a global upper solution, that is, $\mathfrak{b}'(\omega) \geq \mathfrak{h}(\omega, \mathfrak{b}(\omega))$ for all $\omega \in \Omega$. Then, $\mathfrak{b}'(\omega) > \mathfrak{h}(\omega, \mathfrak{b}(\omega)) - \varepsilon$ for all $\omega \in \Omega$ and $\varepsilon > 0$, that is, $\mathfrak{b}$ is a strict global upper solution for $x' = \mathfrak{h}(\omega{\cdot}t, x) - \varepsilon$ for all $\varepsilon > 0$. Let v_ε be the cocycle of solutions of $x' = \mathfrak{h}(\omega{\cdot}t, x) - \varepsilon$. Then, (i) ensures that $\mathfrak{b}(\omega{\cdot}t) \geq v_\varepsilon(t, \omega, \mathfrak{b}(\omega))$ for all $\omega \in \Omega$ and $t \geq 0$, so the continuous dependence of solutions as $\varepsilon \downarrow 0$ ensures that $\mathfrak{b}(\omega{\cdot}t) \geq v(t, \omega, \mathfrak{b}(\omega))$ for all $\omega \in \Omega$ and $t \geq 0$, as it was looked for. □

The following two useful propositions, which are quite of technical nature, explore the relations between different semiequilibria which are somehow connected. In the first one, it is proved that, if two semicontinuous semiequilibria coincide on a residual set of continuity points of both maps and one of them is strong, then the τ-orbits starting on a point of the graph of the strong semiequibrium "strongly go through" the graph of the other semiequilibrium. The second proposition shows that, if a region is filled by a continuous parametric family of strong semiequilibria, then the τ-orbits completely traverse that region in a limited amount of time.

Proposition 1.25. *Let (Ω, σ) be minimal. Let $\mathfrak{b}_1 \colon \Omega \to \mathbb{R}$ be a semicontinuous strong superequilibrium (resp. subequilibrium), let $\mathfrak{b}_2 \colon \Omega \to \mathbb{R}$ be a semicontinuous superequilibrium (resp. subequilibrium) and let us assume that there exists a residual set $\mathcal{R}$ of continuity points of both maps such that $\mathfrak{b}_1(\omega) = \mathfrak{b}_2(\omega)$ for all $\omega \in \mathcal{R}$. Then, there exist $e > 0$ and $s_* > 0$ such that $\mathfrak{b}_2(\omega{\cdot}s) - e > v(s, \omega, \mathfrak{b}_1(\omega))$ (resp. $\mathfrak{b}_2(\omega{\cdot}s) + e < v(s, \omega, \mathfrak{b}_1(\omega))$) for all $s \geq s_*$ and $\omega \in \Omega$.*

Proof. Let us work in the case of superequilibria. Note that the nonincreasing monotonicity of $s \mapsto v(s, \omega{\cdot}(-s), \mathfrak{b}_1(\omega{\cdot}(-s)))$ for any $\omega \in \Omega$ given by Proposition 1.20 ensures that it suffices to prove the inequality of the statement for $s = s_*$. Proposition 1.22 ensures that there exist $s_0 > 0$ and $e_0 > 0$ such that $v(s_0, \omega, \mathfrak{b}_1(\omega)) < \mathfrak{b}_1(\omega{\cdot}s_0) - e_0$ for all $\omega \in \Omega$. Let $\omega_0 \in \mathcal{R} \cap \sigma_{-s_0}(\mathcal{R})$. Then, ω_0 is a continuity point of $\mathfrak{b}_1$, $\mathfrak{b}_2$, $\mathfrak{b}_1 \circ \sigma_{s_0}$ and $\mathfrak{b}_2 \circ \sigma_{s_0}$ and, in addition, $\mathfrak{b}_1(\omega_0{\cdot}s_0) = \mathfrak{b}_2(\omega_0{\cdot}s_0)$. Consequently, $v(s_0, \omega_0, \mathfrak{b}_1(\omega_0)) < \mathfrak{b}_1(\omega_0{\cdot}s_0) - e_0 = \mathfrak{b}_2(\omega_0{\cdot}s_0) - e_0$.

By the continuity of the involved semiequilibria at ω_0 and the continuous dependence of solutions on initial data, there exists $\rho > 0$ such that $v(s_0, \omega, \mathfrak{b}_1(\omega)) < \mathfrak{b}_2(\omega{\cdot}s_0) - e_0$ for all $\omega \in B_\Omega(\omega_0, \rho)$. Lemma 1.8 ensures that there exists $s_1 > 0$ such that for all $\omega \in \Omega$ there exists $0 < s_\omega \leq s_1$ such that $\omega{\cdot}s_\omega \in B_\Omega(\omega_0, \rho)$. Therefore, using the cocycle property, the fiber-monotonicity with the definition of superequilibrium and the previous inequality, we have, for all $\omega \in \Omega$,

$$\begin{aligned} v(s_0 + s_\omega, \omega, \mathfrak{b}_1(\omega)) &= v(s_0, \omega{\cdot}s_\omega, v(s_\omega, \omega, \mathfrak{b}_1(\omega))) \\ &\leq v(s_0, \omega{\cdot}s_\omega, \mathfrak{b}_1(\omega{\cdot}s_\omega)) < \mathfrak{b}_2(\omega{\cdot}(s_0 + s_\omega)) - e_0\,. \end{aligned}$$

By fiber-monotonicity, evolving both sides $s_1 - s_\omega > 0$ we obtain

$$v(s_0 + s_1, \omega, \mathfrak{b}_1(\omega)) < v(s_1 - s_\omega, \omega{\cdot}(s_0 + s_\omega), \mathfrak{b}_2(\omega{\cdot}(s_0 + s_\omega)) - e_0)\,. \tag{1.5}$$

Since $v(t,\omega,x)-v(t,\omega,x-e_0)>0$ for all $(t,\omega,x)\in[0,s_1]\times\Omega\times\mathrm{closure}_{\mathbb{R}}(\mathfrak{b}_2(\Omega))$, which is a compact set, there exists $e>0$ such that $v(t,\omega,x-e_0)<v(t,\omega,x)-e$ for all $(t,\omega,x)\in[0,s_1]\times\Omega\times\mathrm{closure}_{\mathbb{R}}(\mathfrak{b}_2(\Omega))$. Then, (1.5) yields

$$v(s_0+s_1,\omega,\mathfrak{b}_1(\omega))<v(s_1-s_\omega,\omega{\cdot}(s_0+s_\omega),\mathfrak{b}_2(\omega{\cdot}(s_0+s_\omega)))-e\le\mathfrak{b}_2(\omega{\cdot}(s_0+s_1))-e$$

for all $\omega\in\Omega$, since $\mathfrak{b}_2$ is a also a superequilibrium. Rewriting $s_*=s_0+s_1$, we obtain $v(s_*,\omega,\mathfrak{b}_1(\omega))<\mathfrak{b}_2(\omega{\cdot}s_*)-e$ for all $\omega\in\Omega$, as we wanted to show. The subequilibrium case is proved analogously. □

Proposition 1.26. *Let (Ω,σ) be minimal. Let $\mathfrak{b}\colon[0,1]\times\Omega\to\mathbb{R}$, $(\lambda,\omega)\mapsto\mathfrak{b}_\lambda(\omega)$ be a continuous map such that $\mathfrak{b}_\lambda$ is a strong superequilibrium (resp. subequilibrium) for every $\lambda\in[0,1]$ and $\mathfrak{b}_\lambda(\omega)\le\mathfrak{b}_\xi(\omega)$ for all $\omega\in\Omega$ if $\lambda\le\xi$. Then, there exist $e_0>0$ and $s_0\ge0$ such that $v(s,\omega,\mathfrak{b}_1(\omega))\le\mathfrak{b}_0(\omega{\cdot}s)-e_0$ (resp. $\mathfrak{b}_1(\omega{\cdot}s)+e_0\le v(s,\omega,\mathfrak{b}_0(\omega))$) for all $s\ge s_0$ and $\omega\in\Omega$.*

Proof. We work in the superequilibrium case. Note again that the nonincreasing monotonicity of $s\mapsto v(s,\omega{\cdot}(-s),\mathfrak{b}_1(\omega{\cdot}(-s)))$ for any $\omega\in\Omega$ given by Proposition 1.20 ensures that it suffices to prove the inequality of the statement for $s=s_0$. We denote $(\mathfrak{b}_\lambda)_s(\omega)=v(s,\omega{\cdot}(-s),\mathfrak{b}_\lambda(\omega{\cdot}(-s)))$. Let us define $\mathcal{C}=\{\lambda\in[0,1]\mid$ there exist $e_\lambda>0$ and $s_\lambda\ge0$ such that $(\mathfrak{b}_1)_{s_\lambda}(\omega)\le\mathfrak{b}_\lambda(\omega)-e_\lambda$ for all $\omega\in\Omega\}$. Since $\mathfrak{b}_1$ is a continuous strong superequilibrium, Proposition 1.22 ensures that $\mathcal{C}$ is nonempty, so let us define $\lambda_0=\inf\mathcal{C}$. As $(\lambda,\omega)\mapsto\mathfrak{b}_\lambda(\omega)$ is continuous, for each $\lambda\in[0,1]$, there exists a neighborhood $\mathcal{V}_\lambda\subseteq[0,1]$ of λ such that $|\mathfrak{b}_\lambda(\omega)-\mathfrak{b}_\xi(\omega)|<e_\lambda/2$ for every $\xi\in\mathcal{V}_\lambda$ and $\omega\in\Omega$. This shows that $\mathcal{C}$ is open in $[0,1]$.

Now, let us prove that $\lambda_0\in\mathcal{C}$. Since $\mathfrak{b}_{\lambda_0}$ is a continuous strong superequilibrium, Proposition 1.22 ensures that there exist $e>0$ and $s_*>0$ such that $(\mathfrak{b}_{\lambda_0})_{s^*}(\omega{\cdot}s_*)=v(s_*,\omega,\mathfrak{b}_{\lambda_0}(\omega))\le\mathfrak{b}_{\lambda_0}(\omega{\cdot}s_*)-e$ for all $\omega\in\Omega$. Fixed $0<e_0<e$, we deduce from the uniform continuity of the cocycle of solutions v on a compact neighborhood of $\mathfrak{b}_{\lambda_0}(\Omega)$ the existence of $\delta_0>0$ such that

$$v(s_*,\omega,x)\le\mathfrak{b}_{\lambda_0}(\omega{\cdot}s_*)-e_0 \tag{1.6}$$

for every $\omega\in\Omega$ and $x\in B_{\mathbb{R}}(\mathfrak{b}_{\lambda_0}(\omega),\delta_0)$. Now, let us take $\lambda_1\in\mathcal{C}$ with $|\mathfrak{b}_{\lambda_1}(\omega)-\mathfrak{b}_{\lambda_0}(\omega)|<\delta_0$ for all $\omega\in\Omega$. Then, by definition of $\mathcal{C}$, there exists $s_{\lambda_1}\ge0$ such that

$$v(s_{\lambda_1},\omega,\mathfrak{b}_1(\omega))<\mathfrak{b}_{\lambda_1}(\omega{\cdot}s_{\lambda_1})$$

for every $\omega\in\Omega$. Evolving the last inequality by monotonicity a time step $s_*>0$ and applying (1.6), we obtain

$$\begin{aligned}(\mathfrak{b}_1)_{s_*+s_{\lambda_1}}(\omega{\cdot}(s_*+s_{\lambda_1}))&=v(s_*+s_{\lambda_1},\omega,\mathfrak{b}_1(\omega))\\&<v(s_*,\omega{\cdot}s_{\lambda_1},\mathfrak{b}_{\lambda_1}(\omega{\cdot}s_{\lambda_1}))\le\mathfrak{b}_{\lambda_0}(\omega{\cdot}(s_*+s_{\lambda_1}))-e_0\end{aligned}$$

for all $\omega\in\Omega$. Changing ω by $\omega{\cdot}(-s_*-s_{\lambda_1})$ shows that $\lambda_0\in\mathcal{C}$. Hence, since $\mathcal{C}$ is open, $0\in\mathcal{C}$, which means the existence of $e_0>0$ and $s_0\ge0$ such that $v(s_0,\omega,\mathfrak{b}_1(\omega))=(\mathfrak{b}_1)_{s_0}(\omega{\cdot}s_0)\le\mathfrak{b}_0(\omega{\cdot}s_0)-e_0$, as asserted. The subequilibrium case is proved analogously. □

1.2.2 Compact invariant sets and global attractor

In this section, some of the main features of compact invariant sets and minimal sets for the skewproduct flow τ induced by (1.2) are described.

Definition 1.27 (Section of a set)**.** Let $\omega \in \Omega$. The ω-*section* of a set $\mathcal{K} \subseteq \Omega \times \mathbb{R}$ is defined as

$$(\mathcal{K})_\omega = \{x \in \mathbb{R} \mid (\omega, x) \in \mathcal{K}\}.$$

A compact τ-invariant set $\mathcal{K} \subset \Omega \times \mathbb{R}$ is said to be *pinched* if there exists $\omega \in \Omega$ such that the section $(\mathcal{K})_\omega$ is a singleton.

Definition 1.28 (Ordered invariant sets)**.** A compact set $\mathcal{K} \subset \Omega \times \mathbb{R}$ is said to *project onto* Ω if the continuous map $\pi\colon \mathcal{K} \to \Omega$, $(\omega, x) \mapsto \omega$ is surjective, that is, if $(\mathcal{K})_\omega$ is nonempty for all $\omega \in \Omega$.

Two disjoint compact sets $\mathcal{K}_1, \mathcal{K}_2 \subset \Omega \times \mathbb{R}$ which project onto Ω are said to be *fiber-ordered* (or simply *ordered*), and this is denoted by $\mathcal{K}_1 < \mathcal{K}_2$, if $x < y$ for every $(\omega, x) \in \mathcal{K}_1$ and $(\omega, y) \in \mathcal{K}_2$.

Notice that, if $\mathcal{K}$ is a compact τ-invariant set, then π maps orbits onto orbits preserving its direction (i.e., it is a *flow epimorphism*). Consequently, if (Ω, σ) is minimal, then any τ-invariant compact set projects onto Ω.

Lemma 1.29. *Let $\mathcal{K} \subset \Omega \times \mathbb{R}$ be a bounded τ-invariant set projecting onto Ω. Then,*

$$\mathfrak{l}_\mathcal{K}(\omega) = \inf\{x \in \mathbb{R} \mid (\omega, x) \in \mathcal{K}\} \quad \textit{and} \quad \mathfrak{u}_\mathcal{K}(\omega) = \sup\{x \in \mathbb{R} \mid (\omega, x) \in \mathcal{K}\} \qquad (1.7)$$

are τ-equilibria. Moreover, if $\mathcal{K}$ is compact, then $\mathfrak{u}_\mathcal{K}$ is upper semicontinuous and $\mathfrak{l}_\mathcal{K}$ is lower semicontinuous. In this case, if $\mathcal{K}_\omega$ is a singleton for all $\omega \in \Omega$, then $\mathcal{K}$ is a τ-copy of the base.

Proof. We will proceed with $\mathfrak{l}_\mathcal{K}$. First, let check that $t \mapsto v(t, \omega, \mathfrak{l}_\mathcal{K}(\omega))$ is bounded and hence globally defined for any $\omega \in \Omega$. If $(\omega, \mathfrak{l}_\mathcal{K}(\omega)) \in \mathcal{K}$, then it is obvious. If $(\omega, \mathfrak{l}_\mathcal{K}(\omega)) \notin \mathcal{K}$, then there exists a sequence (x_n) such that $\mathfrak{l}_\mathcal{K}(\omega) = \lim_{n\to\infty} x_n$ with $(\omega, x_n) \in \mathcal{K}$ for all $n \in \mathbb{N}$. Hence, $v(t, \omega, \mathfrak{l}_\mathcal{K}(\omega)) = \lim_{n\to\infty} v(t, \omega, x_n) \in \mathrm{closure}_{\Omega\times\mathbb{R}}(\mathcal{K})$, so it is bounded for all $t \in \mathbb{R}$, as we wanted to see. Now, let us check that $v(t, \omega, \mathfrak{l}_\mathcal{K}(\omega)) \le y$ for all $(\omega{\cdot}t, y) \in \mathcal{K}$, which means that $v(t, \omega, \mathfrak{l}_\mathcal{K}(\omega)) \le \mathfrak{l}_\mathcal{K}(\omega{\cdot}t)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$. If $(\omega{\cdot}t, y) \in \mathcal{K}$, then $(\omega, v(-t, \omega{\cdot}t, y)) \in \mathcal{K}$, so $v(-t, \omega{\cdot}t, y) \ge \mathfrak{l}_\mathcal{K}(\omega)$ and hence $y \ge v(t, \omega, \mathfrak{l}_\mathcal{K}(\omega))$, as we wanted to see. Finally, assume for contradiction that there exist $\bar\omega \in \Omega$, $\bar t \in \mathbb{R}$ and $\varepsilon > 0$ such that $v(\bar t, \bar\omega, \mathfrak{l}_\mathcal{K}(\bar\omega)) \le \mathfrak{l}_\mathcal{K}(\bar\omega{\cdot}\bar t) - \varepsilon$. Then, $\mathfrak{l}_\mathcal{K}(\bar\omega) \le v(-\bar t, \bar\omega{\cdot}\bar t, \mathfrak{l}_\mathcal{K}(\bar\omega{\cdot}\bar t) - \varepsilon) < v(-\bar t, \bar\omega{\cdot}\bar t, \mathfrak{l}_\mathcal{K}(\bar\omega{\cdot}\bar t)) \le \mathfrak{l}_\mathcal{K}(\bar\omega)$, a contradiction (in the last inequality it has been used the just previously proved assertion). Hence, $\mathfrak{l}_\mathcal{K}$ is a τ-equilibrium. The case of $\mathfrak{u}_\mathcal{K}$ is analogous.

To end the proof, let us assume that $\mathcal{K}$ is compact and check the lower semicontinuity of $\mathfrak{l}_\mathcal{K}$. First note that $\{\mathfrak{l}_\mathcal{K}\} \subseteq \mathcal{K}$ since $\mathcal{K}$ is closed. Take $\omega_n \to \omega_0$ and assume without loss of generality that $\mathfrak{l}_\mathcal{K}(\omega_n)$ converges to $x \in \mathbb{R}$. By compactness, $(\omega_0, x) \in \mathcal{K}$, so $x \ge \mathfrak{l}_\mathcal{K}(\omega_0)$. Thus $\liminf_{\omega\to\omega_0} \mathfrak{l}_\mathcal{K}(\omega) \ge \mathfrak{l}_\mathcal{K}(\omega_0)$ for any $\omega_0 \in \Omega$, that is, $\mathfrak{l}_\mathcal{K}$ is lower semicontinuous. The case of $\mathfrak{u}_\mathcal{K}$ is analogous. If $\mathcal{K}_\omega$ is a singleton for all $\omega \in \Omega$, then $\mathfrak{l}_\mathcal{K} = \mathfrak{u}_\mathcal{K}$, so it is continuous and $\mathcal{K} = \{\mathfrak{l}_\mathcal{K}\} = \{\mathfrak{u}_\mathcal{K}\}$. □

Definition 1.30. Let $\mathcal{K} \subset \Omega \times \mathbb{R}$ be a bounded τ-invariant set. The maps $\mathfrak{l}_\mathcal{K}$ and $\mathfrak{u}_\mathcal{K}$ defined by (1.7) are called the *lower* and *upper equilibria of* $\mathcal{K}$, respectively.

In particular, $\mathfrak{l}_\mathcal{K}$ and $\mathfrak{u}_\mathcal{K}$ are m-measurable for all $m \in \mathfrak{M}_{\rm inv}(\Omega, \sigma)$, which we will often use. Observe that although

$$\mathcal{K} \subseteq \bigcup_{\omega\in\Omega} \left(\{\omega\} \times [\mathfrak{l}_\mathcal{K}(\omega), \mathfrak{u}_\mathcal{K}(\omega)]\right),$$

$\mathcal{K}$ does not necessarily fill all the space between $\mathfrak{l}_\mathcal{K}$ and $\mathfrak{u}_\mathcal{K}$, not even if (Ω, σ) is minimal and $\mathcal{K}$ is a τ-minimal set.

The following definition provides the concept of the global attractor of a flow. This concept, fundamental throughout the document, corresponds to an object which captures the forward dynamics of the flow τ. In Chapter 3 it will be one of the elements for which bifurcations are studied when considering parametric variations of the families of equations that generate the skewproduct flow.

Definition 1.31 (Global attractor)**.** Assume that all the forward τ-semiorbits are globally defined. A compact τ-invariant set $\mathcal{A} \subset \Omega \times \mathbb{R}$ is said to be the *global attractor for* τ if

$$\lim_{t\to\infty} \operatorname{dist}(\tau_t(\mathcal{C}), \mathcal{A}) = 0$$

for every bounded set $\mathcal{C} \subset \Omega \times \mathbb{R}$, where $\tau_t(\mathcal{C}) = \{(\omega{\cdot}t, v(t,\omega,x)) \mid (\omega, x) \in \mathcal{C}\}$ and $\operatorname{dist}(\mathcal{C}_1, \mathcal{C}_2)$ stands for the *Hausdorff semidistance* from $\mathcal{C}_1$ to $\mathcal{C}_2$:

$$\operatorname{dist}(\mathcal{C}_1, \mathcal{C}_2) = \sup_{(\omega_1,x_1)\in\mathcal{C}_1} \left(\inf_{(\omega_2,x_2)\in\mathcal{C}_2} \left(\operatorname{dist}_{\Omega\times\mathbb{R}}\big((\omega_1, x_1), (\omega_2, x_2)\big)\right)\right). \tag{1.8}$$

It is well known that the global attractor for τ, if it exists, is unique. To prove it, it suffices to note that, if both $\mathcal{A}, \mathcal{B} \subset \Omega\times\mathbb{R}$ are global attractors, then $\operatorname{dist}(\mathcal{A},\mathcal{B}) + \operatorname{dist}(\mathcal{B},\mathcal{A}) = 0$, and hence $\mathcal{A} = \mathcal{B}$ since $\operatorname{dist}(\mathcal{A},\mathcal{B}) + \operatorname{dist}(\mathcal{B},\mathcal{A})$ is the Hausdorff distance between $\mathcal{A}$ and $\mathcal{B}$. Morevoer, [21, Theorem 1.7] ensures that, if the global attractor exists, it is the union of all the globally defined and bounded τ-orbits.

1.2.3 Minimal sets in the case of minimal base flow

In this section, let (Ω, σ) be minimal. The following results give a way to construct τ-minimal sets departing from semicontinuous τ-equilibria and provide some properties about the set of continuity points of semicontinuous τ-equilibria in this framework. Recall (see the paragraph after Definition 1.19) that semicontinuous maps have a residual set of continuity points.

Proposition 1.32. *Let* (Ω, σ) *be minimal, let* $\mathfrak{b}\colon \Omega \to \mathbb{R}$ *be a semicontinuous equilibrium and let* ω_0 *be any continuity point of* $\mathfrak{b}$*. Then,*

$$\mathcal{M} = \operatorname{closure}_{\Omega\times\mathbb{R}}\{(\omega_0{\cdot}t, \mathfrak{b}(\omega_0{\cdot}t)) \mid t \in \mathbb{R}\} \tag{1.9}$$

is a τ*-minimal set, it is independent of the choice of* ω_0*, and* $(\mathcal{M})_\omega = \{\mathfrak{b}(\omega)\}$ *for any continuity point* ω *of* $\mathfrak{b}$*. Moreover, the sections* $(\mathcal{N})_\omega$ *of any* τ*-minimal set* $\mathcal{N} \subset \Omega\times\mathbb{R}$ *are singletons for all the points* ω *in a residual* σ*-invariant subset of* Ω*.*

Proof. Since $\mathcal{M}$ is the closure of a τ-invariant set (an orbit), $\mathcal{M}$ is τ-invariant, and it is compact since $\mathfrak{b}$ is bounded. Let us deduce from the minimality of (Ω, σ) that $(\mathcal{M})_\omega = \{\mathfrak{b}(\omega)\}$ for any continuity point ω of $\mathfrak{b}$. Given $x \in (\mathcal{M})_\omega$, we write

$(\omega,x)=\lim_{n\to\infty}(\omega_0{\cdot}t_n,\mathfrak{b}(\omega_0{\cdot}t_n))$ for a suitable sequence (t_n). Since $\mathfrak{b}$ is continuous at ω, then $x=\mathfrak{b}(\omega)$, as asserted. In particular, if $\mathcal{N}\subseteq\mathcal{M}$ is minimal, then $(\mathcal{N})_{\omega_0}=\{\mathfrak{b}(\omega_0)\}$ and hence, since $\mathcal{N}$ must contain the adherence of the τ-orbit of $(\omega_0,\mathfrak{b}(\omega_0))$, $\mathcal{M}\subseteq\mathcal{N}$, which shows the minimality. The independence of the choice of ω_0 follows from $\text{closure}_{\Omega\times\mathbb{R}}\{(\omega{\cdot}t,\mathfrak{b}(\omega{\cdot}t))\mid t\in\mathbb{R}\}\subseteq\mathcal{M}$ for every continuity point ω of $\mathfrak{b}$. Finally, the last assertion is deduced by applying the previous properties to the lower or upper equilibrium of any τ-minimal set $\mathcal{N}$. □

Corollary 1.33. *Let (Ω,σ) be minimal. Then, two different τ-minimal sets $\mathcal{M}$ and $\mathcal{N}$ are always fiber-ordered.*

Proof. Proposition 1.32 ensures that there exists $\omega_0\in\Omega$ such that $(\mathcal{M})_{\omega_0}=\{x_0\}$ and $(\mathcal{N})_{\omega_0}=\{y_0\}$, assume with no loss of generality that $x_0<y_0$, and take $(\omega,x)\in\mathcal{M}$ and $(\omega,y)\in\mathcal{N}$. The minimal character provides a sequence (t_n) such that $(\omega,x)=\lim_{n\to\infty}\tau(t_n,\omega_0,x_0)$ and there is no restriction in assuming that $(\omega,y)=\lim_{n\to\infty}\tau(t_n,\omega_0,y_0)$, so by monotonicity $x=\lim_{n\to\infty}v(t_n,\omega_0,x_0)\le\lim_{n\to\infty}v(t_n,\omega_0,y_0)=y$. Since any τ-minimal set is the closure of the τ-orbit of any of its points, $x<y$ is needed to ensure that $\mathcal{M}$ and $\mathcal{N}$ are distinct. □

Proposition 1.32 also leads to prove the following proposition, which, if (Ω,σ) is minimal, explains that two ordered semicontinuous equilibria with the suitable semicontinuity can only be equal at the continuity points of both maps.

Proposition 1.34. *Let (Ω,σ) be minimal and let $\mathfrak{b}_1,\mathfrak{b}_2\colon\Omega\to\mathbb{R}$ be, respectively, lower and upper semicontinuous equilibria such that $\mathfrak{b}_1(\omega)\le\mathfrak{b}_2(\omega)$ for every $\omega\in\Omega$. If there exists $\omega_0\in\Omega$ such that $\mathfrak{b}_1(\omega_0)=\mathfrak{b}_2(\omega_0)$, then ω_0 is a continuity point of both maps. In particular, $\mathfrak{b}_1$ and $\mathfrak{b}_2$ give rise to the same τ-minimal set through (1.9).*

Proof. If $\omega_n\to\omega_0$ as $n\to\infty$, then

$$\mathfrak{b}_1(\omega_0)\le\liminf_{n\to\infty}\mathfrak{b}_1(\omega_n)\le\limsup_{n\to\infty}\mathfrak{b}_1(\omega_n)\le\limsup_{n\to\infty}\mathfrak{b}_2(\omega_n)\le\mathfrak{b}_2(\omega_0)=\mathfrak{b}_1(\omega_0)\,.$$

The third term can be replaced by $\liminf_{n\to\infty}\mathfrak{b}_2(\omega_n)$. This shows the assertion. □

1.2.4 Variational equations and hyperbolicity

In this section, we use the tools and definitions of Section 1.1.3 in the framework defined by the skewproduct flow τ induced by (1.2), with $\mathfrak{h}\in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$. Given a fixed globally defined solution $t\mapsto v(t,\omega,x)$ of (1.2), we consider its *variational equation*

$$z'=\mathfrak{h}_x\big(\omega{\cdot}t,v(t,\omega,x)\big)\,z=\mathfrak{h}_x\big(\tau_t(\omega,x)\big)\,z\,.$$

So, when we consider a compact τ-invariant set $\mathcal{K}\subset\Omega\times\mathbb{R}$ and the restriction of τ to $\mathcal{K}$, we get a family of variational equations

$$z'=\mathfrak{h}_x\big(\omega{\cdot}t,v(t,\omega,x)\big)\,z=\mathfrak{h}_x\big(\tau_t(\omega,x)\big)\,z\,,\quad(\omega,x)\in\mathcal{K}\,,$$

which plays the role of (1.1) when applying the definitions of Section 1.1.3. In what follows, we will be simultaneously interested both in the invariant and ergodic measures of (Ω,σ) and those of $(\mathcal{K},\tau)$.

Definition 1.35 (Lyapunov exponent of a compact invariant set)**.** The value $\gamma \in \mathbb{R}$ is a *Lyapunov exponent* of a compact τ-invariant set $\mathcal{K} \subset \Omega \times \mathbb{R}$ if it is a Lyapunov exponent of the restriction $\mathfrak{h}_x \colon \mathcal{K} \to \mathbb{R}$, that is, if there exists $(\omega, x) \in \mathcal{K}$ such that

$$\gamma = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathfrak{h}_x\big(\omega{\cdot}s, v(s,\omega,x)\big)\, ds = \lim_{t\to-\infty} \frac{1}{t} \int_0^t \mathfrak{h}_x\big(\omega{\cdot}s, v(s,\omega,x)\big)\, ds\,.$$

The set of all the Lyapunov exponents of $\mathcal{K}$ will be denoted by $\mathrm{Lyap}(\mathcal{K})$. The values $\inf \mathrm{Lyap}(\mathcal{K})$ and $\sup \mathrm{Lyap}(\mathcal{K})$ are the *lower and upper Lyapunov exponents* of $\mathcal{K}$.

Given a compact τ-invariant set $\mathcal{K} \subset \Omega \times \mathbb{R}$ and a Borel measure ν on $\mathcal{K}$, the *marginal measure m of ν on Ω* is defined as $m(\mathcal{B}) = \nu((\mathcal{B} \times \mathbb{R}) \cap \mathcal{K})$ for every Borel set $\mathcal{B} \subseteq \Omega$. It is easy to check that, if $\nu \in \mathfrak{M}_{\mathrm{inv}}(\mathcal{K},\tau)$, then $m \in \mathfrak{M}_{\mathrm{inv}}(\Omega,\sigma)$ and, if $\nu \in \mathfrak{M}_{\mathrm{erg}}(\mathcal{K},\tau)$, then $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$. A measure $\nu \in \mathfrak{M}_{\mathrm{erg}}(\mathcal{K},\tau)$ is said to *project onto* a measure $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$, if m is the marginal measure of ν on Ω.

Theorem 1.36. *Let $\mathcal{K} \subset \Omega \times \mathbb{R}$ be a compact τ-invariant set projecting onto Ω.*

(i) *Let $\nu \in \mathfrak{M}_{\mathrm{erg}}(\mathcal{K},\tau)$ project onto $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$. Then, there exists an m-measurable equilibrium $\mathfrak{b} \colon \Omega \to \mathbb{R}$ with $\{\mathfrak{b}\} \subseteq \mathcal{K}$ such that, for every continuous function $\mathfrak{g} \colon \Omega \times \mathbb{R} \to \mathbb{R}$,*

$$\int_{\mathcal{K}} \mathfrak{g}(\omega,x)\, d\nu = \int_{\Omega} \mathfrak{g}(\omega,\mathfrak{b}(\omega))\, dm\,. \tag{1.10}$$

(ii) *Let $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$ and let $\mathfrak{b} \colon \Omega \to \mathbb{R}$ be an m-measurable τ-equilibrium with $\{\mathfrak{b}\} \subseteq \mathcal{K}$. Then, (1.10) defines $\nu \in \mathfrak{M}_{\mathrm{erg}}(\mathcal{K},\tau)$ projecting on m. In particular, $\int_{\mathcal{K}} \mathfrak{h}_x(\omega,\mathfrak{b}(\omega))\, dm$ is a Lyapunov exponent of $\mathcal{K}$.*

(iii) *There exist $m^l, m^u \in \mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$, an m^l-measurable equilibrium $\mathfrak{b}^l \colon \Omega \to \mathbb{R}$ and an m^u-measurable equilibrium $\mathfrak{b}^u \colon \Omega \to \mathbb{R}$ such that*

$$\inf \mathrm{Lyap}(\mathcal{K}) = \int_{\Omega} \mathfrak{h}_x(\omega,\mathfrak{b}^l(\omega))\, dm^l \;\; \textit{and} \;\; \sup \mathrm{Lyap}(\mathcal{K}) = \int_{\Omega} \mathfrak{h}_x(\omega,\mathfrak{b}^u(\omega))\, dm^u\,.$$

Besides, $[\inf \mathrm{Lyap}(\mathcal{K}),\, \sup \mathrm{Lyap}(\mathcal{K})] = \{\int_{\mathcal{K}} \mathfrak{h}_x(\omega,x)\, d\nu \mid \nu \in \mathfrak{M}_{\mathrm{inv}}(\mathcal{K},\tau)\}$.

Proof. (i) It can be found in [43, Theorem 4.1] and [11, Theorem 1.8.4].

(ii) Riesz Representation Theorem (see [117, Theorem 6.3]) ensures that (1.10) defines a normalized Borel measure ν on $\mathcal{K}$. Since $(\omega,\mathfrak{b}(\omega)) \in (\mathcal{B} \times \mathbb{R}) \cap \mathcal{K}$ if and only if $\omega \in \mathcal{B}$, μ projects on m. To prove that ν is ergodic, it is enough to check that, if $f \in L^1(\mathcal{K},\nu)$ satisfies $f(\omega,x) = f(\omega{\cdot}t, v(t,\omega,x))$ for all $(\omega,x) \in \mathcal{K}$ and $t \in \mathbb{R}$, then f is ν-a.e. constant (see [54, Proposition 1.2 and Theorem 1.6]). Since $f(\omega,\mathfrak{b}(\omega)) = f(\omega{\cdot}t,\mathfrak{b}(\omega{\cdot}t))$ and m is ergodic, the map $f(\omega,\mathfrak{b}(\omega))$ is m-a.e. constant (see again [54, Theorem 1.6]). And this ensures that $f(\omega,x)$ is ν-a.e. constant, since $\nu(\{\mathfrak{b}\}) = \inf\{\nu(\mathcal{V}) \mid \{\mathfrak{b}\} \subset \mathcal{V} \text{ open}\} = \inf\{\int_{\Omega} \chi_{\mathcal{V}}(\omega,\mathfrak{b}(\omega))\, dm \mid \{\mathfrak{b}\} \subset \mathcal{V} \text{ open}\} = 1$, where $\chi_{\mathcal{V}}(\omega,x) = 1$ if $(\omega,x) \in \mathcal{V}$ and $\chi_{\mathcal{V}}(\omega,x) = 0$ otherwise. The last assertion in (ii) follows from the first one and Proposition 1.12(i).

(iii) According to Proposition 1.12(iii), the upper and lower Lyapunov exponents of $\mathcal{K}$ are $\sup \mathrm{Lyap}(\mathcal{K}) = \int_{\mathcal{K}} \mathfrak{h}_x(\omega,x)\, d\nu^l$ and $\inf \mathrm{Lyap}(\mathcal{K}) = \int_{\mathcal{K}} \mathfrak{h}_x(\omega,x)\, d\nu^u$ for some $\nu^l, \nu^u \in \mathfrak{M}_{\mathrm{erg}}(\mathcal{K},\tau)$. Hence, (i) ensures the existence of m^l, m^u, $\mathfrak{b}^l$ and $\mathfrak{b}^u$ as in the statement. The last assertion follows from Proposition 1.12(iii). □

Now, the definitions of hyperbolic τ-copy of the base and hyperbolic τ-minimal set are introduced. To this end, the classical definition of uniformly exponentially stable set at $+\infty$ or $-\infty$ (on the fiber) is introduced.

Definition 1.37 (Uniformly exponentially stable)**.** A compact τ-invariant set $\mathcal{K} \subset \Omega\times\mathbb{R}$ which projects on Ω is said to be *uniformly exponentially stable at* $+\infty$ (resp. at $-\infty$) if there exist a *radius of uniform stability* $\rho > 0$, and a *dichotomy constant pair* (k,γ) with $k \geq 1$ and $\gamma > 0$ such that, if $(\omega, x) \in \mathcal{K}$ and $(\omega, y) \in \Omega\times\mathbb{R}$ satisfy $|x-y| < \delta$, then $v(t,\omega,y)$ is defined for all $t \geq 0$ (resp. $t \leq 0$) and

$$|v(t,\omega,x) - v(t,\omega,y)| \leq k\, e^{-\gamma\, t}|x-y|\,, \quad \text{for all } t \geq 0\,,$$
$$(\text{resp. } |v(t,\omega,x) - v(t,\omega,y)| \leq \ k\, e^{\gamma\, t}\,|x-y|\,, \quad \text{for all } t \leq 0)\,.$$

Definition 1.38 (Hyperbolic copy of the base)**.** A τ-copy of the base is said to be *hyperbolic attractive* (resp. *hyperbolic repulsive*) if it is uniformly exponentially stable at $+\infty$ (resp. $-\infty$). It is said to be *nonhyperbolic* if it is neither hyperbolic attractive nor hyperbolic repulsive.

The following theorem, whose proof is postponed until Section 1.3.2, shows the persistence of hyperbolic copies of the base. To this end, given $m > 0$ and $\mathfrak{h} \in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$, we define the seminorm

$$\|\mathfrak{h}\|_{1,m} = \sup_{(\omega,x)\in\Omega\times[-m,m]} |\mathfrak{h}(\omega,x)| + \sup_{(\omega,x)\in\Omega\times[-m,m]} |\mathfrak{h}_x(\omega,x)|\,,$$

and, given $\mathfrak{b} \in C(\Omega,\mathbb{R})$, we define $\|\mathfrak{b}\|_\infty = \sup_{\omega\in\Omega} |\mathfrak{b}(\omega)|$.

Theorem 1.39 (Persistence of hyperbolic copies of the base)**.** *Let $\mathfrak{h} \in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$, let $\mathfrak{b}_\mathfrak{h}\colon \Omega \to \mathbb{R}$ be a continuous equilibrium for* (1.2) *such that $\{\mathfrak{b}_\mathfrak{h}\}$ is an attractive (resp. repulsive) hyperbolic copy of the base for* (1.2) *with dichotomy constant pair (k_0,γ_0), and take $m > \|\mathfrak{b}_\mathfrak{h}\|_\infty$. Then, for every $\gamma \in (0,\gamma_0)$ and $\varepsilon > 0$, there exists $\rho_\varepsilon > 0$ and $\delta_\varepsilon > 0$ such that, if $\mathfrak{g} \in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$ and $\|\mathfrak{h}-\mathfrak{g}\|_{1,m} < \delta_\varepsilon$, then there exists a continuous equilibrium $\mathfrak{b}_\mathfrak{g}\colon \Omega \to \mathbb{R}$ for $x' = \mathfrak{g}(\omega{\cdot}t, x)$ such that $\{\mathfrak{b}_\mathfrak{g}\}$ is an attractive (resp. repulsive) hyperbolic copy of the base for $x' = \mathfrak{g}(\omega{\cdot}t,x)$ with radius of uniform stability ρ_ε and dichotomy constant pair (k_0,γ), and which satisfies $\|\mathfrak{b}_\mathfrak{h} - \mathfrak{b}_\mathfrak{g}\|_\infty < \varepsilon$.*

The following theorem states that a τ-copy of the base or a τ-minimal set projecting on Ω is hyperbolic attractive (resp. repulsive) if and only if all their Lyapunov exponents are strictly negative (resp. positive). Its proof is also postponed until Section 1.3.2.

Theorem 1.40. *Let $\mathfrak{h} \in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$. Let $\mathcal{K} \subset \Omega\times\mathbb{R}$ be a compact τ-invariant set projecting onto Ω. Assume that its upper and lower equilibria coincide (at least) on a point of each minimal subset $\mathcal{M} \subseteq \Omega$. Then, the upper (resp. lower) Lyapunov exponent of $\mathcal{K}$ is strictly negative (resp. positive) if and only if $\mathcal{K}$ is an attractive (resp. repulsive) hyperbolic copy of the base.*

In addition, if either $\mathcal{K}$ (and hence Ω) is minimal or its upper and lower equilibria coincide on a τ-invariant subset $\Omega_0 \subseteq \Omega$ with $m(\Omega_0) = 1$ for all $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$, then the condition on its upper and lower equilibria holds.

In Section 4.3.4 we will construct an example of a uniformly exponentially stable (with strictly negative upper Lyapunov exponent), pinched, compact (and connected), and τ-invariant set which projects onto the whole base and which is not a τ-copy of the base. That is, the hypotheses of Theorem 1.40 are not redundant.

Remark 1.41. Assume that (Ω, σ) is minimal, and let $\mathcal{M} \subset \Omega \times \mathbb{R}$ be a τ-minimal set. Then all its Lyapunov exponents are strictly negative (resp. positive) if and only if $\mathcal{M}$ is an attractive (resp. repulsive) hyperbolic τ-copy of the base. In this case, we talk about a *hyperbolic* minimal set. Otherwise, $\mathcal{M}$ is *nonhyperbolic*. Note also that, if (Ω, σ) is minimal, then any τ-copy of the base is a τ-minimal set.

The hyperbolic character of a τ-minimal set $\mathcal{M} \subset \Omega \times \mathbb{R}$ admits a useful characterization in terms of the dynamical spectrum of the map $\mathfrak{h}_x \colon \mathcal{M} \to \mathbb{R}$, which we represent by $\mathrm{sp}_{\mathcal{M}}(\mathfrak{h}_x)$:

Proposition 1.42. *Let (Ω, σ) be minimal. Let $\mathfrak{h} \in C^{0,1}(\Omega \times \mathbb{R}, \mathbb{R})$ and let $\mathcal{M} \subset \Omega \times \mathbb{R}$ be a τ-minimal set. Then, $\mathrm{sp}_{\mathcal{M}}(\mathfrak{h}_x) = [\inf \mathrm{Lyap}(\mathcal{M}), \sup \mathrm{Lyap}(\mathcal{M})]$, and $\mathcal{M}$ is hyperbolic attractive (resp. repulsive) if and only if $\mathrm{sp}_{\mathcal{M}}(\mathfrak{h}_x) \subset (-\infty, 0)$ (resp. $\mathrm{sp}_{\mathcal{M}}(\mathfrak{h}_x) \subset (0, \infty)$); and it is nonhyperbolic if and only if $0 \in \mathrm{sp}_{\mathcal{M}}(\mathfrak{h}_x)$.*

Proof. Theorem 1.36(iii) (for $\mathfrak{h}_x \colon \mathcal{M} \to \mathbb{R}$ instead of $\mathfrak{a} \colon \Theta \to \mathbb{R}$) and Remark 1.14.1 prove the first equality, and the remaining assertions follow from Theorem 1.40. □

1.3 Admissible processes and hull extensions

In this section, the *processes formulation* is introduced and its hull extension is described. A detailed account on this formulation and its relation with the skew-product formulation can be found in [63, Chapter 2]. The processes formulation and related concepts deal with individual equations instead of with a family of them. Conversely, suitable ordinary differential equations give rise to families of them and to a skewproduct flow by means of the hull construction.

Definition 1.43 (Admissible map)**.** Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set. A continuous map $h \colon \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ is *admissible*, $h \in C^{0,0}(\mathbb{R} \times \mathcal{U}, \mathbb{R})$, if the restriction of h to $\mathbb{R} \times \mathcal{J}$ is bounded and uniformly continuous for any compact set $\mathcal{J} \subset \mathcal{U}$. A map $h \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1*-admissible* (resp. C^2*-admissible*), $h \in C^{0,1}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ (resp. $h \in C^{0,2}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$), if h is admissible, there exists its derivative h_x with respect to the second variable and h_x is admissible (resp. there exist h_x and h_{xx} and they are admissible).

In most of the cases, we will work with $\mathcal{U} = \mathbb{R}$. Given $h \in C^{0,1}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, we consider the scalar differential equation

$$x' = h(t, x) \tag{1.11}$$

and represent by $t \mapsto x_h(t, s, x)$ the maximal solution of (1.11) with $x_h(s, s, x) = x$, defined on the interval $\mathcal{I}_{s,x} = (\alpha_{s,x}, \beta_{s,x})$. By uniqueness of solutions, the expression $x_h(t, r, x) = x_h(t, s, x_h(s, r, x))$ holds whenever the right-hand term is defined. Then, the map $(t, s, x) \mapsto x_h(t, s, x)$ is a *process*, and is said to be an *admissible process* because h is C^1-admissible. We will also refer to (1.11) as an *admissible ordinary differential equation.*

We define two concepts which we will frequently manage. Two solutions $b_1(t)$ and $b_2(t)$ of (1.11) are *uniformly separated* if they are bounded and $\inf_{t\in\mathbb{R}} |b_2(t)-b_1(t)| > 0$. The solutions $b_1(t), b_2(t), \ldots, b_n(t)$ are *uniformly separated* if any pair of them is uniformly separated.

1.3.1 The hull construction

Let us describe the already mentioned *hull construction*. The proof of the basic properties stated just below can be found in [111, Part I, Theorem 3.1] and [110, Theorem IV.3]. Admissible processes give rise to families of scalar differential equations which can be treated using the tools of Section 1.2.

Given an admissible map $h\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$, we define $h{\cdot}t(s,x) = h(t+s,x)$. The *hull* Ω_h of h is the closure of the set $\{h{\cdot}t \mid t\in\mathbb{R}\}$ on the set $C(\mathbb{R}\times\mathbb{R},\mathbb{R})$,

$$\Omega_h = \mathrm{closure}_{C(\mathbb{R}\times\mathbb{R},\mathbb{R})}\{h{\cdot}t \mid t\in\mathbb{R}\}\,,$$

provided with the compact-open topology. The set Ω_h is a compact metric space, the time-shift map

$$\sigma_h\colon \mathbb{R}\times\Omega_h\to\Omega_h\,,\quad (t,\omega)\mapsto \omega{\cdot}t$$

defines a global continuous flow, and the map

$$\mathfrak{h}\colon \Omega_h\times\mathbb{R}\to\mathbb{R}\,,\quad (\omega,x)\mapsto \mathfrak{h}(\omega,x)=\omega(0,x) \tag{1.12}$$

is continuous, and will be usually called the *extension to the hull* of h. In addition, if h is C^1-admissible, then $\Omega_h\subset C^{0,1}(\mathbb{R}\times\mathbb{R},\mathbb{R})$, and the continuous map $\mathfrak{h}_x(\omega,x) = \omega_x(0,x)$ is the derivative of $\mathfrak{h}$ with respect to x; and, if h is C^2-admissible, then $\Omega_h\subset C^{0,2}(\mathbb{R}\times\mathbb{R},\mathbb{R})$, and the continuous map $\mathfrak{h}_{xx}(\omega,x)=\omega_{xx}(0,x)$ is the second derivative of $\mathfrak{h}$ with respect to x.

In what follows, we will consider both processes and skewproduct flows. Observe that $(t,x)\mapsto\mathfrak{h}(\omega{\cdot}t,x)$ is C^1-admissible for all $\omega\in\Omega$ if $\mathfrak{h}\in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$.

Note that (Ω_h,σ_h) is a transitive flow (see Definition 1.7), i.e., there exists a dense σ_h-orbit: that of the point $h\in\Omega_h$. More precisely, if Ω_h^α and Ω_h^ω are the α-limit set and ω-limit set for σ_h of the element $h\in\Omega_h$, then

Lemma 1.44. $\Omega_h = \Omega_h^\alpha\cup\{h{\cdot}t\mid t\in\mathbb{R}\}\cup\Omega_h^\omega$.

Proof. We can write any $\omega\in\Omega_h$ as $\omega=\lim_{n\to\infty} h{\cdot}t_n$ in the compact-open topology for a suitable sequence (t_n). If a subsequence (t_k) has limit $-\infty$ or $+\infty$, then ω belongs to Ω_h^α or Ω_h^ω, respectively. Otherwise, there exists a subsequence (t_k) with limit $t_0\in\mathbb{R}$, so $\omega=\lim_{n\to\infty} h{\cdot}t_n = h{\cdot}t_0\in\{h{\cdot}t\mid t\in\mathbb{R}\}$. □

In large parts of this document, we deal with minimal flows (Ω,σ). When coming from a single equation $x'=h(t,x)$, this property is known as recurrency of h:

Definition 1.45 (Recurrent map)**.** An admissible map $h\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is *recurrent* if (Ω_h,σ_h) is a minimal flow, i.e., if every σ_h-orbit is dense in Ω_h.

The functions which are almost periodic uniformly on compact sets, defined below, are admissible functions and provide a broad class of recurrent maps (see e.g. [76, Theorems 2.43 and 2.44]). Periodic, quasiperiodic and almost periodic functions fulfill Definition 1.46, as well as, for instance, polynomials with periodic, quasiperiodic or almost periodic coefficients.

Definition 1.46 (Almost periodic function uniformly on compact sets)**.** Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set. The map $h\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is *almost periodic uniformly on compact subsets of* $\mathbb{R}$ if, for any $\varepsilon > 0$ and any compact $\mathcal{K}\subset\mathbb{R}$, the set $\{T\in\mathbb{R}$ such that $|h(t+T,x)-h(t,x)|\le\varepsilon$ for all $t\in\mathbb{R}$ and $x\in\mathcal{K}\}$ is relatively dense in $\mathbb{R}$.

Assume that h is (at least) C^1-admissible, and let us call τ_h the skewproduct flow defined on $\Omega_h\times\mathbb{R}$ by the family of equations (1.2) corresponding to the extension to the hull $\mathfrak{h}$ introduced in (1.12). Note that this family includes (1.11): it is given by the element $\omega = h \in \Omega_h$. In addition, it is not hard to check that, if $\tau_h(t,\omega,x) = (\omega{\cdot}t, v_h(t,\omega,x))$, then $x_h(t,s,x) = v_h(t-s, h{\cdot}s, x)$. This is the *skew-product flow induced by h on its hull.*

Proposition 1.47. *Let $h\in C^{0,1}(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and let Ω_h be its hull. If $x' = h(t,x)$ has a bounded solution $b\colon\mathbb{R}\to\mathbb{R}$ (resp. n uniformly separated solutions $b_1 < b_2 < \ldots < b_n$), then $x' = \omega(t,x)$ has a bounded solution (resp. n uniformly separated solutions) for all $\omega\in\Omega_h$.*

Proof. We write $\omega = \lim_{n\to\infty} h{\cdot}t_n$ in the compact-open topology for a sequence (t_n). Let x_0 be the limit of a suitable subsequence $(b(t_k))$ of $(b(t_n))$. Then, the solution $v_h(t,\omega,x_0)$ of $x' = \omega(t,x)$ (with value x_0 at 0) is bounded, since by continuity of the map v_h, we get $v_h(t,\omega,x_0) = \lim_{k\to\infty} v_h(t,\omega_k,b(t_k)) = \lim_{k\to\infty} b(t+t_k)$. The same argument proves the other assertion. □

1.3.2 Hyperbolic solutions

In this section, we consider the linearized equation around a solution of an ordinary differential equation: its *exponential dichotomy* characterizes the *hyperbolicity* of the solution. The persistence of such solutions under small perturbations and some useful related properties are proved, and the proofs of Theorems 1.39 and 1.40 are provided.

Definition 1.48 (Exponential dichotomy on $\mathbb{R}$)**.** Let $a\colon\mathbb{R}\to\mathbb{R}$ be a continuous map. The linear system

$$z' = a(t)\, z \tag{1.13}$$

has an *exponential dichotomy on* $\mathbb{R}$ if there exist $k\ge 1$ and $\gamma > 0$

$$\exp\int_s^t a(r)\,dr \le k\, e^{-\gamma\,(t-s)} \text{ whenever } t\ge s \tag{1.14}$$

or

$$\exp\int_s^t a(r)\,dr \le k\, e^{\gamma\,(t-s)} \quad \text{whenever } t\le s\,. \tag{1.15}$$

The (non unique) pair (k,γ) is called a *dichotomy constant pair*. If (1.14) holds, then (1.13) is said to be *Hurwitz at* ∞, while if (1.15) holds, then (1.13) is said to be *Hurwitz at* $-\infty$.

Remark 1.49. It is known that, if the equation (1.13) has an exponential dichotomy on $\mathbb{R}$, then there are no other bounded solutions of (1.13) apart from the trivial one: see e.g. [54, Proposition 1.56].

Given a bounded solution $b\colon \mathbb{R} \to \mathbb{R}$ of (1.11), we consider its *variational* or *linearized equation*

$$z' = h_x(t, b(t))\, z\,. \tag{1.16}$$

Definition 1.50 (Hyperbolic solution)**.** A bounded solution $\tilde b\colon \mathbb{R} \to \mathbb{R}$ of (1.11) is *hyperbolic attractive* (resp. *repulsive*) if there exists $k \ge 1$ and $\gamma > 0$ such that

$$\exp \int_s^t h_x(r, \tilde b(r))\, dr \le k\, e^{-\gamma\,(t-s)} \text{ whenever } t \ge s \tag{1.17}$$

$$\Big(\text{resp. } \exp \int_s^t h_x(r, \tilde b(r))\, dr \le k\, e^{\gamma\,(t-s)} \text{ whenever } t \le s\Big)\,. \tag{1.18}$$

that is, if its variational equation (1.16) is of Hurwitz type at ∞ (resp. $-\infty$). A dichotomy constant pair of the exponential dichtomy is called a *dichotomy constant pair* for $\tilde b$. If (1.17) (resp. (1.18)) holds, then $\tilde b$ is said to be *attractive* (resp. *repulsive*).

Remark 1.51. Notice that $\tilde b(t)$ cannot be at the same time hyperbolic attractive and repulsive: if this were the case, we would have $\exp \int_0^t h_x(r, \tilde b(r))\, dr \le k_1\, e^{-\gamma_1 t}$ and $\exp \int_t^0 h_x(r, \tilde b(r))\, dr \le k_2\, e^{-\gamma_2 t}$ for all $t > 0$, with positive constants k_1, k_2, γ_1 and γ_2. Hence, $1 \le k_1\, k_2\, e^{-(\gamma_1+\gamma_2)\, t}$ for all $t \ge 0$, which is impossible.

Theorem 1.52 shows the persistence of the existence of hyperbolic solutions under small variations on the coefficient function h. It is a classical result (see e.g. [7, Lemma 3.3] or [95, Theorem 3.8]), but we include a proof in our particular setting.

Given $m > 0$ and $h \in C^{0,1}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, we define the seminorm

$$\|h\|_{1,m} = \sup_{(t,x)\in\mathbb{R}\times[-m,m]} |h(t,x)| + \sup_{(t,x)\in\mathbb{R}\times[-m,m]} |h_x(t,x)|\,,$$

and, given $b \in C(\mathbb{R}, \mathbb{R})$ we define $\|b\|_\infty = \sup_{t\in\mathbb{R}} |b(t)|$.

Theorem 1.52 (Persistence of hyperbolic solutions)**.** *Let $h \in C^{0,1}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, let $\tilde b_h$ be an attractive (resp. repulsive) hyperbolic solution of (1.11) with dichotomy constant pair (k_0, γ_0), and take $m > \|\tilde b_h\|_\infty$. Then, for every $\gamma \in (0, \gamma_0)$ and $\varepsilon > 0$, there exists a constant $\delta_\varepsilon > 0$ and a radius of uniform stability $\rho_\varepsilon > 0$ such that, if $g \in C^{0,1}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\|h - g\|_{1,m} < \delta_\varepsilon$, then*

(i) *there exists an attractive (resp. repulsive) hyperbolic solution $\tilde b_g$ of $x' = g(t,x)$ with dichotomy constant pair (k_0, γ) which satisfies $\big\|\tilde b_h - \tilde b_g\big\|_\infty < \varepsilon$, and it is the unique bounded solution satisfying $\big\|\tilde b_h - \tilde b_g\big\|_\infty < \varepsilon$;*

(ii) *if $|\tilde b_g(t_0) - x| \le \rho_\varepsilon$, then*

$$|\tilde b_g(t) - x_g(t, t_0, x)| \le k_0\, e^{-\gamma\,(t-t_0)} |\tilde b_g(t_0) - x| \quad \textit{for all } t \ge t_0\,,$$

$$(\textit{resp. } |\tilde b_g(t) - x_g(t, t_0, x)| \le k_0\, e^{\gamma\,(t-t_0)}\, |\tilde b_g(t_0) - x| \quad \textit{for all } t \le t_0)\,.$$

Proof. Let us define $\delta_0 = (\gamma_0 - \gamma)/k_0$. Recall that [25, Lecture 3] ensures that every continuous map $a\colon \mathbb{R} \to \mathbb{R}$ with $\|h_x(\cdot, \tilde b_h(\cdot)) - a(\cdot)\|_\infty < \delta_0$ determines a new Hurwitz equation, being (k_0, γ) a dichotomy constant pair for its hyperbolic solution 0. This fact will be used at the end of the proof.

We will work in the case in which $\tilde{b}_h$ is hyperbolic attractive. The change of variables $x = \tilde{b}_h(t) + y$ takes the equation $x' = g(t,x)$ to

$$y' = h_x(t, \tilde{b}_h(t))\, y + r_g(t,y)\,,$$

where

$$r_g(t,y) = g(t, \tilde{b}_h(t) + y) - h_x(t, \tilde{b}_h(t))\, y - h(t, \tilde{b}_h(t))\,.$$

Let $\varepsilon_0 \in (0,m]$ satisfy $\|\tilde{b}_h\|_\infty \le m - \varepsilon_0$. Since $h \in C^{0,1}(\mathbb{R}\times\mathbb{R},\mathbb{R})$, there exists $\varepsilon \in (0, \min\{1,\varepsilon_0\})$ such that $|h_x(t,\tilde{b}_h(t)+y) - h_x(t,\tilde{b}_h(t))| \le \delta_0/4$ for every $y \in [-\epsilon,\varepsilon]$ and $t \in \mathbb{R}$. Consequently, for any $g \in C^{0,1}(\mathbb{R}\times\mathbb{R},\mathbb{R})$ such that $\|h-g\|_{1,m} \le \delta_0/4$,

$$\begin{aligned}
&|g_x(t,\tilde{b}_h(t)+y) - h_x(t,\tilde{b}_h(t))| \\
&\quad \le |g_x(t,\tilde{b}_h(t)+y) - h_x(t,\tilde{b}_h(t)+y)| + |h_x(t,\tilde{b}_h(t)+y) - h_x(t,\tilde{b}_h(t))| \\
&\quad \le \|h-g\|_{1,m} + \frac{\delta_0}{4} \le \frac{\delta_0}{2}
\end{aligned} \tag{1.19}$$

for every $y \in [-\varepsilon,\varepsilon]$ and $t \in \mathbb{R}$, since $\|\tilde{b}_h + y\|_\infty \le m$. Notice that r_g can be rewritten by means of the Fundamental Theorem of Calculus

$$r_g(t,y) = y \int_0^1 \big(g_x(t,\tilde{b}_h(t)+s\,y) - h_x(t,\tilde{b}_h(t))\big)\, ds + g(t,\tilde{b}_h(t)) - h(t,\tilde{b}_h(t))\,,$$

so (1.19) ensures that $|r_g(t,y)| \le \delta_0\,\varepsilon/2 + \|h-g\|_{1,m} \le 3\,\delta_0\,\varepsilon/4$ for every $y \in [-\varepsilon,\varepsilon]$ and $t \in \mathbb{R}$, if $\|h-g\|_{1,m} \le \delta_0\,\varepsilon/4$. Since

$$r_g(t,y_1) - r_g(t,y_2) = (y_1 - y_2) \int_0^1 \Big(g_x(t,\tilde{b}_h(t) + s\,y_1 + (1-s)\,y_2) - h_x(t,\tilde{b}_h(t))\Big)\, ds\,,$$

(1.19) also ensures that $|r_g(t,y_1) - r_g(t,y_2)| \le (\delta_0/2)\,|y_1 - y_2|$ for every $y_1, y_2 \in [-\varepsilon,\varepsilon]$ and $t \in \mathbb{R}$, if $\|h-g\|_{1,m} \le \delta_0/4$. Let us take $g \in C^{0,1}(\mathbb{R}\times\mathbb{R},\mathbb{R})$ with $\|h-g\|_{1,m} \le \delta_0\,\varepsilon/4 \le \delta_0/4$. The results of [25, Lecture 3] ensure that, for any $y_0 \in C(\mathbb{R},\mathbb{R})$, there is a unique bounded solution Ty_0 of $y' = h_x(t,\tilde{b}_h(t))\,y + r_g(t,y_0(t))$, given by

$$Ty_0(t) = \int_{-\infty}^t \exp\left(\int_s^t h_x(r,\tilde{b}_h(r))\, dr\right) r_g(s,y_0(s))\, ds\,.$$

We take y_0 with $\|y_0\|_\infty \le \varepsilon$. Since (k_0,γ) is a dichotomy constant pair for $\tilde{b}_h$, we get $\|Ty_0\|_\infty \le (3\varepsilon/4)(\delta_0 k_0/\gamma) < 3\varepsilon/4$ and $\|Ty_1 - Ty_2\|_\infty \le (1/2)(\delta_0\,k_0/\gamma)\,\|y_1-y_2\|_\infty < (1/2)\,\|y_1 - y_2\|_\infty$. The map $T\colon C(\mathbb{R},[-\varepsilon,\varepsilon]) \to C(\mathbb{R},[-\varepsilon,\varepsilon])$ is a contraction and thus it has a unique fixed point y_g. It follows easily that $\tilde{b}_g = \tilde{b}_h + y_g$ is a bounded solution of $x' = g(t,x)$, and it satisfies $\|\tilde{b}_h - \tilde{b}_g\|_\infty \le \varepsilon$. The uniqueness of the fixed point ensures that this is the unique bounded solution with this property. The bound (1.19) and the choice of δ_0 at the beginning of the proof ensure that $\tilde{b}_g$ is an attractive hyperbolic solution of $x' = g(t,x)$ with dichotomy constant pair (k_0,γ), from where (i) follows.

To check (ii), we use again the change the change of variables $x = \tilde{b}_g(t) + y$ to rewrite $x' = g(t,x)$ and use the Fundamental Theorem of Calculus to obtain:

$$y' = g_x(t,\tilde{b}_g(t))\,y + y \int_0^1 \Big(g_x(t,\tilde{b}_g(t)+s\,y) - g_x(t,\tilde{b}_g(t))\Big)\, ds\,.$$

Given $\nu \in (0, k_0/\gamma)$, since

$$
\begin{aligned}
&|g_x(t,\tilde b_g(t)+s\,y)-g_x(t,\tilde b_g(t))|\\
&\qquad\le |g_x(t,\tilde b_g(t)+s\,y)-h_x(t,\tilde b_g(t)+s\,y)|+|h_x(t,\tilde b_g(t)+s\,y)-h_x(t,\tilde b_g(t))|\\
&\qquad\quad+|h_x(t,\tilde b_g(t))-g_x(t,\tilde b_g(t))|\,,
\end{aligned}
$$

there exist $\bar\delta \in (0, \delta_0/4)$ and $\bar\varepsilon \in (0, \varepsilon)$ such that $|g_x(t,\tilde b_g(t)+s\,y)-g_x(t,\tilde b_g(t))| \le \nu$ for every $y \in [-\bar\varepsilon, \bar\varepsilon]$ and $t \in \mathbb{R}$ if $\|h-g\|_{1,m} \le \bar\delta$. Since (k_0, γ) is a common dichotomy constant pair for the hyperbolic solution $\tilde b_g$ of $x' = g(t,x)$ for all these maps g, the First Approximation Theorem (see [46, Chapter III, Theorem 2.4] and its proof) provides $\rho_\varepsilon > 0$ satisfying the statement. The proof is analogous in the repulsive case. □

Theorem 1.52 allows us to identify hyperbolic solutions of (1.11) with uniformly exponentially stable solutions. Recall that a solution $\tilde b_h$ of (1.11) is *uniformly exponentially stable as time increases* (resp. *decreases*) if there exists a *radius of uniform stability* $\rho > 0$ and a dichotomy constant pair (k, γ) with $k \ge 1$ and $\gamma > 0$ such that, if $|\tilde b_h(s) - x| \le \rho$, then

$$
\begin{aligned}
|\tilde b_h(t)-x_h(t,s,x)| &\le k_0\, e^{-\gamma\,(t-s)}|\tilde b_h(s)-x| \quad \text{for all } t\ge s\,,\\
(\text{resp. } |\tilde b_h(t)-x_h(t,s,x)| &\le k_0\, e^{\gamma\,(t-s)}\,|\tilde b_h(s)-x| \quad \text{for all } t\le s)\,.
\end{aligned}
$$

The following proposition provides uniform separation of any other solution from hyperbolic solutions in halflines.

Corollary 1.53. *Let $h \in C^{0,1}(\mathbb{R}\times\mathbb{R},\mathbb{R})$. A bounded solution $\tilde b_h$ of (1.11) is hyperbolic attractive (resp. repulsive) if and only if it is uniformly exponentially stable as time increases (resp. decreases).*

Proof. Theorem 1.52(ii) shows that an attractive (resp. repulsive) hyperbolic solution of (1.11) is uniformly exponentially stable as time increases (resp. decreases).

Conversely, let $\tilde b_h$ be uniformly exponentially stable as time increases. Let ρ be a radius of uniform stability and (k, γ) be a dichotomy constant pair. Hence,

$$
\frac{\partial}{\partial x}\,x_h(t,s,x)\Big|_{x=\tilde b_h(s)} = \lim_{\varepsilon\to 0}\frac{x_h(t,s,\tilde b_h(s)+\varepsilon)-x_h(t,s,\tilde b_h(s))}{\varepsilon} \le k\,e^{-\gamma\,(t-s)}
$$

for $t \ge s$. This derivative solves the variational equation $z' = h_x(t,\tilde b_\omega(t))\,z$ and has value 1 at $t = s$, so $(\partial/\partial x)\,x_h(t,s,x)|_{x=\tilde b_h(s)} = \exp\int_s^t h_x(r,\tilde b_\omega(r))\,dr$ from where the assertion follows. If $\tilde b_h$ is uniformly exponentially stable as time decreases, the proof is analogous. □

The next useful result will be combined with the previous one in the proofs of some of the main results.

Proposition 1.54. *Let $\mathfrak{h} \in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$, let $\tilde b\colon \mathbb{R}\to\mathbb{R}$ be an attractive (resp. repulsive) hyperbolic solution of $x' = \mathfrak{h}(\omega_0{\cdot}t, x)$ for a point $\omega_0 \in \Omega$, and let*

$$
\mathcal{K} = \mathrm{closure}_{\Omega\times\mathbb{R}}\{(\omega_0{\cdot}t,\tilde b(t)) \mid t\in\mathbb{R}\}\,.
$$

(i) *If* $(\bar\omega,\bar x)\in\mathcal{K}$, *then* $\tilde c(t)=v(t,\bar\omega,\bar x)$ *is an attractive (resp. repulsive) hyperbolic solution of* $x'=\mathfrak{h}(\bar\omega{\cdot}t,x)$, *with the same dichotomy constant pairs as* $\tilde b$.

(ii) $\sup\mathrm{Lyap}(\mathcal{K})<0$ *(resp.* $\inf\mathrm{Lyap}(\mathcal{K})>0$*) and* $\mathcal{K}$ *is uniformly exponentially stable at* $+\infty$ *(resp.* $-\infty$*).*

(iii) *Let* $\mathcal{O}\subseteq\mathcal{K}$ *be the* α*-limit set (resp.* ω*-limit set) for* τ *of* $(\omega_0,\tilde b(0))$. *If there exists* $\omega_1\in\Omega$ *such that* $\mathcal{O}_{\omega_1}=\mathcal{K}_{\omega_1}$, *then* $\mathcal{K}_{\omega_0\cdot s}$ *is a singleton for all* $s\in\mathbb{R}$.

(iv) *If* (Ω,σ) *is minimal, then* $\mathcal{K}$ *is a hyperbolic* τ*-copy of the base.*

Proof. (i)-(ii) Let $(\bar\omega,\bar x)\in\mathcal{K}$ and let (t_n) be such that $(\bar\omega,\bar x)=\lim_{n\to\infty}(\omega{\cdot}t_n,\tilde b(t_n))$. Notice that we have $\tilde c(t)=v(t,\bar\omega,\bar x)=\lim_{n\to\infty}v(t,\omega_0{\cdot}t_n,\tilde b(t_n))=\lim_{n\to\infty}v(t+t_n,\omega_0,\tilde b(0))=\lim_{n\to\infty}\tilde b(t+t_n)$ for all $t\in\mathbb{R}$. In particular, $\tilde c$ is bounded. Let us reason in the attractive case. Lebesgue's Dominated Convergence Theorem shows that $\exp\int_s^t\mathfrak{h}_x(\bar\omega{\cdot}r,\tilde c(r))\,dr=\lim_{n\to\infty}\exp\int_s^t\mathfrak{h}_x(\omega_0{\cdot}(t_n+r),\tilde b(t_n+r))\,dr=\lim_{n\to\infty}\exp\int_{s-t_n}^{t-t_n}\mathfrak{h}_x(\omega_0{\cdot}r,\tilde b(r))\,dr\le k\,e^{-\gamma(t-s)}$ if $t\ge s$, where (k,γ) is a dichotomy constant pair for $\tilde b$. This proves (i). In particular, for all $(\omega,x)\in\mathcal{K}$, we have $\lim_{t\to\infty}\exp\int_0^t\mathfrak{h}_x(\tau(r,\omega,x))\,dr\le-\gamma$, and hence $\sup\mathrm{Lyap}(\mathcal{K})\le-\gamma$. The First Approximation Theorem (see [46, Chapter III, Theorem 2.4] and its proof) used as at the end of the proof of Theorem 1.52 shows (ii). The arguments are the same in the repulsive case.

(iii) Since $y\in\mathcal{K}_{\omega_0\cdot s}$ if and only if $y=v(s,\omega_0,x)$ for an $x\in\mathcal{K}_{\omega_0}$, it suffices to check that $\mathcal{K}_{\omega_0}$ is a singleton. Let us work in the attractive case, as the repulsive one is analogous. Let $\mathfrak{l}$ and $\mathfrak{u}$ be the lower and upper τ-equilibria of $\mathcal{K}$, let $\rho>0$ and (k,γ) be a radius of uniform stability and a dichotomy constant pair for $\mathcal{K}$ (provided by (ii): see Definition 1.37). The hypothesis $\mathcal{O}_{\omega_1}=\mathcal{K}_{\omega_1}$ guarantees that $(\omega_1,\mathfrak{u}(\omega_1))\in\mathcal{O}$. Let $(t_n)\downarrow-\infty$ be such that $(\omega_1,\mathfrak{u}(\omega_1))=\lim_{n\to\infty}(\omega_0{\cdot}t_n,\tilde b(t_n))$. Since $\tilde b(t_n)\le\mathfrak{u}(\omega_0{\cdot}t_n)$, and since we can assume without restriction that $(\omega_0{\cdot}t_n,\mathfrak{u}(\omega_0{\cdot}t_n))$ converges to an element of $\mathcal{K}$, $\mathfrak{u}(\omega_1)=\lim_{n\to\infty}\tilde b(t_n)\le\lim_{n\to\infty}\mathfrak{u}(\omega_0{\cdot}t_n)\le\mathfrak{u}(\omega_1)$. Then, there exists $n_0\in\mathbb{N}$ such that $|\mathfrak{u}(\omega_0{\cdot}t_n)-\tilde b(t_n)|<\rho$ for all $n\ge n_0$, so taking $n\to\infty$ in $|\mathfrak{u}(\omega_0)-\tilde b(0)|=|v(-t_n,\omega_0{\cdot}t_n,\mathfrak{u}(\omega_0{\cdot}t_n))-v(-t_n,\omega_0{\cdot}t_n,\tilde b(t_n))|\le k\,e^{\gamma t_n}|\mathfrak{u}(\omega_0{\cdot}t_n)-\tilde b(t_n)|<k\,\rho\,e^{\gamma t_n}$ provides $\mathfrak{u}(\omega_0)=\tilde b(0)$. Analogously, $\mathfrak{l}(\omega_0)=\tilde b(0)$, so $\mathcal{K}_{\omega_0}=\{\tilde b(0)\}$.

(iv) Let us work in the attractive case. Let $\mathcal{M}\subset\Omega\times\mathbb{R}$ be a τ-minimal set contained in the α-limit set for τ of $(\omega_0,\tilde b(0))$. Then $\mathcal{M}\subseteq\mathcal{K}$, so that $\sup\mathrm{Lyap}(\mathcal{M})\le\sup\mathrm{Lyap}(\mathcal{K})<0$. Proposition 1.42 ensures that $\mathcal{M}$ is an attractive hyperbolic τ-minimal set. Corollary 1.58, which will be proved without using this result, ensures that $(\omega_0,\tilde b(0))\in\mathcal{M}$. Therefore, $\mathcal{K}=\mathcal{M}$, that is, $\mathcal{K}$ is a hyperbolic τ-copy of the base. □

The following proposition relates the concept of hyperbolic copy of the base with that of hyperbolic solution.

Proposition 1.55. *Let* $\mathfrak{h}\in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$, *and let* $\mathfrak{b}\colon\Omega\to\mathbb{R}$ *determine an attractive (resp. repulsive) copy of the base for* (1.2) *with dichotomy constant pair* (k,γ). *For any* $\omega\in\Omega$, *the function* $\tilde b_\omega$ *defined by* $\tilde b_\omega(t)=\mathfrak{b}(\omega{\cdot}t)$ *is an attractive (resp. repulsive) hyperbolic solution of* $(1.2)_\omega$ *with dichotomy constant pair* (k,γ).

Proof. Let us reason in the attractive case, fixing $\omega\in\Omega$. Let us define $\omega^*(t,x)=\mathfrak{h}(\omega{\cdot}t,x)$ and v as in (1.3). Then, the solution $x_\omega(t,s,x)$ of $x'=\omega^*(t,x)$ (i.e., of

$(1.2)_\omega$), coincides with $v(t-s,\omega{\cdot}s,x)$, and $\tilde b_\omega(t)=v(t-s,\omega{\cdot}s,\tilde b_\omega(s))$. Let $\rho>0$ be a radius of uniform stability for $\mathfrak{b}$. Hence, the hyperbolicity of $\mathfrak{b}$ ensures that, if $|x-\tilde b_\omega(s)|\le\rho$ for an $s\in\mathbb{R}$, then $x_\omega(t,s,x)$ exists for all $t\ge s$ and it satisfies $|x_\omega(t,s,x)-\tilde b_\omega(t)|\le k\,e^{-\gamma(t-s)}\,|x-\tilde b_\omega(s)|$. Therefore, Corollary 1.53 proves the assertion. □

The following proposition provides uniform separation as time decreases (resp. increases) of any solution from an attractive (resp. repulsive) hyperbolic solution.

Proposition 1.56. *Let $\tilde b(t)$ be an attractive (resp. repulsive) hyperbolic solution of (1.11) for $h\in C^{0,1}(\mathbb{R}\times\mathbb{R},\mathbb{R})$. Then, $\inf_{t<t_0}|\tilde b(t)-\bar x(t)|>0$ (resp. $\inf_{t>t_0}|\tilde b(t)-\bar x(t)|>0$) for any $t_0\in\mathbb{R}$ and any solution $\bar x(t)\ne\tilde b(t)$ defined on $(-\infty,t_0]$ (resp on $[t_0,\infty)$).*

Proof. We reason in the attractive case, assuming for contradiction the existence of $(t_n)\downarrow-\infty$ such that $\lim_{n\to\infty}|\tilde b(t_n)-\bar x(t_n)|=0$. Corollary 1.53 provides $k\ge1$ and $\gamma>0$ such that, for large enough n,

$$|\tilde b(t_0)-\bar x(t_0)|=|x_h(t_0,t_n,\tilde b(t_n))-x_h(t_0,t_n,\bar x(t_n))|\le k\,e^{-\gamma(t_0-t_n)}|\tilde b(t_n)-\bar x(t_n)|\,.$$

The contradiction follows, since the last term tends to 0 as $n\to\infty$. □

We complete this section with a skewproduct version of Proposition 1.56 and a useful corollary about the τ-minimal sets generated by (1.9) when the base flow is minimal.

Proposition 1.57. *Let the family (1.2) be given by $\mathfrak{h}\in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$, and assume that the α-limit set (resp. ω-limit set) of $(\bar\omega,b_0)$ for τ is an attractive (resp. repulsive) hyperbolic τ-copy of the α-limit set $\Omega^\alpha_{\bar\omega}$ (resp. ω-limit set $\Omega^\omega_{\bar\omega}$) of $\bar\omega$, say $\{\mathfrak{b}\}$. Then, $\{\mathfrak{b}\}$ does not intersect the α-limit set (resp. ω-limit set) of any $(\bar\omega,x)$ with $x\ne b_0$ and bounded backward semiorbit (resp. bounded forward semiorbit).*

Proof. We reason in the attractive case. Assume the existence of the α-limit set $\mathcal{K}$ of a point $(\bar\omega,x)$ with $x\ne b_0$, and, for contradiction, the existence of $(\omega,\mathfrak{b}(\omega))\in\mathcal{K}$. We write $(\omega,\mathfrak{b}(\omega))=\lim_{n\to\infty}(\bar\omega{\cdot}t_n,v(t_n,\bar\omega,x))$ for a suitable sequence $(t_n)\downarrow-\infty$, assume without restriction the existence of $\lim_{n\to\infty}v(t_n,\bar\omega,b_0)$, observe that this limit is also $\mathfrak{b}(\omega)$, and note that this contradicts Proposition 1.56. □

Corollary 1.58. *Let (Ω,σ) be minimal, and let $\mathfrak{h}\in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$. Then,*

(i) *an attractive (resp. repulsive) hyperbolic τ-minimal set does not intersect the α-limit (resp. ω-limit set) of any orbit outside itself.*

(ii) *Let $\mathfrak{b}\colon\Omega\to\mathbb{R}$ be a semicontinuous τ-equilibrium and let $\mathcal{M}$ be the τ-minimal set given by (1.9). If $\mathcal{M}$ is hyperbolic, then $\mathfrak{b}$ is continuous and $\mathcal{M}=\{\mathfrak{b}\}$.*

(iii) *If there exists the global attractor $\mathcal{A}$ for τ and there exists a unique τ-minimal set $\mathcal{M}$, which is a hyperbolic attractive copy of the base, then $\mathcal{A}=\mathcal{M}$.*

Proof. (i) The assertions follow from Proposition 1.57, since a hyperbolic minimal set is always a copy of the base (see Remark 1.41), and it is the α-limit and ω-limit set of any of its orbits.

(ii) We assume that $\mathcal{M}$ is attractive (resp. repulsive), and, for contradiction, the existence of $(\omega_1, \mathfrak{b}(\omega_1)) \notin \mathcal{M}$. Let $\mathcal{K}$ be the α-limit set (resp. ω-limit set) of $(\omega_1, \mathfrak{b}(\omega_1))$ for τ. It is easy to check that that $(\omega, \mathfrak{b}(\omega)) \in \mathcal{K}$ for any continuity point ω of $\mathfrak{b}$. Proposition 1.32 shows that $\mathcal{M} \cap \mathcal{K}$ is nonempty. This contradicts (i). So, the graph of $\mathfrak{b}$ is contained in $\mathcal{M}$, which is the graph of a continuous τ-equilibrium.

(iii) If there exists $(\omega, x) \in \mathcal{A} \setminus \mathcal{M}$, then its α-limit set for τ contains the unique τ-minimal set $\mathcal{M}$, which contradicts (i). □

The proofs of the two Theorems 1.39 and 1.40 of Section 1.2.4 that had been postponed complete this section.

Proof of Theorem 1.39. In the attractive case, we repeat step by step the proof of Theorem 1.52, using the uniform properties on Ω to replace those for $t \in \mathbb{R}$. For any continuous $\mathfrak{b} \colon \Omega \to \mathbb{R}$, let $T\mathfrak{b} \colon \Omega \to \Omega$ be the map defined by

$$T\mathfrak{b}(\omega) = \int_{-\infty}^{0} \exp\left(\int_{s}^{0} \mathfrak{h}_x(\omega{\cdot}r, \mathfrak{b}_{\mathfrak{h}}(\omega{\cdot}r))\, dr\right) r_{\mathfrak{g}}(\omega{\cdot}s, \mathfrak{b}(\omega{\cdot}s))\, ds$$

where

$$r_{\mathfrak{g}}(\omega, y) = \mathfrak{g}(\omega, \mathfrak{b}_{\mathfrak{h}}(\omega) + y) - \mathfrak{h}_x(\omega, \mathfrak{b}_h(\omega))\, y - \mathfrak{h}(\omega, \mathfrak{b}_{\mathfrak{h}}(\omega))\,.$$

Since the integrand can be bounded by $k_0\, e^{\gamma_0 s} \max_{\omega\in\Omega} r_{\mathfrak{g}}(\omega, \mathfrak{b}(\omega))$ for $s \le 0$, the map $T\mathfrak{b}$ is continuous on Ω. Hence $T \colon C(\Omega, [-\varepsilon, \varepsilon]) \mapsto C(\Omega, [-\varepsilon, \varepsilon])$ is a well defined operator, which turns out to be a contraction. In this case, the unique fixed point $\mathfrak{b}_{\mathfrak{g}}$ of T is a continuous equilibrium for $x' = \mathfrak{g}(\omega{\cdot}t, x)$. It is easy to check that $t \mapsto \mathfrak{b}_{\mathfrak{g}}(\omega{\cdot}t)$ is an attractive hyperbolic solution for each $\omega \in \Omega$.

Finally, given $\nu \in (0, k_0/\gamma)$, there exist $\bar\delta \in (0, \delta_0/4)$ and $\bar\varepsilon \in (0, \varepsilon)$ such that $|\mathfrak{g}_x(\omega, \mathfrak{b}_{\mathfrak{g}}(\omega) + s\, y) - \mathfrak{g}_x(\omega, \mathfrak{b}_{\mathfrak{g}}(\omega))| \le \nu$ for every $y \in [-\bar\varepsilon, \bar\varepsilon]$, $s \in [0, 1]$ and $\omega \in \Omega$ if $\|\mathfrak{h} - \mathfrak{g}\|_{1,m} \le \bar\delta$, from where the existence of a radius of uniform stability ρ_ε follows. □

Proof of Theorem 1.40. The proof relies on that of [18, Proposition 2.8]. We reason in the attractive case. Let $\mathfrak{l}_{\mathcal{K}}$ and $\mathfrak{u}_{\mathcal{K}}$ be the lower and upper equilibria of $\mathcal{K}$ (given by (1.7)), respectively. Theorem 1.36(ii) and (iii) and Proposition 1.12(iv) show the existence of $k \ge 1$ and $\gamma > 0$ such that

$$\exp \int_0^t \mathfrak{h}_x(\omega{\cdot}r, \mathfrak{l}_{\mathcal{K}}(\omega{\cdot}r))\, dr \le k\, e^{-\gamma t}\,, \qquad \text{for all } \omega \in \Omega \text{ and } t \ge 0\,,$$

which ensures that

$$\exp \int_s^t \mathfrak{h}_x(\omega{\cdot}r, \mathfrak{l}_{\mathcal{K}}(\omega{\cdot}r))\, dr = \exp \int_0^{t-s} \mathfrak{h}_x(\omega{\cdot}s{\cdot}r, \mathfrak{l}_{\mathcal{K}}(\omega{\cdot}s{\cdot}r))\, dr \le k\, e^{-\gamma\,(t-s)}$$

for $\omega \in \Omega$ and $t \ge s$. That is, for all $\omega \in \Omega$, the map $t \mapsto \mathfrak{l}_{\mathcal{K}}(\omega{\cdot}t)$ is an attractive hyperbolic solution of $x' = \mathfrak{h}(\omega{\cdot}t, x)$.

We take $\bar\omega \in \Omega$, a point ω_0 in a minimal subset of the α-limit set of $\bar\omega$ for σ with $\mathfrak{l}_{\mathcal{K}}(\omega_0) = \mathfrak{u}_{\mathcal{K}}(\omega_0)$, and a sequence $(t_n) \downarrow -\infty$ such that there exist $(\omega_0, x_0) = \lim_{n\to\infty} \tau(t_n, \bar\omega, \mathfrak{l}_{\mathcal{K}}(\bar\omega))$ and $(\omega_0, y_0) = \lim_{n\to\infty} \tau(t_n, \bar\omega, \mathfrak{u}_{\mathcal{K}}(\bar\omega))$. Then, we get $\mathfrak{l}_{\mathcal{K}}(\omega_0) \le x_0, y_0 \le \mathfrak{u}_{\mathcal{K}}(\omega_0) = \mathfrak{l}_{\mathcal{K}}(\omega_0)$; i.e., $x_0 = y_0 = \mathfrak{u}_{\mathcal{K}}(\omega_0) = \mathfrak{l}_{\mathcal{K}}(\omega_0)$. Let us check that $\mathfrak{l}_{\mathcal{K}}(\bar\omega) = \mathfrak{u}_{\mathcal{K}}(\bar\omega)$. Let $\rho > 0$ be a radius of uniform stability and $(k, \bar\gamma)$ a dichotomy constant pair of $t \mapsto \mathfrak{l}_{\mathcal{K}}(\bar\omega{\cdot}t)$ (recall Corollary 1.53 and Theorem 1.52(ii)). Since, as checked above, $\lim_{n\to\infty}(\mathfrak{u}_{\mathcal{K}}(\bar{\omega}{\cdot}t_n) - \mathfrak{l}_{\mathcal{K}}(\bar{\omega}{\cdot}t_n)) = y_0 - x_0 = 0$, there exists $j \in \mathbb{N}$ such that $|\mathfrak{u}_{\mathcal{K}}(\bar{\omega}{\cdot}t_n) - \mathfrak{l}_{\mathcal{K}}(\bar{\omega}{\cdot}t_n)| \le \rho$ for all $n \ge j$. Hence,

$$|\mathfrak{u}_{\mathcal{K}}(\bar{\omega}) - \mathfrak{l}_{\mathcal{K}}(\bar{\omega})| \le k\, e^{-\bar{\gamma}\, t_n} |\mathfrak{u}_{\mathcal{K}}(\bar{\omega}{\cdot}t_n) - \mathfrak{l}_{\mathcal{K}}(\bar{\omega}{\cdot}t_n)| \le k\, e^{-\bar{\gamma}\, t_n}\rho\,.$$

Taking $n \to \infty$ gives $\mathfrak{u}_{\mathcal{K}}(\bar{\omega}) = \mathfrak{l}_{\mathcal{K}}(\bar{\omega})$, as asserted. Hence, Lemma 1.29 ensures that $\mathcal{K}$ is a copy of the base. Since $\mathfrak{h} \in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$, given $\nu \in (0, k/\bar{\gamma})$, there exists a sufficiently small $\varepsilon > 0$ such that $|\mathfrak{h}_x(\omega{\cdot}t, \mathfrak{l}_{\mathcal{K}}(\omega{\cdot}t) + sy) - \mathfrak{h}_x(\omega{\cdot}t, \mathfrak{l}_{\mathcal{K}}(\omega{\cdot}t))| \le \nu$ for all $y \in [-\varepsilon,\varepsilon]$, $s \in [0,1]$, $\omega \in \Omega$ and $t \in \mathbb{R}$, and hence a careful revision of the proof of Theorem 1.52(ii) ensures that a common radius of uniform stability $\rho > 0$ and dichotomy constant pair $(k,\bar{\gamma})$ can be taken for $t \mapsto \mathfrak{l}_{\mathcal{K}}(\omega{\cdot}t)$ for all $\omega\in\Omega$. So, $\mathcal{K}$ is hyperbolic attractive.

Now, assume that $\mathcal{K}$ is an attractive hyperbolic copy of the base $\mathcal{K} = \{\mathfrak{b}\}$ for a continuous map $\mathfrak{b}\colon \Omega \to \mathbb{R}$. Since $t \mapsto \mathfrak{b}(\omega{\cdot}t)$ is a hyperbolic solution of $x' = \mathfrak{h}(\omega{\cdot}t, x)$ (see Proposition 1.55), there exists $(k_\omega, \gamma_\omega)$ with $k_\omega \ge 1$ and $\gamma_\omega > 0$ such that $\exp\int_0^t \mathfrak{h}_x(\omega{\cdot}r, \mathfrak{b}(\omega{\cdot}r))\,dr \le k_\omega\, e^{-\gamma_\omega t}$ for all $t \ge 0$. Hence, we get $\limsup_{t\to\infty}(1/t)\int_0^t \mathfrak{h}_x(\omega{\cdot}r, \mathfrak{b}(\omega{\cdot}r))\,dr \le -\gamma_\omega < 0$ for all $\omega\in\Omega$, and Birkhoff's Ergodic Theorem 1.10 ensures that $\int_\Omega \mathfrak{h}_x(\omega, \mathfrak{b}(\omega))\,dm < 0$ for all $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$. Theorem 1.36(iii) ensures that $\mathrm{Lyap}(\mathcal{K}) \subset (-\infty, 0)$.

The proofs are analogous in the repulsive case. The unique meaningful change is taking $(t_n)\uparrow\infty$ instead of $(t_n)\downarrow -\infty$.

The last assertion about the minimal case follows from Proposition 1.32. In the other one, given any σ-minimal set $\mathcal{M}\subseteq\Omega$, there exists $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$ such that $m(\mathcal{M}) = 1$, so $m(\mathcal{M}\cap\Omega_0) = 1$, and hence $\mathcal{M}\cap\Omega_0$ is nonempty. □

1.3.3 Pullback attractive and repulsive properties

The concepts of pullback attraction and repulsion play a fundamental role in the description of the dynamics induced by a process. In fact, the existence of a (local or global) pullback attractor is, in general, quite less demanding than the existence of a global attractor for the induced skewproduct flow.

Definition 1.59 (Locally pullback attractive or repulsive solution)**.** A solution $b\colon(-\infty,\beta)\to\mathbb{R}$ of (1.11) is *locally pullback attractive* if there exists $s_0 < \beta$ and $\delta > 0$ such that, if $s \le s_0$ and $|x - b(s)| < \delta$, then $x_h(t,s,x)$ is defined for $t \in [s, s_0]$ and

$$\lim_{s\to-\infty}\ \max_{x\in[b(s)-\delta,b(s)+\delta]} |b(t) - x_h(t,s,x)| = 0 \qquad \text{for all } t \le s_0\,.$$

A solution $b\colon(\alpha,\infty)\to\mathbb{R}$ of (1.11) is *locally pullback repulsive* if and only if the solution $\bar{b}(t) = b(-t)$ of $y' = -h(-t,y)$ is locally pullback attractive.

In the scalar case in which this document is developed, the definition of locally pullback attractive solution is equivalent to the existence of $s_0 < \beta$ and $\delta > 0$ such that, if $s \le s_0$, then $x_h(t,s,b(s)\pm\delta)$ is defined for $t\in[s,s_0]$, and

$$\lim_{s\to-\infty} |b(t) - x_h(t,s,b(s)\pm\delta)| = 0 \qquad \text{for all } t \le s_0\,.$$

Analogously, the definition of locally pullback repulsive solution is equivalent to the existence of $s_0 > \alpha$ and $\delta > 0$ such that, if $s_0 \le s$, then $x_h(t,s,b(s)\pm\delta)$ is defined for $t \in [s_0, s]$ and

$$\lim_{s\to\infty} |b(t) - x_h(t,s,b(s)\pm\delta)| = 0 \qquad \text{for all } t \ge s_0\,.$$

The next definition requires the notion of *Hausdorff semidistance* (which was already introduced in Definition 1.31 for subsets of $\Omega \times \mathbb{R}$) between subsets $\mathcal{C}_1$ and $\mathcal{C}_2$ of $\mathbb{R}$,

$$\mathrm{dist}(\mathcal{C}_1, \mathcal{C}_2) = \sup_{x_1 \in \mathcal{C}_1} \left(\inf_{x_2 \in \mathcal{C}_2} |x_1 - x_2| \right) .$$

Definition 1.60 (Pullback attractor)**.** A family $\mathscr{A} = \{\mathscr{A}(t) \mid t \in \mathbb{R}\}$ is the *pullback attractor* of (1.11) if

(i) $\mathscr{A}(t)$ is a compact subset of $\mathbb{R}$ for each $t \in \mathbb{R}$;

(ii) $\mathscr{A}$ is invariant for (1.11), i.e., $\mathscr{A}(t) = x_h(t, s, \mathscr{A}(s))$ for all $s, t \in \mathbb{R}$;

(iii) $\mathscr{A}$ pullback attracts bounded subsets of $\mathbb{R}$, that is, for any bounded $\mathcal{C} \subset \mathbb{R}$ and any $t \in \mathbb{R}$,
$$\lim_{s \to -\infty} \mathrm{dist}(x_h(t, s, \mathcal{C}), \mathscr{A}(t)) = 0\,;$$

(iv) $\mathscr{A}$ is the minimal family of closed sets with property (iii).

Property (iv) in the definition ensures the uniqueness of the pullback attractor. The pullback attractor $\mathscr{A}$ is said to be *globally forward attractive* if, for every bounded subset $\mathcal{C} \subset \mathbb{R}$ and every $s \in \mathbb{R}$,

$$\lim_{t \to \infty} \mathrm{dist}(x_h(t, s, \mathcal{C}), \mathscr{A}(t)) = 0\,, \tag{1.20}$$

and it is said to be *locally forward attractive* if there exists $\delta > 0$ such that (1.20) holds for every $s \in \mathbb{R}$ and every bounded subset $\mathcal{C} \subseteq \{x_1 + x_2 \mid \text{exists } s \in \mathbb{R}$ such that $x_1 \in \mathscr{A}(s)$ and $x_2 \in [-\delta, \delta]\}$. We recall that, in general, the pullback attraction property of Definition 1.60 does not imply local forward attraction (see e.g. [66]). However, sometimes, pullback attractors will also be locally or globally forward attractive.

The following proposition describes a hypothesis on coercivity which ensures that all the solutions of the equation are globally forward defined and bounded, as well as the existence of the pullback attractor.

Proposition 1.61. *Let $h \in C^{0,1}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ satisfy $\limsup_{x \to \pm\infty}(\pm h(t, x)) < 0$ uniformly in $t \in \mathbb{R}$, and take $\delta_1, \delta_2 > 0$ and $m_1, m_2 \in \mathbb{R}$ with $h(t, x) \geq \delta_1$ if $x \leq m_1$ and $h(t, x) \leq -\delta_2$ if $x \geq m_2$. Then, all the maximal solutions of (1.11) are globally forward defined and bounded, and the equation has a bounded pullback attractor $\mathscr{A} = \{\mathscr{A}(t) \mid t \in \mathbb{R}\}$, with $\mathscr{A}(t) \subseteq [m_1, m_2]$ composed by the values at t of all the globally bounded solutions for any $t \in \mathbb{R}$.*

Proof. If $y(t)$ solves (1.11), then $y'(t) \leq -\delta_2 < 0$ if $y(t) \geq m_2$ and $y'(t) \geq \delta_1 > 0$ if $y(t) \leq m_1$. Hence, first, $x_h(t, s, [m_1, m_2]) \subseteq [m_1, m_2]$ for all $t \geq s$; second, if $x > m_2$, then $x_h(t, s, x) = x + \int_s^t h(l, x_h(l, s, x))\, dl \leq x - \delta_2(t - s)$ for those values of $t \geq s$ for which $x_h(t, s, x) > m_2$ (note that, since $x_h(t, s, [m_1, m_2]) \subseteq [m_1, m_2]$ for all $t \geq s$, $x_h(t, s, x) > m_2$ if and only if $x_h(l, s, x) > m_2$ for all $l \in [s, t]$), which ensures the existence of $t_s \leq s + (x - m_2)/\delta_2$ such that $x_h(t_s, s, x) = m_2$; and third, analogously, if $x < m_1$, then there exists $t_s \leq s - (x - m_1)/\delta_1$ such that $x_h(t_s, s, x) = m_1$. This yields $\lim_{s \to -\infty} \mathrm{dist}(x_h(t, s, \mathcal{D}), [m_1, m_2]) = 0$ for all bounded set $\mathcal{D} \subset \mathbb{R}$: $\mathcal{B}(t) = [m_1, m_2]$ pullback attracts bounded sets in time t

(see Definition 1.60(ii)). Therefore, [21, Theorem 2.12] ensures the existence of the pullback attractor $\mathscr{A} = \{\mathscr{A}(t) \mid t \in \mathbb{R}\}$ with $\mathscr{A}(t) \subseteq [m_1, m_2]$ for all $t \in \mathbb{R}$. The last assertion follows from [21, Corollary 1.18], since the pullback attractor is bounded. □

1.4 Functions of bounded primitive

Let (Ω, σ) be a continuous flow on a compact metric space.

Definition 1.62. Let us consider

- $C(\Omega, \mathbb{R})$, the space of continuous functions $\mathfrak{a}\colon \Omega \to \mathbb{R}$,
- $C_0(\Omega, \mathbb{R})$, the subspace of functions $\mathfrak{a} \in C(\Omega, \mathbb{R})$ such that $\int_\Omega \mathfrak{a}(\omega)\, dm = 0$ for all $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$,
- $C^1(\Omega, \mathbb{R})$, the subspace of functions $\mathfrak{a} \in C(\Omega, \mathbb{R})$ such that $t \mapsto \mathfrak{a}_\omega(t) = \mathfrak{a}(\omega{\cdot}t)$ is continuously differentiable on $\mathbb{R}$, where we represent $\mathfrak{a}'(\omega) = \mathfrak{a}'_\omega(0)$,
- $CP(\Omega, \mathbb{R})$, the subspace of functions $\mathfrak{a} \in CP(\Omega, \mathbb{R})$ with continuous primitive, that is, such that there exists $\mathfrak{b} \in C^1(\Omega, \mathbb{R})$ with $\mathfrak{b}' = \mathfrak{a}$.

It is clear that any $\mathfrak{a} \in C^1(\Omega, \mathbb{R})$ is C^1 along the base orbits (see Section 1.2.1). It is frequent to refer to a function $\mathfrak{a} \in CP(\Omega)$ as "with bounded primitive". This terminology is supported by the following classical proposition, whose proof can be found e.g. in [52, Lemma 2.7] or [57, Proposition A.1].

Proposition 1.63. *Let (Ω, σ) be minimal and let $\mathfrak{a} \in C(\Omega, \mathbb{R})$. The following assertions are equivalent:*

(a) *there exists $\omega_0 \in \Omega$ such that $\int_0^t \mathfrak{a}(\omega_0{\cdot}s)\, ds$ is bounded either for all $t \geq 0$ or for all $t \leq 0$,*

(b) *$\mathfrak{a} \in CP(\Omega, \mathbb{R})$.*

The following proposition compiles some classical results concerning the function spaces presented in Definition 1.62. We say that a flow (Ω, σ) is *periodic* if there exists $T > 0$ such that $\omega{\cdot}T = \omega$ for all $\omega \in \Omega$. Otherwise, we say that (Ω, σ) is *nonperiodic*.

Proposition 1.64. *The following statements hold:*

(i) *$CP(\Omega) \subseteq C_0(\Omega)$,*

(ii) *if (Ω, σ) is periodic, then $CP(\Omega) = C_0(\Omega)$,*

(iii) *if (Ω, σ) is a minimal nonperiodic flow, then $CP(\Omega)$ is a dense subset of $C_0(\Omega)$ of first category in $C_0(\Omega)$,*

(iv) *$C^1(\Omega)$ is dense in $C(\Omega)$.*

Proof. The application of Birkhoff's Ergodic Theorem to both sides of $\mathfrak{b}' = \mathfrak{a}$ gives (i). To prove (ii), notice that if (Ω, σ) has period $T > 0$, then it is uniquely ergodic and $0 = \int_\Omega \mathfrak{a}(\omega)\, dm = \int_0^T \mathfrak{a}(\omega{\cdot}s)\, ds$ for any $\omega \in \Omega$. Once fixed $\omega_0 \in \Omega$, let $t_\omega \in [0, T]$ be such that $\omega = \omega_0{\cdot}t_\omega$. It is not hard to check that $\mathfrak{b}(\omega) = \int_0^{t_\omega} \mathfrak{a}(\omega_0{\cdot}s)\, ds$ is in $C^1(\Omega, \mathbb{R})$ and $\mathfrak{b}' = \mathfrak{a}$. The proof of (iii) can be found in [19, Lemma 5.1] and the proof of (iv) can be found in [109, Section 2, Kakutani Theorem]. □

Chapter 2

D-concave nonautonomous differential equations

In [114], Tineo considered scalar differential equations

$$x' = h(t,x)\,, \tag{2.1}$$

where $h(t,\cdot)$ admits derivative with respect the state variable x for every $t \in \mathbb{R}$, the maps $h, h_x\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous and h satisfies a certain condition related to the concavity of $x \mapsto h_x(t,x)$, and it was shown that there can exist at most three bounded and uniformly separated solutions of (2.1).

In this chapter, we consider a continuous flow on a compact metric space (Ω,σ) and a family of scalar differential equations of the form

$$x' = \mathfrak{h}(\omega{\cdot}t, x)\,, \quad \omega \in \Omega\,, \tag{2.2}$$

where $\mathfrak{h} \in C^{0,2}(\Omega\times\mathbb{R},\mathbb{R})$ (see Section 1.2). We will prove that, if the set of all $\omega \in \Omega$ for which $x \mapsto \mathfrak{h}_x(\omega,x)$ is concave has full measure for each ergodic measure on (Ω,σ) and that the set of all $\omega \in \Omega$ for which $x \mapsto \mathfrak{h}_x(\omega,x)$ is strictly concave has positive measure for each ergodic measure on (Ω,σ), then the skewproduct flow induced by (2.2) on $\Omega\times\mathbb{R}$ can have at most three ordered disjoint compact invariant sets and at most three ordered bounded measurable equilibria for each ergodic measure on (Ω,σ). Moreover, if there exist three ordered disjoint compact invariant sets, then they are hyperbolic copies of the base and the dynamics of the solutions is completely determined.

The natural framework for the application of these skewproduct flow results is when Ω is the hull of an admissible function (see Section 1.3.1). In this case, the upper bound on the number of ordered disjoint compact invariant sets for (2.2) (at most three) implies an upper bound on the number of bounded and uniformly separated solutions for $x' = h(t,x)$ (at most three).

The approach of this chapter improves some of the results reported in the literature. For instance, in Chapter 4, we will deal with certain families of functions $h \in C^{0,2}(\mathbb{R},\mathbb{R})$ which do not necessarily have concave derivative $x \mapsto h_x(t,x)$ for any $t \in \mathbb{R}$ but which satisfy the hypotheses on the concavity of the derivative of its hull extension $\mathfrak{h}$ required for the analysis.

This chapter is divided into three sections. In Section 2.1, we sum up the basic properties about concavity and divided differences needed in the rest of the chapter.

In Section 2.2, which contains the main results, some dynamical features of a d-concave family of ordinary differential equations are described in the skewproduct formulation. In particular, we discuss the existence of the mentioned upper bound on the number of ordered bounded measurable equilibria and of ordered disjoint compact invariant sets which project onto the whole base (which is three), the shape and properties of the global attractor of the associated skewproduct flow, and the complete dynamical picture in the case of existence of three hyperbolic copies of the base. Finally, in Section 2.3, we relate the property of having concave derivative of scalar equations $x' = \mathfrak{h}(\omega{\cdot}t, x)$ with the property of having negative Schwarzian derivative of scalar discrete dynamical systems. The dynamical features described in this chapter will be the foundation on which to build the bifurcation results of Chapter 3 and the mathematical theory of critical transitions of Chapter 4.

2.1 D-concave functions and divided differences

The definitions and results of this section refer to functions $h\colon \mathbb{R} \to \mathbb{R}$. In the following sections they will be applied to the maps $x \mapsto \mathfrak{h}(\omega, x)$ for each fixed $\omega \in \Omega$.

Definition 2.1. A map $h \in C^1(\mathbb{R}, \mathbb{R})$ is *d-concave* if it has concave derivative h', that is, if

$$h'\big(\alpha\, x_1 + (1-\alpha)\, x_2\big) \geq \alpha\, h'(x_1) + (1-\alpha)\, h'(x_2) \tag{2.3}$$

for all $x_1, x_2 \in \mathbb{R}$ and $\alpha \in [0, 1]$, and it is *strictly d-concave* if it has strictly concave derivative h', that is, if the inequality (2.3) is strict for all $x_1, x_2 \in \mathbb{R}$ and $\alpha \in (0, 1)$.

In other words, h is (strictly) d-concave if the convex combinations of any pair of points of the graph of h' lie (strictly) below the graph of h'. If $h \in C^2(\mathbb{R}, \mathbb{R})$, then h is (strictly) d-concave if and only if its derivative h'' is (strictly) decreasing.

The next result explains how to characterize d-concavity in terms of *divided differences of second order* of h, defined for three different real values x_1, x_2, x_3 by

$$h[x_1, x_2, x_3] = \frac{h[x_2, x_3] - h[x_1, x_2]}{x_3 - x_1}\,, \quad \text{where} \quad h[x_1, x_2] = \frac{h(x_2) - h(x_1)}{x_2 - x_1}\,.$$

We recall that $h[x_1, x_2, x_3]$ is the leading coefficient of the unique quadratic polynomial which interpolates the data $(x_1, h(x_1))$, $(x_2, h(x_2))$ and $(x_3, h(x_3))$. In particular, divided differences are invariant under any permutation of their nodes. The following identities for $h \in C^1(\mathbb{R}, \mathbb{R})$ will prove useful later. The first one is trivial and the second one follows from the Fundamental Theorem of Calculus and the expression of $(\partial/\partial s)\, h(s\, x_1 + (1-s)\, x_2)$:

$$\lim_{x_2 \to x_1} h[x_1, x_2] = h'(x_1)\,, \tag{2.4}$$

$$h[x_1, x_2] = \int_0^1 h'\big(s\, x_1 + (1-s)\, x_2\big)\, ds\,. \tag{2.5}$$

Proposition 2.2. *A map $h \in C^1(\mathbb{R}, \mathbb{R})$ is d-concave if and only if $h[x_1, x_0, x_2] \geq h[x_1, x_0, x_3]$ whenever $x_1 < x_2 < x_3$ and $x_0 \neq x_i$ for $i \in \{1, 2, 3\}$.*

Proof. For completeness, we include the proof, which can be basically found in [114, Lemma 2.1 and the remark after it]. Assume that $h[x_1,x_0,x_2] \ge h[x_1,x_0,x_3]$ whenever $x_1 < x_2 < x_3$ and $x_0 \ne x_i$ for $i \in \{1,2,3\}$. Rewriting $h[x_1,x_0,x_2] \ge h[x_1,x_0,x_3]$ in terms of first order divided differences gives

$$(x_3 - x_1)\, h[x_0,x_2] \ge (x_3 - x_2)\, h[x_1,x_0] + (x_2 - x_1)\, h[x_0,x_3]\,. \tag{2.6}$$

Taking $x_0 \to x_1$, $x_0 \to x_2$ and $x_0 \to x_3$ in (2.6) and using (2.4), we get

$$\begin{array}{lllllll} (x_3 - x_1) & h[x_1,x_2] & \ge & (x_3 - x_2) & h'(x_1) & + & (x_2 - x_1) \;\; h[x_1,x_3]\,, \\ (x_3 - x_1) & h'(x_2) & \ge & (x_3 - x_2) & h[x_1,x_2] & + & (x_2 - x_1) \;\; h[x_2,x_3]\,, \\ (x_3 - x_1) & h[x_3,x_2] & \ge & (x_3 - x_2) & h[x_1,x_3] & + & (x_2 - x_1) \;\; h'(x_3)\,. \end{array}$$

Since $(x_2 - x_1)\, h[x_1,x_2] + (x_3 - x_2)\, h[x_2,x_3] = (x_3 - x_1)\, h[x_1,x_3]$, adding the three previous inequalities yields

$$(x_3 - x_1)\, h'(x_2) \ge (x_3 - x_2)\, h'(x_1) + (x_2 - x_1)\, h'(x_3)\,.$$

Since (2.3) is obvious when $\alpha \in \{0,1\}$ or when $y_1 = y_2$, we take $y_1 \ne y_2$ and $\alpha \in (0,1)$. Let $x_1 = \min\{y_1,y_2\}$, $x_3 = \max\{y_1,y_2\}$, $\beta = \alpha$ if $y_1 < y_2$ or $\beta = 1-\alpha$ if $y_1 > y_2$, and $x_2 = \beta\, x_1 + (1-\beta)\, x_3$. Then, $(x_3 - x_2)/(x_3 - x_1) = \beta$, $(x_2 - x_1)/(x_3 - x_1) = 1 - \beta$ and the previous inequality yields

$$h'\big(\beta\, x_1 + (1-\beta)\, x_3\big) \ge \beta\, h'(x_1) + (1-\beta)\, h'(x_3)\,,$$

which corresponds to (2.3) in any case.

Conversely, we assume that h is d-concave. Let us take $y_1 < y_2 < y_3$ and $y_0 \ne y_i$ for $i \in \{1,2,3\}$, and define $\beta = (y_3 - y_2)/(y_3 - y_1) \in (0,1)$ and $x_i(s) = s\, y_0 + (1-s)\, y_i$ for $i \in \{1,2,3\}$ and $s \in [0,1]$. We write (2.3) for h' with $\alpha = \beta$, $x_1 = x_1(s)$ and $x_2 = x_3(s)$, and get

$$\begin{aligned} &(y_3 - y_1)\, h'\big(s\, y_0 + (1-s)\, y_2\big) \\ &\qquad\qquad \ge (y_3 - y_2)\, h'\big(s\, y_0 + (1-s)\, y_1\big) + (y_2 - y_1)\, h'\big(s\, y_0 + (1-s)\, y_3\big)\,. \end{aligned}$$

Integrating s from 0 to 1 in the previous inequality and using (2.5) gives

$$(y_3 - y_1)\, h[y_0,y_2] \ge (y_3 - y_2)\, h[y_1,y_0] + (y_2 - y_1)\, h[y_0,y_3]\,,$$

which is equivalent to $h[y_1,y_0,y_2] \ge h[y_1,y_0,y_3]$. The proof is complete. □

In the previous proposition, it has been proved that the nonnegative character of the differences $h[x_1,x_0,x_2] - h[x_1,x_0,x_3]$ for $x_1 < x_2 < x_3$ and $x_0 \ne x_i$ for $i \in \{1,2,3\}$ is equivalent to the d-concavity of h. Since d-concavity is not a sufficiently demanding condition for stating the subsequent theory, a relation between strict concavity on an interval and the positivity of divided differences is sought.

To this end, for $h \in C^1(\mathbb{R},\mathbb{R})$, $x_1 < x_2 < x_3$ and $x_0 \ne x_i$ for $i \in \{1,2,3\}$, we define

$$\begin{aligned} b(x_0,x_1,x_2,x_3) &= h[x_1,x_0,x_2] - h[x_1,x_0,x_3]\,. \\ b_i(x_1,x_2,x_3) &= \lim_{x_0 \to x_i} b(x_0,x_1,x_2,x_3) \end{aligned} \tag{2.7}$$

for $i \in \{1,2,3\}$. The existence of the limits b_i follow from (2.4). The next result establishes an equivalence between the sign of b_i and the decreasing character of h''.

Theorem 2.3. *Let $h \in C^2(\mathbb{R},\mathbb{R})$ be d-concave and $x_1 < x_2 < x_3$. Then, for $i \in \{1,2,3\}$, $b_i(x_1,x_2,x_3) \geq 0$, and $b_i(x_1,x_2,x_3) > 0$ if and only if $h''(x_1) > h''(x_3)$.*

Proof. The first assertion follows from Proposition 2.2. For the second one, we work in the case $i = 2$. According to (2.4) and (2.7)

$$\begin{aligned} b_2(x_1,x_2,x_3) &= \lim_{x_0\to x_2}\left(\frac{h[x_0,x_2]-h[x_1,x_0]}{x_2-x_1} - \frac{h[x_0,x_3]-h[x_1,x_0]}{x_3-x_1}\right) \\ &= \frac{1}{x_2-x_1}\left(h'(x_2) - \frac{x_3-x_2}{x_3-x_1}\,h[x_1,x_2] - \frac{x_2-x_1}{x_3-x_1}\,h[x_2,x_3]\right). \end{aligned}$$

Using the previous equality, (2.5) and Fundamental Theorem of Calculus, we obtain

$$\begin{aligned} b_2(x_1,x_2,x_3) &= \frac{1}{x_2-x_1}\left(\frac{x_3-x_2}{x_3-x_1}\int_0^1 \Big(h'(x_2) - h'(s\,x_1 + (1-s)\,x_2)\Big)\,ds \right. \\ &\qquad \left. + \frac{x_2-x_1}{x_3-x_1}\int_0^1 \Big(h'(x_2) - h'(s\,x_3+(1-s)\,x_2)\Big)\,ds\right) \\ &= \frac{x_3-x_2}{x_3-x_1}\int_0^1\int_0^1 s\Big(h''\big(s\,x_1+(1-s)\,x_2+t\,s\,(x_2-x_1)\big) \\ &\qquad - h''\big(s\,x_3+(1-s)\,x_2 - t\,s\,(x_3-x_2)\big)\Big)\,dt\,ds\,. \end{aligned} \tag{2.8}$$

Since $h \in C^2(\mathbb{R},\mathbb{R})$, the integrand is continuous on s,t. It can be checked that $s\,x_1 + (1-s)\,x_2 + t\,s\,(x_2-x_1) \leq s\,x_3 + (1-s)\,x_2 - t\,s\,(x_3-x_2)$ is equivalent to $s\,(1-t)(x_3-x_1) \geq 0$. So, since the C^2 and d-concave character of h ensures that h'' is nonincreasing, the integrand is nonnegative for all $s,t \in [0,1]$. If $h''(x_1) > h''(x_3)$, then the integrand is strictly positive at $(t,s) = (0,1)$, and hence $b_2(x_1,x_2,x_3) > 0$. Conversely, if $h''(x_1) = h''(x_3)$, then h'' is constant on $[x_1,x_3]$, and hence the integrand is identically zero. We proceed analogously with b_1 and b_3. □

In short, the previous theorem states that, given a d-concave function $h \in C^2(\mathbb{R},\mathbb{R})$, the function $b_i(x_1,x_2,x_3)$ for $x_1 < x_2 < x_3$ and $i \in \{1,2,3\}$ is strictly positive if and only if h'' is nonconstant on the interval $[x_1,x_3]$, that is, if h is not a quadratic polynomial on $[x_1,x_3]$. In Section 2.2.1, this will be used to find the sign of the Lyapunov exponents of compact invariant sets of the skewproduct flow.

The following result shows that, if a function with a concave derivative satisfies a certain coercivity condition, then it is a concave-convex function.

Proposition 2.4. *Let $h \in C^2(\mathbb{R},\mathbb{R})$ be d-concave. Assume $\lim_{x\to\pm\infty} h(x)/x = -\infty$. Then, there exists $x_0 \leq x_1$ such that $h''(x) > 0$ for all $x < x_0$, $h''(x) = 0$ for all $x \in [x_0,x_1]$ and $h''(x) < 0$ for all $x > x_1$.*

Proof. Since h is C^2 and d-concave, h'' is nondecreasing. If $h''(x) = 0$ on a halfline, then h is linear on that halfline, so $\lim_{x\to\pm\infty} h(x)/x = -\infty$ is not fulfilled. So, if h'' vanishes on $[x_0,x_1]$, the situation is that of the statement. For contradiction, assume that $h''(x) > 0$ for all $x \in \mathbb{R}$. Then, h' is nondecreasing. This fact and the Mean Value Theorem ensure the existence of $\xi_x \in (0,x)$ such that for any $x > 0$

$$\frac{h(x)-h(0)}{x} = h'(\xi_x) \geq h'(0)\,,$$

which contradicts $\lim_{x\to\infty} h(x)/x = -\infty$. An analogous argument shows that $h'' < 0$ is neither possible. □

2.2 Nonautonomous d-concave dynamics

Let (Ω, σ) be a continuous flow on a compact metric space. The family of nonautonomous scalar ordinary differential equations

$$x' = \mathfrak{h}(\omega{\cdot}t, x)\,, \quad \omega \in \Omega\,, \tag{2.9}$$

is considered. Let τ be the scalar skewproduct flow associated to (2.9) (see Definition 1.15). That is, $\tau(t, \omega, x) = (\omega{\cdot}t, v(t, \omega, x))$, where $t \mapsto v(t, \omega, x)$ stands for the solution of $(2.9)_\omega$ satisfying $v(0, \omega, x) = x$. In the following sections, it will be assumed that $\mathfrak{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies (all or part of) the next conditions:

d1 $\mathfrak{h} \in C^{0,2}(\Omega \times \mathbb{R}, \mathbb{R})$,

d2 $\limsup_{x\to\pm\infty} \big(\pm\, \mathfrak{h}(\omega, x) \big) < 0$ uniformly on Ω,

d3 $m\big(\{\omega \in \Omega \,|\, x \mapsto \mathfrak{h}_x(\omega, x) \text{ is concave}\}\big) = 1$ for all $m \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$,

d4 $m\big(\{\omega \in \Omega \,|\, x \mapsto \mathfrak{h}_x(\omega, x) \text{ is strictly concave on } \mathcal{J}\}\big) > 0$ for all compact interval $\mathcal{J} \subset \mathbb{R}$ and all $m \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$.

In Section 2.2.1, the conditions about d-concavity **d3** and **d4**, together with **d1**, will play a key role in finding the upper bound on the number of ordered bounded m-measurable τ-equilibria (for any $m \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$) and, as a consequence, the upper bound on the number of ordered disjoint compact τ-invariant subsets of $\Omega \times \mathbb{R}$ which project onto Ω. The coercivity property **d2**, together with **d1**, will ensure the existence of a τ-global attractor (see Section 2.2.3).

Remark 2.5. Let $\Omega_0 \subset \Omega$ be a nonempty compact σ-invariant set. Then, any $m_0 \in \mathfrak{M}_{\text{erg}}(\Omega_0, \sigma)$ can be extended to $m \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$ by $m(\mathcal{U}) = m_0(\mathcal{U} \cap \Omega_0)$. So, if $\mathfrak{h}$ satisfies **dj** for $\mathbf{j} \in \{1, 2, 3, 4\}$, also the restriction $\mathfrak{h}\colon \Omega_0 \times \mathbb{R} \to \mathbb{R}$ satisfies **dj**.

The following lemma helps to understand property **d3** whenever $\mathfrak{h}$ satisfies **d1**, and will prove useful later. In particular, it will be used in the next chapter to delve into the relations between different coercivity hypotheses.

Lemma 2.6. *Let $\mathfrak{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy* **d1** *and let us fix $m \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$. Assume that $m(\{\omega \in \Omega \,|\, x \mapsto \mathfrak{h}_x(\omega, x) \text{ is concave}\}) = 1$. Then, there exists a σ-invariant compact set $\Omega_d \subseteq \Omega$ with $m(\Omega_d) = 1$ such that $x \mapsto \mathfrak{h}_x(\omega, x)$ is concave for all $\omega \in \Omega_d$.*

Proof. Define $\mathcal{A} = \{\omega \in \Omega \mid x \mapsto \mathfrak{h}_x(\omega, x) \text{ is concave}\}$, and $\Omega_d = \cap_{s\in\mathbb{Q}} \mathcal{A}{\cdot}s$. Let us check that Ω_d satisfies the desired properties. Given $x_1, x_2 \in \mathbb{R}$, $s \in [0, 1]$ and (ω_n) in $\mathcal{A}$ with $\omega_n \to \omega$ as $n \to \infty$, we get that $\mathfrak{h}_x(\omega_n, \alpha\, x_1 + (1-\alpha)\, x_2) \geq \alpha\, \mathfrak{h}_x(\omega_n, x_1) + (1-\alpha)\mathfrak{h}_x(\omega_n, x_2)$, so **d1** ensures that taking limits, $\mathfrak{h}_x(\omega, \alpha\, x_1 + (1-\alpha)\, x_2) \geq \alpha\, \mathfrak{h}_x(\omega, x_1) + (1-\alpha)\mathfrak{h}_x(\omega, x_2)$, that is, $\omega \in \mathcal{A}$. So, $\mathcal{A}$ is closed, and hence Ω_d is a compact set. For any $m \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$, since m is invariant and **d3** ensures that $m(\mathcal{A}) = 1$, it is satisfied that $m(\mathcal{A}{\cdot}s) = 1$ for all $s \in \mathbb{R}$. Since Ω_d is a countable intersection of full m-measure sets, $m(\Omega_d) = 1$. As $\omega \in \Omega_d$ if and only if $\omega{\cdot}s \in \mathcal{A}$ for all $s \in \mathbb{Q}$, the closedness of $\mathcal{A}$ ensures that $\omega \in \Omega_d$ if and only if $\omega{\cdot}s \in \mathcal{A}$ for all $s \in \mathbb{R}$, from where it follows that Ω_d is σ-invariant. The last assertion follows from the fact that $\Omega_d \subseteq \mathcal{A}$. $\square$

Remark 2.7. Since the definition of Ω_d in the proof of Lemma 2.6 is independent of m, we deduce that, if **d1** holds, then **d3** is equivalent to the existence of a σ-invariant closed set $\Omega_d \subseteq \Omega$ such that $m(\Omega_d) = 1$ for all $m \in \mathfrak{M}_{\rm erg}(\Omega,\sigma)$ and such that $x \mapsto \mathfrak{h}_x(\omega,x)$ is concave for all $\omega \in \Omega_d$. Moreover, if (Ω,σ) is minimal, then $\Omega_d = \Omega$, so assuming that **d1** holds, **d3** is equivalent to $x \mapsto \mathfrak{h}_x(\omega,x)$ being concave for all $\omega \in \Omega$.

Therefore, if (Ω,σ) is minimal, a map h satisfying **d1** and **d3** fulfills the hypotheses of the next lemma, which asks for a coercivity hypothesis stronger than **d2**, under which the concave-convex character of h provided by Proposition 2.4 is uniform on Ω.

Lemma 2.8. *Let $\mathfrak{h}\colon \Omega\times\mathbb{R}\to\mathbb{R}$ satisfy* **d1** *with $x \mapsto \mathfrak{h}_x(\omega,x)$ concave for all $\omega\in\Omega$. Assume that $\lim_{x\to\pm\infty}\mathfrak{h}(\omega,x)/x = -\infty$ for all $\omega\in\Omega$. Then, there exist $x_l \le x_u$ such that $\mathfrak{h}_{xx}(\omega,x) > 0$ for all $x < x_l$ and all $\omega\in\Omega$, and $\mathfrak{h}_{xx}(\omega,x) < 0$ for all $x > x_u$ and all $\omega\in\Omega$.*

Proof. Proposition 2.4 ensures that, for each $\omega\in\Omega$, there exists $x_1^\omega\in\mathbb{R}$ such that $(x_1^\omega,\infty) = \{x\in\mathbb{R} \mid \mathfrak{h}_{xx}(\omega,x) < 0\}$. Let us check that $x_u = \sup_{\omega\in\Omega} x_1^\omega$ is finite. For each $\omega_0\in\Omega$, we take $y_0 > x_1^{\omega_0}$, so $\mathfrak{h}_{xx}(\omega_0,y_0) < 0$. The continuity of $\mathfrak{h}_{xx}$ provides $\mathfrak{h}_{xx}(\omega,y_0) < 0$ and hence $y_0 > x_1^\omega$ for all ω in an open neighborhood of ω_0. The compactness of Ω proves that $x_u\in\mathbb{R}$. If $x > x_u$, then $x > x_1^\omega$ for all $\omega\in\Omega$, and hence $\mathfrak{h}_{xx}(\omega,x) < 0$ for all $\omega\in\Omega$, as asserted. We define x_l analogously. □

2.2.1 An upper bound on the number of equilibria and compact invariant sets

The two theorems of this section establish conditions under which the maximum number of ordered bounded m-measurable τ-equilibria (for any fixed $m\in\mathfrak{M}_{\rm erg}(\Omega,\sigma)$) and of ordered disjoint compact τ-invariant subsets of $\Omega\times\mathbb{R}$ is three.

Theorem 2.9. *Let $\mathfrak{h}\colon \Omega\times\mathbb{R}\to\mathbb{R}$ satisfy* **d1**, *let us fix $m\in\mathfrak{M}_{\rm erg}(\Omega,\sigma)$, and let $\mathfrak{b}_1,\mathfrak{b}_2,\mathfrak{b}_3\colon\Omega\to\mathbb{R}$ be bounded m-measurable τ-equilibria with $\mathfrak{b}_1(\omega)<\mathfrak{b}_2(\omega)<\mathfrak{b}_3(\omega)$ for m-a.e. $\omega\in\Omega$. Assume that $m(\{\omega\in\Omega \mid x\mapsto\mathfrak{h}_x(\omega,x) \text{ is concave}\}) = 1$ and $m(\{\omega\in\Omega \mid \mathfrak{h}_{xx}(\omega,\mathfrak{b}_1(\omega)) > \mathfrak{h}_{xx}(\omega,\mathfrak{b}_3(\omega))\}) > 0$. Then,*

$$\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}_2(\omega))\,dm > 0 \quad \text{and} \quad \int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}_i(\omega))\,dm < 0 \;\; \text{for} \;\; i=1,3\,.$$

In particular, if $\mathfrak{h}$ satisfies **d1**, **d3** *and* **d4**, *then the previous result holds for any $m\in\mathfrak{M}_{\rm erg}(\Omega,\sigma)$ and any triad of bounded m-measurable τ-equilibria which are strictly ordered m-a.e. Consequently, there are at most three bounded m-measurable τ-equilibria which are strictly ordered m-a.e.*

Proof. Some steps of the following proof are based on the algebraic methods of [114, Part II, Theorem 3.2]. Let us define

$$\Omega_0 = \Omega_d \cap \{\omega\in\Omega \mid \mathfrak{b}_1(\omega)<\mathfrak{b}_2(\omega)<\mathfrak{b}_3(\omega)\}\,,$$

where Ω_d stands for the σ-invariant set of points $\omega\in\Omega$ for which $x\mapsto\mathfrak{h}_x(\omega,x)$ is concave given by Lemma 2.6. Since $\{\omega\in\Omega \mid \mathfrak{b}_1(\omega)<\mathfrak{b}_2(\omega)<\mathfrak{b}_3(\omega)\}$ is σ-invariant (recall that $v(t,\omega,\mathfrak{b}_i(\omega)) = \mathfrak{b}_i(\omega{\cdot}t)$ for $i\in\{1,2,3\}$ and $t\in\mathbb{R}$) and has full m-measure, it follows that Ω_0 is σ-invariant and $m(\Omega_0)=1$.

For each $\omega\in\Omega_0$, we represent by $b_i(\omega,x_1,x_2,x_3)$ for $i\in\{1,2,3\}$ the expression $b_i(\omega,x_1,x_2,x_3)$ of (2.7) associated to the d-concave map $x\mapsto\mathfrak{h}(\omega,x)$ and observe that $(x_1,x_2,x_3)\mapsto b_i(x_1,x_2,x_3)$ is continuous on $\{(x_1,x_2,x_3)\mid x_1<x_2<x_3\}$ for every $\omega\in\Omega_0$: see (2.8) for $i=2$. For $i\in\{1,2,3\}$, we define $\mathfrak{b}_i^*\colon\Omega\to\mathbb{R}$ by

$$\mathfrak{b}_i^*(\omega)=b_i\big(\omega,\mathfrak{b}_1(\omega),\mathfrak{b}_2(\omega),\mathfrak{b}_3(\omega)\big) \tag{2.10}$$

for $\omega\in\Omega_0$ and $\mathfrak{b}_i^*(\omega)=0$ if $\omega\notin\Omega_0$, and observe that $\mathfrak{b}_i^*$ is m-measurable and that $\mathfrak{b}_i^*\geq 0$ (see Theorem 2.3). Let us take $i=1$ and rewrite (2.10),

$$\begin{aligned}\mathfrak{b}_1^*(\omega)-\left(\frac{\mathfrak{h}(\omega,\mathfrak{b}_2(\omega))-\mathfrak{h}(\omega,\mathfrak{b}_1(\omega))}{(\mathfrak{b}_2(\omega)-\mathfrak{b}_1(\omega))^2}-\frac{\mathfrak{h}(\omega,\mathfrak{b}_3(\omega))-\mathfrak{h}(\omega,\mathfrak{b}_1(\omega))}{(\mathfrak{b}_3(\omega)-\mathfrak{b}_1(\omega))^2}\right)&\\ =-\left(\frac{1}{\mathfrak{b}_2(\omega)-\mathfrak{b}_1(\omega)}-\frac{1}{\mathfrak{b}_3(\omega)-\mathfrak{b}_1(\omega)}\right)\mathfrak{h}_x(\omega,\mathfrak{b}_1(\omega))\,,&\end{aligned} \tag{2.11}$$

for $\omega\in\Omega_0$. Let us define

$$\mathfrak{v}_1(\omega)=\frac{1}{\mathfrak{b}_2(\omega)-\mathfrak{b}_1(\omega)}-\frac{1}{\mathfrak{b}_3(\omega)-\mathfrak{b}_1(\omega)}$$

for $\omega\in\Omega_0$, and note that $\mathfrak{v}_1(\omega)>0$ for all $\omega\in\Omega_0$. Writing (2.11) for $\omega{\cdot}t$ and using $\mathfrak{h}(\omega{\cdot}t,\mathfrak{b}_i(\omega{\cdot}t))=\mathfrak{b}_i'(\omega{\cdot}t)$, where $\mathfrak{b}_i'(\omega{\cdot}t)$ is the derivative of $t\mapsto\mathfrak{b}_i(\omega{\cdot}t)$, it is obtained

$$\mathfrak{h}_x\big(\omega{\cdot}t,\mathfrak{b}_1(\omega{\cdot}t)\big)=-\frac{\mathfrak{v}_1'(\omega{\cdot}t)}{\mathfrak{v}_1(\omega{\cdot}t)}-\frac{\mathfrak{b}_1^*(\omega{\cdot}t)}{\mathfrak{v}_1(\omega{\cdot}t)}$$

for all $\omega\in\Omega_0$ and $t\in\mathbb{R}$. This yields

$$\frac{1}{t}\int_0^t\mathfrak{h}_x\big(\omega{\cdot}s,\mathfrak{b}_1(\omega{\cdot}s)\big)\,ds=-\frac{1}{t}\log\left(\frac{\mathfrak{v}_1(\omega{\cdot}t)}{\mathfrak{v}_1(\omega)}\right)-\frac{1}{t}\int_0^t\frac{\mathfrak{b}_1^*(\omega{\cdot}s)}{\mathfrak{v}_1(\omega{\cdot}s)}\,ds \tag{2.12}$$

for $\omega\in\Omega_0$ and $t>0$. Lusin's Theorem provides a compact subset $\Delta\subsetneq\Omega_0$ with $m(\Delta)>0$ such that $\mathfrak{v}_1|_\Delta\colon\Delta\to\mathbb{R}$ is continuous. Since $\mathfrak{h}_x(\cdot,\mathfrak{b}_1(\cdot))$ is bounded and $\mathfrak{b}_1^*(\cdot)/\mathfrak{v}_1(\cdot)$ is nonnegative, Birkhoff's Ergodic Theorem 1.10 ensures the existence of a σ-invariant subset $\Omega_0^*\subseteq\Omega_0$ with $m(\Omega_0^*)=1$ such that, for every $\omega\in\Omega_0^*$,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\mathfrak{h}_x\big(\omega{\cdot}s,\mathfrak{b}_1(\omega{\cdot}s)\big)\,ds=\int_\Omega\mathfrak{h}_x\big(\omega,\mathfrak{b}_1(\omega)\big)\,dm\in\mathbb{R}\,, \tag{2.13}$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{\mathfrak{b}_1^*(\omega{\cdot}s)}{\mathfrak{v}_1(\omega{\cdot}s)}\,ds=\int_\Omega\frac{\mathfrak{b}_1^*(\omega)}{\mathfrak{v}_1(\omega)}\,dm\in[0,\infty]\,, \tag{2.14}$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\chi_\Delta(\omega{\cdot}s)\,ds=m(\Delta)>0\,. \tag{2.15}$$

Property (2.15) ensures that, if $\omega\in\Omega_0^*$, then there exists a sequence $(t_n)\uparrow\infty$ such that $\omega{\cdot}t_n\in\Delta$. Hence, the sequence $(\log(\mathfrak{v}_1(\omega{\cdot}t_n)/\mathfrak{v}_1(\omega))$ is bounded. We write (2.12) for $t=t_n$ and take limit as $n\to\infty$ to deduce from (2.13) and (2.14) that

$$\int_\Omega\mathfrak{h}_x(\omega,\mathfrak{b}_1(\omega))\,dm=-\int_\Omega\frac{\mathfrak{b}_1^*(\omega)}{\mathfrak{v}_1(\omega)}\,dm\,.$$

So, $\int_\Omega\mathfrak{h}_x(\omega,\mathfrak{b}_1(\omega))\,dm<0$ follows from $\int_\Omega\mathfrak{b}_1^*(\omega)/\mathfrak{v}_1(\omega)\,dm>0$. To prove this last inequality, we deduce from Theorem 2.3 that $\mathfrak{b}_1^*(\omega)>0$ if and only if $\mathfrak{h}_{xx}(\omega,\mathfrak{b}_1(\omega))>$

$\mathfrak{h}_{xx}(\omega,\mathfrak{b}_2(\omega))$. Hence, the last hypothesis on $\mathfrak{h}_x$ in the statement combined with $m(\Omega_0^*)=1$ and the positiveness of $\mathfrak{v}_1$ yields $m(\{\omega\in\Omega^c \mid \mathfrak{b}_1^*(\omega)/\mathfrak{v}_1(\omega)>0\})>0$, as asserted.

In the case of $i=2$, the definition of $\mathfrak{v}_2(\omega)=1/(\mathfrak{b}_2(\omega)-\mathfrak{b}_1(\omega))+1/(\mathfrak{b}_3(\omega)-\mathfrak{b}_2(\omega))$ for $\omega\in\Omega_0$, leads to

$$\mathfrak{h}_x(\omega{\cdot}t,\mathfrak{b}_2(\omega{\cdot}t))=\big(\mathfrak{b}_2(\omega{\cdot}t)-\mathfrak{b}_1(\omega{\cdot}t)\big)\,\mathfrak{b}_2^*(\omega{\cdot}t)+\frac{\mathfrak{v}_2'(\omega{\cdot}t)}{\mathfrak{v}_2(\omega{\cdot}t)}\,.$$

Arguments analogous to those used with $i=1$ show that $\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}_2(\omega))\,dm>0$. In the case of $i=3$, we define $\mathfrak{v}_3(\omega)=1/(\mathfrak{b}_3(\omega)-\mathfrak{b}_2(\omega))-1/(\mathfrak{b}_3(\omega)-\mathfrak{b}_1(\omega))$ and deduce $\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}_3(\omega))\,dm<0$ from

$$\mathfrak{h}_x(\omega{\cdot}t,\mathfrak{b}_3(\omega{\cdot}t))=-\big(\mathfrak{b}_3(\omega{\cdot}t)-\mathfrak{b}_1(\omega{\cdot}t)\big)\,\mathfrak{b}_3^*(\omega{\cdot}t)+\frac{\mathfrak{v}_3'(\omega{\cdot}t)}{\mathfrak{v}_3(\omega{\cdot}t)}\,.$$

Now, assume that **d1**, **d3** and **d4** hold, and let us check the last assertions. Note that hypothesis **d4**, together with **d1**, ensures that, given any $m\in\mathfrak{M}_{\text{erg}}(\Omega,\sigma)$ and any pair of bounded m-measurable τ-equilibria $\mathfrak{b}_i,\mathfrak{b}_j\colon\Omega\to\mathbb{R}$ satisfying $\mathfrak{b}_i(\omega)<\mathfrak{b}_j(\omega)$ for m-a.e. $\omega\in\Omega$,

$$m(\{\omega\in\Omega \mid \mathfrak{h}_{xx}(\omega,\mathfrak{b}_i(\omega))>\mathfrak{h}_{xx}(\omega,\mathfrak{b}_j(\omega))\})>0\,.$$

Consequently, if **d1**, **d3** and **d4** hold, then all the conclusions of previous part of the theorem hold for any $m\in\mathfrak{M}_{\text{erg}}(\Omega,\sigma)$ and any triad of ordered bounded m-measurable τ-equilibria. Therefore, if there exist four bounded m-measurable τ-equilibria $\mathfrak{b}_1,\mathfrak{b}_2,\mathfrak{b}_3,\mathfrak{b}_4\colon\Omega\to\mathbb{R}$ such that $\mathfrak{b}_1(\omega)<\mathfrak{b}_2(\omega)<\mathfrak{b}_3(\omega)<\mathfrak{b}_4(\omega)$ for m-a.e. $\omega\in\Omega$, then we can apply the previous result to the triads $\mathfrak{b}_1,\mathfrak{b}_2,\mathfrak{b}_3$ and $\mathfrak{b}_2,\mathfrak{b}_3,\mathfrak{b}_4$ to reach a contradiction about the sign of $\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}_2(\omega))\,dm$ (and that of $\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}_3(\omega))\,dm$). □

Remark 2.10. Note that, given $m\in\mathfrak{M}_{\text{erg}}(\Omega,\sigma)$ and two bounded m-measurable equilibria $\mathfrak{b}_1,\mathfrak{b}_2\colon\Omega\to\mathbb{R}$, the subsets $\{\omega\in\Omega \mid \mathfrak{b}_1(\omega)>\mathfrak{b}_2(\omega)\}$, $\{\omega\in\Omega \mid \mathfrak{b}_1(\omega)<\mathfrak{b}_2(\omega)\}$ and $\{\omega\in\Omega \mid \mathfrak{b}_1(\omega)=\mathfrak{b}_2(\omega)\}$ of Ω are σ-invariant, so they have m-measure 0 or 1. Since Ω is the disjoint union of these three sets, one of these sets has m-measure 1 and the other two have m-measure 0. That is, two bounded m-measurable equilibria are distinct m-a.e. if and only if they are ordered m-a.e.

The following theorem achieves the target of giving an upper bound on the number of ordered disjoint compact τ-invariant sets on $\Omega\times\mathbb{R}$ which project onto Ω, as announced at the beginning of the chapter.

Theorem 2.11. *Let $\mathfrak{h}$ satisfy* **d1**, **d3** *and* **d4**. *Then, there exist three ordered disjoint τ-invariant compact sets $\mathcal{K}_1<\mathcal{K}_2<\mathcal{K}_3$ projecting onto Ω if and only if there exist three hyperbolic copies of the base $\{\mathfrak{l}\}$, $\{\mathfrak{m}\}$ and $\{\mathfrak{u}\}$ with $\mathfrak{l}<\mathfrak{m}<\mathfrak{u}$. In this case, $\mathcal{K}_1=\{\mathfrak{l}\}$ and $\mathcal{K}_3=\{\mathfrak{u}\}$ and they are attractive; $\mathcal{K}_2=\{\mathfrak{m}\}$ and it is repulsive; and $\mathcal{B}=\{(\omega,x)\in\Omega\times\mathbb{R} \mid \mathfrak{l}(\omega)\le x\le\mathfrak{u}(\omega)\}$ is the set of globally bounded orbits. In particular, there are at most three disjoint and ordered τ-invariant compact sets projecting onto Ω.*

Proof. Sufficiency is obvious. To check necessity, assume that there exist three disjoint and ordered τ-invariant compact sets $\mathcal{K}_1 < \mathcal{K}_2 < \mathcal{K}_3$ projecting onto Ω. Theorem 1.36(iii) ensures that there exist $m^u_{\mathcal{K}_1} \in \mathfrak{M}_{\rm erg}(\Omega,\sigma)$ and a τ-equilibrium $\mathfrak{b}^u_{\mathcal{K}_1}\colon \Omega \to \mathbb{R}$ such that $\sup \mathrm{Lyap}(\mathcal{K}_1) = \int_\Omega \mathfrak{h}_x(\omega, \mathfrak{b}^u_{\mathcal{K}_1}(\omega))\, dm^u_{\mathcal{K}_1}$, where $\inf \mathrm{Lyap}(\mathcal{K}_i)$ and $\sup \mathrm{Lyap}(\mathcal{K}_i)$ are the lower and upper Lyapunov exponents of $\mathcal{K}_i$ for $i \in \{1,2,3\}$ (see Definition 1.35). Let $\mathfrak{l}_{\mathcal{K}_i}$ and $\mathfrak{u}_{\mathcal{K}_i}$ be the lower and upper τ-equilibria of $\mathcal{K}_i$ given by Lemma 1.29 for $i \in \{1,2,3\}$, respectively. Recall that they are $m^u_{\mathcal{K}_1}$-measurable, since they are semicontinuous. Theorem 2.9 applied to $\mathfrak{b}^u_{\mathcal{K}_1} < \mathfrak{u}_{\mathcal{K}_2} < \mathfrak{u}_{\mathcal{K}_3}$ allows us to conclude that $\sup \mathrm{Lyap}(\mathcal{K}_1) < 0$. Analogous arguments show that $\inf \mathrm{Lyap}(\mathcal{K}_2) > 0$ and $\sup \mathrm{Lyap}(\mathcal{K}_3) < 0$. Moreover, for any $m \in \mathfrak{M}_{\rm erg}(\Omega,\sigma)$, since $\mathfrak{l}_{\mathcal{K}_1} \le \mathfrak{u}_{\mathcal{K}_1} < \mathfrak{l}_{\mathcal{K}_2} \le \mathfrak{u}_{\mathcal{K}_2} < \mathfrak{l}_{\mathcal{K}_3} \le \mathfrak{u}_{\mathcal{K}_3}$ are at least three different m-measurable τ-equilibria, the last assertion in Theorem 2.9 ensures that $\mathfrak{l}_{\mathcal{K}_i}(\omega) = \mathfrak{u}_{\mathcal{K}_i}(\omega)$ for m-a.e. $\omega \in \Omega$, and hence Theorem 1.40 ensures that $\mathcal{K}_1$ and that $\mathcal{K}_3$ are attractive hyperbolic copies of Ω and $\mathcal{K}_2$ is a repulsive hyperbolic copy of Ω. This fact precludes the existence of more than three disjoint and ordered τ-invariant compact sets projecting onto Ω: a contradiction analogous to the one at the end of the proof of Theorem 2.9 would be found, changing the role of the sign of the integrals by the attractive or repulsive character of the copy of the base.

Let us write $\mathcal{K}_1 = \{\mathfrak{l}\}$ and $\mathcal{K}_3 = \{\mathfrak{u}\}$ for continuous maps $\mathfrak{l}, \mathfrak{u}\colon \Omega \to \mathbb{R}$. Clearly, $\bigcup_{\omega\in\Omega}(\{\omega\} \times [\mathfrak{l}(\omega), \mathfrak{u}(\omega)]) \subseteq \mathcal{B}$. To prove the converse inclusion, we assume for contradiction the existence of (ω_0, x_0) with $x_0 > \mathfrak{u}(\omega_0)$ and with globally defined and bounded τ-orbit. Then, the α-limit set $\mathcal{K}$ of this orbit exists and is a compact τ-invariant set projecting onto a compact set $\Omega_\mathcal{K} \subseteq \Omega$. Since $\{\mathfrak{u}\}$ is attractive, Proposition 1.57 restricted to $\Omega_\mathcal{K}$ (see Remark 2.5) ensures that $\mathcal{K} > \mathcal{K}_3|_{\Omega_\mathcal{K}} > \mathcal{K}_2|_{\Omega_\mathcal{K}} > \mathcal{K}_1|_{\Omega_\mathcal{K}}$, which contradicts the last assertion of the previous paragraph. A similar argument with ω-limit sets shows that $x_0 \ge \mathfrak{l}(\omega_0)$ for all $(\omega_0, x_0) \in \mathcal{B}$. □

Remark 2.12. Theorem 2.11 shows that, if there exist three ordered disjoint copies of the base, then the set $\mathcal{B}$ of bounded τ-orbits is nonempty, bounded, and the upper and lower equilibria of $\mathcal{B}$ give the pair of attractive hyperbolic copies of the base. Assume now that we previously know that $\mathcal{B}$ is nonempty and bounded (as it will be the case when **d2** is imposed, see Section 2.2.2), with $\mathcal{B} \subseteq \Omega \times \mathcal{J}_\mathcal{B}$ for a compact interval $\mathcal{J}_\mathcal{B} = [a,b] \subset \mathbb{R}$. Then, all the conclusions of Theorem 2.11 apply if, for all $m \in \mathfrak{M}_{\rm erg}(\Omega,\sigma)$, $m(\{\omega \in \Omega \,|\, x \mapsto \mathfrak{h}_x(\omega,x) \text{ is concave on } \mathcal{J}_\mathcal{B}\}) = 1$ and $m(\{\omega \in \Omega \,|\, x \mapsto \mathfrak{h}_x(\omega,x) \text{ is strictly concave on } \mathcal{J}_\mathcal{B}\}) > 0$. To check this claim, it suffices to consider the d-concave extension $\widetilde{\mathfrak{h}}$ of $\mathfrak{h}$ outside $\mathcal{J}_\mathcal{B}$,

$$\widetilde{\mathfrak{h}}(\omega,x) = \begin{cases} \mathfrak{h}(\omega,a) + \mathfrak{h}_x(\omega,a)(x-a) + \dfrac{\mathfrak{h}_{xx}(\omega,a)}{2}(x-a)^2 - (x-a)^3, & x < a\,, \\ \mathfrak{h}(\omega,x)\,, & a \le x \le b, \\ \mathfrak{h}(\omega,b) + \mathfrak{h}_x(\omega,b)(x-b) + \dfrac{\mathfrak{h}_{xx}(\omega,b)}{2}(x-b)^2 - (x-b)^3, & b < x\,, \end{cases}$$

apply Theorem 2.11 to $\widetilde{\mathfrak{h}}$, which satisfies **d1**, **d3** and **d4**, and deduce the conclusions for $\mathfrak{h}$.

2.2.2 The existence and properties of the global attractor

Now, the coercivity property **d2** comes into play. Recall that, if all the forward τ-semiorbits are globally defined, then a compact τ-invariant set $\mathcal{A} \subset \Omega \times \mathbb{R}$ is the *global attractor of* τ if it attracts every bounded set $\mathcal{C} \subset \Omega \times \mathbb{R}$; that is, if $\lim_{t\to\infty} \operatorname{dist}(\tau_t(\mathcal{C}), \mathcal{A}) = 0$, where $\operatorname{dist}(\mathcal{C}_1, \mathcal{C}_2)$ is the Hausdorff semidistance from $\mathcal{C}_1$ to $\mathcal{C}_2$ and $\tau_t(\mathcal{C}) = \{\tau(t,\omega,x) \mid (\omega,x) \in \mathcal{C}\}$ (see Definition 1.31).

Theorem 2.13. *Let* $\mathfrak{h} \in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$ *satisfy* **d2**, *and take* $\delta_1, \delta_2 > 0$ *and* $m_1, m_2 \in \mathbb{R}$ *with* $\mathfrak{h}(\omega,x) \geq \delta_1$ *if* $x \leq m_1$ *and* $\mathfrak{h}(\omega,x) \leq -\delta_2$ *if* $x \geq m_2$ *for all* $\omega \in \Omega$. *Then,*

(i) $v(t,\omega,x)$ *exists for* $(t,\omega,x) \in [0,\infty)\times\Omega\times\mathbb{R}$, *and* $m_1 \leq \liminf_{t\to\infty} v(t,\omega,x) \leq \limsup_{t\to\infty} v(t,\omega,x) \leq m_2$*: any forward* τ*-semiorbit is bounded.*

(ii) *There exists the global attractor for* τ, *it is of the form*

$$\mathcal{A} = \bigcup_{\omega\in\Omega} \big(\{\omega\} \times [\mathfrak{l}(\omega), \mathfrak{u}(\omega)]\big)\,, \tag{2.16}$$

it is the union of all the globally defined and bounded τ*-orbits, and it is contained in* $\Omega \times [m_1, m_2]$. *Moreover,*

$$\mathfrak{l}(\omega) = \lim_{t\to\infty} v(t, \omega{\cdot}(-t), m_1)\,,$$
$$\mathfrak{u}(\omega) = \lim_{t\to\infty} v(t, \omega{\cdot}(-t), m_2)\,.$$

(iii) *The maps* $\mathfrak{l}$ *and* $\mathfrak{u}$ *are, respectively, lower and upper semicontinuous* τ*-equilibria.*

(iv) $v(t,\omega,x)$ *is bounded from below if and only if* $x \geq \mathfrak{l}(\omega)$, *and from above if and only if* $x \leq \mathfrak{u}(\omega)$.

(v) *If, for a point* $\omega \in \Omega$, *there exists a bounded* C^1 *function* $b\colon \mathbb{R} \to \mathbb{R}$ *such that* $b'(t) \leq \mathfrak{h}(\omega{\cdot}t, b(t))$ *(resp.* $b'(t) \geq \mathfrak{h}(\omega{\cdot}t, b(t))$*) for all* $t \in \mathbb{R}$, *then* $b(t) \leq \mathfrak{u}(\omega{\cdot}t)$ *(resp.* $b(t) \geq \mathfrak{l}(\omega{\cdot}t)$*) for all* $t \in \mathbb{R}$. *If* $b'(t) < \mathfrak{h}(\omega{\cdot}t, b(t))$ *(resp.* $b'(t) > \mathfrak{h}(\omega{\cdot}t, b(t))$*) for all* $t \in \mathbb{R}$, *then* $b(t) < \mathfrak{u}(\omega{\cdot}t)$ *(resp.* $\mathfrak{l}(\omega{\cdot}t) < b(t)$*) for all* $t \in \mathbb{R}$.

(vi) *Assume that* $\mathfrak{h}$ *satisfies also* **d1**, **d3** *and* **d4**, *and that* $\{\mathfrak{l}\}$, $\{\mathfrak{m}\}$ *and* $\{\mathfrak{u}\}$ *are three hyperbolic copies of the base with* $\mathfrak{l} < \mathfrak{m} < \mathfrak{u}$. *Then,* $\mathcal{A} = \bigcup_{\omega\in\Omega}(\{\omega\} \times [\mathfrak{l}(\omega), \mathfrak{u}(\omega)])$; $\{\mathfrak{l}\}$ *and* $\{\mathfrak{u}\}$ *are attractive and* $\{\mathfrak{m}\}$ *is repulsive;* $\lim_{t\to\infty}(v(t,\omega,x) - \mathfrak{u}(\omega{\cdot}t)) = 0$ *if and only if* $x > \mathfrak{m}(\omega)$; $\lim_{t\to\infty}(v(t,\omega,x) - \mathfrak{l}(\omega{\cdot}t)) = 0$ *if and only if* $x < \mathfrak{m}(\omega)$; $\lim_{t\to-\infty}(v(t,\omega,x) - \mathfrak{m}(\omega{\cdot}t)) = 0$ *if and only if* $x \in (\mathfrak{l}(\omega), \mathfrak{u}(\omega))$; *and* $t \mapsto \mathfrak{l}(\omega{\cdot}t), \mathfrak{m}(\omega{\cdot}t), \mathfrak{u}(\omega{\cdot}t)$ *define the three unique hyperbolic solutions of* $(2.9)_\omega$.

(vii) *If the upper Lyapunov exponent* $\sup \operatorname{Lyap}(\mathcal{A})$ *of* $\mathcal{A}$ *is strictly negative, then the* τ*-global attractor is an attractive hyperbolic* τ*-copy of the base.*

Proof. (i)-(iv) The existence of m_1, m_2 and $\delta_1, \delta_2 > 0$ is ensured by hypothesis **d2**: $\limsup_{x\to-\infty} -\mathfrak{h}(\omega,x) < 0$ ensures the existence of m_1 and $\delta_1 > 0$ such that $\mathfrak{h}(\omega,x) \geq \delta_1$ if $x \leq m_1$, $\limsup_{x\to\infty} \mathfrak{h}(\omega,x) < 0$ ensures the existence of m_2 and $\delta_2 > 0$ such that $\mathfrak{h}(\omega,x) \leq -\delta_2$ if $x \geq m_2$. So, it easily follows that, for any $x \in [m_1, m_2]$ and $\omega \in \Omega$, the solution $v(t,\omega,x)$ is defined for all $t \geq 0$ and $v(t,\omega,x) \in [m_1,m_2]$. To prove (i) and (ii), we take $n_1 < m_1$ and $n_2 > m_2$ and check that $v(t,\omega,n_i) \in [m_1,m_2]$ for all $\omega \in \Omega$ and $i = 1,2$ if $t \geq \max\{(1/\delta_1)(m_1 - n_1),(1/\delta_2)(n_2 - m_2)\}$. We deduce from this fact that (i) holds and that $\lim_{t\to\infty} \operatorname{dist}(\tau_t(\mathcal{C}),\Omega\times[m_1,m_2]) = 0$ for every bounded set $\mathcal{C} \subset \Omega\times\mathbb{R}$; i.e., $\Omega\times[m_1,m_2]$ is a compact absorbing set. This property and [22, Theorem 2.2] prove the existence of the global attractor $\mathcal{A} \subseteq \Omega\times[m_1,m_2]$, and [21, Theorem 1.7] ensures that $\mathcal{A}$ is the union of all the globally defined and bounded τ-orbits. Lemma 1.29 proves (iii). To prove the last assertion in (ii), note that the previous properties show that the constant map $\omega \mapsto m_1$ is a global strict lower solution, since $\mathfrak{h}(\omega,m_1) > 0$, and hence Proposition 1.24(i) ensures that it is a strong subequilibrium. Therefore, Proposition 1.20 shows that $\mathfrak{b}(\omega) = \lim_{t\to\infty} v(t,\omega{\cdot}(-t),m_1)$ is a bounded τ-equilibrium, and hence $\{\mathfrak{b}\} \subseteq \mathcal{A}$. For contradiction, we assume that $\mathfrak{l}(\omega_0) < \mathfrak{b}(\omega_0)$. Then, $\mathfrak{l}(\omega_0) < v(t_0,\omega_0{\cdot}(-t_0),m_1)$ for large enough t_0, and hence $\mathfrak{l}(\omega_0{\cdot}(-t_0)) = v(-t_0,\omega_0,\mathfrak{l}(\omega_0)) < m_1$, which is not the case. Hence, $\mathfrak{b}(\omega) = \mathfrak{l}(\omega)$ for all $\omega \in \Omega$. An analogous argument works for $\mathfrak{u}$ and m_2. The assertions in (iv) follow from (i) and (ii).

(v) We consider the case of $b'(t) \leq \mathfrak{h}(\omega{\cdot}t,b(t))$ for all $t \in \mathbb{R}$. The (non strict) conditions on b and a standard comparison argument ensure that $v(t,\omega{\cdot}s,b(s)) \leq b(t+s)$ for any $t \leq 0$, and hence $v(t,\omega{\cdot}s,b(s))$ remains bounded from above as time decreases. Therefore, by (iv), $b(s) \leq \mathfrak{u}(\omega{\cdot}s)$ for all $s \in \mathbb{R}$. Now, we assume $b'(t) < \mathfrak{h}(\omega{\cdot}t,b(t))$ for all $t \in \mathbb{R}$, and, for contradiction, that $b(s) = \mathfrak{u}(\omega{\cdot}s)$ for some $s \in \mathbb{R}$. Then, $b(t+s) > v(t,\omega{\cdot}s,b(s)) = v(t,\omega{\cdot}s,\mathfrak{u}(\omega{\cdot}s)) = \mathfrak{u}(\omega{\cdot}(t+s))$ if $t < 0$, which is not possible. We proceed analogously in the other case.

(vi) Theorem 2.11 and point (ii) show that $\mathcal{A} = \bigcup_{\omega\in\Omega}(\{\omega\}\times[\mathfrak{l}(\omega),\mathfrak{u}(\omega)])$, and that $\{\mathfrak{l}\}$ and $\{\mathfrak{u}\}$ are attractive and $\{\mathfrak{m}\}$ is repulsive. Now we fix (ω_0,x_0) with $x_0 > \mathfrak{m}(\omega_0)$ and deduce from $v(t,\omega_0,x_0) > \mathfrak{m}(\omega_0{\cdot}t)$ for $t \in \mathbb{R}$ that $\bar{x} \geq \mathfrak{m}(\bar{\omega})$ for any $(\bar{\omega},\bar{x})$ in the $\boldsymbol{\omega}$-limit set $\mathcal{O}$ of (ω_0,x_0) for τ. In addition, Proposition 1.57 ensures that $\bar{x} > \mathfrak{m}(\bar{\omega})$. Let $\Omega_{\mathcal{O}} \subseteq \Omega$ be the projection of $\mathcal{O}$. Since $\mathcal{O}$ is a compact τ-invariant set placed strictly above $\{\mathfrak{m}|_{\Omega_{\mathcal{O}}}\}$, which in turn is placed strictly above $\{\mathfrak{l}|_{\Omega_{\mathcal{O}}}\}$, Theorem 2.11 (restricted to $\Omega_{\mathcal{O}}$: see Remark 2.5) ensures that $\mathcal{O} = \{\mathfrak{u}|_{\Omega_{\mathcal{O}}}\}$. Let us now assume for contradiction that it is not true that $\lim_{t\to\infty}(v(t,\omega_0,x_0) - \mathfrak{u}(\omega_0{\cdot}t)) = 0$. Then, since $\mathfrak{u}$ is continuous, there exists $(t_n) \uparrow \infty$ such that $(\omega_0{\cdot}t_n)$ tends to $\bar{\omega}$ and $(v(t_n,\omega_0,x_0))$ tends to $\bar{x} \neq \mathfrak{u}(\bar{\omega})$. But this is impossible, since $(\bar{\omega},\bar{x}) \in \mathcal{O} \subseteq \{\mathfrak{u}\}$. Conversely, if $\lim_{t\to\infty}(v(t,\omega,x) - \mathfrak{u}(\omega{\cdot}t)) = 0$, then the τ-invariance of $\{\mathfrak{m}\}$ ensures that $x > \mathfrak{m}(\omega)$. The same arguments prove the statements for $x_0 < \mathfrak{u}(\omega_0)$ and $x_0 \in (\mathfrak{l}(\omega_0),\mathfrak{u}(\omega_0))$. Finally, Proposition 1.55 shows that $t \mapsto \mathfrak{l}(\omega{\cdot}t),\mathfrak{m}(\omega{\cdot}t),\mathfrak{u}(\omega{\cdot}t)$ are three hyperbolic solutions of $(2.9)_\omega$: attractive the upper and lower ones, and repulsive the middle one. Assume that, for some $\omega \in \Omega$, there exists one more, say $\tilde{b}(t)$. Then, (ii) and the just proved properties yield $\mathfrak{l}(\omega{\cdot}t) < \tilde{b}(t) < \mathfrak{u}(\omega{\cdot}t)$ for all $t \in \mathbb{R}$, and hence $\lim_{t\to-\infty}(\tilde{b}(t) - \mathfrak{m}(\omega{\cdot}t)) = 0$ and either $\lim_{t\to\infty}(\tilde{b}(t) - \mathfrak{u}(\omega{\cdot}t)) = 0$ or $\lim_{t\to\infty}(\tilde{b}(t) - \mathfrak{l}(\omega{\cdot}t)) = 0$. If $\tilde{b}$ is hyperbolic attractive (resp. repulsive), then Proposition 1.56 provides a contradiction, since it cannot approach a solution different from itself as time decreases (resp. increases).

(vii) Let $\mathcal{S} \subseteq \Omega$ be a σ-minimal set, and consider the family (2.9) for $\omega \in \mathcal{S}$. The corresponding attractor is $\mathcal{A}^{\mathcal{S}} = \{(\omega,x) \in \mathcal{A} \mid \omega \in \mathcal{S}\}$. Since all its Lyapunov exponents are negative, [18, Theorem 3.4] guarantees that $\mathcal{A}^{\mathcal{S}}$ is an attractive hyperbolic copy of the base of $(\mathcal{S}\times\mathbb{R},\tau)$. Then, Theorem 1.40 shows that $\mathcal{A}$ is an attractive hyperbolic copy of the base. □

Theorem 2.13(i) ensures the existence of the ω-limit set for τ of any $(\omega, x) \in \Omega\times\mathbb{R}$. Theorem 2.13(iii) ensures that $\mathfrak{l}$ and $\mathfrak{u}$ are m-measurable τ-equilibria for any $m \in \mathfrak{M}_{\rm erg}(\Omega,\sigma)$. We will use these facts throughout the document without further reference.

The following proposition is valid in the minimal base framework. It proves that the existence of either one repulsive hyperbolic τ-minimal set or two hyperbolic τ-minimal sets implies the existence of exactly three τ-minimal sets. The dynamics in that case of three hyperbolic τ-minimal sets will be completely described by Theorem 2.18.

Proposition 2.14. *Let (Ω,σ) be minimal and let $\mathfrak{h}\colon \Omega\times\mathbb{R}\to\mathbb{R}$ satisfy* **d1** *and* **d2**. *Then,*

(i) *if τ admits a repulsive hyperbolic minimal set $\mathcal{M}\subset\Omega\times\mathbb{R}$, then it admits at least three different minimal sets $\mathcal{M}^l<\mathcal{M}<\mathcal{M}^u$.*

(ii) *If τ admits two distinct hyperbolic minimal sets, then it admits at least three different minimal sets.*

Proof. Recall that we say that a τ-minimal set is hyperbolic if it is a τ-copy of the base: see Remark 1.41. Let $\mathfrak{l}$ and $\mathfrak{u}$ be the lower and upper τ-equilibria of $\mathcal{A}$ given by (2.16).

(i) Let $\mathfrak{b}:\Omega\to\mathbb{R}$ provide the repulsive hyperbolic copy of the base $\mathcal{M}=\{\mathfrak{b}\}$, and let $\mathcal{M}^u$ be the τ-minimal set defined from $\mathfrak{u}$ by (1.9). Assume for contradiction the existence of $(\omega,x)\in\mathcal{M}^u$ with $x\le\mathfrak{b}(\omega)$. Then, since $\mathcal{M}^u$ is minimal, for any continuity point ω_0 of $\mathfrak{u}$ there exists $(t_n)\uparrow\infty$ such that $\lim_{n\to\infty}(\omega{\cdot}t_n, v(t_n,\omega,x)) = (\omega_0,\mathfrak{u}(\omega_0))\in\mathcal{M}^u$, so $\mathfrak{u}(\omega_0)=\lim_{n\to\infty}v(t_n,\omega,x)\le\lim_{n\to\infty}v(t_n,\omega,\mathfrak{b}(\omega))=\mathfrak{b}(\omega_0)\le\mathfrak{u}(\omega_0)$. Therefore, $\mathfrak{u}(\omega_0)=\mathfrak{b}(\omega_0)$. So, by the definition of τ-global attractor, $\lim_{t\to\infty}(v(t,\omega_0,x_0)-\mathfrak{b}(\omega_0{\cdot}t))=0$ for $x_0>\mathfrak{b}(\omega_0)$. However, Proposition 1.55 ensures that $t\mapsto\mathfrak{b}(\omega_0{\cdot}t)$ is a repulsive hyperbolic solution, and hence Proposition 1.56 ensures that $\inf_{t>0}|v(t,\omega_0,x_0)-\mathfrak{b}(\omega_0{\cdot}t)|>0$, a contradiction. Therefore $\mathcal{M}<\mathcal{M}^u$. An analogous argument shows that $\mathcal{M}^l<\mathcal{M}$, where $\mathcal{M}^l$ is defined from $\mathfrak{l}$. So, there exist three different τ-minimal sets $\mathcal{M}^l<\mathcal{M}<\mathcal{M}^u$.

(ii) If there were exactly two minimal sets and they were hyperbolic attractive, then Theorem 1.40 would ensure that both of them have strictly negative upper Lyapunov exponent and [18, Theorem 3.4] guarantees that the global attractor $\mathcal{A}$ is a hyperbolic τ-copy of the base, contradicting the existence of two different τ-minimal sets. Consequently, at least one of the two hyperbolic τ-minimal sets is repulsive, and (i) concludes the proof. □

2.2.3 Properties related to Lyapunov exponents

The following result ensures that the sum of two integrals of the type (1.10) given by two bounded measurable equilibria is always negative. Recall that Theorem 1.36(ii) ensures that these integrals correspond to Lyapunov exponents of suitable compact τ-invariant sets.

Theorem 2.15. *Let $\mathfrak{h}\colon\Omega\times\mathbb{R}\to\mathbb{R}$ satisfy* **d1**, *let us fix $m\in\mathfrak{M}_{\rm erg}(\Omega,\sigma)$, and let $\mathfrak{b}_1,\mathfrak{b}_2\colon\Omega\to\mathbb{R}$ be bounded m-measurable τ-equilibria with $\mathfrak{b}_1(\omega)<\mathfrak{b}_2(\omega)$ for*

m-a.e. $\omega\in\Omega$. Assume that $m(\{\omega\in\Omega\,|\,x\mapsto \mathfrak{h}_x(\omega,x)$ is concave$\})=1$. Then,

$$\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}_1(\omega))\,dm+\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}_2(\omega))\,dm\le 0\,.$$

In addition, if $m(\{\omega\in\Omega\,|\,\mathfrak{h}_{xx}(\omega,\mathfrak{b}_1(\omega))>\mathfrak{h}_{xx}(\omega,\mathfrak{b}_2(\omega))\})>0$, then

$$\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}_1(\omega))\,dm+\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}_2(\omega))\,dm< 0\,.$$

Proof. Let Ω_d be the σ-invariant set with $m(\Omega_d)=1$ given by Lemma 2.6, and let $\Omega_0=\Omega_d\cap\{\omega\in\Omega\,|\;\mathfrak{b}_1(\omega)<\mathfrak{b}_2(\omega)\}$. Then, Ω_0 is σ-invariant and $m(\Omega_0)=1$. Let us define $\mathfrak{c}(\omega)=\mathfrak{b}_2(\omega)-\mathfrak{b}_1(\omega)$ for $\omega\in\Omega_0$ and note that $\mathfrak{c}(\omega)>0$ for all $\omega\in\Omega_0$. Then,

$$\mathfrak{c}'(\omega{\cdot}t)=\mathfrak{h}(\omega{\cdot}t,\mathfrak{c}(\omega{\cdot}t)+\mathfrak{b}_1(\omega{\cdot}t))-\mathfrak{h}(\omega{\cdot}t,\mathfrak{b}_1(\omega{\cdot}t))=\mathfrak{c}(\omega{\cdot}t)F(\omega{\cdot}t,\mathfrak{c}(\omega{\cdot}t))\,,\tag{2.17}$$

where $F(\omega,y)=\int_0^1\mathfrak{h}_x(\omega,s\,y+\mathfrak{b}_1(\omega))\,ds$. As $x\mapsto\mathfrak{h}_x(\omega,x)$ is concave for all $\omega\in\Omega_0$, $y\mapsto F_y(\omega,y)=\int_0^1 s\,\mathfrak{h}_{xx}(\omega,s\,y+\mathfrak{b}_1(\omega))\,ds$ is nonincreasing for all $\omega\in\Omega_0$, so

$$F(\omega,\mathfrak{c}(\omega))=F(\omega,0)+\int_0^1\mathfrak{c}(\omega)F_y(\omega,s\,\mathfrak{c}(\omega))\,ds\ge F(\omega,0)+\mathfrak{c}(\omega)F_y(\omega,\mathfrak{c}(\omega))\tag{2.18}$$

for all $\omega\in\Omega_0$. Derivating the equality $yF(\omega,y)=\mathfrak{h}(\omega,y+\mathfrak{b}_1(\omega))-\mathfrak{h}(\omega,\mathfrak{b}_1(\omega))$ with respect to y and evaluating at $y=\mathfrak{c}(\omega)$ yields

$$F(\omega,\mathfrak{c}(\omega))+\mathfrak{c}(\omega)F_y(\omega,\mathfrak{c}(\omega))=\mathfrak{h}_x(\omega,\mathfrak{b}_2(\omega))\,.$$

This equality combined with $F(\omega,0)=\mathfrak{h}_x(\omega,\mathfrak{b}_1(\omega))$ and (2.18) provides

$$\begin{aligned}\int_\Omega \mathfrak{h}_x(\omega,&\mathfrak{b}_1(\omega))\,dm+\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}_2(\omega))\,dm\\&=\int_\Omega\big(F(\omega,0)+F(\omega,\mathfrak{c}(\omega))+\mathfrak{c}(\omega)F_y(\omega,\mathfrak{c}(\omega))\big)\,dm\\&\le 2\int_\Omega F(\omega,\mathfrak{c}(\omega))\,dm\,.\end{aligned}\tag{2.19}$$

According to (2.17), $\mathfrak{c}'(\omega)/\mathfrak{c}(\omega)=F(\omega,\mathfrak{c}(\omega))$ for all $\omega\in\Omega_0$. Since $\mathfrak{c}$ is bounded, so it is $\omega\mapsto F(\omega,\mathfrak{c}(\omega))$, and hence it is in $L^1(\Omega,m)$. Therefore, Birkhoff's Ergodic Theorem 1.10 ensures that

$$\int_\Omega F(\omega,\mathfrak{c}(\omega))\,dm=\lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{\mathfrak{c}'(\omega_0{\cdot}s)}{\mathfrak{c}(\omega_0{\cdot}s)}\,ds=\lim_{t\to\infty}\frac{1}{t}\log\left(\frac{\mathfrak{c}(\omega_0{\cdot}t)}{\mathfrak{c}(\omega_0)}\right)=0\,,\tag{2.20}$$

for some $\omega_0\in\Omega_0$, so the right-hand side of (2.19) vanishes. This proves the first assertion.

Now, let us check that the inequality is strict if, in addition, $m(\Omega_s)>0$, where $\Omega_s=\{\omega\in\Omega\,|\;\mathfrak{h}_{xx}(\omega,\mathfrak{b}_1(\omega))>\mathfrak{h}_{xx}(\omega,\mathfrak{b}_2(\omega))\}$. To this end, we fix $\omega\in\Omega_0$ and note that $F_y(\omega,0)=\int_0^1 s\,\mathfrak{h}_{xx}(\omega,\mathfrak{b}_1(\omega))\,ds$, that $F_y(\omega,\mathfrak{c}(\omega))=\int_0^1 s\,\mathfrak{h}_{xx}(\omega,s\,\mathfrak{b}_2(\omega)+(1-s)\mathfrak{b}_1(\omega))\,ds$, and that $s\,\mathfrak{h}_{xx}(\omega,\mathfrak{b}_1(\omega))\ge s\,\mathfrak{h}_{xx}(\omega,s\,\mathfrak{b}_2(\omega)+(1-s)\mathfrak{b}_1(\omega))$ for all $s\in[0,1]$ and $\omega\in\Omega_0$. If, in addition, $\omega\in\Omega_s$, the continuity of $\mathfrak{h}_{xx}$ ensures that the previous inequality is strict for s close to 1, and hence that $F_y(\omega,0)>F_y(\omega,\mathfrak{c}(\omega))$. Since this happens on the set $\Omega_0\cap\Omega_s$, which is of positive m-measure, we conclude that (2.19) is strict. The assertion follows from here and (2.20). □

The following result gives an important property of the global attractor $\mathcal{A}$ under conditions **d1** and **d2**: the integrals of the type (1.10) given by the lower and upper τ-equilibria $\mathfrak{l},\mathfrak{u}\colon \Omega\to\mathbb{R}$ of the global attractor (see (2.16)) and any $m\in\mathfrak{M}_{\text{erg}}(\Omega,\sigma)$ are always nonpositive. As a consequence, Theorem 1.36(ii) ensures that the lower Lyapunov exponent of $\mathcal{A}$ is always nonpositive, that is, $\inf\text{Lyap}(\mathcal{A})\le 0$.

Proposition 2.16. *Let $\mathfrak{h}\colon\Omega\times\mathbb{R}\to\mathbb{R}$ satisfy* **d1** *and* **d2***, let us fix $m\in\mathfrak{M}_{\text{erg}}(\Omega,\sigma)$, and let $\mathfrak{l}$ and $\mathfrak{u}$ be given by (2.16). Then,*

$$\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{l}(\omega))\,dm\le 0\quad\textit{and}\quad \int_\Omega \mathfrak{h}_x(\omega,\mathfrak{u}(\omega))\,dm\le 0\,.$$

Proof. We reason with $\mathfrak{u}$, assuming for contradiction that $\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{u}(\omega))\,dm=\rho>0$. Birkhoff's Ergodic Theorem 1.10 provides a nonempty σ-invariant subset $\Omega_0\subseteq\Omega$ and, for each $\omega\in\Omega_0$, a time $t_\omega>0$ such that

$$\int_0^t \mathfrak{h}_x(\omega{\cdot}s,\mathfrak{u}(\omega{\cdot}s))\,ds\ge(\rho/2)\,t \tag{2.21}$$

for all $t\ge t_\omega$. Let $L>0$ satisfy the Lipschitz condition

$$|\mathfrak{h}_x(\omega,x)-\mathfrak{h}_x(\omega,\mathfrak{u}(\omega))|\le L\,|x-\mathfrak{u}(\omega)| \tag{2.22}$$

for all $\omega\in\Omega$ and $x\in[\mathfrak{u}(\omega),\mathfrak{u}(\omega)+1]$. Let us take $k\in(0,1)$ with $L\,k\le\rho/4$, and fix $\omega_0\in\Omega_0$ and $x_0>\mathfrak{u}(\omega_0)$. Since $\mathcal{A}$ is the τ-global attractor, there exists $t_1>0$ such that

$$v(t,\omega_0,x_0)-\mathfrak{u}(\omega_0{\cdot}t)\le k<1 \tag{2.23}$$

for all $t\ge t_1$. Then, for $\omega_1=\omega_0{\cdot}t_1$ (which belongs to Ω_0) and $t>0$, there exists $\xi_t\in[\mathfrak{u}(\omega_1),x_1]$ such that, if $x_1=v(t_1,\omega_0,x_0)$, then

$$v(t+t_1,\omega_0,x_0)-\mathfrak{u}(\omega_0{\cdot}(t+t_1))=v_x(t,\omega_1,\xi_t)\,(x_1-\mathfrak{u}(\omega_1))\,. \tag{2.24}$$

Since $v(s,\omega_1,\xi_t)-\mathfrak{u}(\omega_1{\cdot}s)\le v(s+t_1,\omega_0,x_0)-\mathfrak{u}(\omega_0{\cdot}(s+t_1))$, (2.23) and (2.22) ensure that $|\mathfrak{h}_x(\omega_1{\cdot}s,v(s,\omega_1,\xi_t))-\mathfrak{h}_x(\omega_1{\cdot}s,\mathfrak{u}(\omega_1{\cdot}s))|\le L\,k\le\rho/4$ for all $s>0$. So, it follows from (2.21) that $v_x(t,\omega_1,\xi_t)=\exp\int_0^t\mathfrak{h}_x(\omega_1{\cdot}s,v(s,\omega_1,\xi_t))\,ds\ge e^{(\rho/4)\,t}$ if $t\ge t_{\omega_1}$. Hence, the left-hand term of (2.24) cannot converge to 0 as $t\to\infty$, which is the sought-for contradiction. The argument is similar for $\mathfrak{l}$. □

The following result, which refers to the case of minimal base flow and will be often used, makes use of the previous one and Proposition 1.32. The notion of hyperbolic minimal set appears in Remark 1.41.

Proposition 2.17. *Let (Ω,σ) be minimal. Let $\mathfrak{h}\in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$ satisfy* **d1** *and* **d2***, and let $\mathfrak{l}$ and $\mathfrak{u}$ be given by Theorem 2.13. Then,*

- (i) *the expression (1.9) applied to $\mathfrak{l}$ (resp. $\mathfrak{u}$) provides the least (resp. greatest) element $\mathcal{M}^l$ (resp. $\mathcal{M}^u$) of the set of τ-minimal sets equipped with the total order of Definition 1.28 and Corollary 1.33.*
- (ii) *$\mathcal{M}^l$ (resp. $\mathcal{M}^u$) is hyperbolic if and only if it is hyperbolic attractive, in which case $\mathcal{M}^l=\{\mathfrak{l}\}$ (resp. $\mathcal{M}^u=\{\mathfrak{u}\}$).*

Proof. (i) Let ω be a continuity point for $\mathfrak{l}$ and $\mathfrak{u}$. Then, $\mathfrak{l}(\omega)\le x\le\mathfrak{u}(\omega)$ for any point (ω,x) in any minimal set, and (i) follows from here.

(ii) Corollary 1.58(ii) shows that, if $\mathcal{M}^l$ is hyperbolic, then $\mathcal{M}^l=\{\mathfrak{l}\}$. Proposition 2.16 and Theorem 1.36 show that $\inf\text{Lyap}(\mathcal{M}^l)\le 0$, so Theorem 1.40 ensures that $\mathcal{M}^l$ is attractive. The converse is trivial, and the proof for $\mathcal{M}^u$ is analogous. □

2.2.4 The dynamics with three hyperbolic solutions

The next result establishes equivalences regarding the existence of three uniformly separated hyperbolic solutions of a given equation $(2.9)_{\bar\omega}$, for some fixed $\bar\omega \in \Omega$, in terms of the existence of three ordered hyperbolic τ-copies of the corresponding hull $\Omega_{\bar\omega} = \text{closure}\{\bar\omega{\cdot}t \mid t \in \mathbb{R}\} \subseteq \Omega$, and describes the dynamics of any orbit.

Theorem 2.18. *Let $\mathfrak{h}\colon \Omega\times\mathbb{R}\to\mathbb{R}$ satisfy* **d1**, **d2**, **d3** *and* **d4**. *Let us fix $\bar\omega \in \Omega$. Then, the following assertions are equivalent:*

(a) *Equation $(2.9)_{\bar\omega}$ has three hyperbolic solutions.*

(b) *Equation $(2.9)_{\bar\omega}$ has three uniformly separated hyperbolic solutions.*

(c) *Equation $(2.9)_{\bar\omega}$ has three uniformly separated bounded solutions.*

(d) *There exist three hyperbolic copies of the base for the restriction of the family (2.9) to the closure $\Omega_{\bar\omega}$ of $\{\bar\omega{\cdot}t \mid t\in\mathbb{R}\}$, given by $\mathfrak{l}<\mathfrak{m}<\mathfrak{u}$.*

In this case, $t\mapsto \tilde l(t) = \mathfrak{l}(\bar\omega{\cdot}t)$, $t\mapsto \tilde m(t) = \mathfrak{m}(\bar\omega{\cdot}t)$ and $t\mapsto \tilde u(t) = \mathfrak{u}(\bar\omega{\cdot}t)$ are the three unique uniformly separated solutions of $(2.9)_{\bar\omega}$, they are hyperbolic, and there are no more hyperbolic solutions. In addition, if $x_{\bar\omega}(t,s,x)$ is the solution of $(2.9)_{\bar\omega}$ with $x_{\bar\omega}(s,s,x)=x$, then: $\lim_{t\to\infty}(x_{\bar\omega}(t,s,x)-\tilde u(t))=0$ if and only if $x>\tilde m(s)$; $\lim_{t\to\infty}(x_{\bar\omega}(t,s,x)-\tilde l(t))=0$ if and only if $x<\tilde m(s)$; and $\lim_{t\to-\infty}(x_{\bar\omega}(t,s,x)-\tilde m(t))=0$ if and only if $x\in(\tilde l(s),\tilde u(s))$.

Proof. The assertions after the equivalences follow from (d) and Theorem 2.13(iv) and (vi). We will check (b) ⇒ (c) ⇒ (d) ⇒ (a) ⇒ (b). Recall that the hypotheses on $\mathfrak{h}$ are also valid for its restriction to $\Omega_{\bar\omega}\times\mathbb{R}$: see Remark 2.5.

(b) ⇒ (c) ⇒ (d). Obviously, (b) implies (c). Now we assume (c). Let $\mathcal{A}$ be the global attractor and let $\mathfrak{l}$ and $\mathfrak{u}$ be its lower and upper τ-equilibria: see Theorem 2.13(ii). Clearly, there is no restriction in assuming that the three uniformly separated solutions are $l(t)=\mathfrak{l}(\bar\omega{\cdot}t)$, $u(t)=\mathfrak{u}(\bar\omega{\cdot}t)$, and $m(t)$ with $l(t)<m(t)<u(t)$. We call $\delta=\min\{\inf_{t\in\mathbb{R}}(u(t)-m(t)),\,\inf_{t\in\mathbb{R}}(m(t)-l(t))\}>0$. Let $\mathcal{K}_m$ be the closure of the τ-orbit of $(\bar\omega,m(0))$, which projects on $\Omega_{\bar\omega}$. Let us check that $x_0\ge\mathfrak{l}(\omega_0)+\delta$ for all $(\omega_0,x_0)\in\mathcal{K}_m$. We write $(\omega_0,x_0)=\lim_{n\to\infty}(\bar\omega{\cdot}t_n,m(t_n))$ and assume without restriction the existence of $(\omega_0,x^0)=\lim_{n\to\infty}(\bar\omega{\cdot}t_n,l(t_n))$, which belongs to $\mathcal{A}$ (closed). If $x_0<\mathfrak{l}(\omega_0)+\delta$, then $x^0\le x_0-\delta<\mathfrak{l}(\omega_0)+\delta-\delta=\mathfrak{l}(\omega_0)$, impossible. Analogously, $x_0\le\mathfrak{u}(\omega_0)-\delta$ for all $(\omega_0,x_0)\in\mathcal{K}_m$. Let us consider the restriction $\bar\tau$ of τ to $\Omega_{\bar\omega}\times\mathbb{R}$. Since any $\bar\tau$-equilibria with graph in $\mathcal{K}_m$ is strictly below $\mathfrak{u}$ and strictly above $\mathfrak{l}$, Theorems 2.9 and 1.36 show that all the Lyapunov exponents of $\mathcal{K}_m$ are strictly positive, and that its upper and lower equilibria coincide on a σ-invariant set Ω_0 with $m_0(\Omega_0)=1$ for all $m_0\in\mathfrak{M}_{\text{erg}}(\Omega_{\bar\omega},\sigma)$. Hence, Theorem 1.40 ensures that $\mathcal{K}_m$ is a repulsive hyperbolic copy of $\Omega_{\bar\omega}$, in particular, there exists a continuous map $\mathfrak{m}\colon\Omega_{\bar\omega}\to\mathbb{R}$ such that $\mathcal{K}_m=\{\mathfrak{m}\}$. Then, $\mathcal{K}_m$ is strictly above the closure $\mathcal{K}_l$ of $\{(\bar\omega{\cdot}t,\mathfrak{l}(\bar\omega{\cdot}t))\mid t\in\mathbb{R}\}$: given $(\omega_0,x_0)\in\mathcal{K}_l$, we write $(\omega_0,x_0)=\lim_{n\to\infty}(\bar\omega{\cdot}t_n,\mathfrak{l}(\bar\omega{\cdot}t_n))$, and from $\mathfrak{l}(\bar\omega{\cdot}t_n)+\delta\le\mathfrak{m}(\bar\omega{\cdot}t_n)$ it is obtained that $x_0+\delta\le\mathfrak{m}(\omega_0)$. Analogously, $\mathcal{K}_m$ is strictly below the closure $\mathcal{K}_u$ of $\{(\bar\omega{\cdot}t,\mathfrak{u}(\bar\omega{\cdot}t))\mid t\in\mathbb{R}\}$. Hence, Theorem 2.11 ensures that both $\mathcal{K}_l$ and $\mathcal{K}_u$ are attractive hyperbolic copies of $\Omega_{\bar\omega}$: (d) holds.

(d) ⇒ (a) ⇒ (b). If (d) holds, then $t\mapsto\mathfrak{l}(\bar\omega{\cdot}t)$, $t\mapsto\mathfrak{m}(\bar\omega{\cdot}t)$ and $t\mapsto\mathfrak{u}(\bar\omega{\cdot}t)$ are three hyperbolic solutions of $(2.9)_{\bar\omega}$ (see Proposition 1.55), so (a) holds. Let us assume (a), and let $\tilde x_1<\tilde x_2<\tilde x_3$ be the three hyperbolic solutions of $(2.9)_{\bar\omega}$. Let us first eliminate the possibility that $\tilde{x}_2$ is attractive, assuming it for contradiction. Let us call $l(t) = \mathfrak{l}(\bar{\omega}{\cdot}t)$ and $u(t) = \mathfrak{u}(\bar{\omega}{\cdot}t)$. Proposition 1.56 provides $\delta > 0$ such that $\inf_{t\le 0}(u(t) - \tilde{x}_2(t)) > \delta$ and $\inf_{t\le 0}(\tilde{x}_2(t) - l(t)) > \delta$. Let $\mathcal{M}_l$, $\mathcal{M}_2$ and $\mathcal{M}_u$ be the α-limit sets of $(\bar{\omega}, l(0))$, $(\bar{\omega}, \tilde{x}_2(0))$ and $(\bar{\omega}, u(0))$, which project on the α-limit set $\Omega_- \subseteq \Omega_{\bar{\omega}}$ of $\bar{\omega}$. As in the previous paragraph, we check that $\mathfrak{l}(\omega_0)+\delta \le x_0 \le \mathfrak{u}(\omega_0)-\delta$ if $(\omega_0, x_0) \in \mathcal{M}_2$; and deduce that $\mathcal{M}_2$ is a repulsive copy of Ω_-. Proposition 1.55 shows that any orbit in $\mathcal{M}_2$ corresponds to a repulsive hyperbolic solution, Proposition 1.54(i) shows that it also corresponds to an attractive hyperbolic solution, and Remark 1.51 provides the sought-for contradiction.

Hence, $\tilde{x}_2$ is repulsive. Proposition 1.56 yields $\delta > 0$ such that $\inf_{t\ge 0}(u(t) - \tilde{x}_2(t)) > \delta$ and $\inf_{t\ge 0}(\tilde{x}_2(t) - l(t)) > \delta$. Let $\bar{\mathcal{M}}_2$ be the ω-limit set of $(\bar{\omega}, \tilde{x}_2(0))$, which projects on the ω-limit set $\Omega_+ \subseteq \Omega_{\bar{\omega}}$ of $\bar{\omega}$. As in the proof of (c)$\Rightarrow$(d), we check that $\mathfrak{l}(\omega_0) + \delta \le x_0 \le \mathfrak{u}(\omega_0) - \delta$ whenever $(\omega_0, x_0) \in \bar{\mathcal{M}}_2$; and we deduce that $\bar{\mathcal{M}}_2$ is a repulsive copy of Ω_+. Hence, $\bar{\mathcal{M}}_2$ does not intersect the ω-limit sets $\bar{\mathcal{M}}_1$ of $(\bar{\omega}, \tilde{x}_1(0))$ and $\bar{\mathcal{M}}_3$ of $(\bar{\omega}, \tilde{x}_3(0))$: see Proposition 1.57. So, we have $\bar{\mathcal{M}}_1 < \bar{\mathcal{M}}_2 < \bar{\mathcal{M}}_3$. Theorem 2.11 applied to $\Omega_+ \times \mathbb{R}$ ensures that $\bar{\mathcal{M}}_1$ and $\bar{\mathcal{M}}_3$ are attractive hyperbolic copies of Ω_+ and, Proposition 1.54(i) shows that this is only possible if $\tilde{x}_1$ and $\tilde{x}_3$ are attractive. Proposition 1.56 ensures that the three solutions are uniformly separated: (b) holds. □

2.2.5 Possible orders of three minimal sets

The purpose of this section, which is achieved in Theorem 2.20, is to show that, if (Ω, σ) is minimal and $\mathfrak{h}_0(\omega, x) \le \mathfrak{h}_1(\omega, x)$ are two ordered d-concave functions such that both $x' = \mathfrak{h}_0(\omega{\cdot}t, x)$ and $x' = \mathfrak{h}_1(\omega{\cdot}t, x)$ have the maximum possible number (three) of copies of the base, then there are only two possible orders among the copies of the base. The proof of Theorem 2.20 is based on the next result, which does not require (Ω, σ) to be minimal.

Proposition 2.19. *Let $\mathfrak{h}_0, \mathfrak{h}_1 \colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy* **d1** *and* **d2***, with $\mathfrak{h}_0(\omega, x) \le \mathfrak{h}_1(\omega, x)$ for all $(\omega, x) \in \Omega \times \mathbb{R}$.*

(i) *Let $\mathfrak{l}_i$ (resp. $\mathfrak{u}_i$) be the lower (resp. upper) bounds of the global attractor (2.16) of $x' = \mathfrak{h}_i(\omega{\cdot}t, x)$ for $i = 0, 1$. Then, $\mathfrak{l}_0 \le \mathfrak{l}_1$ and $\mathfrak{u}_0 \le \mathfrak{u}_1$.*

(ii) *Assume also that both $\mathfrak{h}_0$ and $\mathfrak{h}_1$ satisfy* **d3** *and* **d4***, and that there exists $\bar{\omega} \in \Omega$ such that $x' = \mathfrak{h}_i(\bar{\omega}{\cdot}t, x)$ has three hyperbolic solutions $\tilde{l}_i < \tilde{m}_i < \tilde{u}_i$ for $i = 0, 1$. Then, $\inf_{t\in\mathbb{R}}(\tilde{m}_0(t) - \tilde{l}_1(t)) > 0$ if and only if $\inf_{t\in\mathbb{R}}(\tilde{u}_0(t) - \tilde{m}_1(t)) > 0$, in which case $\tilde{l}_0 \le \tilde{l}_1 < \tilde{m}_1 \le \tilde{m}_0 < \tilde{u}_0 \le \tilde{u}_1$. If, in addition, $\mathfrak{h}_0(\bar{\omega}{\cdot}t, x) < \mathfrak{h}_1(\bar{\omega}{\cdot}t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$, then all the inequalities are strict.*

Proof. (i) Since $\mathfrak{u}_0'(\omega{\cdot}t) \le \mathfrak{h}_1(\omega{\cdot}t, \mathfrak{u}_0(\omega{\cdot}t))$ and $\mathfrak{l}_1'(\omega{\cdot}t) \ge \mathfrak{h}_0(\omega{\cdot}t, \mathfrak{l}_1(\omega{\cdot}t))$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$, Theorem 2.13(v) proves (i).

(ii) As only a fixed $\bar{\omega} \in \Omega$ is involved, let us use processes language to simplify the notation: let $t \mapsto x_i(t, s, x)$ stand for the solution of $x' = \mathfrak{h}_i(\bar{\omega}{\cdot}t, x)$ satisfying $x_i(s, s, x) = x$ for $i = 0, 1$. Let us assume $\inf_{t\in\mathbb{R}}(\tilde{m}_0(t) - \tilde{l}_1(t)) > 0$: the argument is analogous if $\inf_{t\in\mathbb{R}}(\tilde{u}_0(t) - \tilde{m}_1(t)) > 0$. We define $m_s(t) = x_1(t, s, \tilde{m}_0(s))$ (which always exists since $\tilde{l}_1 < \tilde{m}_0 \le \tilde{u}_0 \le \tilde{u}_1$) and observe that $m_s(t) > \tilde{l}_1(t)$ for all $s, t \in \mathbb{R}$. A standard comparison argument shows that $x_1(t, s, \tilde{m}_0(s)) \le \tilde{m}_0(t)$ if $t \le s$. In addition, if $s_0 < s_1$, then $m_{s_0} \ge m_{s_1}$: $m_{s_1}(t) = x_1(t, s_1, \tilde{m}_0(s_1)) =$

$x_1(t, s_0, x_1(s_0, s_1, \tilde{m}_0(s_1))) \le x_1(t, s_0, \tilde{m}_0(s_0)) = m_{s_0}(t)$. Therefore, there exists $m_\infty(t) = \lim_{s\to\infty} m_s(t) \in [\tilde{l}_1(t), \tilde{m}_0(t)]$ for all $t \in \mathbb{R}$. It is easy to check that $t \mapsto m_\infty(t)$ is a bounded solution of $x' = \mathfrak{h}_1(\bar\omega{\cdot}t, x)$. Our goal is to check that $\lim_{t\to\infty} |m_\infty(t) - \tilde{m}_1(t)| = 0$, which, according to Theorem 2.18, ensures that $m_\infty = \tilde{m}_1$ and hence that $\tilde{m}_1 \le \tilde{m}_0$. This inequality combined with (i) proves that $\tilde{l}_0 \le \tilde{l}_1 < \tilde{m}_1 \le \tilde{m}_0 < \tilde{u}_0 \le \tilde{u}_1$. In turn, this chain of inequalities combined with the uniform separation of the hyperbolic solutions $\tilde{l}_i < \tilde{m}_i < \tilde{u}_i$ (see Theorem 2.18) ensures that $\inf_{t\in\mathbb{R}}(\tilde{u}_0(t) - \tilde{m}_1(t)) > 0$.

The inequalities $m_\infty \le \tilde{m}_0 < \tilde{u}_0 \le \tilde{u}_1$ and $\inf_{t\in\mathbb{R}}(\tilde{u}_0(t) - \tilde{m}_0(t)) > 0$ preclude $\lim_{t\to\infty} |m_\infty(t) - \tilde{u}_1(t)| = 0$. Hence, the unique possibility to be excluded (see again Theorem 2.18) is $\lim_{t\to\infty} |m_\infty(t) - \tilde{l}_1(t)| = 0$, which we assume for contradiction. Corollary 1.53 provides a radius of uniform stability $\rho > 0$ and a dichotomy constant pair (k, β) for the hyperbolic solution $\tilde{l}_1$ of $x' = \mathfrak{h}_1(\bar\omega{\cdot}t, x)$. We look for $t_0 \in \mathbb{R}$ such that $|m_\infty(t_0) - \tilde{l}_1(t_0)| \le \rho/2$, and by the definition of m_∞ and the monotonicity on the limit, we take $s_0 \ge t_0$ such that $|\tilde{l}_1(t_0) - x_1(t_0, s, \tilde{m}_0(s)))| \le \rho$ for all $s \ge s_0$. Then, $|\tilde{l}_1(s) - \tilde{m}_0(s)| = |\tilde{l}_1(s) - x_1(s, t_0, x_1(t_0, s, \tilde{m}_0(s)))| \le k e^{-\beta(s-t_0)}\rho$ for all $s \ge s_0$. Taking limit as $s \to \infty$ yields $\inf_{t\in\mathbb{R}}(\tilde{m}_0(t) - \tilde{l}_1(t)) = 0$, which is the sought-for contradiction.

The last assertion follows easily from contradiction and comparison. For instance, if $m_0(s) = m_1(s)$, then $m_0(t) = x_0(t, s, m_0(s)) < x_1(t, s, m_1(s)) = m_1(t)$ for $t > s$, which is not the case. □

Theorem 2.20. *Assume that (Ω, σ) is minimal. Let $\mathfrak{h}_0, \mathfrak{h}_1 \colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy* **d1**, **d2**, **d3** *and* **d4**, *as well as $\mathfrak{h}_0(\omega, x) < \mathfrak{h}_1(\omega, x)$ for all $(\omega, x) \in \Omega \times \mathbb{R}$. Assume that the family $x' = \mathfrak{h}_i(\omega{\cdot}t, x)$ has three hyperbolic copies of the base $\mathfrak{l}_i < \mathfrak{m}_i < \mathfrak{u}_i$ for $i = 0, 1$. Then, one of the following orders holds:*

(1) $\mathfrak{l}_0 < \mathfrak{l}_1 < \mathfrak{m}_1 < \mathfrak{m}_0 < \mathfrak{u}_0 < \mathfrak{u}_1$,

(2) $\mathfrak{l}_0 < \mathfrak{m}_0 < \mathfrak{u}_0 < \mathfrak{l}_1 < \mathfrak{m}_1 < \mathfrak{u}_1$.

Proof. Let us assume (2) does not hold. Hence, there exists $\bar\omega \in \Omega$ with $\mathfrak{l}_1(\bar\omega) \le \mathfrak{u}_0(\bar\omega)$. A standard comparison argument ensures that $\mathfrak{l}_1(\bar\omega{\cdot}t) < \mathfrak{u}_0(\bar\omega{\cdot}t)$ for all $t < 0$, and hence the minimality of Ω ensures that $\mathfrak{l}_1 \le \mathfrak{u}_0$. If, in addition, $\mathfrak{l}_1(\omega_0) = \mathfrak{u}_0(\omega_0)$ for an $\omega_0 \in \Omega$, then a new comparison argument shows that $\mathfrak{u}_0(\omega_0{\cdot}t) < \mathfrak{l}_1(\omega_0{\cdot}t)$ for all $t > 0$, impossible. Therefore, $\mathfrak{l}_1 < \mathfrak{u}_0$. Now, for contradiction, we assume that (1) is also not satisfied, and deduce from Proposition 2.19(ii) and the minimality of Ω the existence of $\bar\omega \in \Omega$ with $\mathfrak{m}_0(\bar\omega) \le \mathfrak{l}_1(\bar\omega)$. Hence, $\mathfrak{m}_0(\bar\omega{\cdot}t) < \mathfrak{l}_1(\bar\omega{\cdot}t)$ for all $t > 0$. We fix $t_0 > 0$ and call $\omega_0 = \bar\omega{\cdot}t_0$. Theorem 2.13(vi) yields $\lim_{t\to\infty}(\mathfrak{u}_0(\omega_0{\cdot}t) - v_0(t, \omega_0, \mathfrak{l}_1(\omega_0))) = 0$, where v_0 stands for the solutions of $x' = \mathfrak{h}_0(\omega{\cdot}t, x)$. Since $v_0(t, \omega_0, \mathfrak{l}_1(\omega_0)) < \mathfrak{l}_1(\omega_0{\cdot}t)$ for all $t > 0$, we deduce that $\limsup_{t\to\infty}(\mathfrak{u}_0(\omega_0{\cdot}t) - \mathfrak{l}_1(\omega_0{\cdot}t))) \le 0$, which combined with the minimality of Ω contradicts $\mathfrak{l}_1 < \mathfrak{u}_0$. □

So, if (Ω, σ) is minimal, then the only two possible orders between two triads of copies of Ω of $\mathfrak{h}_0 < \mathfrak{h}_1$ are as follows: the three copies of Ω for $\mathfrak{h}_0$ are below and uniformly separated from the three copies of Ω for $\mathfrak{h}_1$ (possibility (2) of Theorem 2.20) or else the two upper copies of Ω for $\mathfrak{h}_0$ are between the two upper copies of Ω for $\mathfrak{h}_1$ and the two lower copies of Ω for $\mathfrak{h}_1$ are between the two lower copies of Ω for $\mathfrak{h}_0$ (possibility (1) of Theorem 2.20).

2.3 Negative Schwarzian derivative

In [50], results of the type discussed earlier in this chapter for quasiperiodically forced increasing (on the fiber) maps $T\colon \mathbb{S}^1\times[a,b]\to\mathbb{S}^1\times[a,b]$ with strictly negative Schwarzian derivative can be found. In [56, 58], pitchfork bifurcations of this type of maps are investigated.

In order to find the link between d-concavity and negative Schwarzian derivative, we will consider, for each fixed time $t\in\mathbb{R}$, the Schwarzian derivative with respect to the state variable x of the nonautonomous map $\Omega\times\mathbb{R}\to\Omega\times\mathbb{R}$, $(\omega,x)\mapsto\tau_t(\omega,x)=(\omega{\cdot}t, v(t,\omega,x))$ given by the evolution at time t according to $x'=\mathfrak{h}(\omega{\cdot}t,x)$.

Recall that $\mathcal{V}\supset\{0\}\times\Omega\times\mathbb{R}$ is the (open) domain of the flow τ: see Section 1.2. Since, for all $(\omega,x)\in\Omega\times\mathbb{R}$, $t\mapsto v_x(t,\omega,x)$ solves the variational equation $z'=\mathfrak{h}_x(\omega{\cdot}t, v(t,\omega,x))\,z$ of $x'=\mathfrak{h}(\omega{\cdot}t,x)$ along $t\mapsto v(t,\omega,x)$ and satisfies $v_x(0,\omega,x)=1$, we have

$$v_x(t,\omega,x)=\exp\left(\int_0^t \mathfrak{h}_x\big(\omega{\cdot}s, v(s,\omega,x)\big)\,ds\right)$$

for all $(t,\omega,x)\in\mathcal{V}$. A repeated usage of continuous dependence on initial conditions and the theorem of derivation under the integral sign lead to the expressions

$$v_{xx}(t,\omega,x)=v_x(t,\omega,x)\int_0^t \mathfrak{h}_{xx}(\omega{\cdot}s, v(s,\omega,x))\,v_x(s,\omega,x)\,ds\,,$$

$$\begin{aligned}v_{xxx}(t,\omega,x)=v_{xx}(t,\omega,x)&\int_0^t \mathfrak{h}_{xx}(\omega{\cdot}s, v(s,\omega,x))\,v_x(s,\omega,x)\,ds\\&+v_x(t,\omega,x)\int_0^t\Big(\mathfrak{h}_{xxx}(\omega{\cdot}s, v(s,\omega,x))\,\big(v_x(s,\omega,x)\big)^2\\&\qquad\qquad+\mathfrak{h}_{xx}(\omega{\cdot}s, v(s,\omega,x))\,v_{xx}(s,\omega,x)\Big)\,ds\,,\end{aligned}$$

$$\begin{aligned}v_{xxxt}(t,\omega,x)=v_{xxt}(t,\omega,x)&\int_0^t \mathfrak{h}_{xx}(\omega{\cdot}s, v(s,\omega,x))\,v_x(s,\omega,x)\,ds\\&+v_{xx}(t,\omega,x)\,\mathfrak{h}_{xx}(\omega{\cdot}t, v(t,\omega,x))\,v_x(t,\omega,x)\\&+v_{xt}(t,\omega,x)\int_0^t\Big(\mathfrak{h}_{xxx}(\omega{\cdot}s, v(s,\omega,x))\,(v_x(s,\omega,x))^2\\&\qquad\qquad+\mathfrak{h}_{xx}(\omega{\cdot}s, v(s,\omega,x))\,v_{xx}(s,\omega,x)\Big)\,ds\\&+v_x(t,\omega,x)\Big(\mathfrak{h}_{xxx}(\omega{\cdot}t, v(t,\omega,x))\,\big(v_x(t,\omega,x)\big)^2\\&\qquad\qquad+\mathfrak{h}_{xx}(\omega{\cdot}t, v(t,\omega,x))\,v_{xx}(t,\omega,x)\Big)\,,\end{aligned}$$

from where it easily follows that $v_{xx}(0,\omega,x)=0$, $v_{xxx}(0,\omega,x)=0$ and $v_{xxxt}(0,\omega,x)=\mathfrak{h}_{xxx}(\omega,x)$ for every $(\omega,x)\in\Omega\times\mathbb{R}$. These expressions will be used in the proofs of the following propositions.

Since $v_x(t,\omega,x)$ never vanishes, the *Schwarzian derivative* with respect to the state variable x, given by

$$S_xv(t,\omega,x)=\frac{v_{xxx}(t,\omega,x)}{v_x(t,\omega,x)}-\frac{3}{2}\left(\frac{v_{xx}(t,\omega,x)}{v_x(t,\omega,x)}\right)^2\,, \tag{2.25}$$

is well defined on $\mathcal{V}$.

Proposition 2.21. *Let* $\mathfrak{h} \in C^{0,3}(\Omega \times \mathbb{R}, \mathbb{R})$. *Then,*

(i) $S_x v(0, \omega, x) = 0$ *for all* $(\omega, x) \in \Omega \times \mathbb{R}$.

(ii) *The partial derivative of* $S_x v$ *with respect to* t *exists and is continuous on* $\mathcal{V}$, *and* $(S_x v)_t(0, \omega, x) = \mathfrak{h}_{xxx}(\omega, x)$ *for every* $(\omega, x) \in \Omega \times \mathbb{R}$.

Proof. Property (i) follows directly from the previous equalities, and the existence and continuity of $(S_x v)_t$ on $\mathcal{V}$ follows from the regularity of all the involved functions. The last assertion is proved by straight computation from (2.25)

$$(S_x v)_t = \frac{v_{xxxt}\, v_x - v_{xxx}\, v_{xt}}{v_x^2} - 3\, \frac{v_{xx}\,(v_{xxt} v_x - v_{xx} v_{xt})}{v_x^3}\,,$$

and evaluation at $t = 0$. □

The following proposition assumes that $x' = \mathfrak{h}(\omega{\cdot}t, x)$ satisfies a regularity assumption stronger than **d1** and stronger d-concavity assumptions than **d3** and **d4**: $\mathfrak{h} \in C^{0,3}(\Omega \times \mathbb{R}, \mathbb{R})$ and $\mathfrak{h}_{xxx}(\omega, x) < 0$ for every $(\omega, x) \in \Omega \times \mathbb{R}$ (which is slightly stronger that $x \mapsto \mathfrak{h}_x(\omega, x)$ being strictly concave for all $\omega \in \Omega$ and then ensures **d3** and **d4**). In this situation, it shows that the fixed time evolution map $\Omega \times \mathbb{R} \to \Omega \times \mathbb{R}$, $(\omega, x) \mapsto \tau_t(\omega, x) = (\omega{\cdot}t, v(t, \omega, x))$ given by $x' = \mathfrak{h}(\omega{\cdot}t, x)$ has strictly negative Schwarzian derivative for all $t > 0$.

Proposition 2.22. *Let* $\mathfrak{h} \in C^{0,3}(\Omega \times \mathbb{R}, \mathbb{R})$. *Let us suppose that* $\mathfrak{h}_{xxx}(\omega, x) < 0$ *for every* $(\omega, x) \in \Omega \times \mathbb{R}$. *Then,* $S_x v(t, \omega, x) < 0$ *for every* $(t, \omega, x) \in \mathcal{V}$ *with* $t > 0$.

Proof. Let us fix $(t_0, \omega_0, x_0) \in \mathcal{V}$ with $t_0 > 0$. We set $k = \sup_{t \in [0, t_0]} |v(t, \omega_0, x_0)|$ and note that $x_0 \in [-k, k]$. Proposition 2.21(ii) ensures that $(S_x v)_t(0, \omega, x) < 0$ for every $(\omega, x) \in \Omega \times [-k, k]$. Combining the continuity of $s \mapsto (S_x v)_t(s, \omega, x)$ with the openness of $\mathcal{V}$ and the compactness of $\Omega \times [-k, k]$, we find $0 < \delta \le t_0$ and $l < 0$ such that $(S_x v)_t(s, \omega, x) \le l$ for every $(s, \omega, x) \in [0, \delta] \times \Omega \times [-k, k]$. Consequently, it follows from Proposition 2.21(i) that

$$S_x v(s, \omega, x) = \int_0^s (S_x v)_t(r, \omega, x)\, dr \le ls < 0 \tag{2.26}$$

for every $(s, \omega, x) \in (0, \delta] \times \Omega \times [-k, k]$. Let $s_0 \in (0, \delta]$ and $n_0 \in \mathbb{N}$ be fixed with $s_0 n_0 = t_0$. [30, Chapter 2, Section 6] gives the formula of Schwarzian derivative of a composition, which yields

$$\begin{aligned} S_x v(ns_0, \omega_0, x_0) &= S_x\big(v(s_0, \omega_0{\cdot}(n-1)s_0, v((n-1)s_0, \omega_0, x_0))\big) \\ &= S_x v(s_0, \omega_0{\cdot}(n-1)s_0, v((n-1)s_0, \omega_0, x_0)) \cdot (v_x((n-1)s_0, \omega_0, x_0))^2 \\ &\quad + S_x v((n-1)s_0, \omega_0, x_0) \end{aligned}$$

for $n \in \{1, 2, \ldots, n_0\}$. Let us show by induction that $S_x v(ns_0, \omega_0, x_0) < 0$ for every $n \in \{1, 2, \ldots, n_0\}$. Equality (2.26) shows it for $n = 1$; and, since $v((n-1)s_0, \omega_0, x_0) \in [-k, k]$, (2.26) (resp. the induction hypothesis) ensures that the first (resp. second) term in the sum is strictly negative. In particular, $S_x v(t_0, \omega_0, x_0) = S_x v(n_0 s_0, \omega_0, x_0) < 0$, as asserted. □

The following proposition explores the converse relation: if the fixed time evolution map $\Omega\times\mathbb{R}\to\Omega\times\mathbb{R}$, $(\omega,x)\mapsto\tau_t(\omega,x)=(\omega{\cdot}t,v(t,\omega,x))$ given by $x'=\mathfrak{h}(\omega{\cdot}t,x)$ has nonnegative Schwarzian for $t>0$ sufficiently close to 0, then $x\mapsto\mathfrak{h}_x(\omega,x)$ is concave for all $\omega\in\Omega$ (which is stronger than **d3**).

Proposition 2.23. *Let $\mathfrak{h}\in C^{0,3}(\Omega\times\mathbb{R},\mathbb{R})$. Let us suppose that, given any $(\omega,x)\in\Omega\times\mathbb{R}$, there exists a sequence $(t_n)\downarrow 0$ with $t_n>0$ for all $n\in\mathbb{N}$ such that $(t_n,\omega,x)\in\mathcal{V}$ and $S_xv(t_n,\omega,x)\le 0$ for every $n\in\mathbb{N}$. Then, $x\mapsto\mathfrak{h}_x(\omega,x)$ is concave for all $\omega\in\Omega$.*

Proof. Let $(\omega,x)\in\Omega\times\mathbb{R}$ be fixed. Proposition 2.21 ensures that $S_xv(t,\omega,x)>0$ for $t>0$ small enough if $(S_xv)_t(0,\omega,x)=\mathfrak{h}_{xxx}(\omega,x)>0$. Since our hypotheses ensure that this is not the case, $\mathfrak{h}_{xxx}(\omega,x)\le 0$ and hence $x\mapsto\mathfrak{h}_x(\omega,x)$ is concave for all $\omega\in\Omega$. □

Comments on Chapter 2

1. The cubic equation with smooth 1-periodic coefficients $a_0,a_1,a_2,a_3\colon\mathbb{R}\to\mathbb{R}$

$$x'=a_3(t)\,x^3+a_2(t)\,x^2+a_1(t)\,x+a_0(t) \tag{2.27}$$

is called the *Abel equation*. Several papers (see [74], [44], [92]) have studied the existence of a bound on the number of periodic solutions of (2.27). In [92], it was proved that, if $a_3(t)>0$ (or $a_3(t)<0$) for all $t\in[0,1]$, then (2.27) can have at most three periodic solutions. And in [74], it was proved that there is no upper bound on the number of periodic solutions of (2.27) if the condition on the sign of a_3 is removed. Notice that the concavity of the derivative $x\mapsto h_x(t,x)$, that is, d-concavity, naturally generalizes the condition of positiveness of $-a_3$ in (2.27).

2. In [34], one can find definitions of various properties of strict d-concavity of a function given in terms of so-called "standardized moduli of d-concavity of a function on a given interval" as well as constructions of examples of functions that satisfy some of the definitions and not others [34, Examples 3.12 and 3.13]. The property **d4** given in this chapter is close to what is there called the $(\mathrm{SDC})_*$ property (see [34, Proposition 3.9]), which is the strongest of those described in that paper, although in [34] the concavity of $x\mapsto\mathfrak{h}_x(\omega,x)$ is asked for all $\omega\in\Omega$ instead of a full measure set as in **d3**. Consequently, it should be noted that some of the results in this chapter (e.g. Theorem 2.11) and later sections also hold for slightly weaker d-concavity conditions than those presented here. In this document, we have chosen to limit ourselves to this definition of strict d-concavity for the sake of clarity and expository coherence.

Chapter 3

Some bifurcation patterns for d-concave nonautonomous differential equations

This chapter addresses the bifurcation theory of d-concave nonautonomous scalar differential equations. Specifically, it examines global bifurcation diagrams for three distinct problems. In certain instances, these diagrams reproduce scenarios previously observed in the autonomous context, yet they may incorporate elements of dynamical complexity. Conversely, other bifurcation diagrams, which may also feature elements of dynamical complexity, are inherently nonautonomous. The bifurcation diagrams, that is, the main results of this chapter, are studied under the assumption that the base flow (Ω, σ) is minimal, that is, every σ-orbit is dense in Ω (recall Definition 1.6). In this way, we study bifurcations of minimal sets and bifurcations of the family of global attractors. On the other hand, some of the precedent results do not require this minimality hypothesis, as it will be indicated below.

Three different one-parameter bifurcation problems of nonautonomous differential equations with concave derivative

$$x' = \mathfrak{f}(\omega{\cdot}t, x, \lambda)\,, \quad \omega \in \Omega\,, \tag{3.1}$$

for $\lambda \in \mathbb{R}$ are considered. More precisely, the map $(\omega, x) \mapsto \mathfrak{f}(\omega, x, \lambda)$ satisfies **d1**, **d2**, **d3** and **d4** (recall Section 2.2) for each $\lambda \in \mathbb{R}$ and takes the form

- $\mathfrak{f}(\omega, x, \lambda) = \mathfrak{h}(\omega, x) + \lambda$, in Section 3.2,
- $\mathfrak{f}(\omega, x, \lambda) = \mathfrak{h}(\omega, x) + \lambda\, x$, with $\mathfrak{h}(\omega, 0) = 0$ for all $\omega \in \Omega$, in Section 3.3,
- $\mathfrak{f}(\omega, x, \lambda) = \mathfrak{h}(\omega, x) + \lambda\, x^2$, with $\mathfrak{h}(\omega, 0) = 0$ for all $\omega \in \Omega$, in Section 3.4.

So, for each fixed value of $\lambda \in \mathbb{R}$, the family (3.1) fits in the framework analyzed in Chapter 2. In continuity with the notation used before, the family of equations (3.1) for a fixed $\lambda \in \mathbb{R}$ is denoted by $(3.1)_\lambda$, and a particular equation of (3.1) with $\lambda \in \mathbb{R}$ and $\omega \in \Omega$ by $(3.1)_{\lambda,\omega}$. The maximal solution of $(3.1)_{\lambda,\omega}$ with initial value $v_\lambda(0, \omega, x) = x$ is $\mathcal{I}^\lambda_{\omega,x} \to \mathbb{R}$, $t \mapsto v_\lambda(t, \omega, x)$, and τ_λ is the corresponding (local) skewproduct flow induced by $(3.1)_\lambda$; i.e.,

$$\tau_\lambda \colon \mathcal{V}_\lambda \subseteq \mathbb{R} \times \Omega \times \mathbb{R} \to \Omega \times \mathbb{R}\,, \quad (t, \omega, x) \mapsto \big(\omega{\cdot}t,\, v_\lambda(t, \omega, x)\big)\,,$$

where $\mathcal{V}_\lambda = \bigcup_{(\omega,x)\in\Omega\times\mathbb{R}}(\mathcal{I}^\lambda_{\omega,x}\times\{(\omega,x)\})$. Hypotheses **d1** and **d2** for each fixed $\lambda\in\mathbb{R}$ ensure the existence of the global attractor $\mathcal{A}_\lambda$ of $(3.1)_\lambda$ for all $\lambda\in\mathbb{R}$: Theorem 2.13 describes the structure of $\mathcal{A}_\lambda$, written as

$$\mathcal{A}_\lambda = \bigcup_{\omega\in\Omega}\Big(\{\omega\}\times[\mathfrak{l}_\lambda(\omega),\mathfrak{u}_\lambda(\omega)]\Big)\,,$$

where $\mathfrak{l}_\lambda\colon\Omega\to\mathbb{R}$ and $\mathfrak{u}_\lambda\colon\Omega\to\mathbb{R}$ are, respectively, lower and upper semicontinuous τ_λ-equilibria. It also describes its variation with respect to λ.

To describe the bifurcation diagrams of (3.1), we define what we mean by two parametric families of bounded τ_λ-equilibria colliding as the parameter λ varies; and, in the case of minimal base flow (Ω,σ), what we mean by two parametric families of τ_λ-minimal sets colliding as λ varies.

Definition 3.1 (Collision of equilibria)**.** Let $(\lambda_-,\lambda_+)\subseteq\mathbb{R}$ be a finite interval, and let $\mathfrak{b}^1_\lambda,\mathfrak{b}^2_\lambda\colon\Omega\to\mathbb{R}$ be bounded semicontinuous τ_λ-equilibria defined for $\lambda\in(\lambda_-,\lambda_+)$ such that $\mathfrak{b}^1_\lambda(\omega)<\mathfrak{b}^2_\lambda(\omega)$ for all $\omega\in\Omega$ and $\lambda\in(\lambda_-,\lambda_+)$, and $\lim_{\lambda\uparrow\lambda_+}\mathfrak{b}^i_\lambda(\omega)$ exists for all $\omega\in\Omega$ and $i\in\{1,2\}$. We shall say that $\mathfrak{b}^1_\lambda$ and $\mathfrak{b}^2_\lambda$ collide on a set $\Omega_0\subseteq\Omega$ as $\lambda\uparrow\lambda_+$ if

$$\lim_{\lambda\uparrow\lambda_+}\mathfrak{b}^1_\lambda(\omega)=\lim_{\lambda\uparrow\lambda_+}\mathfrak{b}^2_\lambda(\omega)$$

for all $\omega\in\Omega_0$. Moreover, a compact τ-invariant set $\mathcal{K}\subset\Omega\times\mathbb{R}$ is said to be *enclosed* in such collision if

$$\mathcal{K}\subseteq\bigcup_{\omega\in\Omega}\big(\{\omega\}\times[\lim_{\lambda\uparrow\lambda_+}\mathfrak{b}^1_\lambda(\omega),\lim_{\lambda\uparrow\lambda_+}\mathfrak{b}^2_\lambda(\omega)]\big)\,.$$

The definition is symmetric when $\lambda\downarrow\lambda_-$.

We remark that, in the previous definition, $\mathfrak{b}^i_{\lambda_+}(\omega)=\lim_{\lambda\uparrow\lambda_+}\mathfrak{b}^i_\lambda(\omega)$ is a τ_{λ_+}-equilibrium for $i=1,2$: it suffices to take limits as $\lambda\uparrow\lambda_+$ on $\mathfrak{b}^i_\lambda(\omega{\cdot}t)=v_\lambda(t,\omega,\mathfrak{b}^i_\lambda(\omega))$. Recall that, if (Ω,σ) is minimal, we call hyperbolic minimal sets to the hyperbolic copies of the base: see Remark 1.41.

Definition 3.2 (Collision of hyperbolic minimal sets)**.** Assume that (Ω,σ) is minimal. Let $(\lambda_-,\lambda_+)\subseteq\mathbb{R}$ be a finite interval, and let $\mathcal{M}^1_\lambda<\mathcal{M}^2_\lambda$ be hyperbolic τ_λ-minimal sets for $\lambda\in(\lambda_-,\lambda_+)$ which are uniformly bounded for all $\lambda\in(\lambda_-,\lambda_+)$. We shall say that $\mathcal{M}^1_\lambda$ and $\mathcal{M}^2_\lambda$ collide on a set $\Omega_0\subseteq\Omega$ as $\lambda\uparrow\lambda_+$ if the continuous τ_λ-equilibria $\mathfrak{b}^1_\lambda,\mathfrak{b}^2_\lambda\colon\Omega\to\mathbb{R}$ which give rise to $\mathcal{M}^1_\lambda=\{\mathfrak{b}^1_\lambda\}$ and $\mathcal{M}^2_\lambda=\{\mathfrak{b}^2_\lambda\}$ collide on Ω_0 as $\lambda\uparrow\lambda_+$. A symmetric definition holds when $\lambda\downarrow\lambda_-$.

To get bifurcation diagrams as simple as possible, in the always highly complicated nonautonomous setting, most of the results of this chapter are formulated for a minimal base. They give information about the number of τ_λ-minimal sets and their hyperbolic or nonhyperbolic structure for each $\lambda\in\mathbb{R}$, and the relation of these properties with the shape of the global attractor $\mathcal{A}_\lambda$, whose existence is guaranteed by our hypotheses. Despite of this, some of the results of this section are formulated on a general base to give the most general theory possible.

Recall that two τ_λ-minimal subsets of $\Omega\times\mathbb{R}$ are always ordered if the base is minimal: see Corollary 1.33. We will use this fact often, without further reference.

The following is a brief description of some of the possible bifurcation diagrams analyzed in this chapter for the case of minimal (Ω,σ). The criterion employed to say that a parameter value $\lambda \in \mathbb{R}$ is a *bifurcation point* relies either on changes in the number of τ_λ-minimal sets or on their hyperbolic structure in a neighborhood of λ. A more detailed description of all these bifurcation diagrams can be found in the theorems scattered throughout the chapter.

In Section 3.2,

$$x' = \mathfrak{h}(\omega{\cdot}t, x) + \lambda\,, \quad \omega \in \Omega\,, \tag{3.2}$$

is considered. The main results are Theorems 3.8 and 3.14, which describe in detail two different bifurcation diagrams, depicted in Figures 3.1 and 3.2:

- *One minimal bifurcation diagram.* There is exactly one τ_λ-minimal set for all $\lambda \in \mathbb{R}$. This pattern corresponds to the nonautonomous analog of the bifurcation diagram of $x' = -x^3 + \lambda$.

- *Double saddle-node bifurcation diagram.* There exist two bifurcation points $\lambda_- < \lambda_+$ such that: there exist three hyperbolic τ_λ-minimal sets $\mathcal{M}^l_\lambda < \mathcal{N}_\lambda < \mathcal{M}^u_\lambda$ for all $\lambda \in (\lambda_-, \lambda_+)$, only one hyperbolic τ_λ-minimal set for all $\lambda \notin [\lambda_-, \lambda_+]$, and at both λ_- and λ_+ a *saddle-node bifurcation of minimal sets* takes place, that is, $\mathcal{M}^l_\lambda$ and $\mathcal{N}_\lambda$ (resp. $\mathcal{N}_\lambda$ and $\mathcal{M}^u_\lambda$) collide on a residual σ-invariant subset of Ω as $\lambda \uparrow \lambda_+$ (resp. $\lambda \downarrow \lambda_-$) and disappear for $\lambda > \lambda_+$ (resp. $\lambda < \lambda_-$). This bifurcation diagram, which roughly speaking has the shape of an "S", corresponds to the nonautonomous analog of that of $x' = -x^3 + x + \lambda$.

To some extent, the analysis in this section is the d-concave version of the saddle-node bifurcation pattern studied in [6], [86] and [88] for the concave case.

In addition, we show that these two ones are the unique possible bifurcation diagrams in the case that (Ω, σ) is uniquely ergodic (as in the autonomous case).

Section 3.3 deals with the second bifurcation problem of this chapter:

$$x' = \mathfrak{h}(\omega{\cdot}t, x) + \lambda\, x\,, \quad \omega \in \Omega\,, \tag{3.3}$$

with $\mathfrak{h}(\omega, 0) = 0$ for all $\omega \in \Omega$. Here, the τ_λ-copy of the base $\mathcal{M}_0 = \Omega \times \{0\}$ is a τ_λ-minimal set for all $\lambda \in \mathbb{R}$, and it is a nonhyperbolic τ_λ-minimal set for an interval $[\lambda_-, \lambda_+]$ of values of the parameter which can reduce to a point. In fact, it indeed reduces to a point if (Ω, σ) is uniquely ergodic, which is the situation in the autonomous case. The main results in this section are Theorems 3.20, 3.21 and 3.22, which describe in detail the unique three possibilities for the bifurcation diagram, briefly described in the following list and depicted in Figures 3.3, 3.4 and 3.5:

- *Classical pitchfork bifurcation.* $\lambda_- \le \lambda_+$ and λ_+ is the unique bifurcation point of change of number of τ_λ-minimal sets. There exist three hyperbolic τ_λ-minimal sets $\mathcal{M}^l_\lambda < \mathcal{M}_0 < \mathcal{M}^u_\lambda$ if $\lambda > \lambda_+$, $\mathcal{M}_0$ is the unique τ_λ-minimal set if $\lambda \le \lambda_+$, and both $\mathcal{M}^l_\lambda$ and $\mathcal{M}^u_\lambda$ collide with $\mathcal{M}_0$ on a residual σ-invariant subset of Ω as $\lambda \downarrow \lambda_+$. This bifurcation diagram corresponds to the nonautonomous analog of $x' = -x^3 + \lambda\, x$, although, in the non uniquely ergodic case, λ_- can be strictly smaller than λ_+.

- *Local saddle-node and transcritical bifurcations.* With $\lambda_- \le \lambda_+$, the bifurcation points are λ_0, λ_- and λ_+, with $\lambda_0 < \lambda_-$; that is, there are two or three of them. There are two possible bifurcation diagrams which are symmetric with respect to the horizontal axis $x = 0$. Let us describe one of them. For

$\lambda > \lambda_+$, there exist three τ_λ-minimal sets $\mathcal{M}^l_\lambda < \mathcal{M}_0 < \mathcal{M}^u_\lambda$; as $\lambda \downarrow \lambda+$, $\mathcal{M}_0$ and $\mathcal{M}^u_\lambda$ collide on a residual σ-invariant subset of Ω; for $\lambda \in [\lambda_-, \lambda_+]$, the unique τ_λ-minimal sets are $\mathcal{M}^l_\lambda < \mathcal{M}_0$, with $\mathcal{M}^l_\lambda$ hyperbolic attractive and $\mathcal{M}_0$ nonhyperbolic; for $\lambda \in (\lambda_0, \lambda_-)$, there are again three hyperbolic τ_λ-minimal sets $\mathcal{M}^l_\lambda < \mathcal{N}_\lambda < \mathcal{M}_0$; as $\lambda \uparrow \lambda_-$, $\mathcal{N}_\lambda$ and $\mathcal{M}_0$ collide on a residual σ-invariant subset of Ω; as $\lambda \downarrow \lambda_0$, $\mathcal{M}^l_\lambda$ and $\mathcal{N}_\lambda$ collide on a residual σ-invariant subset of Ω; for λ_0, the unique τ_{λ_0}-minimal sets are $\mathcal{M}^l_{\lambda_0} < \mathcal{M}_0$, with $\mathcal{M}^l_{\lambda_0}$ hyperbolic attractive and $\mathcal{M}_0$ nonhyperbolic; and $\mathcal{M}_0$ is the unique (attractive hyperbolic) τ_λ-minimal set for $\lambda < \lambda_0$. So, we have a local saddle-node bifurcation of minimal sets at λ_0, and a *generalized* transcritical bifurcation at $[\lambda_-, \lambda_+]$, which is *classical* if $\lambda_- = \lambda_+$. In the case of $\lambda_- = \lambda_+$, this bifurcation diagram is a nonautonomous analog of that of $x' = -x^3 + 2x^2 + \lambda\, x$.

- *Generalized pitchfork bifurcation.* This one is only possible if $\lambda_- < \lambda_+$. There are two bifurcation points λ_+ and $\lambda_0 \in [\lambda_-, \lambda_+)$ of change of the number of τ_λ-minimal sets. There are two possible bifurcation diagrams which are symmetric with respect to the horizontal axis $x = 0$. Let us describe one of them. There exist three hyperbolic τ_λ-minimal sets $\mathcal{M}^l_\lambda < \mathcal{M}_0 < \mathcal{M}^u_\lambda$ if $\lambda > \lambda_+$; as $\lambda \downarrow \lambda+$, $\mathcal{M}_0$ and $\mathcal{M}^u_\lambda$ collide on a residual σ-invariant subset of Ω; there are two τ_λ-minimal sets $\mathcal{M}^l_\lambda < \mathcal{M}_0$ if $\lambda \in (\lambda_0, \lambda_+]$, where $\mathcal{M}^l_\lambda$ is hyperbolic attractive; and $\mathcal{M}_0$ is the unique τ_λ-minimal set if $\lambda < \lambda_0$.

In Section 3.4, the third and last bifurcation problem is considered:

$$x' = \mathfrak{h}(\omega{\cdot}t, x) + \lambda\, x^2\,, \quad \omega \in \Omega\,, \tag{3.4}$$

with $\mathfrak{h}(\omega, 0) = 0$ for all $\omega \in \Omega$. As in (3.3), $\mathcal{M}_0 = \Omega \times \{0\}$ is a τ_λ-minimal set for all $\lambda \in \mathbb{R}$. However, in this case, $\mathcal{M}_0$ has the same hyperbolic attractive, hyperbolic repulsive or nonhyperbolic character for all the values of the parameter. Indeed, the type of bifurcation diagram displayed depends on this character. The main results in this section are Theorems 3.43, 3.44 and 3.45, which describe in detail the unique three possibilities for the bifurcation diagram, briefly described in the following list and depicted in Figures 3.6 and 3.7:

- *No bifurcation.* If $\mathcal{M}_0$ is a repulsive hyperbolic τ_λ-minimal set for all $\lambda \in \mathbb{R}$, then for all $\lambda \in \mathbb{R}$ there are three hyperbolic τ_λ-minimal sets $\mathcal{M}^l_\lambda < \mathcal{M}_0 < \mathcal{M}^u_\lambda$.

- *Two saddle-node bifurcations.* If $\mathcal{M}_0$ is an attractive hyperbolic τ_λ-minimal set for all $\lambda \in \mathbb{R}$, then there exist $\lambda_1 < \lambda_2$ such that: for all $\lambda > \lambda_2$ (resp. $\lambda < \lambda_1$), there exist three hyperbolic τ_λ-copies of the base $\mathcal{M}_0 < \mathcal{N}_\lambda < \mathcal{M}^u_\lambda$), (resp. $\mathcal{M}^l_\lambda < \mathcal{N}_\lambda < \mathcal{M}_0$ and there are two saddle-node bifurcations: $\mathcal{N}_\lambda$ and $\mathcal{M}^u_\lambda$ (resp. $\mathcal{M}^l_\lambda$ and $\mathcal{N}_\lambda$) collide as $\lambda \downarrow \lambda_2$ (resp. $\lambda \uparrow \lambda_1$) on a σ-invariant residual subset of Ω.

- *Weak generalized transcritical bifurcation.* If $\mathcal{M}_0$ is a nonhyperbolic τ_λ-minimal set for all $\lambda \in \mathbb{R}$, then there exist $\lambda_1 \le \lambda_2$ such that: for all $\lambda > \lambda_2$ (resp. $\lambda < \lambda_1$) there exist exactly two τ_λ-minimal sets $\mathcal{M}_0 < \mathcal{M}^u_\lambda$ (resp. $\mathcal{M}^l_\lambda < \mathcal{M}_0$), where $\mathcal{M}^u_\lambda$ (resp. $\mathcal{M}^l_\lambda$) is hyperbolic attractive; and if $\mathcal{M}_0$ is the unique τ_{λ_2}-minimal set (resp. τ_{λ_1}-minimal set), then $\mathcal{M}_0$ and $\mathcal{M}^u_\lambda$ (resp. $\mathcal{M}^l_\lambda$ and $\mathcal{M}_0$) collide on a residual σ-invariant subset of Ω as $\lambda \downarrow \lambda_2$ (resp. $\lambda \uparrow \lambda_1$).

So, the core of Chapter 3 is found in Sections 3.2, 3.3 and 3.4, especially in the aforementioned theorems describing bifurcation diagrams. In the short Section 3.1, of technical character, some relations between different coercivity conditions are established. In this first section, the base flow is not required to be minimal. The same occurs at the beginnings of the last three sections, where some general properties on the variation of the global attractor are obtained without assuming minimality. To avoid confusion, we will mention the minimality hypothesis in the statements of all the results which require it.

3.1 On the coercivity hypotheses

Let us recall the conditions **d1**-**d4** described in Chapter 2, fundamental for the forthcoming analysis.

d1 $\mathfrak{h} \in C^{0,2}(\Omega \times \mathbb{R}, \mathbb{R})$,

d2 $\limsup_{x\to\pm\infty} \big(\pm \mathfrak{h}(\omega, x) \big) < 0$ uniformly on Ω,

d3 $m\big(\{\omega \in \Omega \,|\, x \mapsto \mathfrak{h}_x(\omega, x) \text{ is concave}\}\big) = 1$ for all $m \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$,

d4 $m\big(\{\omega \in \Omega \,|\, x \mapsto \mathfrak{h}_x(\omega, x) \text{ is strictly concave on } \mathcal{J}\}\big) > 0$ for all compact interval $\mathcal{J} \subset \mathbb{R}$ and all $m \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$.

Notice that to ask hypotheses **d1**, **d3** and **d4** to be fulfilled by $(\omega, x) \mapsto \mathfrak{h}(\omega, x) + \lambda$, $(\omega, x) \mapsto \mathfrak{h}(\omega, x) + \lambda\, x$ or $(\omega, x) \mapsto \mathfrak{h}(\omega, x) + \lambda\, x^2$ for all $\lambda \in \mathbb{R}$ is equivalent to ask **d1**, **d3** and **d4** to be fulfilled by $\mathfrak{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$. But the same is not true for the coercivity hypothesis **d2**. To deal with this, we consider the following more restrictive coercivity properties:

d2$_\lambda$ $\displaystyle\lim_{x\to\pm\infty} \mathfrak{h}(\omega, x) = \mp\infty$ uniformly on Ω,

d2$_{\lambda x}$ $\displaystyle\lim_{x\to\pm\infty} \frac{\mathfrak{h}(\omega, x)}{x} = -\infty$ uniformly on Ω,

d2$_{\lambda x^2}$ $\displaystyle\lim_{x\to\pm\infty} \frac{\mathfrak{h}(\omega, x)}{x^2} = \mp\infty$ uniformly on Ω.

The following proposition states relations between the different coercivity properties and unravels their meaning:

Proposition 3.3. *Let* $\mathfrak{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$. *Then,*

(i) $\mathfrak{h}$ *satisfies* **d2**$_\lambda$ *if and only if* $(\omega, x) \mapsto \mathfrak{h}(\omega, x) + \lambda$ *satisfies* **d2** *for all* $\lambda \in \mathbb{R}$,

(ii) $\mathfrak{h}$ *satisfies* **d2**$_{\lambda x}$ *if and only if* $(\omega, x) \mapsto \mathfrak{h}(\omega, x) + \lambda\, x$ *satisfies* **d2** *for all* $\lambda \in \mathbb{R}$,

(iii) $\mathfrak{h}$ *satisfies* **d2**$_{\lambda x^2}$ *if and only if* $(\omega, x) \mapsto \mathfrak{h}(\omega, x) + \lambda\, x^2$ *satisfies* **d2** *for all* $\lambda \in \mathbb{R}$,

(iv) $\mathfrak{h}$ *satisfies* **d1** *and* $x \mapsto \mathfrak{h}_x(\omega, x)$ *is concave for all* $\omega \in \Omega$, *then* $\mathfrak{h}$ *satisfies* **d2**$_{\lambda x}$ *if and only if* $\lim_{x\to\pm\infty} \mathfrak{h}_x(\omega, x) = -\infty$ *uniformly on* Ω,

(v) *if* $\mathfrak{h}$ *satisfies* **d2**$_{\lambda x}$, *then it satisfies* **d2**$_\lambda$,

(vi) *if* $\mathfrak{h}$ *satisfies* $\mathbf{d2}_{\lambda x^2}$, *then it satisfies* $\mathbf{d2}_{\lambda x}$.

Proof. (i) If $\mathfrak{h}$ satisfies $\mathbf{d2}_\lambda$, then $\lim_{x\to\pm\infty}(\pm\mathfrak{h}(\omega,x)+\lambda)=-\infty$ uniformly on Ω for all $\lambda\in\mathbb{R}$, from where the result follows. If $(\omega,x)\mapsto\mathfrak{h}(\omega,x)+\lambda$ satisfies $\mathbf{d2}$ for all $\lambda\in\mathbb{R}$, $\limsup_{x\to\pm\infty}(\pm\mathfrak{h}(\omega,x))<-\lambda$ uniformly on Ω for all $\lambda\in\mathbb{R}$, so $\lim_{x\to\pm\infty}(\pm\mathfrak{h}(\omega,x))=-\infty$ uniformly on Ω, and hence $\mathbf{d2}_\lambda$ holds.

(ii) If $\mathfrak{h}$ satisfies $\mathbf{d2}_{\lambda x}$, then $\lim_{x\to\pm\infty}(\mathfrak{h}(\omega,x)+\lambda\,x)/x=-\infty$ uniformly on Ω for all $\lambda\in\mathbb{R}$, so $\lim_{x\to\pm\infty}(\mathfrak{h}(\omega,x)+\lambda\,x)=\mp\infty$ uniformly on Ω for all $\lambda\in\mathbb{R}$. If $(\omega,x)\mapsto\mathfrak{h}(\omega,x)+\lambda\,x$ satisfies $\mathbf{d2}$ for all $\lambda\in\mathbb{R}$, then for each $\lambda\in\mathbb{R}$ there exists $\rho_\lambda>0$ such that $\mathfrak{h}(\omega,x)/x<-\lambda$ for all $\omega\in\Omega$ and $x\geq\rho_\lambda$, so $\lim_{x\to\infty}\mathfrak{h}(\omega,x)/x=-\infty$ uniformly on Ω. An analogous argument as $x\to-\infty$ shows that $\mathbf{d2}_{\lambda x}$ holds.

(iii) This proof is analogous to that of (ii).

(iv) Assume first that $\lim_{x\to\pm\infty}\mathfrak{h}_x(\omega,x)=-\infty$ uniformly on Ω. Let us reason with $x\to\infty$. Given $m>0$, there exists $\rho>0$ such that $\mathfrak{h}_x(\omega,x)<-2m$ for all $\omega\in\Omega$ and $x\geq\rho$. Then, the Mean Value Theorem ensures that, for $x>\rho$, there exists $\xi_{\omega,x}\in(\rho,x)$ such that

$$\frac{\mathfrak{h}(\omega,x)}{x}=\frac{\mathfrak{h}(\omega,\rho)+\mathfrak{h}_x(\omega,\xi_{\omega,x})(x-\rho)}{x}<\frac{\mathfrak{h}(\omega,\rho)}{x}-2m\left(1-\frac{\rho}{x}\right).$$

Since $\omega\mapsto\mathfrak{h}(\omega,\rho)$ is bounded, there exists $\rho'>\rho$ such that $\mathfrak{h}(\omega,x)/x<-m$ for all $\omega\in\Omega$ and $x\geq\rho'$. The argument for $x\to-\infty$ is analogous.

Now, assume that $\mathfrak{h}$ satisfies $\mathbf{d2}_{\lambda x}$. Let us take $a<x_l$ and $b>x_u$, where x_l and x_u are defined by Lemma 2.8. Since $x\mapsto\mathfrak{h}_x(\omega,x)$ is concave for all $\omega\in\Omega$, $x\mapsto\mathfrak{h}_{xx}(\omega,x)$ is nonincreasing for all $\omega\in\Omega$. Consequently, $\inf\{\mathfrak{h}_{xx}(\omega,x)\mid(\omega,x)\in\Omega\times(-\infty,a]\}=\inf\{\mathfrak{h}_{xx}(\omega,a)\mid\omega\in\Omega\}>0$, due to the compactness of Ω. That is, $\mathfrak{h}_{xx}$ has a strictly positive lower bound on $\Omega\times(-\infty,a]$ and, analogously, it has a strictly negative upper bound on $\Omega\times[b,\infty)$. The Fundamental Theorem of Calculus and the previous bound show that, for any $x\leq a$,

$$\begin{aligned}\mathfrak{h}_x(\omega,x)&=\mathfrak{h}_x(\omega,a)+\int_x^a\mathfrak{h}_{xx}(\omega,y)\,dy\\&\leq\sup\{\mathfrak{h}_x(\omega,a)\mid\omega\in\Omega\}+(x-a)\inf\{\mathfrak{h}_{xx}(\omega,a)\mid\omega\in\Omega\},\end{aligned}$$

so $\lim_{x\to-\infty}\mathfrak{h}_x(\omega,x)=-\infty$ uniformly on Ω. The other limit is proved analogously.

(v)-(vi) These properties follow immediately from the definitions. □

Notice that, for the "if part" in the statement of (iv) of the previous proposition, the fact of $x\mapsto\mathfrak{h}_x(\omega,x)$ being concave for all $\omega\in\Omega$ is not needed, and that $\mathbf{d1}$ can be replaced by $\mathfrak{h}\in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$.

3.2 Bifurcations of $x'=\mathfrak{h}(\omega{\cdot}t,x)+\lambda$

As said in the introduction of this chapter, this section deals with the one-parametric bifurcation problem

$$x'=\mathfrak{h}(\omega{\cdot}t,x)+\lambda\,,\quad\omega\in\Omega\,. \tag{3.5}$$

Recall that Proposition 3.3(i) ensures that $\mathfrak{h}+\lambda$ satisfies $\mathbf{d1}$, $\mathbf{d2}$, $\mathbf{d3}$ and $\mathbf{d4}$ for all $\lambda\in\mathbb{R}$ if (and only if) $\mathfrak{h}$ satisfies $\mathbf{d1}$, $\mathbf{d2}_\lambda$, $\mathbf{d3}$ and $\mathbf{d4}$. In part of the results of this section, with a minimal base (Ω, σ), $\mathfrak{h}$ will be asked to satisfy $\mathbf{d2}_{\lambda x}$ instead of the weaker condition $\mathbf{d2}_\lambda$ (see Proposition 3.3(v)), since the consequence $\lim_{x\to\pm\infty} \mathfrak{h}_x(\omega, x) = -\infty$ uniformly on Ω (see Remark 2.7 and Proposition 3.3(iv)) will be required.

The following result establishes relations between global upper and lower solutions, equilibria and semiequilibria for different values of the parameter in (3.5).

Proposition 3.4. *Let $\mathfrak{h} \in C^{0,1}(\Omega \times \mathbb{R}, \mathbb{R})$. Every global upper (resp. lower) solution of $(3.5)_\lambda$ is a strict global upper (resp. lower) solution of $(3.5)_\xi$ if $\xi < \lambda$ (resp. $\lambda < \xi$). Particularly, any equilibrium for $(3.5)_\lambda$ is a strong superequilibrium for $(3.5)_\xi$ if $\xi < \lambda$, as well as a strong subequilibrium for $(3.5)_\xi$ if $\lambda < \xi$.*

Proof. Let $\mathfrak{b}\colon \Omega \to \mathbb{R}$ be a global upper solution of $(3.5)_\lambda$, and let $\xi < \lambda$. So, $\mathfrak{b}'(\omega) \geq \mathfrak{h}(\omega, \mathfrak{b}(\omega)) + \lambda > \mathfrak{h}(\omega, \mathfrak{b}(\omega)) + \xi$ for all $\omega \in \Omega$, that is, $\mathfrak{b}$ is a strict global upper solution of $(3.5)_\xi$. Then, Proposition 1.24(i) proves the last assertion in the superequilibrium case. The case of $\lambda < \xi$ is analogous. □

Theorem 2.13 proves the existence of the global attractor $\mathcal{A}_\lambda$ of $(3.5)_\lambda$ if $\mathfrak{h} \in C^{0,1}(\Omega \times \mathbb{R}, \mathbb{R})$ satisfies $\mathbf{d2}_\lambda$, and explains part of its properties. The following result provides information on how the global attractor $\mathcal{A}_\lambda$ changes as the parameter λ varies.

Proposition 3.5. *Let $\mathfrak{h} \in C^{0,1}(\Omega \times \mathbb{R}, \mathbb{R})$ satisfy $\mathbf{d2}_\lambda$ and let*

$$\mathcal{A}_\lambda = \bigcup_{\omega\in\Omega} \big(\{\omega\} \times [\mathfrak{l}_\lambda(\omega), \mathfrak{u}_\lambda(\omega)]\big) \tag{3.6}$$

be the global attractor for the skewproduct flow τ_λ induced by $(3.5)_\lambda$. Then,

(i) *for every $\omega \in \Omega$, the maps $\mathbb{R} \to \mathbb{R}$, $\lambda \mapsto \mathfrak{l}_\lambda(\omega)$ and $\mathbb{R} \to \mathbb{R}$, $\lambda \mapsto \mathfrak{u}_\lambda(\omega)$ are strictly increasing on $\mathbb{R}$ and they are, respectively, left- and right-continuous.*

(ii) $\lim_{\lambda\to\pm\infty} \mathfrak{l}_\lambda(\omega) = \lim_{\lambda\to\pm\infty} \mathfrak{u}_\lambda(\omega) = \pm\infty$ *uniformly on Ω.*

(iii) *If $\lim_{x\to\pm\infty} \mathfrak{h}_x(\omega, x) = -\infty$ uniformly on a Borel subset $\Omega_0 \subseteq \Omega$ such that $m(\Omega_0) = 1$ for all $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$, then there exists $\lambda_* > 0$ such that $\mathcal{A}_\lambda$ is an attractive hyperbolic copy of the base if $|\lambda| \geq \lambda_*$. In particular, this is the case if $\mathfrak{h}$ satisfies $\mathbf{d1}$, $\mathbf{d2}_{\lambda x}$ and $\mathbf{d3}$.*

Proof. (i) Let $\xi < \lambda$. Proposition 3.4 ensures that $\mathfrak{u}_\xi$ is a strict global lower solution of $(3.5)_\lambda$. Hence, Theorem 2.13(v) yields $\mathfrak{u}_\xi(\omega) < \mathfrak{u}_\lambda(\omega)$ for all $\omega \in \Omega$. Analogously, $\mathfrak{l}_\lambda$ is a strict global upper solution for $(3.5)_\xi$ and Theorem 2.13(v) ensures that $\mathfrak{l}_\xi(\omega) < \mathfrak{l}_\lambda(\omega)$ for all $\omega \in \Omega$.

Let $\omega_0 \in \Omega$ be fixed. We take a strictly decreasing sequence (λ_n) with $\lambda_n \downarrow \lambda_0$, take into account that the previous part ensures that $(\mathfrak{u}_{\lambda_n}(\omega_0))$ is strictly decreasing and bounded by below by $\mathfrak{u}_{\lambda_0}(\omega_0)$, and call $y_0 = \lim_{n\to\infty} \mathfrak{u}_{\lambda_n}(\omega_0) \geq \mathfrak{u}_{\lambda_0}(\omega_0)$. Then, for any $t \in \mathcal{I}^{\lambda_0}_{\omega_0,y_0}$, we have $v_{\lambda_0}(t, \omega_0, y_0) = \lim_{n\to\infty} v_{\lambda_n}(t, \omega_0, \mathfrak{u}_{\lambda_n}(\omega_0)) = \lim_{n\to\infty} \mathfrak{u}_{\lambda_n}(\omega_0{\cdot}t) \leq \mathfrak{u}_{\lambda_1}(\omega_0{\cdot}t)$, so that $v_{\lambda_0}(t, \omega_0, y_0)$ remains bounded from above as time decreases. Theorem 2.13(iv) ensures that $y_0 \leq \mathfrak{u}_{\lambda_0}(\omega_0)$, and hence that $\mathfrak{u}_{\lambda_0}(\omega_0) = \lim_{n\to\infty} \mathfrak{u}_{\lambda_n}(\omega_0)$. The proof is analogous for $\mathfrak{l}_\lambda$.

(ii) We use $\mathbf{d2}_\lambda$ to find $\rho > 0$ such that $\mathfrak{h}(\omega, x) > 1$ for all $\omega \in \Omega$ if $x \leq -\rho$. Let us take $n > 0$ and $\lambda_n > 1 - \inf\{\mathfrak{h}(\omega, x) \mid (\omega, x) \in \Omega \times [-\rho, n]\}$ with $\lambda_n > 0$.

Then, $\mathfrak{h}(\omega,x)+\lambda>1$ for all $\omega\in\Omega$, $x\le n$ and $\lambda\ge\lambda_n$. Theorem 2.13(ii) applied to $m_1=n$ shows that $n\le\mathfrak{l}_\lambda(\omega)\le\mathfrak{u}_\lambda(\omega)$ for all $\omega\in\Omega$ and $\lambda\ge\lambda_n$. Then, $\lim_{\lambda\to\infty}\mathfrak{l}_\lambda(\omega)=\lim_{\lambda\to\infty}\mathfrak{u}_\lambda(\omega)=\infty$ uniformly on Ω. Analogous arguments prove the limits to $-\infty$ as $\lambda\to-\infty$.

(iii) The additional hypothesis on $\mathfrak{h}$ provides $\rho>0$ such that $\mathfrak{h}_x(\omega,x)<0$ if $|x|>\rho$ and $\omega\in\Omega_0$. By (ii), there exists λ_* such that $\mathcal{A}_\lambda\subset\Omega\times[\rho,\infty)$ and $\mathcal{A}_{-\lambda}\subset\Omega\times(-\infty,-\rho]$ if $\lambda>\lambda_*$. We fix λ with $|\lambda|>\lambda_*$, Theorem 1.36(iii) ensures that there exist $m\in\mathfrak{M}_{\text{erg}}(\Omega,\sigma)$ and an m-measurable τ_λ-equilibrium $\mathfrak{b}\colon\Omega\to\mathbb{R}$ with graph contained in $\mathcal{A}_\lambda\subset\Omega\times[\rho,\infty)$ such that $\sup\text{Lyap}(\mathcal{A}_\lambda)=\int_\Omega\mathfrak{h}_x(\omega,\mathfrak{b}(\omega))\,dm$. Since $m(\Omega_0)=1$, we get that $\sup\text{Lyap}(\mathcal{A}_\lambda)=\int_{\Omega_0}\mathfrak{h}_x(\omega,\mathfrak{b}(\omega))\,dm<0$. Then, Theorem 2.13(vii) ensures that $\mathcal{A}_\lambda$ is an attractive hyperbolic copy of the base for $|\lambda|>\lambda_*$.

To check the last assertion, if $\mathfrak{h}$ satisfies **d1**, **d2**$_{\lambda x}$ and **d3**, then Remark 2.7 ensures the existence of a σ-invariant closed set $\Omega_d\subseteq\Omega$ such that $m(\Omega_d)=1$ for all $m\in\mathfrak{M}_{\text{erg}}(\Omega,\sigma)$ and such that $x\mapsto\mathfrak{h}_x(\omega,x)$ is concave for all $\omega\in\Omega_d$. Proposition 3.3(iv) applied to the restriction $\mathfrak{h}\colon\Omega_0\times\mathbb{R}\to\mathbb{R}$ ensures that $\lim_{x\to\pm\infty}\mathfrak{h}_x(\omega,x)=-\infty$ uniformly on Ω_0 and completes the proof. □

Figures 3.1 and 3.2 depict the evolution with respect to λ of the global attractor $\mathcal{A}_\lambda$ in some particular cases.

The family $\{\mathcal{A}_\lambda\}$ of global attractors is said to be *upper semicontinuous as* $\lambda\to\lambda_0$ if $\lim_{\lambda\to\lambda_0}\text{dist}(\mathcal{A}_\lambda,\mathcal{A}_{\lambda_0})=0$, where dist is the Hausdorff semidistance, defined by (1.8). The family $\{\mathcal{A}_\lambda\}$ of global attractors is said to be *lower semicontinuous as* $\lambda\to\lambda_0$ if $\lim_{\lambda\to\lambda_0}\text{dist}(\mathcal{A}_{\lambda_0},\mathcal{A}_\lambda)=0$. The upper semicontinuity of a family of global attractors given by a λ-parametric family of equations which depends continuously on the parameter is a usual property, but this is not the case with the lower semicontinuity. The *continuity of* $\{\mathcal{A}_\lambda\}$ as $\lambda\to\lambda_0$ consists of the simultaneous upper and lower semicontinuity as $\lambda\to\lambda_0$. Our next result analyzes this continuity in our case:

Proposition 3.6. *Let* $\mathfrak{h}\in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$ *satisfy* **d2**$_\lambda$, *and let* $\mathcal{A}_\lambda=\bigcup_{\omega\in\Omega}(\{\omega\}\times[\mathfrak{l}_\lambda(\omega),\mathfrak{u}_\lambda(\omega)])$ *be the global attractor of* (3.5)$_\lambda$. *Then,*

(i) $\{\mathcal{A}_\lambda\}$ *is upper semicontinuous as* $\lambda\to\lambda_0$ *for all* $\lambda_0\in\mathbb{R}$,

(ii) *if the maps* $\lambda\mapsto\mathfrak{l}_\lambda(\omega)$ *and* $\lambda\mapsto\mathfrak{u}_\lambda(\omega)$ *are continuous at* λ_0 *for every* $\omega\in\Omega$, *then* $\{\mathcal{A}_\lambda\}$ *is continuous as* $\lambda\to\lambda_0$.

Proof. (i) For any λ_0, any sequence (λ_n) with limit λ_0 and $(\omega_n,x_n)\in\mathcal{A}_{\lambda_n}$, we get that $\mathfrak{l}_{\lambda_n}(\omega_n)\le x_n\le\mathfrak{u}_{\lambda_n}(\omega_n)$ for all $n\in\mathbb{N}$. The monotonicity and semicontinuity provided by Proposition 3.5(i) ensures that $\mathfrak{l}_{\lambda_n}(\omega_n)\ge\inf_{\omega\in\Omega}\mathfrak{l}_{\inf(\lambda_n)}(\omega)=\rho_1\in\mathbb{R}$ and that $\mathfrak{u}_{\lambda_n}(\omega_n)\le\sup_{\omega\in\Omega}\mathfrak{u}_{\sup(\lambda_n)}(\omega)=\rho_2\in\mathbb{R}$. So, (x_n) is bounded, and hence it admits a subsequence (x_m) which converges to some $x\in[\rho_1,\rho_2]$. In addition, for any $t\in\mathbb{R}$, $\rho_1\le\mathfrak{l}_{\lambda_n}(\omega_n{\cdot}t)\le v_{\lambda_n}(t,\omega_n,x_n)\le\mathfrak{u}_{\lambda_n}(\omega_n{\cdot}t)\le\rho_2$. So, $\lim_{n\to\infty}v_{\lambda_n}(t,\omega_n,x_n)=v_{\lambda_0}(t,\omega,x)$ is bounded, that is, $x\in\mathcal{A}_{\lambda_0}$. The characterization of upper semicontinuity as $\lambda\to\lambda_0$ given in [21, Lemma 3.2(1)] shows that $\{\mathcal{A}_\lambda\}$ is upper semicontinuous.

(ii) Assume that both maps $\lambda\mapsto\mathfrak{l}_\lambda(\omega)$ and $\lambda\mapsto\mathfrak{u}_\lambda(\omega)$ are continuous at λ_0 for all $\omega\in\Omega$. For any $(\omega,x)\in\mathcal{A}_{\lambda_0}$, let $s\in[0,1]$ be such that $x=s\,\mathfrak{l}_{\lambda_0}(\omega)+(1-s)\,\mathfrak{u}_{\lambda_0}(\omega)$. It is clear that $\lim_{n\to\infty}(\omega,x_n)=(\omega,x)$, where $x_n=s\,\mathfrak{l}_{\lambda_n}(\omega)+(1-s)\,\mathfrak{u}_{\lambda_n}(\omega)$, so the characterization of lower semicontinuity as $\lambda \to \lambda_0$ given in [21, Lemma 3.2(2)] shows that $\{\mathcal{A}_\lambda\}$ is lower semicontinuous at λ_0, and hence, combining this with (i), we conclude that $\{\mathcal{A}_\lambda\}$ is continuous at λ_0. □

Recall that $\int_{\mathcal{K}} \mathfrak{h}_x(\omega, \mathfrak{b}(\omega))\, dm$ is one of the Lyapunov exponents of a compact τ-invariant set $\mathcal{K} \subset \Omega \times \mathbb{R}$ if $\mathfrak{b}\colon \Omega \to \mathbb{R}$ is an m-measurable τ-equilibrium with graph in $\mathcal{K}$ and $m \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$: see Theorem 1.36(ii). The last result of this section establishes a relation between two Lyapunov exponents of two suitable compact sets which are τ_λ-invariant for two different values of the parameter λ, which will be extremely useful in the proofs of the main results. Note that, as in Theorem 2.15, no coercivity condition is required.

Proposition 3.7. *Let $\mathfrak{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy* **d1**, *let us fix $m \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$ and $\lambda_1 < \lambda_2$, and let $\mathfrak{b}_i\colon \Omega \to \mathbb{R}$ be a bounded m-measurable τ_{λ_i}-equilibrium for $i = 1, 2$, with $\mathfrak{b}_1(\omega) < \mathfrak{b}_2(\omega)$ for m-a.e. $\omega \in \Omega$. Assume that $m(\{\omega \in \Omega \mid x \mapsto \mathfrak{h}_x(\omega, x)$ is concave$\}) = 1$. Then,*

$$\int_\Omega \mathfrak{h}_x(\omega, \mathfrak{b}_1(\omega))\, dm + \int_\Omega \mathfrak{h}_x(\omega, \mathfrak{b}_2(\omega))\, dm < 0\,.$$

Proof. The argument is similar to that of Theorem 2.15. We define $\Omega_0 = \Omega_d \cap \{\omega \in \Omega \mid \mathfrak{b}_1(\omega) < \mathfrak{b}_2(\omega)\}$, where Ω_d is the σ-invariant set with $m(\Omega_d) = 1$ given by Lemma 2.6. Hence, $m(\Omega_0) = 1$. A standard comparison argument shows that $\mathfrak{b}_1(\omega{\cdot}t) = v_{\lambda_1}(t, \omega, \mathfrak{b}_1(\omega)) \le v_{\lambda_2}(t, \omega, \mathfrak{b}_1(\omega)) < v_{\lambda_2}(t, \omega, \mathfrak{b}_2(\omega)) = \mathfrak{b}_2(\omega{\cdot}t)$ for all $t \ge 0$ if $\mathfrak{b}_1(\omega) < \mathfrak{b}_2(\omega)$. Hence, $\omega{\cdot}t \in \Omega_0$ for all $t \ge 0$ if $\omega \in \Omega_0$. The function $\mathfrak{c}(\omega) = \mathfrak{b}_2(\omega) - \mathfrak{b}_1(\omega)$ satisfies

$$\frac{\mathfrak{c}'(\omega{\cdot}t)}{\mathfrak{c}(\omega{\cdot}t)} = F(\omega{\cdot}t, \mathfrak{c}(\omega{\cdot}t)) + \frac{\lambda_2 - \lambda_1}{\mathfrak{c}(\omega{\cdot}t)} \tag{3.7}$$

for all $\omega \in \Omega_0$ and $t \ge 0$, where $F(\omega, y) = \int_0^1 \mathfrak{h}_x(\omega, s\,y + \mathfrak{b}_1(\omega))\, ds$. Since $\omega \mapsto F(\omega, \mathfrak{c}(\omega))$ is bounded and hence it is in $L^1(\Omega, m)$, and since $\omega \mapsto (\lambda_2 - \lambda_1)/\mathfrak{c}(\omega)$ is strictly positive on Ω_0, the application of Birkhoff's Ergodic Theorem 1.10 to (3.7) (see the application in the proof of Theorem 2.15) yields

$$\int_\Omega F(\omega, \mathfrak{c}(\omega))\, dm = -(\lambda_2 - \lambda_1) \int_\Omega \frac{1}{\mathfrak{c}(\omega)}\, dm < 0\,,$$

which combined with (2.19), which also holds in this case, completes the proof. □

3.2.1 Bifurcation diagrams with minimal base flow

The results of this section describe two possible bifurcation diagrams of τ_λ-minimal sets for (3.5) under the assumption of minimal base flow (Ω, σ): the double saddle-node bifurcation of Theorem 3.8 (see Figure 3.1) and the one minimal bifurcation diagram of Theorem 3.14 (see Figure 3.2). The bifurcations of the family of global attractors, that is, the points of discontinuity of the global attractors, will be deduced. Moreover, it will be checked that, if (Ω, σ) is uniquely ergodic, then these two bifurcation diagrams are the only possible ones. Finally, following the ideas of Remark 2.12, Theorem 3.16 provides local saddle-node bifurcations.

The next result establishes conditions under which the global bifurcation diagram of (3.2) presents two local saddle-node bifurcation points of τ_λ-minimal sets which are also points of discontinuity of the global attractor $\mathcal{A}_\lambda$. For each $\lambda \in \mathbb{R}$, the maps $\mathfrak{l}_\lambda$ and $\mathfrak{u}_\lambda$ of its statement are those provided by $(3.6)_\lambda$.

Theorem 3.8 (Double saddle-node bifurcation). *Let (Ω, σ) be minimal. Let $\mathfrak{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy* **d1**, **d2**$_{\lambda x}$, **d3** *and* **d4**. *Assume that there exists $\lambda_0 \in \mathbb{R}$ such that there exist three different τ_{λ_0}-minimal sets for (3.5). Then, there exists a finite interval $\mathcal{I} = (\lambda_-, \lambda_+)$ with $\lambda_0 \in \mathcal{I}$ such that*

(i) *for every $\lambda \in \mathcal{I}$, there exist exactly three τ_λ-minimal sets $\mathcal{M}^l_\lambda < \mathcal{N}_\lambda < \mathcal{M}^u_\lambda$ which are hyperbolic τ_λ-copies of the base, given by the graphs of $\mathfrak{l}_\lambda < \mathfrak{m}_\lambda < \mathfrak{u}_\lambda$. In addition, $\mathcal{N}_\lambda$ is repulsive and $\mathcal{M}^l_\lambda$, $\mathcal{M}^u_\lambda$ are attractive, and $\lambda \mapsto \mathfrak{m}_\lambda$ is strictly decreasing on $\mathcal{I}$.*

(ii) *The τ_λ-equilibria $\mathfrak{m}_\lambda$ and $\mathfrak{u}_\lambda$ (resp. $\mathfrak{l}_\lambda$ and $\mathfrak{m}_\lambda$) collide on a σ-invariant residual subset of Ω as $\lambda \downarrow \lambda_-$ (resp. $\lambda \uparrow \lambda_+$), enclosing a nonhyperbolic τ_{λ_-}-minimal set $\mathcal{M}^u_{\lambda_-}$ (resp. τ_{λ_+}-minimal set $\mathcal{M}^l_{\lambda_+}$). In addition, there is exactly other minimal set for τ_{λ_-} (resp. τ_{λ_+}) and it is an attractive hyperbolic copy of the base given by the graph $\mathcal{M}^l_{\lambda_-}$ of $\mathfrak{l}_{\lambda_-}$ (resp. $\mathcal{M}^u_{\lambda_+}$ of $\mathfrak{u}_{\lambda_+}$).*

(iii) *For $\lambda \in (-\infty, \lambda_-) \cup (\lambda_+, \infty)$, $\mathcal{A}_\lambda$ is an attractive hyperbolic τ_λ-copy of the base, given by the graph of the map $\mathfrak{l}_\lambda = \mathfrak{u}_\lambda$.*

In particular, local saddle-node bifurcations of minimal sets and discontinuous bifurcations of attractors occur at λ_- and λ_+.

Proof. Since (Ω, σ) is minimal, Corollary 1.33 allows us to apply Theorem 2.11 to deduce that the three τ_{λ_0}-minimal sets are three hyperbolic τ_{λ_0}-copies of the base, and that the upper and lower ones bound the set of bounded τ_{λ_0}-orbits. Hence, Theorem 2.13(ii) ensures that these τ_{λ_0}-copies of the base are the graphs of $\mathfrak{l}_{\lambda_0} < \mathfrak{m}_{\lambda_0} < \mathfrak{u}_{\lambda_0}$, for a continuous τ_{λ_0}-equilibrium $\mathfrak{m}_{\lambda_0}\colon \Omega \to \mathbb{R}$. Theorem 1.39 ensures that there exists a maximal open interval $\mathcal{I} \ni \lambda_0$ such that, for any $\lambda \in \mathcal{I}$ there are three hyperbolic τ_λ-copies of the base $\mathcal{M}^l_\lambda < \mathcal{N}_\lambda < \mathcal{M}^u_\lambda$, and the just used arguments show that they are the unique τ_λ-minimal sets, given by the graphs of $\mathfrak{l}_\lambda < \mathfrak{m}_\lambda < \mathfrak{u}_\lambda$ for a continuous τ_λ-equilibrium $\mathfrak{m}_\lambda\colon \Omega \to \mathbb{R}$, with $\{\mathfrak{l}_\lambda\}$ and $\{\mathfrak{u}_\lambda\}$ attractive and $\{\mathfrak{m}_\lambda\}$ repulsive. Theorem 1.39 also ensures the continuity of the maps $\mathcal{I} \to C(\Omega, \mathbb{R})$, $\lambda \mapsto \mathfrak{l}_\lambda, \mathfrak{m}_\lambda, \mathfrak{u}_\lambda$ with respect to the uniform topology, and Proposition 3.5(i) shows that $\lambda \mapsto \mathfrak{l}_\lambda$ and $\lambda \mapsto \mathfrak{u}_\lambda$ are strictly increasing on $\mathcal{I}$.

Now, let us check that $\lambda \mapsto \mathfrak{m}_\lambda$ is strictly decreasing on $\mathcal{I}$. The continuous variation on $\mathcal{I}$ allows us to take $\xi > \lambda$ in $\mathcal{I}$ close enough to ensure $\mathfrak{m}_\lambda > \mathfrak{l}_\xi$, and fix $\omega \in \Omega$. Then, the τ_ξ-orbit of $(\omega, \mathfrak{m}_\lambda(\omega))$ is above $\{\mathfrak{l}_\xi\}$. In addition, $v_\xi(t, \omega, \mathfrak{m}_\lambda(\omega)) < v_\lambda(t, \omega, \mathfrak{m}_\lambda(\omega)) = \mathfrak{m}_\lambda(\omega{\cdot}t)$ for all $t < 0$. So, the α-limit set for τ_ξ of $(\omega, \mathfrak{m}_\lambda(\omega))$ exists, and it is below the graph of $\mathfrak{m}_\lambda$. Let $\mathcal{N}$ be a τ_ξ-minimal contained in this α-limit set. Corollary 1.58(i) ensures that $\mathcal{N}$ is neither $\{\mathfrak{l}_\xi\}$ nor $\{\mathfrak{u}_\xi\}$, and hence $\mathcal{N} = \{\mathfrak{m}_\xi\}$. This ensures that $\mathfrak{m}_\xi \le \mathfrak{m}_\lambda$. And, if $\mathfrak{m}_\xi(\bar\omega) = \mathfrak{m}_\lambda(\bar\omega)$, then $\mathfrak{m}_\xi(\bar\omega{\cdot}1) = v_\xi(1, \bar\omega, \mathfrak{m}_\lambda(\omega)) > v_\lambda(1, \bar\omega, \mathfrak{m}_\lambda(\bar\omega)) = \mathfrak{m}_\lambda(\bar\omega{\cdot}1)$, impossible. The assertion is proved.

Proposition 3.5(iii) provides λ_* such that $\mathcal{A}_\lambda$ is an attractive hyperbolic τ_λ-copy of the base if $|\lambda| \ge \lambda_*$. Then, $\mathcal{I} \subset (-\lambda_*, \lambda_*)$. We define $\lambda_- = \inf \mathcal{I} \in [-\lambda_*, \lambda_0)$. Since $\lambda_- \notin \mathcal{I}$, there are at most two τ_{λ_-}-minimal sets. We define $\mathfrak{u}_{\lambda_-}(\omega) = \lim_{\lambda \downarrow \lambda_-} \mathfrak{u}_\lambda(\omega)$,

$\mathfrak{l}_{\lambda_-}(\omega) = \lim_{\lambda\downarrow\lambda_-} \mathfrak{l}_\lambda(\omega)$ and $\mathfrak{m}_{\lambda_-}(\omega) = \lim_{\lambda\downarrow\lambda_-} \mathfrak{m}_\lambda(\omega)$. As monotone limits of continuous functions, they are all semicontinuous on Ω. In particular, $\mathfrak{l}_{\lambda_-}$ and $\mathfrak{u}_{\lambda_-}$ are upper semicontinuous and $\mathfrak{m}_{\lambda_-}$ is lower semicontinuous. The continuous variation with respect to λ ensures that

$$\mathfrak{u}_{\lambda_-}(\omega{\cdot}t) = \lim_{\lambda\downarrow\lambda_-} \mathfrak{u}_\lambda(\omega{\cdot}t) = \lim_{\lambda\downarrow\lambda_-} v_\lambda(t,\omega,\mathfrak{u}_\lambda(\omega)) = v_{\lambda_-}(t,\omega,\mathfrak{u}_{\lambda_-}(\omega))\,,$$

that is, $\mathfrak{u}_{\lambda_-}$ is a τ_{λ_-}-equilibrium. The same holds for $\mathfrak{l}_{\lambda_-}$ and $\mathfrak{m}_{\lambda_-}$. Let $\mathcal{M}^l_{\lambda_-} = \text{closure}_{\Omega\times\mathbb{R}}\{(\omega_0{\cdot}t, \mathfrak{l}_{\lambda_-}(\omega_0{\cdot}t)) \mid t\in\mathbb{R}\}$ for a continuity point ω_0 of λ_-. Proposition 1.32 shows that $\mathcal{M}^l_{\lambda_-}$ is a τ_{λ_-}-minimal set. Since $\mathfrak{l}_{\lambda_-} < \mathfrak{l}_{\lambda_0} < \mathfrak{m}_{\lambda_0}$, we get $\mathcal{M}^l_{\lambda_-} < \{\mathfrak{m}_{\lambda_0}\}$. Since $\{\mathfrak{m}_{\lambda_0}\}$ is a repulsive hyperbolic copy of the base, $\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{m}_{\lambda_0}(\omega))\,dm > 0$ for all $m\in\mathfrak{M}_{\text{erg}}(\Omega,\sigma)$ (see Theorem 1.40), and hence Proposition 3.7 and Theorem 1.36(iii) show that the upper Lyapunov exponent of $\mathcal{M}^l_{\lambda_-}$ is strictly negative. Theorem 1.40 shows that $\mathcal{M}^l_{\lambda_-}$ is an attractive hyperbolic τ_{λ_-}-copy of the base, and Corollary 1.58(ii) shows that $\mathfrak{l}_{\lambda_-}$ is continuous and $\mathcal{M}^l_{\lambda_-}$ is its graph. Proposition 3.5(i) and Theorem 1.39 show that $\mathfrak{l}_{\lambda_-}$ and $\mathfrak{u}_{\lambda_-}$ are the lower and upper bounds of $\mathcal{A}_{\lambda_-}$, so the notation is coherent.

Let $\mathcal{M}_{\lambda_-}$ be the minimal set associated to $\mathfrak{m}_{\lambda_-}$ by (1.9). Proposition 2.14(ii) shows that $\mathcal{M}_{\lambda_-}$ is nonhyperbolic. The strict monotonicity properties of $\mathfrak{l}_\lambda$ and $\mathfrak{m}_\lambda$ ensure that $\mathfrak{l}_{\lambda_-} < \mathfrak{l}_{\lambda_0} < \mathfrak{m}_{\lambda_0} < \mathfrak{m}_{\lambda_-}$, so $\mathcal{M}^l_{\lambda_-} < \mathcal{M}_{\lambda_-}$. Therefore, $\mathcal{M}^l_{\lambda_-}$ and $\mathcal{M}_{\lambda_-}$ are the two unique τ_{λ_-}-minimal sets. Since $\mathfrak{u}_{\lambda_-} \ge \mathfrak{m}_{\lambda_-}$ and there is not a τ_{λ_-}-minimal set above $\mathcal{M}_{\lambda_-}$, we deduce that $\mathcal{M}_{\lambda_-}$ coincides with the τ_{λ_-}-minimal set $\mathcal{M}^u_{\lambda_-}$ defined from $\mathfrak{u}_{\lambda_-}$ by (1.9). Proposition 1.32 shows that $\mathfrak{u}_{\lambda_-}(\omega) = \mathfrak{m}_{\lambda_-}(\omega)$ for all ω in the residual set $\mathcal{R}\subseteq\Omega$ of their common continuity points, at which $(\mathcal{M}^u_{\lambda_-})_\omega$ is a singleton. Note also that $\mathcal{M}^u_{\lambda_-}$ is contained in the set $\bigcup_{\omega\in\Omega}\big(\{\omega\}\times[\mathfrak{m}_{\lambda_-}(\omega),\mathfrak{u}_{\lambda_-}(\omega)]\big)$, which is a compact τ_{λ_-}-invariant pinched set (see Definition 1.27). That is, the collision of $\mathfrak{m}_\lambda$ and $\mathfrak{u}_\lambda$ as $\lambda\downarrow\lambda_-$ encloses the nonhyperbolic minimal set $\mathcal{M}^u_{\lambda_-}$.

The hyperbolic attractive character of $\{\mathfrak{l}_{\lambda_-}\}$ and Theorems 1.40 and 1.36(iii) yield the first inequality of

$$\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{l}_{\lambda_-}(\omega))\,dm < 0, \quad \int_\Omega \mathfrak{h}_x(\omega,\mathfrak{m}_{\lambda_-}(\omega))\,dm \ge 0 \quad \text{and} \quad \int_\Omega \mathfrak{h}_x(\omega,\mathfrak{u}_{\lambda_-}(\omega))\,dm \le 0\,. \tag{3.8}$$

The other ones follow from Theorems 1.40 and 1.36(iii) applied to $\mathfrak{m}_\lambda$ and $\mathfrak{u}_\lambda$ for $\lambda\in\mathcal{I}$, and from Lebesgue's Monotone Convergence Theorem. Let us check that $\mathcal{A}_\lambda$ is an attractive hyperbolic τ_λ-copy of the base for all $\lambda<\lambda_-$. If $\lambda<\lambda_-$, then Propositions 1.24(i), 1.25 and 3.5(i) provide $e>0$ and $s>0$ such that

$$\mathfrak{m}_{\lambda_-}(\omega) - e > v_\lambda(s,\omega{\cdot}(-s),\mathfrak{u}_{\lambda_-}(\omega{\cdot}(-s))) > v_\lambda(s,\omega{\cdot}(-s),\mathfrak{u}_\lambda(\omega{\cdot}(-s))) = \mathfrak{u}_\lambda(\omega)$$

for all $\omega\in\Omega$. In particular, any bounded τ_λ-equilibrium $\mathfrak{b}\colon\Omega\to\mathbb{R}$, whose graph is for sure contained in $\mathcal{A}_\lambda$, is strictly below $\mathfrak{m}_{\lambda_-}$. Proposition 3.7 and the second inequality of (3.8) ensure that $\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}(\omega))\,dm < 0$ for all $m\in\mathfrak{M}_{\text{erg}}(\Omega,\sigma)$, which combined with Theorem 1.36(iii) ensures that all the Lyapunov exponents of $\mathcal{A}_\lambda$ are strictly negative. Hence, Theorem 2.13(vii) proves the assertion.

The same arguments for $\lambda_+ = \sup\mathcal{I}$ complete the proof of (i), (ii) and (iii). Note that a local saddle-node bifurcation of minimal sets takes place at λ_- (resp. λ_+), as the two minimal sets which collide at that value of the parameter actually disappear for $\lambda < \lambda_-$ (resp. $\lambda > \lambda_+$). Proposition 3.6 shows the continuity of $\{\mathcal{A}_\lambda\}$ as $\lambda \to \lambda_0$ for $\lambda_0 \neq \lambda_-, \lambda_+$. Let us check the lower discontinuity of $\{\mathcal{A}_\lambda\}$ as $\lambda \to \lambda_-$. The argument is, as usual, analogous for λ_+. We take a sequence $(\lambda_n) \uparrow \lambda_-$ and assume for contradiction the existence of $(\omega_n, x_n) \in \mathcal{A}_{\lambda_n}$ such that $\lim_{n\to\infty}(\omega_n, x_n) = (\omega, \mathfrak{u}_{\lambda_-}(\omega))$ for a point $\omega \in \Omega$. Then, $x_n = \mathfrak{l}_{\lambda_n}(\omega_n)$, and hence $\mathfrak{u}_{\lambda_-}(\omega) = \lim_{n\to\infty} \mathfrak{l}_{\lambda_n}(\omega_n) \leq \lim_{n\to\infty} \mathfrak{l}_{\lambda_-}(\omega_n) = \mathfrak{l}_{\lambda_-}(\omega)$, which is precluded by the existence of two τ_{λ_-}-minimal sets. The characterization of [21, Lemma 3.2(2)] shows the asserted lower discontinuity. □

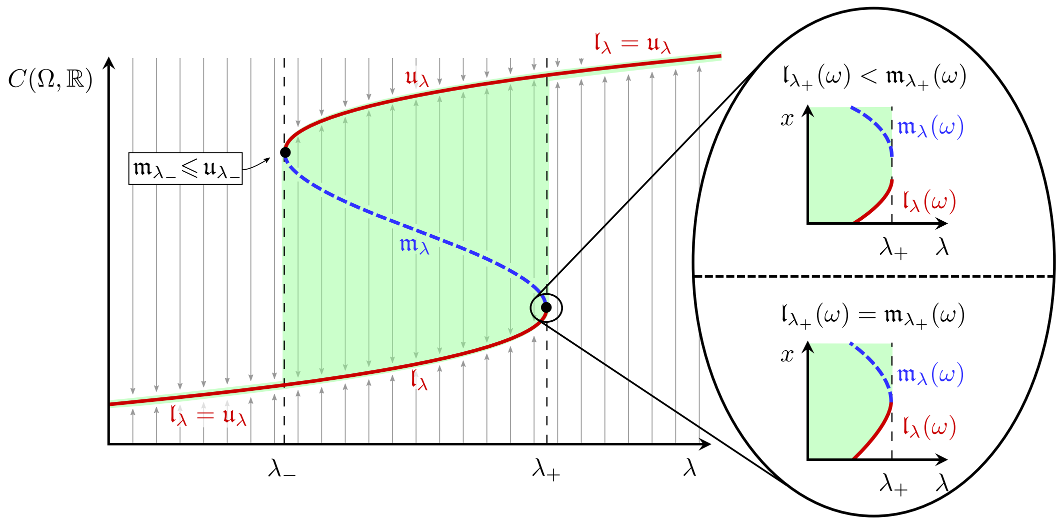


Figure 3.1: Double saddle-node bifurcation diagram described in Theorem 3.8. The strictly increasing solid red curves represent the families of attractive hyperbolic solutions of the λ-parametric family (3.5): $\mathfrak{l}_\lambda$ for $\lambda \neq \lambda_+$ and $\mathfrak{u}_\lambda$ for $\lambda \neq \lambda_-$. The strictly decreasing dashed blue curve represents the family of repulsive hyperbolic solutions of (3.5): $\mathfrak{m}_\lambda$ for $\lambda \in (\lambda_-, \lambda_+)$. A large black point over λ_+ represents the complex possibilities which arise for the collision of $\mathfrak{l}_\lambda$ and $\mathfrak{m}_\lambda$ as $\lambda \uparrow \lambda_+$, which is partly explained in the right zoom: the limit maps $\mathfrak{l}_{\lambda_+}$ and $\mathfrak{m}_{\lambda_+}$ are not necessarily continuous, but lower and upper semicontinuous; for a residual invariant set of points ω, they take the same value; but this residual set may coexist with an invariant set $\Omega_0 \subset \Omega$, at whose points $\mathfrak{l}_{\lambda_+}(\omega) < \mathfrak{m}_{\lambda_+}(\omega)$; and, given $m \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$, nothing allows us to determine a priori if we are in the case of $m(\Omega_0) = 0$ or in the case of $m(\Omega_0) = 1$. The situation is analogous for λ_-, and simply represented by "$\mathfrak{m}_{\lambda_-} \leqslant \mathfrak{u}_{\lambda_-}$". The hyperbolic τ_λ-minimal sets are given by the graphs of the curves $\mathfrak{l}_\lambda$, $\mathfrak{m}_\lambda$ and $\mathfrak{u}_\lambda$ whenever they are hyperbolic. A nonhyperbolic minimal set $\mathcal{M}^l_{\lambda_+}$ exists for λ_+, lying in the region delimited by the graphs of $\mathfrak{l}_{\lambda_+}$ and $\mathfrak{m}_{\lambda_+}$, and with a possibly highly complex shape. The situation is, again, analogous for λ_-, and no more minimal sets exist for any λ. The green-shadowed area represents the global attractor $\mathcal{A}_\lambda$, and the light grey arrows just try to depict the attracting and repulsive properties of $\mathfrak{l}_\lambda$, $\mathfrak{m}_\lambda$ and $\mathfrak{u}_\lambda$. (We will use "large black points" and analogous inequalities in the remaining figures to depict similar situations, as well as red and blue "hyperbolic" curves, green-shadowing, and grey arrows.)

Figure 3.1 depicts the bifurcation diagram described by Theorem 3.8. As we point out in the figure description, the dynamics at the nonhyperbolic minimal set occurring at any of the bifurcation points of the previous theorem can be highly complicated (see e.g. [18], [20], [78], [116] for the possible dynamical complexity that can arise: Li-Yorke chaos, several ergodic measures, sensitive dependence on initial conditions, strange nonchaotic attractors, etc). The next result contributes to understand the possibilities for this complex dynamics. In particular, strictly positive and strictly negative Lyapunov exponents can coexist on that set.

Proposition 3.9. *Assume the hypotheses of Theorem 3.8 and let* $m \in \mathfrak{M}_{\rm erg}(\Omega,\sigma)$ *be fixed. Then,*

(i) *one of the following situations holds:*

(a) $m(\{\omega\in\Omega \mid \mathfrak{m}_{\lambda_-}(\omega)=\mathfrak{u}_{\lambda_-}(\omega)\})=0$,

$$\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{u}_{\lambda_-}(\omega))\,dm<0 \quad \text{and} \quad \int_\Omega \mathfrak{h}_x(\omega,\mathfrak{m}_{\lambda_-}(\omega))\,dm>0\,.$$

(b) $m(\{\omega\in\Omega \mid \mathfrak{m}_{\lambda_-}(\omega)=\mathfrak{u}_{\lambda_-}(\omega)\})=1$ *and*

$$\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{u}_{\lambda_-}(\omega))\,dm=\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{m}_{\lambda_-}(\omega))\,dm=0\,,$$

and this is the situation if and only if $\int_{\mathcal{K}_{\lambda_-}}\mathfrak{h}_x(\omega,x)\,d\nu=0$ *for* $\mathcal{K}_{\lambda_-}=\bigcup_{\omega\in\Omega}\big(\{\omega\}\times[\mathfrak{m}_{\lambda_-}(\omega),\mathfrak{u}_{\lambda_-}(\omega)]\big)$ *and any ergodic measure* $\nu\in\mathfrak{M}_{\rm erg}(\mathcal{K}_{\lambda_-},\tau)$ *projecting onto* m.

(ii) *Let* $\mathfrak{l}_\mathcal{M}$ *and* $\mathfrak{u}_\mathcal{M}$ *be the lower and upper equilibria of* $\mathcal{M}=\mathcal{M}^u_{\lambda_-}$ *respectively. Then,* $\mathfrak{l}_\mathcal{M}$ *(resp.* $\mathfrak{u}_\mathcal{M}$*) coincides* m*-a.e. either with* $\mathfrak{m}_{\lambda_-}$ *or with* $\mathfrak{u}_{\lambda_-}$. *If, in addition,* $\mathfrak{M}_{\rm erg}(\Omega,\sigma)=\{m\}$, *then* $\mathfrak{l}_\mathcal{M}=\mathfrak{m}_{\lambda_-}$ *and* $\mathfrak{u}_\mathcal{M}=\mathfrak{u}_{\lambda_-}$ m*-a.e.*

Analogous properties hold for λ_+.

Proof. (i) As $\{\omega\in\Omega\mid \mathfrak{m}_{\lambda_-}(\omega)=\mathfrak{u}_{\lambda_-}(\omega)\}$ is σ-invariant and $m\in\mathfrak{M}_{\rm erg}(\Omega,\sigma)$, we have $m(\{\omega\in\Omega\mid \mathfrak{m}_{\lambda_-}(\omega)=\mathfrak{u}_{\lambda_-}(\omega)\})\in\{0,1\}$. If $m(\{\omega\in\Omega\mid \mathfrak{m}_{\lambda_-}(\omega)=\mathfrak{u}_{\lambda_-}(\omega)\})=1$, then (3.8) yields $0\le\int_\Omega\mathfrak{h}(\omega,\mathfrak{m}_{\lambda_-}(\omega))\,dm=\int_\Omega\mathfrak{h}_x(\omega,\mathfrak{u}_{\lambda_-}(\omega))\,dm\le 0$, and hence (b) holds. If $m(\{\omega\in\Omega\mid \mathfrak{m}_{\lambda_-}(\omega)=\mathfrak{u}_{\lambda_-}(\omega)\})=0$, then $\mathfrak{l}_{\lambda_-}$, $\mathfrak{m}_{\lambda_-}$ and $\mathfrak{u}_{\lambda_-}$ are three ordered τ_{λ_-}-equilibria, satisfying the conditions of Theorem 2.9. Since all its hypotheses are fulfilled, this result shows that $\int_\Omega\mathfrak{h}_x(\omega,\mathfrak{u}_{\lambda_-}(\omega))\;dm<0$ and $\int_\Omega\mathfrak{h}_x(\omega,\mathfrak{m}_{\lambda_-}(\omega))\;dm>0$, as asserted in (a). To check the "only if" in (b), we assume that $m(\{\omega\in\Omega\mid \mathfrak{m}_{\lambda_-}(\omega)=\mathfrak{u}_{\lambda_-}(\omega)\})=1$ holds, observe that $\mathfrak{b}_{\lambda_-}=\mathfrak{m}_{\lambda_-}=\mathfrak{u}_{\lambda_-}$ m-a.e for any τ_{λ_-}-equilibrium $\mathfrak{b}_{\lambda_-}$ with graph in $\mathcal{K}_{\lambda_-}$, and apply Theorem 1.36(i). The "if" part in (b) follows from Theorem 1.36(ii), wich precludes (a).

(ii) According to Theorems 2.9 and 1.36(i), $\mathcal{A}_{\lambda_-}$ concentrates at most three ergodic measures projecting onto m. Since Theorem 1.36(ii) shows that $\mathcal{M}^l_{\lambda_-}$ concentrates one, the four τ_{λ_-}-equilibria $\mathfrak{m}_{\lambda_-}\le\mathfrak{l}_\mathcal{M}\le\mathfrak{u}_\mathcal{M}\le\mathfrak{u}_{\lambda_-}$ can define at most two (by (1.10)), which ensures the first assertion in (ii). Now, let (Ω,σ) be uniquely ergodic. For contradiction, we assume that $\mathfrak{m}_{\lambda_-}\ne\mathfrak{l}_\mathcal{M}$ m-a.e., which ensures that $\mathfrak{l}_\mathcal{M}=\mathfrak{u}_\mathcal{M}=\mathfrak{u}_{\lambda_-}$ m-a.e.: otherwise Theorem 1.36(ii) would provide more than three elements of $\mathfrak{M}_{\rm erg}(\mathcal{A}_{\lambda_-},\tau_{\lambda_-})$ projecting onto m. Reasoning as in (i), we deduce from $\int_\Omega\mathfrak{h}_x(\omega,\mathfrak{u}_{\lambda_-}(\omega))\,dm<0$ (which holds since (a) holds) and from Theorem 1.36 that $\mathcal{M}$ has a unique Lyapunov exponent, which is strictly negative. This and Theorem 1.40 contadict the nonhyperbolicity of $\mathcal{M}$ ensured by Theorem 3.8(ii). This means that $\mathfrak{m}_{\lambda_-}=\mathfrak{l}_\mathcal{M}$ m-a.e., and similar arguments show that $\mathfrak{u}_\mathcal{M}=\mathfrak{u}_{\lambda_-}$ m-a.e.. □

The next result provides two alternative hypotheses to get the bifurcation diagram of Theorem 3.8.

Theorem 3.10. *Let (Ω,σ) be minimal. Let $\mathfrak{h}\colon \Omega\times\mathbb{R}\to\mathbb{R}$ satisfy* **d1**, **d2**$_{\lambda x}$, **d3** *and* **d4**. *Assume that there exists $\xi\in\mathbb{R}$ such that the flow τ_ξ induced by (3.5)$_\xi$ admits exactly two different minimal sets $\mathcal{M}^1_\xi<\mathcal{M}^2_\xi$ with $\mathcal{M}^1_\xi$ (resp. $\mathcal{M}^2_\xi$) hyperbolic. Then, the bifurcation diagram of (3.5) is that described by Theorem 3.8, with $\lambda_-=\xi$ (resp. $\lambda_+=\xi$). In particular, $\mathcal{M}^1_\xi=\{\mathfrak{l}_\xi\}$ (resp. $\mathcal{M}^2_\xi=\{\mathfrak{u}_\xi\}$).*

Proof. We reason in the case that the lower τ_ξ-minimal set, $\mathcal{M}^1_\xi$, is hyperbolic. Proposition 2.17(ii) ensures that $\mathcal{M}^1_\xi$ is the graph of $\mathfrak{l}_\xi$. Theorem 1.39 guarantees the existence of an attractive hyperbolic τ_{λ_0}-copy of the base $\mathcal{M}^1_{\lambda_0}$ strictly below $\mathcal{M}^2_\xi$ for a $\lambda_0>\xi$ close enough. Recall that $\mathfrak{u}_\lambda$ is strictly increasing with respect to λ: see Proposition 3.5(i). Recall also that the upper minimal set (which is $\mathcal{M}^2_\xi$ for $\lambda=\xi$) is defined from $\mathfrak{u}_\xi$ via (1.9) (see Proposition 2.17(i)), and that the sections of $\mathcal{M}^2_\xi$ and the upper τ_{λ_0}-minimal set $\mathcal{M}^u_{\lambda_0}$ reduce to a point for the residual set $\mathcal{R}\subset\Omega$ of common continuity points of $\mathfrak{u}_\xi$ and $\mathfrak{u}_{\lambda_0}$. Let us take $(\omega,x)\in\mathcal{M}^2_\xi$. Then, the τ_{λ_0}-orbit of (ω,x) is above $\mathcal{M}^1_{\lambda_0}$, and $v_{\lambda_0}(t,\omega,x)<v_\xi(t,\omega,x)\le\mathfrak{u}_\xi(\omega{\cdot}t)$ for all $t<0$. Hence, the α-limit set for τ_{λ_0} of (ω,x) exists and contains a minimal set $\mathcal{N}$. We look for $(\bar\omega,\bar x)\in\mathcal{N}$ with $\bar\omega\in\mathcal{R}$, and write $(\bar\omega,\bar x)=\lim_{n\to\infty}(\omega{\cdot}t_n,v_{\lambda_0}(t_n,\omega,x))$ for $(t_n)\downarrow-\infty$. Then $\bar x\le\mathfrak{u}_\xi(\bar\omega)<\mathfrak{u}_{\lambda_0}(\bar\omega)$ and hence $\mathcal{N}<\mathcal{M}^u_{\lambda_0}$. Since Corollary 1.58(i) ensures that $\mathcal{N}$ cannot be $\mathcal{M}^1_{\lambda_0}$, we conclude that there are three τ_{λ_0}-minimal sets: all the hypotheses of Theorem 3.8 hold. □

Although the next result is stated for τ_0 for the sake of simplicity, it can be applied for any value of the parameter $\lambda\in\mathbb{R}$ whenever the hypothesis **d2**$_\lambda$ holds.

Theorem 3.11. *Let (Ω,σ) be minimal. Let $\mathfrak{h}\colon\Omega\times\mathbb{R}\to\mathbb{R}$ satisfy* **d1**, **d2**, **d3** *and* **d4**. *Assume that the flow τ_0 defined by (3.5)$_0$ admits at least two minimal sets $\mathcal{M}_1$ and $\mathcal{M}_2$. Then,*

(i) *if* $\mathrm{Lyap}(\mathcal{M}_1)\subset[0,\infty)$*, then $\mathcal{M}_2$ is hyperbolic attractive.*

(ii) *If either* $\mathrm{Lyap}(\mathcal{M}_1)$ *or* $\mathrm{Lyap}(\mathcal{M}_2)$ *reduces to a point, then either $\mathcal{M}_1$ or $\mathcal{M}_2$ is hyperbolic attractive.*

(iii) *If (Ω,σ) is uniquely ergodic, then either $\mathcal{M}_1$ or $\mathcal{M}_2$ is hyperbolic attractive.*

Proof. (i) We will prove that $\sup\mathrm{Lyap}(\mathcal{M}_2)<0$, since in this case Theorem 1.40 ensures that $\mathcal{M}_2$ is hyperbolic attractive. Theorem 1.36(iii) ensures that there exist $m\in\mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$ and an m-measurable τ_0-equilibrium $\mathfrak{b}_2\colon\Omega\to\mathbb{R}$ with graph contained in $\mathcal{M}_2$ such that $\sup\mathrm{Lyap}(\mathcal{M}_2)=\int_\Omega\mathfrak{h}_x(\omega,\mathfrak{b}_2(\omega))\,dm$. Let $\mathfrak{b}_1$ be the upper τ_0-equilibrium of $\mathcal{M}_1$. Theorem 1.36(ii) shows that $\int_\Omega\mathfrak{h}_x(\omega,\mathfrak{b}_1(\omega))\,dm\in\mathrm{Lyap}(\mathcal{M}_1)$, and hence, by hypothesis, $\int_\Omega\mathfrak{h}_x(\omega,\mathfrak{b}_1(\omega))\,dm\ge0$. Conditions **d3** and **d4** allow us to apply the second assertion in Theorem 2.15 to conclude that $\sup\mathrm{Lyap}(\mathcal{M}_2)<0$, as asserted.

(ii) Suppose without loss of generality that $\mathrm{Lyap}(\mathcal{M}_1)=\{a\}$ with $a\in\mathbb{R}$. If $a<0$, then Theorem 1.40 ensures that it is hyperbolic attractive. If not, then $\{a\}\subset[0,\infty)$, so (i) proves that $\mathcal{M}_2$ is hyperbolic attractive.

(iii) If $\mathcal{M}_1$ is not an attractive hyperbolic τ_0-minimal set, then Theorem 1.40 ensures that $\sup\mathrm{Lyap}(\mathcal{M}_1)\ge0$. If $\mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)=\{m\}$, Theorem 1.36(iii) ensures that, for $i\in\{1,2\}$, $\sup\mathrm{Lyap}(\mathcal{M}_i)=\int_\Omega\mathfrak{h}_x(\omega,\mathfrak{b}_i(\omega))\,dm$ for an m-measurable τ_0-equilibrium $\mathfrak{b}_i\colon\Omega\to\mathbb{R}$ with graph contained in $\mathcal{M}_i$. As in (i), Theorem 2.15 allows us to conclude that $\sup\mathrm{Lyap}(\mathcal{M}_2)<0$, and Theorem 1.40 completes the proof. □

Remark 3.12. Note that Proposition 1.42 shows that the statements of Theorem 3.11 remain valid if we replace $\mathrm{Lyap}(\mathcal{M}_i)$ for $\mathrm{sp}_{\mathcal{M}_i}(\mathfrak{h}_x)$ for $i \in \{1,2\}$.

The "simplest" global bifurcation diagram of minimal sets (see Figure 3.2) occurs when the flow τ_λ admits only one minimal set for every value of the parameter $\lambda \in \mathbb{R}$. In other words, when there are no bifurcation values, at least from the point of view of the number of τ_λ-minimal sets. The next two results analyze this situation. Recall that the τ_λ-global attractor $\mathcal{A}_\lambda$ is pinched if at least one of its sections is a singleton (see Definition 1.27). The maps $\mathfrak{l}_\lambda$ and $\mathfrak{u}_\lambda$ in the statement are the semicontinuous ones provided by $(3.6)_\lambda$.

Theorem 3.13. *Let (Ω,σ) be minimal. Let $\mathfrak{h} \in C^{0,1}(\Omega\times\mathbb{R},\mathbb{R})$ satisfy* **d2**$_\lambda$*. Assume that $(3.5)_\lambda$ has only one τ_λ-minimal set for all $\lambda \in \mathbb{R}$. Then, $\mathcal{A}_\lambda$ is pinched for all $\lambda \in \mathbb{R}$, and $\mathcal{A}_\lambda < \mathcal{A}_\xi$ (i.e., $\mathfrak{u}_\lambda < \mathfrak{l}_\xi$) if $\lambda < \xi$.*

Proof. Let $\lambda \in \mathbb{R}$ be fixed. According to Proposition 2.17(i), the unique τ_λ-minimal set $\mathcal{M}_\lambda$ can be defined by (1.9) from $\mathfrak{l}_\lambda$ and from $\mathfrak{u}_\lambda$, and $(\mathcal{M}_\lambda)_\omega = \{\mathfrak{l}_\lambda(\omega)\} = \{\mathfrak{u}_\lambda(\omega)\}$ at any point ω in the residual set $\mathcal{R}_\lambda \subseteq \Omega$ of common continuity points of $\mathfrak{l}_\lambda$ and $\mathfrak{u}_\lambda$. Hence, these maps coincide on $\mathcal{R}_\lambda$, so $\mathcal{A}_\lambda$ is pinched.

Now, let $\lambda < \xi$. Proposition 3.4 shows that the semicontinuous maps $\mathfrak{l}_\lambda$ and $\mathfrak{u}_\lambda$ are strong τ_ξ-subequilibria, and we have just checked that they coincide at the residual set $\mathcal{R}_\lambda$. So, Proposition 1.25 provides $e > 0$ and $s_* > 0$ such that $\mathfrak{u}_\lambda(\omega) + e < v_\xi(s_*, \omega{\cdot}(-s_*), \mathfrak{l}_\lambda(\omega{\cdot}(-s_*))) < v_\xi(s_*, \omega{\cdot}(-s_*), \mathfrak{l}_\xi(\omega{\cdot}(-s_*))) = \mathfrak{l}_\xi(\omega)$ for all $\omega \in \Omega$. Here, we have used that $\mathfrak{l}_\lambda(\omega) < \mathfrak{l}_\xi(\omega)$ for all $\omega \in \Omega$ (see Proposition 3.5(i)) and, once more, the monotonicity of the flow. The proof is complete. □

Theorem 3.14 (One minimal bifurcation diagram). *Let (Ω,σ) be minimal. Let $\mathfrak{h}\colon \Omega\times\mathbb{R} \to \mathbb{R}$ satisfy* **d1**, **d2**$_\lambda$ *and* **d3**. *Assume that $(3.5)_\lambda$ has only one τ_λ-minimal set $\mathcal{M}_\lambda$ for all $\lambda \in \mathbb{R}$. Then,*

(i) *for $\lambda \in \mathbb{R}$, if $\mathcal{M}_\lambda$ is hyperbolic, then it is hyperbolic attractive and $\mathcal{A}_\lambda = \mathcal{M}_\lambda$.*

(ii) *If (Ω,σ) is uniquely ergodic (resp. finitely ergodic), then there exists at most a value (resp. a finite number of values) of the parameter at which the τ_λ-minimal set is nonhyperbolic. So, there exists at most a value (resp. a finite number of values) of discontinuous bifurcation of attractors.*

(iii) *If there exists $\lambda_0 \in \mathbb{R}$ such that the dynamical spectrum of $\mathfrak{h}_x$ on $\mathcal{A}_{\lambda_0}$ is $\{0\}$, then λ_0 is the only value of the parameter at which the minimal set is nonhyperbolic.*

Proof. (i) Proposition 2.17(i) and (ii) ensure that $\mathcal{M}_\lambda$ is attractive if it is hyperbolic. In this case, Corollary 1.58(iii) proves that $\mathcal{A}_\lambda = \mathcal{M}_\lambda$.

(ii) Assume that $\mathcal{M}_{\lambda_i}$ is a nonhyperbolic τ_{λ_i}-minimal set for $i \in \{1,2\}$, with $\lambda_1 < \lambda_2$. Theorems 1.40 and 1.36(iii) ensure the existence of $m_i \in \mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$ and an m_i-measurable map $\mathfrak{b}_i\colon \Omega \to \mathbb{R}$ with graph contained in $\mathcal{M}_{\lambda_i}$ such that $\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}_i(\omega))\,dm_i \geq 0$ for $i \in \{1,2\}$. Theorem 3.13 ensures that $\mathcal{A}_{\lambda_1} < \mathcal{A}_{\lambda_2}$, and hence $\mathfrak{b}_1(\omega) < \mathfrak{b}_2(\omega)$ for all $\omega \in \Omega$. Consequently, since $\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}_1(\omega))\,dm_1 + \int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}_2(\omega))\,dm_2 \geq 0$, Proposition 3.7 ensures that $m_1 \neq m_2$. Therefore, the number of parameter values λ for which the minimal set is nonhyperbolic is less than or equal to the number of distinct ergodic measures in $\mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$, so the first assertion in (ii) holds. The second one is a consequence of Proposition 3.6, point (i) and Theorem 1.39.

(iii) Proposition 1.42 ensures that the only τ_{λ_0}-minimal set $\mathcal{M}_{\lambda_0}$ is nonhyperbolic. Let $\lambda > \lambda_0$ (resp. $\lambda < \lambda_0$) be fixed and let $\mathcal{M}_\lambda$ be the only τ_λ-minimal set. Theorem 1.36(iii) provides $m \in \mathfrak{M}_{\text{erg}}(\Omega,\sigma)$ and an m-measurable τ_λ-equilibrium $\mathfrak{b}\colon \Omega \to \mathbb{R}$ with graph contained in $\mathcal{M}_\lambda$ such that $\sup \text{Lyap}(\mathcal{M}_\lambda) = \int_\Omega \mathfrak{h}_x(\omega,\mathfrak{b}(\omega))\,dm$. Since Theorem 1.36(ii) and (iii) ensure that $\int_\Omega \mathfrak{h}_x(\omega,\mathfrak{u}_{\mathcal{M}_{\lambda_0}}(\omega))\,dm = 0$, where $\mathfrak{u}_{\mathcal{M}_{\lambda_0}}$ is the upper τ_{λ_0}-equilibrium of $\mathcal{M}_{\lambda_0}$, and Theorem 3.13 ensures that $\mathfrak{u}_{\mathcal{M}_{\lambda_0}} < \mathfrak{b}$ (resp. $\mathfrak{u}_{\mathcal{M}_{\lambda_0}} > \mathfrak{b}$), Proposition 3.7 ensures that $\sup \text{Lyap}(\mathcal{M}_\lambda) < 0$. Hence, Theorem 1.40 concludes the proof. □

Figure 3.2 depicts the "reasonably simple" variation of $\mathcal{A}_\lambda$ with respect to λ under the hypotheses of Theorem 3.14(iii) and in the uniquely ergodic case of (ii).

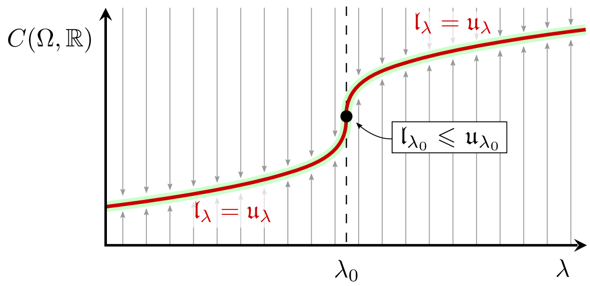


Figure 3.2: Evolution of the attractor $\mathcal{A}_\lambda$ for (3.5) in the cases of a unique value λ_0 of nonhyperbolicity (of $\mathcal{M}_{\lambda_0}$) described in Theorem 3.14. See Figure 3.1 to understand the meaning of the different elements.

Theorem 3.15. *Let (Ω,σ) be minimal and uniquely ergodic. Let $\mathfrak{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy* **d1**, **d2**$_{\lambda x}$, **d3** *and* **d4**. *Then, the bifurcation diagrams of Theorems 3.8 and 3.14(ii) exhaust all the possibilities of (3.5).*

Proof. If there exist $\lambda_0 \in \mathbb{R}$ such that τ_{λ_0} admits three τ_{λ_0}-minimal sets, then the bifurcation diagram of Theorem 3.8 appears. If there exists $\xi \in \mathbb{R}$ such that τ_ξ admits exactly two τ_ξ-minimal sets, then Theorem 3.11(iii) ensures that at least one of them is hyperbolic attractive. If both of them are hyperbolic attractive, then Proposition 2.14(ii) ensures that there are three τ_ξ-minimal sets, a contradiction. So, we are in the framework of Theorem 3.10, and hence the bifurcation diagram of Theorem 3.8 appears. In other case, there is a unique minimal set for all the values of the parameter, so the bifurcation diagram of Theorem 3.14(ii) takes place. □

The last result of this section is a local bifurcation theorem. Following the ideas of Remark 2.12, the properties of hypotheses **d3** and **d4** are only required on certain compact interval $\mathcal{J} \subset \mathbb{R}$ such that $\Omega \times \mathcal{J}$ contains two τ_λ-minimal sets (again, for $\lambda = 0$ in the statement for simplicity). Three different possibilities for these τ_λ-minimal sets ensure the existence of at least a local saddle-node bifurcation of τ_λ-minimal sets.

Theorem 3.16 (Local saddle-node bifurcations)**.** *Let (Ω,σ) be minimal. Let $\mathcal{J} \subset \mathbb{R}$ be a compact interval and let $\mathfrak{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy* **d1** *and*

d3$_{\mathcal{J}}$ $m\big(\{\omega \in \Omega \,|\, x \mapsto \mathfrak{h}_x(\omega,x)$ *is concave on* $\mathcal{J}\}\big) = 1$ *for all* $m \in \mathfrak{M}_{\text{erg}}(\Omega,\sigma)$,

d4$_{\mathcal{J}}$ *$m\big(\{\omega\in\Omega\,|\,x\mapsto \mathfrak{h}_x(\omega,x)$ is strictly concave on $\mathcal{J}\}\big)>0$ for all $m\in\mathfrak{M}_{\rm erg}(\Omega,\sigma)$.*

Assume that the flow τ_0 defined by (3.5)$_0$ admits two minimal sets $\mathcal{M}_1<\mathcal{M}_2$ contained on $\Omega\times\mathcal{J}$. Then,

(i) *if $\mathcal{M}_1$ is hyperbolic attractive or $\mathcal{M}_2$ is hyperbolic repulsive, then there exists $\lambda_+>0$ such that (3.5) exhibits a local saddle-node bifurcation of minimal sets at λ_+ (analogous to that occurring at λ_+ in Theorem 3.8).*

(ii) *If $\mathcal{M}_2$ is hyperbolic attractive or $\mathcal{M}_1$ is hyperbolic repulsive, then there exists $\lambda_-<0$ such that (3.5) exhibits a local saddle-node bifurcation of minimal sets at λ_- (analogous to that occurring at λ_- in Theorem 3.8).*

(iii) *If both $\mathcal{M}_1$ and $\mathcal{M}_2$ are hyperbolic attractive, then there exists an intermediate repulsive hyperbolic minimal set $\mathcal{M}$, and two local saddle-node bifurcations of minimal sets take place at λ_- and λ_+, with $\lambda_-<0<\lambda_+$ (analogous to these occurring at λ_- and λ_+ in Theorem 3.8).*

Proof. Let $\mathcal{J}=[a,b]$. We define $\widetilde{\mathfrak{h}}\colon\Omega\times\mathbb{R}\to\mathbb{R}$ outside $\Omega\times\mathcal{J}$ as in Remark 2.12. It is easy to check that $\widetilde{\mathfrak{h}}$ satisfies **d1**, **d2**$_\lambda$, **d3** and **d4**. We consider the parametric family of differential equations

$$x'=\widetilde{\mathfrak{h}}(\omega{\cdot}t,x)+\lambda\,,\quad \omega\in\Omega\,,$$

and the corresponding local skewproduct flow $\widetilde{\tau}_\lambda$ for each value of the parameter. So, the $\widetilde{\tau}_\lambda$-minimal sets contained in $\Omega\times\mathcal{J}$ are also τ_λ-minimal sets (the flow τ_λ is given by $\mathfrak{h}+\lambda$) and vice versa. Hence, a local saddle-node bifurcation of minimal sets (*lsnb* for short) for $\widetilde{\tau}_\lambda$ at some $\lambda_0\in\mathbb{R}$ taking place on $\Omega\times{\rm int}\,\mathcal{J}$ ensures an lsnb for τ_λ at λ_0: we will look for bifurcations of $\widetilde{\tau}_\lambda$. Since, given a τ_λ-minimal (or equivalently $\widetilde{\tau}_\lambda$-minimal) set $\mathcal{M}\subseteq\Omega\times\mathcal{J}$, we have $\mathfrak{M}_{\rm inv}(\mathcal{M},\tau)=\mathfrak{M}_{\rm inv}(\mathcal{M},\widetilde{\tau})$ and $\mathfrak{h}(\omega,x)=\widetilde{\mathfrak{h}}(\omega,x)$ for all $(\omega,x)\in\mathcal{M}$, the dynamical spectrum of $\mathfrak{h}_x$ on $\mathcal{M}$ coincides with that of $\widetilde{\mathfrak{h}}_x$ on $\mathcal{M}$. Therefore, Proposition 1.42 ensures that $\mathcal{M}$ is hyperbolic attractive (resp. repulsive) for τ_λ if and only if it is so for $\widetilde{\tau}_\lambda$.

Assume that $\widetilde{\tau}_0$ admits three minimal sets $\mathcal{N}_1<\mathcal{N}_2<\mathcal{N}_3$ (so they are hyperbolic copies of the base $\{\mathfrak{b}_1\}<\{\mathfrak{b}_2\}<\{\mathfrak{b}_3\}$, as stated by Theorem 2.11). Theorem 3.8 provides an lsnb at $\lambda_-<0$ on the open band $\bigcup_{\omega\in\Omega}\big(\{\omega\}\times(\mathfrak{b}_2(\omega),\mathfrak{b}_3(\omega))\big)\subset\Omega\times\mathbb{R}$ delimited by $\mathcal{N}_2$ and $\mathcal{N}_3$, as well as an lsnb at $\lambda_+>0$ on the open band delimited by $\mathcal{N}_1$ and $\mathcal{N}_2$. Under the hypotheses of (iii), Proposition 2.14(ii) and Theorem 2.11 provide a repulsive hyperbolic $\widetilde{\tau}_0$-minimal set $\mathcal{M}$ with $\mathcal{M}_1<\mathcal{M}<\mathcal{M}_2$, and hence we have two lsnb on $\Omega\times{\rm int}\,\mathcal{J}$ at $\lambda_-<0$ and $\lambda_+>0$. On the other hand, if $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) is hyperbolic repulsive, Proposition 2.14(i) provides a $\widetilde{\tau}_0$-minimal set $\mathcal{M}$ (possibly not contained in $\Omega\times\mathcal{J}$) with $\mathcal{M}<\mathcal{M}_1<\mathcal{M}_2$ (resp. $\mathcal{M}_1<\mathcal{M}_2<\mathcal{M}$). So, Theorem 3.8 ensures the existence of $\lambda_-<0$ (resp. $\lambda_+>0$) such that a collision of minimal sets occurs as $\lambda\downarrow\lambda_-$ (resp. as $\lambda\uparrow\lambda_+$) in the open region of $\Omega\times\mathbb{R}$ bounded by $\mathcal{M}_1$ and $\mathcal{M}_2$, which is contained in $\Omega\times{\rm int}\mathcal{J}$. Hence, we have at least an lsnb on $\Omega\times{\rm int}\,\mathcal{J}$ at $\lambda_-<0$ (resp. $\lambda_+>0$). If $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) is hyperbolic attractive but $\mathcal{M}_2$ (resp. $\mathcal{M}_1$) is nonhyperbolic, Theorems 3.10 and 3.8 show that $\widetilde{\tau}_\lambda$ has an lsnb on $\Omega\times{\rm int}\,\mathcal{J}$ at $\lambda_+>0$ (resp. $\lambda_-<0$). These properties prove (i) and (ii). □

3.3 Bifurcations of $x' = \mathfrak{h}(\omega{\cdot}t, x) + \lambda\, x$

This section copes with the following bifurcation problem:

$$x' = \mathfrak{h}(\omega{\cdot}t, x) + \lambda\, x\,, \quad \omega \in \Omega\,, \tag{3.9}$$

whose induced skewproduct flow is represented by τ_λ, with $\tau_\lambda(\omega, x) = (\omega{\cdot}t, v_\lambda(t, \omega, x))$. We work under the fundamental assumption

d5 $\mathfrak{h}(\omega, 0) = 0$ for all $\omega \in \Omega$.

Hence, $\mathcal{M}_0 = \Omega \times \{0\}$ is always a τ_λ-copy of the base for every $\lambda \in \mathbb{R}$ and, if (Ω, σ) is minimal, as in most of the results of this section, then $\mathcal{M}_0$ is a τ_λ-minimal set for every $\lambda \in \mathbb{R}$. Recall that Proposition 3.3(ii) ensures that $(\omega, x) \mapsto \mathfrak{h}(\omega, x) + \lambda\, x$ satisfies **d1**, **d2**, **d3**, **d4** and **d5** for all $\lambda \in \mathbb{R}$ if (and only if) $\mathfrak{h}$ satisfies **d1**, **d2**$_{\lambda x}$, **d3**, **d4** and **d5**. As in the previous section, in this case, if (Ω, σ) is minimal, Remark 2.7 and Proposition 3.3(iv) ensure that $\lim_{x\to\pm\infty} \mathfrak{h}_x(\omega, x) = -\infty$ uniformly on Ω.

The next result describes some useful consequences of **d5**.

Proposition 3.17. *Let $\mathfrak{h} \in C^{0,1}(\Omega \times \mathbb{R}, \mathbb{R})$ satisfy* **d5**. *Then,*

(i) *any strictly positive (resp. negative) global upper solution of $(3.9)_\lambda$ is a strict global upper solution of $(3.9)_\xi$ whenever $\xi < \lambda$ (resp. $\lambda < \xi$). In particular, any strictly positive (resp. negative) equilibrium for $(3.9)_\lambda$ is a strong superequilibrium for $(3.9)_\xi$ whenever $\xi < \lambda$ (resp. $\lambda < \xi$).*

(ii) *Any strictly positive (resp. negative) global lower solution of $(3.9)_\lambda$ is a strict global lower solution of $(3.9)_\xi$ whenever $\lambda < \xi$ (resp. $\xi < \lambda$). In particular, any strictly positive (resp. negative) equilibrium for $(3.9)_\lambda$ is a strong subequilibrium for $(3.9)_\xi$ whenever $\lambda < \xi$ (resp. $\xi < \lambda$).*

Proof. The same arguments of the proof of Propostion 3.4 hold in this case, taking into account that the increasing or decreasing character of $\lambda \mapsto \lambda\, \mathfrak{b}(\omega)$ depends on the halfplane on which $\mathfrak{b}$ is placed. □

Theorem 2.13 proves the existence of the global attractor $\mathcal{A}_\lambda$ of $(3.9)_\lambda$ if $\mathfrak{h} \in C^{0,1}(\Omega \times \mathbb{R}, \mathbb{R})$ satisfies **d2**$_{\lambda x}$, since Proposition 3.3(ii) ensures that all the required hypotheses hold; and it explains part of its properties. In the line of Proposition 3.5, the next result analyzes the variation of $\mathcal{A}_\lambda$ with respect to λ. We remark that, within Section 3.3, $\mathcal{A}_\lambda$, τ_λ, $v_\lambda(t, \omega, x)$, $\mathfrak{l}_\lambda$ and $\mathfrak{u}_\lambda$ refer to the dynamical elements of $(3.9)_\lambda$, not of $(3.5)_\lambda$.

Proposition 3.18. *Let $\mathfrak{h} \in C^{0,1}(\Omega \times \mathbb{R}, \mathbb{R})$ satisfy* **d2**$_{\lambda x}$ *and* **d5**, *and let*

$$\mathcal{A}_\lambda = \bigcup_{\omega\in\Omega} \big(\{\omega\} \times [\mathfrak{l}_\lambda(\omega), \mathfrak{u}_\lambda(\omega)]\big)$$

be the global attractor for the skewproduct flow τ_λ induced by $(3.9)_\lambda$. Then,

(i) *$\mathfrak{l}_\lambda(\omega) \le 0 \le \mathfrak{u}_\lambda(\omega)$ for every $\omega \in \Omega$ and $\lambda \in \mathbb{R}$.*

(ii) *For every $\omega \in \Omega$, the maps $\lambda \mapsto \mathfrak{l}_\lambda(\omega)$ and $\lambda \mapsto \mathfrak{u}_\lambda(\omega)$ are respectively nonincreasing and nondecreasing on $\mathbb{R}$ and both are right-continuous. Moreover, if $\mathfrak{l}_{\lambda_0}(\omega_0) < 0$ (resp. $\mathfrak{u}_{\lambda_0}(\omega_0) > 0$) for some $\lambda_0 \in \mathbb{R}$ and $\omega_0 \in \Omega$, then $\mathfrak{l}_{\lambda_2}(\omega_0) < \mathfrak{l}_{\lambda_0}(\omega_0) < \mathfrak{l}_{\lambda_1}(\omega_0)$ for all $\lambda_1 < \lambda_0 < \lambda_2$ (resp. $\mathfrak{u}_{\lambda_1}(\omega_0) < \mathfrak{u}_{\lambda_0}(\omega_0) < \mathfrak{u}_{\lambda_2}(\omega_0)$).*

(iii) $\lim_{\lambda\to\infty} \mathfrak{l}_\lambda(\omega) = -\infty$ *and* $\lim_{\lambda\to\infty} \mathfrak{u}_\lambda(\omega) = \infty$ *uniformly on* Ω. *In particular, if* (Ω, σ) *is minimal, then* τ_λ *admits at least three minimal sets for* λ *large enough.*

(iv) *There exists* $\lambda_0 \in \mathbb{R}$ *such that* $\mathcal{A}_\lambda = \mathcal{M}_0 = \Omega \times \{0\}$ *for every* $\lambda < \lambda_0$ *and it is an attractive hyperbolic* τ_λ*-copy of the base.*

Proof. (i) It follows directly from $\mathcal{M}_0 = \Omega \times \{0\} \subseteq \mathcal{A}_\lambda$, in turn ensured by **d5**.

(ii) Let $\lambda_0 < \lambda_2$. Since $\mathfrak{u}_{\lambda_0}(\omega) \geq 0$ for all $\omega \in \Omega$, $\mathfrak{u}'_{\lambda_0}(\omega{\cdot}t) = \mathfrak{h}(\omega{\cdot}t, \mathfrak{u}_{\lambda_0}(\omega{\cdot}t)) + \lambda_0\, \mathfrak{u}_{\lambda_0}(\omega{\cdot}t) \leq \mathfrak{h}(\omega{\cdot}t, \mathfrak{u}_{\lambda_0}(\omega{\cdot}t)) + \lambda_2\, \mathfrak{u}_{\lambda_0}(\omega{\cdot}t)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$, and hence Theorem 2.13(v) proves that $\mathfrak{u}_{\lambda_0}(\omega) \leq \mathfrak{u}_{\lambda_2}(\omega)$ for all $\omega \in \Omega$. So, $\lambda \mapsto \mathfrak{u}_\lambda(\omega)$ is nondecreasing for all $\omega \in \Omega$. If $\mathfrak{u}_{\lambda_0}(\omega_0) > 0$ for some $\omega_0 \in \Omega$, then $\mathfrak{u}_{\lambda_0}(\omega_0{\cdot}t) > 0$ for all $t \in \mathbb{R}$. Hence, $\mathfrak{u}'_{\lambda_0}(\omega_0{\cdot}t) < \mathfrak{h}(\omega_0{\cdot}t, \mathfrak{u}_{\lambda_0}(\omega_0{\cdot}t)) + \lambda_2\, \mathfrak{u}_{\lambda_0}(\omega_0{\cdot}t)$ for all $t \in \mathbb{R}$, so Theorem 2.13(v) ensures that $\mathfrak{u}_{\lambda_0}(\omega_0) < \mathfrak{u}_{\lambda_2}(\omega_0)$. Given $\lambda_1 < \lambda_0$, if, in addition, $\mathfrak{u}_{\lambda_1}(\omega_0) = 0$, then $\mathfrak{u}_{\lambda_1}(\omega_0) < \mathfrak{u}_{\lambda_0}(\omega_0)$; and if $\mathfrak{u}_{\lambda_1}(\omega_0) > 0$, then the previous argument proves that $\mathfrak{u}'_{\lambda_1}(\omega_0{\cdot}t) < \mathfrak{h}(\omega_0{\cdot}t, \mathfrak{u}_{\lambda_1}(\omega_0{\cdot}t)) + \lambda_0\, \mathfrak{u}_{\lambda_1}(\omega_0{\cdot}t)$ for all $t \in \mathbb{R}$, so $\mathfrak{u}_{\lambda_1}(\omega_0) < \mathfrak{u}_{\lambda_0}(\omega_0)$. The case of $\mathfrak{l}_\lambda$ is analogous, and the assertion on right-continuity follows from the arguments used to prove Proposition 3.5(i).

(iii) For each $\rho > 0$, we take $\lambda_\rho > -\inf\{\mathfrak{h}(\omega, \rho)/\rho \mid \omega \in \Omega\}$. Then, $\mathfrak{h}(\omega, \rho) + \lambda_\rho\, \rho > 0$ for all $\omega \in \Omega$ and hence $\mathfrak{h}(\omega, \rho) + \lambda\, \rho > 0$ for all $\lambda \geq \lambda_\rho$ and $\omega \in \Omega$. Then, Theorem 2.13(v) ensures that $\rho < \mathfrak{u}_\lambda(\omega)$ for all $\lambda \geq \lambda_\rho$ and $\omega \in \Omega$, that is, $\lim_{\lambda\to\infty} \mathfrak{u}_\lambda(\omega) = \infty$ uniformly on Ω. The proof for $\mathfrak{l}_\lambda$ is symmetrical. We have also checked that the τ_λ-minimal set defined from $\mathfrak{u}_\lambda$ by (1.9) is strictly above $\mathcal{M}_0$ if $\lambda \geq \lambda_\rho$. Similarly, that defined from $\mathfrak{l}_\lambda$ by (1.9) is strictly below $\mathcal{M}_0$ if λ is large enough. These facts prove the second assertion in (iii).

(iv) Note that (ii) ensures that $\mathcal{A}_\lambda \subseteq \mathcal{A}_\xi$ if $\lambda < \xi$. Let us fix $\xi \in \mathbb{R}$, take $r > 0$ such that $\mathcal{A}_\xi \subseteq \Omega \times [-r, r]$ and define $\lambda_0 = \min\{\xi, -\sup\{\mathfrak{h}_x(\omega, x) \mid (\omega, x) \in \Omega \times [-r, r]\}\}$. Then, $\mathfrak{h}_x(\omega, x) + \lambda < 0$ for all $\lambda < \lambda_0$ and $(\omega, x) \in \Omega \times [-r, r]$, so Theorem 1.36 ensures that every Lyapunov exponent of $\mathcal{A}_\lambda$ is strictly negative if $\lambda < \lambda_0$. Theorem 2.13(vii) guarantees that $\mathcal{A}_\lambda$ is an attractive hyperbolic τ_λ-copy of the base if $\lambda < \lambda_0$, so it coincides with $\mathcal{M}_0$. □

The results of Proposition 3.6 also hold for the family of global attractors of (3.9), with an analogous proof.

3.3.1 Bifurcation diagrams with minimal base flow

In this section, assuming that the base flow (Ω, σ) is minimal, all the possible bifurcation diagrams of τ_λ-minimal sets for (3.9) are described. An important role in all the bifurcation diagrams is played by the dynamical spectrum $\mathrm{sp}_{\mathcal{M}_0}(\mathfrak{h}_x) = [-\lambda_+, -\lambda_-]$ of $\mathfrak{h}_x \colon \mathcal{M}_0 \to \mathbb{R}$, where $\mathcal{M}_0 = \Omega \times \{0\}$ (see Definition 1.13): it is directly related with the hyperbolic or nonhyperbolic character of $\mathcal{M}_0$, and it determines the endpoints λ_- and λ_+ of an interval of values of the parameter on which most of the bifurcations take place. In particular, when $\lambda_- < \lambda_+$, bifurcations may arise that have no autonomous analog: it is what we will call generalized transcritical bifurcations and generalized pitchfork bifurcations. Throughout Sections 3.3.1 and 3.3.2, special attention will be paid to generalized pitchfork bifurcations, since the construction of some of them is nontrivial.

The bifurcation diagram of (3.9) can be described depending on the relative position of $\lambda_- \le \lambda_+$ and the following two parameters:

$$\begin{aligned} \mu_- &= \inf\{\lambda\colon \forall\,\xi > \lambda \text{ the graph of } \mathfrak{l}_\xi \text{ is a hyperb. minimal set } \mathcal{M}^l_\xi < \mathcal{M}_0\}\,,\\ \mu_+ &= \inf\{\lambda\colon \forall\,\xi > \lambda \text{ the graph of } \mathfrak{u}_\xi \text{ is a hyperb. minimal set } \mathcal{M}^u_\xi > \mathcal{M}_0\}\,. \end{aligned} \tag{3.10}$$

The following proposition gives some information which is common to the three possible bifurcation diagrams of (3.9).

Proposition 3.19. *Let (Ω,σ) be minimal. Let $\mathfrak{h}\colon \Omega\times\mathbb{R}\to\mathbb{R}$ satisfy* **d1**, **d2**$_{\lambda x}$, **d3**, **d4** *and* **d5**. *Let $[-\lambda_+,-\lambda_-]$ be the dynamical spectrum of $\mathfrak{h}_x$ on $\mathcal{M}_0=\Omega\times\{0\}$, with $\lambda_-\le\lambda_+$. Then,*

(i) $[-\lambda_+,-\lambda_-] = \{\int_\Omega \mathfrak{h}_x(\omega,0)\,dm \mid m\in\mathfrak{M}_{\rm inv}(\Omega,\sigma)\}$, *and there exist $m_-,m_+\in\mathfrak{M}_{\rm erg}(\Omega,\sigma)$ such that $-\lambda_- = \int_\Omega \mathfrak{h}_x(\omega,0)\,dm_-$ and $-\lambda_+ = \int_\Omega \mathfrak{h}_x(\omega,0)\,dm_+$.*

(ii) *The τ_λ-minimal set $\mathcal{M}_0$ is hyperbolic attractive if $\lambda<\lambda_-$, nonhyperbolic if $\lambda\in[\lambda_-,\lambda_+]$ and hyperbolic repulsive if $\lambda>\lambda_+$.*

(iii) *τ_λ admits three different hyperbolic minimal sets $\mathcal{M}^l_\lambda<\mathcal{M}_0<\mathcal{M}^u_\lambda$ for $\lambda>\lambda_+$, where $\mathcal{M}^l_\lambda$ and $\mathcal{M}^u_\lambda$ are attractive and given by the graphs of $\mathfrak{l}_\lambda$ and $\mathfrak{u}_\lambda$ respectively.*

(iv) *$\mu_-,\mu_+\in(-\infty,\lambda_+]$, and $\mu_-=\lambda_+$ and/or $\mu_+=\lambda_+$. Moreover, $\mu_-=\lambda_+$ (resp. $\mu_+=\lambda_+$) if and only if $\mathfrak{l}_\lambda$ (resp. $\mathfrak{u}_\lambda$) collides with 0 on a residual σ-invariant subset of Ω as $\lambda\downarrow\lambda_+$. In this case, there are no τ_λ-minimal sets below $\mathcal{M}_0$ (resp. above $\mathcal{M}_0$) for any $\lambda\le\lambda_+$.*

(v) *If $\lambda_-<\lambda_+$, then neither $\mathfrak{l}_\lambda$ nor $\mathfrak{u}_\lambda$ are identically zero for $\lambda\in(\lambda_-,\lambda_+]$.*

Proof. (i) It follows from Theorem 1.36(i) and (iii).

(ii) The dynamical spectrum of $\mathfrak{h}_x+\lambda$ on $\mathcal{M}_0$ is $[-\lambda_++\lambda,-\lambda_-+\lambda]$. Then, $[-\lambda_++\lambda,-\lambda_-+\lambda]\subset(0,\infty)$ if $\lambda>\lambda_+$, $[-\lambda_++\lambda,-\lambda_-+\lambda]\subset(-\infty,0)$ if $\lambda<\lambda_-$ and $0\in[-\lambda_++\lambda,-\lambda_-+\lambda]$ otherwise. Hence, Proposition 1.42 ensures the stated hyperbolicity properties of $\mathcal{M}_0$.

(iii) For $\lambda>\lambda_+$, since $\mathcal{M}_0$ is hyperbolic repulsive, Proposition 2.14(i) ensures that there exist three τ_λ-minimal sets $\mathcal{M}^l_\lambda<\mathcal{M}_0<\mathcal{M}^u_\lambda$, and Theorem 2.11 ensures that $\mathcal{M}^l_\lambda$ and $\mathcal{M}^u_\lambda$ are hyperbolic attractive and that $\mathcal{M}^l_\lambda=\{\mathfrak{l}_\lambda\}$ and $\mathcal{M}^u_\lambda=\{\mathfrak{u}_\lambda\}$.

(iv) Let us check that $\mu_-,\mu_+\in(-\infty,\lambda_+]$. From (iii) and the definition of μ_- and μ_+ in (3.10), it follows that $\mu_+,\mu_-\le\lambda_+$. Proposition 3.18(iv) ensures that μ_+ and μ_- are finite, and hence $\mu_-,\mu_+\in(-\infty,\lambda_+]$. Now, let us check that $\mu_-=\lambda_+$ and/or $\mu_+=\lambda_+$. For contradiction, assume that $\mu_-<\lambda_+$ and $\mu_+<\lambda_+$. Then, $\mathcal{M}^l_{\lambda_+}<\mathcal{M}_0<\mathcal{M}^u_{\lambda_+}$ are three τ_{λ_+}-minimal sets and Theorem 2.11 contradicts the nonhyperbolic character of $\mathcal{M}_0$. So, $\mu_-=\lambda_+$ and/or $\mu_+=\lambda_+$.

Let us check now that $\mu_+=\lambda_+$ if and only if $\mathfrak{u}_\lambda$ collides with 0 on a residual σ-invariant subset of Ω as $\lambda\downarrow\lambda_+$. Assume that $\mu_+=\lambda_+$, and assume for contradiction that $\mathfrak{u}_{\lambda_+}(\omega)>0$ for all $\omega\in\Omega$. Then, Proposition 1.32 ensures that there exists a τ_{λ_+}-minimal set $\mathcal{M}_0<\mathcal{M}^u_{\lambda_+}$. Since the dynamical spectrum $[0,\lambda_+-\lambda_-]$ of $\mathfrak{h}_x+\lambda_+$ on $\mathcal{M}_0$ is contained in $[0,\infty)$, Theorem 3.11(i) and Remark 3.12 ensure that $\mathcal{M}^u_{\lambda_+}$ is hyperbolic attractive, and Proposition 2.17 shows that $\mathcal{M}^u_{\lambda_+}=\{\mathfrak{u}_{\lambda_+}\}$.

Hence, $\mathcal{M}^u_{\lambda_+}$ has an attractive hyperbolic continuation $\mathcal{M}^u_\lambda$ for $\lambda < \lambda_+$ close enough (see Theorem 1.39). If $\lambda_- < \lambda_+$, then the last assertion in Theorem 3.10 shows that $\mathcal{M}^u_\lambda = \{\mathfrak{u}_\lambda\}$ for $\lambda < \lambda_+$ close enough; if $\lambda_- = \lambda_+$, then Proposition 2.14(ii) applied to the attractive hyperbolic τ_λ-minimal sets $\mathcal{M}_0 < \mathcal{M}^u_\lambda$, for $\lambda < \lambda_+$ close enough, ensures the existence of another τ_λ-minimal set $\mathcal{N}_\lambda$, and Theorem 2.11 guarantees that $\mathcal{M}_0 < \mathcal{N}_\lambda < \mathcal{M}^u_\lambda$ and $\mathcal{M}^u_\lambda = \{\mathfrak{u}_\lambda\}$. That is, in both cases, $\{\mathfrak{u}_\lambda\}$ is an attractive hyperbolic τ_λ-copy of the base for $\lambda < \lambda_+$ close enough, which leads us to the contradiction $\mu_+ < \lambda_+$. Hence, there exists $\omega_0 \in \Omega$ such that $\mathfrak{u}_{\lambda_+}(\omega_0) = 0$. Proposition 1.34 ensures that $\mathfrak{u}_{\lambda_+}(\omega) = 0$ on the residual set of the continuity points of $\mathfrak{u}_{\lambda_+}$, that is, $\mathfrak{u}_\lambda$ collide with 0 on a residual σ-invariant set as $\lambda \downarrow \lambda_+$. Conversely, if $\mu_+ < \lambda_+$, then it is clear that $\mathfrak{u}_\lambda$ does not collide with 0 on a residual σ-invariant subset of Ω as $\lambda \downarrow \lambda_+$ since $\lim_{\lambda\downarrow\lambda_+} \mathfrak{u}_\lambda(\omega) = \mathfrak{u}_{\lambda_+}(\omega)$ whose graph is a hyperbolic τ_{λ_+}-minimal set with $\{\mathfrak{u}_{\lambda_+}\} > \mathcal{M}_0$. This proves the equivalence stated in (iv). Now, with $\mu_+ = \lambda_+$, the monotonicity in Proposition 3.18(ii) and Proposition 1.34 ensure that $\mathfrak{u}_\lambda(\omega) = 0$ on the residual set of the continuity points of $\mathfrak{u}_\lambda$ for all $\lambda \le \lambda_+$, and hence (1.9) applied to $\mathfrak{u}_\lambda$ provides $\mathcal{M}_0$ for all $\lambda \le \lambda_+$. Then, Proposition 2.17(i) proves the last assertion in (iv). The proof in the case of μ_- and $\mathfrak{l}_\lambda$ is analogous.

(v) Given $\lambda \in (\lambda_-, \lambda_+]$, since (i) ensures that $\int_\Omega (\mathfrak{h}_x(\omega, 0) + \lambda)\, dm_- = \lambda - \lambda_- > 0$ and Proposition 2.16 ensures that $\int_\Omega (\mathfrak{h}_x(\omega, \mathfrak{l}_\lambda(\omega)) + \lambda)\, dm_- \le 0$ and $\int_\Omega (\mathfrak{h}_x(\omega, \mathfrak{u}_\lambda(\omega)) + \lambda)\, dm_- \le 0$, if we assume that $\mathfrak{l}_\lambda$ or $\mathfrak{u}_\lambda$ is identically 0, then we reach a contradiction. □

The next three theorems describe the three possible bifurcation diagrams of (3.9), which are depicted in Figures 3.3, 3.4 and 3.5. Moreover, Theorem 3.24 ensures that they exhaust all the possibilities of bifurcation diagrams.

Theorem 3.20 (Global classical pitchfork bifurcation)**.** *Let (Ω, σ) be minimal. Let $\mathfrak{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy* **d1**, **d2**$_{\lambda x}$, **d3**, **d4** *and* **d5**. *Let $[-\lambda_+, -\lambda_-]$ be the dynamical spectrum of $\mathfrak{h}_x$ on $\mathcal{M}_0 = \Omega \times \{0\}$, with $\lambda_- \le \lambda_+$. In addition to the conclusions of Proposition 3.19, if $\mu_+ = \mu_- = \lambda_+$, then*

(i) *both $\mathfrak{l}_\lambda$ and $\mathfrak{u}_\lambda$ collide with 0 on a residual σ-invariant subset of Ω as $\lambda \downarrow \lambda_+$.*

(ii) *$\mathcal{M}_0$ is the unique τ_λ-minimal set if $\lambda \le \lambda_+$.*

(iii) *$\mathcal{A}_\lambda = \mathcal{M}_0$ if $\lambda < \lambda_-$.*

That is, a classical pitchfork bifurcation of minimal sets arises around $\mathcal{M}_0$ at λ_+.

Proof. The equivalences in Proposition 3.19(iv) show (i) and the last assertion in Proposition 3.19(iv) shows (ii). Since $\mathcal{M}_0$ is an attractive hyperbolic τ_λ-copy of the base for $\lambda < \lambda_-$ (see Proposition 3.19(ii)), Corollary 1.58(iii) shows that $\mathcal{A}_\lambda = \mathcal{M}_0$ for all $\lambda < \lambda_-$. □

Theorem 3.21 (Local saddle-node and transcritical bifurcation)**.** *Let (Ω, σ) be minimal. Let $\mathfrak{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy* **d1**, **d2**$_{\lambda x}$, **d3**, **d4** *and* **d5**. *Let $[-\lambda_+, -\lambda_-]$ be the dynamical spectrum of $\mathfrak{h}_x$ on $\mathcal{M}_0 = \Omega \times \{0\}$, with $\lambda_- \le \lambda_+$. In addition to the conclusions of Proposition 3.19, if $\mu_+ = \lambda_+$ and $\mu_- < \lambda_-$, then*

(i) *τ_λ admits exactly two minimal sets $\mathcal{M}^l_\lambda < \mathcal{M}_0$ for $\lambda \in [\lambda_-, \lambda_+]$, where $\mathcal{M}^l_\lambda$ is hyperbolic attractive and given by the graph of $\mathfrak{l}_\lambda$.*

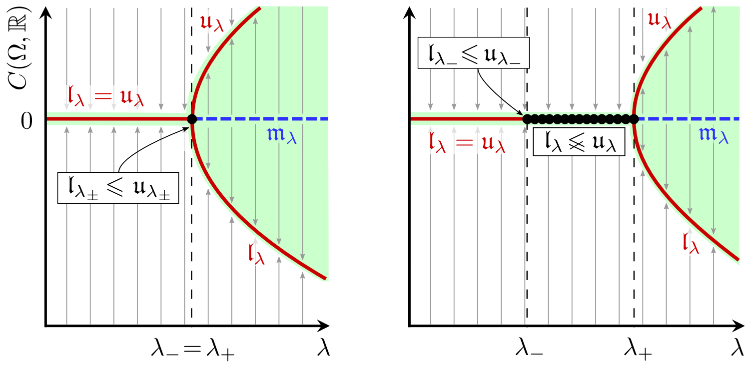


Figure 3.3: The classical pitchfork bifurcation diagrams described in Proposition 3.19 and Theorem 3.20. In the second diagram, the large black points at 0 for $\lambda \in (\lambda_-, \lambda_+]$ represent the fact that the global attractor $\mathcal{A}_\lambda$ does not reduce to the unique τ_λ-minimal set $\mathcal{M}_0$. This may also be the situation at λ_-. See Figure 3.1 to understand the meaning of the remaining elements.

(ii) *τ_λ admits three hyperbolic minimal sets $\mathcal{M}^l_\lambda < \mathcal{N}_\lambda < \mathcal{M}_0$ for $\lambda \in (\mu_-, \lambda_-)$, where $\mathcal{M}^l_\lambda$ is attractive and given by the graph of $\mathfrak{l}_\lambda$, and $\mathcal{N}_\lambda$ is repulsive and given by the graph of a continuous map $\mathfrak{m}_\lambda \colon \Omega \to \mathbb{R}$, the map $\lambda \mapsto \mathfrak{m}_\lambda$ is strictly increasing on (μ_-, λ_-), and $\mathfrak{m}_\lambda$ collides with $\mathfrak{l}_\lambda$ (resp. with 0) on a residual σ-invariant set as $\lambda \downarrow \mu_-$ (resp. $\lambda \uparrow \lambda_-$).*

(iii) *τ_{μ_-} admits exactly two different minimal sets $\mathcal{M}^l_{\mu_-} < \mathcal{M}_0$, where $\mathcal{M}^l_{\mu_-}$ is nonhyperbolic.*

(iv) *$\mathcal{A}_\lambda = \mathcal{M}_0$ for $\lambda < \mu_-$.*

In particular, μ_-, λ_- and λ_+ are the unique bifurcation points: a local saddle-node bifurcation of minimal sets occurs around $\mathcal{M}_{\mu_-}$ at μ_-, as well as a discontinuous bifurcation of attractors; and a classical (resp. generalized) transcritical bifurcation of minimal sets arises around $\mathcal{M}_0$ at λ_- (resp. on $[\lambda_-, \lambda_+]$) if $\lambda_- = \lambda_+$ (resp. $\lambda_- < \lambda_+$).

A global bifurcation diagram which is symmetric to the one described with respect to the horizontal axis arises if $\mu_- = \lambda_+$ and $\mu_+ < \lambda_-$.

Proof. (i) Recall that the definition of μ_- ensures that $\mathcal{M}^l_\lambda = \{\mathfrak{l}_\lambda\}$ is hyperbolic attractive for $\lambda \in (\mu_-, \infty)$. Proposition 3.19(ii) ensures the nonhyperbolicity of $\mathcal{M}_0$ for $\lambda \in [\lambda_-, \lambda_+]$, and then Theorem 2.11 precludes the existence of other τ_λ-minimal set apart from $\mathcal{M}_0$ and $\mathcal{M}^l_\lambda$ for $\lambda \in [\lambda_-, \lambda_+]$.

(ii)-(iii) Again, Proposition 3.19(ii) and the definition of μ_- ensure that $\mathcal{M}_0$ and $\mathcal{M}^l_\lambda$ are hyperbolic attractive for $\lambda \in (\mu_-, \lambda_-)$, and hence Proposition 2.14(ii) and Theorem 2.11 ensure the existence of a repulsive hyperbolic minimal set $\mathcal{N}_\lambda$ between $\mathcal{M}^l_\lambda$ and $\mathcal{M}_0$ for $\lambda \in (\mu_-, \lambda_-)$, given by the graph of a continuous τ_λ-equilibrium $\mathfrak{m}_\lambda \colon \Omega \to \mathbb{R}$. Reasoning as the second paragraph of the proof of Theorem 3.8, but taking into account that in this case the kind of monotonicity depends on the halfplane in which we are working (and $\mathfrak{m}_\lambda < 0$ for $\lambda \in (\mu_-, \lambda_-)$), it can be checked that $\lambda \mapsto \mathfrak{m}_\lambda(\omega)$ is strictly increasing on (μ_-, λ_-) for all $\omega \in \Omega$. Consequently, $\mathfrak{m}_{\lambda_-}(\omega) = \lim_{\lambda \uparrow \lambda_-} \mathfrak{m}_\lambda(\omega)$ defines a lower semicontinuous τ_{λ_-}-equilibrium, since it is the limit of an increasing family of continuous τ_λ-equilibria. Since $\mathfrak{m}_{\lambda_-} > \mathfrak{m}_\lambda > \mathfrak{l}_\lambda > \mathfrak{l}_{\lambda_-}$ for $\lambda \in (\mu_-, \lambda_-)$ and $\mathcal{M}_0$ and $\mathcal{M}^l_{\lambda_-}$ are the unique τ_{λ_-}-minimal sets, Proposition 1.32 ensures that the τ_{λ_-}-minimal set (1.9) provided by $\mathfrak{m}_{\lambda_-}$ is $\mathcal{M}_0$ and that $\mathfrak{m}_{\lambda_-}$ vanishes at the residual set of its continuity points.

Combining the previously obtained information, we observe that: the two hyperbolic τ_λ-minimal sets $\mathcal{N}_\lambda < \mathcal{M}_0$ for $\lambda < \lambda_-$ close to λ_- collide (on a residual set subset of Ω) at λ_- and the two hyperbolic minimal sets $\mathcal{M}_0 < \mathcal{M}^u_\lambda$ for $\lambda > \lambda_+$ close to λ_+ collide (on a residual set subset of Ω) at λ_+, giving rise to a unique nonhyperbolic minimal set $\mathcal{M}_0$ on the interval $[\lambda_-, \lambda_+]$. This is the generalized local transcritical bifurcation of minimal sets around $\mathcal{M}_0$ mentioned in the statement. In the case of $\lambda_- = \lambda_+$, it is a classical transcritical bifurcation.

Now, we define $\mathfrak{m}_{\mu_-}(\omega) = \lim_{\lambda\downarrow\mu_-} \mathfrak{m}_\lambda(\omega)$, which is an upper semicontinuous τ_{μ_-}-equilibrium since it is the (bounded from below) limit of a decreasing family of continuous τ_λ-equilibria, and note that $\mathfrak{m}_{\mu_-}(\omega) < 0$ for all $\omega \in \Omega$. Analogously, $\mathfrak{l}_{\mu_-}(\omega) = \lim_{\lambda\downarrow\mu_-} \mathfrak{l}_\lambda(\omega)$ is a lower semicontinuous τ_{μ_-}-equilibrium (see Proposition 3.18(ii)). Then, $\mathfrak{m}_{\mu_-}(\omega) \geq \mathfrak{l}_{\mu_-}(\omega)$ for all $\omega \in \Omega$, since $\mathfrak{m}_\lambda > \mathfrak{l}_\lambda$ for $\lambda \in (\mu_-, \lambda_-)$. It follows from the definition of μ_- and from Theorem 1.39 that the τ_{μ_-}-minimal set $\mathcal{M}^l_{\mu_-}$ determined from $\mathfrak{l}_{\mu_-}$ by (1.9) is nonhyperbolic. Since Theorem 2.11 ensures that there cannot exist any other τ_{μ_-}-minimal set apart from $\mathcal{M}^l_{\mu_-} < \mathcal{M}_0$ to not contradict the nonhyperbolic character of $\mathcal{M}^l_{\mu_-}$, the fact that $\mathfrak{m}_{\mu_-} < 0$ ensures that the τ_{μ_-}-minimal set given by (1.9) for $\mathfrak{m}_{\mu_-}$ is $\mathcal{M}^l_{\mu_-}$. So, Proposition 1.32 ensures that $\mathfrak{m}_{\mu_-}$ and $\mathfrak{l}_{\mu_-}$ coincide at the residual set of their common continuity points, giving rise to the nonhyperbolic τ_{μ_-}-minimal set $\mathcal{M}^l_{\mu_-}$. That is, $\mathcal{M}^l_{\mu_-} < \mathcal{N}_\lambda < \mathcal{M}_0$ for $\lambda \in (\mu_-, \lambda_-)$; and $\mathcal{M}^l_{\mu_-} < \mathcal{M}_0$ are the unique τ_{μ_-}-minimal sets.

(iv) Let $\lambda < \mu_-$ and let us check that $\mathcal{A}_\lambda = \mathcal{M}_0$. Since $\mathcal{M}_0$ is an attractive hyperbolic τ_λ-copy of the base, it suffices to check that it is the unique τ_λ-minimal set and apply Corollary 1.58(iii). Note first that Proposition 3.19(iv) ensures that there are no τ_λ-minimal sets above $\mathcal{M}_0$ for $\lambda < \mu_- < \lambda_- \leq \lambda_+$. Let us take a τ_λ-minimal set $\mathcal{N}_\lambda \leq \mathcal{M}_0$ and prove that $\mathcal{N}_\lambda = \mathcal{M}_0$. We fix $(\omega_0, x_0) \in \mathcal{N}_\lambda$. Since $\mathfrak{l}_{\mu_-} \leq \mathfrak{m}_{\mu_-}$ are semicontinuous strong τ_λ-subequilibria (see Proposition 3.17(i)) which coincide at their continuity points, Proposition 1.25 provides $s > 0$ and $e > 0$ such that $\mathfrak{m}_{\mu_-}(\omega_0) + e < v_\lambda(s, \omega_0{\cdot}(-s), \mathfrak{l}_{\mu_-}(\omega_0{\cdot}(-s))) \leq v_\lambda(s, \omega_0{\cdot}(-s), \mathfrak{l}_\lambda(\omega_0{\cdot}(-s))) = \mathfrak{l}_\lambda(\omega_0) \leq x_0$, where the second inequality follows from the monotonicity and from $\mathfrak{l}_{\mu_-} < \mathfrak{l}_\lambda$. So, $x_0 > \mathfrak{m}_{\mu_-}(\omega) = \lim_{\lambda\downarrow\mu_-} \mathfrak{m}_\lambda(\omega)$, and hence there exists $\xi_1 > \mu_-$ such that $\mathfrak{m}_{\xi_1}(\omega_0) < x_0$. We take any $\xi \in (\xi_1, \lambda_-)$ and apply Proposition 1.26 to the continuous family of strong τ_λ-subequilibria $\mathfrak{m}_\mu$ with $\mu \in [\xi_1, \xi]$ to conclude that there exists $s_\xi > 0$ such that $\mathfrak{m}_\xi(\omega_0{\cdot}s) \leq v_\lambda(s, \omega_0, \mathfrak{m}_{\xi_1}(\omega_0)) < v_\lambda(s, \omega_0, x_0)$ for all $s \geq s_\xi$. Since $\mathcal{N}_\lambda$ is the $\boldsymbol{\omega}$-limit set for τ_λ of (ω_0, x_0), given any $(\omega_1, x_1) \in \mathcal{N}_\lambda$, there exists $(t_n) \uparrow \infty$ such that $\omega_1 = \lim_{n\to\infty} \omega_0{\cdot}t_n$ and $x_1 = \lim_{n\to\infty} v_\lambda(t_n, \omega_0, x_0)$, and hence $\mathfrak{m}_\xi(\omega_1) = \lim_{n\to\infty} \mathfrak{m}_\xi(\omega_0{\cdot}t_n) \leq \lim_{n\to\infty} v_\lambda(t_n, \omega_0, x_0) = x_1$. In particular, we get $\mathfrak{m}_{\lambda_-}(\omega_1) = \lim_{\xi\uparrow\lambda_-} \mathfrak{m}_\xi(\omega_1) \leq x_1 \leq 0$. Hence, $(\mathcal{N}_\lambda)_\omega = \{0\}$ for all the points ω of the residual set at which $\mathfrak{m}_{\lambda_-}$ coincides with 0, which yields $\mathcal{N}_\lambda = \mathcal{M}_0$. This completes the proof. Note that a local saddle-node bifurcation of minimal sets occurs at μ_- around $\mathcal{M}^l_{\mu_-}$, due to the collision of $\mathcal{M}^l_\lambda < \mathcal{N}_\lambda$ on a residual subset of Ω as $\lambda \downarrow \mu_-$.

Let us check the lower discontinuity of $\{\mathcal{A}_\lambda\}$ as $\lambda \to \mu_-$. We take a sequence $\lambda_n \uparrow \mu_-$ and assume for contradiction the existence of $(\omega_n, x_n) \in \mathcal{A}_{\lambda_n}$ such that $\lim_{n\to\infty}(\omega_n, x_n) = (\omega, \mathfrak{l}_{\mu_-}(\omega))$ for a point $\omega \in \Omega$. Since $\mathcal{A}_{\lambda_n} = \Omega \times \{0\}$ for all $n \in \mathbb{N}$, $x_n = 0$ for all $n \in \mathbb{N}$, leading to $\mathfrak{l}_{\mu_-}(\omega) = 0$, a contradiction. The characterization of [21, Lemma 3.2(2)] shows the asserted lower discontinuity. □

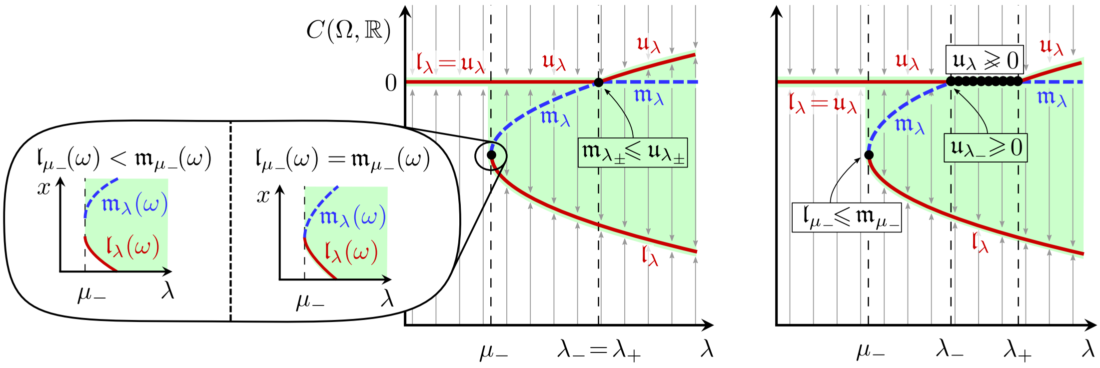


Figure 3.4: Local saddle-node and transcritical bifurcation diagrams described in Proposition 3.19 and Theorem 3.21. The left panel corresponds to $\lambda_- = \lambda_+$ and the right one to $\lambda_- < \lambda_+$. The solid red curves represent the families of attractive hyperbolic solutions of the λ-parametric family (3.9): $\mathfrak{l}_\lambda$ for $\lambda \neq \lambda_0$ and $\mathfrak{u}_\lambda$ for $\lambda \notin [\lambda_-, \lambda_+]$. The blue curve represents the family of repulsive hyperbolic solutions of (3.9): $\mathfrak{m}_\lambda$ for $\lambda \in (\mu_-, \lambda_-) \cup (\lambda_+, \infty)$. In both diagrams, as in Figure 3.1, a large black point with abscissa μ_- represents the complex possibilities which arise for the collision of $\mathfrak{l}_\lambda$ and $\mathfrak{m}_\lambda$ as $\lambda \downarrow \mu_-$, which is partly explained in the left zoom: see the caption of Figure 3.1. The situation is analogous for λ_-, with the collision of $\mathfrak{m}_\lambda$ and $\mathfrak{u}_\lambda \equiv 0$ as $\lambda \uparrow \lambda_-$; and for λ_+, with the collision of $\mathfrak{m}_\lambda \equiv 0$ and $\mathfrak{u}_\lambda$ as $\lambda \downarrow \lambda_+$. In the second diagram the large black points at 0 for $\lambda \in (\lambda_-, \lambda_+]$ represent the fact that $\mathfrak{u}_\lambda$ is not identically equal to 0. This may also be the situation at λ_-. The hyperbolic minimal sets are given by the graphs of the curves $\mathfrak{l}_\lambda$, $\mathfrak{m}_\lambda$ and $\mathfrak{u}_\lambda$ whenever they are hyperbolic. A nonhyperbolic minimal set $\mathcal{M}^l_{\mu_-}$ exists for μ_-, lying in the region delimited by the graphs of $\mathfrak{l}_{\mu_-}$ and $\mathfrak{m}_{\mu_-}$, and with a possibly highly complex shape. For $\lambda \in [\lambda_-, \lambda_+]$, $\mathcal{M}_0$ is nonhyperbolic. And no more τ_λ-minimal sets exist for any λ. As in Figure 3.1, the green-shadowed area represents the global attractor $\mathcal{A}_\lambda$, and the light grey arrows just try to depict the attracting and repelling properties of $\mathfrak{l}_\lambda$, $\mathfrak{m}_\lambda$ and $\mathfrak{u}_\lambda$.

Theorem 3.22 (Global generalized pitchfork bifurcation). *Let (Ω, σ) be minimal. Let $\mathfrak{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy* **d1**, **d2**$_{\lambda x}$, **d3**, **d4** *and* **d5**. *Let $[-\lambda_+, -\lambda_-]$ be the dynamical spectrum of $\mathfrak{h}_x$ on $\mathcal{M}_0 = \Omega \times \{0\}$, with $\lambda_- \le \lambda_+$. In addition to the conclusions of Proposition 3.19, if $\mu_+ = \lambda_+$, $\lambda_- < \lambda_+$ and $\mu_- \in [\lambda_-, \lambda_+)$, then*

(i) *τ_λ-admits two minimal sets $\mathcal{M}^l_\lambda < \mathcal{M}_0$ for $\lambda \in (\mu_-, \lambda_+]$, where $\mathcal{M}^l_\lambda$ is hyperbolic attractive and given by the graph of $\mathfrak{l}_\lambda$.*

(ii) *$\mathcal{M}_0$ is the unique τ_λ-minimal set if $\lambda < \mu_-$.*

(iii) *$\mathcal{A}_\lambda = \mathcal{M}_0$ if $\lambda < \lambda_-$.*

That is, a generalized pitchfork bifurcation of minimal sets around $\mathcal{M}_0$ on $[\lambda_-, \lambda_+]$ arises, with μ_-, λ_- and λ_+ as bifurcation points: the number of τ_λ-minimal sets changes at μ_- and λ_+ and its hyperbolic structure changes at λ_-.

A global bifurcation diagram which is symmetric to the one described with respect to the horizontal axis arises if $\mu_- = \lambda_+$, $\lambda_- < \lambda_+$ and $\mu_+ \in [\lambda_-, \lambda_+)$.

The following technical lemma is needed in our proof of the theorem.

Lemma 3.23. *Let (Ω, σ) be minimal. Let $\mathfrak{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy* **d1**, **d2**$_{\lambda x}$, **d3**, **d4** *and* **d5**. *Assume that the flow τ_0 defined by $(3.5)_0$ admits exactly two minimal sets $\mathcal{M}^l < \mathcal{M}_0 = \Omega \times \{0\}$ (resp. $\mathcal{M}^u > \mathcal{M}_0 = \Omega \times \{0\}$), with $\mathcal{M}^l$ (resp. $\mathcal{M}^u$) hyperbolic attractive. Assume also that $\int_\Omega \mathfrak{h}_x(\omega, 0)\, dm \neq 0$ for an $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$. Then,*

$\mathcal{M}^l = \{\mathfrak{l}_0\}$ (resp. $\mathcal{M}^u = \{\mathfrak{u}_0\}$), $\int_\Omega \mathfrak{h}_x(\omega, \mathfrak{u}_0(\omega))\, dm < 0$ (resp. $\int_\Omega \mathfrak{h}_x(\omega, \mathfrak{l}_0(\omega))\, dm < 0$), and

$$\mathfrak{m}_0(\omega) = \sup\{x \in \mathbb{R}\colon\ \lim_{t\to\infty} (v_0(t,\omega,x) - \mathfrak{l}_0(\omega{\cdot}t)) = 0\} \in (\mathfrak{l}_0(\omega), 0]$$
$$(\textit{resp. } \mathfrak{m}_0(\omega) = \inf\{x \in \mathbb{R}\colon\ \lim_{t\to\infty} (v_0(t,\omega,x) - \mathfrak{u}_0(\omega{\cdot}t)) = 0\} \in [0, \mathfrak{u}_0(\omega))) \,,$$

defines a lower (resp. upper) semicontinuous τ_0-equilibrium which vanishes at the residual set of its continuity points and such that $\int_\Omega \mathfrak{h}_x(\omega, \mathfrak{m}_0(\omega))\, dm > 0$.

Proof. Proposition 2.14(ii) ensures the nonhyperbolicity of $\mathcal{M}_0$. We reason in the case of $\mathcal{M}^l < \mathcal{M}_0$. Theorem 3.10 ensures that the global ξ-bifurcation diagram for

$$x' = \mathfrak{h}(\omega{\cdot}t, x) + \xi \tag{3.11}$$

is described by Theorem 3.8 and that $\xi_- = 0$ is the lower bifurcation value. In particular, $\mathcal{M}^l = \{\mathfrak{l}_0\}$ and it is hyperbolic attractive. In addition, since $\mathfrak{g} \mapsto \int_\Omega \mathfrak{g}(\omega, 0)\, dm$ defines a τ-ergodic measure on any τ-invariant compact set containing $\mathcal{M}_0$ (see Theorem 1.36(ii)), our hypotheses preclude the situation (b) of Proposition 3.9. Hence, situation (a) holds, and this means that $\int_\Omega \mathfrak{h}_x(\omega, \mathfrak{u}_0(\omega))\, dm < 0$ and that $\int_\Omega \mathfrak{h}_x(\omega, \mathfrak{n}_0(\omega))\, dm > 0$, where $\mathfrak{n}_0(\omega) \le 0$ is the lower semicontinuous equilibrium for $(3.11)_0$ (i.e., the τ_0-equilibrium) given by the limit as $\xi \downarrow 0$ of the maps $\mathfrak{n}_\xi < 0$ determining the repulsive hyperbolic copies of the base for $(3.11)_\xi$, which satisfies $\mathfrak{n}_0(\omega) > \mathfrak{l}_0(\omega)$ for all $\omega \in \Omega$. Proposition 1.32 ensures that $\mathfrak{n}_0(\omega) = 0$ at the residual set of its continuity points.

It remains to check that $\mathfrak{n}_0$ coincides with the map $\mathfrak{m}_0$ of the statement. Clearly, $\mathfrak{n}_0 \ge \mathfrak{m}_0$. So, it suffices to take $x < \mathfrak{n}_0(\omega_0) = \lim_{\xi\downarrow 0} \mathfrak{n}_\xi(\omega_0)$ and check that $\lim_{t\to\infty}(v_0(t,\omega,x) - \mathfrak{l}_0(\omega{\cdot}t)) = 0$. We take $\xi > 0$ such that $x < \mathfrak{n}_\xi(\omega_0)$. Let $\bar v_\xi(t,\omega,x)$ represent the cocycle of solutions of $(3.11)_\xi$. Then, $v_0(t,\omega_0,x) = \bar v_0(t,\omega_0,x) < \bar v_\xi(t,\omega_0,x) < \bar v_\xi(t,\omega_0,\mathfrak{n}_\xi(\omega_0)) = \mathfrak{n}_\xi(\omega_0{\cdot}t)$ for $t > 0$, and hence the $\boldsymbol{\omega}$-limit set for τ_0 of (ω_0, x) does not contain $\mathcal{M}_0$. Hence, it contains $\{\mathfrak{l}_0\}$. The assertion follows from the uniform (exponential) stability of $\{\mathfrak{l}_0\}$. □

Proof of Theorem 3.22. Notice that, if $\lambda \in (\mu_-, \lambda_+]$, then there exist only two τ_λ-minimal sets $\mathcal{M}^l_\lambda = \{\mathfrak{l}_\lambda\} < \mathcal{M}_0$: otherwise Theorem 2.11 would contradict the nonhyperbolicity of $\mathcal{M}_0$. So, (i) is proved. Our goal will be two prove that, for all $\lambda < \mu_-$, the τ_λ-equilibrium $\mathfrak{l}_\lambda$ vanishes at one of its continuity points. Then, Proposition 2.17(i) ensures that $\mathcal{M}_0$ is the lower τ_λ-minimal set, which combined with Proposition 3.19(iv) ensures that $\mathcal{M}_0$ is the unique τ_λ-minimal set, which is assertion (ii). Finally, Corollary 1.58(iii) will ensure that $\mathcal{A}_\lambda = \mathcal{M}_0$ for $\lambda < \lambda_-$, which is assertion (iii).

So, let us check that, for all $\lambda < \mu_-$, the τ_λ-equilibrium $\mathfrak{l}_\lambda$ vanishes at some continuity point. We will check it in two different cases. First, let us assume that there exists $\omega_0 \in \Omega$ such that $\mathfrak{l}_{\mu_-}(\omega_0) = 0$. The nonincreasing character of $\lambda \mapsto \mathfrak{l}_\lambda(\omega)$ for all $\omega \in \Omega$ and its non positiveness given by Proposition 3.18(i) and (ii) ensure that $\mathfrak{l}_\lambda(\omega_0) = 0$ for all $\lambda < \mu_-$, and Proposition 1.34 ensures that ω_0 is a continuity point of $\mathfrak{l}_\lambda$ for all $\lambda < \mu_-$, as we wanted to see.

So, we work in the case that $\mathfrak{l}_{\mu_-}(\omega) < 0$ for all $\omega \in \Omega$. Then, Proposition 1.32 ensures the existence of a τ_{μ_-}-minimal set $\mathcal{M}^l_{\mu_-} < \mathcal{M}_0$ described by (1.9) for $\mathfrak{l}_{\mu_-}$. Notice that $\mathcal{M}^l_{\mu_-}$ is nonhyperbolic: otherwise the definition of μ_- would be contradicted by hyperbolic continuation (see Theorem 1.39) and Proposition 2.17(ii).

Let us take $\lambda \in (\mu_-, \lambda_+]$. Since the dynamical spectrum of $\mathfrak{h}_x + \lambda$ on $\mathcal{M}_0$ is the nondegenerate interval $[-\lambda_+ + \lambda, -\lambda_- + \lambda]$, Lemma 3.23 (applied to τ_λ instead of to τ_0) shows that

$$\mathfrak{m}_\lambda(\omega) = \sup\big\{x \in \mathbb{R} \,\big|\, \lim_{t\to\infty}(v_\lambda(t,\omega,x) - \mathfrak{l}_\lambda(\omega{\cdot}t)) = 0\big\} \tag{3.12}$$

is a lower semicontinuous τ_λ-equilibrium $\mathfrak{m}_\lambda\colon \Omega \to (-\infty, 0]$ which vanishes at its continuity points, with $\mathfrak{m}_\lambda > \mathfrak{l}_\lambda$. In addition, it ensures that $\int_\Omega(\mathfrak{h}_x(\omega,\mathfrak{m}_\lambda(\omega)) + \lambda)\,dm > 0$ if $\int_\Omega(\mathfrak{h}_x(\omega,0)+\lambda)\,dm < 0$ for some $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$. In this case, since m is ergodic and $\{\omega\in\Omega \mid \mathfrak{m}_\lambda(\omega) < 0\}$ is σ-invariant, $\mathfrak{m}_\lambda(\omega) < 0$ for m-a.e. $\omega \in \Omega$: otherwise $\int_\Omega(\mathfrak{h}_x(\omega,\mathfrak{m}_\lambda(\omega)) + \lambda)\,dm = \int_\Omega(\mathfrak{h}_x(\omega,0)+\lambda)\,dm$, a contradiction. Let us check that the map $\lambda \mapsto \mathfrak{m}_\lambda(\omega)$ is nondecreasing on $(\mu_-, \lambda_+]$ for all $\omega \in \Omega$. For any $\lambda \in (\mu_-, \lambda_+]$, since $\mathcal{M}^l_\lambda$ is the only τ_λ-minimal set strictly below $\mathcal{M}_0$, (3.12) guarantees that $x < \mathfrak{m}_\lambda(\omega)$ if and only if $\mathcal{M}^l_\lambda = \{\mathfrak{l}_\lambda\}$ is the $\boldsymbol{\omega}$-limit for τ_λ of (ω,x). We take $\mu_- < \lambda_1 < \lambda_2 \le \lambda_+$, $\omega\in\Omega$ and $x < \mathfrak{m}_{\lambda_1}(\omega)$. Then, a standard comparison argument ensures that $v_{\lambda_2}(t,\omega,x) < v_{\lambda_1}(t,\omega,x) < v_{\lambda_1}(t,\omega,\mathfrak{m}_{\lambda_1}(\omega)) = \mathfrak{m}_{\lambda_1}(\omega{\cdot}t)$ for all $t > 0$, which precludes the possibility of $\mathcal{M}_0$ being contained in the $\boldsymbol{\omega}$-limit set for τ_{λ_2} of (ω,x), since $\lim_{t\to\infty}(v_{\lambda_1}(t,\omega,x) - \mathfrak{l}_{\lambda_1}(\omega{\cdot}t)) = 0$. Hence, $x < \mathfrak{m}_{\lambda_2}(\omega)$, and this proves the nondecreasing character of $\lambda\mapsto\mathfrak{m}_\lambda(\omega)$ for all $\omega\in\Omega$. Since $\mathfrak{l}_{\lambda_+} \le \mathfrak{l}_\lambda < \mathfrak{m}_\lambda$ for all $\lambda\in(\mu_-,\lambda_+]$, $\mathfrak{m}_{\mu_-}(\omega) = \lim_{\lambda\downarrow\mu_-}\mathfrak{m}_\lambda(\omega)$ defines an m-measurable τ_{μ_-}-equilibrium for every $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$. Hence, Theorem 2.13(ii) and the monotonicity of $\lambda\mapsto\mathfrak{m}_\lambda$ ensure that $\mathfrak{l}_{\mu_-}(\omega) \le \mathfrak{m}_{\mu_-}(\omega) \le \mathfrak{m}_\lambda(\omega)$ for all $\omega\in\Omega$ if $\lambda\in(\mu_-,\lambda_+]$.

Now, let us check that there exists $\omega_0\in\Omega$ such that $\mathfrak{m}_{\mu_-}(\omega_0) \le \bar{\mathfrak{u}}(\omega_0)$, where $\bar{\mathfrak{u}}$ is the upper τ_{μ_-}-equilibrium of $\mathcal{M}^l_{\mu_-}$ (recall (1.7)). Since $\mathcal{M}^l_{\mu_-}$ is nonhyperbolic, Theorems 1.40 and 1.36(iii) ensure that there exists $m\in\mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$ and an m-measurable τ_{μ_-}-equilibrium $\bar{\mathfrak{b}}\colon\Omega\to\mathbb{R}$ with graph contained in $\mathcal{M}^l_{\mu_-}$ such that $\int_\Omega(\mathfrak{h}_x(\omega,\bar{\mathfrak{b}}(\omega)) + \mu_-)\,dm \ge 0$. Theorem 2.15 ensures that $\int_\Omega(\mathfrak{h}_x(\omega,0)+\mu_-)\,dm < 0$, so $\int_\Omega(\mathfrak{h}_x(\omega,0)+\lambda)\,dm < 0$ for $\lambda \ge \mu_-$ close enough. So, as seen in the previous paragraph, $\mathfrak{m}_\lambda(\omega) < 0$ for m-a.e. $\omega\in\Omega$ for these values of the parameter, and hence $\mathfrak{m}_{\mu_-}(\omega) < 0$ for m-a.e. $\omega\in\Omega$. Assume for contradiction that $\mathfrak{m}_{\mu_-}(\omega) > \bar{\mathfrak{u}}(\omega)$ for all $\omega\in\Omega$, and hence that $\mathfrak{m}_{\mu_-}(\omega) > \bar{\mathfrak{b}}(\omega)$ for all $\omega\in\Omega$. Then, Theorem 2.9 applied to the τ_{μ_-}-equilibria 0, $\mathfrak{m}_{\mu_-}$ and $\bar{\mathfrak{b}}$ ensures that $\int_\Omega(\mathfrak{h}_x(\omega,\bar{\mathfrak{b}}(\omega))+\mu_-)\,dm < 0$, which is not the case. So, there exists $\omega_0\in\Omega$ such that $\mathfrak{m}_{\mu_-}(\omega_0)\le\bar{\mathfrak{u}}(\omega_0)$.

Finally, let us fix $\lambda < \mu_-$ and check that $\mathfrak{l}_\lambda$ vanishes at one of its continuity points. As mentioned before, this completes the proof of the theorem. Note that Proposition 1.32 ensures that $\mathfrak{l}_{\mu_-}$ and $\bar{\mathfrak{u}}$ coincide on the residual set of its common continuity points. Then, Proposition 3.17(ii) ensures that Proposition 1.25 can be applied to show the existence of $s^* > 0$ such that $\bar{\mathfrak{u}}(\omega) < v_\lambda(s^*,\omega{\cdot}(-s^*),\mathfrak{l}_{\mu_-}(\omega{\cdot}(-s^*))) \le \mathfrak{l}_\lambda(\omega)$ for all $\omega\in\Omega$, where the last inequality follows from the flow monotonicity and the nonincreasing character of $\lambda\mapsto\mathfrak{l}_\lambda(\omega)$ given by Proposition 3.18(ii). We take $\omega_0\in\Omega$ with $\mathfrak{m}_{\mu_-}(\omega_0)\le\bar{\mathfrak{u}}(\omega_0)$. Then, $\lim_{\lambda\downarrow\mu_-}\mathfrak{m}_\lambda(\omega_0) = \mathfrak{m}_{\mu_-}(\omega_0) \le \bar{\mathfrak{u}}(\omega_0) < \mathfrak{l}_\lambda(\omega_0)$. Therefore, there exists $\xi > \mu_-$ close enough to get $\mathfrak{m}_\xi(\omega_0) < \mathfrak{l}_\lambda(\omega_0)$. Since $\xi > \lambda$, a standard comparison argument and the flow monotonicity show that $\mathfrak{m}_\xi(\omega_0{\cdot}t) \le v_\lambda(t,\omega_0,\mathfrak{m}_\xi(\omega_0)) < v_\lambda(t,\omega_0,\mathfrak{l}_\lambda(\omega_0)) = \mathfrak{l}_\lambda(\omega_0{\cdot}t)$ for all $t>0$. Let ω_1 be a common continuity point of $\mathfrak{m}_\xi$ and $\mathfrak{l}_\lambda$. As (Ω,σ) is minimal, there exists a sequence $(t_n)\uparrow\infty$ such that $\omega_0{\cdot}t_n\to\omega_1$ as $n\to\infty$. Since $\mathfrak{m}_\xi$ vanishes at ω_1, $0 = \mathfrak{m}_\xi(\omega_1) = \lim_{n\to\infty}\mathfrak{m}_\xi(\omega_0{\cdot}t_n) \le \lim_{n\to\infty}\mathfrak{l}_\lambda(\omega_0{\cdot}t_n) = \mathfrak{l}_\lambda(\omega_1) \le 0$. That is, $\mathfrak{l}_\lambda(\omega_1) = 0$, and the proof is complete. □

Theorem 3.24. *Let (Ω, σ) be minimal. Let $\mathfrak{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy* **d1**, **d2**$_{\lambda x}$, **d3**, **d4** *and* **d5**. *Then, the bifurcation diagrams of Theorems 3.20, 3.21 and 3.22 exhaust all the possibilities of (3.9). Moreover, the bifurcation diagram of Theorem 3.22 is only possible if $\lambda_- < \lambda_+$, where $[-\lambda_+, -\lambda_-]$ is the dynamical spectrum of $\mathfrak{h}_x$ on $\mathcal{M}_0 = \Omega \times \{0\}$.*

Proof. The three theorems exhaust all the possible configurations of the parameters $\lambda_- \le \lambda_+$ and $\mu_-, \mu_+ \le \lambda_+$: global classical pitchfork bifurcation if $\mu_- = \mu_+ = \lambda_+$ (Theorem 3.20); local saddle-node and transcritical bifurcation if either $\mu_+ = \lambda_+$ and $\mu_- < \lambda_-$, or $\mu_- = \lambda_+$ and $\mu_+ < \lambda_-$, which is classical if $\lambda_- = \lambda_+$ (Theorem 3.21); and generalized pitchfork bifurcation if $\lambda_- < \lambda_+$ and either $\mu_+ = \lambda_+$ and $\mu_- \in [\lambda_-, \lambda_+)$, or $\mu_- = \lambda_+$ and $\mu_+ \in [\lambda_-, \lambda_+)$ (Theorem 3.22). □

The diagrams in Figure 3.4 (resp. in Figure 3.3) depict the two cases of local saddle-node and transcritical bifurcations (resp. classical pitchfork bifurcation) described in detail in Proposition 3.19 and Theorem 3.21 (resp. Theorem 3.20) when the dynamical spectrum of $\mathfrak{h}_x$ on $\mathcal{M}_0$ is a point (the first one) and a band (the second one). The diagram in Figure 3.5 depicts the case of a generalized pitchfork bifurcation, described in Theorem 3.22, which requires the dynamical spectrum of $\mathfrak{h}_x$ on $\mathcal{M}_0$ to be a band (and hence never happens in the autonomous case, which is uniquely ergodic). Notice that we have examples of one, two, and three bifurcation points. In particular, the band case of local saddle-node and transcritical bifurcation diagram exhibits three of these values.

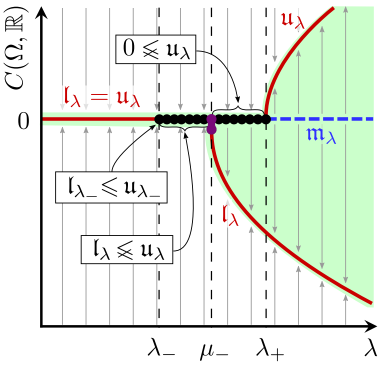


Figure 3.5: The generalized pitchfork bifurcation diagram described in Proposition 3.19 and Theorem 3.22. The possible existence of one or two nonhyperbolic minimal sets at μ_- (a result not included in this document) is depicted by a solid-filled purple eight. See Figures 3.1, 3.3 and 3.4 to understand the meaning of the remaining elements.

There are simple autonomous examples of classical pitchfork bifurcation (as $x' = -x^3 + \lambda x$, with $\lambda_\pm = 0$ as bifurcation point) and local saddle-node and transcritical bifurcation (as $x' = -x^3 \pm 2x^2 + \lambda x$, with $\lambda_0 = -1$ as local saddle-node bifurcation point and $\lambda_\pm = 0$ as local classical transcritical bifurcation point; the two signs of the second-order term correspond to the two possible sings for the curve $\mathfrak{m}_\lambda$ of nonhyperbolic critical points). We will go deeper in this matter in Sections 3.3.2 and 3.4, where we will show that all the possibilities realize for suitable families (3.9).

We complete this subsection by describing a simple situation in which the bifurcation diagram for (3.9) is that of Theorem 3.20: a global pitchfork bifurcation.

Proposition 3.25. *Let (Ω, σ) be minimal. Let $\mathfrak{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy* **d1**, **d2**$_{\lambda x}$, **d3** *and* **d5**. *Let $[-\lambda_+, -\lambda_-]$ be the dynamical spectrum of $\mathfrak{h}_x$ on $\mathcal{M}_0 = \Omega \times \{0\}$, with $\lambda_- \le \lambda_+$. Then,*

(i) *if $\mathfrak{h}_{xx}(\omega, 0) \ge 0$ (resp. $\mathfrak{h}_{xx}(\omega, 0) \le 0$) for all $\omega \in \Omega$, then $\mathfrak{l}_\lambda$ (resp. $\mathfrak{u}_\lambda$) takes the value 0 at its continuity points for all $\lambda < \lambda_+$.*

(ii) *If* $\mathfrak{h}$ *also satisfies* **d4** *and* $\mathfrak{h}_{xx}(\omega,0)=0$ *for all* $\omega\in\Omega$*, then the bifurcation diagram is that described in Theorem* 3.20.

Proof. (i) Remark 2.7 ensures that $x\mapsto\mathfrak{h}_x(\omega,x)$ is concave for all $\omega\in\Omega$, and thus $x\mapsto\mathfrak{h}_{xx}(\omega,x)$ is nonincreasing for all $\omega\in\Omega$. Consequently, if $\mathfrak{h}_{xx}(\omega,0)\ge 0$, then $\mathfrak{h}_{xx}(\omega,x)\ge 0$ for all $x\le 0$. We assume for contradiction the existence of $\lambda<\lambda_+$ such that $\mathfrak{l}_\lambda(\omega)<0$ for all $\omega\in\Omega$. Taylor's Theorem, **d5** and $\mathfrak{h}_{xx}(\omega,x)\ge 0$ for all $x\le 0$ ensure that there exist $x_\omega\in[\mathfrak{l}_\lambda(\omega),0]$ such that

$$\frac{\mathfrak{l}'_\lambda(\omega)}{\mathfrak{l}_\lambda(\omega)}=\frac{\mathfrak{h}(\omega,\mathfrak{l}_\lambda(\omega))}{\mathfrak{l}_\lambda(\omega)}+\lambda=\mathfrak{h}_x(\omega,0)+\frac{\mathfrak{l}_\lambda(\omega)}{2}\,\mathfrak{h}_{xx}(\omega,x_\omega)+\lambda\le\mathfrak{h}_x(\omega,0)+\lambda\,,$$

and hence Birkhoff's Ergodic Theorem 1.10 ensures that $0=\int_\Omega(\mathfrak{l}'_\lambda(\omega)/\mathfrak{l}_\lambda(\omega))\,dm\le\int_\Omega(\mathfrak{h}_x(\omega,0)+\lambda)\,dm$ for all $m\in\mathfrak{M}_{\rm erg}(\Omega,\sigma)$. So, $-\lambda\le\int_\Omega\mathfrak{h}_x(\omega,0)\,dm$ for all $m\in\mathfrak{M}_{\rm erg}(\Omega,\sigma)$, contradicting the existence of m_+ such that $\int_\Omega\mathfrak{h}_x(\omega,0)\,dm_+=-\lambda_+<-\lambda$ (see Proposition 3.19(i)). Hence, there exists $\omega_0\in\Omega$ such that $\mathfrak{l}_\lambda(\omega_0)=0$ and Proposition 1.34 proves the claim. The proof for $\mathfrak{u}_\lambda$ is analogous.

(ii) Property (i) and Proposition 3.19(iv) show that $\mu_-=\mu_+=\lambda_+$. Hence, the bifurcation diagram of Theorem 3.20 holds. □

3.3.2 Criteria for cubic polynomial equations with minimal (Ω,σ)

In this section, also with minimal base flow (Ω,σ), we provide several criteria in cubic polynomials (the simplest d-concave functions) that give rise to each of the three global bifurcation diagrams described in Theorems 3.20, 3.21 and 3.22. Let us consider families of cubic polynomial ordinary differential equations

$$x'=-\mathfrak{a}_3(\omega{\cdot}t)\,x^3+\mathfrak{a}_2(\omega{\cdot}t)\,x^2+(\mathfrak{a}_1(\omega{\cdot}t)+\lambda)\,x\,,\quad \omega\in\Omega\,, \tag{3.13}$$

where $\mathfrak{a}_i\in C(\Omega,\mathbb{R})$ for $i\in\{1,2,3\}$, $\mathfrak{a}_3$ is strictly positive and $\lambda\in\mathbb{R}$. It is easy to check that the function $\mathfrak{h}(\omega,x)=-\mathfrak{a}_3(\omega)\,x^3+\mathfrak{a}_2(\omega)\,x^2+\mathfrak{a}_1(\omega)\,x$ satisfies **d1**, $\mathbf{d2}_{\lambda x^2}$ (and hence $\mathbf{d2}_{\lambda x}$, $\mathbf{d2}_{\lambda}$ and **d2**, as proved in Proposition 3.3(i)-(iii)), **d3**, **d4** and **d5**. Then, Theorem 3.24 shows that the three possible bifurcation diagrams of (3.13) are those of Theorems 3.20, 3.21 and 3.22 and the conclusions of Proposition 3.19 are in force in this section. Our first goal is to describe conditions on the coefficients $\mathfrak{a}_i$ determining the specific diagram. The last subsection is devoted to explain how to get actual patterns satisfying the previously established conditions.

As in Section 3.3.1, $\tau_\lambda(t,\omega,x)=(\omega{\cdot}t,v_\lambda(t,\omega,x))$ repesents the local skewproduct flow induced by $(3.13)_\lambda$ on $\Omega\times\mathbb{R}$, and $\mathcal{A}_\lambda$ represents the global attractor of $(3.13)_\lambda$ with lower and upper equilibria $\mathfrak{l}_\lambda$ and $\mathfrak{u}_\lambda$. Notice that Theorem 1.36(i) ensures that the dynamical spectrum of $\mathfrak{h}_x$ on $\mathcal{M}_0=\Omega\times\{0\}$ (which is τ_λ-minimal for all $\lambda\in\mathbb{R}$) coincides with the dynamical spectrum $\mathrm{sp}(\mathfrak{a}_1)$ of $\mathfrak{a}_1$. This will be used hereafter without further mention.

The case of $\mathfrak{a}_1$ with continuous primitive

The notation for the different sets of functions is established in Definition 1.62, and the notion of dynamical spectrum of a real continuous map on Ω is given in Definition 1.13. Throughout this section, we assume that $\mathfrak{a}_1\in CP(\Omega,\mathbb{R})$. Since

Proposition 1.64(i) ensures that $CP(\Omega,\mathbb{R})\subseteq C_0(\Omega,\mathbb{R})$, the dynamical spectrum of $\mathfrak{a}_1$ is $\mathrm{sp}(\mathfrak{a}_1)=\{0\}$. With the notation of the previous section, this means that $\lambda_-=\lambda_+=0$, and hence in this case the bifurcation diagram of (3.13) fits either Theorem 3.20 or Theorem 3.21 and is given by the left panels in Figures 3.3 or 3.4 (or by the symmetric of this last one). Our goal is to provide conditions on $\mathfrak{a}_1$ (or, more precisely, on $e^{\mathfrak{b}}\mathfrak{a}_1$ for a continuous primitive $\mathfrak{b}$ of $\mathfrak{a}_1$) characterizing each one of these bifurcation possibilities. A key point to distinguish between the two diagrams is to count the number of τ_0-minimal sets: there is either one τ_0-minimal set in Theorem 3.20 or two τ_0-minimal sets in Theorem 3.21.

Proposition 3.27 provides the classification of the possibilities for (3.13) in this case. It is based on the previous bifurcation analysis made in Proposition 3.26 of

$$x'=-\mathfrak{a}_3(\omega{\cdot}t)\,x^3+(\mathfrak{a}_2(\omega{\cdot}t)+\xi)\,x^2\,,\quad \omega\in\Omega\,, \tag{3.14}$$

where $\mathfrak{a}_2,\mathfrak{a}_3\in C(\Omega,\mathbb{R})$ and $\mathfrak{a}_3>0$. Let $\check\tau_\xi$ the local skewproduct flow induced by $(3.14)_\xi$ on $\Omega\times\mathbb{R}$, let $\check v_\xi(t,\omega,x)$ be the cocycle of solutions of $(3.14)_\xi$, and let $\check{\mathfrak{l}}_\xi$ and $\check{\mathfrak{u}}_\xi$ stand for the lower and upper equilibria of the global attractor $\check{\mathcal{A}}_\xi$ of $(3.14)_\xi$, respectively.

Proposition 3.26. *Let (Ω,σ) be minimal. Let $\mathfrak{a}_i\in C(\Omega,\mathbb{R})$ for $i\in\{2,3\}$ and let $\mathfrak{a}_3>0$. Then, the $\check\tau_\xi$-minimal set $\mathcal{M}_0=\Omega\times\{0\}$ is nonhyperbolic for all $\xi\in\mathbb{R}$. In addition, if $\mathrm{sp}(\mathfrak{a}_2)=[-\xi_+,-\xi_-]$, with $\xi_-\le\xi_+$, then*

(i) *$\check\tau_\xi$ admits exactly two minimal sets $\mathcal{M}_0<\mathcal{M}^u_\xi$ respectively given by the graphs of 0 and $\check{\mathfrak{u}}_\xi$ for $\xi>\xi_+$, where $\mathcal{M}^u_\xi$ is hyperbolic attractive; and $\check{\mathfrak{u}}_\xi$ collides with 0 on a residual σ-invariant subset of Ω as $\xi\downarrow\xi_+$.*

(ii) *$\mathcal{M}_0$ is the unique $\check\tau_\xi$-minimal set for $\xi\in[\xi_-,\xi_+]$.*

(iii) *$\check\tau_\xi$ admits exactly two minimal sets $\mathcal{M}^l_\xi<\mathcal{M}_0$ respectively given by the graphs of $\check{\mathfrak{l}}_\xi$ and 0 for $\xi<\xi_-$, where $\mathcal{M}^l_\xi$ is hyperbolic attractive; and $\check{\mathfrak{l}}_\xi$ collides with 0 on a residual σ-invariant subset of Ω as $\xi\uparrow\xi_-$.*

Proof. First note that the linearized equation of $(3.14)_\xi$ around the τ_ξ-minimal set $\mathcal{M}_0$ is $z'=0$ for every $\xi\in\mathbb{R}$. That is, the dynamical spectrum of the linearized equation of $(3.14)_\xi$ around $\mathcal{M}_0$ is $\{0\}$ for all $\xi\in\mathbb{R}$, and hence Proposition 1.42 ensures that $\mathcal{M}_0$ is nonhyperbolic for all $\xi\in\mathbb{R}$.

Let us fix $\xi<\xi_+$. Let us check the existence of $\omega_0\in\Omega$ such that $\check v_\xi(t,\omega_0,x)$ is unbounded for any $x>0$, which ensures the absence of $\check\tau_\xi$-minimal sets above $\mathcal{M}_0$. Let $m_+\in\mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$ satisfy $\int_\Omega(\mathfrak{a}_2(\omega)+\xi_+)\,dm_+=0$. Birkhoff's Ergodic Theorem 1.10 provides $\Omega_0\subseteq\Omega$ with $m_+(\Omega_0)=1$ such that $\int_\Omega(\mathfrak{a}_2(\omega)+\xi)\,dm_+=\lim_{t\to-\infty}(1/t)\int_0^t(\mathfrak{a}_2(\omega{\cdot}s)+\xi)\,ds$ for all $\omega\in\Omega_0$. Then, $\sup_{t\le0}\int_0^t(\mathfrak{a}_2(\omega{\cdot}s)+\xi)\,ds=\infty$ for all $\omega\in\Omega_0$: otherwise $0=\int_\Omega(\mathfrak{a}_2(\omega)+\xi_+)\,dm_+>\int_\Omega(\mathfrak{a}_2(\omega)+\xi)\,dm_+=\lim_{t\to-\infty}(1/t)\int_0^t(\mathfrak{a}_2(\omega{\cdot}s)+\xi)\,ds\ge\lim_{t\to-\infty}(1/t)\sup_{r\le0}\int_0^r(\mathfrak{a}_2(\omega{\cdot}s)+\xi)\,ds=0$ for a point $\omega\in\Omega_0$, a contradiction. We take $\omega_0\in\Omega_0$. Let $\check u_\xi(t,\omega_0,x)$ solve $x'=(\mathfrak{a}_2(\omega_0{\cdot}t)+\xi)\,x^2$ with $\check u_\xi(0,\omega_0,x)=x$. That is, $\check u_\xi(t,\omega_0,x)=(1/x-\int_0^t(\mathfrak{a}_2(\omega_0{\cdot}s)+\xi)\,ds)^{-1}$ for all $x\neq0$. For $x>0$, let $\check\alpha_x=\sup\{t\le0\mid\int_0^t(\mathfrak{a}_2(\omega_0{\cdot}s)+\xi)\,ds=1/x\}\in\mathbb{R}$: it is easy to check that $\check u_\xi(t,\omega_0,x)$ is well defined on $(\check\alpha_x,0]$ and that $\lim_{t\downarrow\check\alpha_x}\check u_\xi(t,\omega_0,x)=\infty$. A standard comparison argument shows that $\check v_\xi(t,\omega_0,x)>\check u_\xi(t,\omega_0,x)$ if $t<0$ and $x>0$, from where the claim follows: $\check v_\xi(t,\omega_0,x)$ is unbounded as time decreases for any $x > 0$. Hence, there are no $\check\tau_\xi$-minimal sets above $\mathcal{M}_0$. Analogously, it can be proved that there are no $\check\tau_\xi$-minimal sets below $\mathcal{M}_0$ for $\xi > \xi_-$. In particular, $\check{\mathcal{M}}_0$ is the unique $\check\tau_\xi$-minimal set for $\xi \in (\xi_-, \xi_+)$.

Let us now fix $\xi > \xi_+$ and prove the existence of a $\check\tau_\xi$-minimal set $\mathcal{M}^u_\xi$ strictly above $\mathcal{M}_0$. Let $\omega_0 \in \Omega$ be fixed. Take $\alpha \in (0, \xi - \xi_+)$ with $\alpha < \sqrt{2r}$ (or equivalently $\alpha/2 < r/\alpha$) for $r = \max_{\omega\in\Omega} \mathfrak{a}_3(\omega) > 0$. Theorem 2.13(i) ensures that $\check v_\xi(t, \omega_0, 1/\alpha)$ is defined and bounded for $t \in [0, \infty)$. We will check below that $\check v_\xi(t, \omega_0, 1/\alpha)$ is bounded away from zero for $t \geq 0$, thus ensuring the existence of a $\check\tau_\xi$-minimal set $\mathcal{M}^u_\xi > \mathcal{M}_0$ contained in the $\boldsymbol{\omega}$-limit of $(\omega_0, 1/\alpha)$. To this end, we consider the function $w(t) = (\check v_\xi(t, \omega_0, 1/\alpha))^{-1}$, which satisfies

$$w' = -(\mathfrak{a}_2(\omega_0{\cdot}t) + \xi) + \frac{\mathfrak{a}_3(\omega{\cdot}t)}{w}$$

with $w(0) = \alpha < 2r/\alpha$. It suffices to check that w is bounded on $[0, \infty)$ to prove that $\check v_\xi(t, \omega_0, 1/\alpha)$ is bounded away from 0 for $t \geq 0$. An analogous argument to that in the proof of Proposition 1.12(iv) ensures that there exists $t_\alpha > 0$ such that $\int_0^{t_\alpha} (\mathfrak{a}_2(\omega{\cdot}s) + \xi)\, ds \geq \alpha\, t_\alpha$ for all $\omega \in \Omega$. We define $t_1 = \sup\{t > 0 \mid w(s) \leq 2\,r/\alpha + l\, t_\alpha$ for all $s \in [0, t]\}$, where $l = \max_{\omega\in\Omega} |\mathfrak{a}_2(\omega) + \xi| + r/\alpha$. We assume for contradiction that $t_1 < \infty$ and define $t_0 = \inf\{t < t_1 \mid w(s) \geq 2r/\alpha$ for all $s \in [t, t_1]\}$. Then, $t_0 < t_1 - t_\alpha$: otherwise

$$w(t_1) = w(t_0) + \int_{t_0}^{t_1} \left(-\big(\mathfrak{a}_2(\omega_0{\cdot}s) + \xi\big) + \frac{\mathfrak{a}_3(\omega_0{\cdot}s)}{w(s)} \right) ds < \frac{2r}{\alpha} + l\, t_\alpha\,,$$

which is not the case. The integrand has been bounded by l since $|\mathfrak{a}_3(\omega_0{\cdot}s)|/w(s) \leq r/(2r/\alpha) = \alpha/2 < r/\alpha$. In particular, $w(t) \geq 2r/\alpha$ for $t \in [t_1 - t_\alpha, t_1]$, and hence

$$\begin{aligned} w(t_1) &= w(t_1 - t_\alpha) - \int_0^{t_\alpha} \big(\mathfrak{a}_2(\omega_0{\cdot}(t_1 - t_\alpha){\cdot}s) + \xi\big)\, ds + \int_{t_1 - t_\alpha}^{t_1} \frac{\mathfrak{a}_3(\omega_0{\cdot}s)}{w(s)}\, ds \\ &\leq w(t_1 - t_\alpha) - \alpha\, t_\alpha + \frac{\alpha}{2}\, t_\alpha < w(t_1 - t_\alpha)\,, \end{aligned}$$

which contradicts the definition of t_1. Hence, $t_1 = \infty$, that is, $w(t) \leq 2r/\alpha + l t_\alpha$ for all $t \geq 0$. Therefore, as explained before, there exists a $\check\tau_\xi$-minimal set $\mathcal{M}^u_\xi > \mathcal{M}_0$.

Analogous arguments show the existence of a $\check\tau_\xi$-minimal set $\mathcal{M}^l_\xi < \mathcal{M}_0$ for $\xi < \xi_-$. Since the dynamical spectrum of the variational equation of $(3.14)_\xi$ on $\mathcal{M}_0$ reduces to $\{0\}$ for any $\xi \in \mathbb{R}$, Theorem 3.11(i) ensures the attractive hyperbolicity of $\mathcal{M}^u_\xi$ for all $\xi > \xi_+$ and $\mathcal{M}^l_\xi$ for all $\xi < \xi_-$. Since Theorem 2.11 and the nonhyperbolicity of $\mathcal{M}_0$ ensure that $\check\tau_\xi$ admits at most two minimal sets for any $\xi \in \mathbb{R}$, Proposition 2.17 ensures that $\mathcal{M}^u_\xi = \{\check{\mathfrak{u}}_\xi\}$ for all $\xi > \xi_+$ and $\mathcal{M}^l_\xi = \{\check{\mathfrak{l}}_\xi\}$ for all $\xi < \xi_-$. Theorem 3.11(i) also precludes the existence of a second minimal set for $\check\tau_{\xi_-}$ (resp. $\check\tau_{\xi_+}$) apart from $\mathcal{M}_0$, since it would be hyperbolic and hence persisting (see Theorem 1.39) for $\xi \in (\xi_-, \xi_+)$ close enough to ξ_+ (resp. ξ_-), which is not possible. Proposition 3.41(ii) will provide the nondecreasing character of $\xi \mapsto \check{\mathfrak{l}}_\xi(\omega)$ and $\xi \mapsto \check{\mathfrak{u}}_\xi(\omega)$ for all $\omega \in \Omega$. In turn, this means that the semicontinuous $\check\tau_{\xi_+}$-equilibrium $\check{\mathfrak{u}}_{\xi_+}(\omega) = \lim_{\xi\downarrow\xi_+} \check{\mathfrak{u}}_\xi(\omega)$ (resp. $\check{\mathfrak{l}}_{\xi_-}(\omega) = \lim_{\xi\uparrow\xi_-} \check{\mathfrak{l}}_\xi(\omega)$) given by the limit of a decreasing (resp. increasing) family of continuous $\check\tau_\xi$-equilibria coincides with 0 at its continuity points: otherwise, Proposition 1.32 would provide a second $\check\tau_{\xi_+}$ (resp. $\check\tau_{\xi_-}$) minimal set. This completes the proof of all the assertions. □

See the right panel in Figure 3.7 for a depiction of the bifurcation diagram described in Proposition 3.26.

Proposition 3.27. *Let (Ω, σ) be minimal. Let $\mathfrak{a}_1 \in CP(\Omega, \mathbb{R})$, $\mathfrak{a}_i \in C(\Omega, \mathbb{R})$ for $i \in \{2, 3\}$ and let $\mathfrak{a}_3 > 0$. Let $\mathfrak{b}$ be a continuous primitive of $\mathfrak{a}_1$. Then,*

(i) $\mathrm{sp}(e^{\mathfrak{b}}\mathfrak{a}_2) \subset (0, \infty)$ *if and only if* (3.13) *exhibits the local saddle-node and classical transcritical bifurcations of minimal sets described in Theorem* 3.21*, with $\mathfrak{l}_\lambda$ colliding with* 0 *on a residual σ-invariant subset of Ω as $\lambda \downarrow \lambda_+$. In particular, this situation holds if $0 \not\equiv \mathfrak{a}_2 \geq 0$.*

(ii) $\mathrm{sp}(e^{\mathfrak{b}}\mathfrak{a}_2) \subset (-\infty, 0)$ *if and only if* (3.13) *exhibits the local saddle-node and classical transcritical bifurcations of minimal sets described in Theorem* 3.21*, with $\mathfrak{u}_\lambda$ colliding with* 0 *on a residual σ-invariant subset of Ω as $\lambda \downarrow \lambda_+$. In particular, this situation holds if $0 \not\equiv \mathfrak{a}_2 \leq 0$.*

(iii) $0 \in \mathrm{sp}(e^{\mathfrak{b}}\mathfrak{a}_2)$ *if and only if* (3.13) *exhibits the classical pitchfork bifurcation of minimal sets described in Theorem* 3.20*.*

Proof. The family of changes of variable $y(t) = e^{-\mathfrak{b}(\omega\cdot t)}x(t)$ takes (3.13) to

$$y' = -e^{2\mathfrak{b}(\omega\cdot t)}\,\mathfrak{a}_3(\omega\cdot t)\,y^3 + e^{\mathfrak{b}(\omega\cdot t)}\,\mathfrak{a}_2(\omega\cdot t)\,y^2 + (\mathfrak{a}_1(\omega\cdot t) - \mathfrak{b}'(\omega\cdot t) + \lambda)\,y\,,$$

which since $\mathfrak{b}' = \mathfrak{a}_1$, coincides with

$$y' = -e^{2\mathfrak{b}(\omega\cdot t)}\,\mathfrak{a}_3(\omega\cdot t)\,y^3 + e^{\mathfrak{b}(\omega\cdot t)}\mathfrak{a}_2(\omega\cdot t)\,y^2 + \lambda\,y\,. \tag{3.15}$$

The possibilities for the bifurcation diagram of (3.13) follow from here, since $\mathcal{N} \subset \Omega \times \mathbb{R}$ is a minimal set for $(3.15)_\lambda$ if and only if $\mathcal{M} = \{(\omega, e^{\mathfrak{b}(\omega)}x) \mid (\omega, x) \in \mathcal{N}\}$ is minimal for $(3.13)_\lambda$. If we define $\mathfrak{f}(\omega, y) = -e^{2\mathfrak{b}(\omega)}\,\mathfrak{a}_3(\omega)\,y^3 + e^{\mathfrak{b}(\omega)}\,\mathfrak{a}_2(\omega)\,y^2$, then the dynamical spectrum of $\mathfrak{f}_x$ on $\mathcal{M}_0$ is $\{0\} = [-\lambda_+, -\lambda_-]$, so $\lambda_- = \lambda_+ = 0$. Therefore, the bifurcation diagram of (3.15) is either the one described by Theorem 3.20 or the one of Theorem 3.21 (see Theorem 3.24). According to Proposition 3.26, the flow induced by $y' = \mathfrak{f}(\omega\cdot t, y)$, that is, $(3.15)_0$, admits just one minimal set if and only if $0 \in \mathrm{sp}(e^{\mathfrak{b}}\mathfrak{a}_2)$; two minimal sets, being $\mathcal{M}_0 = \Omega \times \{0\}$ the lower one, if and only if $\mathrm{sp}(e^{\mathfrak{b}}\mathfrak{a}_2) \subset (0, \infty)$; and two minimal sets, being $\mathcal{M}_0$ the upper one, if and only if $\mathrm{sp}(e^{\mathfrak{b}}\mathfrak{a}_2) \subset (-\infty, 0)$. As said before, this determines the global bifurcation diagram. The last assertions in (i) and (ii) are trivial. □

As a consequence of the previous result, given any strictly positive $\mathfrak{a}_3$ and any changing-sign $\mathfrak{a}_2$, we are able to construct $\mathfrak{a}_1$ with bounded primitive in such a way that (3.13) exhibits the classical pitchfork bifurcation of minimal sets described in Theorem 3.20.

Proposition 3.28. *Let (Ω, σ) be minimal. Let $\mathfrak{a}_i \in C(\Omega, \mathbb{R})$ for $i \in \{2, 3\}$ and let $\mathfrak{a}_3 > 0$. Assume that $\mathfrak{a}_2$ changes sign. Then, there exists $\mathfrak{a}_1 \in CP(\Omega, \mathbb{R})$ such that* (3.13) *exhibits the classical pitchfork bifurcation described in Theorem* 3.20*.*

Proof. Let $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ be arbitrarily fixed. We define the nonempty open sets $\mathcal{U}_1 = \{\omega \in \Omega \mid \mathfrak{a}_2(\omega) > 0\}$ and $\mathcal{U}_2 = \{\omega \in \Omega \mid \mathfrak{a}_2(\omega) < 0\}$. As (Ω, σ) is minimal, $m(\mathcal{U}_1) > 0$ and $m(\mathcal{U}_2) > 0$. A suitable application of Urysohn's Lemma provides nonnegative and not identically zero continuous functions $\mathfrak{c}_1, \mathfrak{c}_2 \colon \Omega \to \mathbb{R}$

such that $\operatorname{supp}(\mathfrak{c}_1) \subseteq \mathcal{U}_1$ and $\operatorname{supp}(\mathfrak{c}_2) \subseteq \mathcal{U}_2$. Then, $\int_\Omega \mathfrak{c}_1(\omega)\,\mathfrak{a}_2(\omega)\,dm > 0$ and $\int_\Omega \mathfrak{c}_2(\omega)\,\mathfrak{a}_2(\omega)\,dm < 0$, so there exists $\varepsilon > 0$ such that $\int_\Omega(\mathfrak{c}_1(\omega)+\varepsilon)\,\mathfrak{a}_2(\omega)\,dm > 0$ and $\int_\Omega(\mathfrak{c}_2(\omega)+\varepsilon)\,\mathfrak{a}_2(\omega)\,dm < 0$. The density of $C^1(\Omega,\mathbb{R})$ on $C(\Omega,\mathbb{R})$ (see Proposition 1.64(iv)) and the strict positiveness of $\mathfrak{c}_1+\varepsilon$ and $\mathfrak{c}_2+\varepsilon$ ensure the existence of strictly positive functions $\hat{\mathfrak{c}}_1, \hat{\mathfrak{c}}_2 \in C^1(\Omega,\mathbb{R})$ such that $\int_\Omega \hat{\mathfrak{c}}_1(\omega)\,\mathfrak{a}_2(\omega)\,dm > 0$ and $\int_\Omega \hat{\mathfrak{c}}_2(\omega)\,\mathfrak{a}_2(\omega)\,dm < 0$. Therefore, there exists $s \in (0,1)$ such that $\int_\Omega(s\,\hat{\mathfrak{c}}_1(\omega) + (1-s)\,\hat{\mathfrak{c}}_2(\omega))\,\mathfrak{a}_2(\omega)\,dm = 0$. Since $s\,\hat{\mathfrak{c}}_1(\omega)+(1-s)\,\hat{\mathfrak{c}}_2(\omega)$ is strictly positive, $\mathfrak{b}(\omega) = \log(s\,\hat{\mathfrak{c}}_1(\omega)+(1-s)\,\hat{\mathfrak{c}}_2(\omega))$ is well defined and $\mathfrak{b} \in C^1(\Omega,\mathbb{R})$. So, $\int_\Omega e^{\mathfrak{b}(\omega)}\mathfrak{a}_2(\omega)\,dm = 0$, and hence $0 \in \operatorname{sp}(e^{\mathfrak{b}}\mathfrak{a}_2)$. We take $\mathfrak{a}_1 = \mathfrak{b}'$ and apply Proposition 3.27 to complete the proof. □

The case of sign-preserving $\mathfrak{a}_2$

The possibilities for the bifurcation diagram of (3.13) are more complicated when $\operatorname{sp}(\mathfrak{a}_1)$ is a nondegenerated interval: it is given by the right panel of Figures 3.3 or 3.4, or by Figure 3.5 (or the symmetric of the last ones). The goal of this section and of the following one is to check that these three bifurcation diagrams, described in Theorems 3.20, 3.21 and 3.22, indeed occur for nonautonomous families of the type (3.13), even under the restriction that $\mathfrak{a}_2$ does not change sign. Our starting point are the functions $\mathfrak{a}_1, \mathfrak{a}_3 \in C(\Omega,\mathbb{R})$, with $\mathfrak{a}_3 > 0$, which are considered fixed. From them, we define six constants:

p1 let $\operatorname{sp}(\mathfrak{a}_1) = [-\lambda_+, -\lambda_-]$, with $\lambda_- \le \lambda_+$, be the dynamical spectrum of $\mathfrak{a}_1$,

p2 let $k_1 < k_2$ be such that $k_1 \le \mathfrak{a}_1(\omega) \le k_2$ for all $\omega \in \Omega$,

p3 let $0 < r_1 \le r_2$ be such that $r_1 \le \mathfrak{a}_3(\omega) \le r_2$ for all $\omega \in \Omega$.

Our goal in this section is to give bounds for $\mathfrak{a}_2$ in terms of these six constants in order to provide each of the different bifurcation diagrams for (3.13). A remarkable fact is that, in the cases studied this section, $\mathfrak{a}_2$ never changes sign, in contrast with the situation of Proposition 3.28 and those that will be analyzed at the end of Section 3.4.

The next result establishes relations between λ_-, λ_+, k_1 and k_2 which we will use in the proofs of the main results of this section.

Lemma 3.29. *Let (Ω,σ) be minimal, let $\mathfrak{a}_1 \in C(\Omega,\mathbb{R})$ and let λ_-, λ_+, k_1 and k_2 be given by* **p1** *and* **p2***. Then,*

(i) $k_1 \le \min_{\omega\in\Omega}\mathfrak{a}_1(\omega) \le -\lambda_+ \le -\lambda_- \le \max_{\omega\in\Omega}\mathfrak{a}_1(\omega) \le k_2$.

(ii) $\mathfrak{a}_1$ *is nonconstant if and only if* $\min_{\omega\in\Omega}\mathfrak{a}_1(\omega) < -\lambda_+$ *(or equivalently if and only if* $\max_{\omega\in\Omega}\mathfrak{a}_1(\omega) > -\lambda_-$*).*

(iii) $k_1 \le \min_{\omega\in\Omega}\mathfrak{a}_1(\omega) < -\lambda_+ < -\lambda_- < \max_{\omega\in\Omega}\mathfrak{a}_1(\omega) \le k_2$ *if and only if* $\lambda_- < \lambda_+$*, i.e. if and only if* $\mathfrak{a}_1$ *has band spectrum.*

Proof. (i) It follows from $\min_{\omega\in\Omega}\mathfrak{a}_1(\omega) \le \int_\Omega \mathfrak{a}_1(\omega)\,dm \le \max_{\omega\in\Omega}\mathfrak{a}_1(\omega)$ for all $m \in \mathfrak{M}_{\mathrm{inv}}(\Omega,\sigma)$, which ensures that $[-\lambda_+,-\lambda_-] \subseteq [\min_{\omega\in\Omega}\mathfrak{a}_1(\omega), \max_{\omega\in\Omega}\mathfrak{a}_1(\omega)]$. The inequalities $k_1 \le \min_{\omega\in\Omega}\mathfrak{a}_1(\omega)$ and $\max_{\omega\in\Omega}\mathfrak{a}_1(\omega) \le k_2$ follow directly from **p2**.

(ii) Obviously, $\mathfrak{a}_1$ is nonconstant if $\min_{\omega\in\Omega}\mathfrak{a}_1(\omega) < -\lambda_+$ or $\max_{\omega\in\Omega}\mathfrak{a}_1(\omega) > -\lambda_-$. Let us now assume that $\mathfrak{a}_1$ is nonconstant and check that $\min_{\omega\in\Omega}\mathfrak{a}_1(\omega) < -\lambda_+$. We take $0 < \varepsilon < \max_{\omega\in\Omega} \mathfrak{a}_1(\omega) - \min_{\omega\in\Omega} \mathfrak{a}_1(\omega)$. We define the nonempty open set $\mathcal{U}_1 = \{\omega \in \Omega \mid \mathfrak{a}_1(\omega) > \min_{\omega\in\Omega} \mathfrak{a}_1(\omega) + \varepsilon\}$, and note that $m(\mathcal{U}_1) > 0$ for all $m \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$, since (Ω, σ) is minimal. Proposition 1.12(iii) and Remark 1.14.1 provide $m_1 \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$ such that $\inf \text{sp}(\mathfrak{a}_1) = \int_\Omega \mathfrak{a}_1(\omega)\, dm_1$. Then,

$$\begin{aligned}\inf \text{sp}(\mathfrak{a}_1) &= \int_\Omega \mathfrak{a}_1(\omega)\, dm_1 = \int_{\Omega\setminus\mathcal{U}_1} \mathfrak{a}_1(\omega)\, dm_1 + \int_{\mathcal{U}_1} \mathfrak{a}_1(\omega)\, dm_1 \\ &\geq m_1(\Omega \setminus \mathcal{U}_1) \min_{\omega\in\Omega} \mathfrak{a}_1(\omega) + m_1(\mathcal{U}_1) \left(\min_{\omega\in\Omega} \mathfrak{a}_1(\omega) + \varepsilon \right) > \min_{\omega\in\Omega} \mathfrak{a}_1(\omega)\,.\end{aligned}$$

To check that $\max_{\omega\in\Omega} \mathfrak{a}_1(\omega) > -\lambda_-$ if $\mathfrak{a}_1$ is nonconstant, we work analogously with $\mathcal{U}_2 = \{\omega \in \Omega \mid \mathfrak{a}_1(\omega) < \max_{\omega\in\Omega} \mathfrak{a}_1(\omega) - \varepsilon\}$.

(iii) It follows from (ii), since $\mathfrak{a}_1$ is nonconstant if $\mathfrak{a}_1$ has band spectrum, i.e. if $\lambda_- < \lambda_+$. □

Now, we recall and complete the statement of Proposition 3.25 when applied to our current model (3.13), which gives a sufficient criterium for the classical pitchfork bifurcation.

Proposition 3.30. *Let (Ω, σ) be minimal. Let $\mathfrak{a}_i \in C(\Omega, \mathbb{R})$ for $i \in \{1, 2, 3\}$, and let $\mathfrak{a}_3 > 0$. Let λ_+ be given by* **p1**. *Then,*

(i) (A criterium ensuring classical pitchfork bifurcation). *If $\mathfrak{a}_2(\omega) = 0$ for all $\omega \in \Omega$, then (3.13) exhibits the classical pitchfork bifurcation of minimal sets described in Theorem 3.20.*

(ii) *If $\mathfrak{a}_2(\omega) \geq 0$ (resp. $\mathfrak{a}_2(\omega) \leq 0$) for all $\omega \in \Omega$, then $\mathfrak{l}_\lambda$ (resp. $\mathfrak{u}_\lambda$) takes the value 0 at its continuity points for all $\lambda \leq \lambda_+$.*

Proof. Proposition 3.25(ii) ensures that (i) holds. Proposition 3.25(i) guarantees that the assertions of (ii) concerning $\lambda < \lambda_+$ hold. To check it for λ_+, it suffices to observe that in the three possible bifurcation diagrams (see Theorem 3.24), whenever $\mathfrak{l}_\lambda$ (resp. $\mathfrak{u}_\lambda$) takes the value 0 at its continuity points for all $\lambda < \lambda_+$, also $\mathfrak{l}_{\lambda_+}$ (resp. $\mathfrak{u}_{\lambda_+}$) takes the value 0 at its continuity points. (This fact can be observed in Figures 3.3, 3.4 and 3.5 in the case of $\mathfrak{u}_\lambda$.) □

Propositions 3.32, 3.33 and 3.34 are the main results of this section, and their proofs use the next technical result, which deals with some of the conditions in the statement of Propositions 3.32 and 3.33.

Lemma 3.31. *Let (Ω, σ) be minimal. Let $\mathfrak{a}_i \in C(\Omega, \mathbb{R})$ for $i \in \{1, 2, 3\}$, and let $\mathfrak{a}_3 > 0$. Let k_1 and r_2 be given by* **p2** *and* **p3**, *and let $\lambda < -k_1$.*

(i) *If $\mathfrak{a}_2(\omega) > 2\sqrt{r_2(-\lambda - k_1)}$ for all $\omega \in \Omega$, then $\rho_1 = \sqrt{(-\lambda - k_1)/r_2} > 0$ is a global strict lower solution of $(3.13)_\lambda$. Consequently, τ_λ admits a minimal set $\mathcal{M}^u_\lambda > \mathcal{M}_0$.*

(ii) *If $\mathfrak{a}_2(\omega) < -2\sqrt{r_2(-\lambda - k_1)}$ for all $\omega \in \Omega$, then $-\rho_1$ is a global strict upper solution of $(3.13)_\lambda$. Consequently, τ_λ admits a minimal set $\mathcal{M}^l_\lambda < \mathcal{M}_0$.*

Proof. Let us prove (i). We define $g(\rho) = r_2\,\rho - (\lambda + k_1)/\rho$. Then, $g(\rho_1) = 2\sqrt{r_2(-\lambda - k_1)}$, and, using that $-r_2 \le -\mathfrak{a}_3(\omega)$ and $k_1 \le \mathfrak{a}_1(\omega)$ for all $\omega \in \Omega$,

$$-\mathfrak{a}_3(\omega)\,\rho_1^3 + \mathfrak{a}_2(\omega)\,\rho_1^2 + (\mathfrak{a}_1(\omega) + \lambda)\,\rho_1 \ge \rho_1^2\left(\mathfrak{a}_2(\omega) - \left(r_2\,\rho_1 - \frac{\lambda + k_1}{\rho_1}\right)\right)$$
$$= \rho_1^2\,(\mathfrak{a}_2(\omega) - g(\rho_1)) > 0$$

for all $\omega \in \Omega$, which proves the first assertion. In turn, this property ensures that $\rho_1 < \mathfrak{u}_\lambda(\omega)$ for all $\omega \in \Omega$ (see Theorem 2.13(v)), and hence Proposition 2.17(i) proves the existence of a τ_λ-minimal set $\mathcal{M}^u_\lambda$ above $\mathcal{M}_0$. The proof of (ii) is analogous. □

Proposition 3.32 (A criterium ensuring saddle-node and transcritical bifurcations)**.** *Let (Ω, σ) be minimal. Let $\mathfrak{a}_i \in C(\Omega, \mathbb{R})$ for $i \in \{1, 2, 3\}$, and let $\mathfrak{a}_3 > 0$. Let λ_-, λ_+, k_1 and r_2 be given by* **p1**, **p2** *and* **p3**. *If $k_1 < -\lambda_+$ and*

$$\mathfrak{a}_2(\omega) > 2\sqrt{r_2(-\lambda_- - k_1)} \qquad (\textit{resp.}\ \ \mathfrak{a}_2(\omega) < -2\sqrt{r_2(-\lambda_- - k_1)}\,)$$

for all $\omega \in \Omega$, then (3.13) exhibits the local saddle-node and transcritical bifurcations of minimal sets described in Theorem 3.21, with $\mathfrak{l}_\lambda$ (resp. $\mathfrak{u}_\lambda$) colliding with 0 on a residual σ-invariant subset of Ω as $\lambda \downarrow \lambda_+$.

Proof. Note that if $k_1 < -\lambda_+$, then $-\lambda_- - k_1 \ge -\lambda_+ - k_1 > 0$. Take $\delta > 0$ such that $\mathfrak{a}_2(\omega) > 2\sqrt{r_2(-\lambda_- + \delta - k_1)}$ (resp. $\mathfrak{a}_2(\omega) < -2\sqrt{r_2(-\lambda_- + \delta - k_1)}\,$) for all $\omega \in \Omega$. Hence, Lemma 3.31(i) (resp. (ii)) applied with $\lambda = \lambda_- - \delta$ ensures the existence of a $\tau_{\lambda_- - \delta}$-minimal set $\mathcal{M}^u_{\lambda_- - \delta} > \mathcal{M}_0$ (resp. $\mathcal{M}^l_{\lambda_- - \delta} < \mathcal{M}_0$), and this situation only arises in the bifurcation diagram of Theorem 3.21: recall Theorem 3.24 and note that in Theorems 3.20 and 3.22 we have that $\mathcal{A}_{\lambda_- - \delta} = \mathcal{M}_0$. □

Proposition 3.33. *Let (Ω, σ) be minimal. Let $\mathfrak{a}_i \in C(\Omega, \mathbb{R})$ for $i \in \{1, 2, 3\}$, and let $\mathfrak{a}_3 > 0$. Let λ_-, λ_+, k_1, k_2, r_1 and r_2 be given by* **p1**, **p2** *and* **p3**.

(i) (A criterium precluding classical pitchfork bifurcation). *If $k_1 < -\lambda_+$ and*

$$\mathfrak{a}_2(\omega) > 2\sqrt{r_2(-\lambda_+ - k_1)} \qquad (\textit{resp.}\ \ \mathfrak{a}_2(\omega) < -2\sqrt{r_2(-\lambda_+ - k_1)}\,)$$

for all $\omega \in \Omega$, then (3.13) does not exhibit the classical pitchfork bifurcation of minimal sets described in Theorem 3.20.

(ii) (A criterium precluding local saddle-node and transcritical bifurcations). *If $\lambda_- < \lambda_+$ and*

$$0 \le \mathfrak{a}_2(\omega) < \frac{\sqrt{r_1}\,(\lambda_+ - \lambda_-)}{\sqrt{\lambda_+ + k_2}} \qquad \left(\textit{resp.}\ \ -\frac{\sqrt{r_1}\,(\lambda_+ - \lambda_-)}{\sqrt{\lambda_+ + k_2}} < \mathfrak{a}_2(\omega) \le 0\right)$$

for all $\omega \in \Omega$, then (3.13) does not exhibit the saddle-node and transcritical bifurcations of minimal sets described in Theorem 3.21.

Proof. (i) Lemma 3.31(i) (resp. (ii)) applied with $\lambda = \lambda_+$ shows the existence of a τ_{λ_+}-minimal set $\mathcal{M}^u_{\lambda_+} > \mathcal{M}_0$ (resp. $\mathcal{M}^l_{\lambda_+} < \mathcal{M}_0$), precluding the occurrence of the bifurcation diagram of Theorem 3.20.

(ii) We reason in the case of $\mathfrak{a}_2\ge 0$. We consider an auxiliary family obtained by suppressing $\mathfrak{a}_2$ in (3.13):

$$x' = -\mathfrak{a}_3(\omega{\cdot}t)\,x^3 + (\mathfrak{a}_1(\omega{\cdot}t)+\lambda)\,x\,, \quad \omega\in\Omega\,. \tag{3.16}$$

During this proof, let $\hat\tau_\lambda$ the skewproduct flow induced by $(3.16)_\lambda$ on $\Omega\times\mathbb{R}$, let $\hat v_\lambda(t,\omega,x)$ be the cocycle of solutions of $(3.16)_\lambda$, let $\hat{\mathcal{A}}_\lambda$ be its global attractor, and let $\hat{\mathfrak{l}}_\lambda$ and $\hat{\mathfrak{u}}_\lambda$ be the lower and upper equilibria of $\hat{\mathcal{A}}_\lambda$. Proposition 3.30(i) ensures that the bifurcation diagram of (3.16) is that of Theorem 3.20.

Let us fix $\lambda>\lambda_+$, so that Lemma 3.29(iii) ensures that $\lambda+k_2>\lambda_++k_2>0$. If $\rho>\sqrt{(\lambda+k_2)/r_1}$, then

$$-\mathfrak{a}_3(\omega)\,\rho^3+(\mathfrak{a}_1(\omega)+\lambda)\,\rho\le -r_1\,\rho^3+(\lambda+k_2)\,\rho<0\,;$$

that is, the constant map ρ is a global strict upper solution of $(3.16)_\lambda$, and hence a strong $\hat\tau_\lambda$-superequilibrium (see Proposition 1.24(i)). Since $\rho\ge\hat v(t,\omega,\rho)>0$ for all $t\ge 0$, the $\boldsymbol{\omega}$-limit set of (ω,ρ) for $\hat\tau_\lambda$ contains a $\hat\tau_\lambda$-minimal set $\mathcal{M}$ which is contained in $\Omega\times[0,\rho]$. Corollary 1.58(i) precludes $\mathcal{M}=\mathcal{M}_0$, since Proposition 3.19(ii) ensures that $\mathcal{M}_0$ is hyperbolic repulsive. Proposition 2.14(i) ensures the existence of three $\hat\tau_\lambda$-minimal sets, and Theorem 2.11 ensures that $\mathcal{M}$ is the hyperbolic attractive graph of $\hat{\mathfrak{u}}_\lambda$, which is hence continuous. In particular, $0<\hat{\mathfrak{u}}_\lambda(\omega)\le\rho$ for all $\omega\in\Omega$. It follows that $0<\hat{\mathfrak{u}}_\lambda(\omega)\le\sqrt{(\lambda+k_2)/r_1}$ for all $\omega\in\Omega$ and $\lambda>\lambda_+$.

Now, we return to the complete equation $(3.13)_\lambda$ and make use of the information just obtained for $(3.16)_\lambda$. The inequality in the statement ensures the existence of $\delta>0$ such that, if $\lambda\in[\lambda_+,\lambda_++\delta]$, then $\mathfrak{a}_2(\omega)\le\sqrt{r_1}\,(\lambda_+-\lambda_-)/\sqrt{\lambda+k_2}$ for all $\omega\in\Omega$. If $\lambda\in(\lambda_+,\lambda_++\delta]$, the bound obtained in the previous paragraph combined with the previous inequality gives

$$\mathfrak{a}_2(\omega)\,\hat{\mathfrak{u}}_\lambda(\omega)\le\frac{\sqrt{r_1}\,(\lambda_+-\lambda_-)}{\sqrt{\lambda+k_2}}\sqrt{\frac{\lambda+k_2}{r_1}}=\lambda_+-\lambda_-$$

for all $\omega\in\Omega$, and hence

$$\begin{aligned}\hat{\mathfrak{u}}_\lambda'(\omega)&=-\mathfrak{a}_3(\omega)\,\hat{\mathfrak{u}}_\lambda(\omega)^3+(\mathfrak{a}_1(\omega)+\lambda)\,\hat{\mathfrak{u}}_\lambda(\omega)\\&>-\mathfrak{a}_3(\omega)\,\hat{\mathfrak{u}}_\lambda(\omega)^3+(\lambda_+-\lambda_-)\,\hat{\mathfrak{u}}_\lambda(\omega)+(\mathfrak{a}_1(\omega)+\lambda_-)\,\hat{\mathfrak{u}}_\lambda(\omega)\\&\ge-\mathfrak{a}_3(\omega)\hat{\mathfrak{u}}_\lambda(\omega)^3+\mathfrak{a}_2(\omega)\,\hat{\mathfrak{u}}_\lambda(\omega)^2+(\mathfrak{a}_1(\omega)+\lambda_-)\,\hat{\mathfrak{u}}_\lambda(\omega)\end{aligned}$$

for all $\omega\in\Omega$. That is, for $\lambda\in(\lambda_+,\lambda_++\delta]$, we have that $\hat{\mathfrak{u}}_\lambda$ is a global strict upper τ_{λ_-}-solution, and, in particular, a continuous strong τ_{λ_-}-superequilibrium (see Proposition 1.24(i)). We recall that, since the bifurcation diagram of (3.16) is that of Theorem 3.20, $\hat{\mathfrak{u}}_{\lambda_+}=\lim_{\lambda\downarrow\lambda_+}\hat{\mathfrak{u}}_\lambda$ collides with 0 on a residual set $\mathcal{R}_1\subseteq\Omega$.

Let us assume for contradiction that the situation for (3.13) is that described in Theorem 3.21. This theorem includes two possibilities. In one of them, $\mathfrak{l}_\lambda$ is strictly negative for all $\lambda\in[\lambda_-,\lambda_+]$, but this is precluded by Proposition 3.30(ii) since $\mathfrak{a}_2\ge 0$. So, the other possibility holds, which means that $\mathfrak{u}_{\lambda_-}$ is a continuous strictly positive τ_{λ_-}-equilibrium whose graph is the attractive hyperbolic τ_{λ_-}-minimal set $\mathcal{M}^u_{\lambda_-}$. Since $\mathrm{sp}(\mathfrak{a}_1+\lambda_-)=[-(\lambda_+-\lambda_-),0]$, Proposition 1.12(iii) and Remark 1.14.1 ensure that there exists $m\in\mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$ such that $\int_\Omega(\mathfrak{a}_1(\omega)+\lambda_-)\,dm<0$, and hence, since $\mathfrak{h}_x(\omega,0)=\mathfrak{a}_1(\omega)$ for all $\omega\in\Omega$, Lemma 3.23 ensures that

$$\mathfrak{m}_{\lambda_-}(\omega)=\inf\big\{x\in\mathbb{R}\,\big|\,\lim_{t\to\infty}(v_{\lambda_-}(t,\omega,x)-\mathfrak{u}_{\lambda_-}(\omega{\cdot}t))=0\big\}\in[0,\mathfrak{u}_{\lambda_-}(\omega))$$

vanishes on the residual set of its continuity points $\mathcal{R}_2 \subseteq \Omega$.

Let us take $\omega_0 \in \mathcal{R}_1 \cap \mathcal{R}_2$, so that $\mathfrak{m}_{\lambda_-}(\omega_0) = 0$ and $\hat{\mathfrak{u}}_\lambda(\omega_0)$ decreases to 0 as $\lambda \downarrow \lambda_+$. Since $\mathfrak{u}_{\lambda_-}(\omega_0) > 0$, we can take $\lambda \in (\lambda_+, \lambda_+ + \delta]$ such that $\mathfrak{m}_{\lambda_-}(\omega_0) = 0 < \hat{\mathfrak{u}}_\lambda(\omega_0) < \mathfrak{u}_{\lambda_-}(\omega_0)$. Hence, $\lim_{n\to\infty}(v_{\lambda_-}(t_n, \omega_0, \hat{\mathfrak{u}}_\lambda(\omega_0)) - \mathfrak{u}_{\lambda_-}(\omega_0{\cdot}t_n)) = 0$ for a sequence $(t_n) \uparrow \infty$ such that $\omega_0 = \lim_{n\to\infty} \omega_0{\cdot}t_n$. Since the map $\hat{\mathfrak{u}}_\lambda$ is a continuous τ_{λ_-}-superequilibrium, $v_{\lambda_-}(t_n, \omega_0, \hat{\mathfrak{u}}_\lambda(\omega_0)) \le \hat{\mathfrak{u}}_\lambda(\omega_0{\cdot}t_n)$ for all $n \in \mathbb{N}$, and hence $0 \le \lim_{n\to\infty}(\hat{\mathfrak{u}}_\lambda(\omega_0{\cdot}t_n) - \mathfrak{u}_{\lambda_-}(\omega_0{\cdot}t_n)) = \hat{\mathfrak{u}}_\lambda(\omega_0) - \mathfrak{u}_{\lambda_-}(\omega_0) < 0$. This is the sought-for contradiction, which completes the proof. □

Observe that Proposition 3.33(ii) requires $\mathfrak{a}_1$ to have band spectrum. The same happens with the next result: although $\lambda_- < \lambda_+$ is not explicitly required in the statement of Proposition 3.34, it must be fulfilled for condition (3.17) to be satisfied.

Proposition 3.34 (A criterium ensuring generalized pitchfork bifurcation)**.** *Let (Ω, σ) be minimal. Let $\mathfrak{a}_i \in C(\Omega, \mathbb{R})$ for $i \in \{1, 2, 3\}$, and let $\mathfrak{a}_3 > 0$. Let λ_-, λ_+, k_1, k_2, r_1 and r_2 be given by* **p1**, **p2** *and* **p3**. *If*

$$r_1(\lambda_+ - \lambda_-)^2 + 4\, r_2(\lambda_+ + k_1)(\lambda_+ + k_2) > 0 \tag{3.17}$$

and

$$\begin{gathered} 2\sqrt{r_2(-\lambda_+ - k_1)} < \mathfrak{a}_2(\omega) < \frac{\sqrt{r_1}\,(\lambda_+ - \lambda_-)}{\sqrt{\lambda_+ + k_2}} \\ \left(\textit{resp.} \quad -\frac{\sqrt{r_1}\,(\lambda_+ - \lambda_-)}{\sqrt{\lambda_+ + k_2}} < \mathfrak{a}_2(\omega) < -2\sqrt{r_2(-\lambda_+ - k_1)} \right) \end{gathered} \tag{3.18}$$

for all $\omega \in \Omega$, then (3.13) exhibits the generalized pitchfork bifurcation of minimal sets described in Theorem 3.22, with $\mathfrak{l}_\lambda$ (resp. $\mathfrak{u}_\lambda$) colliding with 0 on a residual σ-invariant subset of Ω as $\lambda \downarrow \lambda_+$.

Proof. Lemma 3.29(i) ensures that $(\lambda_+ + k_1)(\lambda_+ + k_2) \le 0$. Therefore, condition (3.17) yields $\lambda_- < \lambda_+$, and hence Lemma 3.29(iii) and (3.17) ensure that the intervals given in (3.18) in which $\mathfrak{a}_2$ can take values are finite and nondegenerate. Proposition 3.33(i) and (ii) respectively preclude the diagrams of Theorems 3.20 and 3.21, so Theorem 3.24 ensures that the bifurcation diagram of Theorem 3.22 takes place. Proposition 3.30(ii) ensures the stated collision property for $\mathfrak{l}_\lambda$ (resp. for $\mathfrak{u}_\lambda$). □

Cases of generalized pitchfork bifurcation

As said after Theorem 3.24, there are simple autonomous examples presenting either the classical pitchfork bifurcation of Theorem 3.20 or the local saddle-node and classical transcritical bifurcations of Theorem 3.21. The requirement of band spectrum ($\lambda_- < \lambda_+$) in the statement of Theorem 3.22 ensures that the two previous possibilities are also the unique ones in nonautonomous examples if $\mathfrak{a}_1$ has point spectrum. This is the case of $\mathfrak{a}_1 \in CP(\Omega, \mathbb{R})$, in which case Proposition 3.27 gives necessary and sufficient conditions for the bifurcation diagrams of Theorems 3.20 and 3.21 to take place. In Proposition 3.30(i), we have observed that the classical pitchfork bifurcation diagram of Theorem 3.20 can also occur in cases of $\mathfrak{a}_1$ with band spectrum. Proposition 3.32 proves the same with the bifurcation diagram of

Theorem 3.21, in which case there is a generalized transcritical bifurcation. Moreover, Proposition 3.30(i) and 3.32 provide simple ways of construction of examples by choosing a suitable $\mathfrak{a}_2$ once fixed $\mathfrak{a}_1$ and $\mathfrak{a}_3$. In the same line, Proposition 3.34 establishes conditions ensuring the generalized pitchfork diagram of Theorem 3.22. However, in this case the existence of nonautonomous cubic polynomials (and constants given by **p1**, **p2** and **p3**) satisfying condition (3.17) is not so obvious.

Therefore, our next objective is to develop systematic ways of constructing nonautonomous third degree polynomials giving rise to families (3.13) for which the global bifurcation diagram is that of Theorem 3.22. The conclusion is that all the situations described in Theorem 3.24 actually realize.

Lemma 3.35. *Let $m_1,\ldots,m_n$ be different elements of $\mathfrak{M}_{\text{erg}}(\Omega,\sigma)$ with $n\ge 1$, and let $0<\varepsilon<1$ be fixed. For every $i\in\{1,\ldots,n\}$, there exists a continuous $\mathfrak{c}_i\colon\Omega\to[0,1]$ with $\min_{\omega\in\Omega}\mathfrak{c}_i(\omega)=0$ and $\max_{\omega\in\Omega}\mathfrak{c}_i(\omega)=1$ such that $\mathfrak{c}_i\mathfrak{c}_j\equiv 0$ and*

$$1-\varepsilon<\int_\Omega \mathfrak{c}_i(\omega)\,dm_i\le 1\,,\qquad 0\le\int_\Omega \mathfrak{c}_i(\omega)\,dm_j<\varepsilon \tag{3.19}$$

for every $i,j\in\{1,\ldots,n\}$ with $j\ne i$.

Proof. Let $\varepsilon>0$ be fixed. As explained in [54, Remark 1.10] and [75, Chapter II, Section 6], there exist disjoint σ-invariant Borel sets $\Omega_1,\ldots,\Omega_n\subseteq\Omega$ such that $m_i(\Omega_j)=\delta_{ij}$ for $i,j\in\{1,\ldots,n\}$, where δ_{ij} is the Kronecker delta: just take Ω_i as the so-called *ergodic component* of m_i. Since m_i for $i\in\{1,\ldots,n\}$ are regular, we take compact sets $\mathcal{K}_i\subseteq\Omega_i$ such that $m_i(\mathcal{K}_i)>1-\varepsilon$ for $i\in\{1,\ldots,n\}$. Let $d=\min_{1\le i<j\le n}(\inf\{d_\Omega(\omega_i,\omega_j)\mid \omega_i\in\mathcal{K}_i,\,\omega_j\in\mathcal{K}_j\})$ and let $\mathcal{U}_i$ be an open set such that $\mathcal{K}_i\subset\mathcal{U}_i\subseteq B_\Omega(\mathcal{K}_i,d/3)$, $m_i(\mathcal{U}_i\setminus\mathcal{K}_i)<\varepsilon$ and $m_j(\mathcal{U}_i)=m_j(\mathcal{U}_i\setminus\mathcal{K}_i)<\varepsilon$ for $i,j\in\{1,\ldots,n\}$ with $i\ne j$. The choice of d ensures that $\mathcal{U}_1,\ldots,\mathcal{U}_n$ are pairwise disjoint. Finally, Urysohn's Lemma provides continuous functions $\mathfrak{c}_i\colon\Omega\to[0,1]$ with $\mathfrak{c}_i(\omega)=1$ for all $\omega\in\mathcal{K}_i$ and $\mathfrak{c}_i(\omega)=0$ for all $\omega\notin\mathcal{U}_i$, for $i\in\{1,\ldots,n\}$. Then, $\mathfrak{c}_i\mathfrak{c}_j\equiv 0$ follows from $\mathcal{U}_i$ and $\mathcal{U}_j$ being disjoint; $0\le\int_\Omega\mathfrak{c}_i(\omega)\,dm_j\le 1$ for all $i,j\in\{1,\ldots,n\}$ follows from $\mathfrak{c}_i\colon\Omega\to[0,1]$; $\int_\Omega\mathfrak{c}_i(\omega)\,dm_i\ge\int_{\mathcal{K}_i}\mathfrak{c}_i(\omega)\,dm_i=m_i(\mathcal{K}_i)>1-\varepsilon$; and $\int_\Omega\mathfrak{c}_i(\omega)\,dm_j=\int_{\mathcal{U}_i}\mathfrak{c}_i(\omega)\,dm_j=\int_{\mathcal{U}_i\setminus\mathcal{K}_i}\mathfrak{c}_i(\omega)\,dm_j\le m_j(\mathcal{U}_i\setminus\mathcal{K}_i)<\varepsilon$ for every $i,j\in\{1,\ldots,n\}$ with $j\ne i$. □

Proposition 3.36. *Let (Ω,σ) be minimal. Let $m_1,\ldots,m_n$ be different elements of $\mathfrak{M}_{\text{erg}}(\Omega,\sigma)$ with $n\ge 2$. Take $r\ge 1$ and $\varepsilon>0$ with*

$$\varepsilon<\varepsilon_1=\frac{n+2r(n-1)-2\sqrt{r(n-1)\big(r(n-1)+n\big)}}{n^2}$$

(so that $0<\varepsilon<1/n$). Let $\mathfrak{c}_1\ldots,\mathfrak{c}_n:\Omega\to[0,1]$ be the continuous functions given by Lemma 3.35 for $m_1,\ldots,m_n$ and ε. Take constants $\alpha_1\le\alpha_2\le\cdots\le\alpha_n$ with $\alpha_1<0$ and $\alpha_n>0$, and define $\mathfrak{a}_1=\sum_{i=1}^n\alpha_i\mathfrak{c}_i$, so that $\alpha_1\le\mathfrak{a}_1(\omega)\le\alpha_n$ for all $\omega\in\Omega$. Then, $\mathfrak{a}_1$ has band spectrum $\operatorname{sp}(\mathfrak{a}_1)=[-\lambda_+,-\lambda_-]\subset(\alpha_1,\alpha_n)$ and

$$(\lambda_+-\lambda_-)^2+4\,r\,(\lambda_++\alpha_1)(\lambda_++\alpha_n)>0\,. \tag{3.20}$$

Consequently, if $\mathfrak{a}_3\in C(\Omega,\mathbb{R})$ takes values in $[r_1,r_2]$ for $r_1>0$ and $r_2=r\,r_1$, and if $\mathfrak{a}_2\in C(\Omega,\mathbb{R})$ satisfies (3.18) for $k_1=\alpha_1$ and $k_2=\alpha_n$, then (3.13) exhibits the generalized pitchfork bifurcation of minimal sets described in Theorem 3.22.

Proof. It is easy to check that

$$2r(n-1) < 2\sqrt{r(n-1)(r(n-1)+n)} < n + 2r(n-1)\,,$$

and hence $0 < \varepsilon_1 < 1/n$. In addition, according to Lemma 3.35,

$$\int_\Omega \mathfrak{a}_1(\omega)\,dm_1 = \alpha_1 \int_\Omega \mathfrak{c}_1(\omega)\,dm_1 + \sum_{i=2}^{n} \alpha_i \int_\Omega \mathfrak{c}_i(\omega)\,dm_1 < \alpha_1(1-\varepsilon) + (n-1)\,\alpha_n\,\varepsilon\,,$$
$$\int_\Omega \mathfrak{a}_1(\omega)\,dm_n = \alpha_n \int_\Omega \mathfrak{c}_n(\omega)\,dm_n + \sum_{i=1}^{n-1} \alpha_i \int_\Omega \mathfrak{c}_i(\omega)\,dm_n > \alpha_n(1-\varepsilon) + (n-1)\,\alpha_1\,\varepsilon\,.$$

Since the supports of $\mathfrak{c}_1, \ldots, \mathfrak{c}_n$ are pairwise disjoint, $\alpha_1 = \min_{\omega\in\Omega} \mathfrak{a}_1(\omega)$ and $\alpha_n = \max_{\omega\in\Omega} \mathfrak{a}_n(\omega)$, so, in particular $\mathfrak{a}_1$ is nonconstant. Hence, the previous inequalities, the definition of $\mathrm{sp}(\mathfrak{a}_1)$ and Lemma 3.29(i) ensure that $\alpha_1 \le -\lambda_+ < \alpha_1(1-\varepsilon) + (n-1)\,\alpha_n\,\varepsilon$ and $\alpha_n(1-\varepsilon)+(n-1)\,\alpha_1\,\varepsilon < -\lambda_- \le \alpha_n$. Then, $\lambda_+ - \lambda_- > (\alpha_n - \alpha_1)(1-n\,\varepsilon) > 0$ (which shows the nondegeneracy of $\mathrm{sp}(\mathfrak{a}_1)$). Now, Lemma 3.29(iii) shows that $\alpha_1 < -\lambda_+ < -\lambda_- < \alpha_n$, and hence $0 > \lambda_+ + \alpha_1 > \varepsilon\,(\alpha_1 - (n-1)\,\alpha_n)$ and $0 < \lambda_+ + \alpha_n < \alpha_n - \alpha_1$, which in turn yields

$$(\lambda_+ - \lambda_-)^2 + 4\,r\,(\lambda_+ + \alpha_1)(\lambda_+ + \alpha_n)$$
$$> (\alpha_n - \alpha_1)^2(1-n\,\varepsilon)^2 + 4\,r\,\varepsilon\,(\alpha_n - \alpha_1)\big(\alpha_1 - (n-1)\,\alpha_n\big)\,.$$

So, in order to check that (3.20) holds with $k_1 = \alpha_1$ and $k_2 = \alpha_n$, it is enough to check that the right-hand side is strictly positive, that is, it suffices to check that

$$\alpha_n\,\big((1-n\,\varepsilon)^2 - 4\,r\,\varepsilon\,(n-1)\big) > \alpha_1\,\big((1-n\,\varepsilon)^2 - 4\,r\,\varepsilon\big)\,.$$

Since $(1-n\,\varepsilon)^2 - 4\,r\,\varepsilon\,(n-1) \le (1-n\,\varepsilon)^2 - 4\,r\,\varepsilon$ and $\alpha_1 < 0 < \alpha_n$, it suffices to check that $(1-n\,\varepsilon)^2 - 4\,r\,\varepsilon\,(n-1) > 0$. That is, $n^2\,\varepsilon^2 - 2\,(n + 2\,r\,(n-1))\,\varepsilon + 1 > 0$. And this follows from $\varepsilon < \varepsilon_1$, since ε_1 is the smallest root of this quadratic polynomial on the variable ε. Then, (3.20) holds. Finally, note that all the hypotheses of Proposition 3.34 are fulfilled with $k_1 = \alpha_1$ and $k_2 = \alpha_n$. This proves the last assertion. □

Note that every function $\mathfrak{a}_1$ constructed by the procedure of Proposition 3.36 takes both positive and negative values, that is, it changes sign. However, this is not a real restriction to get a generalized pitchfork bifurcation diagram through Proposition 3.36, since replacing $\mathfrak{a}_1$ by $\mathfrak{a}_1 + \mu$ for any constant $\mu \in \mathbb{R}$ induces the same type of bifurcation diagram.

Proposition 3.36 shows that no more than two different ergodic measures in $\mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ are required to prove the existence of families (3.13) showing the generalized pitchfork bifurcation diagram of Theorem 3.22. The functions $\mathfrak{a}_1$ constructed as there indicated are intended to satisfy (3.17); that is, their extremal Lyapunov exponents (see Remark 1.14) are near its maximum and minimum. But in fact this is not a necessary condition for a function $\mathfrak{a}_1$ to be the first order coefficient of a polynomial giving rise to a generalized pitchfork bifurcation. Theorem 3.39 proves this assertion in the case of a finitely ergodic base flow. Its proof in based on Proposition 3.37 and Corollary 3.38.

Proposition 3.37. *Assume that $\mathfrak{M}_{\rm erg}(\Omega,\sigma)=\{m_1,\dots,m_n\}$ with $n\ge 1$. There exists $\varepsilon_2>0$ such that, if $\varepsilon\in(0,\varepsilon_2]$ and $\mathfrak{c}_1,\dots,\mathfrak{c}_n\colon\Omega\to[0,1]$ are the continuous functions of Lemma 3.35 corresponding to $m_1,\dots,m_n$ and ε, then*

$$C(\Omega,\mathbb{R})=\langle\mathfrak{c}_1,\dots,\mathfrak{c}_n\rangle\oplus C_0(\Omega,\mathbb{R})$$

as topological direct sum of vector spaces, where $C(\Omega,\mathbb{R})$ is endowed with the uniform topology, given by $\|\mathfrak{a}\|=\max_{\omega\in\Omega}|\mathfrak{a}(\omega)|$. In particular, the dynamical spectrum of $\mathfrak{a}\in C(\Omega,\mathbb{R})$ coincides with that of its projection onto $\langle\mathfrak{c}_1,\dots,\mathfrak{c}_n\rangle$.

Proof. Let $M_{n\times n}(\mathbb{R})$ be the linear space of $n\times n$ real matrices, which we endow with the norm $\|C\|_\infty=\max_{1\le i,j\le n}|c_{ij}|$, where $C=\{c_{ij}\}_{1\le i,j\le n}$. The set of regular $n\times n$ real matrices $GL_n(\mathbb{R})$ is an open subset of $M_{n\times n}(\mathbb{R})$, and the $n\times n$ identity matrix I belongs to $GL_n(\mathbb{R})$. Hence, there exists $\varepsilon_2\in(0,1)$ such that, if $\|C-I\|_\infty\le\varepsilon_2$, then C is regular. Therefore, if $\varepsilon\in(0,\varepsilon_2]$, then the functions $\mathfrak{c}_1,\dots,\mathfrak{c}_n\colon\Omega\to[0,1]$ of Lemma 3.35 corresponding to $m_1,\dots,m_n$ and ε provide a regular matrix

$$C=\begin{pmatrix}\int_\Omega\mathfrak{c}_1(\omega)\,dm_1 & \dots & \int_\Omega\mathfrak{c}_n(\omega)\,dm_1\\ \vdots & \ddots & \vdots\\ \int_\Omega\mathfrak{c}_1(\omega)\,dm_n & \dots & \int_\Omega\mathfrak{c}_n(\omega)\,dm_n\end{pmatrix}.$$

Let us consider the continuous linear functionals $T_i\colon C(\Omega,\mathbb{R})\to\mathbb{R}$, $\mathfrak{b}\mapsto\int_\Omega\mathfrak{b}(\omega)\,dm_i$ for $i\in\{1,\dots,n\}$, and note that $\mathrm{Ker}(T_i)$ has codimension 1. Therefore, the codimension of the set $C_0(\Omega,\mathbb{R})=\bigcap_{i\in\{1,\dots,n\}}\mathrm{Ker}(T_i)$, is at most n. In addition, the linear space $\langle\mathfrak{c}_1,\dots,\mathfrak{c}_n\rangle$ has dimension n, since the supports of $\mathfrak{c}_1,\dots,\mathfrak{c}_n$ are pairwise disjoint. Let us check that $\langle\mathfrak{c}_1,\dots,\mathfrak{c}_n\rangle\cap C_0(\Omega,\mathbb{R})=\{0\}$: if $\mathfrak{c}=\sum_{i=1}^n\alpha_i\mathfrak{c}_i\in C_0(\Omega,\mathbb{R})$, then $0=\int_\Omega\mathfrak{c}(\omega)\,dm_j=\sum_{i=1}^n\alpha_i\int_\Omega\mathfrak{c}_i(\omega)\,dm_j$ for every $j\in\{1,\dots,n\}$. These n equations provide a homogeneous linear system for $\alpha_1,\dots,\alpha_n$ with regular coefficient matrix C; so $\alpha_1=\dots=\alpha_n=0$ and hence $\mathfrak{c}\equiv 0$. Consequently, $C(\Omega,\mathbb{R})$ is the algebraic direct sum of $\langle\mathfrak{c}_1,\dots,\mathfrak{c}_n\rangle$ and $C_0(\Omega,\mathbb{R})$. We will check that the projections of $C(\Omega,\mathbb{R})$ onto $\langle\mathfrak{c}_1,\dots,\mathfrak{c}_n\rangle$ and $C_0(\Omega,\mathbb{R})$ are continuous, which will complete the proof of the first assertion. Given $\mathfrak{a}\in C(\Omega,\mathbb{R})$, its projection $P_{\langle\mathfrak{c}_1,\dots,\mathfrak{c}_n\rangle}\mathfrak{a}=\sum_{i=1}^n\alpha_i\mathfrak{c}_i$ onto $\langle\mathfrak{c}_1,\dots,\mathfrak{c}_n\rangle$ is given by

$$\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}=C^{-1}\begin{pmatrix}\int_\Omega\mathfrak{a}(\omega)\,dm_1\\ \vdots\\ \int_\Omega\mathfrak{a}(\omega)\,dm_n\end{pmatrix},$$

since, for these $\alpha_1,\dots,\alpha_n$, we have that $\int_\Omega(\mathfrak{a}(\omega)-\sum_{i=1}^n\alpha_i\mathfrak{c}_i(\omega))\,dm_j=0$ for $j\in\{1,\dots,n\}$. Therefore, $\|\alpha_i\mathfrak{c}_i\|=|\alpha_i|\le n\,\|C^{-1}\|_\infty\|\mathfrak{a}\|$ for every $i\in\{1,\dots,n\}$, and hence $\|P_{\langle\mathfrak{c}_1,\dots,\mathfrak{c}_n\rangle}\mathfrak{a}\|=\|\sum_{i=1}^n\alpha_i\mathfrak{c}_i\|\le n^2\|C^{-1}\|_\infty\|\mathfrak{a}\|$. So, $P_{\langle\mathfrak{c}_1,\dots,\mathfrak{c}_n\rangle}\colon C(\Omega,\mathbb{R})\to\langle\mathfrak{c}_1,\dots,\mathfrak{c}_n\rangle$ is continuous. Finally, as $P_{C_0(\Omega,\mathbb{R})}\mathfrak{a}=\mathfrak{a}-P_{\langle\mathfrak{c}_1,\dots,\mathfrak{c}_n\rangle}\mathfrak{a}$, also the projection $P_{C_0(\Omega,\mathbb{R})}$ is continuous, so $C(\Omega,\mathbb{R})$ is the topological direct sum of $\langle\mathfrak{c}_1,\dots,\mathfrak{c}_n\rangle$ and $C_0(\Omega,\mathbb{R})$. Since $\int_\Omega\mathfrak{a}(\omega)\,dm_i=\int_\Omega(P_{\langle\mathfrak{c}_1,\dots,\mathfrak{c}_n\rangle}\mathfrak{a})(\omega)\,dm_i+\int_\Omega(P_{C_0(\Omega,\mathbb{R})}\mathfrak{a})(\omega)\,dm_i=\int_\Omega(P_{\langle\mathfrak{c}_1,\dots,\mathfrak{c}_n\rangle}\mathfrak{a})(\omega)\,dm_i$ for all $i\in\{1,\dots,n\}$, the second assertion follows. □

Corollary 3.38. *Let (Ω,σ) be minimal. Assume that $\mathfrak{M}_{\rm erg}(\Omega,\sigma)=\{m_1,\dots,m_n\}$ with $n\ge 2$, and take $r\ge 1$. Let $\mathfrak{a}_1\in C(\Omega,\mathbb{R})$ have dynamical spectrum $\mathrm{sp}(\mathfrak{a}_1)=[-\lambda_+,-\lambda_-]$ with $\lambda_-<0<\lambda_+$. Then, there exist $\tilde{\mathfrak{a}}_1\in C(\Omega,\mathbb{R})$ with dynamical* *spectrum* $\mathrm{sp}(\tilde{\mathfrak{a}}_1) = [-\lambda_+, -\lambda_-]$ *and* $\mathfrak{a}_1 - \tilde{\mathfrak{a}}_1 \in C_0(\Omega, \mathbb{R})$*, and* $k_1, k_2 \in \mathbb{R}$ *such that* $k_1 \le \tilde{\mathfrak{a}}_1(\omega) \le k_2$ *for all* $\omega \in \Omega$ *and*

$$(\lambda_+ - \lambda_-)^2 + 4\, r\, (\lambda_+ + k_1)(\lambda_+ + k_2) > 0\,. \tag{3.21}$$

Proof. We take $0 < \varepsilon < \min(\varepsilon_1, \varepsilon_2)$, with ε_1 and ε_2 respectively provided by Propositions 3.36 and 3.37. Let $\mathfrak{c}_1, \dots, \mathfrak{c}_n$ be the functions given by Lemma 3.35 for $m_1, \dots, m_n$ and ε. Proposition 3.37 provides $\alpha_1, \cdots, \alpha_n$ such that the map $\tilde{\mathfrak{a}}_1 = \alpha_1\mathfrak{c}_1 + \cdots + \alpha_n\mathfrak{c}_n$ satisfies $\mathrm{sp}(\tilde{\mathfrak{a}}_1) = \mathrm{sp}(\mathfrak{a}_1)$. Consider a permutation $\pi\colon \{1, \dots, n\} \to \{1, \dots, n\}$ such that $\alpha_{\pi(1)} \le \alpha_{\pi(2)} \le \cdots \le \alpha_{\pi(n)}$. Hence, since $\lambda_- < 0 < \lambda_+$, $\tilde{\mathfrak{a}}_1$ takes positive and negative values, and therefore $\alpha_{\pi(1)} < 0 < \alpha_{\pi(n)}$. Proposition 3.36 proves the claim when applied to $m_{\pi(1)}, \dots, m_{\pi(n)}$ and ε. □

Theorem 3.39. *Let* (Ω, σ) *be minimal. Assume that* $\mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma) = \{m_1, \dots, m_n\}$ *with* $n \ge 2$*. Let* $\mathfrak{a}_1 \in C(\Omega, \mathbb{R})$ *have dynamical spectrum* $\mathrm{sp}(\mathfrak{a}_1) = [-\lambda_+, -\lambda_-]$ *with* $\lambda_- < \lambda_+$*. Then, there exist* $\mathfrak{a}_i \in C(\Omega, \mathbb{R})$ *with* $\mathfrak{a}_i > 0$ *for* $i \in \{2, 3\}$ *such that* (3.13) *exhibits the generalized pitchfork bifurcation of minimal sets described in Theorem* 3.22*.*

Proof. We take any strictly positive $\tilde{\mathfrak{a}}_3 \in C(\Omega, \mathbb{R})$ and $0 < r_1 \le r_2$ with $r_1 \le \tilde{\mathfrak{a}}_3(\omega) \le r_2$ for all $\omega \in \Omega$, and call $r = r_2/r_1$. There is no loss of generality in assuming that $\lambda_- < 0 < \lambda_+$, since the type of bifurcation diagram for $\mathfrak{a}_1$ and $\mathfrak{a}_1 + \mu$ coincide for any $\mu \in \mathbb{R}$. We associate $\tilde{\mathfrak{a}}_1$ to $\mathfrak{a}_1$ and r by Corollary 3.38 and also $k_1, k_2 \in \mathbb{R}$ such that $k_1 \le \tilde{\mathfrak{a}}_1(\omega) \le k_2$ for all $\omega \in \Omega$, so (3.21) holds. Note that (3.21) and $\lambda_- < \lambda_+$ ensure that there exists $\delta_0 > 0$ such that, if $|\lambda_+ - \mu_+| < \delta_0$ and $|\lambda_- - \mu_-| < \delta_0$, then $\mu_- < \mu_+$ and

$$(\mu_+ - \mu_-)^2 + 4\, r(\mu_+ + k_1 - \delta_0)(\mu_+ + k_2 + \delta_0) > 0\,. \tag{3.22}$$

Then, for any $\mathfrak{c} \in C(\Omega, \mathbb{R})$ with $|\mathfrak{c}(\omega)| < \delta_0$ for all $\omega \in \Omega$ and any $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$,

$$-\lambda_+ - \delta_0 \le \int_\Omega \tilde{\mathfrak{a}}_1(\omega)\, dm - \delta_0 < \int_\Omega (\tilde{\mathfrak{a}}_1(\omega) + \mathfrak{c}(\omega))\, dm < \int_\Omega \tilde{\mathfrak{a}}_1(\omega)\, dm + \delta_0 \le -\lambda_- + \delta_0\,.$$

So, if we denote $\mathrm{sp}(\tilde{\mathfrak{a}}_1 + \mathfrak{c}) = [-\mu_+, -\mu_-]$, then $|\lambda_+ - \mu_+| < \delta_0$ and $|\lambda_- - \mu_-| < \delta_0$.

Note that (Ω, σ) is not a periodic flow, since it is not uniquely ergodic. Hence, $CP(\Omega, \mathbb{R})$ is dense in $C_0(\Omega, \mathbb{R})$ (see Proposition 1.64(iii)). By construction, $\mathfrak{a}_1 - \tilde{\mathfrak{a}}_1 \in C_0(\Omega, \mathbb{R})$, and hence there exists $\mathfrak{b} \in C^1(\Omega, \mathbb{R})$ such that $\max_{\omega \in \Omega} |\mathfrak{a}_1(\omega) - \tilde{\mathfrak{a}}_1(\omega) - \mathfrak{b}'(\omega)| < \delta_0$. We take $\mathfrak{c} = \mathfrak{a}_1 - \tilde{\mathfrak{a}}_1 - \mathfrak{b}'$, so that $|\mathfrak{c}(\omega)| < \delta_0$ for all $\omega \in \Omega$. We also take $\tilde{\mathfrak{a}}_2 \in C(\Omega, \mathbb{R})$ satisfying $2\sqrt{r_2\,(\mu_+ - k_1 + \delta_0)} < \tilde{\mathfrak{a}}_2(\omega) < \sqrt{r_1}\,(\mu_+ - \mu_-)/(\sqrt{(\mu_+ + k_2 + \delta_0)}\,)$ for all $\omega \in \Omega$. Then, since (3.22) holds for $[-\mu_+, -\mu_-] = \mathrm{sp}(\tilde{\mathfrak{a}}_1 + \mathfrak{c})$, Proposition 3.34 ensures that the parametric family

$$x' = -\tilde{\mathfrak{a}}_3(\omega{\cdot}t)\, x^3 + \tilde{\mathfrak{a}}_2(\omega{\cdot}t)\, x^2 + (\tilde{\mathfrak{a}}_1(\omega{\cdot}t) + \mathfrak{c}(\omega{\cdot}t) + \lambda)\, x\,, \quad \omega \in \Omega \tag{3.23}$$

presents a generalized pitchfork bifurcation of minimal sets. The same change of variables $y(t) = e^{\mathfrak{b}(\omega\cdot t)}x(t)$ used in the proof of Proposition 3.27 takes (3.23) to

$$y' = -e^{-2\mathfrak{b}(\omega\cdot t)}\, \tilde{\mathfrak{a}}_3(\omega{\cdot}t)\, y^3 + e^{-\mathfrak{b}(\omega\cdot t)}\, \tilde{\mathfrak{a}}_2(\omega{\cdot}t)\, y^2 + (\tilde{\mathfrak{a}}_1(\omega{\cdot}t) + \mathfrak{c}(\omega{\cdot}t) + \mathfrak{b}'(\omega{\cdot}t) + \lambda)\, y\,,$$

which, since $\mathfrak{c} = \mathfrak{a}_1 - \tilde{\mathfrak{a}}_1 - \mathfrak{b}'$, coincides with

$$y' = -e^{-2\mathfrak{b}(\omega\cdot t)}\, \tilde{\mathfrak{a}}_3(\omega{\cdot}t)\, y^3 + e^{-\mathfrak{b}(\omega\cdot t)}\, \tilde{\mathfrak{a}}_2(\omega{\cdot}t)\, y^2 + (\mathfrak{a}_1(\omega{\cdot}t) + \lambda)\, y\,,$$

and it does not change the global structure of the bifurcation diagram, as it was explained in the proof of Proposition 3.27. That is, the strictly positive functions $\mathfrak{a}_3 = e^{-2\mathfrak{b}}\,\tilde{\mathfrak{a}}_3$ and $\mathfrak{a}_2 = e^{-\mathfrak{b}}\,\tilde{\mathfrak{a}}_2$ fulfill the statement. □

3.4 Bifurcations of $x' = \mathfrak{h}(\omega{\cdot}t, x) + \lambda\, x^2$

The ideas and methods developed in Sections 3.2 and 3.3 allow us to classify and describe all the possibilities for the bifurcation diagram of a third problem,

$$x' = \mathfrak{h}(\omega{\cdot}t, x) + \lambda\, x^2\,, \quad \omega \in \Omega\,, \tag{3.24}$$

which will always be studied under assumption **d5**. Let τ_λ be the skewproduct flow induced by $(3.24)_\lambda$ in $\Omega \times \mathbb{R}$, with $\tau_\lambda(\omega, x) = (\omega{\cdot}t, v_\lambda(t, \omega, x))$. Hence, as in Section 3.3, $\mathcal{M}_0 = \Omega \times \{0\}$ is a τ_λ-copy of the base for all $\lambda \in \mathbb{R}$ and, if (Ω, σ) is minimal (which will be the case in most of the results of this section), then $\mathcal{M}_0 = \Omega \times \{0\}$ is a τ_λ-minimal set for every $\lambda \in \mathbb{R}$. However, a relevant difference arises with Section 3.3: Theorem 1.36(i) ensures that the dynamical spectrum of $\mathfrak{h}_x + 2\lambda\, x$ on $\mathcal{M}_0$ coincides with the dynamical spectrum $\mathrm{sp}(\mathfrak{h}_x(\cdot, 0))$ of $\omega \mapsto \mathfrak{h}_x(\omega, 0)$, which is independent of λ. Therefore, Proposition 1.42 guarantees that $\mathcal{M}_0$ exhibits the same hyperbolic or nonhyperbolic character for all the values of the parameter λ. We recall that Proposition 3.3(iii) ensures that $\mathfrak{h}(\omega, x) + \lambda\, x^2$ satisfies **d1**, **d2**, **d3**, **d4** and **d5** for all $\lambda \in \mathbb{R}$ if (and only if) $\mathfrak{h}$ satisfies **d1**, $\mathbf{d2}_{\lambda x^2}$, **d3**, **d4** and **d5**. As in the previous section, in this case, if (Ω, σ) is minimal, then Remark 2.7 and Proposition 3.3(iv) ensure that $\lim_{x\to\pm\infty} \mathfrak{h}_x(\omega, x) = -\infty$ uniformly on Ω.

Besides its own interest, the analysis of (3.24) allows us to go deeper in the construction of patterns for the three bifurcation possibilities described in Theorem 3.24 for (3.9), as explained at the end of this section.

Proposition 3.40. *Let $\mathfrak{h} \in C^{0,1}(\Omega \times \mathbb{R}, \mathbb{R})$ satisfy* **d5**. *Every strictly positive or negative global upper (resp. lower) solution of $(3.24)_\lambda$ is a strict global upper (resp. lower) solution of $(3.24)_\xi$ if $\xi < \lambda$ (resp. $\lambda < \xi$). Particularly, any strictly positive or negative equilibrium for $(3.24)_\lambda$ is a strong superequilibrium for $(3.24)_\xi$ if $\xi < \lambda$, as well as a strong subequilibrium for $(3.24)_\xi$ if $\lambda < \xi$.*

Proof. It is analogous to the proof of Propositions 3.4 and 3.17. □

The following proposition describes the parametric variation of the global attractor $\mathcal{A}_\lambda$, whose existence under the assumed conditions is guaranteed by Theorem 2.13, and whose lower and upper equilibria are represented by $\mathfrak{l}_\lambda$ and $\mathfrak{u}_\lambda$. We emphasize that, within this section, $\mathcal{A}_\lambda$, τ_λ, $v_\lambda(t, \omega, x)$, $\mathfrak{l}_\lambda$ and $\mathfrak{u}_\lambda$ refer to the dynamical elements of $(3.24)_\lambda$, not of $(3.5)_\lambda$ or $(3.9)_\lambda$.

Proposition 3.41. *Let $\mathfrak{h} \in C^{0,1}(\Omega \times \mathbb{R}, \mathbb{R})$ satisfy* $\mathbf{d2}_{\lambda x^2}$ *and* **d5**, *and let*

$$\mathcal{A}_\lambda = \bigcup_{\omega\in\Omega} \big(\{\omega\} \times [\mathfrak{l}_\lambda(\omega), \mathfrak{u}_\lambda(\omega)]\big)$$

be the global attractor for the skewproduct flow τ_λ induced by $(3.24)_\lambda$. Then,

(i) $\mathfrak{l}_\lambda(\omega) \le 0 \le \mathfrak{u}_\lambda(\omega)$ *for every $\omega \in \Omega$ and $\lambda \in \mathbb{R}$.*

(ii) *For any $\omega \in \Omega$, the maps $\lambda \mapsto \mathfrak{u}_\lambda(\omega)$ and $\lambda \mapsto \mathfrak{l}_\lambda(\omega)$ are nondecreasing on $\mathbb{R}$ and, respectively, right- and left-continuous. Moreover, if $\mathfrak{u}_{\lambda_0}(\omega_0) > 0$ (resp. $\mathfrak{l}_{\lambda_0}(\omega_0) < 0$) for some $\lambda_0 \in \mathbb{R}$ and $\omega_0 \in \Omega$, then $\mathfrak{u}_{\lambda_1}(\omega_0) < \mathfrak{u}_{\lambda_0}(\omega_0) < \mathfrak{u}_{\lambda_2}(\omega_0)$ (resp. $\mathfrak{l}_{\lambda_1}(\omega_0) < \mathfrak{l}_{\lambda_0}(\omega_0) < \mathfrak{l}_{\lambda_2}(\omega_0)$) for all $\lambda_1 < \lambda_0 < \lambda_2$.*

(iii) $\lim_{\lambda\to-\infty}\mathfrak{l}_\lambda(\omega)=-\infty$ *and* $\lim_{\lambda\to\infty}\mathfrak{u}_\lambda(\omega)=\infty$ *uniformly on* Ω.

Proof. The arguments are analogous to those of Proposition 3.18(i)-(iii). □

The results of Proposition 3.6 also hold for the family of global attractors of (3.24), with an analogous proof.

The proof of Theorem 3.45, which describes one of the possible bifurcation diagrams for (3.24), requires the next technical result, similar to Theorem 2.15 and Proposition 3.7.

Proposition 3.42. *Let* $\mathfrak{h}\colon\Omega\times\mathbb{R}\to\mathbb{R}$ *satisfy* **d1** *and* **d5**, *let us fix* $m\in\mathfrak{M}_{\rm erg}(\Omega,\sigma)$, $\mu>0$ *and* $\lambda_1\le\lambda_2$ *(resp.* $\lambda_2\le\lambda_1$*), let* $\mathfrak{b}_1\colon\Omega\to\mathbb{R}$ *be a bounded* m*-measurable equilibrium for* $x'=\mathfrak{h}(\omega{\cdot}t,x)-\mu\,x+\lambda_1\,x^2$, *and let* $\mathfrak{b}_2\colon\Omega\to\mathbb{R}$ *be a bounded* m*-measurable equilibrium for* $x'=\mathfrak{h}(\omega{\cdot}t,x)+\lambda_2\,x^2$, *such that* $0<\mathfrak{b}_1(\omega)<\mathfrak{b}_2(\omega)$ *(resp.* $\mathfrak{b}_2(\omega)<\mathfrak{b}_1(\omega)<0$*) for* m*-a.e.* $\omega\in\Omega$. *Assume that* $m(\{\omega\in\Omega\mid x\mapsto\mathfrak{h}_x(\omega,x)$ *is concave*$\})=1$. *Then,*

$$\int_\Omega\big(\mathfrak{h}_x(\omega,\mathfrak{b}_1(\omega))-\mu+2\lambda_1\,\mathfrak{b}_1(\omega)\big)\,dm+\int_\Omega\big(\mathfrak{h}_x(\omega,\mathfrak{b}_2(\omega))+2\lambda_2\,\mathfrak{b}_2(\omega)\big)\,dm<0\,.$$

Proof. Let Ω_d be the σ-invariant set with $m(\Omega_d)=1$ given by Lemma 2.6, and let $\Omega_0=\Omega_d\cap\{\omega\in\Omega\mid 0<\mathfrak{b}_2(\omega)<\mathfrak{b}_1(\omega)\}$. Then, $m(\Omega_0)=1$. Since $\mathfrak{b}_1'(\omega{\cdot}t)=\mathfrak{h}(\omega{\cdot}t,\mathfrak{b}_1(\omega{\cdot}t))-\mu\,\mathfrak{b}_1(\omega{\cdot}t)+\lambda_1\,\mathfrak{b}_1(\omega{\cdot}t)^2<\mathfrak{h}(\omega{\cdot}t,\mathfrak{b}_1(\omega{\cdot}t))+\lambda_2\,\mathfrak{b}_1(\omega{\cdot}t)^2$ for all $t\in\mathbb{R}$ and $\omega\in\Omega$ such that $\mathfrak{b}_1(\omega)>0$, a standard comparison argument shows that $0<\mathfrak{b}_1(\omega{\cdot}t)<\mathfrak{b}_2(\omega{\cdot}t)$ for all $t\ge0$ if $0<\mathfrak{b}_1(\omega)<\mathfrak{b}_2(\omega)$, and hence $\omega{\cdot}t\in\Omega_0$ if $\omega\in\Omega_0$ and $t\ge0$. The function $\mathfrak{c}(\omega)=\mathfrak{b}_2(\omega)-\mathfrak{b}_1(\omega)$ satisfies

$$\begin{aligned}\mathfrak{c}'(\omega{\cdot}t)&=\mathfrak{h}(\omega{\cdot}t,\mathfrak{b}_2(\omega{\cdot}t))+\lambda_2\,\mathfrak{b}_2(\omega{\cdot}t)^2-\mathfrak{h}(\omega{\cdot}t,\mathfrak{b}_1(\omega{\cdot}t))+\mu\,\mathfrak{b}_1(\omega{\cdot}t)-\lambda_1\,\mathfrak{b}_1(\omega{\cdot}t)^2\\ &=\mathfrak{c}(\omega{\cdot}t)\int_0^1\mathfrak{h}_x\big(\omega{\cdot}t,s\,\mathfrak{c}(\omega{\cdot}t)+\mathfrak{b}_1(\omega{\cdot}t)\big)\,ds\\ &\qquad+(\lambda_2-\lambda_1)\,\mathfrak{b}_2(\omega{\cdot}t)^2+\lambda_1\big(\mathfrak{b}_2(\omega{\cdot}t)^2-\mathfrak{b}_1(\omega{\cdot}t)^2\big)+\mu\,\mathfrak{b}_1(\omega{\cdot}t)\end{aligned}$$

for all $\omega\in\Omega$ and all $t\in\mathbb{R}$, and hence

$$\frac{\mathfrak{c}'(\omega{\cdot}t)}{\mathfrak{c}(\omega{\cdot}t)}=F(\omega{\cdot}t,\mathfrak{c}(\omega{\cdot}t))+(\lambda_2-\lambda_1)\frac{\mathfrak{b}_2(\omega{\cdot}t)^2}{\mathfrak{c}(\omega{\cdot}t)}+\lambda_1\big(\mathfrak{b}_1(\omega{\cdot}t)+\mathfrak{b}_2(\omega{\cdot}t)\big)+\mu\,\frac{\mathfrak{b}_1(\omega{\cdot}t)}{\mathfrak{c}(\omega{\cdot}t)}\tag{3.25}$$

for all $\omega\in\Omega_0$ and $t\ge0$, where $F(\omega,y)=\int_0^1\mathfrak{h}_x(\omega,s\,y+\mathfrak{b}_1(\omega))\,ds$. Since $\omega\mapsto F(\omega,\mathfrak{c}(\omega))+\lambda_1(\mathfrak{b}_1(\omega)+\mathfrak{b}_2(\omega))$ is bounded, and hence it is in $L^1(\Omega,m)$ and $\omega\mapsto(\lambda_2-\lambda_1)\,\mathfrak{b}_2^2(\omega)/\mathfrak{c}(\omega)+\mu\,\mathfrak{b}_1(\omega)/\mathfrak{c}(\omega)$ is strictly positive on Ω_0, Birkhoff's Ergodic Theorem 1.10 applied to (3.25) (see the proof of Theorem 2.15) yields

$$\begin{aligned}0=\int_\Omega F(\omega,\mathfrak{c}(\omega))\,dm&+(\lambda_2-\lambda_1)\int_\Omega\frac{\mathfrak{b}_2(\omega)^2}{\mathfrak{c}(\omega)}\,dm\\ &+\lambda_1\int_\Omega\big(\mathfrak{b}_1(\omega)+\mathfrak{b}_2(\omega)\big)\,dm+\mu\int_\Omega\frac{\mathfrak{b}_1(\omega)}{\mathfrak{c}(\omega)}\,dm\,.\end{aligned}\tag{3.26}$$

Equation (2.19), which also holds in this case, and (3.26) yield

$$\begin{aligned}\int_\Omega&\big(\mathfrak{h}_x(\omega,\mathfrak{b}_1(\omega))-\mu+2\lambda_1\,\mathfrak{b}_1(\omega)\big)\,dm+\int_\Omega\big(\mathfrak{h}_x(\omega,\mathfrak{b}_2(\omega))+2\lambda_2\,\mathfrak{b}_2(\omega)\big)\,dm\\ &\le2\int_\Omega F(\omega,\mathfrak{c}(\omega))\,dm+2\int_\Omega\big(\lambda_1\,\mathfrak{b}_1(\omega)+\lambda_2\,\mathfrak{b}_2(\omega)\big)\,dm-\mu\\ &=-2\,(\lambda_2-\lambda_1)\int_\Omega\frac{\mathfrak{b}_1(\omega)\mathfrak{b}_2(\omega)}{\mathfrak{c}(\omega)}\,dm-\mu\int_\Omega\frac{\mathfrak{b}_1(\omega)+\mathfrak{b}_2(\omega)}{\mathfrak{c}(\omega)}\,dm<0\,,\end{aligned}$$

which proves the statement. The other case is analogous. □

3.4.1 Bifurcation diagrams with minimal base flow

In this section, under the assumption that (Ω,σ) is minimal, all the possible bifurcation diagrams of τ_λ-minimal sets for (3.24) are presented. As remarked previously, in this case, the hyperbolic or nonhyperbolic character of the τ_λ-minimal set $\mathcal{M}_0=\Omega\times\{0\}$ does not change with the parameter λ. The three possible bifurcation diagrams are classified in terms of $\mathcal{M}_0$ being hyperbolic repulsive, hyperbolic attractive or nonhyperbolic, or equivalently, in terms of the dynamical spectrum $\mathrm{sp}(\mathfrak{h}_x(\cdot,0))$ either being contained in $(0,\infty)$, contained in $(-\infty,0)$ or containing 0.

Theorem 3.43 (No bifurcation)**.** *Let (Ω,σ) be minimal. Let $\mathfrak{h}\colon\Omega\times\mathbb{R}\to\mathbb{R}$ satisfy* **d1**, **d2**$_{\lambda x^2}$, **d3**, **d4** *and* **d5**. *If $\mathrm{sp}(\mathfrak{h}_x(\cdot,0))\subset(0,\infty)$, then*

(i) *for all $\lambda\in\mathbb{R}$ there exist three hyperbolic τ_λ-minimal sets $\mathcal{M}^l_\lambda<\mathcal{M}_0<\mathcal{M}^u_\lambda$, where $\mathcal{M}^l_\lambda$ and $\mathcal{M}^u_\lambda$ are attractive and given by the graphs of $\mathfrak{l}_\lambda$ and $\mathfrak{u}_\lambda$ respectively, and $\mathcal{M}_0$ is repulsive.*

(ii) *The maps $\mathbb{R}\to C(\Omega,\mathbb{R})$, $\lambda\mapsto\mathfrak{u}_\lambda$ and $\mathbb{R}\to C(\Omega,\mathbb{R})$, $\lambda\mapsto\mathfrak{l}_\lambda$ are continuous in the uniform topology, and $\lim_{\lambda\to\infty}\mathfrak{l}_\lambda(\omega)=\lim_{\lambda\to-\infty}\mathfrak{u}_\lambda(\omega)=0$ uniformly on Ω.*

Proof. (i) Theorem 1.36(i) and Proposition 1.42 ensure that $\mathcal{M}_0$ is a repulsive hyperbolic τ_λ-minimal set for every $\lambda\in\mathbb{R}$. Consequently, Proposition 2.14(i) and Theorem 2.11 ensure that, for every $\lambda\in\mathbb{R}$, there exist three different hyperbolic τ_λ-minimal sets $\mathcal{M}^l_\lambda<\mathcal{M}_0<\mathcal{M}^u_\lambda$, where $\mathcal{M}^l_\lambda$ and $\mathcal{M}^u_\lambda$ are hyperbolic attractive and respectively given by the graphs of $\mathfrak{l}_\lambda<\mathfrak{u}_\lambda$.

(ii) The hyperbolic continuation of minimal sets (see Theorem 1.39) guarantees the continuity of the maps $\mathbb{R}\to C(\Omega,\mathbb{R})$, $\lambda\mapsto\mathfrak{u}_\lambda$ and $\mathbb{R}\to C(\Omega,\mathbb{R})$, $\lambda\mapsto\mathfrak{l}_\lambda$ in the uniform topology. To check that $\lim_{\lambda\to-\infty}\mathfrak{u}_\lambda(\omega)=0$ uniformly on Ω, we take any $\varepsilon>0$. Hypothesis **d2**$_{\lambda x^2}$ provides $\rho>\varepsilon>0$ such that $\mathfrak{h}(\omega,x)\le 0$ for all $x\ge\rho$ and $\omega\in\Omega$. Let us choose $\lambda_\varepsilon<-\sup\{\mathfrak{h}(\omega,x)/x^2\mid(\omega,x)\in\Omega\times[\varepsilon,\rho]\}$ with $\lambda_\varepsilon<0$, so $\mathfrak{h}(\omega,x)+\lambda_\varepsilon x^2\le 0$ for all $x\in[\varepsilon,\rho]$ and $\omega\in\Omega$. Then, $\mathfrak{h}(\omega,x)+\lambda x^2\le 0$ for all $\lambda\le\lambda_\varepsilon$, $x\ge\varepsilon$ and $\omega\in\Omega$. According to Theorem 2.13(v), $\mathfrak{u}_\lambda(\omega)\le\varepsilon$ for all $\omega\in\Omega$ if $\lambda\le\lambda_\varepsilon$, which proves the assertion. The argument is analogous for $\mathfrak{l}_\lambda$. □

The left panel of Figure 3.6 depicts the absence of bifurcation described in Theorem 3.43 (and Proposition 3.41).

Theorem 3.44 (Two local saddle-node bifurcations)**.** *Let (Ω,σ) be minimal. Let $\mathfrak{h}\colon\Omega\times\mathbb{R}\to\mathbb{R}$ satisfy* **d1**, **d2**$_{\lambda x^2}$, **d3**, **d4** *and* **d5**. *If $\mathrm{sp}(\mathfrak{h}_x(\cdot,0))\subset(-\infty,0)$, then*

(i) *$\mathcal{M}_0$ is an attractive hyperbolic τ_λ-copy of the base for all $\lambda\in\mathbb{R}$.*

In addition, there exist $\lambda_1<\lambda_2$ such that

(ii) *for all $\lambda>\lambda_2$ (resp. $\lambda<\lambda_1$), there exist three hyperbolic τ_λ-minimal sets $\mathcal{M}_0<\mathcal{N}_\lambda<\mathcal{M}^u_\lambda$ (resp. $\mathcal{M}^l_\lambda<\mathcal{N}_\lambda<\mathcal{M}_0$) which are τ_λ-copies of the base, given by the graphs of $\mathfrak{l}_\lambda=0<\mathfrak{m}_\lambda<\mathfrak{u}_\lambda$ (resp. $\mathfrak{l}_\lambda<\mathfrak{m}_\lambda<0=\mathfrak{u}_\lambda$), where $\mathcal{M}^u_\lambda$ (resp. $\mathcal{M}^l_\lambda$) is attractive and $\mathcal{N}_\lambda$ is repulsive, and $\lambda\mapsto\mathfrak{m}_\lambda$ is strictly decreasing on (λ_2,∞) (resp. $(-\infty,\lambda_1)$); and the graphs of $\mathfrak{m}_\lambda$ and $\mathfrak{u}_\lambda$ (resp. $\mathfrak{l}_\lambda$ and $\mathfrak{m}_\lambda$) collide on a residual σ-invariant set as $\lambda\downarrow\lambda_2$ (resp. $\lambda\uparrow\lambda_1$), giving rise to a nonhyperbolic τ_{λ_2}-minimal set $\mathcal{M}^u_{\lambda_2}$ (resp. τ_{λ_1}-minimal set $\mathcal{M}^l_{\lambda_1}$).*

(iii) *For* $\lambda\in(\lambda_1,\lambda_2)$, $\mathcal{A}_\lambda=\mathcal{M}_0$.

(iv) $\lim_{\lambda\to\pm\infty}\mathfrak{m}_\lambda(\omega)=0$ *uniformly on* Ω.

In particular, two local saddle-node bifurcations of minimal sets occur at λ_1 *and* λ_2*, which are points of discontinuity of the global attractor.*

Proof. Theorem 1.36(i) and Proposition 1.42 ensure that $\mathcal{M}_0$ is a attractive hyperbolic τ_λ-minimal set for every $\lambda\in\mathbb{R}$. We fix $\rho>0$ and $\lambda_\rho^+>-\inf\{\mathfrak{h}(\omega,\rho)/\rho^2\mid\omega\in\Omega\}$ with $\lambda_\rho^+>0$, and take $\lambda>\lambda_\rho^+$. Then, $\mathfrak{h}(\omega,\rho)+\lambda\,\rho^2>0$ for all $\omega\in\Omega$. A standard comparison argument ensures that $v_\lambda(t,\omega,\rho)\in(0,\rho)$ for all $t<0$, and hence the α-limit set for τ_λ of a point (ω,ρ) contains a τ_λ-minimal set $\mathcal{N}_\lambda\subset\Omega\times[0,\rho]$. Corollary 1.58(i) guarantees that $\mathcal{M}_0<\mathcal{N}_\lambda$. On the other hand, Theorem 2.13(v) yields $\rho<\mathfrak{u}_\lambda(\omega)$ for all $\omega\in\Omega$, and hence Proposition 2.17(i) ensures that the upper minimal set $\mathcal{M}_\lambda^u$ is strictly above $\mathcal{N}_\lambda$. Altogether, we conclude that for any $\lambda>\lambda_\rho^+$, there exist three minimal sets $\mathcal{M}_0<\mathcal{N}_\lambda<\mathcal{M}_\lambda^u$, and hence Theorem 2.11 ensures that the three of them are hyperbolic τ_λ-copies of the base with $\mathcal{M}_\lambda^u=\{\mathfrak{u}_\lambda\}$ attractive and $\mathcal{N}_\lambda$ repulsive. An analogous argument works for $-\rho$ and $\lambda_\rho^-<-\sup\{\mathfrak{h}(\omega,-\rho)/\rho^2\mid\omega\in\Omega\}$ with $\lambda_\rho^-<0$, providing three different hyperbolic τ_λ-copies of the base $\mathcal{M}_\lambda^l<\mathcal{N}_\lambda<\mathcal{M}_0$ for $\lambda\le\lambda_\rho^-$, with $\mathcal{M}_\lambda^l=\{\mathfrak{l}_\lambda\}$ attractive and $\mathcal{N}_\lambda$ repulsive. Let $\mathfrak{m}_\lambda$ be the continuous τ_λ-equilibrium whose graph is $\mathcal{N}_\lambda$, both for $\lambda\ge\lambda_\rho^+$ and $\lambda\le\lambda_\rho^-$. Since the initially fixed $\rho>0$ is as small as desired and $|\mathfrak{m}_\lambda|\le\rho$ in both cases, $\lim_{\lambda\to\pm\infty}\mathfrak{m}_\lambda(\omega)=0$ uniformly on Ω. The claims (i) and (iv) have been proved. Let us define

$$\begin{aligned}\mathcal{I}_1&=\{\lambda\colon\forall\,\xi<\lambda\text{ the graph of }\mathfrak{l}_\xi\text{ is a hyperb. minimal set }\mathcal{M}_\xi^l<\mathcal{M}_0\}\,,\\ \mathcal{I}_2&=\{\lambda\colon\forall\,\xi>\lambda\text{ the graph of }\mathfrak{u}_\xi\text{ is a hyperb. minimal set }\mathcal{M}_\xi^u>\mathcal{M}_0\}\,,\end{aligned}\tag{3.27}$$

and observe that $\lambda_\rho^-\in\mathcal{I}_1$ and $\lambda_\rho^+\in\mathcal{I}_2$ for any $\rho>0$. We also define $\lambda_1=\sup\mathcal{I}_1$ and $\lambda_2=\inf\mathcal{I}_2$ and note that $\lambda_1\notin\mathcal{I}_1$ and $\lambda_2\notin\mathcal{I}_2$: otherwise Proposition 2.14(ii) would ensure that there exist three τ_{λ_1}- or τ_{λ_2}-minimal sets, and Theorems 1.39 and 2.11 would contradict the definition of λ_1 or λ_2. Moreover, $\lambda_1\le\lambda_2$: otherwise there exists $\lambda\in(\lambda_2,\lambda_1)$ such that $\{\mathfrak{l}_\lambda\}<\mathcal{M}_0<\{\mathfrak{u}_\lambda\}$ are three attractive hyperbolic τ_λ-minimal sets (see Proposition 2.17(ii)), which is precluded by Theorem 2.11. Hence, both λ_1 and λ_2 are finite. If $\lambda\in\mathcal{I}_1$ (resp. $\lambda\in\mathcal{I}_2$), then Propositions 2.17(ii) and 2.14(ii) and Theorem 2.11 ensure that there exists a repulsive hyperbolic τ_λ-copy of the base $\mathcal{N}_\lambda$ and that $\mathcal{M}_\lambda^l<\mathcal{N}_\lambda<\mathcal{M}_0$ and $\mathfrak{u}_\lambda\equiv0$ (resp. $\mathcal{M}_0<\mathcal{N}_\lambda<\mathcal{M}_\lambda^u$ and $\mathfrak{l}_\lambda\equiv0$). Let $\mathfrak{m}_\lambda$ be the continuous τ_λ-equilibrium whose graph is $\mathcal{N}_\lambda$ for $\lambda\in\mathcal{I}_1\cup\mathcal{I}_2$. Theorem 1.39 guarantees the continuity with respect to the uniform topology of the maps $\lambda\mapsto\mathfrak{l}_\lambda,\mathfrak{m}_\lambda$ on $\mathcal{I}_1$ and $\lambda\mapsto\mathfrak{m}_\lambda,\mathfrak{u}_\lambda$ on $\mathcal{I}_2$. In addition, an analogous argument to that of the second paragraph of the proof of Theorem 3.8 shows that $\lambda\mapsto\mathfrak{m}_\lambda(\omega)$ is strictly decreasing on $\mathcal{I}_1$ and $\mathcal{I}_2$ for all $\omega\in\Omega$.

The maps $\mathfrak{l}_{\lambda_1}(\omega)=\lim_{\lambda\uparrow\lambda_1}\mathfrak{l}_\lambda(\omega)$ and $\mathfrak{m}_{\lambda_1}(\omega)=\lim_{\lambda\uparrow\lambda_1}\mathfrak{m}_\lambda(\omega)$ satisfy $\mathfrak{l}_{\lambda_1}\le\mathfrak{m}_{\lambda_1}<0$ and are lower and upper semicontinuous τ_{λ_1}-equilibria respectively, which must coincide on the residual set of its common continuity points: otherwise Proposition 1.32 would define two different τ_{λ_1}-minimal sets $\mathcal{M}_{\lambda_1}^l<\mathcal{N}_{\lambda_1}<\mathcal{M}_0$, so $\mathcal{M}_{\lambda_1}^l$ would be hyperbolic attractive (see Theorem 2.11) and $\lambda_1\in\mathcal{I}_1$, which is not the case. Proposition 2.14(ii) ensures that the unique τ_{λ_1}-minimal set $\mathcal{M}_{\lambda_1}^l$ that $\mathfrak{l}_{\lambda_1}$ and $\mathfrak{m}_{\lambda_1}$ define by (1.9) is nonhyperbolic. Therefore, λ_1 is a local saddle-node bifurcation point of minimal sets. In addition, notice that $\lambda_1<\lambda_2$: otherwise there would exist three τ_{λ_1}-minimal sets, contradicting (see Theorem 2.11) the nonhyperbolicity of $\mathcal{M}^l_{\lambda_1}$. An analogous argument to that at the end of the proof of Theorem 3.8 shows that both λ_1 and λ_2 are points of lower discontinuity of the global attractor. So (ii) and the final assertions of the statement are proved.

It remains to prove (iii). Let $\lambda \in (\lambda_1, \lambda_2)$, and let $\omega_0 \in \Omega$ be a common continuity point of $\mathfrak{l}_\lambda$, $\mathfrak{l}_{\lambda_1}$ and $\mathfrak{m}_{\lambda_1}$. In particular, $\mathfrak{l}_{\lambda_1}(\omega_0) = \mathfrak{m}_{\lambda_1}(\omega_0)$. We assume for contradiction that $\mathfrak{l}_\lambda(\omega_0) < 0$. Then, Proposition 3.41(ii) ensures that $\mathfrak{m}_{\lambda_1}(\omega_0) = \mathfrak{l}_{\lambda_1}(\omega_0) < \mathfrak{l}_\lambda(\omega_0)$. Since $\lim_{\lambda\to-\infty} \mathfrak{m}_\lambda(\omega_0) = 0$ and $(-\infty, \lambda_1] \to \mathbb{R}$, $\lambda \mapsto \mathfrak{m}_\lambda(\omega_0)$ is continuous, there exists $\xi < \lambda_1$ such that $\mathfrak{m}_\xi(\omega_0) = \mathfrak{l}_\lambda(\omega_0)$. Proposition 3.40 and the definition of λ_1 ensure that $\mathfrak{m}_\xi$ is a strong continuous τ_λ-subequilibrium. Proposition 1.22 provides $s > 0$ and $e > 0$ such that $\mathfrak{l}_\lambda(\omega_0{\cdot}t) \geq \mathfrak{m}_\xi(\omega_0{\cdot}t) + e$ for all $t \geq s$, and we get the contradiction $\mathfrak{l}_\lambda(\omega_0) = \lim_{n\to\infty} \mathfrak{l}_\lambda(\omega_0{\cdot}t_n) \geq \lim_{n\to\infty} \mathfrak{m}_\xi(\omega_0{\cdot}t_n) + e = \mathfrak{m}_\xi(\omega_0) + e$ by taking $(t_n) \uparrow \infty$ with $\omega_0 = \lim_{n\to\infty} \omega_0{\cdot}t_n$. This means that $\mathfrak{l}_\lambda(\omega_0) = 0$, and hence Proposition 2.17(i) shows that $\mathcal{M}^l_\lambda = \mathcal{M}_0$ is the lowest τ_λ-minimal set. An analogous argument shows that there are no τ_λ-minimal sets above $\mathcal{M}_0$. Consequently, $\mathcal{M}_0$ is the unique τ_λ-minimal set for $\lambda \in (\lambda_1, \lambda_2)$ and, since it is hyperbolic attractive, Corollary 1.58(iii) proves that $\mathcal{A}_\lambda = \mathcal{M}_0$ for $\lambda \in (\lambda_1, \lambda_2)$. □

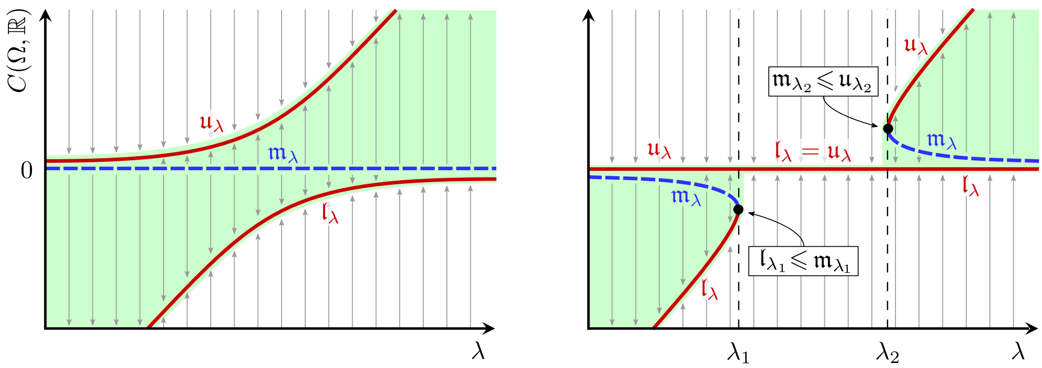


Figure 3.6: The no bifurcation bifurcation diagram (left) described in Theorem 3.43 and two local saddle-node bifurcation diagram (right) described in Theorem 3.44. The meaning of the different elements is explained in Figure 3.4.

The right panel of Figure 3.6 depicts the bifurcation described diagram described by Theorem 3.44 (and Proposition 3.41).

Theorem 3.45 (Weak generalized transcritical bifurcation). *Let (Ω, σ) be minimal. Let $\mathfrak{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy* **d1**, **d2**$_{\lambda x^2}$, **d3**, **d4** *and* **d5**. *If $0 \in \operatorname{sp}(\mathfrak{h}_x(\cdot, 0))$, then*

(i) *$\mathcal{M}_0$ is a nonhyperbolic τ_λ-copy of the base for all $\lambda \in \mathbb{R}$, and there exists at most another τ_λ-minimal set.*

In addition, there exist $\lambda_1 \leq \lambda_2$ such that

(ii) *for all $\lambda > \lambda_2$ (resp. $\lambda < \lambda_1$) there exist exactly two τ_λ-minimal sets $\mathcal{M}_0 < \mathcal{M}^u_\lambda$ (resp. $\mathcal{M}^l_\lambda < \mathcal{M}_0$), where $\mathcal{M}^u_\lambda$ (resp. $\mathcal{M}^l_\lambda$) is an attractive hyperbolic copy of the base given by the graph of $\mathfrak{u}_\lambda$ (resp. $\mathfrak{l}_\lambda$).*

(iii) *If $\mathcal{M}_0$ is the unique τ_{λ_2}-minimal set (resp. τ_{λ_1}-minimal set), then 0 and $\mathfrak{u}_\lambda$ (resp. $\mathfrak{l}_\lambda$ and 0) collide on a residual σ-invariant set as $\lambda \downarrow \lambda_2$ (resp. $\lambda \uparrow \lambda_1$).*

(iv) *If $0 = \inf \operatorname{sp}(\mathfrak{h}_x(\cdot, 0))$, then $\mathcal{M}_0$ is the unique τ_λ-minimal set for all $\lambda \in [\lambda_1, \lambda_2]$.*

(v) *If* $0 \neq \inf \operatorname{sp}(\mathfrak{h}_x(\cdot,0))$, *then* $\lambda_1 < \lambda_2$ *and* $\mathcal{M}_0$ *is the unique* τ_λ*-minimal set for any* $\lambda \in (\lambda_1, \lambda_2)$.

(vi) *If* $0 = \sup \operatorname{sp}(\mathfrak{h}_x(\cdot,0))$, *then there exists a* σ*-invariant subset* $\Omega_0 \subseteq \Omega$ *such that* $m(\Omega_0) = 1$ *for all* $m \in \mathfrak{M}_{\text{erg}}(\Omega,\sigma)$ *and such that* $\mathfrak{l}_\lambda(\omega) = 0$ *for all* $\omega \in \Omega_0$ *and* $\lambda > \lambda_2$ *and* $\mathfrak{u}_\lambda(\omega) = 0$ *for all* $\omega \in \Omega_0$ *and* $\lambda < \lambda_1$.

(vii) *If* $0 \neq \sup \operatorname{sp}(\mathfrak{h}_x(\cdot,0))$, *then* $\mathfrak{u}_\lambda$ *and* $\mathfrak{l}_\lambda$ *are not identically* 0 *for any* $\lambda \in \mathbb{R}$.

Proof. (i) Theorem 1.36(i) and Proposition 1.42 ensure that $\mathcal{M}_0$ is nonhyperbolic for all $\lambda \in \mathbb{R}$, and this fact precludes the existence of three τ_λ-minimal sets (see Theorem 2.11).

(ii) Propositions 3.41(iii) and 2.17 ensure that $\mathfrak{u}_\lambda$ defines a τ_λ-minimal set $\mathcal{M}^u_\lambda > \mathcal{M}_0$ by (1.9) if λ is large enough, say $\lambda > \lambda_0$. Let us check that $\mathcal{M}^u_\lambda$ is an attractive hyperbolic τ_λ-copy of the base if λ is large enough. To this end, we fix $\mu > 0$ such that the dynamical spectrum of $\mathfrak{h}_x(\omega,0) - \mu$ is contained in $(-\infty,0)$. Consequently, the bifurcation diagram of

$$x' = \mathfrak{h}(\omega{\cdot}t, x) - \mu\, x + \xi\, x^2 \tag{3.28}$$

with respect to the parameter ξ is that described in Theorem 3.44. Let $\xi_1 < \xi_2$ be the two saddle-node bifurcation points of (3.28), and, for any $\xi > \xi_2$, let $0 < \hat{\mathfrak{m}}_\xi < \hat{\mathfrak{u}}_\xi$ be the equilibria of $(3.28)_\xi$ giving rise to the hyperbolic copies of the base. We take $\lambda > \max\{\lambda_0, \xi_2\}$. Then,

$$\hat{\mathfrak{u}}'_\lambda(\omega) = \mathfrak{h}(\omega, \hat{\mathfrak{u}}_\lambda(\omega)) - \mu\, \hat{\mathfrak{u}}_\lambda(\omega) + \lambda\, \hat{\mathfrak{u}}_\lambda(\omega)^2 < \mathfrak{h}(\omega, \hat{\mathfrak{u}}_\lambda(\omega)) + \lambda\, \hat{\mathfrak{u}}_\lambda(\omega)^2\,,$$

so Theorem 2.13(v) ensures that $\hat{\mathfrak{u}}_\lambda(\omega) < \mathfrak{u}_\lambda(\omega)$ for all $\omega \in \Omega$. Hence, if follows from the definition (1.9) of $\mathcal{M}^u_\lambda$ that $\hat{\mathfrak{u}}_\lambda \le \mathfrak{b}_\lambda$ for any equilibrium $\mathfrak{b}_\lambda$ with graph contained in $\mathcal{M}^u_\lambda$. Theorem 1.36(iii) provides a τ_λ-equilibrium $\mathfrak{b}_\lambda\colon \Omega \to \mathbb{R}$ with graph contained in $\mathcal{M}^u_\lambda$ and $m \in \mathfrak{M}_{\text{erg}}(\Omega,\sigma)$ such that $\sup \operatorname{Lyap}(\mathcal{M}^u_\lambda) = \int_\Omega \big(\mathfrak{h}_x(\omega, \mathfrak{b}_\lambda(\omega)) + 2\lambda\, \mathfrak{b}_\lambda(\omega)\big)\, dm$. Since $\{\hat{\mathfrak{m}}_\lambda\}$ is a repulsive hyperbolic copy of the base for $(3.28)_\lambda$, Theorems 1.40 and 1.36(iii) ensure that $\int_\Omega \big(\mathfrak{h}_x(\omega, \hat{\mathfrak{m}}_\lambda(\omega)) - \mu + 2\lambda\, \hat{\mathfrak{m}}_\lambda(\omega)\big)\, dm > 0$. Thus, since $\hat{\mathfrak{m}}_\lambda(\omega) < \hat{\mathfrak{u}}_\lambda(\omega) \le \mathfrak{b}_\lambda(\omega)$ for all $\omega \in \Omega$, Proposition 3.42 ensures that $\sup \operatorname{Lyap}(\mathcal{M}^u_\lambda) < 0$. Then, Theorem 1.40 ensures that $\mathcal{M}^u_\lambda$ is an attractive hyperbolic τ_λ-minimal set for these (large enough) values of λ. An analogous argument shows that $\mathcal{M}^l_\lambda$ is an attractive hyperbolic τ_λ-minimal set if $-\lambda$ is large enough. Now, we define $\mathcal{I}_1$ and $\mathcal{I}_2$ as in (3.27), $\lambda_1 = \sup \mathcal{I}_1$ and $\lambda_2 = \inf \mathcal{I}_2$. Note that $\lambda_1 \le \lambda_2$: otherwise there would be three τ_λ-minimal sets for $\lambda \in (\lambda_2, \lambda_1)$, which is not possible, as said before. This proves (ii).

(iii) Proposition 2.17(i) ensures that $\mathfrak{u}_{\lambda_2}$ vanishes on a residual set of continuity points. The continuity of $[\lambda_2, \infty) \to \mathbb{R}$, $\lambda \mapsto \mathfrak{u}_\lambda(\omega)$ for all $\omega \in \Omega$, given by Proposition 3.41(ii) and Theorem 1.39, provides the collision. This and an analogous argument for λ_1 prove the claim.

(iv) Assume that $0 = \inf \operatorname{sp}(\mathfrak{h}_x(\cdot,0))$, which implies that $\int_\Omega \mathfrak{h}_x(\omega,0)\, dm \ge 0$ for all $m \in \mathfrak{M}_{\text{erg}}(\Omega,\sigma)$. Let us check that $\mathcal{M}_0$ is the unique τ_λ-minimal set if $\lambda \in [\lambda_1, \lambda_2]$. Theorems 2.15 and 1.36(iii) ensure that, for any $\lambda \in \mathbb{R}$, the upper Lyapunov exponent of any τ_λ-minimal set distinct from $\mathcal{M}_0$ is strictly negative. Therefore, Theorem 1.40 ensures that, for any $\lambda \in \mathbb{R}$, any τ_λ-minimal set distinct from $\mathcal{M}_0$ is hyperbolic attractive. If $\mathcal{M}^u_{\lambda_2} > \mathcal{M}_0$, then Proposition 2.17(ii) would ensure that $\mathcal{M}^u_{\lambda_2} = \{\mathfrak{u}_{\lambda_2}\}$, which combined with the persistence Theorem 1.39 would contradict the definition of λ_2. Thus, Proposition 2.17(i) ensures that $\mathfrak{u}_{\lambda_2}(\omega) = 0$ on the residual subset of Ω of its continuity points. So, the monotonicity properties of $\mathfrak{u}_\lambda$ established in Proposition 3.41(ii) ensure that there are no τ_λ-minimal sets above $\mathcal{M}_0$ for $\lambda \le \lambda_2$: otherwise, if there exists $\mathcal{M}^u_\lambda > \mathcal{M}_0$, then $\mathfrak{u}_\lambda > 0$, so $\mathfrak{u}_{\lambda_2} > 0$, a contradiction. An analogous reasoning shows that there are no τ_λ-minimal sets below $\mathcal{M}_0$ for $\lambda \ge \lambda_1$. Consequently, $\mathcal{M}_0$ is the unique τ_λ-minimal set for $\lambda \in [\lambda_1, \lambda_2]$.

(v) Assume that $0 \ne \inf \mathrm{sp}(\mathfrak{h}_x(\cdot, 0))$. Since $0 \in \mathrm{sp}(\mathfrak{h}_x(\cdot, 0))$, $\inf \mathrm{sp}(\mathfrak{h}_x(\cdot, 0)) < 0$, so there exists $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ with $\int_\Omega \mathfrak{h}_x(\omega, 0)\, dm < 0$. Let us check that $\lambda_1 < \lambda_2$. For $\lambda > \lambda_2$, Lemma 3.23 ensures that

$$\mathfrak{m}_\lambda(\omega) = \inf \big\{ x \in \mathbb{R} \mid \lim_{t\to\infty} (v_\lambda(t, \omega, x) - \mathfrak{u}_\lambda(\omega{\cdot}t)) = 0 \big\} \in [0, \mathfrak{u}_\lambda(\omega))$$

defines an upper semicontinuous τ_λ-equilibrium which satisfies $\int_\Omega (\mathfrak{h}_x(\omega, \mathfrak{m}_\lambda(\omega)) + 2\lambda\, \mathfrak{m}_\lambda(\omega))\, dm > 0$. Reasoning as in Theorem 3.22, we check that $\lambda \mapsto \mathfrak{m}_\lambda(\omega)$ is nonincreasing on $\mathcal{I}_2$ for all $\omega \in \Omega$. Since $0 \le \mathfrak{m}_{\lambda'} \le \mathfrak{m}_\lambda \le \mathfrak{u}_\lambda$ for any $\lambda_2 < \lambda' \le \lambda$, there exists the limit $\mathfrak{m}_{\lambda_2} = \lim_{\lambda \downarrow \lambda_2} \mathfrak{m}_\lambda \ge 0$ and it defines an m-measurable τ_{λ_2}-equilibrium. Lebesgue's Convergence Theorem ensures that $\int_\Omega (\mathfrak{h}_x(\omega, \mathfrak{m}_{\lambda_2}(\omega)) + 2\lambda\, \mathfrak{m}_{\lambda_2}(\omega))\, dm \ge 0$. In particular, $\mathfrak{m}_{\lambda_2}(\omega) > 0$ for m-a.e. $\omega \in \Omega$: otherwise, since m is ergodic, $m(\{\omega \in \Omega \mid \mathfrak{m}_{\lambda_2}(\omega) = 0\}) = 1$, and hence the previous inequality yields $\int_\Omega \mathfrak{h}_x(\omega, 0)\, dm \ge 0$, which is not the case. A symmetric procedure performed for $\lambda < \lambda_1$, defining $\mathfrak{m}_\lambda(\omega) = \sup\{x \in \mathbb{R} \mid \lim_{t\to\infty} (v_\lambda(t, \omega, x) - \mathfrak{l}_\lambda(\omega{\cdot}t)) = 0\}$, and checking that $\lambda \mapsto \mathfrak{m}^1_\lambda$ is nonincreasing on $\mathcal{I}_1$, shows the existence of a τ_{λ_1}-equilibrium $\mathfrak{m}_{\lambda_1} \le 0$ such that $\mathfrak{m}_{\lambda_1}(\omega) < 0$ for m-a.e. $\omega \in \Omega$. Finally, we assume for contradiction that $\lambda_1 = \lambda_2$, observe that $\mathfrak{m}_{\lambda_1} \le 0 \le \mathfrak{m}_{\lambda_2}$ define three τ_{λ_1}-equilibria which are strictly ordered for m-a.e. $\omega \in \Omega$, and conclude that $\int_\Omega \mathfrak{h}_x(\omega, 0)\, dm > 0$ (see Theorem 2.9), which is not the case. This contradiction proves the assertion.

It remains to check that $\mathcal{M}_0$ is the unique τ_λ-minimal set for $\lambda \in (\lambda_1, \lambda_2)$. To this end, we will check that, for $\lambda > \lambda_1$, there are no τ_λ-minimal sets below $\mathcal{M}_0$ and that, for $\lambda < \lambda_2$, there are no τ_λ-minimal sets above $\mathcal{M}_0$. As in the proof of (iv), we deduce from Proposition 3.41(ii) that $\mathcal{M}^u_\lambda = \mathcal{M}_0$ for all $\lambda < \lambda_2$ if $\mathfrak{u}_{\lambda_2}(\omega) = 0$ for some $\omega \in \Omega$. Let us check the same in the case of $\mathfrak{u}_{\lambda_2}(\omega) > 0$ for all $\omega \in \Omega$. The argument adapts to this situation that of the two last paragraphs of the proof of Theorem 3.22, as we sketch in what follows. First, for $\lambda > \lambda_2$, we consider the τ_λ-equilibrium $\mathfrak{m}_\lambda$ of the previous paragraph, which satisfies $\mathfrak{m}_\lambda(\omega) > 0$ m-a.e. whenever $\int_\Omega \mathfrak{h}_x(\omega, 0)\, dm < 0$ for an $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$. Second, we combine this property with the nonhyperbolicity of the τ_{λ_2}-minimal set $\mathcal{M}^u_{\lambda_2}$ induced by $\mathfrak{u}_{\lambda_2}$ by (1.9) to deduce that there are points in the graph of $\mathfrak{m}_{\lambda_2} = \lim_{\lambda \downarrow \lambda_2} \mathfrak{m}_\lambda$ which are above the lower equilibrium of $\mathcal{M}^u_{\lambda_2}$. And third, we deduce from this fact that $\mathfrak{u}_\lambda(\omega) = 0$ on its residual set of continuity points if $\lambda < \lambda_2$, which combined with Proposition 2.17(i) proves the assertion. The argument is analogous for $\lambda > \lambda_1$, and the proof of (v) is complete.

(vi) Assume that $0 = \sup \mathrm{sp}(\mathfrak{h}_x(\cdot, 0))$ and, by contradiction, that there exist $\lambda < \lambda_1$ and $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ such that $m(\{\omega \in \Omega \mid \mathfrak{u}_\lambda(\omega) > 0\}) = 1$. Then, $\mathfrak{l}_\lambda(\omega) < 0 < \mathfrak{u}_\lambda(\omega)$ for m-a.e. $\omega \in \Omega$, and hence Theorem 2.9 ensures that $\int_\Omega \mathfrak{h}_x(\omega, 0)\, dm > 0$. This contradicts $0 = \sup \mathrm{sp}(\mathfrak{h}_x(\cdot, 0))$. Thus, since m is ergodic, $m(\{\omega \in \Omega \mid \mathfrak{u}_\lambda(\omega) = 0\} = 1$ for every $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ and $\lambda < \lambda_1$. We take $(\lambda_n) \uparrow \lambda_1$ and define $\Omega_0 = \cap_{n\in\mathbb{N}} \{\omega \in \Omega \mid \mathfrak{u}_{\lambda_n}(\omega) = 0\}$. So, Ω_0 is σ-invariant and $m(\Omega_0) = 1$ for all $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$. The monotonicity of $\lambda \mapsto \mathfrak{u}_\lambda(\omega)$ ensured by Proposition 3.41(ii) yields $\mathfrak{u}_\lambda(\omega) = 0$ for every $\omega \in \Omega_0$ and $\lambda < \lambda_1$. The argument is analogous for λ_2.

(vii) If we assume that $\sup \mathrm{sp}(\mathfrak{h}_x(\cdot,0)) > 0$, then there exists $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$ such that $\int_\Omega \mathfrak{h}_x(\omega,0)\,dm > 0$. On the other hand, Proposition 2.16 ensures that $\int_\Omega (\mathfrak{h}_x(\omega,\mathfrak{l}_\lambda(\omega)) + 2\lambda\,\mathfrak{l}_\lambda(\omega))\,dm \le 0$ and $\int_\Omega(\mathfrak{h}_x(\omega,\mathfrak{u}_\lambda(\omega)) + 2\lambda\,\mathfrak{u}_\lambda(\omega))\,dm \le 0$. So, we reach a contradiction if we assume that $\mathfrak{l}_\lambda$ or $\mathfrak{u}_\lambda$ is identically 0 for any $\lambda \in \mathbb{R}$. □

Figure 3.7 depicts some possible bifurcation diagrams corresponding to the global situation described in Theorem 3.45.

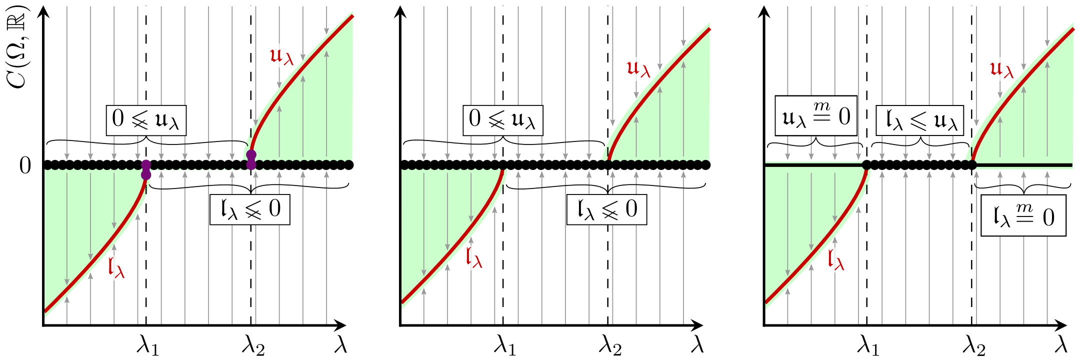


Figure 3.7: Some possibilities for the weak generalized transcritical bifurcation diagrams described in Theorem 3.45. See Figures 3.1, 3.4 and 3.5 to understand the meaning of the different elements. In the first two diagrams, $\mathrm{sp}(\mathfrak{h}_x(\cdot,0))$ is a nondegenerate interval which contains 0. The first one corresponds to $\inf \mathrm{sp}(\mathfrak{h}_x(\cdot,0)) < 0 < \sup \mathrm{sp}(\mathfrak{h}_x(\cdot,0))$ (which yields $\lambda_1 < \lambda_2$). The second one to $\inf \mathrm{sp}(\mathfrak{h}_x(\cdot,0)) = 0 < \sup \mathrm{sp}(\mathfrak{h}_x(\cdot,0))$ with $\lambda_1 < \lambda_2$ (the diagram for $\lambda_1 = \lambda_2$ is obtained by deleting the vertical strip given by (λ_1,λ_2)). The third diagram corresponds to $\mathrm{sp}(\mathfrak{h}_x(\cdot,0)) = \{0\}$ with $\lambda_1 < \lambda_2$ (again, erasing the vertical strip given by (λ_1,λ_2) provides the diagram for $\lambda_1 = \lambda_2$ in this point spectrum case). The notation $\alpha_\mu \overset{m}{=} 0$ and $\beta_\mu \overset{m}{=} 0$ represents the information in Theorem 3.45(vi).

Theorem 3.46. *Let (Ω,σ) be minimal. Let $\mathfrak{h}\colon \Omega\times\mathbb{R}\to\mathbb{R}$ satisfy* **d1**, **d2**$_{\lambda x^2}$, **d3**, **d4** *and* **d5**. *Then, the bifurcation diagrams of Theorems 3.43, 3.44 and 3.45 exhaust all the possibilities of (3.24).*

Proof. As said at the beginning of this section, $\mathrm{sp}(\mathfrak{h}_x(\cdot,0)) \subset (0,\infty)$ (Theorem 3.43), $\mathrm{sp}(\mathfrak{h}_x(\cdot,0)) \subset (-\infty,0)$ (Theorem 3.44) and $0 \in \mathrm{sp}(\mathfrak{h}_x(\cdot,0))$ (Theorem 3.45) exhaust all the possibilities. □

Autonomous cases $x' = h(x) + \lambda\,x^2$ fitting the three possibilities described in Theorem 3.46 are very easy to find, since they just depend on the sign of $h'(0)$. For example, $x' = -x^3 + x + \lambda\,x^2$ for the first one, $x' = -x^3 - x + \lambda\,x^2$ for (ii) for the second one, and $x' = -x^3 + \lambda\,x^2$ for the third one. Note also that the model analyzed in Proposition 3.26 fits in the situation of Theorem 3.45, and that in that case we can determine the values of λ_1 and λ_2.

3.4.2 A two-parameter bifurcation problem

We close this chapter by using the information just obtained in Section 3.4 to go deeper in the analysis of the bifurcation possibilities of the problem (3.9) analyzed in Section 3.3, i.e., $x' = \mathfrak{h}(\omega{\cdot}t,x) + \lambda\,x$. To this end, we will study

$$x' = \mathfrak{h}(\omega{\cdot}t,x) + \lambda\,x + \mu\,x^2\,,\quad \omega\in\Omega\,, \tag{3.29}$$

which is a bifurcation problem of the type analyzed in this section for each fixed value of λ, and a bifurcation problem of the type analyzed in Section 3.3 for each fixed value of μ. We will assume that (Ω,σ) is minimal and that $\mathfrak{h}\colon \Omega\times\mathbb{R}\to\mathbb{R}$ satisfies **d1**, **d2**$_{\lambda x^2}$, **d3**, **d4** and **d5**, which according to Proposition 3.3 provide the required hypotheses in both cases.

Let us conveniently fix the notation. Let $\mathcal{A}_{\lambda,\mu}$ be the global attractor of the local skewproduct flow $\tau_{\lambda,\mu}$ induced on $\Omega\times\mathbb{R}$ by $(3.29)_{\lambda,\mu}$, with lower and upper equilibria $\mathfrak{l}_{\lambda,\mu}$ and $\mathfrak{u}_{\lambda,\mu}$, and let $\mathrm{sp}(\mathfrak{h}_x(\cdot,0))=[-\lambda_+,-\lambda_-]$ with $\lambda_-\le\lambda_+$. We define

$$\begin{aligned}\hat\mu\colon(-\infty,\lambda_+]\to\mathbb{R}\,,\ \lambda\mapsto\inf\{\mu\in\mathbb{R}\mid &\text{ the graph of } \mathfrak{u}_{\lambda,\nu} \text{ defines a}\\ &\text{hyperbolic minimal set } \mathcal{M}^u_{\lambda,\nu}\neq\mathcal{M}_0 \text{ for all } \nu>\mu\}\,,\\ \hat\lambda\colon\mathbb{R}\to(-\infty,\lambda_+]\,,\ \mu\mapsto\inf\{\lambda\in\mathbb{R}\mid &\text{ the graph of } \mathfrak{u}_{\xi,\mu} \text{ defines a}\\ &\text{hyperbolic minimal set } \mathcal{M}^u_{\xi,\mu}\neq\mathcal{M}_0 \text{ for all } \xi>\lambda\}\,.\end{aligned}\tag{3.30}$$

Notice that, $\mathrm{sp}(\mathfrak{h}_x(\cdot,0)+\lambda)=[\lambda-\lambda_+,\lambda-\lambda_-]\not\subset(0,\infty)$ for any $\lambda\le\lambda_+$, which ensures that the bifurcation diagram for the μ-parametric family $(3.29)_\lambda$ (fixing the parameter λ) is not that of Theorem 3.43. So, Theorem 3.46 ensures that is is described by Theorems 3.44 or 3.45, and hence $\hat\mu(\lambda)$ is well defined: it is the upper bifurcation point of the bifurcation diagram of $(3.29)_\lambda$. Theorem 3.24 shows that the bifurcation diagram of $(3.29)_\mu$ is given by Theorems 3.20, 3.21 or 3.22 for any $\mu\in\mathbb{R}$, and in these three cases λ is well defined and satisfies $\hat\lambda(\mu)\le\lambda_+$.

The following proposition establishes properties of the two maps $\hat\mu$ and $\hat\lambda$, and will be used to use Corollary 3.48.

Proposition 3.47. *Let (Ω,σ) be minimal. Let $\mathfrak{h}\colon\Omega\times\mathbb{R}\to\mathbb{R}$ satisfy* **d1**, **d2**$_{\lambda x^2}$, **d3**, **d4** *and* **d5**. *Let $\hat\lambda$ and $\hat\mu$ be the maps defined in (3.30). Then,*

(i) $\hat\lambda\circ\hat\mu=\mathrm{Id}_{(-\infty,\lambda_+]}$, *and consequently, $\hat\lambda$ is onto and $\hat\mu$ is injective.*

(ii) $\hat\lambda$ *is nonincreasing and continuous.*

Proof. (i) We fix $\lambda_0\in(-\infty,\lambda_+]$ and call $\mu_0=\hat\mu(\lambda_0)$ and $\bar\lambda_0=\hat\lambda(\mu_0)$. The goal is to check that $\bar\lambda_0=\lambda_0$. The definition of μ_0 ensures that the graph of $\mathfrak{u}_{\lambda_0,\mu_0}$ does not define a hyperbolic τ_{λ_0,μ_0}-minimal set distinct from $\mathcal{M}_0$, and hence $\lambda_0\le\hat\lambda(\mu_0)=\bar\lambda_0$. For contradiction, we assume that $\lambda_0<\bar\lambda_0$, and fix $\lambda\in(\lambda_0,\bar\lambda_0)$. As said before, μ_0 is the upper bifurcation point of the diagram described by Theorems 3.44 or 3.45 for the μ-family of $(3.29)_{\lambda_0}$. Note also that, for any $\varepsilon>0$, the upper equilibrium $\mathfrak{u}_{\lambda_0,\mu_0+\varepsilon}$ of $\mathcal{A}_{\lambda_0,\mu_0+\varepsilon}$ is continuous and strictly positive. Then, if $\varepsilon\in(0,1)$ satisfies $\varepsilon<(\lambda-\lambda_0)/\sup_{\omega\in\Omega}\mathfrak{u}_{\lambda_0,\mu_0+1}(\omega)$, we have $\lambda_0-\lambda+\varepsilon\,\mathfrak{u}_{\lambda_0,\mu_0+\varepsilon}(\omega)<0$ for all $\omega\in\Omega$, and hence

$$\begin{aligned}\mathfrak{u}'_{\lambda_0,\mu_0+\varepsilon}(\omega)&=\mathfrak{h}(\omega,\mathfrak{u}_{\lambda_0,\mu_0+\varepsilon}(\omega))+\lambda_0\,\mathfrak{u}_{\lambda_0,\mu_0+\varepsilon}(\omega)+(\mu_0+\varepsilon)\,\mathfrak{u}_{\lambda_0,\mu_0+\varepsilon}(\omega)^2\\&=\mathfrak{h}(\omega,\mathfrak{u}_{\lambda_0,\mu_0+\varepsilon}(\omega))+\lambda\,\mathfrak{u}_{\lambda_0,\mu_0+\varepsilon}(\omega)+\mu_0\,\mathfrak{u}_{\lambda_0,\mu_0+\varepsilon}(\omega)^2\\&\qquad+\mathfrak{u}_{\lambda_0,\mu_0+\varepsilon}(\omega)\,(\lambda_0-\lambda+\varepsilon\,\mathfrak{u}_{\lambda_0,\mu_0+\varepsilon}(\omega))\\&<\mathfrak{h}(\omega,\mathfrak{u}_{\lambda_0,\mu_0+\varepsilon}(\omega))+\lambda\,\mathfrak{u}_{\lambda_0,\mu_0+\varepsilon}(\omega)+\mu_0\,\mathfrak{u}_{\lambda_0,\mu_0+\varepsilon}(\omega)^2\,.\end{aligned}$$

That is, $\mathfrak{u}_{\lambda_0,\mu_0+\varepsilon}$ is a global strict lower solution for $(3.29)_{\lambda,\mu_0}$. So, Theorem 2.13(v) ensures that $0<\mathfrak{u}_{\lambda_0,\mu_0+\varepsilon}<\mathfrak{u}_{\lambda,\mu_0}$, which in particular implies the existence of a strictly positive τ_{λ,μ_0}-minimal set: that defined in Proposition 2.17(i) from $\mathfrak{u}_{\lambda,\mu_0}$. But this is not possible: in the three possible bifurcation cases for $(3.29)_{\mu_0}$ described by Theorems 3.20, 3.21 and 3.22, there are no τ_{λ,μ_0}-minimal sets above $\mathcal{M}_0$ if $\lambda < \bar{\lambda}_0$. This contradiction shows that $\bar{\lambda}_0 = \lambda_0$, and hence that $\hat{\lambda} \circ \hat{\mu} = \mathrm{Id}_{(-\infty,\lambda_+]}$.

(ii) For contradiction, we assume that there exist $\mu_1 < \mu_2$ such that $\lambda_1 = \hat{\lambda}(\mu_1) < \hat{\lambda}(\mu_2) = \lambda_2$. We take $\lambda \in (\lambda_1, \lambda_2)$. As $\lambda > \lambda_1$, the definition of λ_1 ensure that the upper equilibrium $\mathfrak{u}_{\lambda,\mu_1}$ of $\mathcal{A}_{\lambda,\mu_1}$ is continuous and strictly positive. Notice that

$$\begin{aligned} \mathfrak{u}'_{\lambda,\mu_1}(\omega) &= \mathfrak{h}(\omega, \mathfrak{u}_{\lambda,\mu_1}(\omega)) + \lambda\, \mathfrak{u}_{\lambda,\mu_1}(\omega) + \mu_2\, \mathfrak{u}_{\lambda,\mu_1}(\omega)^2 + (\mu_1 - \mu_2)\, \mathfrak{u}_{\lambda,\mu_1}(\omega)^2 \\ &< \mathfrak{h}(\omega, \mathfrak{u}_{\lambda,\mu_1}(\omega)) + \lambda\, \mathfrak{u}_{\lambda,\mu_1}(\omega) + \mu_2\, \mathfrak{u}_{\lambda,\mu_1}(\omega)^2 , \end{aligned}$$

so $\mathfrak{u}_{\lambda,\mu_1}$ is a strict global lower solution for $(3.29)_{\lambda,\mu_2}$. Therefore, Theorem 2.13(v) ensures that $0 < \mathfrak{u}_{\lambda,\mu_1} < \mathfrak{u}_{\lambda,\mu_2}$, which in particular implies the existence of a strictly positive τ_{λ,μ_2}-minimal set (see Proposition 2.17(i)). But the definition of λ_2 ensures that there are no τ_{λ,μ_2}-minimal sets above $\mathcal{M}_0$, since $\lambda < \lambda_2$, a contradiction. Hence, $\hat{\lambda}$ is nonincreasing. Finally, a nondecreasing and onto function defined from an interval to an interval is always continuous. □

The following corollary is the main result of this section. It states that, given any function $\mathfrak{h}$ satisfying the assumptions we have been asking for in the previous sections, (3.9) can exhibit any of the bifurcation diagrams described in Theorem 3.24 if a suitable multiple of x^2 is added to $\mathfrak{h}$. (Of course, the generalized pitchfork diagram will only be possible if $\mathfrak{h}$ has band spectrum.)

Corollary 3.48. *Let (Ω, σ) be minimal. Let $\mathfrak{h}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy* **d1**, **d2**$_{\lambda x^2}$, **d3**, **d4** *and* **d5**. *Let $\mathrm{sp}(\mathfrak{h}_x(\cdot,0)) = [-\lambda_+, -\lambda_-]$ and $\lambda_0 \le \lambda_+$. The λ-parametric family*

$$x' = \big(\mathfrak{h}(\omega{\cdot}t, x) + \hat{\mu}(\lambda_0)\, x^2\big) + \lambda\, x \tag{3.31}$$

exhibits

- *the classical pitchfork bifurcation of Theorem 3.20 if $\lambda_0 = \lambda_+$, where λ_0 is the unique point at which the number of minimal sets changes;*
- *the local saddle-node and transcritical bifurcations of Theorem 3.21 if $\lambda_0 < \lambda_-$, with $\lambda_0 = \mu_+$ as local saddle-node bifurcation point and $\mathfrak{l}_\lambda$ colliding with 0 as $\lambda \downarrow \lambda_+$;*
- *and the generalized pitchfork bifurcation of Theorem 3.22 if $\lambda_- < \lambda_+$ and $\lambda_0 \in [\lambda_-, \lambda_+)$, with bifurcation point $\lambda_0 = \mu_+$ and $\mathfrak{l}_\lambda$ colliding with 0 as $\lambda \downarrow \lambda_+$.*

Proof. Proposition 3.47(i) ensures that $\hat{\lambda}(\hat{\mu}(\lambda_0)) = \lambda_0$. That is, $\lambda_0 = \inf\{\lambda \in \mathbb{R}\colon$ the graph of $\mathfrak{u}_{\xi,\hat{\mu}(\lambda_0)}$ defines a hyperbolic minimal set $\mathcal{M}^u_{\xi,\hat{\mu}(\lambda_0)} \neq \Omega \times \{0\}$ for all $\xi > \lambda\}$. The conclusions follow from the descriptions of Theorems 3.20, 3.21 and 3.22 applied to (3.31): notice that in the last two cases the bifurcation diagram is the symmetric of that depicted in Figure 3.4 or in Figure 3.5, since the definition of the map $\bar{\lambda}$ forces $\mathfrak{u}_\lambda$ to determine a hyperbolic copy of the base for all $\lambda > \lambda_0$; and hence $\mathfrak{l}_\lambda$ collides with 0 as $\lambda \downarrow \lambda_+$. □

If analogous definitions were made to (3.30) replacing upper equilibria by lower equilibria, then an analogous result to Corollary 3.48 would be obtained, with $\mathfrak{u}_\lambda$ colliding with 0 at the upper bifurcation points.

Comments on Chapter 3

1. In Theorems 3.8 and 3.10 and Proposition 3.9, the hypothesis on strict concavity **d4** can be asked not for all compact interval $\mathcal{J} \subset \mathbb{R}$ but for a sufficiently large compact interval $\mathcal{J}_0 \subset \mathbb{R}$ such that $\mathcal{A}_\lambda \subseteq \Omega \times \mathcal{J}_0$ for all the values of the parameter for which $\mathcal{A}_\lambda$ may not be a copy of the base (Proposition 3.5(iii) ensures that these values of the parameter are contained on a compact interval $[-\lambda_*, \lambda_*]$). This is the case of the statements of [34, Theorems 5.10 and 5.12, Proposition 5.11]. However, since the size of this compact interval $\mathcal{J}_0$ is not known a priori, for the sake of simplicity, **d4** has been assumed instead in this document.

2. The proof of the results of Section 3.2 can be repeated with small modifications for the bifurcation problem $x' = \mathfrak{h}(\omega{\cdot}t, x) + \lambda\, \mathfrak{c}(\omega{\cdot}t)$, where $\mathfrak{c}\colon \Omega \to \mathbb{R}$ is a strictly positive continuous function.

3. If (Ω, σ) has at least two ergodic measures, then we know that other bifurcation diagram for $x' = \mathfrak{h}(\omega{\cdot}t, x) + \lambda$ different from the two described in Theorems 3.8 and 3.14, and which does not have any autonomous analog, can exist. But it is not referred to in this document, as further research on this subject is still being conducted. This fact is strongly related to the generalized pitchfork bifurcation of Theorem 3.22, which is extensively described in this work.

4. All the bifurcation theorems in this chapter have been developed with minimal base flow (Ω, σ). Of course, this is not the only case in which nonautonomous bifurcations are of interest. For example, in [33, Theorem B.3] an "analog" to Theorem 3.8 in the case of transitive base flow (Ω, σ) can be found. Some additional difficulties arise in this case that must be carefully addressed.

5. An interesting motivation for the study of the bifurcation problem (3.9) can be found in [34, Section 6.2] (it also serves as a motivation for (3.24)). Given a minimal set $\mathcal{M}$ for the skewproduct flow τ_0 induced by $x' = \mathfrak{h}(\omega{\cdot}t, x)$, the classical problem of bifurcation of recurrent solutions "around" a fixed solution $t \mapsto v_0(t, \omega, z)$ can be included in the analysis of bifurcation patterns for the family of equations

$$x' = \mathfrak{h}(\omega{\cdot}t, x) + \lambda \left(x - v_0(t, \omega, z)\right), \tag{3.32}$$

for $(\omega, z) \in \mathcal{M}$. A change of variables and a change of skewproduct base (see [34, Section 6.1]) transforms (3.32) into (3.9). In [34, Section 6.4], the information obtained into the bifurcation theorems is translated into bifurcations of recurrent solutions of (3.32).

6. In [35, Section 5], the ideas of Section 3.3.2 are used to construct examples of all the three possible types of bifurcation diagrams described in Theorem 3.24 for families of differential equations of a slightly more general type:

$$x' = \big(-\mathfrak{a}_3(\omega{\cdot}t) + \mathfrak{h}(\omega{\cdot}t, x)\big)\, x^3 + \mathfrak{a}_2(\omega{\cdot}t)\, x^2 + \big(\mathfrak{a}_1(\omega{\cdot}t) + \lambda\big)\, x\,, \quad \omega \in \Omega\,,$$

where $\mathfrak{a}_i \in C(\Omega)$ for $i \in \{1, 2, 3\}$, $\mathfrak{a}_3 > 0$, $\mathfrak{h} \in C^{0,2}(\Omega \times \mathbb{R}, \mathbb{R})$, and $\mathfrak{h}(\omega, 0) = 0$ for all $\omega \in \Omega$. In this case, new bounds involving $\mathfrak{h}$ are found to guarantee the different bifurcation diagrams.

7. The techniques developed in this document can also be used to analyze certain non scalar triangular systems of ordinary differential equations. For example,

consider a triangular family of two-dimensional systems defined along the orbits of a global flow (Ω, σ),

$$\begin{cases} x' = \mathfrak{h}(\omega{\cdot}t, x)\,, \\ y' = \mathfrak{g}(\omega{\cdot}t, x, y)\,, \end{cases}$$

where $y \mapsto \mathfrak{g}(\omega, x, y)$ satisfies suitable d-concavity hypotheses, and assume that there exists a compact invariant set $\mathcal{K} \subset \Omega \times \mathbb{R}$ for the flow τ induced by $x' = \mathfrak{h}(\omega{\cdot}t, x)$ on $\Omega \times \mathbb{R}$. Then, the two-dimensional skewproduct flow can be partly analyzed by studying the dynamics of the (d-concave) scalar equation $y' = \mathfrak{g}(\tau|_{\mathcal{K}}(t, \omega, x), y)$. This matter will be the subject of further investigation.

Chapter 4

Critical transitions for d-concave nonautonomous differential equations and applications

Critical transitions or *tipping points* are significant nonlinear phenomena characterized by substantial, abrupt, and often irreversible changes in the state of a complex system in response to minor and gradual changes in external conditions or inputs of the physical phenomenon. In recent years, this concept has encompassed phenomena in applied sciences as diverse as earthquakes, sudden desertification of a typically vegetated region, coral reef collapse, habitat invasion caused by just a few individuals, epileptic seizures, outbreak of global pandemics, stock market crashes...

The purpose of this chapter is to develop a theory of critical transitions for phenomena modeled by d-concave nonautonomous equations. In the applications, we will focus on the mathematical modeling of critical transitions in nonautonomous continuous single species population models subject to the Allee effect: our attention is on two phenomena, the critical extinction of a species and the sudden invasion of certain patch. As will be indicated a little further down, we assume that the initial and final states of the transition are governed by evolution laws given by d-concave functions, which is a natural hypothesis in this context. Thus, we assume that the population dynamics during the transition is given by a *transition equation*

$$x' = g(t,x)\,, \tag{4.1}$$

where g is a sufficiently regular, coercive, and admissible function. The existence of two other sufficiently regular, coercive, and d-concave functions g_- and g_+ is assumed, satisfying $\lim_{t\to\pm\infty}(g(t,x) - g_\pm(t,x)) = 0$ uniformly on compact subsets of $\mathbb{R}$. These functions serve to represent the *past equation* $x' = g_-(t,x)$ and the *future equation* $x' = g_+(t,x)$, which govern the dynamics of the system before and after the transition occurs respectively. We remark that the evolution law of the transition equation, g, may not have a concave derivative: this is sometimes the case in population dynamics models which include factors as migration or predation. The theory we develop establishes conditions on the transition and limit equations which determine all the dynamical possibilities for (4.1), achieved in the main Theorem 4.16, which we call Cases A, B and C, and explains the repercussion that each of them has on the global attractor of the skewproduct and on the pullback attractor of the transition equation. Case A, or *tracking*, signifies the seamless connection of past and future states, which is typically the desirable situation in many applications (in population models, it means persistence of the species at risk or continued control of the potentially invasive species); Cases C, or *tipping*, mean that all bounded solutions of the transition equation have the same limit (which usually represents the catastrophic situation of extinction or invasion); and Cases B are typically unstable situations that separate Cases A and C. Of course, there are contexts in which the desired situation may be one of the Cases C. One of the relevant results of the chapter, Theorem 4.21, proves that the variation from Case A to one of the Cases C always occurs through Cases B and can be understood as a nonautonomous saddle-node bifurcation of hyperbolic solutions.

The chapter comprises four sections. The first one serves as motivation by introducing the nonautonomous population models that drive the study. It also includes a nonautonomous approach to the concepts of strong and weak Allee effect, as well as conditions to guarantee it for all the equations of a skewproduct flow. Section 4.2 provides the general theory for equations of the type (4.1): it describes the framework in which the aforementioned classification exhausts the dynamical possibilities. Section 4.3 particularizes the results of Section 4.2 for transition equations of the form $x' = f(t, x, \Gamma(t, x))$, where Γ bears the asymptotic variation. There, we present three distinct mechanisms that can be sources of tipping: rate, phase, and size. It also proves the existence of a safety interval for the range of Γ that preclude tipping, and ensures the presence of tipping when part of the range of Γ goes sufficiently far away from this safety interval. Lastly, Section 4.4 presents four examples, including numerical simulations that illustrate various peculiarities of the theory presented, and giving them a proper ecological sense.

4.1 Nonautonomous d-concave population models

In this section we will briefly motivate the interest in ecology of nonautonomous d-concave models to justify the development of a mathematical theory of critical transitions in this type of models. In addition, a nonautonomous approach to the notions of strong and weak Allee effect is also presented, as well as results that ensure the same type of Allee effect for all the equations of a skewproduct flow.

Numerous aspects of life on Earth undergo temporal fluctuations. Various phenomena, including Earth's rotation, climate variability, and seasonal alternations, exert influences on population environments. As stated in e.g. [100], those populations whose law of evolution (the set of factors affecting their development) is time-dependent can be suitably modeled by nonautonomous equations

$$x' = h(t, x)\,,$$

where the variable t represents time and the state variable x represents the population size. At least, $h \in C^{0,2}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ will always be assumed. This condition is sufficient to allow the construction of the hull presented in Section 1.3.1, which will be a fundamental tool in the subsequent theory. Under this nonautonomous approach, all the model parameters (growth rates, carrying capacities, competitive effects, predation features...) may be continuous time-dependent functions.

Many continuous models of single species populations in mathematical biology are concave or d-concave, that is, the population growth rate h as a function of the population size x either is concave or has concave derivative. One of the most classical (concave) population dynamics models, which incorporates intraspecific competition for resources, is the *autonomous logistic equation*

$$x' = r\,x\left(1-\frac{x}{K}\right), \tag{4.2}$$

where $r>0$ and $K>0$ stand for the intrinsic growth rate of the population and the carrying capacity of the environment, respectively. Changing the parameters r and K by time-dependent, bounded, uniformly continuous, and positively bounded from below maps $r(t)$ and $K(t)$, we obtain the *nonautonomous logistic equation*

$$x' = r(t)\,x\left(1-\frac{x}{K(t)}\right).$$

Here and in the whole chapter, we say that a real map is *positively bounded from below* if its inferior is strictly positive. It is remarkable that, in this model, $K(t)$ does no longer represent the healthy steady population since it is not even a solution of the equation. However, as we will explain in Remark 4.3, there always exists a strictly positive hyperbolic solution $u(t)$ which represents the steady positive population, u takes values in $[\inf_{t\in\mathbb{R}} K(t), \sup_{t\in\mathbb{R}} K(t)]$, and, if the coefficients are recurrent, then $u(t)$ coincides with $K(t)$ at least in the points of a two-sided sequence $(t_n)_{n\in\mathbb{Z}}$, with $\lim_{n\to\pm\infty} t_n = \pm\infty$. The map $r(t)$ retains the meaning that r has in (4.2).

Around 1930, perhaps earlier, W.C. Allee (see [2]) brought attention to the idea that not only intraspecific competition, but also intraspecific cooperation, is important for the evolution of a population: larger group sizes may encourage reproduction or extend survival in adverse conditions. This could imply a correlation between low population density and increased risk of extinction. There are several biological mechanisms related to survival and reproduction which can justify the appearance of the Allee effect on different biological systems: easier mate finding, cooperative breeding, cooperative anti-predator behavior, increased foraging efficiency... (see [14], [27]). This is what nowdays is called *the Allee effect* (see [27], [31], [64]): a positive correlation between the size of a population and its *fitness*, that is, the per capita population growth rate. The study of this phenomenon is an active area of both theoretical and experimental research.

The mathematical modeling of the Allee effect usually provides differential equations given by functions whose derivative with respect the state variable is (globally or locally) concave. Some phenomenological models are obtained by including a multiplicative term to the logistic model. Depending on its features, one of the following multiplicative models (see [5], [27], [120]) may be preferred to the other one:

$$x' = r(t)\,x\left(1-\frac{x}{K(t)}\right)\frac{x-S(t)}{K(t)}, \tag{4.3}$$

$$x' = r(t)\,x\left(1-\frac{x}{K(t)}\right)\frac{x-\mu(t)}{\nu(t)+x}, \tag{4.4}$$

where S, μ and ν are bounded and uniformly continuous maps which determine the strength of the Allee effect, $K(t)+S(t)\geq 0$ for all $t\in\mathbb{R}$, $S \not\geq K$, and ν and $\nu+\mu$ are positively bounded from below.

If the triggering mechanism of the Allee effect is related to predation, an additive term (see [70]) is usually added to the logistic model, often in the form of a *Holling type II functional response*, getting

$$x' = r(t)\,x\left(1 - \frac{x}{K(t)}\right) - \frac{a(t)\,x}{x + b(t)}\,, \tag{4.5}$$

where the maps a and b are assumed to be bounded, uniformly continuous, and positively bounded from below. They depend on the predator density and the average time between attacks of a predator. On the other hand, the coexistence of various mechanisms that give rise to the Allee effect can be studied through a series of models that mix these elements. For instance, adding a *Holling type III functional response* to (4.3) provides

$$x' = r(t)\,x\left(1 - \frac{x}{K(t)}\right)\frac{x - S(t)}{K(t)} - \frac{a(t)\,x^2}{b(t) + x^2}\,. \tag{4.6}$$

The meaning of a and b is the same that in the Holling type II functional response. (The biological meaning of Holling functional responses is detailed in [49]).

Allee effect and d-concavity

Let us display the third derivatives of the right-hand side of the previous models to show its d-concavity properties. In particular, we will look for a strong version of d-concavity: having strictly negative third derivative. To this end, we calculate the third derivative with respect to x of the right-hand side of (4.3),

$$\frac{\partial^3}{\partial x^3}\left(r(t)\,x\left(1 - \frac{x}{K(t)}\right)\frac{x - S(t)}{K(t)}\right) = -\frac{6\,r(t)}{K(t)^2} < 0\,, \tag{4.7}$$

of the right-hand side of (4.4),

$$\frac{\partial^3}{\partial x^3}\left(r(t)\,x\left(1 - \frac{x}{K(t)}\right)\frac{x - \mu(t)}{\nu(t) + x}\right) = -\frac{6\,\nu(t)\,r(t)\,(K(t) + \nu(t))(\nu(t) + \mu(t))}{K(t)\,(\nu(t) + x)^4} < 0\,,$$

of the right-hand side of (4.5),

$$\frac{\partial^3}{\partial x^3}\left(-\frac{a(t)\,x}{x + b(t)}\right) = -\frac{6\,a(t)\,b(t)}{(b(t) + x)^4} < 0\,,$$

of the additive predation term in (4.6),

$$\frac{\partial^3}{\partial x^3}\left(-\frac{a(t)\,x^2}{x^2 + b(t)}\right) = -\frac{24\,a(t)\,b(t)\,x\,(x^2 - b(t))}{(b(t) + x^2)^4}\,, \tag{4.8}$$

which is not forcefully strictly negative. The third derivative of the right-hand side of (4.6) is the sum of (4.7) and (4.8). Let us establish a relation between the coefficient functions ensuring its d-concavity, under the assumption that $b(t)$ is constant: $b(t) \equiv b$.

It is not difficult to check that the real function $g(y) = -y\,(y^2-1)/(1+y^2)^4$ attains its global maximum at $y_0 = \sqrt{1-2/\sqrt{5}}$, with $g(y_0) = (5/128)\sqrt{(25+11\sqrt{5})/2}$. Then,

$$\begin{aligned}\max_{x\in\mathbb{R}} \left(-\frac{24\,a(t)\,b\,x\,(x^2-b)}{(b+x^2)^4}\right) &= \max_{x\in\mathbb{R}} 24\,a(t)\,b^{-3/2} g\left(\frac{x}{\sqrt{b}}\right)\\ &= \frac{15\,a(t)\,b^{-3/2}}{16}\sqrt{\frac{25+11\sqrt{5}}{2}}\,,\end{aligned}$$

and hence, if

$$-\frac{6\,r(t)}{K(t)^2} + \frac{15\,a(t)\,b^{-3/2}}{16}\sqrt{\frac{25+11\sqrt{5}}{2}} < 0\,,$$

for all $t\in\mathbb{R}$, then, the right-hand side of (4.6) is d-concave. This holds whenever

$$a(t) \in \left[0,\; b^{3/2}\frac{64}{5\sqrt{(5-2\sqrt{5})\,(7+3\sqrt{5})}}\;\inf_{t\in\mathbb{R}}\frac{r(t)}{K(t)^2}\right) \tag{4.9}$$

for all $t\in\mathbb{R}$. This condition will be used hereafter.

4.1.1 Nonautonomous approach to the types of Allee effect

In autonomous models, the Allee effect is said to be *weak* if the per capita population growth rate (represented by x'/x in our equation) is lower at low density than at higher densities but positive; and it is said to be *strong* if the per capita population growth rate becomes negative below a certain value, which is called an *Allee threshold* or a *critical population size* (see e.g. [16], [27], [113]) and which corresponds to a strictly positive repulsive fixed point (i.e., a constant solution). That is, populations under the Allee threshold decline to extinction. For example, $x' = r\,x\,(1-x/K)(x-S)/K$ exhibits strong Allee effect if $0<S<K$ and weak Allee effect if $S<0<K$.

To extend these concepts to our nonautonomous setting, we will classify strong and weak Allee effect in terms of the existence or non-existence of such an (nonautonomous) Allee threshold. This approach intends to be valid also for models which are more general than (4.3), (4.4), (4.5) or (4.6), for which 0 may not solve the equation.

In what follows, we assume that $x' = h(t,x)$ has exactly three hyperbolic solutions. It is important to highlight that this property is not guaranteed a priori for any of these equations, so it depends on the choice of the coefficient functions; and in the case that it holds, these three hyperbolic solutions are not necessarily positive, so its biological meaning is not automatically guaranteed. We will be interested in the dynamics above the smallest nonnegative bounded solution, which is the biologically meaningful dynamics. Later in this chapter, Lemma 4.7 will prove that, if $h\in C^{0,2}(\mathbb{R}\times\mathbb{R},\mathbb{R})$ satisfies the coercivity condition $\limsup_{x\to\pm\infty}(\pm h(t,x)) < 0$ uniformly on $\mathbb{R}$ and the strict d-concavity condition $\inf_{t\in\mathbb{R}}(h_{xx}(t,x_1)-h_{xx}(t,x_2)) > 0$ whenever $x_1<x_2$, then the hypotheses of Theorem 2.18 hold for its extension to the hull, and therefore $x'=h(t,x)$ has at most three uniformly separated solutions, in which case they are hyperbolic: attractive the upper and lower ones, which bound

the set of bounded solutions, and repulsive the middle one. These are the hypotheses under which we will work in what follows.

If the three hyperbolic solutions of $x' = h(t,x)$ are nonnegative, then the upper and lower ones, attractive, can be understood as steady population states: the upper one represents a healthy population state while the lower one represent the species extinction (if it is 0) or a sparse (or low density) steady population. The intermediate hyperbolic solution plays the role of a critical population size, separating the domains of attraction of the other two solutions. This situation is what we will call the *strong Allee effect*. If only two of the three hyperbolic solutions of $x' = h(t,x)$ are nonnegative, then the upper one, which is attractive represents again a healthy population state, while the extinct or a sparse steady population is represented by the repulsive hyperbolic solution (typically close to 0). This is what we will call the *weak Allee effect*: there is not a critical population size. There can exist intermediate situations between weak and strong Allee effect, which we will not study in this work.

Now, always under the assumption of existence of three hyperbolic solutions of $x' = h(t,x)$, we will characterize weak and strong Allee effects when 0 is one of the hyperbolic solutions. So, the Allee effect is strong if 0 is attractive but not the upper one, and it is weak if 0 is repulsive. Note that, in this case, the per capita population growth rate at 0 is given by $\lim_{x\to 0}(h(t,x)/x) = h_x(t,0)$ for any $t \in \mathbb{R}$. Let us check that, if the upper bounded (and hyperbolic) solution of $x' = h(t,x)$ is positively bounded from below, then $x' = h(t,x)$ exhibits strong Allee effect if and only if the per capita population growth rate at 0 has negative average, that is,

$$\lim_{l\to\infty}\left(\sup\left\{\frac{1}{t-s}\int_s^t h_x(r,0)\,dr \;\middle|\; t-s\geq l\right\}\right) < 0\,. \tag{4.10}$$

If (4.10) holds, then there exists $\gamma > 0$ and $l_0 > 0$ such that $\int_s^t h_x(r,0)\,dr < -\gamma\,(t-s)$ for all $t-s \geq l_0$, so $\exp\int_s^t h_x(r,0)\,dr < k\,e^{-\gamma\,(t-s)}$ for all $t \geq s$, where $k = \sup\{\exp(\gamma(t-s)+\int_s^t h_x(r,0)\,dr) \mid t-s \in [0,l_0]\}$, which is finite thanks to the C^1-admissibility of h. That is, 0 is hyperbolic attractive, and since the upper bounded solution of $x' = h(t,x)$ is positively bounded from below, 0 is not the upper one. Conversely, it is easy to check (4.10) if there exist $\gamma > 0$ and $k \geq 1$ such that $\exp\int_s^t h_x(r,0)\,dr < k\,e^{-\gamma\,(t-s)}$ for all $t \geq s$. Analogous arguments show that, $x' = h(t,x)$ exhibits weak Allee effect if and only if the per capita population growth rate at 0 has positive average, that is,

$$\lim_{l\to\infty}\left(\inf\left\{\frac{1}{t-s}\int_s^t h_x(r,0)\,dr \;\middle|\; t-s\geq l\right\}\right) > 0\,.$$

We underline that, in autonomous dynamics, these averages of the per capita population growth rate at 0 are equal to the constant value of the per capita population growth rate at 0.

4.1.2 Allee effect in the skewproduct formalism

The C^2-admissibility of the map $\mathfrak{h}$ (which holds when $\mathfrak{h}$ is the right-hand side of (4.3), (4.4), (4.5) and (4.6)) allows us to work with the skewproduct flow defined on $\Omega\times\mathbb{R}$, where Ω is the hull of $\mathfrak{h}$: see Subsection 1.3.1. If, in addition, $\mathfrak{h}$ is recurrent (see Definition 1.45), then Ω is minimal, which we assume in what follows. Our next goal is to establish conditions on this hull extension ensuring the different types of Allee effect for the equations of the family.

So, let (Ω,σ) be a minimal flow on a compact metric space, and let us consider a family of nonautonomous scalar equations (see Section 1.2)

$$x' = \mathfrak{h}(\omega{\cdot}t, x)\,, \quad \omega\in\Omega\,, \tag{4.11}$$

where $\mathfrak{h}\colon \Omega\times\mathbb{R}\to\mathbb{R}$ satisfies **d1**, **d2**, **d3**, **d4** and **d5** (see Sections 3.1 and 3.3). Let τ be the scalar skewproduct flow induced by (4.11) in $\Omega\times\mathbb{R}$ (see Definition 1.15), and let $\mathfrak{l}$ and $\mathfrak{u}$ be the lower and upper τ-equilibria of the global attractor for τ (see Theorem 2.13). Observe that $\mathcal{M}_0=\Omega\times\{0\}$ is always a τ-minimal set. Recall that any τ-copy of the base is a τ-minimal set, since (Ω,σ) is minimal. And that any hyperbolic τ-minimal set is a hyperbolic τ-copy of the base: see Remark 1.41.

Proposition 4.1 provides simple criteria based on some results of Chapter 2 to ensure that, thanks to Proposition 1.55, all the equations $(4.11)_\omega$ of the family (4.11) exhibit the same type of Allee effect: weak or strong.

Proposition 4.1. *Let (Ω,σ) be minimal, and let $\mathfrak{h}\colon\Omega\times\mathbb{R}\to\mathbb{R}$ satisfy* **d1**, **d2**, **d3**, **d4** *and* **d5**. *The following statements hold:*

(i) (Weak Allee effect). *If $\mathcal{M}_0$ is a repulsive hyperbolic τ-minimal set, then there exist exactly three different τ-minimal sets $\mathcal{M}_l<\mathcal{M}_0<\mathcal{M}_u$, which are hyperbolic, with $\mathcal{M}_l=\{\mathfrak{l}\}$ and $\mathcal{M}_u=\{\mathfrak{u}\}$ attractive.*

(ii) (Strong Allee effect). *If $\mathcal{M}_0$ is an attractive hyperbolic τ-minimal set and there exists $\rho>0$ such that $\mathfrak{h}(\omega,\rho)>0$ for all $\omega\in\Omega$, then (4.11) has exactly three different τ-minimal sets $\mathcal{M}_0<\mathcal{M}_m<\mathcal{M}_u$, which are hyperbolic, with $\mathcal{M}_m=\{\mathfrak{m}\}$ repulsive and $\mathcal{M}_u=\{\mathfrak{u}\}$ attractive; and, in addition, $\mathfrak{m}(\omega)<\rho<\mathfrak{u}(\omega)$ for all $\omega\in\Omega$.*

Proof. (i) Proposition 2.14(i) ensures the existence of at least three τ-minimal sets, and Theorem 2.11 proves the remaining assertions.

(ii) Theorem 2.13(v) ensures that $\rho<\mathfrak{u}(\omega)$ for all $\omega\in\Omega$, and $\mathfrak{l}(\omega)\le 0<\rho$ for all $\omega\in\Omega$ since 0 is a bounded solution. Hence $t\mapsto v(t,\omega,\rho)$ is globally defined and bounded for all $\omega\in\Omega$. Take $\omega_0\in\Omega$, and let $\mathcal{M}_u$ and $\mathcal{M}_m$ be two τ-minimal sets contained in the $\boldsymbol{\omega}$-limit set and the $\boldsymbol{\alpha}$-limit set for τ of (ω_0,ρ), respectively. Propositions 1.22 and 1.24 ensure that $\mathcal{M}_m<\Omega\times\{\rho\}<\mathcal{M}_u$, and Corollary 1.58(i) ensures that $\mathcal{M}_0<\mathcal{M}_m$. Hence, there exist three hyperbolic τ-minimal sets, and Theorem 2.11 completes the proof. □

Now, we consider the particular case of (4.11) given by a model with multiplicative Allee effect, a skewproduct version of (4.3),

$$x' = \mathbf{r}(\omega{\cdot}t)\,x\left(1-\frac{x}{\mathbf{K}(\omega{\cdot}t)}\right)\frac{x-\mathbf{S}(\omega{\cdot}t)}{\mathbf{K}(\omega{\cdot}t)}\,,\quad \omega\in\Omega\,, \tag{4.12}$$

where $\mathbf{r},\mathbf{K},\mathbf{S}\colon\Omega\to\mathbb{R}$ are continuous, $\mathbf{r}$ and $\mathbf{K}$ are strictly positive, and $\mathbf{S}$ satisfies $\mathbf{S}(\omega)+\mathbf{K}(\omega)\ge 0$ for all $\omega\in\Omega$. The next result provides conditions ensuring each type of Allee effect, and states a relation between the functions $\mathbf{K}$ and $\mathbf{S}$ and the τ-minimal sets which determine the strength and range of action of the Allee effect.

Proposition 4.2. *Let* (Ω,σ) *be minimal, let* $\mathbf{r},\mathbf{K},\mathbf{S}\colon\Omega\to\mathbb{R}$ *be continuous maps with* $\inf_{\omega\in\Omega}\mathbf{r}(\omega)>0$, $\inf_{\omega\in\Omega}\mathbf{K}(\omega)>0$ *and let* $m\in\mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$.

(i) (Weak Allee effect). *Assume that* $\sup_{\omega\in\Omega}\mathbf{S}(\omega)<0$. *Then,* (4.12) *is in the situation of Proposition* 4.1(i), *and hence it exhibits weak Allee effect. In addition,* $\mathfrak{u}$ *takes values in* $[\inf_{\omega\in\Omega}\mathbf{K}(\omega),\,\sup_{\omega\in\Omega}\mathbf{K}(\omega)]$ *and either* $\mathbf{K}\equiv\mathfrak{u}$ *are constant maps or, for all* $\omega\in\Omega$, *there exists a strictly increasing two-sided sequence* $(t_n)_{n\in\mathbb{Z}}$ *with* $\lim_{n\to\pm\infty}t_n=\pm\infty$ *such that* $t\mapsto\mathbf{K}(\omega{\cdot}t)-\mathfrak{u}(\omega{\cdot}t)$ *changes sign at* t_n *for all* $n\in\mathbb{Z}$.

(ii) (Strong Allee effect). *Assume that* $\inf_{\omega\in\Omega}\mathbf{S}(\omega)>0$ *and* $\sup_{\omega\in\Omega}\mathbf{S}(\omega)<\rho<\inf_{\omega\in\Omega}\mathbf{K}(\omega)$ *for some constant* $\rho>0$. *Then,* (4.12) *is in the situation of Proposition* 4.1(ii), *and hence it exhibits strong Allee effect. In addition,* $\mathfrak{u}$ *(resp.* $\mathfrak{m}$*) takes values in* $[\inf_{\omega\in\Omega}\mathbf{K}(\omega),\,\sup_{\omega\in\Omega}\mathbf{K}(\omega)]$ *(resp.* $[\inf_{\omega\in\Omega}\mathbf{S}(\omega),\,\sup_{\omega\in\Omega}\mathbf{S}(\omega)]$*), and either* $\mathbf{K}\equiv\mathfrak{u}$ *(resp.* $\mathbf{S}\equiv\mathfrak{m}$*) are constant maps or, for all* $\omega\in\Omega$, *there exists a strictly increasing two-sided sequence* $(t_n)_{n\in\mathbb{Z}}$ *with* $\lim_{n\to\pm\infty}t_n=\pm\infty$ *such that* $t\mapsto\mathbf{K}(\omega{\cdot}t)-\mathfrak{u}(\omega{\cdot}t)$ *(resp.* $t\mapsto\mathbf{S}(\omega{\cdot}t)-\mathfrak{m}(\omega{\cdot}t)$*) changes sign at* t_n *for all* $n\in\mathbb{Z}$.

Proof. Let $\mathfrak{h}(\omega,x)=\mathbf{r}(\omega)\,x\,(\mathbf{K}(\omega)-x)(x-\mathbf{S}(\omega))/\mathbf{K}(\omega)^2$.

(i) If $\sup_{\omega\in\Omega}\mathbf{S}(\omega)<0$, then $\mathfrak{h}_x(\omega,0)=-\mathbf{r}(\omega)\,\mathbf{S}(\omega)/\mathbf{K}(\omega)>0$ for all $\omega\in\Omega$, and hence Proposition 1.42 (and Theorem 1.36(i)) shows that $\mathcal{M}_0$ is a repulsive hyperbolic τ-minimal set, which is the situation of Proposition 4.1(i). We define $k_-=\inf_{\omega\in\Omega}\mathbf{K}(\omega)>0$ and $k_+=\sup_{\omega\in\Omega}\mathbf{K}(\omega)\ge k_-$ and observe that $\mathfrak{h}(\omega,k_-)\ge 0$ and $\mathfrak{h}(\omega,k_+)\le 0$ for all $\omega\in\Omega$. A standard comparison argument shows that $k_-\le v(t,\omega_0,k_-)\le v(t,\omega_0,k_+)\le k_+$ for all $t\ge 0$ and $\omega_0\in\Omega$, which ensures that the $\boldsymbol{\omega}$-limit set of (ω_0,k_-) for τ exists and contains a τ-minimal set in turn contained in $\Omega\times[k_-,k_+]$. It follows from Corollary 1.58(i) that this τ-minimal set is $\{\mathfrak{u}\}$. In addition,

$$\frac{\mathfrak{u}'(\omega)}{\mathfrak{u}(\omega)}=\frac{\mathbf{r}(\omega)}{\mathbf{K}(\omega)^2}\,(\mathbf{K}(\omega)-\mathfrak{u}(\omega))(\mathfrak{u}(\omega)-\mathbf{S}(\omega)) \tag{4.13}$$

for all $\omega\in\Omega$, where $\mathfrak{u}'(\omega)=(d/dt)\,\mathfrak{u}(\omega{\cdot}t)|_{t=0}$. Hence, $\mathfrak{u}'$ is continuous. Birkhoff's Ergodic Theorem 1.10 applied to (4.13) yields

$$\int_\Omega\frac{\mathbf{r}(\omega)}{\mathbf{K}(\omega)^2}\,(\mathbf{K}(\omega)-\mathfrak{u}(\omega))\,(\mathfrak{u}(\omega)-\mathbf{S}(\omega))\,dm=0\,. \tag{4.14}$$

If $\mathfrak{u}=\mathbf{K}$, then (4.13) evaluated on $\omega{\cdot}t$ yields $\mathfrak{u}'(\omega{\cdot}t)=0$ for all $\omega\in\Omega$ and $t\in\mathbb{R}$. So, $\mathfrak{u}$ is constant along any orbit in Ω and, since Ω is minimal and $\mathfrak{u}$ is continuous, $\mathfrak{u}$ (and hence $\mathbf{K}$) is constant on Ω. If this is not the case, (4.14) and $\mathbf{r}\,(\mathfrak{u}-\mathbf{S})/\mathbf{K}^2>0$ preclude $\mathfrak{u}>\mathbf{K}$ or $\mathfrak{u}<\mathbf{K}$. Hence, there exist open sets $\mathcal{U}_+,\mathcal{U}_-\subset\Omega$ such that $\mathbf{K}(\omega)-\mathfrak{u}(\omega)>0$ for all $\omega\in\mathcal{U}_+$ and $\mathbf{K}(\omega)-\mathfrak{u}(\omega)<0$ for all $\omega\in\mathcal{U}_-$. We fix $\omega\in\Omega$ and deduce from the minimality of Ω the existence of $(s_n^\pm)\uparrow\infty$ such that $\omega{\cdot}s_n^\pm\in\mathcal{U}^\pm$ for all $n\in\mathbb{N}$. Consequently, there exists a sequence $(t_n)\uparrow\infty$ such that $t\mapsto\mathbf{K}(\omega{\cdot}t)-\mathfrak{u}(\omega{\cdot}t)$ changes sign at t_n for all $n\in\mathbb{N}$. The same argument proves the existence of $(\tilde t_n)\downarrow-\infty$ with the same property.

(ii) Now, $\mathfrak{h}(\omega,0)<0$ for all $\omega\in\Omega$, and hence Proposition 1.42 (and Theorem 1.36(i)) ensures that $\mathcal{M}_0$ is hyperbolic attractive. In addition, $\mathfrak{h}(\omega,\rho)>0$ for all $\omega\in\Omega$, so the situation is that of Proposition 4.1(ii). Similar arguments to that of (i) show the assertions concerning $\mathfrak{u}$, having in mind that $\mathfrak{u}-\mathbf{S}>\rho-\mathbf{S}>0$. To cope with $\mathfrak{m}$, we take $s_-=\inf_{\omega\in\Omega}\mathbf{S}(\omega)>0$ and $s_+=\sup_{\omega\in\Omega}\mathbf{S}(\omega)\geq s_-$ and use a comparison argument to check that $s_-\leq v(t,\omega_0,s_-)\leq v(t,\omega_0,s_+)\leq s_+$ for all $t\leq 0$ and $\omega_0\in\Omega$. From here we can repeat the arguments used for $\mathfrak{u}$, working with the α-limit set for τ of (ω_0,s_-). $\square$

Remark 4.3. For $x'=\mathbf{r}(\omega{\cdot}t)\,x\,(\mathbf{K}(\omega{\cdot}t)-x)/\mathbf{K}(\omega{\cdot}t)$, the arguments used in the proof of Proposition 4.2(i) combined with the existence of a unique positive copy of the base $\{\mathfrak{u}\}$, which is hyperbolic attractive (apply Proposition 1.20 to $\inf_{\omega\in\Omega}\mathbf{K}(\omega)$ and see [88] and [73]), prove the same properties relating $\mathfrak{u}$ and $\mathbf{K}$.

The maps $\mathfrak{u}$ and $\mathfrak{m}$ of Proposition 4.2(ii) respectively represent the maximum population size and the critical population size (Allee threshold). They can be naturally used to determine two more indicators of the strength of strong Allee effect:

$$\inf_{m\in\mathfrak{M}_{\rm erg}(\Omega,\sigma)}\int_\Omega\frac{\mathfrak{m}(\omega)}{\mathfrak{u}(\omega)}\,dm \qquad\text{and}\qquad \sup_{m\in\mathfrak{M}_{\rm erg}(\Omega,\sigma)}\int_\Omega\frac{\mathfrak{m}(\omega)}{\mathfrak{u}(\omega)}\,dm\,,$$

which measure the relative position of the upper and middle minimal sets with respect to 0. If the first quantity is close to 1, it indicates that $\mathfrak{m}$ is very close to $\mathfrak{u}$, so that the strength of strong Allee effect is high: only a population very close to the maximum population size can persist; and if the second quantity is close to 0, it indicates that $\mathfrak{m}$ is much lower than $\mathfrak{u}$, and hence that the strength of the strong Allee effect is low: only very (relatively) small populations become extinct.

4.2 General theory of transition equations

The primary objective of this section is to establish the dynamical possibilities for a class of equations

$$x'=g(t,x)\,, \tag{4.15}$$

which represents a transition between a past and a future, respectively modeled by the equations $x'=g_-(t,x)$ and $x'=g_+(t,x)$. This class is considerably larger than the set of transition equations for which g is strictly d-concave, which, as we will see in the examples of Section 4.4, significantly broadens the scope of applications. Our classification will serve as the key piece to discuss critical transitions. Once obtained, we will characterize the different possibilities in terms of topological properties of the global attractor and forward attraction properties of the pullback attractor. This is achieved in Section 4.2.1. Sections 4.2.2 and 4.2.3 contain some general monotonicity properties that lay the groundwork for the analysis of a particular type of parametric families of transition equations that will appear in Section 4.3.

Let $g\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a C^2-admissible function. The hull construction described in Section 1.3.1 allows us to understand the σ-orbit of g, $\{g{\cdot}t\mid t\in\mathbb{R}\}$, which is dense in the hull Ω_g, as a connection between its α-limit set Ω_g^α and its ω-limit set Ω_g^ω. In fact, the hull Ω_g is the union of these three sets: see Lemma 1.44. Our main goal in this section is to describe the dynamical possibilities for an "asymptotically d-concave" equation (4.15) under conditions which ensure that the families of equations defined over Ω_g^α (α-family) and Ω_g^ω (ω-family) satisfy the regularity, coercivity and strict d-concavity properties **d1**, **d2**, **d3** and **d4**, as well as the existence of three hyperbolic copies of the base for the α-family and the ω-family. This last condition provides the widest possible range of dynamical possibilities for (4.15) under conditions **d1**, **d2**, **d3** and **d4**: the maximum number of uniformly separated solutions for each equation of the α-family or the ω-family is three (see Theorem 2.18); hence, Proposition 1.47 precludes the existence of more than three uniformly separated solutions of (4.15); and, if there are three, then Theorem 2.18 yields three hyperbolic copies of the base for the α-family and the ω-family. That is, a lower number of hyperbolic solutions of the α-family or the ω-family precludes the existence of three hyperbolic solutions of (4.15). Since the structures of Ω_g^α and Ω_g^ω represent the past and future of g, we are understanding (4.15) as a transition between the α-limit and ω-limit families.

We will achieve all the required properties on Ω_g^α and Ω_g^ω by assuming the existence of two strictly d-concave (in x) maps g_- and g_+ such that g and g_- (resp. g and g_+) form an asymptotic pair as $t \to -\infty$ (resp. as $t \to \infty$) in the common hull of g and g_- (resp. g and g_+): recall Definition 1.5. (The common hull of two admissible maps h_1 and h_2 is the compact metric space defined as the closure of $\{h_i{\cdot}t \mid i = 1, 2,\, t \in \mathbb{R}\}$ in the compact-open topology.) The existence of the maps g_- and g_+ does not imply their uniqueness, but Lemma 4.6 below also shows that Ω_g^α and Ω_g^ω respectively coincide with $\Omega_{g_-}^\alpha$ and $\Omega_{g_+}^\omega$, independently of the choice of g_- and g_+, which is a key point in our analysis.

So, we fix g and assume the existence of g_- and g_+ such that:

g1 $g, g_-, g_+ \in C^{0,2}(\mathbb{R}\times\mathbb{R},\mathbb{R})$.

g2 $\lim_{t\to\pm\infty}\big(g(t,x) - g_\pm(t,x)\big) = 0$ uniformly on each compact subset $\mathcal{J} \subset \mathbb{R}$.

g3 $\limsup_{x\to\pm\infty}\big(\pm h(t,x)\big) < 0$ uniformly on $\mathbb{R}$ for $h = g, g_-, g_+$.

g4 $\inf_{t\in\mathbb{R}}\big((g_\pm)_{xx}(t,x_1) - (g_\pm)_{xx}(t,x_2)\big) > 0$ whenever $x_1 < x_2$.

g5 Each one of the equations

$$x' = g_-(t,x) \quad \text{and} \quad x' = g_+(t,x) \tag{4.16}$$

has three hyperbolic solutions, $\tilde{l}_{g_-} < \tilde{m}_{g_-} < \tilde{u}_{g_-}$ and $\tilde{l}_{g_+} < \tilde{m}_{g_+} < \tilde{u}_{g_+}$.

As Lemma 4.7 will prove, conditions **g1**-**g4** provide a setting satisfying the hypotheses of Chapter 2.

Remarks 4.4. 1. Slightly abusing language, we will say that "g satisfies conditions **g1**-**g5**" if there exist g_- and g_+ such that all the listed conditions are satisfied.

2. To simplify the language, we will refer to (4.15) as a *transition equation* between the *past equation* and the *future equation*, which are the first one and the second one in (4.16). That the use of these words is accurate is partly justified by the previously mentioned equalities $\Omega_{g_-}^\alpha = \Omega_g^\alpha$ and $\Omega_{g_+}^\omega = \Omega_g^\omega$, which mean that the hyperbolic structures of the equations (4.16) condition that of (4.15) and viceversa; and it will be better justified by the main results of this section. But observe that the future of the dynamics of the nonautonomous equation $x' = g_-(t,x)$ is not necessarily related to its past (since $\Omega_{g_-}^\alpha$ can be different $\Omega_{g_-}^\omega$), and hence it can be not related to the dynamics of $x' = g(t,x)$. And the same happens with the past dynamics of $x' = g_+(t,x)$ and $x' = g(t,x)$.

3. Note again that the function g giving rise to the transition equation is not required to be d-concave.

As said before, the main (and initial) purpose of this section is to classify the dynamical scenarios for the transition equation (4.15) when g satisfies **g1-g5**. We will check that the dynamical possibilities for (4.15) correspond to the cases described in the following definition, describe the dynamics of each of them and relate their dynamics to that of the past and future equations (4.16).

Definition 4.5. Let g satisfy **g1-g5**. We shall say that equation (4.15) is

- in Case A if it has three hyperbolic solutions, which are the unique three uniformly separated solutions.
- In Case B if it has exactly two uniformly separated solutions, one of which is the only hyperbolic solution, of attractive type, and the other locally pullback attractive and repulsive. If the hyperbolic solution is above the other one, it is in Case B1, and otherwise in Case B2.
- In Case C if it has no uniformly separated solutions and it has exactly two hyperbolic solutions, which are attractive, and a locally pullback repulsive solution defined on a positive halfline, which is either below the hyperbolic solutions (Case C1) or above them (Case C2).

The classification is given in Theorem 4.16, whose proof relies on Theorems 4.17 and 4.18. The statements of these three theorems require the information provided by another fundamental result, Theorem 4.13. The proof of all these results require some previous work. Having a first look to Figures 4.1, 4.2 and 4.3 may provide a global idea of the dynamics of each one of the five cases, although a complete understanding of these drawings is difficult before completing the reading of the statement of Theorem 4.16 (in turn based on the previous results). Throughout this chapter, the numerical integration has been performed using the variable step integration algorithm `ode45` of Matlab2023a with suitable tolerances.

Our first two lemmas, fundamental for the subsequent application of Theorem 2.18, refer to the hull extensions (see Subsection 1.3.1). Recall that we represent by $x_h(t,s,x)$ the maximal solution of $x' = h(t,x)$ which satisfies $x_h(s,s,x) = x$ (see Section 1.3): we will use this notation for h equal to g, g_-, g_+, and some other auxiliary admissible functions. In all these cases, the set Ω_h is the hull of h, and Ω_h^α and Ω_h^ω are the α-limit set and ω-limit set of the element $h \in \Omega_h$. Recall that $h{\cdot}t(s,x) = h(t+s,x)$, and that $\mathfrak{h}(\omega,x) = \omega(0,x)$ if $\omega \in \Omega_h$. We represent by $\mathfrak{g}$, $\mathfrak{g}_-$ and $\mathfrak{g}_+$ the extensions to the corresponding hulls of g, g_- and g_+ and by τ_g, τ_{g_-} and τ_{g_+} the corresponding skewproduct flows on $\Omega_g \times \mathbb{R}$, $\Omega_{g_-} \times \mathbb{R}$ and $\Omega_{g_+} \times \mathbb{R}$ respectively. These auxiliary lemmas do not need all the conditions **g1-g5**: we will specify the required ones.

Lemma 4.6. *Let g and $g_\pm$ satisfy* **g1** *and* **g2**. *Then,*

(i) $\Omega_g^\alpha = \Omega_{g_-}^\alpha$ *and* $\Omega_g^\omega = \Omega_{g_+}^\omega$.

(ii) $\Omega_g = \Omega_{g_-}^\alpha \cup \{g{\cdot}t \mid t \in \mathbb{R}\} \cup \Omega_{g_+}^\omega$.

(iii) *The restriction to $\Omega_g^\alpha \times \mathbb{R}$ (resp. $\Omega_g^\omega \times \mathbb{R}$) of the skewproduct flow given by $x' = \mathfrak{g}(\omega{\cdot}t,x)$ coincides with that of $x' = \mathfrak{g}_-(\omega{\cdot}t,x)$ (resp. $x' = \mathfrak{g}_+(\omega{\cdot}t,x)$). In particular, both restrictions have the same invariant compact sets.*

Proof. Given a sequence (t_n) with limit ∞, it follows from **gc2** that $\lim_{n\to\infty}(g(t+t_n,x)-g_+(t+t_n,x))=0$ uniformly for (t,x) in a compact subset of $\mathbb{R}\times\mathbb{R}$. Consequently, $\omega(t,x)=\lim_{n\to\infty} g(t+t_n,x)$ uniformly on the compact subsets of $\mathbb{R}\times\mathbb{R}$ if and only if $\omega(t,x)=\lim_{n\to\infty} g_+(t+t_n,x)$ uniformly on the compact subsets of $\mathbb{R}\times\mathbb{R}$. This proves $\Omega_g^{\omega}=\Omega_{g_+}^{\omega}$. We complete the proof of (i) by checking in an analogous way that $\Omega_g^{\alpha}=\Omega_{g_-}^{\alpha}$. Combining the two obtained equalities with Lemma 1.44 proves (ii), and (iii) is a trivial consequence of (i). □

Lemma 4.7. *If $h\in C^{0,2}(\mathbb{R}\times\mathbb{R},\mathbb{R})$ then $\mathfrak{h}$ satisfies* **d1** *on Ω_h. If $h\in C^{0,2}(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $\limsup_{x\to\pm\infty}(\pm h(t,x))<0$ uniformly on $\mathbb{R}$, then $\mathfrak{h}$ satisfies* **d2** *on Ω_h. And, if* **g1**, **g2** *and* **g4** *hold, then $\mathfrak{g}$ and $\mathfrak{g}_\pm$ satisfy* **d3** *and* **d4** *on Ω_g and $\Omega_{g_\pm}$, respectively.*

Proof. As explained in Section 1.3.1, the C^2-admissibility of h ensures **d1** for $\mathfrak{h}$. If, in addition, $\limsup_{x\to\pm\infty}(\pm h(t,x))<0$ uniformly on $\mathbb{R}$, then there exists $\delta>0$ and $\rho_\delta>0$ such that $h(t,x)\le-\delta$ if $x\ge\rho_\delta$ and $t\in\mathbb{R}$, and $h(t,x)\ge\delta$ if $x\le-\rho_\delta$ and $t\in\mathbb{R}$. Since any $\omega\in\Omega_h$ satisfies $\omega(0,x)=\lim_{n\to\infty}h(t_n,x)$ for a sequence (t_n), we have $\mathfrak{h}(\omega,x)=\omega(0,x)\le-\delta$ if $x\ge\rho_\delta$ and $\mathfrak{h}(\omega,x)\ge\delta$ if $x\le-\rho_\delta$: **d2** holds on Ω_h.

Now, we assume that **g1**, **g2** and **g4** hold. To prove the last assertion, it is enough to reason with g, since g_- and g_+ satisfy the conditions assumed on g. Let us check that $\mathfrak{g}$ satisfies **d3** and **d4** on Ω_g. Lemma 1.44 ensures that $\Omega_g=\Omega_g^{\alpha}\cup\{g{\cdot}t\,|\,t\in\mathbb{R}\}\cup\Omega_g^{\omega}$. In particular, given $m\in\mathfrak{M}_{\mathrm{erg}}(\Omega_g,\sigma_g)$, $m(\Omega_g^{\alpha})=1$ or $m(\Omega_g^{\omega})=1$ (or both): this is trivial if g is independent of t or t-periodic (since $\Omega_g=\Omega_g^{\alpha}=\Omega_g^{\omega}$); and, in the remaining cases, $\{g{\cdot}t\,|\,t\in\mathbb{R}\}=\bigcup_{n\in\mathbb{Z}}\sigma_n(\{g{\cdot}t\,|\,t\in[0,1)\})$ (where $\sigma_n(\omega)=\omega{\cdot}n$) is a nonfinite union of disjoint sets. Therefore, $m(\sigma_n(\{g{\cdot}t\,|\,t\in[0,1)\}))=0$ for all $n\in\mathbb{N}$, since this measure is independent of n. Hence, it suffices to check that $x\mapsto\mathfrak{g}_x(\omega,x)$ is strictly concave on $\mathbb{R}$ for all $\omega\in\Omega_g^{\alpha}\cup\Omega_g^{\omega}$: this ensures that $m(\{\omega\in\Omega_g\,|\,x\mapsto\mathfrak{g}_x(\omega,x)$ is strictly concave on $\mathbb{R}\})=1$ for all $m\in\mathfrak{M}_{\mathrm{erg}}(\Omega_g,\sigma_g)$, which is stronger than **d3** and **d4**. We reason for $\omega\in\Omega_g^{\omega}$. According to Lemma 4.6(i), $\omega=\lim_{n\to\infty}g_+{\cdot}t_n$ (in the compact-open topology) for a sequence (t_n) with limit ∞. Then, ω_x is the limit of any subsequence of $((g_+)_x{\cdot}t_n)$ which uniformly converges on the compact subsets of $\mathbb{R}\times\mathbb{R}$, and hence $\omega_x=\lim_{n\to\infty}(g_+)_x{\cdot}t_n$ uniformly on the compact subsets of $\mathbb{R}\times\mathbb{R}$ (recall that in a compact metric space, if every convergent sequence converges to the same limit, then the sequence converges to that limit; and that, since $g_+\in C^{0,2}(\mathbb{R}\times\mathbb{R},\mathbb{R})$, $\mathrm{closure}_{\Omega\times\mathbb{R}}\{(g_+)_x{\cdot}t\mid t\in\mathbb{R}\}$ is compact in the compact-open topology). Analogously, $\omega_{xx}=\lim_{n\to\infty}(g_+)_{xx}{\cdot}t_n$. We take $x_1<x_2$, and apply **g4** to get

$$\begin{aligned}\mathfrak{g}_{xx}(\omega,x_1)-\mathfrak{g}_{xx}(\omega,x_2)&=\omega_{xx}(0,x_1)-\omega_{xx}(0,x_2)\\&=\lim_{n\to\infty}\big((g_+)_{xx}(t_n,x_1)-(g_+)_{xx}(t_n,x_2)\big)>0\,,\end{aligned}$$

which completes the proof. □

Remark 4.8. Lemma 4.7 shows that $\mathfrak{g}_-$ satisfies **d1**, **d2**, **d3** and **d4** if g_- satisfies the conditions assumed on it on **g1**, **g3** and **g4**. Hence, in this case, and according to Theorem 2.18, the property corresponding to g_- in condition **g5** can be reformulated as: "the equation $x'=g_-(t,x)$ has three uniformly separated solutions", which determine the global dynamics according to Theorem 2.18. The same applies to g_+. We will use these facts without further reference.

The next result allows us to use the persistence of hyperbolic solutions guaranteed by Theorem 1.52 in the proofs of Proposition 4.12 and of Theorem 4.13.

Lemma 4.9. *If g and $g_\pm$ satisfy* **g1** *and* **g2**, *then* $\lim_{t\to\pm\infty}(g_x(t,x)-(g_\pm)_x(t,x))=0$ *and* $\lim_{t\to\pm\infty}(g_{xx}(t,x)-(g_\pm)_{xx}(t,x))=0$ *uniformly on each compact subset* $\mathcal{J}\subset\mathbb{R}$.

Proof. Let us reason for the map g_-, taking $(t_n)\downarrow-\infty$. Since $h_-=g-g_-$ is C^2-admissible, $\mathrm{closure}_{\Omega\times\mathbb{R}}\{(h_-)_x{\cdot}t\mid t\in\mathbb{R}\}$ is a compact set, so it suffices to check that $\lim_{k\to\infty}(h_-)_x(t_k,x)=0$ for every subsequence (t_k) for which $(h_-)_x{\cdot}t_k$ converges (uniformly on compact sets). Let (t_k) be such a sequence, define $d_-(x)=\lim_{k\to\infty}(h_-)_x(t_k,x)$, observe that the uniform convergence in compact sets ensures that d_- is continuous, and assume for contradiction that $d_-\not\equiv 0$. Then, there is no restriction in assuming that $d_-(x)>0$ for all x in an interval $[x_1,x_2]$, and hence we have $0=\lim_{k\to\infty}(h_-(t_k,x_2)-h_-(t_k,x_1))=\lim_{k\to\infty}\int_{x_1}^{x_2}(h_-)_x(t_k,s)\,ds=\int_{x_1}^{x_2}d_-(s)\,ds>0$, a contradiction. The proofs for g_+ and for the second order derivatives are analogous. □

Remark 4.10. Assume that $h\in C^{0,1}(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $\limsup_{x\to\pm\infty}(\pm h(t,x))<0$ uniformly in $t\in\mathbb{R}$. Hypotheses **g1** and **g3** ensure that this is the case for g, g_- and g_+. Lemma 4.7 and Theorem 2.13 ensure the existence of the global attractor

$$\mathcal{A}_h=\bigcup_{\omega\in\Omega_h}\big(\{\omega\}\times[\mathfrak{l}_h(\omega),\mathfrak{u}_h(\omega)]\big)$$

of the flow τ_h defined by $x'=\mathfrak{h}(\omega{\cdot}t,x)$ on $\Omega_h\times\mathbb{R}$. In particular, if $\omega_0=h$, then the maps $l_h(t)=\mathfrak{l}_h(\omega_0{\cdot}t)$ and $u_h(t)=\mathfrak{u}_h(\omega_0{\cdot}t)$ define the lower and upper bounded solutions of $x'=h(t,x)$, and all the positive (forward) semiorbits of $x'=h(t,x)$ are globally defined and bounded. In addition, since the global attractor corresponds to the set of bounded orbits, Proposition 1.61 ensures that the pullback attractor (see Definition 1.60) of the induced process $x_h(t,s,x)$ is $\mathscr{A}_h=\{\mathscr{A}_h(s)=[l_h(s),u_h(s)]\mid s\in\mathbb{R}\}$. Recall that $x_h(t,s,x)$ is the solution of $x'=h(t,x)$ which satisfies $x_h(s,s,x)=x$.

Definition 4.11. The graph of a solution b of (4.15) defined on a positive halfline (resp. negative halfline) is said to *approach that of a continuous map* $c\colon\mathbb{R}\to\mathbb{R}$ *as time increases* (resp. *as time decreases*) if

$$\lim_{t\to\infty}\big(b(t)-c(t)\big)=0\qquad\Big(\text{resp. }\lim_{t\to-\infty}\big(b(t)-c(t)\big)=0\Big).$$

Theorem 4.13, key in the proof of Theorem 4.16, establishes the existence of three solutions which govern the dynamics of (4.15) if **g1**-**g5** hold: the lower and upper bounded solutions of Remark 4.10, l_g and u_g, which are locally pullback attractive, and a locally pullback repulsive one, m_g. They are characterized in terms of asymptotic approaches to the hyperbolic solutions of the limit equations. Its proof requires the next previous result, which describes the part of the dynamics of the future equation (resp. past equation) that is inherited by the transition equation in the case that only the part of the assumptions **g1**-**g5** concerning the future equation (resp. past equation) and the transition equation are required.

Proposition 4.12. *Assume that* (4.15) *has three uniformly separated hyperbolic solutions* $\tilde l_g<\tilde m_g<\tilde u_g$, *with* $\tilde l_g$ *and* $\tilde u_g$ *attractive and* $\tilde m_g$ *repulsive.*

(i) *If g, g_+ satisfy the part of* **g1**-**g5** *concerning them, then* $\lim_{t\to\infty}(x_g(t,s,x)-\tilde u_{g_+}(t))=0$ *if and only if* $x>\tilde m_g(s)$, $\lim_{t\to\infty}(\tilde m_g(t)-\tilde m_{g_+}(t))=0$, *and* $\lim_{t\to\infty}(x_g(t,s,x)-\tilde l_{g_+}(t))=0$ *if and only if* $x<\tilde m_g(s)$.

(ii) *If g, g_- satisfy the part of* **g1**-**g5** *concerning them, then $t \mapsto x_g(t,s,x)$ is bounded from above (resp. from below) as time decreases if and only if $x \le \tilde{u}_g(s)$ (resp. $x \ge \tilde{l}_g(s)$); and $\lim_{t\to-\infty}(\tilde{u}_g(t) - \tilde{u}_{g_-}(t)) = 0$, $\lim_{t\to-\infty}(x_g(t,s,x) - \tilde{m}_{g_-}(t)) = 0$ if and only if $x \in (\tilde{l}_g(s), \tilde{u}_g(s))$ and $\lim_{t\to-\infty}(\tilde{l}_g(t) - \tilde{l}_{g_-}(t)) = 0$.*

Proof. (i) Remark 4.8 ensures that $\mathfrak{g}_+$ (the extension to the hull of g_+) fulfills **d1**, **d2**, **d3** and **d4**. The part of hypothesis **g5** concerning g_+ ensures that there exist three hyperbolic solutions of the future equation, so Theorem 2.18(d) ensures that there exist three hyperbolic τ_{g_+}-copies of the base $\{\mathfrak{l}_{g_+}\} < \{\mathfrak{m}_{g_+}\} < \{\mathfrak{u}_{g_+}\}$ given by the continuous τ_{g_+}-equilibria $\mathfrak{l}_{g_+}, \mathfrak{m}_{g_+}, \mathfrak{u}_{g_+} : \Omega_{g_+} \to \mathbb{R}$, whose restrictions to $\Omega_g^{\omega} = \Omega_{g_+}^{\omega}$ (see Lemma 4.6(i)) we represent by $\mathfrak{l}_{g+}^{\omega}$, $\mathfrak{m}_{g+}^{\omega}$ and $\mathfrak{u}_{g+}^{\omega}$, and which satisfy $\mathfrak{l}_{g_+}(g_+{\cdot}t) = \tilde{l}_{g_+}(t)$, $\mathfrak{m}_{g_+}(g_+{\cdot}t) = \tilde{m}_{g_+}(t)$ and $\mathfrak{u}_{g_+}(g_+{\cdot}t) = \tilde{u}_{g_+}(t)$. Proposition 1.54(i) ensures that the $\boldsymbol{\omega}$-limit set for τ_g of $(g, \tilde{m}_g(0))$, which projects onto Ω_g^{ω}, is composed by orbits of repulsive hyperbolic solutions, and hence Proposition 1.55 precludes that it intersects $\{\mathfrak{l}_{g+}^{\omega}\}$ and $\{\mathfrak{u}_{g+}^{\omega}\}$. So, this $\boldsymbol{\omega}$-limit set coincides with $\{\mathfrak{m}_{g+}^{\omega}\}$, since Theorem 2.18 proves that for any $\omega \in \Omega_{g_+}^{\omega}$ there are no repulsive hyperbolic solutions of $x' = \mathfrak{g}_+(\omega{\cdot}t, x)$ other than $t \mapsto \mathfrak{m}_{g_+}(\omega{\cdot}t)$. We take $(t_n) \uparrow \infty$ and look for a subsequence (t_m) such that there exists $\lim_{m\to\infty}(g{\cdot}t_m, \tilde{m}_g(t_m)) \in \{\mathfrak{m}_{g+}^{\omega}\} \subseteq \{\mathfrak{m}_{g_+}\}$. So, this limit is $(\omega, \mathfrak{m}_{g_+}(\omega))$ for an element $\omega \in \Omega$. Lemma 4.6(i) (see also its proof) shows that $\omega = \lim_{m\to\infty} g_+{\cdot}t_m$, and hence $\mathfrak{m}_{g_+}(\omega) = \lim_{m\to\infty} \mathfrak{m}_{g_+}(g_+{\cdot}t_m) = \lim_{m\to\infty} \tilde{m}_{g_+}(t_m)$. So, $\lim_{m\to\infty}(\tilde{m}_g(t_m) - \tilde{m}_{g_+}(t_m)) = 0$, and hence $\lim_{t\to\infty}(\tilde{m}_g(t) - \tilde{m}_{g_+}(t)) = 0$.

Now, let us take $x > \tilde{m}_g(s)$ and a point $(\bar{\omega}, \bar{x})$ in the $\boldsymbol{\omega}$-limit set of $(g{\cdot}s, x)$. So, $(\bar{\omega}, \bar{x}) = \lim_{n\to\infty}(g{\cdot}(s+t_n), x_g(t_n, s, x))$ for a sequence $(t_n) \uparrow \infty$ for which there exists $\lim_{n\to\infty} \tilde{m}_g(t_n)$. As seen before, this last limit is $\mathfrak{m}_{g_+}(\bar{\omega})$, and hence Proposition 1.56 shows that $\bar{x} > \mathfrak{m}_{g_+}(\bar{\omega})$. So, $(\bar{\omega}, \bar{x})$ belongs to a compact τ_g-invariant set above $\{\mathfrak{m}_{g+}^{\omega}\}$, which is necessarily $\{\mathfrak{u}_{g+}^{\omega}\}$. Therefore, the $\boldsymbol{\omega}$-limit set for τ_g of $(g{\cdot}s, x)$ is $\{\mathfrak{u}_{g+}^{\omega}\}$. In particular, this also happens for the $\boldsymbol{\omega}$-limit set for τ_g of $(g, \tilde{u}_g(0))$. An analogous argument to that at the end of the previous paragraph shows that $\lim_{t\to\infty}(x_g(t,s,x) - \tilde{u}_{g_+}(t)) = 0$. The proof of the case $x < \tilde{m}_g(s)$ is analogous. The equivalences follow immediately.

(ii) As in (i), let $\{\mathfrak{l}_{g_-}\} < \{\mathfrak{m}_{g_-}\} < \{\mathfrak{u}_{g_-}\}$ be the three hyperbolic τ_{g_+}-copies of the base on Ω_{g_-}. Arguments analogous to those of (i) show that the $\boldsymbol{\alpha}$-limit sets of $(g, \tilde{l}_g(0))$, $(g, \tilde{m}_g(0))$ and $(g, \tilde{u}_g(0))$ for τ_g are $\{\mathfrak{l}_{g_-}^{\alpha}\}$, $\{\mathfrak{m}_{g_-}^{\alpha}\}$ and $\{\mathfrak{u}_{g_-}^{\alpha}\}$ respectively, and also that $\lim_{t\to-\infty}(\tilde{u}_g(t) - \tilde{u}_{g_-}(t)) = 0$, $\lim_{t\to-\infty}(x_g(t,s,x) - \tilde{m}_{g_-}(t)) = 0$ for all $x \in (\tilde{l}_g(s), \tilde{u}_g(s))$ and $\lim_{t\to-\infty}(\tilde{l}_g(t) - \tilde{l}_{g_-}(t)) = 0$. To prove the first assertion, we assume for contradiction that $t \mapsto x_g(t,s,x)$ is bounded for some $x > \tilde{u}_g(s)$. Then, there exists the $\boldsymbol{\alpha}$-limit set for τ_g of $(g{\cdot}s, x)$. Let $(t_n) \downarrow -\infty$ be such that there exists $(\bar{\omega}, \bar{x}) = \lim_{n\to\infty}(g{\cdot}(s+t_n), x_g(t_n, s, x))$ and assume without restriction that $\mathfrak{u}_{g_-}(\bar{\omega}) = \lim_{n\to\infty} \tilde{u}_g(t_n)$. Since $(\bar{\omega}, \bar{x}) \in \mathcal{A}_g$, we get that $\bar{x} \le \mathfrak{u}_{g_-}(\bar{\omega})$. However, Proposition 1.56 ensures that $\bar{x} - \mathfrak{u}_{g_-}(\bar{\omega}) = \lim_{n\to\infty}(x_g(t_n, s, x) - \tilde{u}_g(t_n)) \ge \inf_{t<0}(x_g(t,s,x) - \tilde{u}_g(t)) > 0$, a contradiction. The case $x < \tilde{l}_g(s)$ is analogous. The equivalences follow immediately. □

Theorem 4.13. *Assume that g and g_- satisfy the part of* **g1**-**g5** *concerning them. Let $\tilde{l}_{g_-} < \tilde{m}_{g_-} < \tilde{u}_{g_-}$ be the hyperbolic solutions of $x' = g_-(t,x)$ given by* **g5**, *and let l_g and u_g be the lower and upper bounded solutions of $x' = g(t,x)$. Then,*

(i) *u_g and l_g are the unique solutions of (4.15) satisfying $\lim_{t\to-\infty}(u_g(t) - \tilde{u}_{g_-}(t)) = 0$ and $\lim_{t\to-\infty}(l_g(t) - \tilde{l}_{g_-}(t)) = 0$, and they are locally pullback attractive.*

(ii) *For any* $s\in\mathbb{R}$, $\lim_{t\to-\infty}(x_g(t,s,x)-\tilde m_{g_-}(t))=0$ *if and only if* $x\in(l_g(s),u_g(s))$.

Assume that g *and* g_+ *satisfy the part of* **g1**-**g5** *concerning them. Let* $\tilde l_{g_+}<\tilde m_{g_+}<\tilde u_{g_+}$ *be the hyperbolic solutions given by* **g5**. *Then,*

(iii) *there exists a unique solution* m_g *of* (4.15) *defined at least on a positive half-line and satisfying* $\lim_{t\to\infty}(m_g(t)-\tilde m_{g_+}(t))=0$, *and it is locally pullback repulsive.*

(iv) *For* $s\in\mathbb{R}$ *in the interval of definition of* m_g, $\lim_{t\to\infty}(x_g(t,s,x)-\tilde u_{g_+}(t))=0$ *if and only if* $x>m_g(s)$, *and* $\lim_{t\to\infty}(x_g(t,s,x)-\tilde l_{g_+}(t))=0$ *if and only if* $x<m_g(s)$.

Proof. In this first part of the proof we will prove a general fact to be used in both the proof of (i)-(ii) and of (iii)-(iv). The double sign $\pm$ appears in all the next expressions: those for $-$ (resp. $+$) require the part of the hypotheses concerning g and g_- (resp. g and g_+). We point out now that Lemma 4.7 ensures that the hull extensions of $g_\pm$ and g satisfy **d1** and **d2**, which allows us to apply Theorem 2.13. We use its point (ii) to take $\kappa>0$ such that $\|b\|_\infty\le\kappa$ for any bounded solution of $x'=g(t,x)$ and $x'=g_\pm(t,x)$. Let us define

$$\varepsilon_0=\frac{1}{3}\min\left\{\inf_{t\in\mathbb{R}}\big(\tilde u_{g_\pm}(t)-\tilde m_{g_\pm}(t)\big),\ \inf_{t\in\mathbb{R}}\big(\tilde m_{g_\pm}(t)-\tilde l_{g_\pm}(t)\big)\right\},$$

where $\tilde l_{g_\pm}<\tilde m_{g_\pm}<\tilde u_{g_\pm}$ are the hyperbolic solutions of $x'=g_\pm(t,x)$ provided by **g5**. Given $\varepsilon\in(0,\varepsilon_0]$, Theorem 1.52 provides $\delta_\pm>0$ such that, if $\|g_\pm-h_\pm\|_{1,\kappa}<\delta_\pm$, then each one of the equations $x'=h_\pm(t,x)$ has three hyperbolic solutions, at a uniform distance from those of $x'=g_\pm(t,x)$ bounded by ε. We choose $t^0=t^0(\varepsilon)>0$ such that $|g(t,x)-g_\pm(t,x)|<\delta_\pm/2$ and $|g_x(t,x)-(g_\pm)_x(t,x)|<\delta_\pm/2$ if $\pm t\ge t^0$ and $|x|\le\kappa$ (see Lemma 4.9), and define

$$f_\pm(t,x)=\begin{cases} g(t,x) & \text{if } \pm t>t^0\,,\\ g_\pm(t,x)-g_\pm(\pm t^0,x)+g(\pm t^0,x) & \text{if } \pm t\le t^0\,.\end{cases}\tag{4.17}$$

Therefore,

$$\|g_\pm-f_\pm\|_{1,\kappa}=\sup_{\substack{\pm t\ge t^0\\ x\in[-\kappa,\kappa]}}|g(t,x)-g_\pm(t,x)|+\sup_{\substack{\pm t\ge t^0\\ x\in[-\kappa,\kappa]}}|g_x(t,x)-(g_\pm)_x(t,x)|<\delta_\pm\,,$$

so each of the equations

$$x'=f_\pm(t,x)$$

has three uniformly separated hyperbolic solutions $\tilde l_{f_\pm}<\tilde m_{f_\pm}<\tilde u_{f_\pm}$, satisfying $\|\tilde l_{f_\pm}-\tilde l_{g_\pm}\|_\infty\le\varepsilon$, $\|\tilde m_{f_\pm}-\tilde m_{g_\pm}\|_\infty\le\varepsilon$ and $\|\tilde u_{f_\pm}-\tilde u_{g_\pm}\|_\infty\le\varepsilon$. It is easy to check that f_+ (resp. f_-) satisfies the hypotheses assumed on g in Proposition 4.12(i) (resp. (ii)) with future equation g_+ (resp. with past equation g_-), which provides fundamental information for the next steps. Note also that Theorem 2.13 can also be applied to the hull extension of $f_\pm$.

(i)-(ii) Let us take $\varepsilon\in(0,\varepsilon_0]$ and the value of $t^0=t^0(\varepsilon)$ before described. Let l_ε^- and u_ε^- be the maximal solutions of (4.15) satisfying $l_\varepsilon^-(-t^0)=\tilde l_{f_-}(-t^0)$ and $u_\varepsilon^-(-t^0)=\tilde u_{f_-}(-t^0)$, observe that $l_\varepsilon^-(t)=\tilde l_{f_-}(t)$ and $u_\varepsilon^-(t)=\tilde u_{f_-}(t)$ for all

$t \le -t^0$, and deduce from Theorem 2.13(i) that l_ε^- and u_ε^- are bounded. The characterizations "$x_0 \le \tilde u_{f_-}(-t^0) = u_\varepsilon^-(-t^0)$ if and only if the solution $x_{f_-}(t,-t^0,x_0)$ is bounded from above as time decreases" (given by Theorem 2.18 and Theorem 2.13(iv) applied to f_-) and "$x_0 \le u_g(-t^0)$ if and only if the solution $x_g(t,-t^0,x_0)$ is bounded from above as time decreases" (given by Theorem 2.13(iv) applied to g, since u_g is the upper bounded solution of (4.15)), combined with the equality $x_{f_-}(t,-t^0,x_0) = x_g(t,-t^0,x_0)$ for all $t \le -t^0$ in the (common) interval of definition, ensure that $u_\varepsilon^-(-t^0) = u_g(-t^0)$ and hence the global equality $u_\varepsilon^- = u_g$. Therefore, $|\tilde u_{g_-}(t) - u_g(t)| = |\tilde u_{g_-}(t) - u_\varepsilon^-(t)| = |\tilde u_{g_-}(t) - \tilde u_{f_-}(t)| \le \varepsilon$ for $t \le -t^0(\varepsilon)$. This shows that $\lim_{t\to-\infty}(u_g(t) - \tilde u_{g_-}(t)) = 0$. Analogous arguments show that $l_\varepsilon^- = l_g$ and $\lim_{t\to-\infty}(l_g(t) - \tilde l_{g_-}(t)) = 0$.

Now, let us check that $\lim_{t\to-\infty}(x_g(t,s,x) - \tilde m_{g_-}(t)) = 0$ if and only if $x \in (l_g(s), u_g(s))$. Note that $\bar x = x_g(-t^0,s,x) \in (l_g(-t^0), u_g(-t^0)) = (\tilde l_{f_-}(-t^0), \tilde u_{f_-}(-t^0))$, and that $x_g(t,s,x) = x_g(t,-t^0,\bar x) = x_{f_-}(t,-t^0,\bar x)$ for $t \le -t^0$. So, Proposition 4.12(ii) shows that $\lim_{t\to-\infty}(x_g(t,s,x) - \tilde m_{g_-}(t)) = 0$, as asserted. This property proves (ii) and shows the uniqueness ensured in (i), since any bounded solution different from l_g and u_g approaches $\tilde m_{g_-}$ as time decreases.

The previously proved properties and the attractive hyperbolicity (see Corollary 1.53) of $\tilde u_{f_-}$ provide a radius of uniform stability $\delta \in (0,\varepsilon)$, and a dichotomy constant pair (k,γ) with $k \ge 1$ and $\gamma > 0$ such that

$$|u_g(t) - x_g(t,s,u_g(s) \pm \delta)| = |\tilde u_{f_-}(t) - x_{f_-}(t,s,\tilde u_{f_-}(s) \pm \delta)| \le k\,\delta\,e^{-\gamma(t-s)}$$

if $s \le -t^0$ and $t \in [s,-t^0]$. Taking limit as $s \to -\infty$ proves that u_g is locally pullback attractive. It can be analogously proved that l_g is locally pullback attractive.

(iii)-(iv) Let us fix $\varepsilon \in (0,\varepsilon_0]$ and $t^0 = t^0(\varepsilon)$ as above, and define m_g as the (perhaps locally defined) maximal solution of (4.15) satisfying $m_g(t^0) = \tilde m_{f_+}(t^0)$. We take s in the domain of m_g and $x > m_g(s)$, and we observe that, to prove $\lim_{t\to\infty}(x_g(t,s,x) - \tilde u_{g_+}(t)) = 0$, there is no restriction in assuming that $s \ge t^0$, since all the solutions are globally forward defined. Then, $x_g(t,s,x) = x_{f_+}(t,s,x)$ for all $t \ge s$, and the assertion follows from Proposition 4.12(i). The same argument shows that $\lim_{t\to\infty}(x_g(t,s,x) - \tilde l_{g_+}(t)) = 0$ if $x < m_g(s)$. This duality shows that m_g is unique and independent of the choice of ε. In addition, $m_g(t) = \tilde m_{f_+}(t)$ for all $t \ge t^0$, which ensures that $|m_g(t) - \tilde m_{g_+}(t)| \le \varepsilon$ for all $t \ge t^0(\varepsilon)$, that is, $\lim_{t\to\infty}(m_g(t) - \tilde m_{g_+}(t)) = 0$. This completes the proofs of the asymptotic approaching properties of the solutions. The proof of the locally pullback repulsive character of m_g is analogous to that of the locally pullback attractive character of u_g. □

Remark 4.14. We point out that Theorem 4.13 shows that, if g and g_+ satisfy the part of **g1-g5** concerning them, then any $(s,x) \in \mathbb{R}\times\mathbb{R}$ satisfies one of the following three possibilities

(1) $\lim_{t\to\infty}(x_g(t,s,x) - \tilde l_{g_+}(t)) = 0$,

(2) $\lim_{t\to\infty}(x_g(t,s,x) - \tilde m_{g_+}(t)) = 0$,

(3) $\lim_{t\to\infty}(x_g(t,s,x) - \tilde u_{g_+}(t)) = 0$,

and, in particular, (2) holds if and only if $m_g(s) = x$. Theorem 4.13 also determines the asymptotic approaching as time decreases of the solution starting at (s,x), for $x \in [l_g(s), u_g(s)]$, to one of the hyperbolic solutions of the past equation if g and g_- satisfy the part of **g1-g5** concerning them.

The following technical lemma relates the asymptotic dynamics of the solutions of (4.15) to their hyperbolic character or the lack thereof. We include its proof even though it is standard (and similar to that of [32, Theorem 4.3.6]) for the sake of completeness.

Lemma 4.15. *Assume that g satisfies* **g1** *and* **g2**. *Let b_g be a bounded solution of* (4.15).

(i) *If the graph of b_g approaches that of an attractive (resp. repulsive) hyperbolic solution of the past equation $x' = g_-(t,x)$ as time decreases, then there exist $s_0 > 0$, $k \geq 1$ and $\gamma > 0$ such that $\exp \int_s^t g_x(l, b_g(l))\, dl \leq k\, e^{-\gamma(t-s)}$ if $s \leq t \leq -s_0$ (resp. $\exp \int_s^t (g_x(l, b_g(l))\, dl \leq k\, e^{\gamma(t-s)}$ if $t \leq s \leq -s_0$).*

(ii) *If the graph of b_g approaches that of an attractive (resp. repulsive) hyperbolic solution of the future equation $x' = g_+(t,x)$ as time increases, then there exist $s_0 > 0$, $k \geq 1$ and $\gamma > 0$ such that $\exp \int_s^t (g_x(l, b_g(l))\, dl \leq k\, e^{-\gamma(t-s)}$ if $s_0 \leq s \leq t$ (resp. $\exp \int_s^t g_x(l, b_g(l))\, dl \leq k\, e^{\gamma(t-s)}$ if $s_0 \leq t \leq s$).*

Consequently,

(iii) *if the graph of b_g approaches that of an attractive (resp. repulsive) hyperbolic solution of the past equation $x' = g_-(t,x)$ as time decreases and that of an attractive (resp. repulsive) hyperbolic solution of the future equation $x' = g_+(t,x)$ as time increases, then b_g is an attractive (resp. repulsive) hyperbolic solution of* (4.15).

(iv) *If the graph of b_g approaches that of an attractive (resp. repulsive) hyperbolic solution of the past equation $x' = g_-(t,x)$ as time decreases and that of a repulsive (resp. attractive) hyperbolic solution of the future equation $x' = g_+(t,x)$ as time increases, then b_g is a nonhyperbolic solution of* (4.15).

Proof. (i)-(ii) Assume that $\tilde{b}_{g_-}$ is a hyperbolic attractive solution of the past equation which is approached by b_g as time decreases. Let (k, γ) be a dichotomy constant pair for $\tilde{b}_{g_-}$. Given $\delta \in (0, \gamma)$, Lemma 4.9 ensures that there exists $s_0 > 0$ such that $|g_x(t, b_g(t)) - (g_-)_x(t, \tilde{b}_{g_-}(t))| < \delta$ for all $t \leq -s_0$. So, $\exp \int_s^t g_x(l, b_g(l))\, dl < \exp \int_s^t ((g_-)_x(l, \tilde{b}_{g_-}(l)) + \delta)\, dl \leq k\, e^{-(\gamma-\delta)\,(t-s)}$ for all $s \leq t \leq -s_0$. The proof in the hyperbolic repulsive case and the proof of (ii) are analogous.

(iii) Let us work in the attractive hyperbolic case. From (i), it is deduced (taking the maximum of both k's and minimum of both γ's) that there exist $k \geq 1$, $\gamma > 0$ and $s_1 < s_2$ such that $\exp \int_s^t g_x(l, b_g(l))\, dl \leq k\, e^{-\gamma(t-s)}$ if $s \leq t \leq s_1$ and $\exp \int_s^t g_x(l, b_g(l))\, dl \leq k\, e^{-\gamma(t-s)}$ if $s_2 \leq s \leq t$. As we can take larger k and smaller γ, we can assume without restriction that $\exp \int_s^t g_x(l, b_g(l))\, dl \leq k\, e^{-\gamma(t-s)}$ if $s_1 \leq s \leq t \leq s_2$. Finally, it is easy to check that $\exp \int_s^t g_x(l, b_g(l))\, dl \leq k^3\, e^{-\gamma(t-s)}$ for all $t \leq s$, which proves that b_g is hyperbolic attractive. The proof is analogous in the repulsive hyperbolic case.

(iv) Assume for contradiction that b_g is hyperbolic repulsive and that it approaches the graph of an attractive hyperbolic solution $\tilde{b}_{g_-}$ of the past equation as time increases. Then, (i) ensures that there exist $s_0 > 0$, $k_1 \geq 1$ and $\gamma_1 > 0$ such that $\exp \int_s^t g_x(l, b_g(l))\, dl \leq k_1\, e^{-\gamma_1(t-s)}$ if $s \leq t \leq -s_0$. On the other hand, the repulsive hyperbolic character of b_g provides the existence of $k_2 \geq 1$ and $\gamma_2 > 0$ such that $\exp \int_s^t g_x(l, b_g(l))\, dl \leq k_2\, e^{\gamma_2 (t-s)}$ for $t \leq s$. Interchanging the role of t and s in the second inequality and multiplying both of them: $1 \leq k_1 k_2 e^{-(\gamma_1+\gamma_2)\,(t-s)}$ for $s \leq t \leq -s_0$, a contradiction. In the other case, an analogous contradiction is reached. □

We will denote l_g, m_g and u_g by $\tilde{l}_g$, $\tilde{m}_g$ and $\tilde{u}_g$ when they are hyperbolic. Recall that two uniformly separated solutions are, by definition, bounded. Clearly, there exist (at least) two uniformly separated solutions if and only if l_g and u_g are uniformly separated.

Now, we state the main result of this section.

Theorem 4.16. *Assume that g satisfies* **g1**-**g5***, let $\tilde{l}_{g\pm} < \tilde{m}_{g\pm} < \tilde{u}_{g\pm}$ be the hyperbolic solutions given by* **g5***, and let l_g, u_g and m_g be the solutions of (4.15) provided by Theorem 4.13. Then, the dynamics of the transition equation (4.15) fits in one of the following dynamical scenarios:*

- Case A: *there exist exactly three hyperbolic solutions, $\tilde{l}_g = l_g$ and $\tilde{u}_g = u_g$, which are attractive, and $\tilde{m}_g = m_g$, which is repulsive. In addition, the unique solution uniformly separated from $\tilde{l}_g$ and $\tilde{u}_g$ is $\tilde{m}_g$. In this case, $\tilde{l}_g < \tilde{m}_g < \tilde{u}_g$, $\lim_{t\to\pm\infty}(\tilde{l}_g(t) - \tilde{l}_{g\pm}(t)) = 0$, $\lim_{t\to\pm\infty}(\tilde{m}_g(t) - \tilde{m}_{g\pm}(t)) = 0$ and $\lim_{t\to\pm\infty}(\tilde{u}_g(t) - \tilde{u}_{g\pm}(t)) = 0$.*

- Case B: *there exists exactly one hyperbolic solution, which is attractive, and uniformly separated only from another solution, which is locally pullback attractive and repulsive. There are two possibilities:*
  - Case B1: *$\tilde{u}_g = u_g$ is hyperbolic attractive and uniformly separated of $l_g = m_g$. In this case, $\lim_{t\to\infty}(\tilde{u}_g(t) - \tilde{u}_{g+}(t)) = 0$ and $\lim_{t\to\infty}(l_g(t) - \tilde{m}_{g+}(t)) = 0$.*
  - Case B2: *$\tilde{l}_g = l_g$ is hyperbolic attractive and uniformly separated of $m_g = u_g$. In this case, $\lim_{t\to\infty}(\tilde{l}_g(t) - \tilde{l}_{g+}(t)) = 0$ and $\lim_{t\to\infty}(u_g(t) - \tilde{m}_{g+}(t)) = 0$.*

- Case C: *there are no uniformly separated solutions. In this case, $\tilde{l}_g = l_g$ and $\tilde{u}_g = u_g$ are the unique hyperbolic solutions, they are attractive, and the locally pullback repulsive solution m_g is unbounded. There are two possibilities:*
  - Case C1: *$m_g < \tilde{l}_g$ in its domain of definition. In this case, $\lim_{t\to\infty}(\tilde{l}_g(t) - \tilde{u}_{g+}(t)) = \lim_{t\to\infty}(\tilde{u}_g(t) - \tilde{u}_{g+}(t)) = 0$.*
  - Case C2: *$m_g > \tilde{u}_g$ in its domain of definition. In this case, $\lim_{t\to\infty}(\tilde{l}_g(t) - \tilde{l}_{g+}(t)) = 0 = \lim_{t\to\infty}(\tilde{u}_g(t) - \tilde{l}_{g+}(t)) = 0$.*

Furthermore, if we assume $\lim_{t\to\infty}(u_g(t)-\tilde{u}_{g+}(t)) = 0$ (resp. $\lim_{t\to\infty}(l_g(t)-\tilde{l}_{g+}(t)) = 0$), then Case A *holds if and only if there exists $t_g \in \mathbb{R}$ such that $l_g(t_g) < m_g(t_g)$ (resp. $m_g(t_g) < u_g(t_g)$),* Case B1 *(resp.* B2*) holds if and only if there exists $t_g \in \mathbb{R}$ such that $l_g(t_g) = m_g(t_g)$ (resp. $m_g(t_g) = u_g(t_g)$), and* Case C1 *(resp.* C2*) holds if and only if there exists $t_g \in \mathbb{R}$ such that $l_g(t_g) > m_g(t_g)$ (resp. $m_g(t_g) > u_g(t_g)$).*

The proof of Theorem 4.16 requires the information provided by the next two theorems. Figures 4.1, 4.2 and 4.3 depict these five dynamical possibilities in the case of a map g which is asymptotically periodic with respect to t.

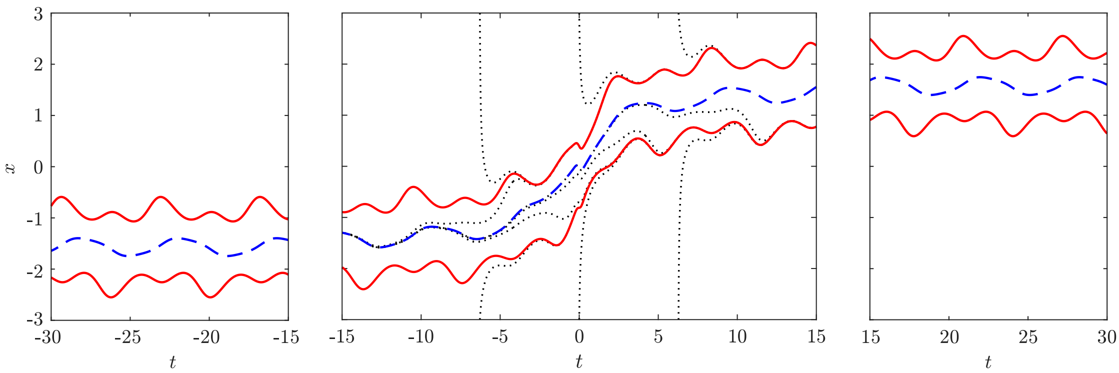


Figure 4.1: Numerical depiction of Case A. The central panel represents solutions of (4.15) with $g(t,x) = -(x-\phi(t))((x-\phi(t))^2 - \sin^2(t)) + 0.2\sin(t) + \phi'(t) + a\exp(-(c\,t)^2)$, where $\phi(t) = \arctan(0.3\,t)$, $a = -1$ and $c = 6$. In red solid line, the attractive hyperbolic solutions $\tilde{l}_g$ and $\tilde{u}_g$ and, in blue dashed line, the repulsive hyperbolic solution $\tilde{m}_g$. In black dotted lines, as in Figures 4.2 and 4.3, typical solutions different from the significant ones. Taking $g_\pm(t,x) = -(x\mp\pi/2)((x\mp\pi/2)^2 - \sin^2(t)) + 0.2\sin(t)$, we obtain g, g_- and g_+ satisfying **g1**-**g5** independently of the values of a and c. The three hyperbolic solutions of the past and the future equations are depicted in the left and right panels: for large values of t in the right one, for large values of $-t$ in the left one. Looking at all three panels simultaneously, the asymptotic approaching of the hyperbolic solutions of the transition equations to the hyperbolic solutions of the limit equations is apparent. Locally pullback attractive (resp. repulsive) solutions have been numerically approximated by forward (resp. backward) integration from close initial data on a large negative (resp. positive) time: their hyperbolicity properties guarantee a reliable representation. This way of approximating locally attractive pullback solutions will be used throughout the rest of the document without further mention.

Theorem 4.17. *Assume that g satisfies* **g1**-**g5**, *let $\tilde{l}_{g\pm} < \tilde{m}_{g\pm} < \tilde{u}_{g\pm}$ be the hyperbolic solutions given by* **g5**, *and let l_g, u_g and m_g be the solutions of (4.15) provided by Theorem 4.13. Assume also that $\lim_{t\to\infty}(u_g(t) - \tilde{u}_{g+}(t)) = 0$. Then, $\tilde{u}_g = u_g$ is an attractive hyperbolic solution, and one of the following cases holds:*

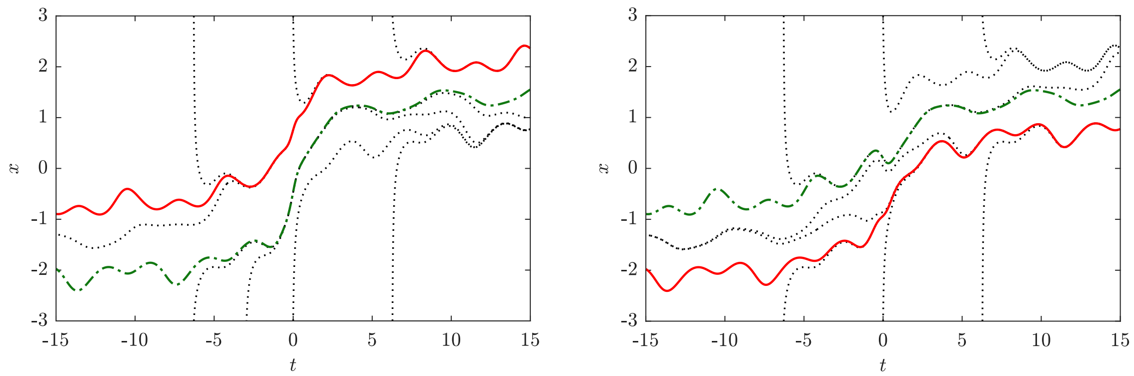
(1) *$\lim_{t\to\infty}(l_g(t) - \tilde{l}_{g+}(t)) = 0$, in which case m_g is globally defined and satisfies $\lim_{t\to-\infty}(m_g(t) - \tilde{m}_{g-}(t)) = 0$; if so, the equation (4.15) is in Case A, being $\tilde{l}_g = l_g$, $\tilde{m}_g = m_g$ and $\tilde{u}_g = u_g$ the three hyperbolic solutions. In addition, the unique solution uniformly separated from $\tilde{l}_g$ and $\tilde{u}_g$ is $\tilde{m}_g$.*

(2) *$\lim_{t\to\infty}(l_g(t) - \tilde{m}_{g+}(t)) = 0$ or, equivalently, $m_g = l_g$; if so, the equation (4.15) is in Case B1, being $\tilde{u}_g = u_g$ the unique hyperbolic solution and being uniformly separated only from $m_g = l_g$;*

(3) *$\lim_{t\to\infty}(l_g(t) - \tilde{u}_{g+}(t)) = 0$, in which case $\lim_{t\to\infty}(b_g(t) - \tilde{u}_{g+}(t)) = 0$ for every bounded solution $b_g(t)$ of (4.15); if so, the equation (4.15) is in Case C1, being $\tilde{u}_g = u_g$ and $\tilde{l}_g = l_g$ the unique hyperbolic solutions. In addition, m_g is unbounded and $m_g < \tilde{l}_g$ in its domain of definition.*

Besides, if $m_g = l_g$ or if $\lim_{t\to\infty}(l_g(t) - \tilde{u}_{g+}(t)) = 0$, then $\lim_{t\to\infty}(u_g(t) - \tilde{u}_{g+}(t)) = 0$.

Proof. Remark 4.14 shows that (1), (2) and (3) exhaust the possibilities for the limiting behavior of l_g as $t \to \infty$, as well as the equivalence stated in (2). In the three cases, the hyperbolicity of u_g follows from the assumed condition $\lim_{t\to\infty}(u_g(t) - \tilde{u}_{g+}(t)) = 0$, the fact that $\lim_{t\to-\infty}(u_g(t) - \tilde{u}_{g-}(t)) = 0$ (see Theorem 4.13(i)), and Lemma 4.15(iii).



(a) Case B1. In red solid line the attractive hyperbolic solution $\tilde{u}_g$, in green dashed-dotted line the locally pullback attractive and repulsive solution $l_g = m_g$. In this case, $a = 1$ and $c = 3.193748430049$.

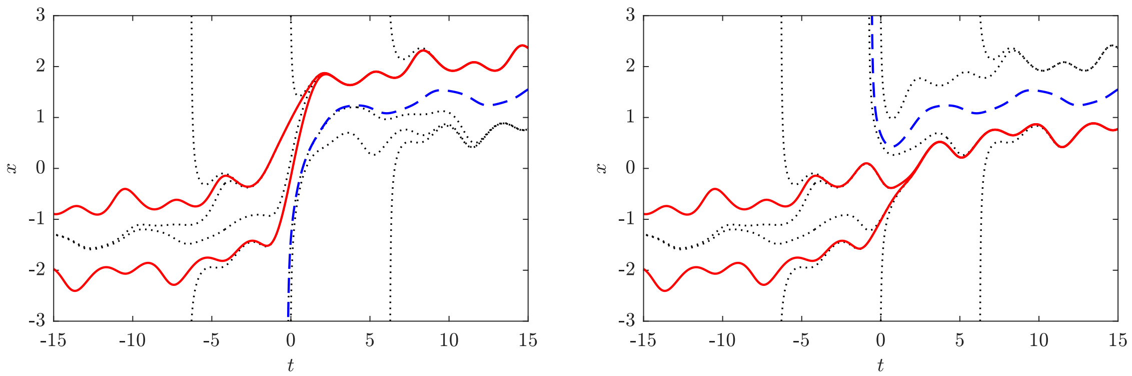
(b) Case B2. In red solid line the attractive hyperbolic solution $\tilde{l}_g$, in green dashed-dotted line the locally pullback attractive and repulsive solution $u_g = m_g$. In this case, $a = -1$ and $c = 2.343278767126$.

Figure 4.2: Aproximations to numerical depictions of Cases B. The high unpersistence of these cases causes them to be numerically undetectable. So, we choose as g a map whose general expression is that written in the caption of Figure 4.1, now with different choices for a and c which ensure that, still in Case A, the solutions l_g and m_g in Figure 4.2a, and m_g and u_g in Figure 4.2b, are at distances of less than 1e-7 from each other in the representation interval. Hence, we do not distinguish them, obtaining in this way suitable representations of Cases B. Here, we do not represent the hyperbolic solutions of the limit equations $x' = g_\pm(t, x)$, but, by having a look to the left and right panels in Figure 4.1, it is easy to observe that these six hyperbolic solutions play a role in the dynamics of the transition equation in both cases: see Remark 4.14.

Let us analyze situation (1): $\lim_{t\to\infty}(l_g(t) - \tilde{l}_{g+}(t)) = 0$ and $\lim_{t\to\infty}(u_g(t) - \tilde{u}_{g+}(t)) = 0$. That is, the graphs of l_g, m_g and u_g respectively approach those of $\tilde{l}_{g+}$, $\tilde{m}_{g+}$ and $\tilde{u}_{g+}$ as time increases (see Theorem 4.13(iii)), which ensures that $m_g(t) \in (l_g(t), u_g(t))$ for large enough t. In particular, m_g is globally defined. In these conditions, and according to Theorem 4.13(i) and (ii), the graphs of l_g, m_g and u_g respectively approach those of $\tilde{l}_{g-}$, $\tilde{m}_{g-}$ and $\tilde{u}_{g-}$ as time decreases. Hence, $l_g < m_g < u_g$ are three uniformly separated solutions, and Lemma 4.15(iii) shows that they are hyperbolic. The dynamics fits in Case A. Since Theorem 4.13(ii) and (iv) ensures that any bounded solution b_g distinct from $\tilde{l}_g < \tilde{m}_g < \tilde{u}_g$ satisfies $\lim_{t\to-\infty}(b_g(t) - \tilde{m}_{g-}(t)) = 0$ and either $\lim_{t\to\infty}(b_g(t) - \tilde{l}_{g+}(t)) = 0$ or $\lim_{t\to\infty}(b_g(t) - \tilde{u}_{g+}(t)) = 0$, the unique solution uniformly separated from $\tilde{l}_g$ and $\tilde{u}_g$ is $\tilde{m}_g$.

Assume that (2) holds (i.e., $m_g = l_g$), and let b_g be a bounded solution of (4.15) with $l_g = m_g < b_g < u_g$. Theorem 4.13(ii) and (iv) ensure that $\lim_{t\to-\infty}(b_g(t) - \tilde{m}_{g-}(t)) = 0$ and $\lim_{t\to\infty}(b_g(t) - \tilde{u}_{g+}(t)) = 0$. Therefore, u_g is only uniformly separated from $l_g = m_g$, with $\tilde{u}_g = u_g$ hyperbolic attractive and, according to Theorem 4.13(i) and (iii), l_g is locally pullback attractive and repulsive; and Lemma 4.15(iv) ensures that b_g and l_g are nonhyperbolic. We conclude that the dynamics fits in Case B1.

Finally, we assume that (3) holds; i.e., that $\lim_{t\to\infty}(l_g(t) - \tilde{u}_{g+}(t)) = 0$. In this case, any bounded solution b_g of (4.15) with $l_g \le b_g \le u_g$ satisfies $\lim_{t\to\infty}(b_g(t) - \tilde{u}_{g+}(t)) = 0$ (and hence Theorem 4.13(iii) ensures that m_g cannot be bounded), and if $l_g < b_g < u_g$, then $\lim_{t\to-\infty}(b_g(t) - \tilde{m}_{g-}(t)) = 0$. This precludes the existence of (bounded) uniformly separated solutions, as well as the existence of hyperbolic solutions different from u_g and l_g. In addition, Lemma 4.15(iii) shows the attractive hyperbolicity of l_g. Since there exists t_0 sufficiently large such that $m_g(t_0) < l_g(t_0)$, $m_g < l_g$ in the domain of definition of m_g. The dynamics fits in Case C1.



(a) Case C1. In red solid line the attractive hyperbolic solutions $\tilde{l}_g$ and $\tilde{u}_g$, in blue dashed line m_g. Here, $a = 1$ and $c = 1$.

(b) Case C2. In red solid line the attractive hyperbolic solutions $\tilde{l}_g$ and $\tilde{u}_g$, in blue dashed line m_g. Here, $a = -1$ and $c = 1$.

Figure 4.3: Numerical depictions of Cases C. Again, we change the values of a and c in the expression of g given in the caption Figure 4.1, and take the same $g_\pm$. And again, the information of Remark 4.14 is observable by having a look to the representation of the hyperbolic solutions of the limit equations made in the left and right panels of Figure 4.1.

Let us check the last assertions. If $m_g = l_g$, then $m_g(-t) < u_g(-t)$ for $t > 0$ sufficiently large. Then, Theorem 4.13(iv) ensures that $\lim_{t\to\infty}(u_g(t) - \tilde{u}_{g+}(t)) = 0$. And if $\lim_{t\to\infty}(l_g(t) - \tilde{u}_{g+}(t)) = 0$, then $m_g(t) < l_g(t) < u_g(t)$ for $t > 0$ sufficiently large, so Theorem 4.13(iv) ensures that $\lim_{t\to\infty}(u_g(t) - \tilde{u}_{g+}(t)) = 0$. □

Figures 4.1, 4.2a and 4.3a represent the situations described by points (1), (2) and (3) of Theorem 4.17. Figures 4.1, 4.2b and 4.3b represent the situations described by points (1), (2) and (3) of the next "symmetrical" result, whose proof is completely analogous to that of the previous theorem.

Theorem 4.18. *Assume that g satisfies* **g1**-**g5**, *let $\tilde{l}_{g\pm} < \tilde{m}_{g\pm} < \tilde{u}_{g\pm}$ be the hyperbolic solutions given by* **g5**, *and let l_g, u_g and m_g be the solutions of (4.15) provided by Theorem 4.13. Assume also that $\lim_{t\to\infty}(l_g(t) - \tilde{l}_{g+}(t)) = 0$. Then, $\tilde{l}_g = l_g$ is an attractive hyperbolic solution, and one of the following cases holds:*

(1) *$\lim_{t\to\infty}(u_g(t) - \tilde{u}_{g+}(t)) = 0$, in which case m_g is globally defined and satisfies $\lim_{t\to-\infty}(m_g(t) - \tilde{m}_{g-}(t)) = 0$; if so, the equation (4.15) is in Case A, being $\tilde{l}_g = l_g$, $\tilde{m}_g = m_g$ and $\tilde{u}_g = u_g$ the three hyperbolic solutions. In addition, the unique solution uniformly separated from $\tilde{l}_g$ and $\tilde{u}_g$ is $\tilde{m}_g$.*

(2) *$\lim_{t\to\infty}(u_g(t) - \tilde{m}_{g+}(t)) = 0$ or, equivalently, $m_g = u_g$; if so, the equation (4.15) is in Case B2, being $\tilde{l}_g = l_g$ the unique hyperbolic solution and being uniformly separated only from $m_g = u_g$;*

(3) *$\lim_{t\to\infty}(u_g(t) - \tilde{l}_{g+}(t)) = 0$, in which case $\lim_{t\to\infty}(b_g(t) - \tilde{l}_{g+}(t))$ for every bounded solution $b_g(t)$ of (4.15); if so, the equation (4.15) is in Case C2, being $\tilde{u}_g = u_g$ and $\tilde{l}_g = l_g$ the unique hyperbolic solutions. In addition, m_g is unbounded and $m_g > \tilde{u}_g$ in its domain of definition.*

Besides, if $m_g = u_g$ or if $\lim_{t\to\infty}(u_g(t) - \tilde{l}_{g+}(t)) = 0$, then $\lim_{t\to\infty}(l_g(t) - \tilde{l}_{g+}(t)) = 0$.

Proof of Theorem 4.16. The description of the cases is that of Theorems 4.17 and 4.18. Let us check that they exhaust the dynamical possibilities of (4.15). Remark 4.14 ensures that one of the three following possibilities hold: $\lim_{t\to\infty}(u_g(t) -$

$\tilde{l}_{g+}(t)) = 0$, $\lim_{t\to\infty}(u_g(t) - \tilde{m}_{g+}(t)) = 0$ or $\lim_{t\to\infty}(u_g(t) - \tilde{u}_{g+}(t)) = 0$. In the third one, Theorem 4.17 exhausts all the dynamical possibilities. In the first and the second one, the last assertion of Theorem 4.18 ensures that it exhausts all the dynamical possibilities of (4.15).

Let us check the last assertion in the case of $\lim_{t\to\infty}(u_g(t) - \tilde{u}_{g+}(t)) = 0$. Theorem 4.13(iv) ensures that $l_g(t_g) < m_g(t_g)$ if and only if $\lim_{t\to\infty}(l_g(t) - \tilde{l}_{g+}(t)) = 0$, $l_g(t_g) = m_g(t_g)$ if and only if $\lim_{t\to\infty}(l_g(t) - \tilde{m}_{g+}(t)) = 0$ and $l_g(t_g) > m_g(t_g)$ if and only if $\lim_{t\to\infty}(l_g(t) - \tilde{u}_{g+}(t)) = 0$. Theorem 4.17 ends the proof. The other case is analogous using Theorem 4.18 instead. □

4.2.1 Global attractor and pullback attractor

In this section, we analyze the continuity of the upper and lower equilibria of the global attractor (see Remark 4.10) in the different dynamical cases described in Theorem 4.16, and we provide a characterization of all the cases in terms of the forward attraction properties of the pullback attractor.

We recall that Theorem 2.13(iii) ensures that $\mathfrak{l}_g$ and $\mathfrak{u}_g$ are respectively lower and upper semicontinuous maps.

Proposition 4.19. *Assume that g satisfies* **g1**-**g5**. *The map $\mathfrak{l}_g$ (resp. $\mathfrak{u}_g$) is continuous if and only if* $\lim_{t\to\infty}(l_g(t) - \tilde{l}_{g+}(t)) = 0$ *(resp.* $\lim_{t\to\infty}(u_g(t) - \tilde{u}_{g+}(t)) = 0$*). Consequently,*

(i) (4.15) *is in* Case A *if and only if both $\mathfrak{l}_g$ and $\mathfrak{u}_g$ are continuous.*

(ii) *If* (4.15) *is in* Case B1 *or* C1, *then $\mathfrak{u}_g$ is continuous.*

(iii) *If* (4.15) *is in* Case B2 *or* C2, *then $\mathfrak{l}_g$ is continuous.*

Proof. Let us consider $\mathfrak{l}_{g\pm}, \mathfrak{u}_{g\pm} : \Omega_{g\pm} \to \mathbb{R}$ (see Remark 4.10) and recall that $\mathfrak{l}_g(g{\cdot}t) = l_g(t)$, $\mathfrak{u}_g(g{\cdot}t) = u_g(t)$, $\mathfrak{l}_{g\pm}(g_\pm{\cdot}t) = l_{g\pm}(t)$ and $\mathfrak{u}_{g\pm}(g_\pm{\cdot}t) = u_{g\pm}(t)$ for all $t \in \mathbb{R}$. Hypothesis **g5** and Theorem 2.18 ensure that $\mathfrak{l}_{g\pm}$ and $\mathfrak{u}_{g\pm}$ are continuous, since their graphs are hyperbolic copies of $\Omega_{g\pm}$: see Remark 4.8. Since both $t \mapsto \mathfrak{l}_g(\omega{\cdot}t)$ and $t \mapsto \mathfrak{l}_{g+}(\omega{\cdot}t)$ stand for the lower bounded solution of $x' = \omega(t, x)$ with $\omega \in \Omega_g^\omega = \Omega_{g+}^\omega$ (recall Lemma 4.6(i)), we get that $\mathfrak{l}_g(\omega) = \mathfrak{l}_{g+}(\omega)$, and analogously $\mathfrak{u}_g(\omega) = \mathfrak{u}_{g+}(\omega)$, for all $\omega \in \Omega_g^\omega = \Omega_{g+}^\omega$. An analogous argument shows that $\mathfrak{l}_g(\omega) = \mathfrak{l}_{g-}(\omega)$ and $\mathfrak{u}_g(\omega) = \mathfrak{u}_{g-}(\omega)$ for all $\omega \in \Omega_g^\alpha = \Omega_{g-}^\alpha$. Hence, the Pasting Lemma (see [80, Theorem 18.3]) ensures that the restrictions of $\mathfrak{l}_g$ and $\mathfrak{u}_g$ to $\Omega_g^\omega \cup \Omega_g^\alpha$ are continuous.

Let us check that $\lim_{t\to\infty}(u_g(t) - \tilde{u}_{g+}(t)) = 0$ is equivalent to the continuity of $\mathfrak{u}_g$ on Ω_g. We begin by assuming that $\mathfrak{u}_g$ is continuous on Ω_g. Take $\omega \in \Omega_g^\omega$, and $(t_n) \uparrow \infty$ such that $g{\cdot}t_n \to \omega$ as $n \to \infty$. The same argument used in the proof of Lemma 4.6 shows that $g_+{\cdot}t_n \to \omega$ as $n \to \infty$. Therefore, $\lim_{n\to\infty} \mathfrak{u}_g(g{\cdot}t_n) = \mathfrak{u}_g(\omega) = \mathfrak{u}_{g+}(\omega) = \lim_{n\to\infty} \mathfrak{u}_{g+}(g_+{\cdot}t_n)$, and hence $\lim_{n\to\infty}(u_g(t_n) - \tilde{u}_{g+}(t_n)) = 0$. Remark 4.14 and the uniform separation between $\tilde{l}_{g+} < \tilde{m}_{g+} < \tilde{u}_{g+}$ prove the claim.

Conversely, assume that $\lim_{t\to\infty}(u_g(t) - \tilde{u}_{g+}(t)) = 0$. If $\{g{\cdot}t \mid t \in \mathbb{R}\} \cap \Omega_{g+}^\omega \neq \emptyset$, then the compactness and τ-invariance of Ω_{g+}^ω ensures that $\{g{\cdot}t \mid t \in \mathbb{R}\} \subseteq \Omega_g \subseteq \Omega_{g+}^\omega$, and therefore $\mathfrak{u}_g = \mathfrak{u}_{g+}$ is continuous on $\Omega_g \subseteq \Omega_{g+}^\omega$. An analogous argument shows the continuity of $\mathfrak{u}_g$ if $\{g{\cdot}t \mid t \in \mathbb{R}\} \cap \Omega_{g-}^\alpha \neq \emptyset$. So, we assume without restriction that Ω is the disjoint union of $\{g{\cdot}t \mid t \in \mathbb{R}\}$ and $\Omega_{g+}^\omega \cup \Omega_{g-}^\alpha$. In particular, $\{g{\cdot}t \mid t \in \mathbb{R}\}$ is open. Since $\mathfrak{u}_g(g{\cdot}t) = u_g(t)$, the restriction of $\mathfrak{u}_g$

to $\{g{\cdot}t \mid t \in \mathbb{R}\}$ is also continuous. Hence, checking that $\mathfrak{u}_g$ is continuous on $\Omega^{\omega}_{g+} \cup \Omega^{\alpha}_{g-}$ proves that $\mathfrak{u}_g$ is globally continuous. We take $\omega \in \Omega^{\omega}_{g+} \cup \Omega^{\alpha}_{g-}$ and (ω_n) with limit ω. To check that $\mathfrak{u}_g(\omega_n) \to \mathfrak{u}_g(\omega)$ as $n \to \infty$, it suffices to prove that every subsequence admits a subsequence that converges to $\mathfrak{u}_g(\omega)$. Given any subsequence $(\mathfrak{u}_g(\omega_m))$ of $(\mathfrak{u}_g(\omega_n))$, we take a subsequence (ω_k) of (ω_m) contained either in $\{g{\cdot}t \mid t \in \mathbb{R}\}$ or in $\Omega^{\omega}_{g+} \cup \Omega^{\alpha}_{g-}$. In the first case, we can assume without loss of generality (taking a suitable subsequence, and taking into account that the case $(t_k) \downarrow -\infty$ is analogous) that there exists $(t_k) \uparrow \infty$ such that $g{\cdot}t_k \to \omega$ as $k \to \infty$. So, we get that $0 = \lim_{k\to\infty}(u_g(t_k) - \tilde{u}_{g_+}(t_k)) = \lim_{k\to\infty}(\mathfrak{u}_g(g{\cdot}t_k) - \mathfrak{u}_{g_+}(g_+{\cdot}t_k)) = \lim_{k\to\infty} \mathfrak{u}_g(g{\cdot}t_k) - \mathfrak{u}_{g_+}(\omega)$, and hence $\lim_{k\to\infty} \mathfrak{u}_g(g{\cdot}t_k) = \mathfrak{u}_{g_+}(\omega) = \mathfrak{u}_g(\omega)$. On the other hand, if (ω_k) is contained in $\Omega^{\omega}_{g+} \cup \Omega^{\alpha}_{g-}$, then the last assertion in the first paragraph ensures that $\mathfrak{u}_g(g{\cdot}t_n) \to \mathfrak{u}_g(\omega)$ as $n \to \infty$.

An analogous argument shows that $\lim_{t\to\infty}(l_g(t) - \tilde{l}_{g_+}(t)) = 0$ is equivalent to the continuity of $\mathfrak{l}_g$ on Ω_g. Points (i)-(iii) follow immediately from the two proved equivalences and Theorem 4.16. □

As said in Subsection 1.3.3, in general, pullback and forward attraction are unrelated (see e.g. [66]). The next result studies the local or global forward attraction properties of the pullback attractor $\mathscr{A}_g = \{\mathscr{A}_g(t) \mid t \in \mathbb{R}\}$ of (4.15), whose existence is guaranteed by Proposition 1.61 under our hypotheses **g1-g5** on g (see again Remark 4.10).

Proposition 4.20. *Assume that g satisfies* **g1-g5**. *Then,*

(i) *$\mathscr{A}_g$ is globally forward attractive if and only if (4.15) is in* Case A.

(ii) *$\mathscr{A}_g$ is not locally forward attractive if and only if (4.15) is in* Case B.

(iii) *$\mathscr{A}_g$ is locally forward attractive but not globally forward attractive if and only if (4.15) is in* Case C.

Proof. Recall that $\mathscr{A}_g(t) = [l_g(t), u_g(t)]$: see Remark 4.10. First, note that

$$\inf_{x_2 \in \mathscr{A}(t)} |x_g(t,s,x_1) - x_2| = \begin{cases} l_g(s) - x_g(t,s,x_1) & \text{if } x_1 < l_g(s)\,, \\ 0 & \text{if } x_1 \in [l_g(s), u_g(s)]\,, \\ x_g(t,s,x_1) - u_g(s) & \text{if } x_1 > u_g(s)\,, \end{cases}$$

so, for any bounded set $\mathcal{C} \subset \mathbb{R}$ and $s \in \mathbb{R}$, it is not hard to check that

$$\operatorname{dist}\big(x_g(t,s,\mathcal{C}), \mathscr{A}_g(t)\big) = \max\Big\{x_g(t,s,\sup \mathcal{C}) - u_g(t)\,,\ l_g(t) - x_g(t,s,\inf \mathcal{C})\,,\ 0\Big\}.$$

Therefore, $\mathscr{A}_g$ is globally forward attractive if and only if $\lim_{t\to\infty}(l_g(t) - x_g(t,s,x)) = 0$ for all $x < l_g(s)$ and $\lim_{t\to\infty}(x_g(t,s,x) - u_g(t)) = 0$ for all $x > u_g(s)$, and is locally forward attractive if and only if there exists $\delta > 0$ such that $\lim_{t\to\infty}(l_g(t) - x_g(t,s,l_g(s) - \delta)) = 0$ and $\lim_{t\to\infty}(x_g(t,s,u_g(s) + \delta) - u_g(t)) = 0$ for all $s \in \mathbb{R}$.

To prove the claims, let us check that, if (4.15) is in Case A, then $\mathscr{A}_g$ is globally forward attractive, that, if (4.15) is in Case B, then $\mathscr{A}_g$ is not locally forward attractive (and therefore not globally forward attractive), and that, if (4.15) is in Case C, then $\mathscr{A}_g$ is locally forward attractive but not globally forward attractive. Then, Theorem 4.16 completes the proof of the equivalences.

Recall that Theorem 4.13(iv) ensures that $\lim_{t\to\infty}(x_g(t,s,x) - \tilde{u}_{g+}(t)) = 0$ for all $x > m_g(s)$ and $\lim_{t\to\infty}(x_g(t,s,x) - \tilde{l}_{g+}(t)) = 0$ for all $x < m_g(s)$. If (4.15) is in Case A, then Theorem 4.16 ensures that $\lim_{t\to\infty}(l_g(t) - \tilde{l}_{g+}(t)) = 0$, and hence $\lim_{t\to\infty}(x_g(t,s,x) - l_g(t)) = \lim_{t\to\infty}(x_g(t,s,x) - \tilde{l}_{g+}(t)) + \lim_{t\to\infty}(\tilde{l}_{g+}(t) - l_g(t)) = 0$ for all $x < l_g(s)$. Analogously, $\lim_{t\to\infty}(u_g(t) - x_g(t,s,x)) = 0$ for all $x > u_g(s)$. Then, $\mathscr{A}_g$ is globally forward attractive. If (4.15) is in Case B1, then, for any $s \in \mathbb{R}$ and any $x < l_g(s) = m_g(s)$, $\lim_{t\to\infty}(x_g(t,s,x) - \tilde{l}_{\gamma+}(t)) = 0$, which combined with the fact that Theorem 4.16 ensures that $\lim_{t\to\infty}(l_g(t) - \tilde{m}_{g+}(t)) = 0$ and the uniform separation of $\tilde{l}_{\gamma+}$ and $\tilde{m}_{\gamma+}$ precludes $\lim_{t\to\infty}(x_g(t,s,x) - l_g(t)) = 0$. Therefore, $\mathscr{A}_g$ is not locally forward attractive. The argument is analogous if (4.15) is in Case B2. If (4.15) is in Case C1, then, an analogous argument to the one in Case B1 precludes $\lim_{t\to\infty}(x_g(t,s,x) - l_g(t)) = 0$ for $s \in \mathbb{R}$ in the domain of definition of m_g and $x < m_g(s)$, so $\mathscr{A}_g$ is not globally forward attractive. In this case, both u_g and l_g are hyperbolic attractive. The minimum of a radius of uniform stability of each of the two solutions (recall Corollary 1.53) provide $\delta > 0$ ensuring that $\mathscr{A}_g$ is locally forward attractive. The argument is analogous if (4.15) is in Case C2. □

4.2.2 Parametric variation of transition equations: What do we understand by a critical transition?

As explained in the Introduction, a *critical transition* (or *tipping point*) occurs when a small variation on the external input of the equation causes a dramatic variation on the dynamics of a system. We will focus on critical transitions associated to one-parametric families of equations which occur when the dynamics moves from Case A to one of the Cases C (see Theorem 4.16) as the parameter crosses a particular *critical value*. While in Case A, which from now on will also be referred to as *tracking*, the global attractor of the transition equation globally connects with that of the past equation as time decreases and with that of the future equation as time increases (see Figure 4.1, and see Proposition 4.19(i)), in Cases C it connects that of the past with only a part of that of the future. To this extent, Cases C can themselves be understood as tipping situations: the dynamics drastically changes as the transition takes place, and for this reason from now on Cases C will also be referred to as *tipping*. That is, if we understand that each of the past attractive hyperbolic solutions has a different physical or biological meaning, then in Cases C there exists a solution of the transition equation which connects states with different meanings (l_g or u_g, see Theorem 4.16).

Theorem 4.21 shows the persistence of Cases A, C1 and C2 under small suitable parametric variations, as well as the occurrence of a saddle-node bifurcation phenomenon when Case A transits to one of the Cases B as the parameter varies: the critical transition occurs as a consequence of the collision of an attractive hyperbolic solution with a repulsive one.

Theorem 4.21. *Let $\mathcal{C} \subseteq \mathbb{R}$ be an open interval, and let $\bar{g}\colon \mathbb{R}\times\mathbb{R}\times\mathcal{C} \to \mathbb{R}$ be a map such that $g^c(t,x) = \bar{g}(t,x,c)$ satisfies* **g1-g5** *for all $c \in \mathcal{C}$. Let $\bar{g}_x$ be the partial derivative with respect to the second variable, and assume that $\bar{g}$ and $\bar{g}_x$ are admissible on $\mathbb{R}\times\mathbb{R}\times\mathcal{C}$. Assume also that* $\limsup_{x\to\pm\infty}(\pm\bar{g}(t,x,c)) < 0$ *uniformly on $\mathbb{R}\times\mathcal{J}$ for any compact interval $\mathcal{J} \subset \mathcal{C}$.*

(i) *Assume that there exist c_1, c_2 in $\mathcal{C}$ with $c_1 < c_2$ such that the dynamics of*

$x' = g^c(t,x)$ is in Case A for $c = c_1$ and not for $c = c_2$. If $c_0 = \inf\{c > c_1 \,|\, \text{Case A does not hold}\}$, then $c_0 > c_1$. Let $\tilde{l}_{g^c} < \tilde{m}_{g^c} < \tilde{u}_{g^c}$ be the three hyperbolic solutions of $x' = g^c(t,x)$ for $c \in [c_1, c_0)$. Then, the dynamics of $x' = g^{c_0}(t,x)$ is either in Case B1, with $\lim_{c\to c_0^-}(\tilde{m}_{g^c}(t) - \tilde{l}_{g^c}(t)) = 0$ for all $t \in \mathbb{R}$, or in Case B2, with $\lim_{c\to c_0^-}(\tilde{u}_{g^c}(t) - \tilde{m}_{g^c}(t)) = 0$ for all $t \in \mathbb{R}$. The results are analogous if $c_1 > c_2$.

(ii) *Assume that there exist c_3, c_4 in $\mathcal{C}$ with $c_3 < c_4$ such that the dynamics of $x' = g^c(t,x)$ is in Case C1 for $c = c_3$ and not for $c = c_4$. If $c_0 = \inf\{c > c_3 \,|\, \text{Case C1 does not hold}\}$, then $c_0 > c_3$, and the dynamics of $x' = g^{c_0}(t,x)$ is in Case B1. The results are analogous by replacing C1 and B1 by C2 and B2, and also if $c_3 > c_4$.*

Proof. The admissibility hypotheses ensure that, for $c \in \mathcal{C}$, $\rho > 0$ and $\delta > 0$ fixed, there exists $\varepsilon_0 > 0$ such that

$$\sup_{(t,x)\in\mathbb{R}\times[-\rho,\rho]} |g^c(t,x) - g^{c+\varepsilon}(t,x)| + \sup_{(t,x)\in\mathbb{R}\times[-\rho,\rho]} |g^c_x(t,x) - g^{c+\varepsilon}_x(t,x)| < \delta$$

if $|\varepsilon| \le \varepsilon_0$. Hence, Theorems 1.52 and 4.16 guarantee the persistence of Case A under small variations of c. Let us check that also Case C1 is persistent, assuming for contradiction that $x' = g^{c_3}(t,x)$ is in Case C1 for a point $c_3 \in \mathcal{C}$, and the existence of a sequence (c_n) with limit c_3 such that $x' = g^{c_n}(t,x)$ is not Case C1 for all $n \in \mathbb{N}$. (The same argument works for Case C2.) Theorem 1.52 shows that $x' = g^{c_n}(t,x)$ has two different attractive hyperbolic solutions for large enough n, which must be $\tilde{l}_{g^{c_n}}$ and $\tilde{u}_{g^{c_n}}$ (see Theorem 4.16), and which satisfy $\lim_{n\to\infty} \big\|\tilde{l}_{g^{c_n}} - \tilde{l}_{g^{c_3}}\big\|_\infty = \lim_{n\to\infty} \big\|\tilde{u}_{g^{c_n}} - \tilde{u}_{g^{c_3}}\big\|_\infty = 0$. This precludes Cases B for large enough n. Let ρ be a radius of uniform exponential attraction provided by Theorem 1.52 for all $\tilde{l}_{g^{c_n}}$ and $\tilde{u}_{g^{c_n}}$ with n large enough, and let us take $n_0 \in \mathbb{N}$ such that $\big\|\tilde{l}_{g^{c_n}} - \tilde{l}_{g^{c_3}}\big\|_\infty < \rho/3$ and $\big\|\tilde{u}_{g^{c_n}} - \tilde{u}_{g^{c_3}}\big\|_\infty < \rho/3$ for all $n \ge n_0$. If $n \ge n_0$, we deduce from $\lim_{t\to\infty}(\tilde{u}_{g^{c_3}}(t) - \tilde{l}_{g^{c_3}}(t)) = 0$ the existence of t_0 such that $|\tilde{u}_{g^{c_n}}(t_0) - \tilde{l}_{g^{c_n}}(t_0)| < \rho$, and hence that $\lim_{t\to\infty}(\tilde{u}_{g^{c_n}}(t) - \tilde{l}_{g^{c_n}}(t)) = 0$, which precludes Case A. That is, $x' = g^{c_n}(t,x)$ is in Case C2 for all $n \ge n_0$.

Let k be a common bound for the $\|\cdot\|_\infty$-norm of the bounded solutions of $x' = g^{c_3}(t,x)$ and $x' = g^{c_3}_+(t,x)$ (see Proposition 1.61), and let $\varepsilon > 0$ be smaller than $\inf_{t\in\mathbb{R}}(\tilde{u}_{g^{c_3}_+}(t) - \tilde{m}_{g^{c_3}_+}(t))$ and $\inf_{t\in\mathbb{R}}(\tilde{m}_{g^{c_3}_+}(t) - \tilde{l}_{g^{c_3}_+}(t))$. Theorem 1.52 applied to $\varepsilon/4$ provides $\delta > 0$ such that, if f is C^1-admissible and $\|f - g^{c_3}_+\|_{1,k} < \delta$, then $x' = f(t,x)$ has three hyperbolic solutions at a $\|\cdot\|_\infty$-distance of those of $x' = g^{c_3}_+(t,x)$ less that $\varepsilon/4$, and hence with a separation between them of at least $\varepsilon/2$. The admissibility of $\bar{g}$ and condition **g2** applied to g^{c_3} and $g^{c_3}_+$ allow us to choose t_0 and n_0 large enough to get

$$\sup_{(t,x)\in[t_0,\infty)\times[-k,k]} |g^{c_n}(t,x) - g^{c_3}_+(t,x)| + \sup_{(t,x)\in[t_0,\infty)\times[-k,k]} |(g^{c_n})_x - (g^{c_3}_+)_x(t,x)| < \delta$$

for all $n \ge n_0$: we just write $|g^{c_n} - g^{c_3}_+| \le |g^{c_n} - g^{c_3}| + |g^{c_3} - g^{c_3}_+|$, do the same with the derivatives, and apply Lemma 4.9. Let us define $f^{c_n}_+(t,x)$ by truncating g^{c_n} at t_0, as in (4.17). Since $\|f^{c_n}_+ - g^{c_3}_+\|_{1,k} < \delta$, the equation $x' = g^{c_n}(t,x)$ has three (possibly locally defined) solutions, $b^{c_n}_1 < b^{c_n}_2 < b^{c_n}_3$, with $|b^{c_n}_i(t)| \le k + \varepsilon/4$ and $b^{c_n}_{i+1}(t) - b^{c_n}_i(t) \ge \varepsilon/2$ for all $t \ge t_0$ and $n \ge n_0$. We define $\bar{b}^{c_3}_i(t) = \lim_{n\to\infty} b^{c_n}_i(t)$ for

$i \in \{1,2,3\}$, and get three solutions of $x' = g^{c_3}(t,x)$ defined and uniformly separated by $\varepsilon/2$ on $[t_0,\infty)$. Since $x' = g^{c_n}(t,x)$ is in Case C2, we have $\bar b_2^{c_n}(t) \ge \tilde u_{g^{c_n}}(t)$ for all $t \in [t_0,\infty)$: there cannot be two different solutions separated on $[t_0,\infty)$ strictly below $\tilde u_{g^{c_n}}$. Hence, $\bar b_3^{c_3}(t) \ge \bar b_2^{c_3}(t) + \varepsilon/2 \ge \tilde u_{g^{c_3}}(t) + \varepsilon/2$ for all $t \in [t_0,\infty)$, which is not possible in Case C1. This is the sought-for contradiction.

Let us complete the proof of (i) in the case $c_1 < c_2$. The persistence of Cases A and C ensures that $c_0 > c_1$ and that $x' = g^{c_0}(t,x)$ is in one of the Cases B, say B1. Let us prove that $\lim_{c\to c_0^-}(\tilde m_{g^c}(t) - \tilde l_{g^c}(t)) = 0$ for all $t \in \mathbb{R}$ by checking that, given $(c_n) \uparrow c_0$, $\lim_{n\to\infty} \tilde m_{g^{c_n}}(t) = \lim_{n\to\infty} \tilde l_{g^{c_n}}(t) = m_{g^{c_0}}(t)$ for all $t \in \mathbb{R}$. The hypothesis on $\limsup_{x\to\pm\infty}(\pm\bar g(t,x,c))$ ensures that there exists a common bound for all the bounded solutions of $x' = g^{c_n}(t,x)$ for all $n \in \mathbb{N}$ (see again Proposition 1.61). Then, for $s \in \mathbb{R}$ fixed, there exists a subsequence (c_k) of (c_n) such that $\lim_{k\to\infty} \tilde m_{g^{c_k}}(s)$ and $\lim_{k\to\infty} \tilde l_{g^{c_k}}(s)$ exist. Moreover, $x_{c_0}(t,s,\lim_{k\to\infty} \tilde m_{g^{c_k}}(s)) = \lim_{k\to\infty} x_{c_k}(t,s,\tilde m_{g^{c_k}}(s)) = \lim_{k\to\infty} \tilde m_{g^{c_k}}(t)$ for all $t \in \mathbb{R}$, that is, $t \mapsto \lim_{k\to\infty} \tilde m_{g^{c_k}}(t)$ is a bounded solution of $x' = g^{c_0}(t,x)$. And analogously with $t \mapsto \lim_{k\to\infty} \tilde l_{g^{c_k}}(t)$. A new application of the last assertion of Theorem 1.52 applied to $\tilde u_{g^{c_0}}$ and its approximants $\tilde u_{g^{c_k}}$ provides a radius of uniform stability $\rho > 0$, and shows that $\tilde u_{g^{c_k}}(t) - \tilde l_{g^{c_k}}(t) > \tilde u_{g^{c_k}}(t) - \tilde m_{g^{c_k}}(t) \ge \rho$ if k is large enough: otherwise $\tilde u_{g^{c_k}}$ and $\tilde m_{g^{c_k}}$ would not be uniformly separated, which is impossible in Case A. And hence both limits are $m_{g^{c_0}}$, which is the unique bounded solution of $x' = g^{c_0}(t,x)$ uniformly separated from $\tilde u_{g^{c_0}} = \lim_{k\to\infty} \tilde u_{g^{c_k}}$ (see Theorem 4.16). Since this is the limit of any convergent subsequence, it is the limit of the original sequence. The remaining situations are proved with similar arguments.

To complete the proof of (ii) in the case $c_3 < c_4$ and with $x' = g^{c_3}(t,x)$ in Case C2, we deduce from the proved persistence that $c_0 > c_3$ and that the dynamics of $x' = g^{c_0}(t,x)$ is in one of the Cases B. Let us assume for contradiction that it is in Case B1, so that $\tilde u_{g^{c_0}}$ is hyperbolic. We take $(c_n) \uparrow c_0$, with $x' = g^{c_n}(t,x)$ in Case C2, and get the sought-for contradiction by repeating the last paragraph of the proof of the persistence of Case C1: just replace c_3 by c_0. The remaining cases are proved with similar arguments. □

We complete this part with a result which ensures the existence and local uniqueness of tipping points for certain parametric families:

Theorem 4.22. *Let $\mathcal{C} \subseteq \mathbb{R}$ be an open interval, and let $\{g^c \mid c \in \mathcal{C}\}$ be a family of functions satisfying* **g1**-**g5** *and such that, if $\bar g(t,x,c) = g^c(t,x)$, then $\bar g$ and $\bar g_x$ are admissible on $\mathbb{R}\times\mathbb{R}\times\mathcal{C}$. Assume that there exists $\bar c \in \mathcal{C}$ such that the dynamics of $x' = g^{\bar c}(t,x)$ is in* Case B1 *(resp.* Case B2*), and such that, for all $c_-, c_+ \in \mathcal{C}$ with $c_- < \bar c < c_+$: $g^{c_-}(t,x) \le g^{\bar c}(t,x) \le g^{c_+}(t,x)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}$; and, for any compact set $\mathcal{K} \subset \mathbb{R}$, there exist $t_{c_-} = t_{c_-}(\mathcal{K})$ and $t_{c_+} = t_{c_+}(\mathcal{K})$ such that the first and second inequalities are strict for $t = t_{c_-}$ and $t = t_{c_+}$ (respectively) and all $x \in \mathcal{K}$. Then, there exists $\rho > 0$ such that $x' = g^c(t,x)$ is in* Case A *(resp.* Case C2*) for $c \in (\bar c - \rho, \bar c)$ and in* Case C1 *(resp.* Case A*) for $c \in (\bar c, \bar c + \rho)$.*

Proof. Let l_c, u_c and m_c be the three solutions of $x' = g^c(t,x)$ given by Theorem 4.13. Let $g_\pm^c$ be the globally bounded and C^2-admissible functions associated to g^c by **g2** at $\pm\infty$, and let $\tilde l_{g_\pm^c} < \tilde m_{g_\pm^c} < \tilde u_{g_\pm^c}$ be the three hyperbolic solutions of $x' = g_\pm^c(t,x)$ provided by **g5**. Let k be a common bound for the $\|\cdot\|_\infty$-norm of these six solutions for $c = \bar c$. We take $2\varepsilon > 0$ smaller than $\inf_{t\in\mathbb{R}}(\tilde u_{g_\pm^{\bar c}}(t) - \tilde m_{g_\pm^{\bar c}}(t))$ and

$\inf_{t\in\mathbb{R}}(\tilde m_{g_\pm^{\bar c}}(t)-\tilde l_{g_\pm^{\bar c}}(t))$. Then, Theorem 1.52 provides $\delta>0$ such that if $\|g_\pm^{\bar c}-g\|_{1,k}<\delta$ for a C^1-admissible map g, then $x'=g(t,x)$ has three hyperbolic solutions at $\|\cdot\|_\infty$-distance of those of $x'=g_\pm^{\bar c}(t,x)$ less than ε. Since $x'=g^{\bar c}(t,x)$ has an attractive hyperbolic solution (see Theorem 4.16), we assume without restriction that if $\|g^{\bar c}-g\|_{1,k}<\delta/3$ for a C^1-admissible map g, then $x'=g(t,x)$ has an attractive hyperbolic solution at $\|\cdot\|_\infty$-distance of that of $x'=g^{\bar c}(t,x)$ less than $\varepsilon/2$.

The admissibility of $\bar g$ and $\bar g_x$ provides $\rho>0$ such that $\|g^{\bar c}-g^c\|_{1,k}<\delta/3$ for all $c\in(\bar c-\rho,\bar c+\rho)$. Our hypotheses provide $t_c\ge 0$ with $\sup_{(t,x)\in[t_c,\infty)\times[-k,k]}|g^c(t,x)-g_+^c(t,x)|+\sup_{(t,x)\in[t_c,\infty)\times[-k,k]}|g_x^c(t,x)-(g_+^c)_x(t,x)|<\delta/3$ for all $c\in(\bar c-\rho,\bar c+\rho)$: see **g2** and Lemma 4.9. We take $c_*\in(\bar c-\rho,\bar c+\rho)$ and $t^0=\max(t_{\bar c},t_{c_*})$. For $h(t,x)$, we denote by $\hat h(t,x)$ the map given by $h(t,x)$ for $t\ge t^0$ and by $g_+^{\bar c}(t,x)-g_+^{\bar c}(t^0,x)+h(t^0,x)$ for $t<t^0$. In this way, we construct $\hat g_+^{c_*}$, $\hat g^{\bar c}$ and $\hat g^{c_*}$ from $g_+^{c_*}$, $g^{\bar c}$ and g^{c_*}, and note that they are C^1-admissible. Then: $\|\hat g^{\bar c}-\hat g^{c_*}\|_{1,k}<\delta/3$, since the difference is $\hat g^{\bar c}(t_0,x)-\hat g^{c_*}(t_0,x)$ for $t<t^0$ and $g^{\bar c}(t,x)-g^{c_*}(t,x)$ for $t\ge t^0$; and $\|\hat g^{c_*}-\hat g_+^{c_*}\|_{1,k}<\delta/3$ and $\|g_+^{\bar c}-\hat g^{\bar c}\|_{1,k}<\delta/3$ for analogous reasons. So, $\|g_+^{\bar c}-\hat g_+^{c_*}\|_{1,k}<\delta$, and hence $x'=\hat g_+^{c_*}(t,x)$ has three hyperbolic solutions at a distance less than ε of those of $x'=g_+^{\bar c}(t,x)$. In addition, since they solve $x'=g_+^{c_*}(t,x)$ on $[t^0,\infty)$, the middle one coincides with $\tilde m_{g_+^{c_*}}$ on $[t^0,\infty)$: $\tilde m_{g_+^{c_*}}$ is the unique solution of $x'=g_+^{c_*}(t,x)$ uniformly separated from two other solutions as t increases. Hence, $\tilde u_{g_+^{\bar c}}(t)\ge\tilde m_{g_+^{c_*}}(t)+\varepsilon$ and $\tilde l_{g_+^{\bar c}}(t)\le\tilde m_{g_+^{c_*}}(t)-\varepsilon$ for $t\ge t^0$. Analogous arguments show, possibly increasing t^0, that $\tilde u_{g_-^{\bar c}}(t)\ge\tilde m_{g_-^{c_*}}(t)+\varepsilon$ and $\tilde l_{g_-^{\bar c}}(t)\le\tilde m_{g_-^{c_*}}(t)-\varepsilon$ for $t\le -t^0$.

Let us assume that $x'=g^{\bar c}(t,x)$ is in Case B1, associate ρ to $\bar c$ as above, and check that $x'=g^{c_*}(t,x)$ is in Case C1 for any $c_*\in(\bar c,\bar c+\rho)$, which we fix. Since $g^{c_*}(t,l_{c_*}(t))\ge g^{\bar c}(t,l_{c_*}(t))$ for all $t\in\mathbb{R}$, Theorem 2.13(v) shows that $l_{\bar c}\le l_{c_*}$. Let $t_0=t_0(\text{closure}(l_{c_*}(\mathbb{R})))$. These inequalities combined with $g^{c_*}(t_0,l_{c_*}(t_0))>g^{\bar c}(t_0,l_{c_*}(t_0))$ yield $l_{\bar c}(t)<l_{c_*}(t)$ for all $t>t_0$, and hence $\lim_{t\to\infty}(x_{\bar c}(t,t_0{+}1,l_{c_*}(t_0{+}1))-\tilde u_{g_+^{\bar c}}(t))=0$: see Theorems 4.13 and 4.16. A standard comparison argument shows that $x_{\bar c}(t,t_0+1,l_{c_*}(t_0+1))\le l_{c_*}(t)$ for $t\ge t_0+1$, and hence $\liminf_{t\to\infty}(l_{c_*}(t)-\tilde u_{g_+^{\bar c}}(t))\ge 0$. Thus, $\liminf_{t\to\infty}(l_{c_*}(t)-\tilde m_{g_+^{c_*}}(t))\ge\varepsilon$, which means Case C1 for c_*: see Theorem 4.16. Let us show now that Case A holds for $c_*\in(\bar c-\rho,\bar c)$. Analogous comparison arguments to those used for $c_*\in(\bar c,\bar c+\rho)$ ensure that $\limsup_{t\to\infty}(l_{c^*}(t)-\tilde m_{g_+^{c_*}}(t))\le -\varepsilon$, so $\lim_{t\to\infty}(l_{c^*}(t)-\tilde l_{g_+^{c_*}}(t))=0$. On the other hand, if $\tilde b_{g^{c_*}}$ is the attractive hyperbolic solution of $x'=g^{c_*}(t,x)$ mentioned at the end of the first paragraph, then $\liminf_{t\to\infty}(\tilde b_{g^{c_*}}(t)-\tilde m_{g_+^{c_*}}(t))\ge\liminf_{t\to\infty}(\tilde b_{g^{c_*}}(t)-\tilde u_{g_+^{\bar c}}(t))+\varepsilon\ge\liminf_{t\to\infty}(\tilde u_{g^{\bar c}}(t)-\tilde u_{g_+^{\bar c}}(t))+\varepsilon/2=\varepsilon/2$, and analogously $\liminf_{t\to-\infty}(\tilde b_{g^{c_*}}(t)-\tilde m_{g_-^{c_*}}(t))\ge\varepsilon/2$. So, Theorem 4.16 ensures that $\tilde b_{g^{c_*}}$ coincides with $u_{c_*}=\tilde u_{c_*}$ and that $x'=g^{c_*}(t,x)$ is in Case A. The stated properties if $x'=g^{\bar c}(t,x)$ is in Case B2 are proved analogously. □

Corollary 4.23. *Assume the hypotheses of Theorem 4.22 and, in addition, that $c\mapsto g^c(t,x)$ is nondecreasing for all $(t,x)\in\mathbb{R}$ and, for any $c_-<c_+$ with $c_-,c_+\in\mathcal{C}$ and any compact $\mathcal{K}\subset\mathbb{R}$ there exists $t_*=t_*(\mathcal{K})$ such that $g^{c_-}(t_*,x)<g^{c_+}(t_*,x)$ for all $x\in\mathcal{K}$. If $x'=g^{\bar c}(t,x)$ is in Case B1 (resp. B2), then $x'=g^c(t,x)$ is in Case C1 (resp. C2) for all $c\in\mathcal{C}$ with $c>\bar c$ (resp. $c<\bar c$). Consequently, there are at most two tipping points in $\mathcal{C}$ and, if they exist, say $\bar c_1<\bar c_2$, then $x'=g^c(t,x)$ is in Case C2 for all $c\in\mathcal{C}$ with $c<\bar c_1$, Case B2 for $c=\bar c_1$, Case A for $c\in(\bar c_1,\bar c_2)$, Case B1 for $c=\bar c_2$ and Case C1 for all $c\in\mathcal{C}$ with $c>\bar c_2$.*

Proof. Assume that there exists $\bar{c} \in \mathcal{C}$ such that $x' = g^{\bar{c}}(t,x)$ is in Case B1. Theorem 4.22 ensures that there exists $\rho > 0$ such that $x' = g^c(t,x)$ is in Case C1 for $c \in (\bar{c}, \bar{c}+\rho)$. Assume for contradiction that $c_0 = \inf\{c \in \mathcal{C}$ with $c > \bar{c}$ such that $x' = g^c(t,x)$ is not in Case C1$\}$ belongs to $\mathcal{C}$. Then, Theorem 4.21(ii) ensures that $x' = g^{c_0}(t,x)$ is in Case B1. A contradiction is reached applying Theorem 4.22 to c_0. So, $x' = g^c(t,x)$ is in Case C1 for all $c \in \mathcal{C}$ with $c > \bar{c}$. The proof is analogous if $x' = g^{\bar{c}}(t,x)$ is in Case B2. In addition, these properties show that neither two points in Case B1 nor two points in Case B2 can exist in $\mathcal{C}$ and that, in case two points $\bar{c}_1 < \bar{c}_2$ in Cases B exist, they are such that $x' = g^c(t,x)$ is in Case C2 for $c < \bar{c}_1$, Case B2 for $c = \bar{c}_1$, in Case B1 for $c = \bar{c}_2$, and in Case C1 for $c > \bar{c}_2$. Theorems 4.22 and 4.21(i) ensure that $x' = g^c(t,x)$ is in Case A for $c \in (\bar{c}_1, \bar{c}_2)$. □

Remarks 4.24. 1. All the hypotheses concerning order in Theorem 4.22 and Corolary 4.23 hold if $c \mapsto g^c(t,x)$ is strictly increasing for all $(t,x) \in \mathbb{R} \times \mathbb{R}$, which is the case in many examples.

2. By changing in all the hypotheses of these two results the type of monotonicity with respect to c, we obtain analogous results.

4.2.3 A property precluding some dynamical possibilities

We complete this section with another monotonicity property, which unlike the previous ones is not related to a parametric variation, and which will be used in the next section. It establishes two conditions precluding some of the five dynamical possibilities for (4.15) under conditions **g1**-**g5** provided by Theorem 4.16.

Proposition 4.25. *Assume that g satisfies* **g1**-**g5**. *Then,*

(i) *if there exists $h_1 \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $h_1(t,x) \le g(t,x)$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}$, and $x' = h_1(t,x)$ has a bounded solution b_1 such that $\liminf_{t\to\infty}(b_1(t) - \tilde{m}_{g_+}(t)) > 0$, then $x' = g(t,x)$ is in Case A, B1 or C1.*

(ii) *If there exists $h_2 \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $h_2(t,x) \ge g(t,x)$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}$, and $x' = h_2(t,x)$ has a bounded solution b_2 such that $\liminf_{t\to\infty}(\tilde{m}_{g_+}(t) - b_2(t)) > 0$, then $x' = g(t,x)$ is in Case A, B2 or C2.*

Proof. Let us check (i): the second proof is analogous. Let u_g be the upper bounded solution of $x' = g(t,x)$. Then, $b_1 \le u_g$ (see Theorem 2.13(v)). Hence, $\liminf_{t\to\infty}(u_g(t) - \tilde{m}_{g_+}(t)) \ge \liminf_{t\to\infty}(b_1(t) - \tilde{m}_{g_+}(t)) > 0$, which according to Theorem 4.16 precludes Cases B2 and C2. □

Note that, by combining both conditions, we guarantee Case A.

The example depicted in Figure 4.4 shows that the hypotheses concerning the relative order of $\tilde{m}_{g_+}$ and the bounded solution b_i in the statement are not superfluous.

4.3 Critical transitions for $x' = f(t,x,\Gamma^c(t,x))$

The main purpose of this section is to analyze the occurrence (or lack) of tipping points for a parametric family of transition equations of the form

$$x' = f(t,x,\Gamma^c(t,x)) \tag{4.18}$$

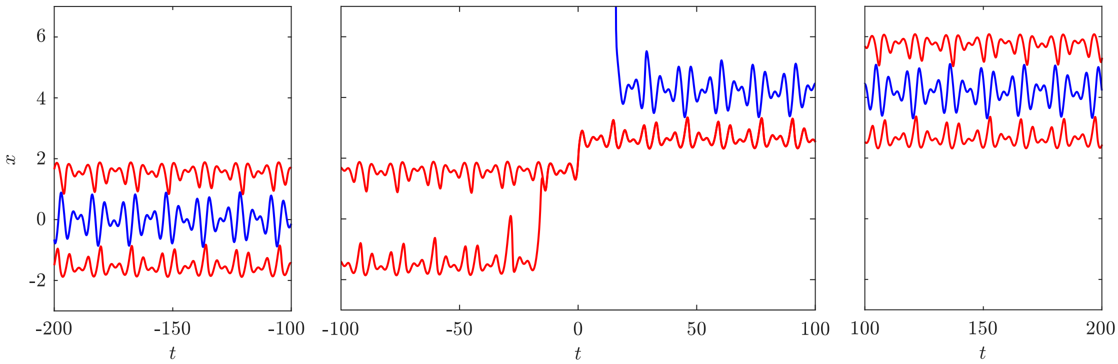


Figure 4.4: We define $\Gamma(t) = \arctan(5t)/\pi + 1/2 \in (0,1)$, $g(t,x) = -x^3 + \sin(t) + \sin(\sqrt{2}\,t) + (5/2)\,x + \Gamma(t)\,a\,(3x^2 - 3ax + a^2 - 5/2)$, and $g_-(t,x)$ and $g_+(t,x)$ by replacing $\Gamma(t)$ by 0 and 1 in $g(t,x)$, respectively. Then, **g1-g5** hold. The central panel shows that the dynamics of $x' = g(t,x)$ for $a = 4.2$ corresponds to CASE C2, as we will check below: we depict in red the two attractive hyperbolic solutions $\tilde l_g$ and $\tilde u_g$, and in blue the unbounded locally pullback repulsive solution m_g. The right (resp. left) panel shows the three hyperbolic solutions of the future equation $x' = g_+(t,x)$ (resp. past equation $x' = g_-(t,x)$). It is easy to check that $g_-(t,x) \le g(t,x)$ for this choice of a. But, as checked below, any bounded solution of $x' = g_-(t,x)$ (which are bounded by the red curves in the left panel) is below $\tilde m_{g_+}$ (depicted in blue in the right panel), and hence neither the hypotheses (using g_- as h_1) nor the thesis of Proposition 4.25(i) are fulfilled.
To prove the previous assertions, we first check that $g_-(t,x-a) = g_+(t,x)$ and that $\pm g_-(t,r) < 0$ for $\pm r > 2$. Hence, $-2 \le \tilde l_{g_-}(t) < \tilde u_{g_-}(t) \le 2$, and $2.2 \le \tilde l_{g_+}(t) < \tilde u_{g_+}(t) \le 6.2$, which implies the assertion. In addition, since $\liminf_{t\to-\infty}(\tilde l_{g_+}(t) - u_g(t)) > \lim_{t\to-\infty}(\tilde u_{g_-}(t) - u_g(t)) + 0.2 = 0.2$, we get $u_g(t) < \tilde l_{g_+}(t)$ for $t \le t_0$; and $u_g'(t) < \tilde l_{g_+}'(t)$ if $u_g(t) = \tilde l_{g_+}(t)$, from where we deduce that $u_g(t) < \tilde l_{g_+}(t)$ for all $t \in \mathbb{R}$. This is only possible in CASE C2.

as c varies. In this case, the name of transition equation assigned to (4.18) is justified by the functions Γ^c, which are assumed to approach a pair of maps $\Gamma^c_\pm(t,x)$ as $t \to \pm\infty$ for each value of c. This configuration of the transition equation enables a distinct treatment of two components within the evolution law: the stationary or permanent component, typically picked up by the explicit dependence on t and x in the function f, and the transient or transitional component, picked up by the function Γ^c. In many instances, the functions $\Gamma^c_\pm$ will typically be independent of c, say $\Gamma_\pm$, giving rise to the (common) past and future equations:

$$x' = f(t,x,\Gamma_-(t,x)) \quad \text{and} \quad x' = f(t,x,\Gamma_+(t,x))\,.$$

As we will explain very soon, the parameter may play different roles: for example, to determine the speed of this transition, or its value at time 0, or the maximum size that certain magnitude reaches during the transition. This formulation is a generalization of that which gave rise to this theory: the parameter shifts studied in [12]. In our case, both the transition function Γ^c and the functions $\Gamma^c_\pm$ that Γ^c asymptotically approximates can explicitly depend on both the state variable x and the time variable t. Clearly, this broadens the field of application of the theory, allowing us to deal with more realistic models: a priori, there is no reason to assume that the future (or the past) will not depend on time and/or state.

Let us mention three of the large variety of physical mechanisms that may cause critical transitions in (4.18):

- *Rate-induced critical transitions:* if $\Gamma^c(t,x) = \Gamma(ct,x)$ for a fixed Γ and any $c > 0$, then the parameter $c > 0$ determines the speed of the transition Γ^c. In order to have a past and a future independent of the rate, we require Γ_- and Γ_+ to be independent of t. So, a larger c means a significant distance from $\Gamma(ct,x)$ to $\Gamma_\pm(x)$ during a shorter period.

- *Phase-induced critical transitions:* if $\Gamma^c(t,x) = \Gamma(c+t,x)$, then the parameter $c \in \mathbb{R}$ represents the initial phase of the transition function. As before, we assume Γ_- and Γ_+ independent of t.

- *Size-induced critical transitions:* with $\Gamma_- \equiv 0$ and $\Gamma^c(t,x) = c\,\Gamma(t,x)$, different values of $c > 0$ mean different sizes of the transition function which "takes" $x' = f(t,x,0)$ to $x' = f(t,x,c\,\Gamma_+(t,x))$.

There are two typical rate-related phenomena: *rate-induced tipping* and the so-called *rate-induced tracking*. The first one occurs when the increase of the transition rate leads the system to tipping, i.e. surpassing a critical speed threshold causes tipping. The second consists of the inverse scenario, i.e., the increase of the transition speed leads the system to tracking. In this case, tipping occurs for speeds below a specific threshold. There are abundant references to both applied phenomena and mathematical formulation on rate-induced tipping [3, 12, 60, 61, 65, 73, 105, 115], while rate-induced tracking (sometimes called overshooting) has so far received less attention [101, 102]. In the numerical simulations discussed in this document we will focus on instances of rate-induced tracking. Regarding the phase-induced critical transitions, we emphasize that the change of variable $s = t + c$ transforms the parametric transition equation into $x' = f(s-c,x,\Gamma(s,x))$, giving rise to the analysis of *total tipping*, *partial tipping* and *total tracking* (see [3] and [71, 72]), i.e., the study of tipping or tracking for a fixed transition function and all the functions of the hull of f with a fixed transition function Γ. This subject will not be dealt with in the present document. On the other hand, we would like to emphasize that, unlike rate-induced and phase-induced critical transitions, our definition of size-induced critical transitions does allow for the future equation to depend on the parameter c. In these transitions, the significance of "size" corresponds to the magnitude referenced by the γ parameter, the third variable of the function f.

The section if composed by several subsections. The first one is devoted to establish conditions on f, on the maps Γ^c (or more precisely, on the map Γ^c for every fixed value of the parameter c) and on the "limits" $\Gamma^c_\pm$ that guarantee that the equation $(4.18)_c$ satisfies the conditions **g1-g5** required in the previous section on $x' = g(t,x)$, its past and its future, so that the classification in Cases A, B1, B2, C1 and C2 holds. Thus, the meaning of a critical transition is that explained at the beginning of Section 4.2.2. The parametric variation is introduced in Section 4.3.2, and conditions ensuring that the general results of Sections 4.2.2 and 4.2.3 can be applied are established. Section 4.3.3 presents some results on the existence of safety intervals such that if the transition function remains in them throughout the transition, the occurrence of critical transitions is ruled out. Furthermore, under the size-induced and rate-induced paradigms, it explains the possible existence of tipping points when the transition function goes out of the safety interval: if it goes out of the safety interval by a large amount (size) or during very long time intervals (rate), a critical transition may occur. Finally, Section 4.3.4 does not contain new results concerning critical transitions, but rather includes two examples that apply the theory from this chapter to demonstrate the optimality of the assumptions employed in two theorems of previous chapters.

4.3.1 Dynamical possibilities for a fixed parameter value

In this section, we provide a set of hypotheses that ensure that, for each value of the parameter c, the dynamical possibilities of equation $(4.18)_c$ are those described by Theorem 4.16. To this end, we consider a single transition equation of the form $x' = f(t, x, \Gamma(t, x))$ and study its dynamical possibilities.

Let $\mathcal{I} \subseteq \mathbb{R}$ be an open interval, and let the maps $f\colon \mathbb{R}\times\mathbb{R}\times\mathcal{I} \to \mathbb{R}$, $(t, x, \gamma) \mapsto f(t, x, \gamma)$ and $\Gamma, \Gamma_-, \Gamma_+ \colon \mathbb{R}\times\mathbb{R} \to \mathcal{I}$ satisfy the following hypotheses:

f1 There exist the derivatives f_x, f_{xx}, f_γ, $f_{\gamma\gamma}$, $f_{x\gamma}$ and $f_{\gamma x}$, and f, f_x, f_{xx}, f_γ, $f_{\gamma\gamma}$, $f_{x\gamma}$ and $f_{\gamma x}$ are admissible on $\mathbb{R}\times\mathbb{R}\times\mathcal{I}$.

f2 Γ, Γ_- and Γ_+ take values in $[a, b] \subset \mathcal{I}$, are C^2-admissible, and $\lim_{t\to\pm\infty}(\Gamma(t, x) - \Gamma_\pm(t, x)) = 0$ uniformly on each compact subset $\mathcal{J} \subset \mathbb{R}$.

f3 $\limsup_{x\to\pm\infty}(\pm f(t, x, \gamma)) < 0$ uniformly in $(t, \gamma) \in \mathbb{R}\times\mathcal{J}$ for all compact interval $\mathcal{J} \subset \mathcal{I}$.

f4 $\inf_{t\in\mathbb{R}} \big((\partial^2/\partial x^2) f(t, x, \Gamma_\pm(t, x))|_{x=x_1} - (\partial^2/\partial x^2) f(t, x, \Gamma_\pm(t, x))|_{x=x_2}\big) > 0$ whenever $x_1 < x_2$.

f5 Each equation $x' = f(t, x, \Gamma_\pm(t, x))$ has three hyperbolic solutions $\tilde{l}_{\Gamma_\pm} < \tilde{m}_{\Gamma_\pm} < \tilde{u}_{\Gamma_\pm}$.

With the same abuse of language as in Remark 4.4.1, we will say that the pair (f, Γ) satisfies **f1-f5** if there exist maps Γ_- and Γ_+ such that the previous conditions are satisfied, and refer to the equations

$$x' = f(t, x, \Gamma_-(t, x)) \quad \text{and} \quad x' = f(t, x, \Gamma_+(t, x)) \tag{4.19}$$

as the "past" and "future" of

$$x' = f(t, x, \Gamma(t, x))\,. \tag{4.20}$$

We can easily prove the next result.

Proposition 4.26. *Assume that (f, Γ) satisfies* **f1-f5**. *Then, the maps g and $g_\pm$ respectively given by $g(t, x) = f(t, x, \Gamma(t, x))$ and $g_\pm(t, x) = f(t, x, \Gamma_\pm(t, x))$ satisfy the conditions* **g1-g5**. *Therefore, the dynamical possibilities for (4.20) are those described in Theorem 4.16.*

Proof. Let $\mathcal{J} \subset \mathbb{R}$ be a compact set. Since **f2** ensures that $\Gamma(\mathbb{R}\times\mathbb{R}) \subseteq [a, b]$ and **f1** ensures that f is bounded on $\mathbb{R}\times\mathcal{J}\times[a, b]$, we get that $g(t, x) = f(t, x, \Gamma(t, x))$ is bounded on $\mathbb{R}\times\mathcal{J}$. Since f is uniformly continuous in $\mathbb{R}\times\mathcal{J}\times[a, b]$, given $\varepsilon > 0$, there exists $\delta' > 0$ such that $|f(t_1, x_1, \gamma_1) - f(t_2, x_2, \gamma_2)| \le \varepsilon$ if $|t_1 - t_2| < \delta'$, $x_1, x_2 \in \mathcal{J}$, $|x_1 - x_2| < \delta'$, $\gamma_1, \gamma_2 \in [a, b]$ and $|\gamma_1 - \gamma_2| < \delta'$. Since Γ is uniformly continuous on $\mathbb{R}\times\mathcal{J}$, there exists $\delta \in (0, \delta')$ such that $|\Gamma(t_1, x_1) - \Gamma(t_2, x_2)| < \delta'$ if $|t_1 - t_2| < \delta$, $x_1, x_2 \in \mathcal{J}$ and $|x_1 - x_2| < \delta$. Hence, $|g(t_1, x_1) - g(t_2, x_2)| = |f(t_1, x_1, \Gamma(t_1, x_1)) - f(t_2, x_2, \Gamma(t_2, x_2))| \le \varepsilon$ if $|t_1 - t_2| < \delta$, $x_1, x_2 \in \mathcal{J}$ and $|x_1 - x_2| < \delta$, that is, g is uniformly continuous on $\mathbb{R}\times\mathcal{J}$. So, g is admissible. Analogous arguments show that $g_x(t, x) = f_x(t, x, \Gamma(t, x)) + f_\gamma(t, x, \Gamma(t, x))\,\Gamma_x(t, x)$ and $g_{xx}(t, x) = f_{xx}(t, x, \Gamma(t, x)) + 2 f_{x\gamma}(t, x, \Gamma(t, x))\,\Gamma_x(t, x) + f_\gamma(t, x, \Gamma(t, x))\,\Gamma_{xx}(t, x) + f_{\gamma\gamma}(t, x, \Gamma(t, x))\,(\Gamma_x(t, x))^2$ are also admissible (note that $f_{x\gamma} = f_{\gamma x}$). The proofs are the same for g_- and g_+. That is, **g1** holds. Using again the uniform continuity of f on $\mathbb{R} \times \mathcal{J} \times [a,b]$, we get that $|f(t,x,\Gamma(t,x)) - f(t,x,\Gamma_+(t,x))| \le \varepsilon$ if $t \in \mathbb{R}$, $x \in \mathcal{J}$ and $|\Gamma(t,x) - \Gamma_+(t,x)| < \delta'$, which **f2** ensures that is uniformly achieved on $\mathcal{J}$ as $t \to \infty$. The proof for g_- is analogous, taking $t \to -\infty$ instead of $t \to \infty$. Therefore, **g2** holds. Conditions **g3**-**g5** follow directly from **f3**-**f5**. □

Remarks 4.27. 1. It can be checked that the proof of Proposition 4.26 can be repeated in the next cases. First, if we remove the boundedness of Γ and $\Gamma_\pm$ from condition **f2** but assume that $\mathcal{I} = \mathbb{R}$ and that the limit in **f3** is uniform in $(t,\gamma) \in \mathbb{R} \times \mathbb{R}$: since Γ is bounded on $\mathbb{R} \times \mathcal{J}$, a compact interval containing $\Gamma(\mathbb{R} \times \mathcal{J})$ plays the role of $[a,b]$ at the beginning of the proof, and **g3** follows from this boundedness and the more demanding hypotheses which replaces **f3**. And second, if we remove the assumptions on the derivatives f_γ, $f_{\gamma\gamma}$, $f_{x\gamma}$ and $f_{\gamma x}$ of **f1** but assume that Γ, and hence $\Gamma_\pm$, depend only on t: in this case, we can repeat the proof and observe that these derivatives do not appear in the expressions of g_x and g_{xx}. Hence, the conclusions of Theorems 4.13 and 4.16 also hold under these conditions.

2. As explained in Remark 4.8, Proposition 4.26 applied to the pairs $(f, \Gamma_\pm)$ allows us to reformulate condition **f5** as: "each equation $x' = f(t,x,\Gamma_\pm(t,x))$ has three uniformly separated solutions", which determines its global dynamics.

4.3.2 Concerning parametric variation

Our purpose in this section is to check that hypotheses **f1**-**f5** on (f,Γ) combined with the independence of t of $\Gamma_\pm$ in the rate-induced and phase-induced cases, suffices to identify the rate-induced and phase-induced tipping points with parametric transitions from Case A to one of the Cases C, which must happen "crossing" one of the Cases B and can be understood as nonautonomous saddle-node bifurcations. That is, that the parametric family (4.18) satisfies all the hypotheses of Theorem 4.21. In the size-induced case, it is also required to reach the same conclusion that conditions **f1**-**f5** hold for $(f, c\Gamma)$ for all c, which is particularly fulfilled if we assume that $\Gamma_- = \Gamma_+ \equiv 0$. Proposition 4.28 will prove that all the hypotheses of the (slightly) more general Theorem 4.30 hold, and this theorem proves our assertion.

Proposition 4.28. *Let Γ satisfy* **f2**. *The following assertions hold:*

(i) *If Γ_+ and Γ_- are independent of t, then $(t,x,c) \mapsto \Gamma^c(t,x) = \Gamma(ct,x)$ is admissible on $\mathbb{R} \times \mathbb{R} \times (0,\infty)$.*

(ii) *If Γ_+ and Γ_- are independent of t, then $(t,x,c) \mapsto \Gamma^c(t,x) = \Gamma(c+t,x)$ is admissible on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$,*

(iii) *$(t,x,c) \mapsto \Gamma^c(t,x) = c\,\Gamma(t,x)$ is admissible on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$.*

In addition, in the three cases, for any $c \in \mathcal{C}$, there exists $\delta_c > 0$ such that $\sup_{(t,x)\in\mathbb{R}\times\mathbb{R},\, |\varepsilon|\le\delta_c} |\Gamma^{c+\varepsilon}(t,x)| < \infty$.

Proof. (i) Note that **f2** ensures that $\Gamma^c(\mathbb{R}\times\mathbb{R}) = \Gamma(\mathbb{R}\times\mathbb{R}) \subseteq [a,b]$ for all $c > 0$. This proves that $(t,x,c) \mapsto \Gamma^c(t,x)$ is bounded on its domain, as well as the last assertion of the theorem in this rate-variation case. It remains to check that $(t,x,c) \mapsto \Gamma^c(t,x)$ is uniformly continuous on $\mathbb{R} \times \mathcal{K} \times \mathcal{J}$ for arbitrarily fixed compact sets $\mathcal{K} \subset \mathbb{R}$ and

$\mathcal{J} \subset (0,\infty)$. Let us fix $\varepsilon > 0$. Since $\lim_{t\to\pm\infty} \Gamma(t,x) = \Gamma_\pm(x)$ uniformly on $\mathcal{K}$, there exists $t_0 > 0$ such that $|\Gamma(t,x) - \Gamma_\pm(x)| \le \varepsilon/3$ if $\pm t \ge t_0$ and $x \in \mathcal{K}$. Since $\Gamma_\pm$ are uniformly continuous on $\mathcal{K}$ (because they are admissible), there exists $\delta_1 > 0$ such that $|\Gamma_\pm(x_1) - \Gamma_\pm(x_2)| \le \varepsilon/3$ if $x_1, x_2 \in \mathcal{K}$ and $|x_1 - x_2| < \delta_1$. So, if $t_1, t_2 \ge t_0$, $x_1, x_2 \in \mathcal{K}$ and $|x_1 - x_2| < \delta_1$, then

$$\begin{aligned} |\,\Gamma(t_1,x_1) - \Gamma(t_2,x_2)| \le |\,\Gamma(t_1,x_1) - \Gamma_+(x_1)| \\ + |\,\Gamma_+(x_1) - \Gamma_+(x_2)| + |\,\Gamma_+(x_2) - \Gamma(t_2,x_2)| \le \varepsilon\,, \end{aligned}$$

and the same happens if $t_1, t_2 \le -t_0$, $x_1, x_2 \in \mathcal{K}$ and $|x_1 - x_2| < \delta_1$. Since, due to its admissibility, Γ is uniformly continuous on $\mathbb{R} \times \mathcal{K}$, there exists $\delta_2 \in (0, \delta_1)$ such that $|\Gamma(t_1,x_1) - \Gamma(t_2,x_2)| \le \varepsilon$ if $|t_1 - t_2| < \delta_2$, $x_1, x_2 \in \mathcal{K}$ and $|x_1 - x_2| < \delta_2$. So, to complete the proof, it suffices to check the existence of $\delta_3 \in (0, \delta_2)$ such that, if $|t_1 - t_2| < \delta_3$, $c_1, c_2 \in \mathcal{J}$ and $|c_1 - c_2| < \delta_3$, then one of these three options holds: $|c_1t_1 - c_2t_2| < \delta_2$; $c_1t_1, c_2t_2 \ge t_0$; $c_1t_1, c_2t_2 \le -t_0$. Let $k = \max\{t_0, t_0/\inf \mathcal{J}\} > 0$. Clearly, if $t_1 \ge 2k$ (resp. $t_1 \le -2k$) and $|t_1 - t_2| < k$, then $t_2 \ge k$ (resp. $t_2 \le -k$), and $c_1t_1, c_2t_2 \ge t_0$ (resp. $c_1t_1, c_2t_2 \le -t_0$) for all $c_1, c_2 \in \mathcal{J}$. Hence, it suffices to find $\delta_3 \in (0, \delta_2)$ with $\delta_3 \le k$ such that, if $t_1, t_2 \in [-2k, 2k]$, $|t_1 - t_2| < \delta_3$, $c_1, c_2 \in \mathcal{J}$ and $|c_1 - c_2| < \delta_3$, then $|c_1t_1 - c_2t_2| < \delta_2$, which is easy.

(ii) The same arguments provide a quite simpler proof in this case.

(iii) Given $\varepsilon > 0$, we use arguments analogous to those in the proof of (i) to find $\delta > 0$ such that $|c_1\,\Gamma(t_1,x_1) - c_2\,\Gamma(t_2,x_2)| \le |c_1|\,|\Gamma(t_1,x_1) - \Gamma(t_2,x_2)| + |c_1 - c_2|\,|\Gamma(t_2,x_2)| \le \varepsilon$ for $|t_1 - t_2| < \delta$, $x_1, x_2 \in \mathcal{K}$, $|x_1 - x_2| < \delta$, $c_1, c_2 \in \mathcal{J}$ and $|c_1 - c_2| < \delta$. The last assertion of the statement is deduced from $\Gamma(\mathbb{R} \times \mathbb{R}) \subseteq [a,b]$. □

Remark 4.29. Note that if, in the considered case of rate and phase variation, with $\Gamma_\pm$ independent of t, all the pairs (f, Γ^c) satisfy **f1-f5** if (f, Γ) does, with the same maps $\Gamma_\pm$. The same occurs in the size-variation case if we also assume $\Gamma_+ \equiv 0$.

Theorem 4.30. *Let $\mathcal{C} \subseteq \mathbb{R}$ be an open interval, and let the maps $\{\Gamma^c \mid c \in \mathcal{C}\}$ be a family of functions such that all the pairs (f, Γ^c) satisfy* **f1-f5** *and such that $\mathbb{R} \times \mathbb{R} \times \mathcal{C} \to \mathbb{R}$, $(t,x,c) \mapsto \Gamma^c(t,x)$ is admissible. Assume also that, for any $c \in \mathcal{C}$, there exists $\delta_c > 0$ such that $\sup_{(t,x)\in\mathbb{R}\times\mathbb{R},\,|\varepsilon|\le\delta_c} |\Gamma^{c+\varepsilon}(t,x)| < \infty$. Then, the map $\bar{g}(t,x,c) = f(t,x,\Gamma^c(t,x))$ satisfies all the hypotheses of Theorem 4.21.*

Proof. Proposition 4.26 ensures that $g^c(t,x) = \bar{g}(t,x,c)$ satisfies **g1-g5** for all $c \in \mathcal{C}$. Let $\mathcal{K}_1 \subset \mathbb{R}$ and $\mathcal{J} \subset \mathcal{C}$ be compact sets. Let δ_c be the constant of the statement for any $c \in \mathcal{C}$. Since $\mathcal{J}$ is covered by a finite amount of balls $B_\mathbb{R}(c, \delta_c)$, there exists a compact set $\mathcal{K}_2 \subset \mathbb{R}$ such that $\Gamma^c(t,x) \in \mathcal{K}_2$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}$ and $c \in \mathcal{J}$. Since f is bounded on $\mathbb{R} \times \mathcal{K}_1 \times \mathcal{K}_2$, we get that $\bar{g}$ is bounded on $\mathbb{R} \times \mathcal{K}_1 \times \mathcal{J}$. Now, let us check that $\bar{g}$ is uniformly continuous on $\mathbb{R} \times \mathcal{K}_1 \times \mathcal{J}$. Given $\varepsilon > 0$, the uniform continuity of f on $\mathbb{R} \times \mathcal{K}_1 \times \mathcal{K}_2$ provides $\delta' > 0$ such that $|f(t_1, x_1, \Gamma^{c_1}(t_1,x_1)) - f(t_2, x_2, \Gamma^{c_2}(t_2,x_2))| \le \varepsilon$ if $|t_1 - t_2| < \delta'$, $x_1, x_2 \in \mathcal{K}_1$, $|x_1 - x_2| < \delta'$ and $|\Gamma^{c_1}(t_1,x_1) - \Gamma^{c_2}(t_2,x_2)| < \delta'$. Since $(t,x,c) \mapsto \Gamma^c(t,x)$ is admissible, it is uniformly continuous on $\mathbb{R} \times \mathcal{K}_1 \times \mathcal{J}$, so there exists $\delta \in (0, \delta')$ such that $|\Gamma^{c_1}(t_1,x_1) - \Gamma^{c_2}(t_2,x_2)| < \delta'$ if $|t_1 - t_2| < \delta$, $x_1, x_2 \in \mathcal{K}_1$, $|x_1 - x_2| < \delta$, $c_1, c_2 \in \mathcal{J}$ and $|c_1 - c_2| < \delta$. Hence, $\bar{g}$ is admissible on $\mathbb{R} \times \mathbb{R} \times \mathcal{C}$. Analogous arguments prove the admissibility of $\bar{g}_x$. Since $\Gamma^c(t,x) \in \mathcal{K}_2$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}$ and $c \in \mathcal{J}$, we deduce from **f3** that $\limsup_{x\to\pm\infty}(\pm\bar{g}(t,x,c)) < 0$ uniformly on $\mathbb{R} \times \mathcal{J}$. This completes the proof. □

4.3.3 Transition functions and safety intervals

In this section, we describe conditions ensuring the lack of rate-induced and phase-induced critical transitions (in Theorem 4.31 and Corollary 4.32), as well as the occurrence of size-induced critical transitions (in Theorem 4.33) and the uniqueness of the tipping point in rate-induced tracking scenarios (in Theorem 4.36).

This collection of results has a common thread. Let us assume that $\gamma \mapsto f(t,x,\gamma)$ is nondecreasing (to be specified below). Roughly speaking, the idea is that, if the transition function Γ and its future asymptotic approximate Γ_+ take values in a suitable *safety interval*, determined by the values of the parameter for with $x' = f(t,x,\gamma)$ has three hyperbolic solutions, then the transition exhibits tracking. The immediate consequence is the absence of rate-induced and phase-induced tipping points. This is what Theorem 4.31 and Corollary 4.32 prove. If, on the contrary, the parametric variation of the transition function forces it to leave any bounded interval, and hence any possible safety interval, then tipping occurs. Theorem 4.33 establishes conditions under which this is precisely the situation. Under its hypotheses, in the size-induced variation analysis, tipping occurs either for all the values of the parameter or outside a compact interval. Finally, Theorem 4.36 uses the previous results to establish conditions ensuring the uniqueness, in case of existence, of a rate-induced critical transition, which is of rate-induced tracking type.

Safety intervals: absence of critical points

The next results show the aforementioned existence of a safety interval for the range of Γ and Γ_+, precluding tipping and hence the occurrence of size-induced or phase-induced critical transitions. The hypotheses of Corollary 4.32 are more demanding than those of Theorem 4.31, but also easier to be verified in the applications.

Theorem 4.31. *Assume that (f,Γ) satisfies* **f1**-**f5**. *Assume also that $\gamma \mapsto f(t,x,\gamma)$ is nondecreasing for all $(t,x) \in \mathbb{R}\times\mathbb{R}$, and that there exist $\gamma_1 \le \gamma_2$ such that: $\Gamma(\mathbb{R}\times\mathbb{R}) \subseteq [\gamma_1,\gamma_2]$, $x' = f(t,x,\gamma_1)$ has a bounded solution b_1 with $\liminf_{t\to\infty}(b_1(t) - \tilde m_{\Gamma_+}(t)) > 0$; and $x' = f(t,x,\gamma_2)$ has a bounded solution b_2 with $\liminf_{t\to\infty}(\tilde m_{\Gamma_+}(t) - b_2(t)) > 0$. Then, (4.20) is in* Case A.

If, in addition, we assume that $\Gamma_\pm$ do not depend on t, then the equations $x' = f(t,x,\Gamma(ct,x))$ and $x' = f(t,x,\Gamma(t+c,x))$ are in Case A *for all $c > 0$ and $c \in \mathbb{R}$, respectively: there are neither rate-induced nor phase-induced critical transitions.*

Proof. Apply Proposition 4.25(i) to $h_1(t,x) = f(t,x,\gamma_1) \le f(t,x,\Gamma(t,x))$ to preclude Cases B2 and C2 for (4.20), and apply Proposition 4.25(ii) to $f(t,x,\Gamma(t,x)) \le h_2(t,x) = f(t,x,\gamma_2)$ to preclude Cases B1 and C1 for (4.20).

Note that $\Gamma^c(\mathbb{R}\times\mathbb{R}) \subseteq \Gamma(\mathbb{R}\times\mathbb{R})$ for $\Gamma^c(t,x) = \Gamma(ct,x)$ and for $\Gamma^c(t,x) = \Gamma(c+t,x)$. This fact, Remark 4.29 and the already checked property prove the second assertion. □

Corollary 4.32. *Assume that (f,Γ) satisfies* **f1**-**f5**. *Assume also that $\gamma \mapsto f(t,x,\gamma)$ is nondecreasing for all $(t,x) \in \mathbb{R}\times\mathbb{R}$, and that there exist $\gamma_1 \le \gamma_2$ such that $\Gamma(\mathbb{R}\times\mathbb{R}) \subseteq [\gamma_1,\gamma_2]$, $\Gamma_+(\mathbb{R}\times\mathbb{R}) \subseteq [\gamma_1,\gamma_2]$, $\inf_{t\in\mathbb{R}}(f_{xx}(t,x_1,\gamma) - f_{xx}(t,x_2,\gamma)) > 0$ whenever $x_1 < x_2$ for all $\gamma \in [\gamma_1,\gamma_2]$, and $x' = f(t,x,\gamma)$ has three hyperbolic solutions for all $\gamma \in [\gamma_1,\gamma_2]$. Then, (4.20) is in* Case A.

If, in addition, we assume that $\Gamma_\pm$ do not depend on t, then the equations $x' = f(t,x,\Gamma(ct,x))$ and $x' = f(t,x,\Gamma(t+c,x))$ are in Case A *for all $c>0$ and $c \in \mathbb{R}$, respectively: there are neither rate-induced nor phase-induced critical transitions.*

Proof. Lemma 4.7 with $g(t,x) = g_\pm(t,x) = f(t,x,\gamma)$ ensures that the extension to the hull of $(t,x) \mapsto f(t,x,\gamma)$ satisfies **d1**, **d2**, **d3** and **d4** for all $\gamma \in [\gamma_1,\gamma_2]$, and hence that the dynamics of $x' = f(t,x,\gamma)$ is that described in Theorem 2.18 for all $\gamma \in [\gamma_1,\gamma_2]$. Let us take $\tilde\gamma \in [\gamma_1,\gamma_2]$. Theorem 1.52 provides $\delta_{\tilde\gamma} > 0$ such that $x' = f(t,x,\gamma)$ has three hyperbolic solutions $\tilde l_\gamma < \tilde m_\gamma < \tilde u_\gamma$ for all $\gamma \in (\tilde\gamma - \delta_{\tilde\gamma}, \tilde\gamma + \delta_{\tilde\gamma})$ with $\inf_{t\in\mathbb{R}}(\tilde u_{\lambda_1}(t) - \tilde m_{\lambda_2}(t)) > 0$ for all $\lambda_1 \le \lambda_2$ in $(\tilde\gamma - \delta_{\tilde\gamma}, \tilde\gamma + \delta_{\tilde\gamma})$. Hence, Proposition 2.19(ii) ensures that $\tilde l_{\lambda_1} \le \tilde l_{\lambda_2} < \tilde m_{\lambda_2} \le \tilde m_{\lambda_1} < \tilde u_{\lambda_1} \le \tilde u_{\lambda_2}$ if $\lambda_1 \le \lambda_2$ in $(\tilde\gamma - \delta_{\tilde\gamma}, \tilde\gamma + \delta_{\tilde\gamma})$. Since $[\gamma_1,\gamma_2]$ is compact, there exist $\tilde\gamma_1, \tilde\gamma_2, \ldots \tilde\gamma_n$ such that $(\tilde\gamma_i - \delta_{\tilde\gamma_i}, \tilde\gamma_i + \delta_{\tilde\gamma_i})$ cover $[\gamma_1,\gamma_2]$ for $i \in \{1,2,\ldots,n\}$. Hence, thanks to the connection of $[\gamma_1,\gamma_2]$, we can combine the previous inequalities to obtain that $\tilde l_{\gamma_1} \le \tilde l_{\gamma_2} < \tilde m_{\gamma_2} \le \tilde m_{\gamma_1} < \tilde u_{\gamma_1} \le \tilde u_{\gamma_2}$. Now, since $\Gamma_+(\mathbb{R}\times\mathbb{R}) \subseteq [\gamma_1,\gamma_2]$, Proposition 2.19(i) ensures that $\tilde u_{\gamma_1} \le \tilde u_{\Gamma_+}$, so $\tilde m_{\gamma_2} \le \tilde m_{\gamma_1} < \tilde u_{\gamma_1} \le \tilde u_{\Gamma_+}$. Then, $\inf_{t\in\mathbb{R}}(\tilde u_{\Gamma_+}(t) - \tilde m_{\gamma_2}(t)) > 0$ and Proposition 2.19(ii) ensures that $\inf_{t\in\mathbb{R}}(\tilde m_{\Gamma_+}(t) - \tilde l_{\gamma_2}(t)) > 0$. Analogously, $\inf_{t\in\mathbb{R}}(\tilde u_{\gamma_1}(t) - \tilde m_{\Gamma_+}(t)) > 0$, so the hypotheses of Theorem 4.31 hold. The proof of the first assertion is complete, and the second one follows from the fact that $\Gamma^c(\mathbb{R}\times\mathbb{R}) \subseteq \Gamma(\mathbb{R}\times\mathbb{R})$ for $\Gamma^c(t,x) = \Gamma(ct,x)$ and for $\Gamma^c(t,x) = \Gamma(c+t,x)$, Remark 4.29 and the already checked property. □

Occurrence of size-induced critical transitions

Theorem 4.33, based on Theorem 4.22, shows either the absence of critical transitions or the occurrence of exactly two tipping points under hypotheses precluding the transition function Γ^d to take values in any fixed interval for all the values of the parameter, where $\Gamma^d = \Gamma_0 + d\,\Gamma$ (the size-variation case is included). Looking for clarity in the statements, we just analyze the situation of nonnegative Γ.

Theorem 4.33. *Let $\Gamma_0\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $\Gamma\colon \mathbb{R}\times\mathbb{R} \to [0,\infty)$ be globally bounded and C^2-admissible, and such that the pair $(f, \Gamma_0 + d\,\Gamma)$ satisfies* **f1**-**f5** *for all $d \in \mathbb{R}$. Assume that $\Gamma(t_0,x) > 0$ for all $x \in \mathbb{R}$ and a $t_0 \in \mathbb{R}$. Assume also that $\gamma \mapsto f(t,x,\gamma)$ is strictly increasing on $\mathbb{R}$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}$, with $\lim_{\gamma\to\pm\infty} f(t,x,\gamma) = \pm\infty$ uniformly on compact sets of $\mathbb{R}\times\mathbb{R}$. Then,*

$$x' = f(t,x,\Gamma_0(t,x) + d\,\Gamma(t,x)) \tag{4.21}$$

is either in Case C1 *for all $d \in \mathbb{R}$, in* Case C2 *for all $d \in \mathbb{R}$, or there exist $d_- < d_+$ such that it is in* Case C2 *for $d < d_-$, in* Case B2 *for $d = d_-$, in* Case A *for $d \in (d_-, d_+)$, in* Case B1 *for $d = d_+$, and in* Case C1 *for $d > d_+$.*

Proof. Proposition 4.28(iii) shows that $(t,x,d) \mapsto d\,\Gamma(t,x)$ is admissible on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, and hence so it is $(t,x,d) \mapsto \Gamma_0(t,x) + d\,\Gamma(t,x)$. It also shows that, for any $d \in \mathbb{R}$, there exists $\delta_d > 0$ such that $\sup_{(t,x)\in\mathbb{R}\times\mathbb{R},\,|\varepsilon|\le\delta_d} |\Gamma_0(t,x) + (d+\varepsilon)\,\Gamma(t,x)| < \infty$. Let us define $g^d(t,x) = f(t,x,\Gamma_0(t,x) + d\,\Gamma(t,x))$. The nonnegative character of Γ and the strictly increasing character of $\gamma \mapsto f(t,x,\gamma)$ ensures that $g^{d_1}(t,x) \le g^{d_0}(t,x) \le g^{d_2}(t,x)$ and $g^{d_1}(t_0,x) < g^{d_0}(t_0,x) < g^{d_2}(t_0,x)$ for all $d_1 < d_0 < d_2$ and $(t,x) \in \mathbb{R}\times\mathbb{R}$. Since Proposition 4.26 ensures that g^d satisfies **g1**-**g5** for all $d \in \mathbb{R}$, Theorems 4.21 and 4.22 can be applied at any point.

We assume that $(4.21)_d$ is in Case A for $d = \bar d$ and define $d_+ = \inf\{d > \bar d \mid$ Case A does not hold$\}$. Theorem 1.52 ensures that $\bar d < d_+$. We assume for contradiction that $d_+ = \infty$. Let $\tilde{l}_d < \tilde{m}_d < \tilde{u}_d$ be the three hyperbolic solutions of $(4.21)_d$ for $d \geq \bar{d}$. Theorem 2.13(v) yields $\tilde{l}_{\bar{d}} \leq \tilde{l}_d$ for all $d \geq \bar{d}$. Let us prove that $\tilde{m}_d \leq \tilde{m}_{\bar{d}}$ for all $d \geq \bar{d}$. Clearly, it suffices to check it for $d \in [\bar{d}, \bar{d}+\rho]$ for any $\rho > 0$, which we deduce from Theorem 1.52 and Proposition 2.19(ii) as in the proof of Corollary 4.32.

Therefore, we can take two constants $m_1 < m_2$ such that $m_1 \leq \tilde{l}_{\bar{d}} \leq \tilde{l}_d < \tilde{m}_d \leq \tilde{m}_{\bar{d}} \leq m_2$ for all $d \geq \bar{d}$. We look for $t_1 < t_0 < t_2$ and $k > 0$ such that $\Gamma(t,x) > k$ if $t \in [t_1,t_2]$ and $x \in [m_1,m_2]$, and call $k_d = \inf_{(t,x)\in[t_1,t_2]\times[m_1,m_2]} f(t,x,\Gamma_0(t,x) + d\,\Gamma(t,x))$ for $d \geq \bar{d}$. Then, $k_d \geq \inf_{(t,x)\in[t_1,t_2]\times[m_1,m_2]} f(t,x,\Gamma_0(t,x)+dk)$ if $d \geq \max(0,\bar{d})$, which combined with the hypothesis on $\lim_{\gamma\to\infty} f(t,x,\gamma)$ ensures that $\lim_{d\to\infty} k_d = \infty$. Hence, $(t_2 - t_1)\,k_d \leq \tilde{l}_d(t_2) - \tilde{l}_d(t_1) \leq m_2 - m_1$ for all $d \geq \bar{d}$, which is impossible. This contradiction shows that $d_+ \in (0,\infty)$. Analogously, $d_- = \sup\{d < \bar{d} \mid$ Case A does not hold$\} < \bar{d}$ is finite. Corollary 4.23 ensures that the variation is the stated one: C2 for all $d < d_-$, B2 at d_-, A for all $d \in (d_-, d_+)$, B1 at d_+, and C1 for all $d > d_+$.

Theorem 4.22 precludes the dynamics to be always in one of the Cases B, and shows the existence of values of the parameter in Case A if one of the Cases B occurs at some value of the parameter. So, we would be in the situation of the previous paragraph. Finally, the absence of Cases A and B is only possible if either Case C1 or Case C2 occurs for all the values of c, as Theorem 4.21(ii) ensures. The proof is complete. □

A meaningful interpretation of the previous statement can be achieved by understanding Γ_0 as a transition function whose range is contained within a safety interval (such as those provided in Theorem 4.31 and Corollary 4.32), while the contribution of $d\,\Gamma$ causes the transition function to deviate from the safety interval for sufficiently large values of d or $-d$: if we move too far away from the safety interval we will obtain a critical transition.

By reviewing the proof of the previous theorem, we observe that we have in fact proved the next result, which we will use in Section 4.4.

Theorem 4.34. *Let $\Gamma_0\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $\Gamma\colon \mathbb{R}\times\mathbb{R} \to [0,\infty)$ be globally bounded and C^2-admissible, and such that the pair $(f, \Gamma_0 + d\,\Gamma)$ satisfies* **f1**-**f5** *for all $d \in \mathbb{R}$. Assume that there exists $\bar{d} \in \mathbb{R}$ such that*

$$x' = f(t,x,\Gamma_0(t,x) + d\,\Gamma(t,x)) \tag{4.22}$$

is in Case A for $d = \bar{d}$, and let $\tilde{l}_{\bar{d}} < \tilde{m}_{\bar{d}} < \tilde{u}_{\bar{d}}$ be its three hyperbolic solutions. Let $m_1 < m_2$ and $m_3 < m_4$ be such that $m_1 \leq \tilde{l}_{\bar{d}}(t) < \tilde{m}_{\bar{d}}(t) \leq m_2$ for all $t \in \mathbb{R}$ and $m_3 \leq \tilde{m}_{\bar{d}}(t) < \tilde{u}_{\bar{d}}(t) \leq m_4$ for all $t \in \mathbb{R}$.

(i) *Assume that there exists t_0 such that $\Gamma(t_0,x) > 0$ for all $x \in [m_1,m_2]$, that $\gamma \mapsto f(t,x,\gamma)$ is nondecreasing for all $(t,x) \in \mathbb{R}\times\mathbb{R}$ and strictly increasing for $(t,x) \in \mathbb{R}\times[m_1,m_2]$, with $\lim_{\gamma\to\infty} f(t,x,\gamma) = \infty$ uniformly on compact sets of $\mathbb{R}\times[m_1,m_2]$. Then, there exists $d_+ > \bar{d}$ such that $(4.22)_d$ is in Case C1 for $d > d_+$, in Case B1 for $d = d_+$, in Case A for $d \in [\bar{d}, d_+)$.*

(ii) *Assume that there exists t_0 such that $\Gamma(t_0,x) > 0$ for all $x \in [m_3,m_4]$, that $\gamma \mapsto f(t,x,\gamma)$ is nondecreasing for all $(t,x) \in \mathbb{R}\times\mathbb{R}$ and strictly increasing for $(t,x) \in \mathbb{R}\times[m_3,m_4]$, with $\lim_{\gamma\to-\infty} f(t,x,\gamma) = -\infty$ uniformly on compact sets* *of* $\mathbb{R}\times[m_3,m_4]$. *Then, there exists* $d_-<\bar d$ *such that* $(4.22)_d$ *is in* Case C2 *for* $d<d_-$, *in* Case B2 *for* $d=d_-$, *in* Case A *for* $d\in(d_-,\bar d]$.

Remark 4.35. Frequently, the limit maps providing condition **f2** for all d are $\Gamma_{0,\pm}+d\,\Gamma_\pm$ for C^2-admissible maps $\Gamma_{0,\pm}$ and $\Gamma_\pm\ge 0$. If so, condition **f5** for all $x'=f(t,x,\Gamma_{0,\pm}(t,x)+d\,\Gamma_\pm(t,x))$ is only possible if, for any $t_0\in\mathbb{R}$, each map $x\mapsto\Gamma_\pm(t_0,x)$ vanishes at least for an $x^\pm_{t_0}\in[m_1,m_2]$: otherwise, Theorem 4.33 precludes the existence of three hyperbolic solutions if d and $-d$ are large enough. Also often, $\Gamma_{0,\pm}\equiv\Gamma_0$ and $\Gamma_-\equiv 0$, and so Theorems 4.33 and 4.34 study the occurrence of size-induced critical transitions: just define $f^*(t,x,d\,\Gamma(t,x))=f(t,x,\Gamma_0(t,x)+d\,\Gamma(t,x))$.

Uniqueness of a rate-induced critical transition

In what follows, with an eye on applications, we deal with a two-parametric transition function $d\,\Gamma^c$: d determines the size and c represents the rate in the examples. The information provided by Theorem 4.34 allows us to establish conditions ensuring the existence of a continuous map $d_+(c)$ determining the unique positive critical value of $x'=f(t,x,d\,\Gamma^c(t,x))$, and to describe a continuous bifurcation function $\varphi(c)$ which vanishes at at most a point and whose sign determines the dynamical situation of $x'=f(t,x,\Gamma^c(t,x))$.

Theorem 4.36. *Let* $\mathcal{C}\subseteq\mathbb{R}$ *be an open interval, and let the maps* $\{\Gamma^c\mid c\in\mathcal{C}\}$ *be a family of functions such that*

(1) *all the pairs* (f,Γ^c) *satisfy* **f1-f5** *with* $\Gamma_-\equiv\Gamma_+\equiv 0$,

(2) $\Gamma^c\ge 0$ *and* $\mathbb{R}\times\mathbb{R}\times\mathcal{C}\to[0,\infty)$, $(t,x,c)\mapsto\Gamma^c(t,x)$ *is admissible,*

(3) *for any* $c\in\mathcal{C}$, *there exists* $\delta_c>0$ *such that* $\sup_{(t,x)\in\mathbb{R}\times\mathbb{R},\,|\varepsilon|\le\delta_c}|\Gamma^{c+\varepsilon}(t,x)|<\infty$,

(4) $\gamma\mapsto f(t,x,\gamma)$ *is strictly increasing for all* $(t,x)\in\mathbb{R}\times\mathbb{R}$.

Let $\tilde l_0<\tilde m_0<\tilde u_0$ *be the three hyperbolic solutions of* $x'=f(t,x,0)$ *provided by* **f5**, *and let* $m_1<m_2$ *be such that* $m_1\le\tilde l_0(t)<\tilde m_0(t)\le m_2$ *for all* $t\in\mathbb{R}$. *Assume that* $\lim_{\gamma\to\infty}f(t,x,\gamma)=\infty$ *uniformly on compact sets of* $\mathbb{R}\times[m_1,m_2]$ *and that, for any* $c\in\mathcal{C}$, *there exists* $t_c\in\mathbb{R}$ *such that* $\Gamma^c(t_c,x)>0$ *for all* $x\in[m_1,m_2]$. *The following assertions hold:*

(i) *If, for each* $c\in\mathcal{C}$, *the value* $d_+(c)>0$ *is provided by Theorem* 4.34(i) *applied to* $x'=f(t,x,d\,\Gamma^c(t,x))$, *then* $\mathcal{C}\to(0,\infty)$, $c\mapsto d_+(c)$ *is continuous.*

(ii) *Besides, if* $c\mapsto\Gamma^c(t,x)$ *is nonincreasing (resp. nondecreasing) for all* $(t,x)\in\mathbb{R}\times\mathbb{R}$ *and, for any* $c_1<c_2$ *with* $c_1,c_2\in\mathcal{C}$ *and any compact set* $\mathcal{K}\subset\mathbb{R}$, *there exists* t_* *such that* $\Gamma^{c_1}(t_*,x)>\Gamma^{c_2}(t_*,x)$ *(resp.* $\Gamma^{c_1}(t_*,x)<\Gamma^{c_2}(t_*,x)$*) for all* $x\in\mathcal{K}$, *then* $\mathcal{C}\to(0,\infty)$, $c\mapsto d_+(c)$ *is strictly decreasing (resp. increasing).*

(iii) *There exists a continuous bifurcation function* $\varphi\colon\mathcal{C}\to\mathbb{R}$ *such that the equation*

$$x'=f(t,x,\Gamma^c(t,x)) \tag{4.23}$$

is in Case A *if* $\varphi(c)<0$, B1 *if* $\varphi(c)=0$, *and* C1 *if* $\varphi(c)>0$. *If the additional hypotheses of* (ii) *hold, then* φ *is strictly decreasing (resp. strictly increasing).*

Proof. For any fixed $d \in \mathbb{R}$, the admissibility of $\mathbb{R} \times \mathbb{R} \times \mathcal{C} \to [0, \infty)$, $(t, x, c) \mapsto d\,\Gamma^c(t, x)$ follows from (2), and condition (3) obviously ensures that, for any $c \in \mathcal{C}$, there exists $\delta_c > 0$ such that $\sup_{(t,x)\in\mathbb{R}\times\mathbb{R},\,|\varepsilon|\le\delta_c} |d\,\Gamma^{c+\varepsilon}(t, x)| < \infty$. Hence, it can be checked that all the pairs $(f, d\,\Gamma^c)$ satisfy **f1-f5** with $\Gamma_- \equiv \Gamma_+ \equiv 0$ for all $c \in \mathcal{C}$ and $d \in \mathbb{R}$. So, Theorem 4.34(i) can be applied: $\mathcal{C} \to (0, \infty)$, $c \mapsto d_+(c)$ is well defined.

(i) For contradiction, we assume that there exist $c_0 \in \mathcal{C}$, (c_n) with $\lim_{n\to\infty} c_n = c_0$ and $\delta > 0$ such that $d_+(c_n) > d_+(c_0) + \delta$ for all $n \in \mathbb{N}$. (An analogous argument proves the same if $d_+(c_n) < d_+(c_0) - \delta$ for all $n \in \mathbb{N}$.) Since $d_+(c_n) > d_+(c_0) + \delta > d_+(c_0)$ for all $n \in \mathbb{N}$, Theorem 4.34(i) ensures that $x' = f(t, x, (d_+(c_0) + \delta)\,\Gamma^{c_0}(t, x))$ is in Case C1 and that $x' = f(t, x, (d_+(c_0)+\delta)\,\Gamma^{c_n}(t, x))$ is in Case A for all $n \in \mathbb{N}$. On the other hand, Theorem 4.30 ensures that the hypotheses of Theorem 4.21(ii) hold, and this result yields $\inf\{c > c_0 \mid x' = f(t, x, (d_+(c_0) + \delta)\,\Gamma^c(t, x))$ is not in Case C1$\} > c_0$, which contradicts the previous assertion. Hence, $\mathcal{C} \to (0, \infty)$, $c \mapsto d_+(c)$ is continuous.

(ii) Assume that $c \mapsto \Gamma^c(t, x)$ is nondecreasing for all $(t, x) \in \mathbb{R} \times \mathbb{R}$. Then, the map $c \mapsto f(t, x, d\,\Gamma^c(t, x))$ is nondecreasing for any $d > 0$ and $(t, x) \in \mathbb{R} \times \mathbb{R}$ fixed and, for any $c_1 < c_2$ with $c_1, c_2 \in \mathcal{C}$ and any compact set $\mathcal{K} \subset \mathbb{R}$, there exists $t_* \in \mathbb{R}$ such that $f(t_*, x, d\,\Gamma^{c_1}(t_*, x)) < f(t_*, x, d\,\Gamma^{c_2}(t_*, x))$ for all $d > 0$ and $x \in \mathcal{K}$. Then, given $c_1 < c_2$ with $c_1, c_2 \in \mathcal{C}$, Corollary 4.23 ensures that $x' = f(t, x, d_+(c_1)\,\Gamma^{c_2}(t, x))$ is in Case C1. Consequently, $d_+(c_2) < d_+(c_1)$, as we wanted to see.

(iii) Note that $x' = f(t, x, d\,\Gamma^c(t, x))$ is in Case A if $d \in [0, d_+(c))$, in B1 if $d = d_+(c)$ and in C1 if $d \in (d_+(c), \infty)$. Hence, $(4.23)_c$ is in Case A if $1 \in [0, d_+(c))$, in B1 if $d_+(c) = 1$, and in C1 if $1 \in (d_+(c), \infty)$. Therefore, the map $\varphi\colon \mathcal{C} \to \mathbb{R}$ given by $\varphi(c) = 1 - d_+(c)$, whose continuity is ensured by (i), satisfies the statement. Property (ii) ensures the strict monotonicity of φ if the additional properties hold. □

Remarks 4.37. 1. Let us explain how to apply Theorem 4.36 in a situation which will appear in one of the examples described in Section 4.4. Let (f, Γ) satisfy **f1-f5** with $\Gamma_- \equiv \Gamma_+ \equiv 0$, and assume that $\Gamma(t, x) = \Gamma(t) \ge 0$ for all $(t, x) \in \mathbb{R}$ with $\Gamma(0) > 0$, that $t \mapsto \Gamma(t)$ is nondecreasing on $(-\infty, 0]$ and nonincreasing on $[0, \infty)$, that $\gamma \mapsto f(t, x, \gamma)$ is strictly increasing for all $(t, x) \in \mathbb{R} \times \mathbb{R}$, and that $\lim_{\gamma\to\infty} f(t, x, \gamma) = \infty$ uniformly on compact sets of $\mathbb{R} \times \mathbb{R}$. Let us define $\Gamma^c(t, x) = \Gamma(c\,t)$ for any $c > 0$ and $(t, x) \in \mathbb{R} \times \mathbb{R}$. Proposition 4.28(i) ensures that (2) and (3) holds, $\Gamma^c(0, x) > 0$ for all $x \in \mathbb{R}$, and Remark 4.29 ensures that (1) also holds. Notice also that $t \mapsto \Gamma(t)$ is nondecreasing on $(-\infty, 0]$ and nonincreasing on $[0, \infty)$ if and only if $(0, \infty) \to \mathbb{R}$, $c \mapsto \Gamma^c(t) = \Gamma(ct)$ is nonincreasing for all $t \in \mathbb{R}$, and that, in this case, $\Gamma(0) = \sup_{t\in\mathbb{R}} \Gamma(t) > 0 = \lim_{t\to\infty} \Gamma(t)$. We take $\gamma_0 \in (0, \Gamma(0))$ and $t_0 = \inf\{t > 0 \mid \Gamma(t) = \gamma_0\}$, so that $\Gamma(t_0) < \Gamma(s)$ for every $0 \le s < t_0$. Given $0 < c_1 < c_2$, we take $t_* = t_0/c_2$. Since $0 < c_1 t_* < c_2 t_* = t_0$, we get $\Gamma^{c_2}(t_*) < \Gamma^{c_1}(t_*)$. Theorem 4.36(iii) provides a strictly decreasing continuous bifurcation map $\varphi\colon (0, \infty) \to \mathbb{R}$ such that $(4.23)_c$ is in Case A if $\varphi(c) < 0$, B1 if $\varphi(c) = 0$, and C1 if $\varphi(c) > 0$.

2. A result on rate-induced critical transitions can be obtained using an analog of Theorem 4.36 involving d_- instead of d_+: let (f, Γ) satisfy **f1-f5** with $\Gamma_- \equiv \Gamma_+ \equiv 0$, and assume that $\Gamma(t, x) = \Gamma(t) \le 0$ for all $(t, x) \in \mathbb{R}$ with $\Gamma(0) > 0$, that $t \mapsto \Gamma(t)$ is nonincreasing on $(-\infty, 0]$ and nondecreasing on $[0, \infty)$, that $\gamma \mapsto f(t, x, \gamma)$ is strictly increasing for all $(t, x) \in \mathbb{R} \times \mathbb{R}$ and that $\lim_{\gamma\to-\infty} f(t, x, \gamma) = -\infty$ uniformly on compact sets of $\mathbb{R}\times\mathbb{R}$. Let us define $\Gamma^c(t, x) = \Gamma(ct)$ for any $c > 0$ and $(t, x) \in \mathbb{R}\times\mathbb{R}$.

In this case, $\mathcal{C} \to (-\infty, 0)$, $c \mapsto d_-(c)$ is strictly decreasing, and there also exists a strictly decreasing continuous bifurcation map $\varphi\colon (0,\infty) \to \mathbb{R}$ such that (4.23) is in Case A if $\varphi(c) < 0$, B2 if $\varphi(c) = 0$, and C2 if $\varphi(c) > 0$.

3. The two previous remarks are still true if we remove the assumption on $\lim_{\gamma\to\infty} f(t,x,\gamma)$. To check it in the situation of Remark 4.37.1, let $[0,b]$ be such that $\text{closure}_{\mathbb{R}}(\Gamma(\mathbb{R}\times\mathbb{R})) \subseteq [0,b]$ (recall hypothesis **f2**) and let us define $g(t,x,\gamma) = f(t,x,\gamma)$ if $\gamma \le b$ and $g(t,x,\gamma) = f(t,x,b) + \gamma - b$ if $\gamma > b$. Then $g(t,x,\Gamma^c(t)) = f(t,x,\Gamma^c(t))$, and it is easy to check that (g,Γ) satisfies all the hypotheses required in Theorem 4.36(iii), from where the assertion follows. An analogous definition is made in the case of Remark 4.37.2.

The situation of Remark 4.37.1 can be read as a possible situation of rate-induced tracking. If, for a small rate $c_1 > 0$, the dynamics of $x' = f(t,x,\Gamma(ct))$ correspond to tipping, then $\varphi(c_1) > 0$. If the strictly decreasing map φ reaches 0 at a certain rate c_2, this means tracking for all rate $c > c_2$. Notice, however, that the existence of a root of the bifurcation function is not guaranteed, only its uniqueness.

All this can be understood in terms of the safety interval $\mathcal{I} \ni 0$ determined by the values of γ such that $x' = f(t,x,\gamma)$ has three hyperbolic solutions if, in addition, $x \mapsto f_x(t,x,\gamma)$ is strictly concave for all $\gamma \in \mathcal{I}$. If $\Gamma(0) \in \mathcal{I}$, then Corollary 4.32 ensures tracking for all c. So, let us assume that $\Gamma(0) > \sup \mathcal{I}$. Then, the rate c is inversely proportional to the length of the period of time during which the range of $t \mapsto \Gamma(ct)$ (with $0 \in \mathcal{I}$ as both asymptotic limits) escapes from $\mathcal{I}$. So, a small value of c may mean that the range of Γ is outside $\mathcal{I}$ during a period large enough to produce tipping, but a larger value of c may revert this situation. In other words, we have either tracking forever, or tipping forever, or a unique phenomenon of rate-induced tracking. Figure 4.5 shows a drawing representing the possibility of a unique rate bifurcation point, i.e., the case of rate-induced tracking.

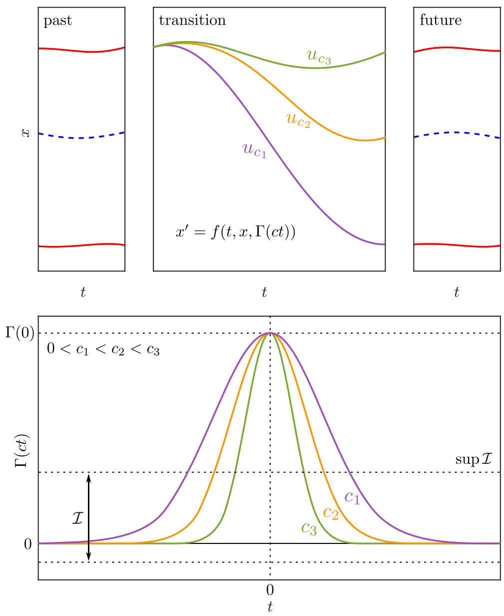


Figure 4.5: Sketch of the possible occurrence of rate-induced tracking when the transition function Γ escapes from a safety interval $\mathcal{I}$. In the top middle panel, the upper locally pullback attractive solutions for three different values $0 < c_1 < c_2 < c_3$ of c, in the bottom panel, the transition functions for these values of c.

4.3.4 Revisiting the necessity of hypotheses in Theorems 1.40 and 2.20 through examples

In this section, we present two examples demonstrating the optimality of the assumptions employed in two theorems of Chapters 1 and 2. Specifically, we address the condition imposed on the lower and upper equilibria of an invariant compact set in Theorem 1.40, as well as the minimal base flow condition assumed in Theorem 2.20. These results are included at this point because the examples are constructed precisely from transition equations such as those introduced in this chapter.

An example related to Theorem 1.40

We present an example of a uniformly exponentially stable (with strictly negative upper Lyapunov exponent), pinched, compact (and connected), and τ-invariant set $\mathcal{K}$ which projects onto the whole base and which is not a τ-copy of the base. So, the thesis of Theorem 1.40 is false, due to existence of a minimal subset of the base on which the sections of $\mathcal{K}$ do not reduce to a point. This is the unique hypothesis of Theorem 1.40 not fulfilled.

To this end, we consider $f(t,x,\gamma) = -x\,(x^2-1)+\gamma$ and $\Gamma(t) = \exp(-t^2)$. Note that $\Gamma_\pm = \lim_{t\to\pm\infty}\Gamma(t) = 0$, and that -1, 0 and 1 are hyperbolic solutions of the (past and future) equation $x' = -x\,(x^2-1)$, which is also the equation $x' = f(t,x,d\,\Gamma(t))$ for $d=0$. So, **fd1-fd5** hold and Theorem 4.33 ensures that there exists $d_- < 0$ such that, for all $d<d_-$, the transition equation $x' = f(t,x,d\,\Gamma(t))$ is in Case C2. Therefore, the upper locally pullback attractive solution u_d for $d<d_-$ satisfies $\lim_{t\to-\infty} u_d(t) = 1$ and $\lim_{t\to\infty} u_d(t) = -1$.

From now on, we fix $d<d_-$ and consider the hull Ω_g of $g(t,x) = f(t,x,d\,\Gamma(t))$ equipped with the usual translation flow $\sigma_g\colon \mathbb{R}\times\Omega_g\to\Omega_g$. Since the map $g_-(x) = g_+(x) = -x(x^2-1)$ does not depend on t, it is a fixed point of σ_g, and the unique element of the α-limit set and the ω-limit set of g. That is, (Ω_g,σ_g) is homeomorphic to a continuous flow on $\mathbb{S}^1$ with only one fixed point. In the words of the decomposition of Lemma 1.44, $\Omega_g = \Omega_g^\alpha\cup\{g{\cdot}t \mid t\in\mathbb{R}\}\cup\Omega_g^\omega$, where $\Omega_g^\alpha = \Omega_g^\omega = \{g_+\}$ is disjoint from the orbit $\{g{\cdot}t \mid t\in\mathbb{R}\}$.

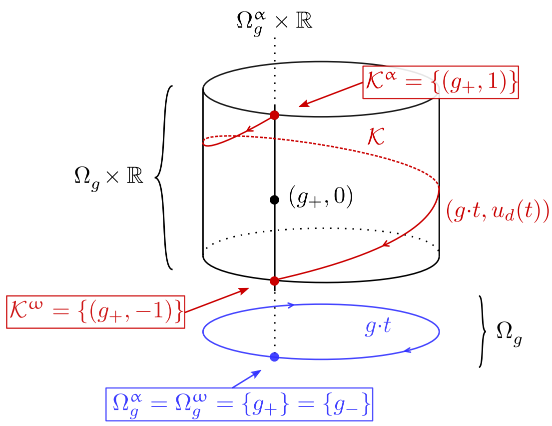


Figure 4.6: Sketch of the structure of the example related to Theorem 1.40 in the skewproduct formalism.

Let us consider the compact (and connected) τ-invariant set

$$\mathcal{K} = \mathrm{closure}_{\Omega\times\mathbb{R}}\{(g{\cdot}t,u_d(t)) \mid t\in\mathbb{R}\}\,.$$

Since $x' = g(t,x)$ is in Case C2, Theorem 4.16 ensures that $t\mapsto u_d(t)$ is an attractive hyperbolic solution. So, Proposition 1.54(ii) ensures that $\sup\mathrm{Lyap}(\mathcal{K})<0$ and that $\mathcal{K}$ is uniformly exponentially stable. An analogous argument to that of Lemma 1.44 shows that $\mathcal{K} = \mathcal{K}^\alpha\cup\{(g{\cdot}t,u_d(t)) \mid t\in\mathbb{R}\}\cup\mathcal{K}^\omega$, where $\mathcal{K}^\alpha$ and $\mathcal{K}^\omega$ are the α-limit and ω-limit sets for τ of $(g,u_d(0))$, and in this case $\mathcal{K}^\alpha\cup\mathcal{K}^\omega$ and $\{(g{\cdot}t,u_d(t)) \mid t\in\mathbb{R}\}$ are disjoint, since they project onto disjoint parts of Ω_g: onto $\Omega_g^\alpha = \Omega_g^\omega = \{g_+\}$ and onto $\{g{\cdot}t \mid t\in\mathbb{R}\}$, respectively. Since $\Omega_g^\alpha = \Omega_g^\omega = \{g_+\}$ is minimal and $\sup\mathrm{Lyap}(\mathcal{K}^\alpha)\le\sup\mathrm{Lyap}(\mathcal{K})<0$, Theorem 1.40 ensures that $\mathcal{K}^\alpha$ is an attractive hyperbolic copy of $\Omega_g^\alpha = \Omega_g^\omega = \{g_+\}$, and analogously $\mathcal{K}^\omega$. Moreover, the asymptotic approaching ensures that $\mathcal{K}^\alpha = \{(g_+,1)\}$ and $\mathcal{K}^\omega = \{(g_+,-1)\}$. Thus,

$\mathcal{K}_\omega$ is a singleton for all $\omega \in \{g{\cdot}t \mid t \in \mathbb{R}\}$ (and hence it is pinched) and has two elements for all $\Omega_g^\alpha = \Omega_g^\omega = \{g_+\}$ (and hence it is not a copy of the base), as we wanted to see. See Figure 4.6 for a depiction of the skewproduct structure of the example.

An example related to Theorem 2.20

We present an example which shows that the minimality of (Ω, σ) is indeed required in Theorem 2.20. We construct a non minimal set Ω and a pair of functions $\mathfrak{h}_1, \mathfrak{h}_2 \colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying **d1**, **d2**, **d3** and **d4** with $\mathfrak{h}_1(\omega, x) > \mathfrak{h}_2(\omega, x)$ for all $(\omega, x) \in \Omega \times \mathbb{R}$, such that $x' = \mathfrak{h}_i(\omega{\cdot}t, x)$ has three hyperbolic copies of the base $\mathfrak{l}_i < \mathfrak{m}_i < \mathfrak{u}_i$ for $i = 1, 2$ which satisfy none of the two possible orders described in Theorem 2.20. We make use of the transition framework of the present chapter to construct the example: $x' = \mathfrak{h}_i(\omega{\cdot}t, x)$ for $i = 1, 2$ will be transition equations, with Ω composed by a heteroclinic orbit connecting its α-limit set to its ω-limit set, which are singletons and hence minimal subsets of Ω. The cornerstone of the example is the fact that we construct three hyperbolic copies of the base Ω for each one of the equations, whose projections over the α-limit set and ω-limit set are ordered following order (1) and order (2) of Theorem 2.20, respectively. This fact and the continuity of the copies of the base precludes one of the two orders to hold over the whole set Ω.

Let $\Gamma \colon \mathbb{R} \to (0, 1)$ be a continuous map with $\lim_{t\to\infty} \Gamma(t) = \Gamma_+ = 1$ and $\lim_{t\to-\infty} \Gamma(t) = \Gamma_- = 0$ (as $\Gamma(t) = \arctan(t)/\pi + 1/2$). We take $a \geq \sqrt{10}$ and

$$h_b(x, \alpha) = -x^3 + x + \alpha\,(3x^2 a - 3xa^2 + a^3 - a) + \alpha\,(1 - \alpha)\,b$$

for some $b \geq 0$ which will be properly fixed later. Note that: $h_b(x, \alpha) = h_0(x, \alpha) + \alpha(1 - \alpha)\,b$; $h_b(x, 0) = -x(x - 1)(x + 1)$; $h_b(x, 1) = -(x - a)(x - a - 1)(x - a + 1)$; and $3x^2 a - 3xa^2 + a^3 - a > 0$ for all $x \in \mathbb{R}$ by the choice of a, so $\alpha \mapsto h_0(x, \alpha)$ is strictly increasing for all $x \in \mathbb{R}$. For each $b \geq 0$, we consider the equation

$$x' = h_b(x, \Gamma(t))\,. \tag{4.24}$$

It is easy to check that (h_b, Γ) satisfies **f1-f5** for any $b \geq 0$: the past equation $x' = h_b(x, 0)$ has three hyperbolic critical points -1, 0 and 1, and the future equation $x' = h_b(x, 1)$, which is a shift of the past one, has three hyperbolic critical points $a - 1$, a and $a + 1$. So, the dynamics of $(4.24)_b$ fits in one of the dynamical cases of Theorem 4.16.

We will check later the existence of $b_0 > 0$ such that $(4.24)_b$ is in Case A for $b = b_0$. Let Ω be the hull of $(t, x) \mapsto h_{b_0}(x, \Gamma(t))$ (see Section 1.3.1), and let $\mathfrak{h}_1 \colon \Omega \times \mathbb{R} \to \mathbb{R}$ be given by $\mathfrak{h}_1(\omega, x) = \omega(0, x)$ for $(\omega, x) \in \Omega \times \mathbb{R}$, that is, the extension of h_{b_0} to Ω. Then, $\mathfrak{h}_1(\omega, x)$ is a cubic polynomial with -1 as leading coefficient for all $\omega \in \Omega$, and hence $\mathfrak{h}_1$ satisfies **d1**, **d2**, **d3** and **d4**. Note that Ω is the union of the (heteroclinic) σ-orbit $\{h_{b_0}(x, \Gamma(t + s)) \mid s \in \mathbb{R}\}$ and its α-limit and ω-limit sets, $\{h_{b_0}(x, 0)\}$ and $\{h_{b_0}(x, 1)\}$: see Lemma 1.44. Theorem 2.18 ensures that $x' = \mathfrak{h}_1(\omega{\cdot}t, x)$ has three hyperbolic copies of the base $\mathfrak{l}_1 < \mathfrak{m}_1 < \mathfrak{u}_1$. In particular, the restrictions of these three copies to the α-limit set $\{h_{b_0}(x, 0)\}$ are -1, 0 and 1, and to the ω-limit set $\{h_{b_0}(x, 1)\}$ are $a - 1$, a and $a + 1$.

Next, we define $\mathfrak{h}_2(\omega,x) = -x^3 + x - \varepsilon$ for $\varepsilon \in (0, 2/(3\sqrt{3}))$, which clearly satisfies **d1**, **d2**, **d3** and **d4** and $\mathfrak{h}_1(\omega,x) > \mathfrak{h}_2(\omega,x)$ for all $(\omega,x) \in \Omega\times\mathbb{R}$. It can be checked that $x' = \mathfrak{h}_2(\omega{\cdot}t, x)$ has three copies of the base: three constant equilibria $\mathfrak{l}_2, \mathfrak{m}_2, \mathfrak{u}_2$ satisfying $\mathfrak{l}_2 < -1 < 0 < \mathfrak{m}_2 < \mathfrak{u}_2 < 1$. So, the order of $\mathfrak{l}_1$, $\mathfrak{m}_1$, $\mathfrak{u}_1$ and $\mathfrak{l}_2$, $\mathfrak{m}_2$, $\mathfrak{u}_2$ is $\mathfrak{l}_2 < -1 < 0 < \mathfrak{m}_2 < \mathfrak{u}_2 < 1$ (like in Theorem 2.20(1)) over the minimal set $\{h_{b_0}(x,0)\} \subset \Omega$, and $\mathfrak{l}_2 < \mathfrak{m}_2 < \mathfrak{u}_2 < a-1 < a < a+1$ (like in Theorem 2.20(2)) over the minimal set $\{h_{b_0}(x,1)\} \subset \Omega$, as asserted.

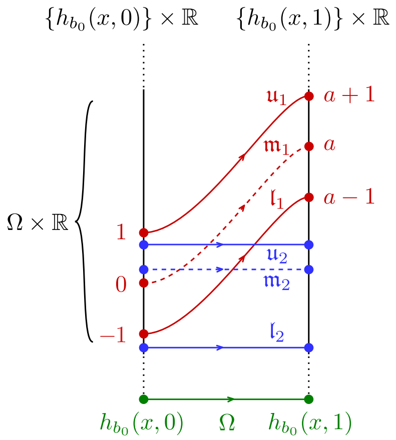


Figure 4.7: Sketch of the structure of the example related to Theorem 2.20 in the skewproduct formalism. In red lines, the copies of the base for $x' = \mathfrak{h}_1(\omega{\cdot}t, x)$, in blue for $x' = \mathfrak{h}_2(\omega{\cdot}t, x)$: in solid lines the attractive ones and in dashed lines the repulsive ones.

It remains to check the existence of $b_0 > 0$ such that $(4.24)_{b_0}$ is in Case A, for which it suffices to check $(4.24)_0$ is in Case C2 and that there exists $b_1 > 0$ such that $(4.24)_{b_1}$ is in Case C1: Theorems 4.21 and 4.22 preclude moving from Case C2 to Case C1 as b varies without crossing A.

We denote by l_b and u_b (resp. m_b) the locally pullback attractive (resp. repulsive) solutions of $(4.24)_b$ provided by Theorem 4.13(i) (resp. (iii)), and recall that $\lim_{t\to-\infty} u_b(t) = 1$, $\lim_{t\to-\infty} l_b(t) = -1$, and $\lim_{t\to\infty} m_b(t) = a$. Since $\Gamma(t) < 1$ for all $t\in\mathbb{R}$ and $\alpha \mapsto h_0(x,\alpha)$ is strictly increasing for all $x\in\mathbb{R}$, we have $h_0(a-1,\Gamma(t)) < h_0(a-1,1) = 0$ for all $t\in\mathbb{R}$, so $\mathbb{R}\times(-\infty, a-1]$ is positively invariant for $(4.24)_0$. Since $\lim_{t\to-\infty} u_0(t) = 1 < a-1$, we have $u_0(t) \in (-\infty, a-1]$ for all $t\in\mathbb{R}$, and hence $\lim_{t\to\infty} u_0(t) = a-1$: the other possible future limits a and $a+1$ are uniformly separated from u_0. That is, $(4.24)_0$ is in Case C2. To look for b_1, we first check that all the bounded solutions of $(4.24)_b$ take values in $[-1,\infty)$ for $b > 0$, since $h_b(x,0) < h_b(x,\Gamma(t))$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}$, and hence any $m_1 < -1$ satisfies the initial hypothesis of Theorem 2.13. Next, we take $t_0 > 0$ in the domain of definition of m_0 with $m_0(t) < a + 1/2$ for all $t \ge t_0$ and assume for contradiction that $l_b(t) \le a + 1/2$ for all $b > 0$ and $t \in [t_0, t_0+1]$. Let γ be a lower bound for $\Gamma(t)(1-\Gamma(t))$ for $t\in[t_0,t_0+1]$. Then $l_b(t_0+1) \ge -2 + \int_{t_0}^{t_0+1}(-(a+1/2)^3 + \gamma\, b)\, ds$ for all $b > 0$, which is impossible. We take b_1 and t_1 with $l_{b_1}(t_1) > a + 1/2 > m_0(t_1)$. Theorem 4.13(iv) ensures that $\lim_{t\to\infty}(x_0(t,t_1,l_{b_1}(t_1)) - (a+1)) = 0$, and a comparison argument yields $l_{b_1}(t) = x_{b_1}(t,t_1,l_{b_1}(t_1)) \ge x_0(t,t_1,l_{b_1}(t_1))$ for $t \ge t_1$. That is, $\liminf_{t\to\infty}(l_{b_1}(t) - (a+1)) \ge 0$, which may only happen in Case C1 (see Theorem 4.16). This completes the proof.

4.4 Numerical simulations in d-concave and asymptotically d-concave models

In this section, we consider four different single species population models whose internal dynamics are driven by nonautonomous cubic equations and which include predation and migration phenomena. The intrinsic cubic dynamics is due to the Allee effect (see Section 4.1), e.g., due to some breeding cooperation mechanism or to an easier mate finding. In both cases, the evolution of the population is modeled by

$$x' = r(t)\, x \left(1 - \frac{x}{K(t)}\right) \frac{x - S(t)}{K(t)} + \Delta(t,x)\,, \tag{4.25}$$

where we assume r, K and S to be quasiperiodic functions with r and K positively bounded from below, and Δ to be C^2-admissible: so, if we define $h(t,x,\delta) = r(t)\, x\,(1 - x/K(t))(x - S(t))/K(t) + \delta$, then h satisfies **f1** and **f3**. In addition, we will assume that (h,Δ) satisfies **f2**, **f4** and **f5** for some maps $\Delta_\pm$. The meaning of r, K and S has been explained in Section 4.1, and Δ models the contribution of two external effects: predation and migration.

The first two examples in this section use transition functions (using the approach of Section 4.3) that do not depend on the state variable x and are asymptotically constant, the third example incorporates asymptotic functions that depend on time, and the fourth example includes explicit dependence of the transition function on the state variable x. Proposition 4.26 ensures that the dynamical possibilities are Cases A, B or C of Theorem 4.16. Throughout this section, we find Cases A, B and C for different values of certain parameters in different models, and we point to certain parametric variations as possible causes of tipping.

The hypotheses on the past and future equations ensure that for each of them there are two possible steady populations, given by the two attractive hyperbolic (positive) solutions. The desirable target of the transition can be either of the two: either the upper one, which would represent a large healthy population, or the lower one, which would represent a sparse population near extinction or of low density. If the upper population is the target one, then the lower population will mean extinction of the species under study or closeness to extinction [81], while in the second case, the upper population will mean habitat invasion [69]. Let us focus on the case in which the desirable target population is the upper one, i.e., we study a population that is under some risk of extinction. We will assume that in the past the population is also the desirable one, represented by the hyperbolic solution $\tilde{u}_{h_-}$. This solution is approached as time decreases by the upper bounded (and locally pullback attractive) solution u_h. So, the desirable dynamics for the transition equation is the stable Case A (tracking) that means that u_h approaches the upper bounded solution $\tilde{u}_{h_+}$ (the desirable large healthy population) of the future equation as time increases; and the catastrophic situation corresponds to the stable Case C2 (tipping), with u_h approaching the extinction state $\tilde{l}_{h_+}$ as time increases. A critical transition from A to C2 (which means crossing the highly unstable Case B2 due to a small change in the predation and/or migration term $\Delta(t,x)$) means a disaster, while a critical transition from C2 to A means the recovery of the desirable situation. If, on the contrary, our population was close to extinction in the past and hence represented by the lower hyperbolic solution $\tilde{l}_{h_-}$, the tipping Case C1 represents the recovery of a healthy state $\tilde{u}_{h_+}$ while the tracking of Case A represents continuing in an endangered state $\tilde{l}_{h_+}$, close to extinction or even extinct. Then, a critical transition from A to C1 (through B1) is desirable, and the converse one is something to avoid. The perspective on the possible advantages and disadvantages of a critical transition is exactly the opposite if we deal with the situation in which the lower steady state is the target one. The study of critical transitions of extinction of a species will appear in all four examples, while critical transitions of invasion will appear only in the first of them.

Example 4.38. Different types of critical transitions due to immigration in the same model. In this example, we consider (4.25) and assume that the term Δ only incorporates migration phenomena. Moreover, as starting point, we assume that $\Delta(t,x) = \gamma\,\phi(t)$, where γ is a nonnegative parameter and ϕ is a quasiperiodic function positively bounded from below, which represents the arrival of new individuals to the habitat (immigration):

$$x' = r(t)\,x\left(1-\frac{x}{K(t)}\right)\frac{x-S(t)}{K(t)} + \gamma\,\phi(t)\,, \tag{4.26}$$

which we rewrite as $x' = f(t,x,\gamma)$. Departures of individuals (emigration), harvesting and hunting could be represented by adding an analogous negative parametric term. For the sake of simplicity, we will not deal with this case.

We consider rate-induced families of x-independent asymptotically constant transition functions $\Gamma\colon \mathbb{R}\to\mathbb{R}$ (that is, $\Gamma(t,x) = \Gamma(t)$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}$ when considering the framework of Section 4.3) which model the variation of the parameter γ through the transition. In all the cases, Γ will have the same constant asymptotic limit $\gamma_+ = \lim_{t\to\pm\infty}\Gamma(t)$ in the past and in the future, which plays the role of Γ_- and Γ_+. So, the transition equation to study, which includes the time varying immigration term $\Delta^c(t,x) = \Gamma(ct)\,\phi(t)$, is

$$x' = r(t)\,x\left(1-\frac{x}{K(t)}\right)\frac{x-S(t)}{K(t)} + \Gamma(ct)\,\phi(t) \tag{4.27}$$

for positive rates $c>0$. Our goal is to show that different choices of Γ result in two of the different types of critical transitions described above: extinction and invasion. We will work under the hypothesis that equation $(4.26)_{\gamma_+}$ shows strong Allee effect (recall Section 4.1.1), that is, the past and future equations, which are the same and given by $(4.26)_{\gamma_+}$, display strong Allee effect, i.e., $(4.26)_{\gamma_+}$ has three nonnegative (biologically meaningful) hyperbolic solutions. Then, it is clear that hypotheses **f1-f5** are satisfied for all $c>0$. It will be shown that equation (4.27) for large values of the rate $c>0$ shows a dynamical behavior very similar to that of equation $(4.26)_{\gamma_+}$, but that for small values of $c>0$ very different dynamical behaviors may appear: the critical transitions representing extinction and invasion.

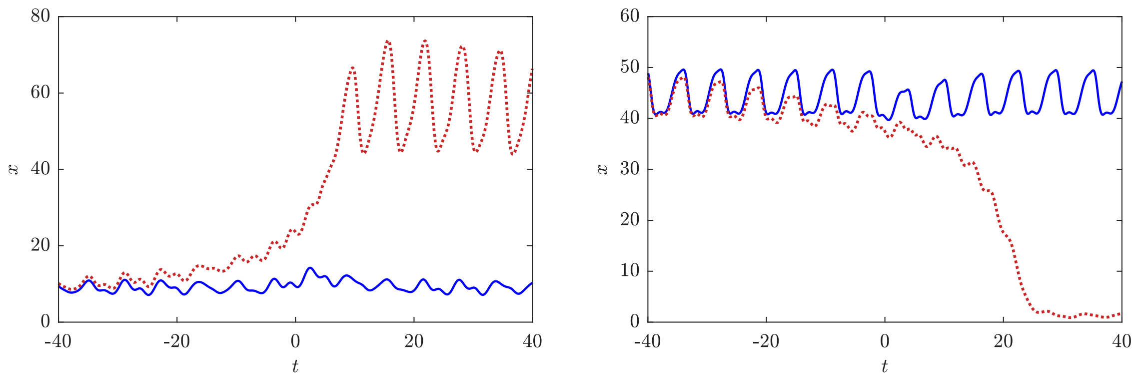
In bird populations (see [96]), there exist several causes of migrant population change: hunting, diseases, adverse winds, storms, orientation errors, changing attractiveness of the breeding colony (availability of nesting sites)... Some of these factors are not persistent in time, especially in populations with continental distribution, which makes it appropriate to model them with a transition function with equal asymptotic limits γ_+. A somehow related real-world example, with foxes as predators, can be found in the work [90], which describes the colonization of Punta de la Banya by the Audouin's gull: an increasing population began to severely decline from a certain time due to the arrival of foxes, whom later were removed; and then the gull population began to increase again.

To model some of these phenomena in a pretty simple way, we choose a Gaussian transition function $\Gamma(t) = \gamma_+ + (\gamma_* - \gamma_+)\exp(-t^2/10)$ with asymptotic limit γ_+ both at $-\infty$ and $+\infty$: it represents an impulse from its asymptotic limits at $\pm\infty$ to a value $\gamma_* \neq \gamma_+$. The fact that $|\Gamma(t)-\gamma_+|$ exponentially decreases as $|t|$ increases implies that the time window in which most of the transition takes place is relatively small. Once the rate $c > 0$ is introduced, the size of the this window decreases as c increases.

The key element in understanding the types of critical transitions that can occur in (4.27) is the bifurcation diagram that underlies (4.26). The proof of Theorem 3.8 and of the results of Chapter 3 leading to it can be repeated with little modification for the bifurcation problem $x' = \mathfrak{h}(\omega{\cdot}t, x) + \lambda\,\mathfrak{c}(\omega{\cdot}t)$, where $\mathfrak{c}\colon \Omega \to \mathbb{R}$ is an strictly positive continuous function. To apply this bifurcation result to (4.26), we consider the hull of the quadruple (r, K, S, ϕ), that is, the closure of $\{(r{\cdot}t, K{\cdot}t, S{\cdot}t, \phi{\cdot}t) \mid t \in \mathbb{R}\}$ in the compact-open topology of $C(\mathbb{R}, \mathbb{R}^4)$. It can be checked that this hull is minimal (see [40, Theorem 2.9] and [76, Theorems 2.43 and 2.44]), and Proposition 2.18 ensures that there exist three τ_{γ_+}-copies of the base. Hence, the extension of Theorem 3.8 ensures that there exist $\gamma_1 < \gamma_+ < \gamma_2$ such that, for all $\gamma \in (\gamma_1, \gamma_2)$, there exist three τ_γ-copies of the base, and two saddle-node bifurcations of minimal sets take place at γ_1 and γ_2 (recall Figure 3.1). Proposition 1.55 provides a reading of these facts in terms of hyperbolic solutions of (4.26): $(4.26)_\gamma$ has three hyperbolic solutions for $\gamma \in (\gamma_1, \gamma_2)$, and exactly one (attractive) hyperbolic solution if $\gamma < \gamma_1$ or $\gamma > \gamma_2$. Moreover, both the lower and upper bounded solutions of $(4.26)_\gamma$ strictly increase as γ increases. And they do so continuously, except for the upper bounded solution at γ_1 and the lower bounded solution at γ_2. Hence, if we choose the extreme value γ_* of Γ also in (γ_1, γ_2), then Corollary 4.32 precludes the existence of tipping for any rate $c \in (0, \infty)$. That is, (γ_1, γ_2) is a safety interval for the range of Γ. To allow critical transitions to take place, we will always choose $\gamma_* \notin (\gamma_1, \gamma_2)$. This implies that the transition function Γ takes values outside (γ_1, γ_2) during a period of time which is determined by the rate c. What we will observe is that, if this period is short (i.e., if c is large), then the dynamics of the transition equation is basically equal to that of the future (and past) equation $x' = f(t, x, \gamma_+)$. But if the period is long enough (given by a sufficiently small c), then the dynamics changes dramatically, in two possible different ways. That is, there is at least one (and, as we will see, just one) positive critical value of the rate and it is of rate-induced tracking type.

To perform the numerical simulations, we fix the functions $r(t) \equiv 1$, $K(t) = 60{+}30\sin(t)$, $S(t) = 38.5{+}10\sin(2t{-}3\pi/2)$ and $\phi(t) = 0.8{+}0.4\sin^2(t\sqrt{5}/2)$ defining the right-hand side of (4.26). With these choices, a simple numerical simulation shows that 0 is the unique bounded solution for $\gamma = 0$. This uniqueness, which is not possible in the autonomous formulation of (4.26), is fundamental in what follows, since it ensures that all the bounded solutions of $(4.26)_\gamma$ for all $\gamma > 0$ are nonnegative, i.e. biologically meaningful. In addition, the robustness of the existence of hyperbolic solutions under small perturbations (see Theorem 1.52) makes it easy to obtain numerical evidences of: $[1.0, 4.0] \subset (\gamma_1, \gamma_2)$, $0.5 \notin (\gamma_1, \gamma_2)$, and $5.0 \notin (\gamma_1, \gamma_2)$. Altogether, we can identify (γ_1, γ_2) with the set of values of γ such that $(4.26)_\gamma$ exhibits strong Allee effect, i.e. has three nonnegative hyperbolic solutions.

Figure 4.8a corresponds to $\gamma_+ = 4.0$ and $\gamma_* = 5.0$. The dotted red line represents the lower locally pullback attractive solution l_c of $(4.27)_c$ (see Theorem 4.13(i)) for $c = 0.1$, and the solid blue one for $c = 1.0$. In both cases (as for any $c > 0$), this solution may correspond to a desirable low density population which is under control for large negative values of t, when $\Gamma(ct)$ is practically equal to γ_+. The value of $\Gamma(ct)$ is also practically equal to γ_+ if t is large enough, but it smoothly increases towards γ_* as $|t|$ approaches 0. This evolution occurs during a period of time which



(a) For $4.0 = \gamma_+ < \gamma_* = 5.0$, a small transition rate $c = 0.1$ leads to habitat invasion (in dotted red line), that is, CASE C1; while a controlled small population persists for $c = 1.0$ (in solid blue), CASE A.

(b) For $1.0 = \gamma_+ > \gamma_* = 0.5$, a small transition rate $c = 0.1$ causes the extinction of the species (in dotted red line), CASE C2; while a healthy large population persists for $c = 1.0$ (in solid blue), CASE A.

Figure 4.8: Numerical simulations of rate-induced critical transitions produced by migration in $(4.27)_c$, with $r(t) \equiv 1$, $K(t) = 60 + 30\sin(t)$, $S(t) = 38.5 + 10\sin(2t - 3\pi/2)$, $\phi(t) = 0.8 + 0.4\sin^2(t\sqrt{5}/2)$ and $\Gamma(t) = \gamma_+ + (\gamma_* - \gamma_+)\exp(-t^2/10)$. Each panel corresponds to a different choice of γ_+ and γ_*. In the left (resp. right) pannel, the lower (resp. upper) locally pullback attractive solution l_c (resp. u_c) given by Theorem 4.13(i) for $(4.27)_c$ is plotted for two different values of the rate $c > 0$: one leading to CASE A (tracking) and the other leading to CASE C (tipping).

increases as the rate c decreases. For the value $c = 1.0$ (as for any large enough c), the previous low population remains under control for always; but, for $c = 0.1$ (as for any small enough c), the smallest steady population undergoes an overgrowth which leads to the invasion of the habitat. So, there exists at least a critical rate (and, as we will see, just one): a threshold which must be exceeded in order to avoid an invasion. A rate above this threshold means increased immigration (more arrivals) for a period of time which is short enough to allow the population to keep its controlled size (CASE A). But if the time is longer, invasion occurs (CASE C1).

Figure 4.8b corresponds to $\gamma_+ = 1.0$ and $\gamma_* = 0.5$. Now, the immigration smoothly decreases from $\gamma_+ = 1.0$ to $\gamma_* = 0.5$ when $|t|$ approaches 0. Here, the dotted red line represents the upper locally pullback attractive solution u_c of $(4.27)_c$ (see Theorem 4.13(i)) for $c = 0.1$, and the solid blue one for $c = 1.0$. Let us understand this upper solution as a healthy population. As before, the behaviour does not depend on c if $-t$ is large enough. As we observe in the figure, this population persists if the rate is large enough, but the population gets basically extinct if c is very small. So, again, there exists a critical rate (and later we will see that exactly one): a threshold which must be exceeded to avoid the critical extinction. A rate above this threshold means a decreased immigration (less arrivals) for a period of time short enough to avoid extinction (CASE A), and a rate below this threshold means extinction (CASE C2).

It is remarkable that these two parametric problems present an opposite behavior to other ones more usual in the literature (see e.g. [12], [73]), in which tracking takes place at low transition rates and tipping appears for high transition rates. That is, *rate-induced tracking* takes place in these two scenarios (see Section 4.3).

The difference between the type of critical transitions in the two analyzed examples can be easily explained by Remarks 4.37 and the comments below them. In the case of $\gamma_+ < \gamma_*$, we are dealing with the situation described in Remark 4.37.1, while in the case of $\gamma_+ > \gamma_*$ it is Remark 4.37.2 which must be considered. In both cases, the dynamical situation is determined by the sign of a strictly decreasing continuous bifurcation function $\varphi\colon (0,\infty) \to \mathbb{R}$, which can have at most a zero (and hence the critical rate is unique if it exists), but the dynamical cases determined by this bifurcation function are different in both cases. If this zero c_0 exists and $\gamma_+ < \gamma_*$ (resp. $\gamma_+ > \gamma_*$), then $(4.27)_c$ is in Case C1 (resp. C2) if $0 < c < c_0$, B1 (resp. B2) if $c = c_0$), and A if $c > c_0$. Herein lies the difference between critical transitions of extinction and of invasion. We point out once again that the occurrence of the (unique) critical transition is due to the fact that $\Gamma(ct) \notin (\gamma_1, \gamma_2)$ for an interval of time which tends to infinity as c tends to cero. The radical difference appearing in the two cases described in Figures 4.8a and 4.8b depends on the relation $\inf \Gamma(ct) < \gamma_1$ (extinction) or $\sup \Gamma(ct) > \gamma_2$ (invasion).

Example 4.39. Persistence or extinction depending on the type of Allee effect involved. In this example, we compare numerical simulations of (4.25) for two different choices of the coefficients. In one of them, the equation exhibits strong Allee effect while in the other it exhibits weak Allee effect. This difference in the type of Allee effect will cause a difference in the type of reaction of the population to predation: critical extinction in the case of strong Allee effect versus continuous decline in the case of weak Allee effect. We assume that the term Δ includes only predation, and that it is modeled by a Holling type III functional response term (see (4.6)). We fix $b > 0$ and consider as starting point the γ-parametric auxiliary model

$$x' = r(t)\,x\left(1 - \frac{x}{K(t)}\right)\frac{x - S(t)}{K(t)} - \gamma\,\frac{x^2}{b + x^2}\,, \tag{4.28}$$

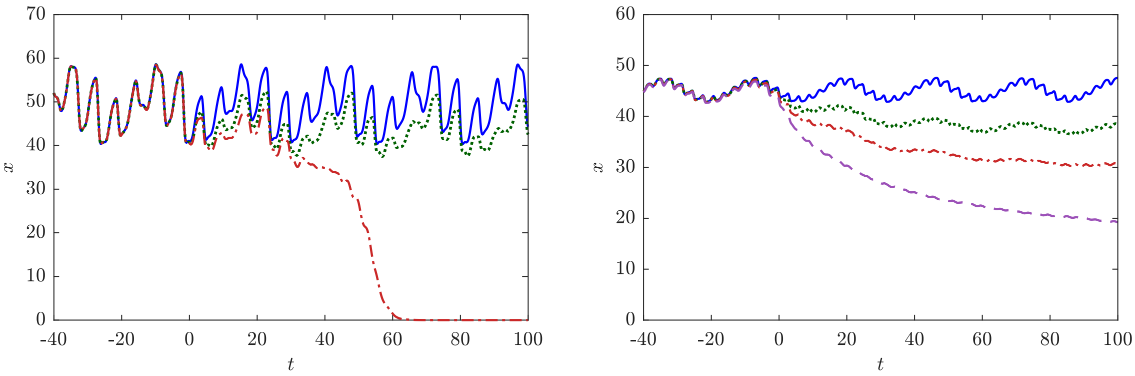
which we rewrite as $x' = f(t, x, \gamma)$. In this case, we take a transition function $\Gamma^d(t) = d\,\Gamma(t)$, with $\Gamma(t) = 1/2 + \arctan(t)/\pi$, depending on the parameter $d \in \mathbb{R}$. That is, we study size-induced critical transitions. Since $0 = \lim_{t\to-\infty} \Gamma^d(t)$ for all $d \in \mathbb{R}$ and $d = \lim_{t\to\infty} \Gamma^d(t)$, the biological meaning of this transition is that the habitat of the species under study is initially free of predation but a group of predators whose size is proportional to d arrives during the transition. We will see that this may (not necessarily) give rise to a size-induced critical transition in the dynamics of the transition equation

$$x' = r(t)\,x\left(1 - \frac{x}{K(t)}\right)\frac{x - S(t)}{K(t)} - d\,\Gamma(t)\,\frac{x^2}{b + x^2} \tag{4.29}$$

as the parameter d increases (i.e., as predation increases). We will always take

$$d \in \mathcal{I}_b = \left[0,\ b^{3/2}\frac{64}{5\sqrt{(5 - 2\sqrt{5})\,(7 + 3\sqrt{5})}}\ \inf_{t\in\mathbb{R}} \frac{r(t)}{K(t)^2}\right) \tag{4.30}$$

to ensure that hypothesis **f4** holds (see (4.9) and Section 4.1). It is plain then that hypotheses **f1**-**f4** are satisfied. We choose r, K, S and b in such a way that the past equation $x' = f(t, x, 0)$ has three hyperbolic solutions. That is, the part of **f5** concerning $\Gamma^d_- \equiv 0$ is satisfied for all $d \in \mathbb{R}$. Hence, the existence of three hyperbolic solutions of $x' = f(t, x, d)$ for $d > 0$ close enough to 0 (see Theorem 1.52) ensures that the part of **f5** concerning $\Gamma^d_+ \equiv d$ is also satisfied for $d > 0$ close enough to 0. Since



(a) A critical extinction (in dashed-dotted red line) occurs as the predation increases for a population with multiplicative strong Allee effect in the absence of predation: $r(t) \equiv 1$, $K(t) = 50 + 20\sin(t) + 20\cos^2(t\sqrt{5}/2)$, $S(t) = 26.7 + 5\sin(2t - 3\pi/2) + 10\cos^2(t\sqrt{5}/2)$, and $b = 800$. The blue solid line depicts u_d for $d = 0$, the green dotted line for $d = 1.1$, and the red dashed-dotted line for $d = 1.5$.

(b) Persistence for several values of d of a single species population which exhibits multiplicative weak Allee effect in the absence of predation. The chosen functions are $r(t) = 0.01 + 0.1\cos^2(t\sqrt{5}/2)$, $K(t) = 30 + 60\sin^2(t)$, $S(t) \equiv -0.01$, and $b = 5 \cdot 10^5$. The blue solid line depicts u_d for $d = 0$, the green dotted line for $d = 100$, the red dashed-dotted line for $d = 200$, and the violet dashed line for $d = 400$.

Figure 4.9: Numerical simulations of extinction or persistence depending on the type of Allee effect in the absence of predation. In each panel, the upper locally pullback attractive solution u_d of $(4.29)_d$ is depicted for different values of d, with $\Gamma(t) = 1/2 + \arctan(t)/\pi$, and certain choices of the parameter functions r, K and S and the parameter b. A critical extinction takes place in the left panel while a smooth decrease occurs in the right panel.

$x' = f(t, x, \Gamma^0(t))$ coincides with $x' = f(t, x, 0)$, $(4.29)_0$ is in Case A. Consequently, the robustness of Case A given by Theorem 4.21(i) (whose hypotheses are fulfilled, as Proposition 4.28 shows) ensures that $(4.29)_d$ is in Case A for $d > 0$ small enough. Note that Theorem 4.13(i) ensures that the upper bounded solution u_d of $(4.29)_d$ is locally pullback attractive and connects with the upper solution of $(4.28)_0$ as time decreases for all d. Note that $f_x(t, 0, \diamondsuit) = -r(t)S(t)/K(t)$ for $\diamondsuit = \gamma$ or $\diamondsuit = \Gamma^d$. Since the maps r, S and K are quasiperiodic, the solution 0 is always hyperbolic attractive (resp. repulsive) if the mean value of rS/K is positive (resp. negative): this mean value is the unique Lyapuov exponent of the minimal set $\Omega \times \{0\}$ in the corresponding hull extensions of $x = f(t, x, \diamondsuit)$, which are made as indicated in Example 4.38.

The two panels of Figure 4.9 show this solution u_d for different values of d. In Figure 4.9a, we take $r(t) \equiv 1$, $K(t) = 50 + 20\sin(t) + 20\cos^2(t\sqrt{5}/2)$, $S(t) = 26.7 + 5\sin(2t - 3\pi/2) + 10\cos^2(t\sqrt{5}/2)$, and $b = 800$. Since $rS/K > 0$, we have that 0 is an attractive hyperbolic solution of $(4.28)_\gamma$ for all γ and also of $(4.29)_d$ for all $d \in \mathbb{R}$. In this case, a numerical simulation shows that $x' = f(t, x, 0)$ has three nonnegative hyperbolic solutions, i.e. the past equation exhibits strong Allee effect. In addition, with these choices of the functions and the parameter, $\sup \mathcal{I}_b > 3.5$ (see (4.30)). The blue solid line represents u_d for $d = 0$, the green dotted line for $d = 1.1$, and the red dashed-dotted line for $d = 1.5$. We observe that, as already known, a small predation ensures the persistence of the target population, while a greater predation causes extinction. In Figure 4.9b, we take $r(t) = 0.01 + 0.1\cos^2(t\sqrt{5}/2)$, $K(t) = 30 + 60\sin^2(t)$, $S(t) \equiv -0.01$, and $b = 5 \cdot 10^5$. In this case $rK/S < 0$, so 0 is a repulsive hyperbolic solution of $(4.28)_\gamma$ for all γ and also of $(4.29)_d$ for all $d \in \mathbb{R}$. Proposition 4.1(i) shows the occurrence of weak Allee effect, which can also be observed numerically. In addition, $\sup \mathcal{I}_b > 560$ (see (4.30)). The blue solid line depicts u_d for $d = 0$, the green dotted line for $d = 100$, the red dashed-dotted line for $d = 200$, and the violet dashed line for $d = 400$. Here, we simply observe a smooth decrease of the population as predation increases.

So, in Figure 4.9a, departing from a population exhibiting strong Allee effect in the absence of predation, we find at least a tipping value d_0 of the parameter in the predation term between 1.1 and 1.5 (which is in fact the unique one in $\mathcal{I}_b$, as we will explain below). In the case of Figure 4.9b, departing from a population with weak Allee effect, the numerical simulation shows that Case A holds as d increases, with u_d decreasing with respect to d. (In this case, we numerically check that $x' = f(t, x, d) = f(t, x, \Gamma_+^d)$ has three hyperbolic solutions for the chosen values of d; so, **f1-f5** are satisfied, and we can properly talk about Case A.)

The reason of this difference can be found in the underlying nonautonomous bifurcation diagram of (4.28). The predation perturbation which has been introduced does not change the hyperbolic character of 0 as the parameter changes; so, as will be explained below, similarity arises with the mirror image (right-left) of the positive halfplane of the bifurcation diagrams of Theorems 3.43 and 3.44. Proposition 1.55 translates those minimal sets into hyperbolic solutions of $(4.28)_\gamma$, and a result analogous to Proposition 3.41(ii) ensures that the upper bounded solution of $(4.28)_\gamma$ strictly decreases as γ increases if it different from 0.

Let us consider the case of Figure 4.9a, with strong Allee effect. We already know that 0 is hyperbolic attractive for $(4.28)_\gamma$ for all $\gamma \in \mathbb{R}$, that there are two more hyperbolic solutions for $\gamma = 0$, which persist as γ increases in a maximal interval $[0, \gamma_0)$, and that the upper hyperbolic solution decreases as γ increases on $[0, \gamma_0)$. Using arguments analogous to those in the proof of Theorem 3.8, we check that the middle one increases, and we check numerically that $\gamma_0 < 2 \in \mathcal{I}_b$: 0 is the unique bounded solution of $x' = f(t, x, 2)$. Altogether, this ensures that $\gamma_0 \in (0, 2) \subset \mathcal{I}_b$ is a bifurcation value for (4.28). That is, the two upper hyperbolic solutions approach each other as the parameter increases until they collapse at γ_0, and there are no strictly positive bounded solutions for $\gamma > \gamma_0 \in \mathcal{I}_b$: we have a saddle-node bifurcation point at γ_0. Let us check that this value of γ_0 coincides with the unique tipping value $d_0 > 0$ for (4.29) in $\mathcal{I}_b$. If $0 \le d < \gamma_0$, then $x' = f(t, x, \gamma)$ has three hyperbolic solutions for all $\gamma \in [0, d]$. Therefore, the pair (f, Γ^d) satisfies all the hypotheses of Corollary 4.32, and this result ensures that the dynamics of $(4.29)_d$ is in Case A. Now, let us take $d \in (\gamma_0, \sup \mathcal{I}_b)$: 0 is the unique nonnegative hyperbolic solution of $(4.28)_d$ (i.e. $(4.28)_\gamma$ with $\gamma = d$), and it exponentially attracts any solution as time increases. Let us check that this ensures that 0 is the asymptotic limit at ∞ of any positive solution $(4.29)_d$, which completes the proof of our assertion concerning the uniqueness of d_0 in $\mathcal{I}_b$. Since 0 is a uniformly exponentially asymptotically stable solution of $(4.29)_d$, a contradiction argument ensures the existence of a positive bounded solution uniformly separated from 0. This means that the corresponding skewproduct flow on the hull has a positive orbit uniformly separated from 0. The ω-limit set of this orbit contains a strictly positive minimal set, and according to Lemma 4.6 this minimal set is also minimal for the future equation $(4.28)_d$, which is impossible.

Notice that, since $f_x(t, 0, \gamma) = -r(t)S(t)/K(t)$ independently of the value of the parameter, we have that, in contrast to the situation described in [99], there can exist a critical transition of the d-parametric family (4.29) without a modification of the indicator of the strength of strong Allee effect (4.10).

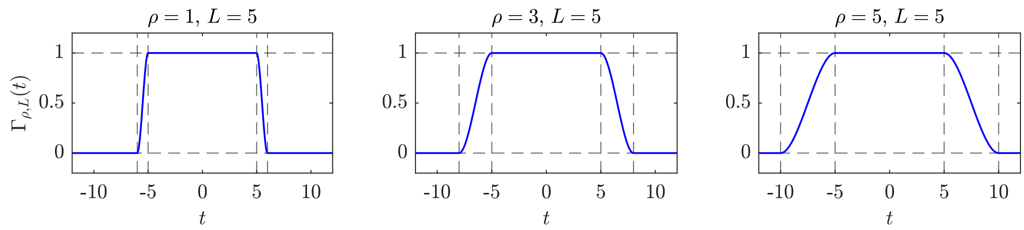


Figure 4.10: The C^1 map $\Gamma_{\rho,L}$ for $L = 5$ and several values of $\rho > 0$. This map is defined as the unique C^1 cubic spline which takes value 1 on $[-L, L]$ and 0 outside $[-L-\rho, L+\rho]$: if $Q(y) = 2y^3 - 3y^2 + 1$, then $\Gamma_{\rho,L}(t) = Q(-(t+L)/\rho)$ for $t \in [-L-\rho, -L]$ and $\Gamma_{\rho,L}(t) = Q((t-L)/\rho)$ for $t \in [L, L+\rho]$. This map is increasing on $[-L-\rho, -L]$ and decreasing on $[L, L+\rho]$, and hence $\Gamma_{\rho,L}(\cdot)$ is nondecreasing with respect to L and with respect to ρ.

In this example of strong Allee effect, we deal with a map Γ with asymptotic limits 0 and 1 which is increasing, and with $d \in \mathcal{I}_b$. This causes the range Γ^d to be contained in $\mathcal{I}_d$, and hence the righ-hand side of (4.29) is d-concave for all t. It turns out that the most significant conclusions remain valid by removing the condition of monotonicity of Γ, and hence, possibly, the previous condition on d-concavity. By assuming that the asymptotic limits of Γ are 0 and 1, we can repeat the previous arguments, excepting one: we cannot apply Corollary 4.32 to check that Case A persists for all $d \in [0, \gamma_0)$. What we know is that Case A persists for small positive values of d, and we can check as before that all bounded solutions converge to 0 for $d \in (\gamma_0, \sup \mathcal{I}_b)$. Corollary 4.23 ensures the uniqueness of the bifurcation point in $[0, \gamma_0)$ if it exists. For example, if we take $\Gamma(t) = \arctan(t)/\pi + 1/2 + 5\exp(-t^2/10)$ and the rest of the functions as in Figure 4.9a, then we numerically find the size-induced tipping point of (4.29) near $d = 0.58294$, while the point where the underlying bifurcation diagram of (4.28) loses the three hyperbolic solutions is $\gamma_0 \approx 1.22740$ (the same as in Figure 4.9a).

In the case of weak Allee effect of Figure 4.9b, 0 is a repulsive hyperbolic solution of $(4.29)_d$ for all $d \in \mathbb{R}$ and therefore $\Omega \times \{0\}$ is a repulsive hyperbolic copy of the base in the skewproduct formalism. Therefore, there exists a strictly positive attractive hyperbolic solution while d remains in $\mathcal{I}_b$ (see Proposition 2.14(i) and recall again Proposition 1.55). Consequently, the upper minimal set continuously decreases when the parameter increases (see Theorem 3.43). The continuous variation of the set of bounded solutions precludes the possibility of a critical extinction (while d is in $\mathcal{I}_b$).

Example 4.40. Seasonal predation: introducing time-dependent asymptotic limits of transition functions. We begin by assuming again that, in (4.25), the migration-predation term Δ only includes predation, which we assume to be suitably modeled by a Holling type III functional response term $-\gamma\, x^2/(b + x^2)$ (see again (4.6)). We have as starting point the γ-parametric auxiliary problem

$$x' = r(t)\, x \left(1 - \frac{x}{K(t)}\right) \frac{x - S(t)}{K(t)} - \gamma\, \frac{x^2}{b + x^2}\,. \tag{4.31}$$

Next, we assume that the population is attacked by a predator species which behaves as follows: the habitat is initially free of predators; at a certain time a group of predators arrives at the ecosystem, which they leave after some time; and this behavior repeats yearly. Such a pattern may correspond to the colonization of a new patch by a migratory species of predators, due to the reproductive, nutritional,

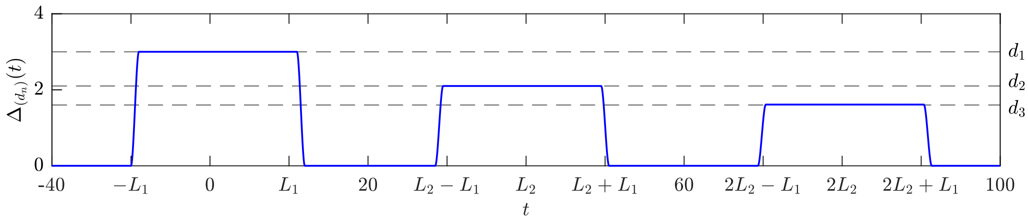


Figure 4.11: The transition map $\Delta_{(d_n)}$ defined in (4.32) for $\rho = 1$, $L_1 = 10$, $L_2 = 40$, $d_+ = 0.5$, $d = 2.5$, $d_n = d_+ + d/((n-1)/4+1)^2$, and $p_n = (n-1)L_2 + (-1)^{(n-1)}/n$ for all $n \in \mathbb{N}$.

breeding or wintering interest of the habitat. (See, e.g., [1] for a study on the evolution of some migration patterns of common swift, an insectivorous bird.)

Let L_2 be the length of the year. We assume that the n-th predation season occurs during the interval of time $[p_n - L_1 - \rho, p_n + L_1 + \rho]$, with $L_2 > 2(L_1 + \rho)$, and that the maximum number of predators, $d_n \geq 0$, acts during $[p_n - L_1, p_n + L_1]$: $\rho > 0$ is the (short) time needed for predators to reach and leave the patch. To avoid superposition of the predation seasons, we assume that $p_{n+1} - p_n > 2(L_1 + \rho)$ for all $n \in \mathbb{N}$. We assume that the sequence (d_n) of the predator group sizes is bounded and has limit d_+, so that the size of the n-th group tends to d_+. We also assume that $\lim_{n\to\infty}(p_n - (n-1)\,L_2) = 0$, which ensures that $\lim_{n\to\infty}(p_{n+1} - p_n) = L_2$: the yearly predation season is eventually almost identical to $[nL_2 - L_1 - \rho, nL_2 + L_1 + \rho]$, and the possible differences between p_n and $(n-1)L_2$ capture variations in the initial date of these seasons. (See [45] for a study on the variation of arrival dates of common swift and barn swallow to the Iberian Peninsula.) The previous hypotheses describe an asymptotically periodic phenomenon, which means that the behavior of the predators becomes as regular as possible over time. Other more complicated types of recurrence in the future equation may also fit in the model. The phenomenon of lack of predators in some occasional years can be described through null elements in the sequence (d_n).

To model the effect of these phenomena in a simple way, we use a C^1 approximation to the characteristic function of $[-L, L]$: the map $\Gamma_{\rho,L}$ is the unique C^1 cubic spline which takes the value 1 on $[-L, L]$ and 0 outside $[-L-\rho, L+\rho]$. Figure 4.10 depicts $\Gamma_{\rho,L}$ for $L = 5$ and some values of ρ, and its caption explains some of its properties. The amount of predators at the ecosystem at time t will be described by a transition function build by disjoint superposition of such maps:

$$\Delta_{(d_n)}(t) = \sum_{n=1}^{\infty} d_n\,\Gamma_{\rho,L_1}(t - p_n)\,, \tag{4.32}$$

which is a bounded continuous function due to the boundedness of (d_n) and to the disjointness of the intervals of predation. Figure 4.11 depicts $\Delta_{(d_n)}$ for $\rho = 1$, $L_1 = 10$, $L_2 = 40$ and certain sequences (d_n) and (p_n). Note that $\Delta_{(d_n)}(t) = 0$ for all $t < p_1 - L_1 - \rho$.

So, we study the transition equation

$$x' = r(t)\,x\left(1 - \frac{x}{K(t)}\right)\frac{x - S(t)}{K(t)} - \Delta_{(d_n)}(t)\,\frac{x^2}{b + x^2}\,, \tag{4.33}$$

which represents the dynamics of the single species population through the repeated

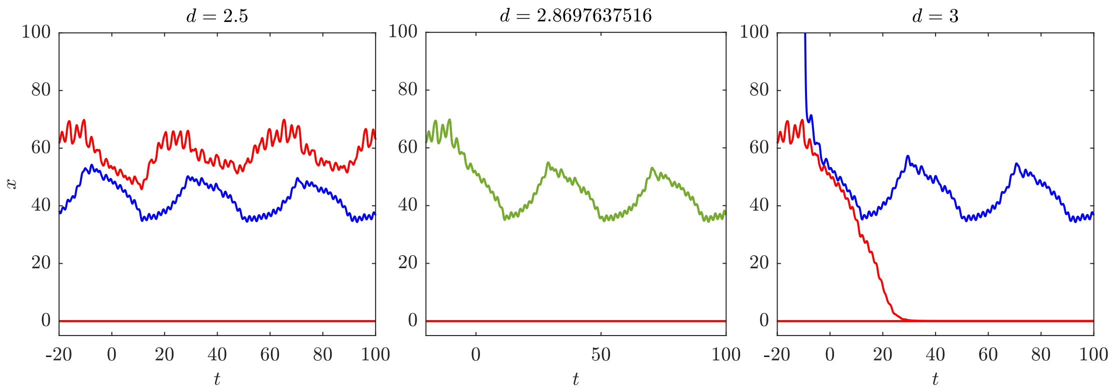


Figure 4.12: Numerical depiction of the existence of a unique size-tipping point for $(4.33)_d$. The central panel shows the dynamics for an accurate approximation to the tipping point d_0: the two upper hyperbolic solutions are so close within the representation window that are a good approximation (green) to the nonhyperbolic solution of Case B2. The left panel depicts Case A, which is the dynamics for any $d \in [0, d_0)$ and means survival: the attractive hyperbolic solutions are drawn in red, and the repulsive one in blue. The right panel depicts Case C2, which is the dynamics for any $d > d_0$ and means extinction: the attractive hyperbolic solutions are drawn in red, and the locally pullback repulsive solution in blue.

passage (which tends to be periodic) of groups of predators. We define $\Delta_- = 0$ and

$$\Delta_+(t) = \sum_{n=-\infty}^{\infty} d_+ \Gamma_{\rho, L_1}\big(t - (n-1)L_2\big)\,,$$

which is bounded, continuous, and L_2-periodic in time. Then, $\lim_{t\to-\infty}(\Delta_{(d_n)}(t) - \Delta_-(t)) = 0$, since $\Delta_{(d_n)}(t) = 0$ for all $t \le p_1 - L_1 - \rho$, and $\lim_{t\to\infty}(\Delta_{(d_n)}(t) - \Delta_+(t)) = 0$, since the uniform continuity of Γ_{ρ,L_1} on compact sets ensures that $\lim_{n\to\infty} \|\Gamma_{\rho,L_1}(t - p_n) - \Gamma_{\rho,L_1}(t-(n-1)L_2)\|_\infty = 0$, and the separation of the supports of the terms of the series guaranteed by the conditions $L_2 > 2(L_1 + \rho)$ and $p_{n+1} - p_n > 2(L_1 + \rho)$ ensures that we can compare the series term-by-term. That is, (4.32) corresponds to a transition between these two limit functions Δ_- and Δ_+, and **f2** is fulfilled. It can be checked that the right-hand side of equation (4.33) is not d-concave if $\max_{t\in\mathbb{R}} \Delta_{(d_n)}(t) = \max_{n\in\mathbb{N}} d_n$ is large enough, while $r(t)\, x\,(1 - x/K(t))(x - S(t))/K(t) - \Delta_+(t)\, x^2/(b + x^2)$ is d-concave if d_+ is not too large, in which case also **f4** is fulfilled: recall (4.9).

Let us choose: $r(t) = 0.7 + 0.3\sin^2(t)$, $K(t) = 70 + 20\cos(\sqrt{5}\,t)$ and $S(t) = 20 + 30\cos^2(\sqrt{3}\,t)$ for the internal dynamics of the species, $b = 200$ for the influence of the predation, and $L_1 = 10$, $L_2 = 40$, $d_+ = 0.3$, $d_n = d_+ + d/((n-1)/20+1)^2$ and $p_n = (n-1)L_2 + (-1)^{(n-1)}/n$ (for all $n \in \mathbb{N}$) for the shape of the transition function. The particular expression of d_n implies that the annual number of predators d_n decreases to d_+. The decreasing attractiveness of the habitat can be indebted to different causes: learning of defensive mechanisms, overpopulation of predators in the previous season, insufficient nesting or breeding space, etc. The constant d of the definition of d_n is a size bifurcation parameter in terms of which we will study the dynamical cases of (4.33). The choice of d_+ (below 0.32) guarantees **f4**. We numerically check **f5**, and hence **f1**-**f5** hold for all $d \ge 0$. In addition, the size of d_n for small n provides a not d-concave equation (4.33) if d is large enough (above 0.96).

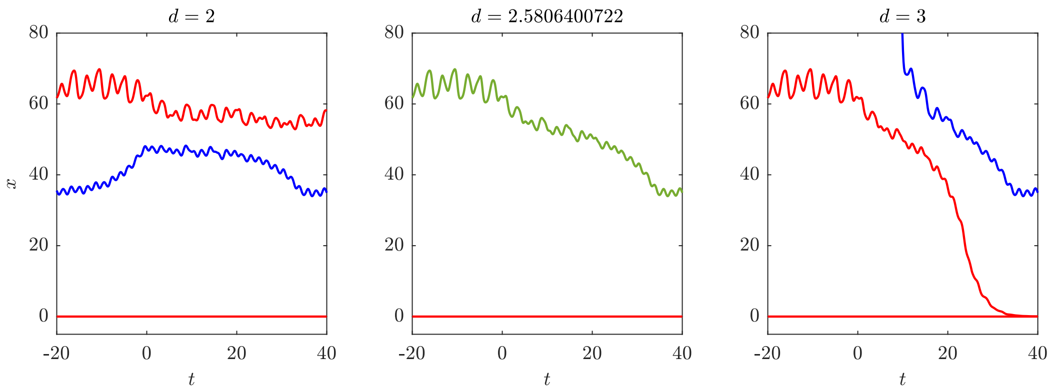


Figure 4.13: Numerical depiction of the existence of a unique size-tipping point for $(4.35)_d$ when L and c are fixed. In this example, $L = 20$ and $c = 0.02$. The central panel corresponds to the approximation for $d(20, 0.02)$ of Table 4.1: Case B2. To its left and right, we find Cases A and C2. See Figure 4.12 to understand the color code.

We can rewrite $\Delta_{(d_n)} = \tilde{\Delta}_+ + d\,\Delta$ for

$$\tilde{\Delta}_+(t) = \sum_{n=1}^{\infty} d_+ \Gamma_{\rho,L_1}\big(t - p_n\big)\,, \quad \Delta(t) = \sum_{n=1}^{\infty} \left(\frac{1}{((n-1)/20+1)^2}\right) \Gamma_{\rho,L_1}(t - p_n)\,,$$

so that Δ is a continuous nonnegative map whose limits as $t \to \pm\infty$ are 0. We define

$$f(t, x, \gamma) = r(t)\, x \left(1 - \frac{x}{K(t)}\right) \frac{x - S(t)}{K(t)} - \tilde{\Delta}_+(t)\, \frac{x^2}{b + x^2} - \gamma\, \frac{x^2}{b + x^2}$$

and $g(t, x, \gamma) = f(t, x, -\gamma)$, and check that the pairs $(g, d\,\Delta)$ satisfy the hypotheses of Theorem 4.34(ii) (with $\Gamma_0 \equiv 0$, $\Gamma \equiv \Delta$, and $\bar{d} = 0$). To this end, we numerically check that $x' = g(t, x, 0)$ has three hyperbolic copies of the base and that the lower one, attractive, is 0 (and hence $\tilde{m}_0$ is positively bounded from below). Hence, Theorem 4.33 ensures the existence of a unique size-induced tipping point $d_0 > 0$ for $x' = f(t, x, d\,\Delta(t))$ (i.e., for (4.33)): Case A holds for $0 \le d < d_0$, and Case C2 holds for $d > d_0$. That is, an excessive increase in the number of predators visiting the habitat leads to the extinction of the species. The existence of this critical transition is depicted in Figure 4.12.

In this example, it would also make sense to study the phase-variation of the global dynamics, i.e. to replace the transition function $\Delta_{(d_n)}(t)$ by $\Delta_{(d_n)}(t + c)$ for $c \in \mathbb{R}$. This parametric variation implies advancing or delaying the arrival of predators every year and would allow the occurrence of phase-induced critical transitions, in a similar way as critical transitions in the L and c parameters will appear in the following example.

Example 4.41. Modeling human actions on a herd: state-dependent transition function. Now, we consider that a flock of x animals described by (4.25) grazes in a patch which is initially free of predators. We assume that at time $t = 0$ a group of predators, which we suppose that have constant density d (due to the time scale in which we work) and whose predation mechanism is assumed to be suitably modeled by a Holling type III functional response term $-d\,x^2/(b(t) + x^2)$ reaches the patch (see again (4.6)). The function b is assumed to be quasiperiodic

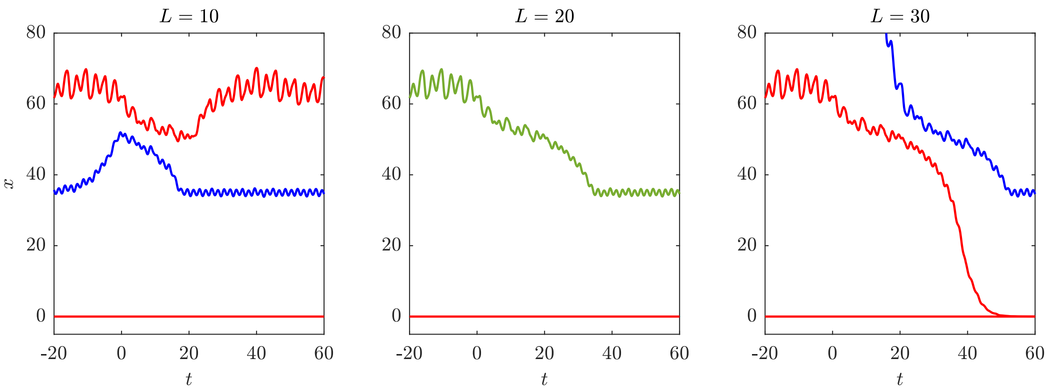


Figure 4.14: Numerical depiction of an L-induced tipping point: for $c = 0.02$ and $d = 2.5806400722$, we find Case A for $L = 10$, Case B2 for $L = 20$ and Case C2 for $L = 30$.

and positively bounded from below. At time $L > 0$, the threat is identified by the flock owner and s shepherds per unit of time are hired to protect the flock: there are $s\,(t-L)$ shepherds at time $t \geq L$, and each shepherd is assumed to be able to protect h heads of livestock. As soon as there are enough shepherds to protect the whole herd, i.e, when $x \leq h\,s\,(t-L)$, predators are not able to attack the flock. That is, predation occurs while $0 \leq t \leq L\,(cx+1)$, where $c = 1/hsL$. So, for $x \geq 0$, we can model the evolution of the flock by the equation

$$x' = r(t)\,x\left(1 - \frac{x}{K(t)}\right)\frac{x - S(t)}{K(t)} - d\,\Gamma_L\left(\frac{2\,t}{cx+1} - L\right)\frac{x^2}{b(t)+x^2}\,, \tag{4.34}$$

where we take $\Gamma_L = \Gamma_{\rho,L}$ for some small fixed $\rho > 0$, with $\Gamma_{\rho,L}$ defined in Example 4.40 (see Figure 4.10). Thus, the predation term practically vanishes when t is outside the interval $[0, L\,(cx+1)]$. By multiplying the Holling type III functional response term by Γ_L, it is implicitly assumed that the search for prey mechanism, i.e. the Holling type III interaction, is not affected by the presence of shepherds as long as there are not enough of them to protect the whole herd. This assumption, made for the sake of simplicity, can be understood as follows: if a shepherd has more sheep in his care than he can protect, then a predator, once it has located its prey, can wait a negligible amount of time on the timescale we are working with until the shepherd moves on to other sheep, far enough away to allow the predator to hunt the prey.

Since $\mathbb{R} \times [0,\infty)$ is an invariant set for the process given by (4.34) and only nonnegative solutions have biological meaning, we can replace the predation term by a globally defined one. To this end, we take a a globally defined C^2-map $k(x)$ which coincides with $1/(cx+1)$ on $[0,\infty)$, and consider the equation

$$x' = r(t)\,x\left(1 - \frac{x}{K(t)}\right)\frac{x - S(t)}{K(t)} - d\,\Gamma_L(2\,t\,k(x) - L)\,\frac{x^2}{b(t)+x^2}\,, \tag{4.35}$$

Let $\Lambda_{L,c}(t,x) = \Gamma_L(2\,t\,k(x) - L)$. Then, for any choices of $d \geq 0$, $L > 0$ and $c > 0$, $d\,\Lambda_{L,c}$ is globally bounded, C^2-admissible on $\mathbb{R}\times\mathbb{R}$, and with $\lim_{t\to\pm\infty} d\,\Lambda_{L,c}(t,x) = 0$ uniformly on each compact set $\mathcal{J} \subset \mathbb{R}$. That is, $d\,\Lambda_{L,c}$ globally satisfies **f2**, with $\Lambda_\pm = 0$. In addition, if f is the right-hand term of (4.31), then it is not difficult to check that **f1**, **f3** and **f4** hold.

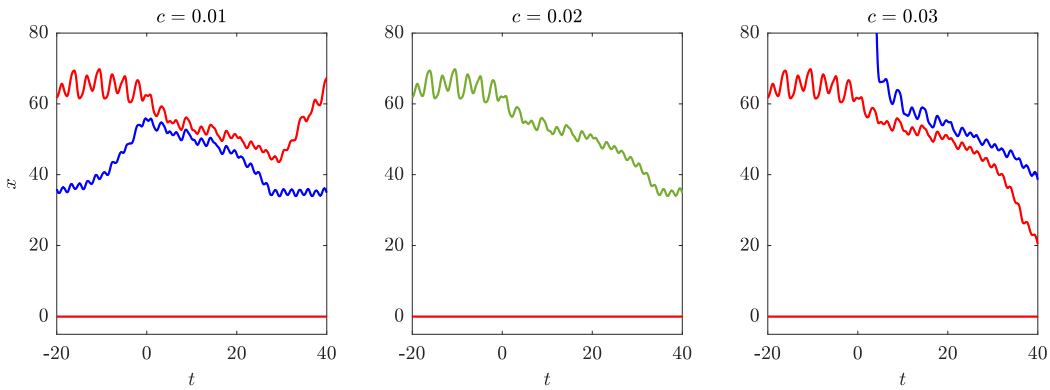


Figure 4.15: Numerical depiction of a c-induced tipping point: for $L = 20$ and $d = 2.5806400722$, we find Case A for $c = 0.01$, Case B2 for $c = 0.02$ and Case C2 for $c = 0.03$.

We choose $r(t) = 0.7 + 0.3\sin^2(t)$, $K(t) = 70 + 20\cos(\sqrt{5}\,t)$, $S(t) = 20 + 30\cos^2(\sqrt{3}\,t)$, and $b(t) = 20 + \cos(t)$ to construct Table 4.1 and Figures 4.13, 4.14 and 4.15, and numerically check that **f5** holds for these choices, being 0 the lower bounded solution of $x' = f(t, x, 0)$. That is $(f, d\,\Lambda_{L,c})$ satisfies **f1**-**f5** for all $d \geq 0$, $L > 0$ and $c > 0$, and hence the dynamics of (4.35) fits in one of the cases described by Theorem 4.16. Moreover, since 0 is the lowest bounded solution for the past and future equations, Cases B1 and C1 are preluded. In addition, if $g(t, x, \gamma) = f(t, x, -\gamma)$, then the pairs $(g, d\,\Lambda_{L,c})$ satisfy all the hypotheses of Theorem 4.34(ii) (with $\Gamma_0 \equiv 0$, $\Gamma \equiv \Lambda_{L,c}$ and $\bar{d} = 0$). This result shows the existence of a unique tipping value $d(L, c) > 0$: $(4.35)_d$ is in Case A for all $d \in [0, d(L, c))$, in Case B2 for $d = d(L, c)$ and in Case C2 for all $d > d(L, c)$. Figure 4.13 depicts the upper locally pullback attractive and the locally pullback repulsive solutions of the transition equation $(4.35)_d$ for d close to the bifurcation point, for some fixed L and c, and Table 4.1 shows numerical approximations to $d(L, c)$ for different $L, c > 0$.

$d(L, c)$	$c = 0.01$	$c = 0.02$	$c = 0.03$
$L = 2$	9.5918417988	7.8400146619	6.6406325271
$L = 10$	3.5156887400	3.1640725896	2.9522195572
$L = 20$	2.7559336044	2.5806400722	2.4622290038
$L = 30$	2.4757094854	2.3677420953	2.3132184604
$L = 40$	2.3543746813	2.2850546293	2.2459305139

Table 4.1: Numerical approximations up to ten places to the bifurcation points $d(L, c)$ of $(4.35)_d$. The displayed number is a value of d for which $(4.35)_d$ is in Case A and such that $(4.35)_{d+1e-10}$ is in Case C. The final integration has been carried out over the interval $[-\texttt{1e4}, \texttt{1e4}]$.

Now, we will let the other parameters, L and c, vary. We will observe the occurrence of critical transitions which do not fit in the rate-induced, phase-induced or size-induced ones described in Section 4.3. It is not hard to check the nondecreasing character of $L \mapsto \Gamma_L(2t/(cx + 1) - L)$ for any $(t, x) \in \mathbb{R} \times [0, \infty)$: $\Gamma_L(s - L) = \Gamma_{L'}(s - L')$ for all $L < L'$ and $s \leq 2L$, $(\partial/\partial L)\,\Gamma_L(s - L) = (\partial/\partial L)\,Q((s - 2L)/\rho) = -(2/\rho)\,Q'((s - 2L)/\rho) > 0$ for all $s \in (2L, 2L + \rho)$ (see the caption of Figure 4.10), and $\Gamma_L(s - L) = 0$ for all $s \geq 2L + \rho$. This monotonicity yields the uniqueness of a possible tipping point L_0 for $(4.35)_L$ for d and c fixed. In fact, if $\tilde{u}_L$ and $\tilde{m}_L$

are the upper and middle hyperbolic solutions when $(4.35)_L$ is in Case A, and if $L_1 < L_2$ provide this case, then Theorem 2.13(iv) shows that $\tilde{u}_{L_1} > \tilde{u}_{L_2}$, and a new comparison argument shows that $\tilde{m}_{L_1} \le \tilde{m}_{L_2}$. So, if Case B2 (the unique possible one) occurs as $L \uparrow L_0$, then they collide, and Case A cannot occur for $L > L_0$.

Analogously, the nondecreasing character of $c \mapsto \Gamma_L(2t/(cx+1) - L)$ for any $(t, x) \in \mathbb{R} \times [0, \infty)$ ensures the uniqueness of the bifurcation for $(4.35)_c$ for d and L fixed in the case of existence. The biological sense of the problem makes it reasonable to expect at most a critical transition as L or c varies: the decrease in L means an earlier detection of the problem and therefore the extinction of the hinders; and the decrease in c means an increase in the rate of recruitment of shepherds, i.e., a faster response to the problem that facilitates survival.

Figures 4.14 and 4.15 represent the behaviour of the locally pullback attractive and locally pullback repulsive solutions of the transition equation $(4.35)_L$ for fixed d and c, and $(4.35)_c$ for fixed d and L, respectively. As in the case of Figure 4.13, the left-hand panel corresponds to the survival of the species (Case A), the right-hand panel corresponds to extinction (Case C2), and the middle panel is an approximation to the intermediate unstable situation between them (Case B2).

Comments on Chapter 4

1. During this chapter we have discussed the Allee effect in d-concave equations, that we have already seen that they provide a natural framework. However, the Allee effect can also be modeled through other equations that are coercive and bistable. This is also the case if the per capita population growth rate $h(t, x)/x$ is concave (that is, x-concave equations; see [34, 35] and Section 5 of [86]).

2. One of the focuses of interest of applied scientists in the study of critical transitions is the search for early-warning signals (see [28], [29], [68], [106]), i.e., signals that indicate the proximity of the catastrophe before it occurs. In [33], the effectiveness of finite-time Lyapunov exponents as early-warning signals of rate-induced tracking scenarios has been explored in the framework of this chapter.

3. In none of the four numerical examples in this chapter do the “limits” Γ_- and Γ_+ of the transition function depend on the state variable. However, [37, Example 4.21] presents a transition in a concave equation undergoing migration and predation that is studied through a transition function whose bounds depend explicitly on x.

4. In the numerical examples, we have not used the Holling type II functional response since it does not modify the d-concavity of the equation, which could lead to critical transitions based on the generalized pitchfork bifurcation. This will be the subject of further study.

5. The assumption that the Holling type III interaction is not affected by the presence of shepherds as long as there are not enough to protect the entire herd made in Example 4.41 may seem somewhat artificial. Another option for further study would be to replace the function Γ_L by a function that has value 1 in $-L$ and decreases until it reaches 0 in L. This would reflect the decrease in the intensity of predation as the number of shepherds increases, which may be more realistic in some cases.

Conclusions

The results presented in this document delve deeper into nonautonomous bifurcation theory with a view towards critical transitions. Nonautonomous d-concave scalar differential equations have been studied due to their significance in modeling various real-world phenomena. Special interest has been placed on their applications in ecology, where d-concave equations are frequently employed to describe single species populations subject to the Allee effect.

It has been found that the skewproduct flow induced by nonautonomous d-concave scalar equations can admit at most three ordered compact invariant sets projecting onto the whole base and at most three ordered bounded measurable equilibria. These constraints, alongside certain identified properties regarding the Lyapunov exponents of compact invariant sets, restrict the dynamical possibilities of these equations, providing the necessary tools for the study of the subsequent nonautonomous bifurcation theory and critical transitions theory.

In our approach to nonautonomous bifurcation theory, three different parametric variations of an equation described by a recurrent function have been analyzed. The recurrence provides a skewproduct flow over a minimal base, and this flow has been the framework for the analysis. Certain obtained bifurcation diagrams serve as nonautonomous counterparts to previously known autonomous bifurcation diagrams, although in some cases nonhyperbolic minimal sets with high dynamical complexity may exist, contrary to the autonomous case. Other bifurcation diagrams depict entirely novel scenarios that do not arise in the autonomous setting. Such is the case with the generalized pitchfork bifurcation and the weak generalized transcritical bifurcation.

In the context of the first problem $x' = \mathfrak{h}(\omega{\cdot}t, x) + \lambda$, two different bifurcation diagrams of minimal sets have been described. It has been demonstrated that these two diagrams represent the only possibilities when considering a uniquely ergodic base flow. As mentioned in the comments at the end of Chapter 3, the existence of another possible bifurcation diagram that is only possible if there are several ergodic measures in the base is known. However, this scenario has not been detailed in the present document and will serve as the focus of further investigation.

For the second problem $x' = \mathfrak{h}(\omega{\cdot}t, x) + \lambda\, x$, three different bifurcation diagrams have been obtained and it has been shown that they are the unique possible ones. Among them is the generalized pitchfork bifurcation, which can only appear in the case where there exist several ergodic measures for the base flow. Furthermore, in the case of cubic polynomials, we have found bounds on the size of the nonautonomous coefficients that allow us to find each of the three bifurcation diagrams.

The third problem $x' = \mathfrak{h}(\omega{\cdot}t, x) + \lambda\, x^2$ also reveals only three possible bifurcation diagrams, among which is the weak generalized transcritical bifurcation. Through the examination of this third bifurcation problem, further insights have been garnered from the second problem: examples showcasing all three types of bifurcation, have been constructed in equations much more general than the cubic equations.

In the domain of critical transitions theory, a novel dynamical formulation has been introduced, wherein all components within the transition equation are allowed to vary over time. Asymptotically d-concave nonautonomous scalar differential equations have been employed as the transition equations in our investigation, always assuming that the (d-concave) past and future equations have the maximum possible number of hyperbolic solutions, that is, three.

In the skewproduct formulation, the orbit of the transition equation has been conceptualized as a heterocline (or homocline) orbit connecting its α-limit set, symbolizing the past, to its ω-limit set, symbolizing the future. This has enabled us to establish the five possible dynamical cases that the transition equation can exhibit. Three of these scenarios, Case A and Cases C, represent robust situations and respectively depict the desired and undesired (or undesired and desired, depending on the case) outcomes for the transition. In our examples, Case A means either the persistence of a species at risk of extinction or the continued control of the population size of an invasive species, while Cases C mean either extinction or habitat invasion. These cases are typically separated by the unstable Cases B. The dynamical objects that enable the construction of the different cases and demonstrate that these three cases encompass all dynamical possibilities are two locally pullback attractive solutions that approach the attractive hyperbolic solutions of the past equation as time decreases, and one locally pullback repulsive solution that approaches the repulsive hyperbolic solution of the future equation as time increases.

Some of the various mechanisms described in the literature of applied sciences as potential triggers for critical transitions (rate, phase, size, etc.) have been incorporated as parametric variations into the transition equation. In this formulation, critical transitions can be understood as saddle-node bifurcations of hyperbolic solutions occurring when the parametric variation shifts the system from Case A to Cases C. Rigorous theorems have been provided, demonstrating the existence, uniqueness, or absence of these critical transitions in various systems, as well as the existence of continuous bifurcation maps.

Finally, numerical simulations have been conducted in the context of population dynamics models with Allee effect, in order to illustrate various results and aspects of the theory, ranging from different types of critical transitions (extinction and invasion) to the temporal dependence of various elements of the transition equation for modeling seasonal phenomena, as well as the different consequences that the Allee effect can have depending on whether it is strong or weak.

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