# Chapter 5
# Local Invariant Hypersurfaces for Singular Foliations

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*Dedicated to Dominique Cerveau*

**Abstract** The question of R. Thom of existence of invariant hypersurface for germs of holomorphic codimension one foliations is a leitmotiv in the theory. In these notes, we give a panorama of the state of art of this question, where the reduction of singularities plays a central role. We start with an elementary and detailed study of the final points expected after reduction of singularities, focusing the attention on concepts and properties concerning invariant hypersurfaces.

## 5.1 Introduction

The Theory of Holomorphic Singular Foliations has foundational papers in the works of Martinet and Ramis [23, 24], Malgrange [22], Mattei and Moussu [25] and Cerveau and Mattei [18]. The problem of existence of invariant hypersurfaces for germs of holomorphic singular foliations of codimension one starts with a question of René Thom. It is a "leitmotiv" in the Theory, from the years 1980s to the present time.

There is a classical positive answer by C. Camacho and P. Sad for ambient dimension two [4]. When the ambient dimension is three or higher the question splits into two natural situations, the non-dicritical and the dicritical case.

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In the non-dicritical case, the answer is also positive as a consequence of the reduction of singularities of non-dicritical foliations in dimension three [6, 8]. More precisely, in ambient dimension three the result is due to F. Cano and D. Cerveau [8] and in ambient dimension equal or bigger to four, the existence is assured by cohomological reasons from the three-dimensional case, as shown by F. Cano and J.F. Mattei [10].

In the dicritical case, there is a classical example of a conical foliation without invariant hypersurface in ambient dimension three, due to J.P. Jouanolou [21]. Anyway, there are situations where we can assure the existence of such invariant hypersurfaces, such as the toric type foliations in dimension three, as shown by F. Cano and B. Molina-Samper [12].

The reduction of singularities of germs of singular holomorphic foliations of codimension one in three-dimensional ambient spaces [7] is the main tool for approaching the problem. For this approaching, we need a good description of simple points obtained after reduction of singularities and the local formal or convergent invariant hypersurfaces around them. These points will be connected by means of the so called *partial separatrices*, in order to create a global invariant hypersurface in the total space obtained after desingularization. This global invariant hypersurface projects over a germ of invariant hypersurface thanks to the properness of the desingularization morphism. The above one is the main argument in the paper [8], that extends to certain dicritical situations.

There are several alternative proofs for Camacho Sad result, namely [3, 15, 16, 19, 28, 31, 34]. Also we can find other results of existence of invariant hypersurfaces in different settings, see for instance [29, 33].

These notes are divided in two parts. The first one concerns a description of simple points of codimension one holomorphic foliations, with almost complete computations, focusing in the properties concerning the invariant hypersurfaces. We do not include the formal classification in terms of abelian Lie algebras and we have tried to get the necessary statements directly from the Frobenius integrability condition.

The second part is of expository nature and intends to give an idea of the state of the art in the problem of the existence of invariant hypersurfaces for codimension one holomorphic foliations.

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## Part I: Invariant Hypersurfaces and Simple Points

In this first part, we develop basic concepts in Singular Foliation theory and we focus in the description of simple points from the viewpoint of reduction of singularities.

## 5.2 Codimension One Foliations

In this section we make a quick introduction to general concepts for codimension one singular holomorphic foliations.

### *5.2.1 Foliated Spaces*

We consider *ambient spaces* $(M, K)$ that are non-singular germs of complex analytic space over a compact subset $K \subset M$. If no confusion arises, we just write $M$ for the ambient space. More precisely, we are interested in ambient spaces issued from sequences of blowing-ups with non-singular centers

$$\pi : (M, K = \pi^{-1}(0)) \to (\mathbb{C}^n, 0).$$

By a *foliated space* $\mathcal{M}$ we mean a triple $\mathcal{M} = (M, E, \mathcal{F})$, where $E$ is a normal crossings divisor in the ambient space $M$ and $\mathcal{F}$ is a codimension one holomorphic singular foliation of $M$. Given a point $P \in M$, we denote by $e_P E$ the number of irreducible components of $E$ through $P$.

There are several ways of defining codimension one holomorphic singular foliations. Without going into details, let us say that an *atlas for a foliation* is a family

$$\{(\omega_i, U_i)\}_{i \in I}$$

where the $U_i$ define an open covering of $M$ and the $\omega_i$ are never zero meromorphic differential 1-forms over $U_i$ such that $\omega_i \wedge d\omega_i = 0$, that is, they are Frobenius integrable. Moreover, the following property holds:

> Given $i, j \in I$, for each $P \in U_i \cap U_j$ there is a germ of meromorphic function $h_{ij}$ at $P$ such that $\omega_i = h_{ij}\omega_j$.

Two atlases for a foliation are compatible if their union is still an atlas. In this way, a *codimension one holomorphic singular foliation* $\mathcal{F}$ is a maximal atlas or, equivalently, a class of compatibility of atlases. A *local holomorphic generator of* $\mathcal{F}$ at a point $P$ is a germ

$$\omega = a_1 dx_1 + a_2 dx_2 + \cdots + a_n dx_n$$

of holomorphic differential 1-form belonging to $\mathcal{F}$, where the coefficients $a_i$ are germs of holomorphic functions without common factor. If $\omega'$ is another local holomorphic generator of $\mathcal{F}$ at $P$, then we have that $\omega' = u\omega$, where $u$ is a unit in the local ring $\mathcal{O}_{M,P}$. Conversely, for any unit $u \in \mathcal{O}_{M,P}$, the germ $u\omega$ is a local holomorphic generator of $\mathcal{F}$ at $P$.

The *singular locus* $\operatorname{Sing}\mathcal{F}$ is the closed analytic subset of $M$ locally defined by

$$a_1 = a_2 = \cdots = a_n = 0.$$

Note that $\operatorname{Sing}\mathcal{F}$ has codimension at least two in the ambient space $M$.

Take a reduced germ of holomorphic function $f \in \mathcal{O}_{M,P}$. We say that the germ of hypersurface $f = 0$ is *invariant for* $\mathcal{F}$ when there is a germ of holomorphic differential 2-form $\eta$ such that

$$\omega \wedge df = f\eta.$$

This definition extends without obstruction to formal hypersurfaces defined by reduced formal functions $\hat{f}$ in the Krull completion $\widehat{\mathcal{O}}_{M,P}$ of the local ring $\mathcal{O}_{M,P}$. Formal invariant hypersurfaces that are not convergent may exist, as shown by Euler's foliation $(y - x)dx - x^2dy = 0$, where $y = \sum_{k=0}^{\infty} k!x^{k+1}$ is a formal (divergent) invariant curve.

Classical Frobenius' Theorem shows that a codimension one singular holomorphic foliation $\mathcal{F}$ defines a regular codimension one foliation on $M \setminus \operatorname{Sing}\mathcal{F}$. A closed analytic hypersurface $H \subset M$ is invariant if, and only if, $H \setminus \operatorname{Sing}\mathcal{F}$ is a finite union of leaves.

Given a foliated space $\mathcal{M} = (M, E, \mathcal{F})$, we say that an irreducible component $D$ of $E$ is *dicritical* if it is not invariant. The divisor $E$ is the union of two normal crossings divisors $E_{\text{inv}}$ and $E_{\text{dic}}$, corresponding respectively to the invariant and dicritical irreducible components.

**Definition 5.2.1** The *adapted singular locus* $\operatorname{Sing}(\mathcal{F}, E) = \operatorname{Sing}\mathcal{M}$ is by definition the union of the singular locus $\operatorname{Sing}(\mathcal{F})$ and the set of non-singular points where $E$ and $\mathcal{F}$ do not have normal crossings (the divisor $E$ does not have normal crossings with the only invariant hypersurface of $\mathcal{F}$ trough $P$).

Let $N$ be a connected non-singular complex analytic space. Given a morphism $\phi : N \to M$, one of the two following possibilities occurs:

- $\phi^*\omega \neq 0$, for any local holomorphic generator $\omega$ of $\mathcal{F}$ around points in the image of $\phi$. We say that $\phi$ is *generically transverse for* $\mathcal{F}$.
- $\phi^*\omega = 0$, for any local holomorphic generator $\omega$ of $\mathcal{F}$ around points in the image of $\phi$. We say that $\phi$ is *invariant for* $\mathcal{F}$.

When $\phi$ is generically transverse we have a well defined foliation $\phi^*\mathcal{F}$ on $N$, that we call the *transform of* $\mathcal{F}$ *by* $\phi$. Particularly interesting examples are the blowing-ups with non-singular centers. If $\pi : M' \to M$ is a blowing-up with a center $Y \subset M$ having normal crossings with $E$, we have a *blowing-up of foliated spaces*

$$\pi : (M', E', \mathcal{F}') \to (M, E, \mathcal{F}),$$

where $\mathcal{F}' = \pi^*\mathcal{F}$ and $E' = \pi^{-1}(E \cup Y)$. We recall that a non-singular subspace $Y$ of $M$ has normal crossings with $E$, when we can find local coordinates at each point $P$ of $Y$ in such a way that the components of $E$ are coordinate hyperplanes and $Y$ is an intersection of coordinate hyperplanes.

A closed analytic subspace $Y \subset M$ is called *invariant for $\mathcal{F}$ at a point $P \in Y$* if each morphism $\phi : (\mathbb{C}, 0) \to (M, P)$ factoring through $(Y, P)$ is invariant. We say that $Y$ is *invariant for $\mathcal{F}$* when the property holds at each point $P \in Y$. Being invariant at a point is an open and closed property on $Y$, hence, an irreducible subspace $Y$ of $M$ is invariant if, and only if, it is invariant at a point. We are mainly interested in blowing-ups with invariant centers.

## 5.2.2 The Dimensional Type

Let $\mathcal{F}$ be a codimension one foliation on $M$ and consider a point $P$ in $M$.

A germ $\chi$ of holomorphic vector field at $P$ is *tangent to $\mathcal{F}$* if $\omega(\chi) = 0$ for the local holomorphic generators $\omega$ of $\mathcal{F}$ at $P$. We define $T_P\mathcal{F}$ to be the $\mathbb{C}$-vector subspace of the tangent space $T_PM$ spanned by the tangent vectors $\chi(P)$, where $\chi$ is a germ of holomorphic vector field at $P$ tangent to $\mathcal{F}$. The *dimensional type $\tau_P\mathcal{F}$ of $\mathcal{F}$ at $P$* is the codimension of $T_P\mathcal{F}$ in $T_PM$.

At a non-singular point, the foliation $\mathcal{F}$ is locally given by $dx = 0$, where $x$ is a function belonging to a local coordinate system. In this case, we can write a local generator of the foliation in a single variable and the dimensional type is equal to one. More generally, next results give the link between the number of variables needed to define a foliation and its dimensional type.

**Lemma 5.2.2** *Take local coordinates $x_1, x_2, \ldots, x_n$ at $P$ and a germ of integrable holomorphic 1-form $\alpha$ written as $\alpha = \sum_{i=1}^{k} b_i dx_i$, for a certain natural number $1 \leq k \leq n$. Then, there is a holomorphic germ $g$ at $P$ such that $\alpha = g\omega$, with $\omega = \sum_{i=1}^{k} a_i dx_i$ and the following properties hold:*

1. *The coefficients $a_i$ have no common factor, for $1 = 1, 2, \ldots, k$.*
2. *$\partial a_i/\partial x_\ell = 0$, for any $i = 1, 2, \ldots, k$ and $k + 1 \leq \ell \leq n$.*

*Moreover, if the coefficients $b_i$ have no common factor, then $g$ is a unit in $\mathcal{O}_{M,P}$.*

***Proof*** We assume without loss of generality that the coefficients $b_i$ have no common factor. By the condition $\alpha \wedge d\alpha = 0$, we deduce that

$$\partial(b_j/b_s)/\partial x_\ell = 0, \quad 1 \leq j, s \leq k, \quad k + 1 \leq \ell \leq n.$$

Working in the field of germs of meromorphic functions, choosing a non-zero coefficient, say $b_1$, and a natural number $2 \leq s \leq k$, we deduce that

$$b_s/b_1 = f_s/h_s, \text{ with } \partial f_s/\partial x_\ell = \partial h_s/\partial x_\ell = 0, \text{ for } k + 1 \leq \ell \leq n.$$

Let us denote $\tilde{a}_1 = \prod_{\ell=2}^{k} h_\ell$ and $\tilde{a}_s = \tilde{a}_1 f_s/h_s$, for $2 \leq s \leq k$. Let $\tilde{g}$ be the maximum common divisor of the $\tilde{a}_s$ and put $\tilde{a}_s = \tilde{g}a_s$. The vectors $(b_1, b_2, \ldots, b_k)$ and $(a_1, a_2, \ldots, a_k)$ are proportional in the field of meromorphic functions. Since both are vectors of holomorphic germs without common factor, we find a unit $g \in$

$\mathcal{O}_{M,P}$ such that

$$(b_1, b_2, \ldots, b_k) = g(a_1, a_2, \ldots, a_k).$$

This ends the proof. □

**Proposition 5.2.3** *Let $\mathcal{M} = (M, E, \mathcal{F})$ be a foliated space. We have the inequality $e_P E_{inv} \leq \tau_P \mathcal{F}$, for any point $P \in M$.*

***Proof*** Take a local coordinate system $x_1, x_2, \ldots, x_n$, such that $E_{\text{inv}}$ is given by $\prod_{i=1}^{e} x_i = 0$. Any tangent to $\mathcal{F}$ germ of vector field $\chi$ is also tangent to $E_{\text{inv}}$ and thus, we can write it as

$$\chi = \sum_{i=1}^{e} f_i x_i \partial/\partial x_i + \sum_{i=e+1}^{n} f_i \partial/\partial x_i.$$

We conclude that $T_P\mathcal{F}$ is contained in the vector space spanned by $\partial/\partial x_i|_P$, for $i = e+1, e+2, \ldots, n$. Then $\dim T_P\mathcal{F} \leq n - e$ and hence $\tau_P\mathcal{F} \geq e = e_P E_{\text{inv}}$. □

**Proposition 5.2.4** *Take a natural number $k$, such that $1 \leq k \leq n = \dim M$. We have that $k \geq \tau_P\mathcal{F}$ if, and only if, there is a local coordinate system $x_1, x_2, \ldots, x_n$ and a local generator of $\mathcal{F}$ of the form $\omega = \sum_{i=1}^{k} a_i dx_i$, where $\partial a_i/\partial x_\ell = 0$, for $1 \leq i \leq k$ and $k+1 \leq \ell \leq n$.*

***Proof*** If there is such an $\omega = \sum_{i=1}^{k} a_i dx_i$, the linearly independent tangent vectors $\partial/\partial x_\ell|_P$, for $k+1 \leq \ell \leq n$, belong to $T_P\mathcal{F}$. Conversely, let us assume that $k \geq \tau_P\mathcal{F}$. If $k = n$, we are done. Let us suppose that $k < n$ and let us make inverse induction on $k$. By induction hypothesis, we have a local coordinate system $y_1, y_2, \ldots, y_n$ and a local generator $\alpha = \sum_{i=1}^{k+1} b_i dy_i$ of $\mathcal{F}$ such that

$$\partial b_i/\partial y_\ell = 0, \quad k+2 \leq \ell \leq n.$$

Since $\alpha(\partial/\partial y_\ell) = 0$, we have that the tangent vectors $\partial/\partial y_\ell|_P$ belong to $T_P\mathcal{F}$, for $k+2 \leq \ell \leq n$. Recalling that $\dim_{\mathbb{C}} T_P\mathcal{F} \geq n-k$ and up to a linear coordinate change in $y_1, y_2, \ldots, y_{k+1}$, we find a tangent to $\mathcal{F}$ germ of vector field $\chi$ such that $\chi(P) = \partial/\partial y_{k+1}|_P$. Up to multiplying $\chi$ by a unit, we can suppose that

$$\chi = \partial/\partial y_{k+1} + \sum_{j \leq k} f_j \partial/\partial y_j + \sum_{\ell \geq k+2} f_\ell \partial/\partial y_\ell.$$

Now, the fact that $\alpha(\chi) = 0$ is equivalent to saying that

$$b_{k+1} + \sum_{j=1}^{k} b_j f_j = 0. \tag{5.1}$$

For each natural number $1 \leq j \leq k$, let us write $f_j = f_j^0 + h_j$, where $h_j$ belongs to the ideal generated by $y_{k+2}, y_{k+3}, \ldots, y_n$ and

$$\partial f_j^0 / \partial y_\ell = 0, \quad k+2 \leq \ell \leq n.$$

By Eq. (5.1), we deduce that $\alpha(\chi^0) = 0$, where $\chi^0 = \partial/\partial y_{k+1} + \sum_{j=1}^{k} f_j^0 \partial/\partial y_j$, just recalling that $\partial b_j/\partial y_\ell = 0$, for $k+2 \leq \ell \leq n$. Let us rectify $\chi^0$ by means of a coordinate change not concerning the variables $y_{k+2}, y_{k+3}, \ldots, y_n$. In this way, we obtain a new local coordinate system $x_1, x_2, \ldots, x_n$ such that $\chi^0 = \partial/\partial x_{k+1}$ and

$$\partial y_j / \partial x_\ell = 0, \quad 1 \leq j \leq k+1, \; k+2 \leq \ell \leq n.$$

As a consequence, we get that $\alpha$ is written as $\alpha = \sum_{i=1}^{k+1} g_i dx_i$. Since $\alpha(\chi^0) = 0$ we have that $g_{k+1} = 0$. Then $\alpha = \sum_{i=1}^{k} g_i dx_i$ and we end by Lemma 5.2.2. □

### 5.2.3 Axes and Transversal Type

Consider a foliated space $\mathcal{M} = (M, E, \mathcal{F})$ and a point $P$ in $M$ of dimensional type $\tau = \tau_P \mathcal{F}$. Denote by pr the linear projection

$$\text{pr} : (\mathbb{C}^n, 0) = (\mathbb{C}^\tau \times \mathbb{C}^{n-\tau}, 0) \to (\mathbb{C}^\tau, 0)$$

over the first $\tau$ coordinates. In view of Proposition 5.2.4, there is an isomorphism $\phi : (M, P) \to (\mathbb{C}^n, 0)$ and a foliation $\mathcal{G}$ on $(\mathbb{C}^\tau, 0)$, such that $\mathcal{F} = (\text{pr} \circ \phi)^* \mathcal{G}$.

**Proposition 5.2.5** *Consider an immersion $\delta : (\mathbb{C}^\tau, 0) \to (M, P)$ transverse to the tangent space $T_P \mathcal{F}$. Then $\delta^* \mathcal{F}$ is isomorphic to $\mathcal{G}$.*

***Proof*** It is enough to consider $\delta$, with target in $(\mathbb{C}^n, 0)$, throughout the isomorphism given by $\phi$ as above. □

*The transversal type of $\mathcal{F}$ at $P$* is any germ of foliation isomorphic to $\mathcal{G}$.

Let us consider $\pi = \text{pr} \circ \phi : (M, P) \to (\mathbb{C}^\tau, 0)$, we say that $A = \pi^{-1}(0)$ is an *axis* for $\mathcal{F}$ at $P$. Note that $\mathcal{F}$ has the same dimensional type and the same transversal type at all the points of the axis. We sometimes refer to this situation by saying that $\mathcal{F}$ is "analytically trivial" along $A$.

The projection $\pi$ is tangent to $E_{\text{inv}}$ and then, we have a normal crossings divisor on $(\mathbb{C}^\tau, 0)$ by direct image. On the other hand, an immersion $\delta$ transverse to $T_P \mathcal{F}$, as above, has necessarily normal crossings with $E_{\text{inv}}$. Hence we obtain a foliated space $((\mathbb{C}^\tau, 0), \delta^{-1}(E_{\text{inv}}), \delta^* \mathcal{F})$, where the normal crossings divisor $\delta^{-1}(E_{\text{inv}})$ is invariant for $\delta^* \mathcal{F}$.

**Remark 5.2.6** Consider a foliated space $\mathcal{M} = (M, E, \mathcal{F})$, where $E = E_{\text{inv}}$ and $n = \dim M$. Let us consider an irreducible closed analytic subset $Z \subset M$, *saturated* *by the local analytic type of* $\mathcal{M}$ in the following sense: if $P \in Z$ and $Q \in M$ is another point such that the germs $\mathcal{M}_P$ and $\mathcal{M}_Q$ are isomorphic, then $Q \in Z$. If we consider a point $P \in Z$, any axis $A$ for $\mathcal{F}$ at $P$ must be contained in $Z$. In particular, if $\dim Z \leq n - \tau_P\mathcal{F}$, then $\dim Z = n - \tau_P\mathcal{F}$ and $Z$ is the only axis of $\mathcal{F}$ at $P$.

There are two examples that will be useful for our study of pre-simple and simple points. Both cases assume that we are looking around a point $P$ that is in the intersection of all the irreducible components of $E$, that we denote $E_i$, for $i = 1, 2, \ldots, \ell$ (this situation may be obtained at a convenient neighborhood of $P$):

(a) We define $Z = \cap_{i=1}^{\ell} E_i$. If $\tau_P = n - \ell$ we get that $Z$ is the axis of $\mathcal{F}$ at $P$.
(b) We define $Z = \cap_{i=1}^{\ell} E_i \cap \{Q;\ \tau_Q\mathcal{F} \geq \ell + 1\}$. If $Z$ is irreducible, $\dim Z = n - \ell - 1$ and $\tau_P\mathcal{F} = \ell + 1$, then $Z$ is the axis of $\mathcal{F}$ at $P$.

## 5.3 Pre-simple Points

Let $\mathcal{M} = (M, E, \mathcal{F})$ be a foliated space and consider a point $P$ in $M$. Take local coordinates $x_1, x_2, \ldots, x_n$ and a local holomorphic generator $\omega = \sum_{i=1}^{n} a_i dx_i$ of $\mathcal{F}$ at $P$. The *order* $\mathrm{ord}_P\,\mathcal{F}$ of $\mathcal{F}$ at $P$ is given by

$$\mathrm{ord}_P\,\mathcal{F} = \min\{\mathrm{ord}_P(a_i);\ i = 1, 2, \ldots, n\},$$

where $\mathrm{ord}_P(a_i)$ denotes the usual Krull order of $a_i \in \mathcal{O}_{M,P}$. Note that the order defines an upper semi-continuous function $M \to \mathbb{Z}_{\geq 0}$. The open set $M \setminus \mathrm{Sing}\,\mathcal{F}$ is given by the points $P$ such that $\mathrm{ord}_P\,\mathcal{F} = 0$.

The coordinate system is *adapted to* $E$ if there is a subset $A$ of $\{1, 2, \ldots, n\}$ such that $E = (\prod_{i\in A} x_i = 0)$, locally at $P$. An irreducible component $x_i = 0$ of $E$ is invariant for $\mathcal{F}$ if, and only if, $x_i$ divides $a_j$ for any $j \neq i$. Let $A_{\mathrm{inv}} \subset A$ be the set of indices corresponding to the invariant components of $E$ at $P$ and put $A_{\mathrm{dic}} = A \setminus A_{\mathrm{inv}}$. Write

$$\eta = (\prod_{i\in A_{\mathrm{inv}}} x_i)^{-1}\omega = \textstyle\sum_{i\in A} b_i dx_i/x_i + \sum_{i\in\{1,2,\ldots,n\}\setminus A} b_i dx_i.$$

The coefficients $b_i$ are germs of holomorphic functions without common factor. Such an $\eta$ is called a *logarithmic generator for* $\mathcal{M}$ *at* $P$. Given $i \in A$, note that $x_i$ divides $b_i$ if, and only if, $i \in A_{\mathrm{dic}}$. The *logarithmic order* $\mathrm{logord}_P\,\mathcal{M}$ of $\mathcal{M}$ at $P$ is

$$\mathrm{logord}_P\,\mathcal{M} = \min\{\mathrm{ord}_P(b_i);\ i = 1, 2, \ldots, n\}.$$

The following inequalities hold: $\mathrm{logord}_P\,\mathcal{M} - 1 \leq \mathrm{ord}_P\,\mathcal{F} - \#A_{\mathrm{inv}} \leq \mathrm{logord}_P\,\mathcal{M}$.

**Definition 5.3.1** We say that $P$ is a *pre-simple point for* $\mathcal{M} = (M, E, \mathcal{F})$ if one of the following properties holds:

(a) $\operatorname{logord}_P \mathcal{M} = 0$.
(b) $\operatorname{logord}_P \mathcal{M} = 1$ and there are natural numbers $k, \ell \in \{1, 2, \ldots, n\}$, with $k \in A$ and $\ell \notin A$, such that $\partial b_k / \partial x_\ell(P) \neq 0$.

**Remark 5.3.2** The properties $\operatorname{logord}_P \mathcal{M} = 0$ and $\operatorname{logord}_P \mathcal{M} = 1$ are independent of the choice of the logarithmic generator $\eta$ and the adapted coordinates $x = (x_1, x_2, \ldots, x_n)$. Let us see why the property (b) is also independent of the choice of $\eta$ and $x$. Another generator is obtained by multiplying $\eta$ by a unit, and hence the property (b) still holds. Let us show the independence of the adapted coordinates. Let us denote:

- $\mathcal{J}_{E,P} \subset \mathcal{O}_{M,P}$, the ideal of $E$ at the point $P$.
- $\Theta_{M,P}$, the module of germs of vector fields at $P$.
- $\Theta_{M,P}[\log E] \subset \Theta_{M,P}$. the germs of vector fields that are tangent to $E$.
- $\mathfrak{m}_{M,P}$, the maximal ideal of $\mathcal{O}_{M,P}$.

There is a natural mapping $\pi : (\mathfrak{m}_{M,P} \Theta_{M,P}) \cap \Theta_{M,P}[\log E] \to \mathcal{O}_{M,P} / \mathcal{J}_{E,P}$, given by $\xi \mapsto \eta(\xi) + \mathcal{J}_{E,P}$. Then, condition b) holds if, and only if, $\operatorname{logord}_P \mathcal{M} = 1$ and there is at least an element in the image of $\pi$ of order equal to one.

**Remark 5.3.3** If $\mathcal{M} = (M, E, \mathcal{F})$ is a foliated space, we can consider the foliated space $\mathcal{M}_{\text{inv}} = (M, E_{\text{inv}}, \mathcal{F})$. We have that $\operatorname{logord}_P \mathcal{M} \geq \operatorname{logord}_P \mathcal{M}_{\text{inv}}$, for any $P \in M$. Thus, a pre-simple point for $\mathcal{M}$ is also a pre-simple point for $\mathcal{M}_{\text{inv}}$, since in the case $1 = \operatorname{logord}_P \mathcal{M} = \operatorname{logord}_P \mathcal{M}_{\text{inv}}$, the condition (b) only holds at indices corresponding to invariant components.

**Remark 5.3.4** The set of pre-simple points for the foliated space $(M, E, \mathcal{F})$ is an open subset of $M$.

### *5.3.1 Trace and Corner Pre-simple Points*

In next Proposition 5.3.5, we describe the two fundamental types of pre-simple points: the trace and corner type points. The number of invariant components of the divisor through a corner point is maximum and thus it coincides with the dimensional type $\tau$. In the case of trace type points, there are $\tau - 1$ invariant components. We end this subsection with the definition of *primary residual vector*, which is a key concept to introducing the notion of simple points.

Along this subsection, we consider a pre-simple point $P$ for a foliated space $\mathcal{M} = (M, E, \mathcal{F})$. We denote by $\tau$ to the dimensional type of $\mathcal{F}$ at $P$ and by $e = e_P E_{\text{inv}}$ the number of invariant components of $E$ through $P$.

**Proposition 5.3.5** *We have that $e \leq \tau \leq e+1$. Moreover, there are a local coordinate system $x_1, x_2, \ldots, x_n$, a logarithmic generator $\eta$ of $\mathcal{M}$ at $P$ and a natural number $\tau+1 \leq k \leq n+1$ such that:*

1. *$E_{inv} = (\prod_{i=1}^{e} x_i = 0)$ and $E_{dic} = (\prod_{i=k}^{n} x_i = 0)$, locally at $P$ (we have that $E_{inv} = \emptyset$, if $e = 0$, and $E_{dic} = \emptyset$, if $k = n+1$).*
2. *The logarithmic generator $\eta$ is written as*

$$\eta = \begin{cases} \sum_{i=1}^{\tau} b_i dx_i/x_i, & \text{if } e = \tau, \\ b_\tau dx_\tau + \sum_{i=1}^{\tau-1} b_i dx_i/x_i, & \text{if } e = \tau - 1, \end{cases}$$

*where $\partial b_i/\partial x_\ell = 0$, for $1 \leq i \leq \tau$ and $\tau+1 \leq \ell \leq n$.*

3. *If $e = \tau$, then $\operatorname{ord}_P(b_i; i = 1, 2, \ldots, \tau) = 0$.*
4. *If $e = \tau - 1$, then $\operatorname{ord}_P(b_i; i = 1, 2, \ldots, \tau-1) \geq 1$ and either $\operatorname{ord}_P(b_\tau) = 0$ or $\partial b_i/\partial x_\tau(P) \neq 0$, for some $i \in \{1, 2, \ldots, \tau-1\}$.*

***Proof*** Choose a local coordinate system $x = (x_1, x_2, \ldots, x_n)$ such that

$$E_{\text{inv}} = (\textstyle\prod_{i=1}^{e} x_i = 0), \quad E_{\text{dic}} = (\prod_{i=k}^{n} x_i = 0), \quad k \geq e+1.$$

Take a local logarithmic generator $\eta$ of the form

$$\eta = \textstyle\sum_{i=1}^{e} b_i dx_i/x_i + \sum_{i=e+1}^{k-1} b_i dx_i + \sum_{i=k}^{n} b_i x_i dx_i/x_i$$

and consider the natural number $m(x)$ defined by

$$m(x) = \min\{\ell;\ 0 \leq \ell \leq n \text{ and } b_{\ell+1} = b_{\ell+2} = \ldots = b_n = 0\}.$$

Note that $\max\{1, e\} \leq m(x) \leq n$, since $x_i$ does not divide the coefficients $b_i$, for $1 \leq i \leq e$. Note also that $\tau \leq m(x)$, in view of Lemma 5.2.2 and Proposition 5.2.4. Let us show that, up to making a coordinate change, we can get that $m(x) \leq e+1$ and hence $e \leq m(x) \leq e+1$. In view of Proposition 5.2.3, we conclude in this way that $e \leq \tau \leq e+1$.

Assume that $m = m(x) \geq e+2$. By Lemma 5.2.2, up to multiplying $\eta$ by a unit, we can suppose that the coefficients $b_i$ of $\eta$ do not depend on the variables $x_j$, for $m+1 \leq j \leq n$. Let us find a new coordinate system $x'$, respecting the equations of $E$, such that $m(x') < m(x)$. We find $x'$ after the rectification of a germ $\chi$ of non-singular vector field tangent to $\mathcal{F}$ (and hence tangent to $E_{\text{inv}}$), with the following properties:

(i) $\chi$ does not depend on the coordinates $x_{m+1}, x_{m+2}, \ldots, x_n$,
(ii) $\chi$ is tangent to $x_j = 0$, for $k \leq j \leq n$, with $j \neq m$,
(iii) $\chi$ is transverse to $x_m = 0$.

Let us find such a vector field $\chi$. If $\operatorname{logord}_P \mathcal{M} = 0$, there are two cases:

$$\operatorname{ord}_P(b_i; i = 1, 2, \ldots, e) = 0 \text{ or } \operatorname{ord}_P(b_i; i = 1, 2, \ldots, e) \geq 1.$$

In the first case, we have that $e \geq 1$ and there is a natural number $i \in \{1, 2, \ldots, e\}$, such that $b_i(P) \neq 0$. We can take $\chi = b_i \partial/\partial x_m - b_m x_i \partial/\partial x_i$. In the second case, we have that $k \geq e + 2$ and there is a natural number $j \neq m$, with $e + 1 \leq j < k$, such that $(b_j(P), b_m(P)) \neq (0, 0)$. Up to exchanging $x_j$ and $x_m$, we can suppose that $b_j(P) \neq 0$. We take $\chi = b_j \partial/\partial x_m - b_m \partial/\partial x_j$.

Suppose now that $\operatorname{logord}_P \mathcal{M} = 1$. We have that $1 \leq e$, $e + 1 < k$ and, up to a reordering of the coordinates, we assume that $\partial b_1/\partial x_{e+1}(P) \neq 0$. Note that $n \geq 3$, since $m \geq e + 2 \geq 3$. The integrability condition $\eta \wedge d\eta = 0$ implies that $0 = b_1\alpha + b_m\beta + b_{e+1}\gamma$, where

$$\alpha = \partial b_{e+1}/\partial x_m - \partial b_m/\partial x_{e+1},$$
$$\beta = \partial b_1/\partial x_{e+1} - x_1 \partial b_{e+1}/\partial x_1,$$
$$\gamma = x_1 \partial b_m/\partial x_1 - \partial b_1/\partial x_m.$$

We take $\chi = \alpha x_1 \partial/\partial x_1 + \beta \partial/\partial x_m + \gamma \partial/\partial x_{e+1}$, noting that $\beta(P) \neq 0$.

When $\operatorname{logord}_P \mathcal{M} = 0$ and $\operatorname{ord}_P(b_i; i = 1, 2, \ldots, e) = 0$, the above arguments allow to reduce the number $m$, even in the case that $m = e + 1$.

Summarizing our arguments, we get that the coefficients $b_i$ depend only on the variables $x_1, x_2, \ldots, x_m$ and we arrive to one of the following situations:

(a) $m = e$ and $\operatorname{logord}_P \mathcal{M} = 0$. In this case $\tau = e = m$.
(b) $m = e + 1$, $\operatorname{logord}_P \mathcal{M} = 0$ and $\operatorname{ord}_P(b_i; i = 1, 2, \ldots, e) \geq 1$. In particular, we have that $b_{e+1}(P) \neq 0$.
(c) $m = e + 1$, $\operatorname{logord}_P \mathcal{M} = 1$ and $\partial b_1/\partial x_{e+1}(P) \neq 0$.

In the cases (b) and (c), we have $\tau = e + 1$. Indeed, if there is a non-singular vector field $\chi$ trivializing the foliation and depending only on the coordinates $x_1, x_2, \ldots, x_{e+1}$, we can take it of the form

$$\chi = \partial/\partial x_{e+1} - \textstyle\sum_{s=1}^{e} f_s x_s \partial/\partial x_s.$$

This implies that $b_{e+1} = \sum_{s=1}^{e} f_s b_s$. This is obviously not possible in case (b). In case (c), after rectification of $\chi$, we would find local coordinates $y_1, y_2, \ldots, y_e, z$ adapted to $E_{\mathrm{inv}}$, such that

$$\eta = g_1 dy_1/y_1 + g_2 dy_2/y_2 + \cdots + g_e dy_e/y_e$$

and $\partial g_1/\partial z(P) \neq 0$, where the $g_i$ are without common factor, for $i = 1, 2, \ldots, e$, see Remark 5.3.2. By Lemma 5.2.2, there would be a unit $u$ such that $u\eta$ does not depend on the variable $z$, in particular $uf_1$ does not depend on $z$ and this is not possible. □

**Definition 5.3.6** We say that $P$ is of *trace type* if $e = \tau - 1$ and that it is of *corner type* if $e = \tau$. If $P$ is of trace type, we say that it is a *transversal saddle-node* when $\operatorname{logord}_P \mathcal{M} = 1$ and that it is a *tangent trace point* when $\operatorname{logord}_P \mathcal{M} = 0$.

In next proposition, we give a characterization of pre-simple corners:

**Proposition 5.3.7** *Consider a foliated space* $\mathcal{M} = (M, E, \mathcal{F})$ *and a point* $P \in M$*. Suppose that there is a logarithmic generator of* $\mathcal{M}$ *of the form*

$$\eta = \sum_{i=1}^{e} b_i dx_i/x_i + \sum_{i=e+1}^{k-1} b_i dx_i + \sum_{i=k}^{n} b_i x_i dx_i/x_i,$$

*where the invariant divisor is given by* $\prod_{i=1}^{e} x_i = 0$*. The point* $P$ *is a pre-simple corner for* $\mathcal{M}$ *if, and only if,* $\mathrm{ord}_P(b_i;\ i = 1, 2, \ldots, e) = 0$.

***Proof*** It follows from the proof of Proposition 5.3.5. □

**Remark 5.3.8** Let $P$ be a pre-simple point of corner type and assume that the dimensional type at $P$ is equal to $n = \dim M$. There are exactly $n$ irreducible components of $E = E_{\mathrm{inv}}$ through $P$. Let $H$ be one of that irreducible components and consider the union $E'$ of the other ones, then $(M, E', \mathcal{F})$ has a pre-simple point of trace type at $P$ and $H$ is an invariant hypersurface through $P$ having normal crossings with $E'$.

**Remark 5.3.9** When we are dealing with pre-simple points, the dicritical components can be written in coordinates not appearing in a suitable choice of local generator. In order to simplify the exposition, we frequently consider the case $\tau_P\mathcal{F} = n = \dim M$ and hence $E_{\mathrm{dic}} = \emptyset$.

Consider a coordinate system $x = (x_1, x_2, \ldots, x_n)$ adapted to $E$ and a logarithmic generator $\eta$ of $\mathcal{M}$ at $P$ as in Proposition 5.3.5, where

$$\eta = \begin{cases} \sum_{i=1}^{\tau} b_i dx_i/x_i, & \text{if } P \text{ is of corner type} \\ b_\tau dx_\tau + \sum_{i=1}^{\tau-1} b_i dx_i/x_i, & \text{if } P \text{ is of trace type} \end{cases}$$

**Definition 5.3.10** *The primary residual vector* $\lambda$ for $\mathcal{M}$ at $P$ associated to $\eta$ and to the chosen coordinate system $x$ is defined by $\lambda = (b_1(P), b_2(P), \ldots, b_\tau(P))$, when $P$ is of corner type and by

$$\lambda = (\partial b_1/\partial x_\tau(P), \partial b_2/\partial x_\tau(P), \ldots, \partial b_{\tau-1}/\partial x_\tau(P), b_\tau(P)),$$

when $P$ is of trace type.

**Remark 5.3.11** If $\lambda$ and $\lambda'$ are primary residual vectors, then $\lambda'$ is of the form $\lambda' = \alpha\tilde{\lambda}$, where $\alpha \neq 0$ and $\tilde{\lambda}$ is obtained by a reordering of the entries of $\lambda$. That is, the primary residual vectors are uniquely defined up to a reordering and up to homothety.

### 5.3.2 Pre-simple Corners

Here we give a useful structure of the coefficients of local logarithmic generators for pre-simple corners. As a consequence, we obtain a description of the singular locus and the notion of *secondary residual vector*.

Let $P$ be a pre-simple corner for a foliated space $\mathcal{M} = (M, E, \mathcal{F})$ such that $\tau = \tau_P \mathcal{F} \geq 2$. Take a local system of coordinates $x_1, x_2, \ldots, x_n$ adapted to $E$, where $E_{\text{inv}} = (\prod_{i=1}^{\tau} x_i = 0)$ and a logarithmic local generator $\eta$ for $\mathcal{M}$ at $P$, given by

$$\eta = \textstyle\sum_{i=1}^{\tau} b_i dx_i/x_i, \quad b_i \in \mathbb{C}\{x_1, x_2, \ldots, x_\tau\}.$$

We know that $x_i$ does not divide $b_i$, for any natural number $i = 1, 2, \ldots, \tau$. Define the subsets $J_1$ and $J_2$ of $\{1, 2, \ldots, \tau\}$ by

$$J_1 = \{i; b_i(P) \neq 0\}, \quad J_2 = \{1, 2, \ldots, \tau\} \setminus J_1.$$

We know that $J_1 \neq \emptyset$, since $\operatorname{logord}_P \mathcal{M} = 0$. Moreover, it is possible that $J_2 = \emptyset$, in this case all the coefficients $b_i$ are units. The primary residual vector is given by $\lambda = (\lambda_i)_{i=1}^{\tau}$, where $\lambda_i = b_i(P)$, for any $i = 1, 2, \ldots, \tau$.

Recall that the integrability condition $\eta \wedge d\eta = 0$ is equivalent to say that

$$\begin{aligned} 0 = {} & b_k \left(x_j \partial b_\ell/\partial x_j - x_\ell \partial b_j/\partial x_\ell\right) + \\ & b_j \left(x_\ell \partial b_k/\partial x_\ell - x_k \partial b_\ell/\partial x_k\right) + \\ & b_\ell \left(x_k \partial b_j/\partial x_k - x_j \partial b_k/\partial x_j\right), \end{aligned} \tag{5.2}$$

for any indices $j, k, \ell \in \{1, 2, \ldots, \tau\}$.

**Lemma 5.3.12** *We have that $x_k$ divides $b_j|_{x_j=0}$, for any $j \in J_2$ and $k \in J_1$.*

***Proof*** Up to multiplying $\eta$ by a unit, we assume that $b_k = 1$. Given a germ of function $f \in \mathbb{C}\{x_1, x_2, \ldots, x_\tau\}$, let us denote $\bar{f} = f|_{x_j=0}$. Since $\operatorname{ord}_P(b_j) > 0$, we have that $x_k$ divides $\bar{b}_j$ if, and only if, $x_k$ divides $\partial \bar{b}_j/\partial x_\ell$, for any $\ell \in \{1, 2, \ldots, \tau\} \setminus \{k, j\}$. If $\tau = 2$, we are done. If $\tau \geq 3$, by restriction of Equation (5.2) to $x_j = 0$, we find that

$$x_\ell \partial \bar{b}_j/\partial x_\ell = x_k(\bar{b}_\ell \partial \bar{b}_j/\partial x_k - \bar{b}_j \partial \bar{b}_\ell/\partial x_k) \tag{5.3}$$

and we are done. □

**Proposition 5.3.13** *If $J_2 \neq \emptyset$, there are a monomial $\mathbf{m} = \prod_{k \in J_1} x_k^{q_k}$, with $q_k \geq 1$ for any $k \in J_1$ and decompositions*

$$b_j = \mu_j \mathbf{m} u_j + x_j f_j, \quad u_j = \phi + g_j, \quad j \in J_2,$$

*satisfying the following properties:*

1. *The germ of function $\phi$ is a unit in $\mathbb{C}\{(x_k)_{k\in J_1}\}$, with $\phi(P) = 1$.*
2. *The coefficients $\mu_j$ are non-zero scalars.*
3. *The germs of function $g_j \in \mathbb{C}\{(x_i)_{i=1}^{\tau}\}$ are in the ideal generated by $\{x_\ell\}_{\ell\in J_2}$ and we have that $\partial g_j/\partial x_j = 0$.*

*Moreover, the primary residual vector $\lambda = (\lambda_k)_{k\in J_1}$ is proportional to $(q_k)_{k\in J_1}$.*

***Proof*** Up to a reordering of the coordinates, there is a natural number $s$, with $1 \leq s < \tau$, such that

$$J_1 = \{1, 2, \dots, s\}, \quad J_2 = \{s+1, s+2, \dots, \tau\}.$$

If $\tau = 2$, the results follows, since $x_2$ does not divide $b_2$. Assume that $\tau \geq 3$. If the result is true for a given $\eta$, it is also true up to multiplying $\eta$ by a unit in $\mathbb{C}\{x_1, x_2, \dots, x_\tau\}$. This allows us to assume that $b_1 = 1$. Consider a natural number $j \in \{s+1, s+2, \dots, \tau\}$. Since $x_j$ does not divide $b_j$, there is a unique decomposition

$$b_j = \mathbf{m}_j u_j + x_j f_j, \quad \mathbf{m}_j = \prod_{i=1, i\neq j}^{\tau} x_i^{\alpha_{j,i}},$$

where $u_j \neq 0$, $\partial u_j/\partial x_j = 0$ and there is no $x_\ell$ dividing $u_j$. Moreover, by Lemma 5.3.12, we know that $\alpha_{j,k} \geq 1$, for any $1 \leq k \leq s$.

*First Step: $u_j$ Is a Unit* We do the proof in the case that $j = \tau$. Recall that $u_\tau \in \mathbb{C}\{x_1, x_2, \dots, x_{\tau-1}\}$. Since $x_1$ does not divide $u_\tau$, we can write $u_\tau = \varphi_1 + v_1$, where $x_1$ divides $v_1$, and $0 \neq \varphi_1 \in \mathbb{C}\{x_2, x_3, \dots, x_{\tau-1}\}$. Let $t$ be the maximum of the natural numbers $\ell$, with $2 \leq \ell \leq \tau$, for which we have a decomposition

$$u_\tau = \varphi_{\ell-1} + v_{\ell-1},$$

where $0 \neq \varphi_{\ell-1} \in \mathbb{C}\{x_\ell, x_{\ell+1}, \dots, x_{\tau-1}\}$ and $v_{\ell-1}$ belongs to the ideal $\mathcal{J}_{\ell-1}$ generated by $x_1, x_2, \dots, x_{\ell-1}$. As we have seen, such a natural number $t$ exists. If $t = \tau$, then $u_\tau$ is a unit, since $\varphi_{\tau-1}$ must be a non-zero scalar. Let us show that $t = \tau$. Assume that $t < \tau$, in order to find a contradiction. Write $u_\tau = \varphi_{t-1} + v_{t-1}$ as before. Let us decompose

$$\varphi_{t-1} = \varphi_t + x_t\psi,$$

where $\varphi_t \in \mathbb{C}\{x_{t+1}, x_{t+2}, \dots, x_{\tau-1}\}$. If $\varphi_t \neq 0$, we contradict the maximality of $t$, since in this case we can put $u_\tau = \varphi_t + v_t$, where $v_t = x_t\psi + v_{t-1}$. Let us show that the property $\varphi_t = 0$ leads also to a contradiction. Thus, we suppose that $\varphi_{t-1} = x_t\psi \neq 0$.

For any $f \in \mathbb{C}\{x_1, x_2, \ldots, x_\tau\}$, we take the notation $\bar{f} = f|_{x_\tau=0}$ and $f^* = f|_{Z_t}$, where $Z_t$ is given by $x_1 = x_2 = \cdots = x_{t-1} = 0$. Note that we have

$$\bar{f} \in \mathbb{C}\{x_1, x_2, \ldots, x_{\tau-1}\}, \quad \bar{f}^* \in \mathbb{C}\{x_t, x_{t+1}, \ldots, x_{\tau-1}\}.$$

By the integrability condition, as in Eq. (5.3), we obtain that

$$x_t \partial \bar{b}_\tau / \partial x_t = \bar{b}_t x_1 \partial \bar{b}_\tau / \partial x_1 - \bar{b}_\tau x_1 \partial \bar{b}_t / \partial x_1. \tag{5.4}$$

Note that $\bar{b}_\tau = \mathbf{m}_\tau u_\tau \neq 0$. Substituting $\bar{b}_\tau$ by $\mathbf{m}_\tau u_\tau$ in Eq. (5.4) and dividing by $\mathbf{m}_\tau$, we get

$$\alpha_{\tau,t} u_\tau + x_t \partial u_\tau / \partial x_t = \bar{b}_t (\alpha_{\tau,1} u_\tau + x_1 \partial u_\tau / \partial x_1) - x_1 u_\tau \partial \bar{b}_t / \partial x_1. \tag{5.5}$$

Up to a restriction to $Z_t$ and dividing by $x_t$, we obtain that

$$(\alpha_{\tau,t} + 1)\psi + x_t \partial \psi / \partial x_t = \bar{b}_t^* \alpha_{\tau,1} \psi. \tag{5.6}$$

Let us expand $\psi = \sum_{s \geq d} \psi_s x_t^s$, where we have that $\partial \psi_s / \partial x_t = 0$ and $\psi_d \neq 0$. Write also $\bar{b}_t^* = \sum_{s=0}^{\infty} B_s x_t^s$, where the conditions $\partial B_s / \partial x_t = 0$ hold. Looking at the coefficient of $x_t^d$ in Eq. (5.6), we get

$$(\alpha_{\tau,t} + 1 + d)\psi_d = \alpha_{\tau,1} B_0 \psi_d.$$

Thus, we get that $B_0 = b_t(P) = (\alpha_{\tau,t} + 1 + d)/\alpha_{\tau,1} \in \mathbb{Q}_{>0}$. Now, we write $u_\tau = f + x_t g$ with $\partial f / \partial x_t = 0$. Note that $f \neq 0$, since $x_t$ does not divide $u_\tau$. If we make $x_t = 0$ in Eq. (5.5), we obtain

$$\alpha_{\tau,t} f = (\bar{b}_t|_{x_t=0})(\alpha_{\tau,1} f + x_1 \partial f / \partial x_1) - x_1 f \partial (\bar{b}_t|_{x_t=0}) / \partial x_1.$$

Let us write $f = \sum_{s \geq \rho} f_s x_1^s$, with $f_\rho \neq 0$ and $\partial f_s / \partial x_1 = 0$. Looking at the coefficient of $x_1^\rho$ as above, we get that $b_t(P) = \alpha_{\tau,t}/(\alpha_{\tau,1} + \rho)$. We obtain the contradiction $\alpha_{\tau,t}/(\alpha_{\tau,1} + \rho) = (\alpha_{\tau,t} + 1 + d)/\alpha_{\tau,1}$.

*Second Step:* $\mathbf{m}_j \in \mathbb{C}\{x_1, x_2, \ldots, x_s\}$ It is enough to show that $b_j \notin \mathcal{I}$, where $\mathcal{I}$ is the ideal of $\mathbb{C}\{x_1, x_2, \ldots, x_\tau\}$ generated by $x_{s+1}, x_{s+2}, \ldots, x_\tau$. Let us prove it for $j = \tau$. If $s = \tau - 1$, we are done, since $x_\tau$ does not divide $b_\tau$. Suppose now that $s \leq \tau - 2$. Assume, by contradiction, that $b_\tau$ belongs to $\mathcal{I}$. Write any $b_t$ as a formal series

$$b_t = \sum_{r=(r_1, r_2, \ldots, r_s)} b_{t,r}(x_{s+1}, x_{s+2}, \ldots, x_\tau) x_1^{r_1} x_2^{r_2} \cdots x_s^{r_s}.$$

By Lemma 5.3.12, we know that $x_k$ divides $b_\tau|_{x_\tau=0}$ for all $1 \leq k \leq s$. This implies that $b_{\tau,0}|_{x_\tau=0}$ is identically zero and hence $x_\tau$ divides $b_{\tau,0}$. Since $x_\tau$ does not divide

$b_\tau$, there is $m = (m_1, m_2, \ldots, m_s) \in \mathbb{Z}^s_{\geq 0}$ and $1 \leq \kappa \leq s$ satisfying the following properties:

- $x_\tau$ divides $b_{\tau,r}$, for any $r = (r_1, r_2, \ldots, r_s)$ such that $r_\ell \leq m_\ell$, for all $1 \leq \ell \leq s$.
- $x_\tau$ does not divide $b_{\tau,m'}$, where $m'$ is given by $m'_\ell = m_\ell$, for $\ell \neq \kappa$ and $m'_\kappa = m_\kappa + 1$.

If we multiply $\eta$ by a unit, the above properties still hold, thus, we may assume that $b_\kappa = 1$ (we lose, for a moment, the hypothesis that $b_1 = 1$). Given $s+1 \leq \ell \leq \tau-1$, by the integrability condition we obtain that

$$x_\tau \partial b_\ell/\partial x_\tau - x_\ell \partial b_\tau/\partial x_\ell = -b_\ell x_\kappa \partial b_\tau/\partial x_\kappa + b_\tau x_\kappa \partial b_\ell/\partial x_\kappa. \tag{5.7}$$

Looking at the coefficient of $x^{m'}$ in this equation, restricted to $x_\tau = 0$, we see that

$$x_\ell \partial \bar{b}_{\tau,m'}/\partial x_\ell = m_\kappa \bar{b}_{\ell,0} \bar{b}_{\tau,m'}, \tag{5.8}$$

where $\bar{f} = f|_{x_\tau=0}$ as before. Recall that we are assuming that $b_{\tau,m'} \in \mathcal{I}$ and hence $\bar{b}_{\tau,m'}$ has positive order. Noting that $\bar{b}_{\ell,0}$ is not a unit, the only positive order solution of Eq. (5.8) is $\bar{b}_{\tau,m'} = 0$. This is the desired contradiction.

*Third Step:* $\mathbf{m}_\ell = \mathbf{m}_\tau$ *and There Is* $\phi$ *such that* $u_\ell = \mu_\ell \phi + g_\ell$*, for any* $s+1 \leq \ell \leq \tau$ We obviously assume that $s \leq \tau - 2$ and $\ell < \tau$. Consider

$$Z = (x_{s+1} = x_{s+2} = \cdots = x_\tau = 0)$$

and put $\phi_t = b_t|_Z$, for $s + 1 \leq t \leq \tau$. We know that $\phi_t = \mathbf{m}_t \psi_t$, where $\psi_t$ is a unit in $\mathbb{C}\{x_1, x_2, \ldots, x_s\}$. Hence, we have that $\phi_t \neq 0$. Let us show now that $\phi_\ell/\phi_\tau$ is a non-null constant meromorphic series. Since we know that $\phi_\ell \neq 0 \neq \phi_\tau$, it is enough to show that

$$\phi_\tau \partial \phi_\ell/\partial x_k - \phi_\ell \partial \phi_\tau/\partial x_k = 0, \quad k = 1, 2, \ldots, s. \tag{5.9}$$

By restriction of Eq. (5.7) to $Z$, we directly obtain Eq. (5.9). We find the equality of monomials $\mathbf{m}_\ell = \mathbf{m}_\tau$ and also we get a non-null series $\phi = \phi_\tau$ and a scalar $\mu_\ell \neq 0$, with $\psi_\ell = \mu_\ell \phi$.

*Fourth Step: The Vectors* $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ *and* $(q_1, q_2, \ldots, q_s)$ *Are Proportional One to Each Other* If $s = 1$, the result is straightforward. Assume that $s \geq 2$ and take a natural number $2 \leq k \leq s$. As in Eq. (5.7), we have that

$$x_\tau \partial b_k/\partial x_\tau - x_k \partial b_\tau/\partial x_k = -b_k x_1 \partial b_\tau/\partial x_1 + b_\tau x_1 \partial b_k/\partial x_1. \tag{5.10}$$

By taking initial parts of degree $d$ in the restriction of Eq. (5.10) to $Z$, we conclude that

$$x_k \partial \mathbf{m}/\partial x_k = \lambda_k x_1 \partial \mathbf{m}/\partial x_1.$$

This implies that $q_k = \lambda_k q_1$, for $k = 2, 3, \ldots, s$. Since $\lambda_1 = 1$, we get the desired proportionality.

□

**Corollary 5.3.14** *Given a pre-simple corner $P$ for $\mathcal{M} = (M, E, \mathcal{F})$, the singular locus* $\mathrm{Sing}\,\mathcal{F}$ *is given, locally at $P$, by the union of the two by two intersections of invariant components of $E$ through $P$.*

***Proof*** Let $Q$ be a point near $P$. If $Q \in M \setminus E_{\mathrm{inv}}$ the fact that $J_1 \neq \emptyset$ implies that $Q \notin \mathrm{Sing}\,\mathcal{F}$. Take an irreducible component $x_i = 0$ of $E_{\mathrm{inv}}$. We have that

$$\mathrm{Sing}\,\mathcal{F} \cap (x_i = 0) = (x_i = 0) \cap (b_i \prod_{j \neq i} x_j = 0).$$

Now, the result follows if we show that $(x_i = 0) \cap (b_i = 0)$ is contained in $\prod_{j \neq i} x_j = 0$. When $i \in J_1$, we conclude by noting that $b_i$ is a unit. In the case $i \in J_2$, we have that $b_i|_{x_i=0} = \mu_i \mathbf{m} u_i$ and we are done, since $u_i$ is a unit. □

**Definition 5.3.15** The vector $\mu = (\mu_j)_{j \in J_2}$ obtained in Proposition 5.3.13 is the *secondary residual vector* attached to $\eta$ and the chosen coordinates.

The secondary residual vector is well defined for $\mathcal{M}$ at the pre-simple corner $P$ up to reordering and to homothety.

In next sections we will see that primary and secondary residual vectors are useful to characterize simple points.

## 5.3.3 *Complete Pre-simple Trace Points*

In this section, we consider a pre-simple trace point $P$ of dimensional type $\tau$ for a foliated space $\mathcal{M} = (M, E, \mathcal{F})$.

**Definition 5.3.16** We say that $P$ is a *complete pre-simple trace point* if, and only if, there is a unique non-singular formal hypersurface $\hat{H}$ through $P$ invariant for $\mathcal{F}$, such that $\hat{H} \not\subset E$ and $E_{\mathrm{inv}} \cup \hat{H}$ defines a formal normal crossings divisor at $P$.

Let us recall that $e = \tau - 1$, where $e$ denotes the number of irreducible invariant components of $E$ through $P$. Along this section we assume for simplicity that $\tau = n$, where $n$ is the dimension of the ambient space $M$.

### 5.3.3.1 Some Complete Trace Points

Take coordinates $x_1, x_2, \ldots, x_n$ at $P$ such that $E_{\mathrm{inv}} = (\prod_{i=1}^{n-1} x_i = 0)$. Fix also a local logarithmic generator

$$\eta = b_n dx_n + \sum_{i=1}^{n-1} b_i dx_i / x_i$$

of $\mathcal{M}$ at $P$. Recall that $\mathrm{ord}_P(b_1, b_2, \ldots, b_{n-1}) \geq 1$ and one of the following holds:

(a) $\mathrm{ord}_P(b_n) \geq 1$ and $\partial b_j/\partial x_n(P) \neq 0$, for a certain natural number $j$ with $j \in \{1, 2, \ldots, n-1\}$. This is the *transversal saddle-node case*.
(b) $\mathrm{ord}_P(b_n) = 0$. This is the *tangent pre-simple trace case*.

The point $P$ is a complete pre-simple trace point if, and only if, there is a unique formal series $f \in \mathbb{C}[[x_1, x_2, \ldots, x_{n-1}]]$ of order greater or equal than one, such that the formal hypersurface $x_n = f$ is invariant. This is equivalent to say that the equation

$$b_i(x_1, x_2, \ldots, x_{n-1}, f) = -b_n(x_1, x_2, \ldots, x_{n-1}, f)x_i\partial f/\partial x_i \tag{5.11}$$

holds, for any $i = 1, 2, \ldots, n-1$.

**Proposition 5.3.17** *If $P$ is a transversal saddle-node, then it is a complete pre-simple trace point.*

***Proof*** Take a natural number $j$ such that $1 \leq j \leq n-1$ and $\partial b_j/\partial x_n(P) \neq 0$. Recalling that $\mathrm{ord}_P(b_n) \geq 1$ and following the method of indeterminate coefficients, we see that Eq. (5.11) has a unique formal solution $f$ in $\mathbb{C}[[x_1, x_2, \ldots, x_{n-1}]]$, with $\mathrm{ord}_P(f) \geq 1$, for the natural number $i = j$. Let us see that $f$ gives also a solution of Eq. (5.11) for any $i = 1, 2, \ldots, n-1$. Let us put $\hat{z} = x_n - f$ and write $\eta$ as

$$\eta = b_n d\hat{z} + \textstyle\sum_{i=1}^{n-1}(b_i + b_n x_i\partial f/\partial x_i)dx_i/x_i. \tag{5.12}$$

We know that $\hat{z}$ divides $b_j + b_n x_j\partial f/\partial x_j$. Since $\partial b_j/\partial x_n(P) \neq 0$, there is a formal unit $\hat{u}$ such that $\hat{z} = \hat{u}(b_j + b_n x_j\partial f/\partial x_j)$. Let us put $\hat{\eta} = \hat{u}\eta$, where we write

$$\hat{\eta} = \hat{b}_n d\hat{z} + \hat{z}dx_j/x_j + \sum_{i=1, i\neq j}^{n-1} \hat{b}_i dx_i/x_i.$$

Equation (5.11) holds for the natural number $i$ if, and only if, the formal series $\hat{z}$ divides $\hat{b}_i$. Looking at the coefficient of $dx_i \wedge dx_j \wedge d\hat{z}$ in the integrability condition $\hat{\eta} \wedge d\hat{\eta} = 0$, evaluated in $\hat{z} = 0$, and denoting $\beta_i = \hat{b}_i|_{\hat{z}=0}$, $\beta_n = \hat{b}_n|_{\hat{z}=0}$, we obtain that

$$\beta_i = \beta_i x_j\partial\beta_n/\partial x_j - \beta_n x_j\partial\beta_i/\partial x_j.$$

The only solution of this equation is $\beta_i = 0$, since $\mathrm{ord}_P(\beta_n) \geq 1$. Hence $\hat{z}$ divides $\hat{b}_i$ as desired. □

Assume now that $P$ is a tangent trace point. We normalize $\eta$ by assuming that $b_n = 1$, that is, we write

$$\eta = dx_n + \textstyle\sum_{i=1}^{n-1} b_i dx_i/x_i. \tag{5.13}$$

In this situation, we say that $P$ is *binary-resonant* if there is a natural number $i \in \{1, 2, \ldots, n-1\}$ such that $\lambda_i \in \mathbb{Z}_{<0}$, where $\lambda_i = \partial b_i/\partial x_n(P)$.

**Proposition 5.3.18** *If $P$ is a tangent trace point that is not binary-resonant, then it is complete.*

***Proof*** We look for an invariant hypersurface of the form $x_n = f$ as before. Recall that it is invariant if, and only if, the formal series $f$ satisfies the conditions of Eq. (5.11), where $b_n = 1$, for any $i = 1, 2, \ldots, n-1$, that is

$$b_i(x_1, x_2, \ldots, x_{n-1}, f) + x_i \partial f/\partial x_i = 0, \quad i = 1, 2, \ldots, n-1. \tag{5.14}$$

We do a decomposition $b_i = \beta_i + x_n(\lambda_i + \phi_i)$, where $\partial\beta_i/\partial x_n = 0$ and $\text{ord}_P(\phi_i) > 0$. Put $\lambda' = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$. We separate two cases $\lambda' \neq 0$ and $\lambda' = 0$.

Suppose first that $\lambda' \neq 0$. Up to a reordering, we can assume that $\lambda_1 \neq 0$. Substituting the previous decomposition in Eq. (5.14) for $i = 1$, we obtain that

$$(\lambda_1 + \phi_1(x_1, x_2, \ldots, x_{n-1}, f))f + x_1\partial f/\partial x_1 = -\beta_1.$$

This equation has a unique formal solution $f(x_1, x_2, \ldots, x_{n-1})$, since we have the hypothesis $\text{ord}_P(\phi_1) \geq 1$ and $\lambda_1 \notin \mathbb{Z}_{\leq 0}$. If $n = 2$, we are done. Let us assume $n \geq 3$, denote $\hat{z} = x_n - f$ and write

$$\eta = d\hat{z} + \textstyle\sum_{i=1}^{n-1} \hat{b}_i dx_i/x_i,$$

where we know that $\hat{b}_1 = \hat{z}(\lambda_1 + \hat{\phi}_1)$, with $\text{ord}_P(\hat{\phi}_1) \geq 1$. Let us see that $\hat{z}$ divides $\hat{b}_i$, for $i = 2, 3, \ldots, n-1$. Write $\hat{b}_i = \hat{\beta}_i + \hat{z}(\lambda_i + \hat{\phi}_i)$ as before. By the integrability condition, we have that

$$x_1\partial\hat{b}_i/\partial x_1 - x_i\partial\hat{b}_1/\partial x_i - \hat{b}_1\partial\hat{b}_i/\partial\hat{z} + \hat{b}_i\partial\hat{b}_1/\partial\hat{z} = 0,$$

for any $i \in \{2, 3, \ldots, n-1\}$. Doing $\hat{z} = 0$, we obtain the equation

$$x_1\partial\hat{\beta}_i/\partial x_1 + \hat{\beta}_i(\lambda_1 + \hat{\phi}_1|_{\hat{z}=0}) = 0.$$

It has the only solution $\hat{\beta}_i = 0$, since $\lambda_1 \notin \mathbb{Z}_{\leq 0}$ and $\text{ord}_P(\hat{\phi}_1) \geq 1$. Thus $\hat{z}$ divides all the coefficients $\hat{b}_i$, for $i = 1, 2, \ldots, n-1$. Hence $\hat{z} = 0$ is the desired formal invariant hypersurface.

Assume now that $\lambda' = 0$. Let us perform a change of coordinates

$$z = x_n - \Phi(x_1, x_2, \ldots, x_{n-1}),$$

where $\Phi \neq 0$ is homogeneous of degree $k \geq 1$. Write

$$\eta = dz + \textstyle\sum_{i=1}^{n-1} \tilde{b}_i dx_i/x_i, \quad \tilde{b}_i = b_i + x_i\partial\Phi/\partial x_i.$$

Let us decompose $\tilde{b}_i = \tilde{\beta}_i(x_1, x_2, \ldots, x_{n-1}) + z\tilde{\phi}_i$, where $\mathrm{ord}_P(\tilde{\phi}_i) \geq 1$. Now, we put $d = \mathrm{ord}_P(\beta_1, \beta_2, \ldots, \beta_{n-1}) \geq 1$. By a direct verification we see that

$$\begin{aligned} &\text{If } k < d, \ \text{then } \ \tilde{d} = k, \\ &\text{If } k > d, \ \text{then } \ \tilde{d} = d, \\ &\text{If } k = d, \ \text{then } \ \tilde{d} \geq d, \end{aligned}$$

where $\tilde{d} = \mathrm{ord}_P(\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_{n-1})$. Thus, in order to show the existence and uniqueness of $f$ satisfying Eq. (5.14), it is enough to show that there is a unique $\Phi$ homogeneous of degree $d$ such that $\tilde{d} > d$. Let us do it. Denote by $B_i$ the homogeneous part of degree $d$ of $\beta_i$, for $i = 1, 2, \ldots, n-1$. We have that $\tilde{d} > d$ if, and only if, the following equation holds:

$$B_i + x_i \partial\Phi/\partial x_i = 0, \quad i = 1, 2, \ldots, n-1.$$

Doing $x_n = 0$ in the integrability condition

$$x_j \partial b_i/\partial x_j - x_i \partial b_j/\partial x_i = b_j \partial b_i/\partial x_n - b_i \partial b_j/\partial x_n$$

and taking the homogeneous components of degree $d$, we obtain that

$$x_j \partial B_i/\partial x_j = x_i \partial B_j/\partial x_i, \quad 1 \leq i, j \leq n-1.$$

The differential 1-form $\Omega = -\sum_{i=1}^{n-1} B_i dx_i/x_i$ is exact and, by a logarithmic version of Poincaré Lemma, we see that it is the differential of a unique homogeneous $\Phi$. □

#### 5.3.3.2 Singular Locus and Secondary Residual Vectors

Let $P$ be a complete pre-simple trace point for $\mathcal{M} = (M, E, \mathcal{F})$. Let $\hat{H}$ be the invariant formal hypersurface at $P$, mentioned in Definition 5.3.16. We can consider the formal foliated space $\widehat{\mathcal{M}}_P = (\hat{M}_P, E \cup \hat{H}, \mathcal{F})$. By Definition 5.3.1, we see that $P$ is a (formal) pre-simple corner for $\widehat{\mathcal{M}}_P$. Let us consider local coordinates $x_1, x_2, \ldots, x_n$ at $P$ and assume that $\hat{H}$ is given by $\hat{z} = 0$, where $\hat{z} = x_n - f(x_1, x_2, \ldots, x_{n-1})$. A consequence of the arguments leading to Corollary 5.3.14, is that the singular locus $\mathrm{Sing}\,\mathcal{F}$ is given by

$$\mathrm{Sing}\,\mathcal{F} = \left(\bigcup_{i=1}^{n-1} (\hat{z} = x_i = 0)\right) \bigcup \left(\bigcup_{1 \leq i < j \leq n-1} (x_i = x_j = 0)\right).$$

Since the irreducible components of $\mathrm{Sing}\,\mathcal{F}$ are convergent, the formal subspaces $\hat{z} = x_i = 0$ are convergent ones. In this situation, up to a convergent coordinate change in $x_n$, we may assume that

$$(\hat{z} = x_i = 0) = (x_n = x_i = 0), \quad 1 \le i \le n-1.$$

That is, we may assume that the convergent coordinates are *adapted to the singular locus* in the above sense.

**Remark 5.3.19** When $x_1, x_2, \ldots, x_n$ is adapted to the singular locus, the formal invariant hypersurface $x_n = f(x_1, x_2, \ldots, x_{n-1})$ satisfies that $x_i$ divides $f$, for any $i = 1, 2, \ldots, n-1$. Then, for another coordinate system $x_1, x_2, \ldots, x_{n-1}, x_n^*$ adapted to the singular locus, we have that $x_n^* - x_n$ is divisible by $x_i$, for any $i = 1, 2, \ldots, n-1$.

Let us look for the primary and secondary residual vectors. We can write a convergent logarithmic generator $\eta$ for $\mathcal{M}$ at $P$ in the formal coordinates $x_1, x_2, \ldots, x_{n-1}, \hat{z}$ as follows

$$\eta = \hat{z}\left(\hat{b}_n d\hat{z}/\hat{z} + \sum_{i=1}^{n-1} \hat{b}_i dx_i/x_i\right).$$

The primary residual vector, considered as a formal pre-simple corner, is given by $(\lambda_i)_{i=1}^n$, where $\lambda_i = \hat{b}_i(P)$. It coincides with the primary residual vector defined for pre-simple trace points in Sect. 5.3.1, having in mind that we can exhibit the formal trace point as

$$\eta = \hat{b}_n d\hat{z} + \sum_{i=1}^{n-1} \hat{z}\hat{b}_i dx_i/x_i. \tag{5.15}$$

Let us decompose $\{1, 2, \ldots, n\} = J_1 \cup J_2$, where

$$J_1 = \{k; \lambda_k \neq 0\}, \quad J_2 = \{j; \lambda_j = 0\}$$

as usual. If $J_2 = \emptyset$, there is no secondary residual vector. Assume that $J_2 \neq \emptyset$. By considering $P$ as a formal corner, there is a secondary residual vector $(\mu_j)_{j \in J_2}$ that we define to be *the secondary residual vector for* $\mathcal{M}$ *at the complete trace point* $P$. It depends on the ordering of the coordinates and it is defined up to homothety.

Let us see now how to visualize the residual vectors in convergent coordinates. Let $\eta = b_n dx_n + \sum_{i=1}^{n-1} b_i dx_i/x_i$ be a local logarithmic generator of $\mathcal{M}$ at $P$, written in coordinates adapted to the singular locus. Let us perform the coordinate change $\hat{z} = x_n - f$ as before, that allows to write $\eta$ as in Eq. (5.15). We have that

$$\hat{z}\hat{b}_i = b_i + b_n x_i \partial f/\partial x_i, \quad i = 1, 2, \ldots, n-1; \quad \hat{b}_n = b_n. \tag{5.16}$$

In particular, the primary residual vector $(\lambda_i)_{i=1}^n$, where $\lambda_i = \hat{b}_i(0)$, may be read as

$$\lambda_n = b_n(P); \quad \lambda_i = \partial b_i/\partial x_n(P), \ 1 \le i \le n-1.$$

Let us describe how to read the secondary residual vector in convergent coordinates. Thus, we assume that $J_2 \neq \emptyset$. We know that $(\lambda_k)_{k\in J_1}$ is proportional to a vector $(q_k)_{k\in J_1}$, where the $q_k$ are positive integer numbers. Moreover, there is a monomial

$$\widehat{\mathbf{m}} = \prod_{k\in J_1} \hat{x}_k^{q_k}, \quad \hat{x}_n = \hat{z},\ \hat{x}_j = x_j,\ 1 \leq j \leq n-1,$$

with the properties expressed in Proposition 5.3.13. Let us apply Eq. (5.16) to the structural results in Proposition 5.3.13. Noting that $x_i$ divides $f$ for any $1 \leq i \leq n-1$, we obtain the following expressions:

$$b_i = \begin{cases} x_n\lambda_i u_i + x_i g_i, & \text{when } i \in J_1 \setminus \{n\}, \\ \lambda_n u_n + x_n g_n, & \text{when } i = n \in J_1, \\ x_n\mu_i \mathbf{m} u_i + x_i g_i, & \text{when } i \in J_2 \setminus \{n\}, \end{cases} \tag{5.17}$$

where $\mathbf{m} = \prod_{k\in J_1} x_k^{q_k}$, $\partial u_i/\partial x_i = 0$ and $u_i(P) = 1$.

In the case that $J_2 \neq \{n\}$, we also have that

$$b_n = \mu_n \mathbf{m} u_n + x_1 x_2 \cdots x_{n-1} h + x_n g_n, \text{ when } n \in J_2, \tag{5.18}$$

where we require that $\partial h/\partial x_n = 0$ and that $\mathbf{m}$ does not divide any monomial in the expression of $x_1x_2\cdots x_{n-1}h$.

**Lemma 5.3.20** *Assume that $J_2 \not\subset \{n\}$ and that we have a presentation of the coefficients of $\eta$ as in Equations (5.17) and (5.18), with respect to a monomial $\mathbf{m}' = \prod_{k\in J_1} x_k^{r_k}$ with $r_k \geq 1$, for $k \in J_1$. Then, this presentation is exactly the one in Eqs. (5.17) and (5.18). In particular, we have that $\mathbf{m}' = \mathbf{m}$ and we detect the primary and secondary residual vectors from it.*

***Proof*** Given $\mathbf{m}'$, there is a unique possible way for writing a decomposition of the coefficients as in Eqs. (5.17) and (5.18). Noting that $J_2 \not\subset \{n\}$, there are no two possible monomials for obtaining the decomposition. □

**Remark 5.3.21** When $J_2 = \{n\}$, we have an automatic knowledge of the secondary residual vector, since it has a single entry. On the other hand, in this case, the statement of Lemma 5.3.20 would not be true. In fact, we may have several possible monomials $\mathbf{m}$ and $\mathbf{m}'$ to get an expression of $b_n$ as in Eq. (5.18), or even there is no such presentation. Take the example given by

$$\eta = \tilde{z}\,(p dx/x + q dy/y) + (\tilde{z} - x^p y^q)d\tilde{z}, \quad p, q \geq 2.$$

If we do the coordinate change $z = \tilde{z} - \mu xy$, the coefficient $b$ of $dz$ in the coordinates $x, y, z$ is $b = \mu xy(1 - x^{p-1}y^{q-1}) + z = -x^p y^q + \mu xy + z$. Thus, we can take $\mathbf{m} = x^p y^q$ and $\mathbf{m}' = xy$. If we do the change $z^* = \tilde{z} - x^p y^q$, the coefficient $b^*$ of $dz^*$ is $b^* = z^*$ and there is no presentation. Anyway, if we take a coordinate

$z'$ where the order of $\tilde{z} - z'$ is greater than $p + q$, we get a presentation with the monomial $x^p y^q$, that is the one associated to the corner.

### 5.3.4 Invariability of Residual Vectors

In this subsection we show that the primary and secondary residual vectors are invariant by isomorphisms, up to a reordering or a homothety. More precisely, we have the following result:

**Proposition 5.3.22** *Let $P$ be a pre-simple corner or a complete pre-simple trace point for the foliated space $\mathcal{M} = (M, E, \mathcal{F})$. Consider a biholomorphism $F : \mathcal{M} \to \mathcal{M}'$ between foliated spaces. Then, the primary and secondary residual vectors for $\mathcal{M}'$ at $P' = F(P)$ are the same ones as for $\mathcal{M}$ at $P$.*

***Proof*** We assume without loss of generality that the dimensional type of $\mathcal{F}$ at $P$ is equal to $n = \dim M$ and that $F$ is a coordinate change in $P$ that respects the components of the divisor. In the trace case, we can add the formal hypersurface to the divisor and consider a formal coordinate change. This unify the complete trace case with the corner case, thus we deal only with corners. The coordinate change is given by

$$x_i' = u_i x_i, \quad i = 1, 2, \ldots, n,$$

where the $u_i$ are units. A local logarithmic generator expressed in the coordinates $x = (x_1, x_2, \ldots, x_n)$ produces residual vectors $(\lambda_i)_{i=1}^n$ and $(\mu_j)_{j \in J_2}$. If we look for the expression of the same local logarithmic generator in the coordinates $x'$, we obtain the residual vectors $(\lambda_i)_{i=1}^n$ and $(\mu_j/c)_{j \in J_2}$, where $c = \prod_{k \in J_1} u_k^{q_k}(P) \neq 0$ (see the structure of the coefficients given in Proposition 5.3.13). □

## 5.4 Simple Foliated Spaces

Simple points define a class of pre-simple points with an added requirement of "non-resonance", that we explain in this section. The reduction of singularities is intended to obtain foliated spaces where all the points are simple.

Given a vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\tau) \in \mathbb{C}^\tau$, the *space of $\mathbb{Q}$-resonances* $\mathrm{Res}_{\mathbb{Q}}(\lambda)$ *of* $\lambda$ is the $\mathbb{Q}$-vectorial subspace of $\mathbb{Q}^\tau$ whose elements are the $q = (q_1, q_2, \ldots, q_\tau) \in \mathbb{Q}^\tau$ such that $\sum_{i=1}^{\tau} q_i \lambda_i = 0$. We say that:

- $\lambda$ is *resonant,* if $\mathrm{Res}_{\mathbb{Q}}(\lambda) \cap \mathbb{Q}^\tau_{\geq 0} \neq \{0\}$.
- $\lambda$ is *strictly resonant,* if $\mathrm{Res}_{\mathbb{Q}}(\lambda) \cap \mathbb{Q}^\tau_{>0} \neq \emptyset$.

**Remark 5.4.1** A vector $\lambda = (\lambda_i)_{i=1}^{\tau}$ is not resonant if, and only if, the vector given by $\lambda_J = (\lambda_i)_{i\in J}$ is not strictly resonant, for any non-empty subset $J \subset \{1, 2, \ldots, \tau\}$.

**Definition 5.4.2** Let $\mathcal{M} = (M, E; \mathcal{F})$ be a foliated space and let $P$ be a pre-simple point for $\mathcal{M}$ with $\tau_P\mathcal{F} = \tau$ and primary residual vector $\lambda$. If $\tau = 1$, the point $P$ is *simple for* $\mathcal{M}$. If $\tau \geq 2$, the point $P$ is *simple* if $\lambda$ is not strictly resonant and there exists an open neighborhood $U$ of $P$ such that any $Q \in U$ with $\tau_Q\mathcal{F} < \tau$ is simple.

**Proposition 5.4.3** *Consider a foliated space* $\mathcal{M} = (M, E, \mathcal{F})$ *and a pre-simple point* $P$ *with* $\tau_P\mathcal{F} = n = \dim M \geq 2$. *There is an open subset* $U \subset M$ *with* $P \in U$, *such that* $\tau_Q\mathcal{F} < n$, *for any* $Q \in U$ *with* $Q \neq P$.

***Proof*** The pre-simple property being open, all the points in $U$ are pre-simple if $U$ is small enough. Consider first the case where $P$ is of corner type. Take a logarithmic generator $\eta = \sum_{i=1}^{n} b_i dx_i/x_i$ of $\mathcal{M}$ at $P$. Up to a permutation of coordinates, we can assume that $1 \in J_1$. The vector fields $\xi_i = b_i x_1 \partial/\partial x_1 - x_i \partial/\partial x_i$ are non-singular for points in $x_i \neq 0$, for any $i = 2, 3, \ldots, n$. Hence the dimensional type drops around $P$, except maybe at the curve $Y$ given by $x_2 = x_3 = \cdots = x_n = 0$. If there is a coefficient $b_i$ not belonging to the ideal $\mathcal{I}$ generated by the coordinates $x_2, x_3, \ldots, x_n$, then $\xi_i$ is non-singular at the points of $Y$ different from $P$ and the dimensional type drops. If $\#J_1 \geq 2$, we are done since $b_k$ is a unit, with $k \in J_1 \setminus \{1\}$, and then $b_k \notin \mathcal{I}$. If $J_1 = \{1\}$, we apply Proposition 5.3.13 to conclude.

Assume now that $P$ is of trace type. We take $\eta$ as follows:

$$\eta = b_n dx_n + \textstyle\sum_{i=1}^{n-1} b_i dx_i/x_i, \quad \mathrm{ord}_P(b_1, b_2, \ldots, b_{n-1}) \geq 1.$$

In view of Proposition 5.3.5, we see that the dimensional type drops except maybe at the points of the curve $Z$ given by $x_1 = x_2 = \ldots = x_{n-1} = 0$. When $P$ is a transversal saddle-node, we have that $\mathrm{ord}_P(b_n) \geq 1$ and, up to a reordering and multiplying $\eta$ by a unit, we can assume that

$$b_1 = x_n - g(x_1, x_2, \ldots, x_{n-1}).$$

Let us place us at a point $Q \neq P$, such that $x_i(Q) = 0$ for $i = 1, 2, \ldots, n-1$ and $x_n(Q) = \epsilon \neq 0$, with $\varepsilon$ being small enough. The tangent vector field

$$b_n x_1 \partial/\partial x_1 - b_1 \partial/\partial x_n$$

is non-singular at $Q$ and hence the dimensional type drops.

Assume now that $b_n$ is a unit, that is $P$ is a tangent trace point. Without loss of generality, we suppose that $b_n = 1$. The vector fields $\xi_i = x_i \partial/\partial x_i - b_i \partial/\partial x_n$ are tangent to the foliation, for $i = 1, 2, \ldots, n-1$. Thus, the dimensional type drops at a generic point of $Z$ if, and only if, there is a coefficient $b_i$, with $1 \leq i \leq n-1$, not belonging to the ideal $\mathcal{J}$ generated by $x_1, x_2, \ldots, x_{n-1}$. Let us see that we obtain a contradiction if $b_i$ belongs to $\mathcal{J}$, for all $i = 1, 2, \ldots, n-1$. If $n = 2$, we are done, since $x_1$ does not divide $b_1$. Assume that $n \geq 3$. For each $i = 1, 2, \ldots, n-1$, let us write

$$b_i = \sum_{s=0}^{\infty} B_{i,s}, \quad B_{i,s} \in \mathbb{C}\{x_n\}[x_1, x_2, \ldots, x_{n-1}],$$

where the $B_{i,s}$ are homogeneous polynomials of degree $s$, with coefficients in $\mathbb{C}\{x_n\}$, in the variables $x_1, x_2, \ldots, x_{n-1}$. Note that $B_{i,0} = 0$, since $b_i \in \mathcal{J}$, for any natural number $i = 1, 2, \ldots, n-1$. In particular $x_1$ divides $B_{1,0}$. Since $x_1$ does not divide $b_1$, there is a maximum $d > 0$, such that $x_1$ divides $B_{1,d'}$, for any $0 \leq d' < d$. Let us find a contradiction with the maximality of $d$, by showing that $x_1$ must divide $B_{1,d}$. Noting that $B_{1,d} \in \mathcal{J}$, it is enough to see that $x_1$ divides $\partial B_{1,d}/\partial x_j$, for any $j = 2, 3, \ldots, n-1$. By the integrability condition $\eta \wedge d\eta = 0$, we have that

$$b_j \partial b_1/\partial x_n - b_1 \partial b_j/\partial x_n = x_j \partial b_1/\partial x_j - x_1 \partial b_j/\partial x_1. \tag{5.19}$$

Looking at the homogeneous component of degree $d$ (with respect to the variables $x_1, x_2, \ldots, x_{n-1}$) in Eq. (5.19), we deduce that $x_1$ divides

$$x_j \partial B_{1,d}/\partial x_j - x_1 \partial B_{j,d}/\partial x_1$$

and hence $x_1$ divides $\partial B_{1,d}/\partial x_j$, as desired. □

**Corollary 5.4.4** *Being a simple point is an open property.*

***Proof*** The result follows from the proposition and the properties of the dimensional type given in Sect. 5.2.3. □

### 5.4.1 Formal Normal Forms

In these notes we intend to focus on the properties concerning invariant hypersurfaces. The formal classification of simple singularities is another very interesting subject. Anyway, here you have a list of formal normal forms for simple singularities of codimension one singular foliations. This list can be found in [8, Introduction, Proposition 4.4] for the case of three-dimensional ambient:

Dimensional type two (or one):

(i) $xy(dx/x + \lambda dy/y)$, $\lambda \in \mathbb{C} \setminus \mathbb{Q}_{<0}$.
(ii) $xyy^s(dx/x + (\varepsilon + (1/y^s))dy/y)$, $\varepsilon \in \mathbb{C}$, $s \geq 1$.
(iii) $xy(x^p y^q)^s(dx/x + (\varepsilon + (1/(x^p y^q)^s))(p dx/x + q dy/y))$, $\varepsilon \in \mathbb{C}$, $s \geq 1$ and g. c. d$(p, q) = 1$.

Dimensional type three:

(iv) $xyz(\alpha dx/x + \beta dy/y + dz/z)$, and there are no resonances $m\alpha + n\beta + r = 0$, where $(m, n, r) \in \mathbb{Z}^3_{\geq 0} \setminus \{0\}$ (in particular, we have $\alpha\beta \neq 0$).

(v) $xyzz^s(dx/x + \beta dy/y + (\varepsilon + (1/z^s))dz/z)$, $\varepsilon \in \mathbb{C}$, $s \geq 1$ and $\beta \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$.
(vi) $xyz(y^p z^q)^s(dx/x + \beta dy/y + (\varepsilon + (1/(y^p z^q)^s))(p dy/y + q dz/z))$, $\varepsilon \in \mathbb{C}$, $s \geq 1$ and $\mathrm{g.c.d}(p, q) = 1$.
(vii) $xyz(x^p y^q z^r)^s(dx/x + \beta dy/y + (\varepsilon + (1/(x^p y^q z^r)^s))(p dx/x + q dy/y + r dz/z))$, $\varepsilon \in \mathbb{C}$, $s \geq 1$ and $\mathrm{g.c.d}(p, q, r) = 1$.

The expressions, for ambient dimensions greater or equal than four, can be deduced, in a straightforward way, from the above ones.

### 5.4.2 Resonances and Simple Points

In this subsection we give a characterization of simple points, in terms of non-resonance conditions for the primary and secondary residual vectors.

**Proposition 5.4.5** *Binary-resonant tangent trace points are not simple.*

***Proof*** Take a simple tangent trace point $P$, a local system of coordinates and a local logarithmic generator of $\mathcal{M}$ at $P$ as in Eq. (5.13). For simplicity, we can consider $\tau = n$. Let us show that $\lambda_i \notin \mathbb{Z}_{<0}$, for any $i = 1, 2, \ldots, n-1$. Take $i = 1$ and suppose that $\lambda_1 \neq 0$. If the coordinate subspace $Z$ given by $x_1 = x_n = 0$ is contained in $\mathrm{Sing}\,\mathcal{F}$, we are done by looking at the transversal type at a generic point of $Z$. Let us see that we always can choose the coordinates in such a way that $Z$ is contained in $\mathrm{Sing}\,\mathcal{F}$. Let us decompose $b_1$ as follows:

$$b_1 = \psi(x_2, x_3, \ldots, x_{n-1}) + x_1 f + x_n g, \quad g(P) = \lambda_1.$$

By Weierstrass preparation, there is a unit $u(x_2, x_3, \ldots, x_n)$ such that

$$b_1|_{x_1=0} = u \cdot (x_n - \phi(x_2, x_3, \ldots, x_{n-1})).$$

We finish by performing the coordinate change $\tilde{x}_n = x_n - \phi$. □

Let $P$ be a pre-simple point for a foliated space $\mathcal{M} = (M, E, \mathcal{F})$ that is not a binary-resonant tangent trace point. If $P$ is a trace type point, then it is complete, in view of Propositions 5.3.17 and 5.3.18. Let $\tau$ be its dimensional type. Consider a local coordinate system $x_1, x_2, \ldots, x_n$ adapted to the singular locus and a local logarithmic generator $\eta$ of $\mathcal{M}$ at $P$ that we write as follows:

$$\eta = \begin{cases} \sum_{i=1}^{\tau} b_i dx_i/x_i, & \text{if } P \text{ is of corner type} \\ b_\tau dx_\tau + \sum_{i=1}^{\tau-1} b_i dx_i/x_i, & \text{if } P \text{ is of trace type} \end{cases} \tag{5.20}$$

where $b_j \in \mathbb{C}\{x_1 x_2, \ldots, x_\tau\}$, see Proposition 5.3.5. Recall that the singular locus is the union of the coordinate sets $x_i = x_j = 0$, for $1 \leq i < j \leq \tau$.

**Remark 5.4.6** The foliation of $(\mathbb{C}^\tau, 0)$ defined by the expression of $\eta$ as in Eq. (5.20) is the transversal type of $\mathcal{M}_{\rm inv}$ at P.

Working in a small enough neighbourhood of $P$, we can stratify the space accordingly to the coordinates $x_1, x_2, \ldots, x_\tau$. The strata $S_J$ are labelled by the subsets $J \subset \{1, 2, \ldots, \tau\}$, with the property that

$$\overline{S_J} = \bigcap_{i \in J}(x_i = 0), \quad S_J = \overline{S_J} \setminus \bigcup_{J' \supset J, J' \neq J} \overline{S_{J'}}.$$

In the case of a corner point, the stratification is determined by $E_{\rm inv}$. In the trace case, the strata $S_\emptyset$ and $S_{\{\tau\}}$ depend on the coordinate choice, but all the other strata are independent of the particular choice of coordinates.

Recall the structure results, the sets of indices $J_1$ and $J_2$ and the notations given in Sects. 5.3.2 and 5.3.3, particularly in Eqs. (5.17) and (5.18). Then, we have a primary residual vector $(\lambda_i)_{i=1}^{\tau}$ and a secondary one $(\mu_j)_{j \in J_2}$ (when $J_2 \neq \emptyset$), defined at the point $P$.

Taking notations as in Remark 5.4.1, we denote $\alpha_J = (\alpha_j)_{j \in J}$, for a given $\alpha = (\alpha_j)_{j=1}^{\tau}$ and a non-empty $J \subset \{1, 2, \ldots, \tau\}$.

**Theorem 5.4.7** *Let $P$ be a pre-simple point for $\mathcal{M} = (M, E, \mathcal{F})$ that is not a binary-resonant tangent trace point. Take local coordinates $x = (x_1, x_2, \ldots, x_n)$ adapted to the singular locus, where the transversal type of $\mathcal{F}$ depends only on the first $\tau$ coordinates. Let $\lambda = (\lambda_i)_{i=1}^{\tau}$ and $\mu = (\mu_j)_{j \in J_2}$ be the primary and secondary residual vectors (the secondary one is defined only when $J_2 \neq \emptyset$). Consider the stratification associated to the coordinate system and take a point $Q \in S_J$, with $\#J \geq 2$. Then, the dimensional type of $\mathcal{F}$ at $Q$ is $\tau_Q \mathcal{F} = \#J$ and $S_J$ is an axis for $\mathcal{F}$ at $Q$. Moreover, we obtain residual vectors at $Q$ as follows:*

*(a) If $J \subset J_1$, a primary residual vector is $\lambda_J$. There is no secondary residual vector.*
*(b) If $J \subset J_2$, a primary residual vector is $\mu_J$. There is no secondary residual vector.*
*(c) If $J \cap J_1 \neq \emptyset \neq J \cap J_2$, the point $Q$ is not a binary-resonant tangent trace, a primary residual vector is $\lambda_J$ and a secondary residual vector is $\mu_{J \cap J_2}$.*

***Proof*** We assume without loss of generality that $\tau = \dim M = n$. Write $Q = Q_\varepsilon$, where $x_\ell(Q_\varepsilon) = \varepsilon_\ell \neq 0$, for $\ell \in J^c = \{1, 2, \ldots, n\} \setminus J$ and $x_j(Q_\varepsilon) = \varepsilon_j = 0$, for $j \in J$. Let us take local coordinates at $Q_\varepsilon$ given by $x_j^* = x_j - \varepsilon_j$, for $j = 1, 2, \ldots, n$ and the transversal subspace at $Q$ given by $\Delta_\varepsilon = (x_\ell = \varepsilon_\ell;\ \ell \in J^c)$.

*Corner Case* We assume that $P$ is a pre-simple corner. The local expression of $\eta$ at $Q_\varepsilon$ is given by

$$\eta = \sum_{j \in J} b_j dx_j / x_j + \sum_{k \in J^c} b_k dx_k^* / (x_k^* + \varepsilon_k).$$

Let us see that $Q_\varepsilon$ is a pre-simple corner by applying Proposition 5.3.7. If there is $j_0 \in J \cap J_1$, we have that $b_{j_0}(Q_\varepsilon) \neq 0$; if $J \subset J_2$, then, the monomial $\mathbf{m}(Q_\varepsilon)$

given in Proposition 5.3.13 is a non-null scalar and $b_j(Q_\varepsilon)\neq 0$, for any $j\in J$. In both cases, we apply Proposition 5.3.7 to see that $Q_\varepsilon$ is a pre-simple corner. Since there are exactly $\#J$ invariant irreducible components of $E$ through $Q_\varepsilon$, we have that $\tau_{Q_\varepsilon}\mathcal{F}=\#J$ (the dimensional type of a pre-simple corner is equal to the number of invariant irreducible components of the divisor through it). We conclude that $S_J$ is the axis for $\mathcal{F}$ at $Q_\varepsilon$, in view of Remark 5.2.6. The transversal type of $\mathcal{M}$ at the points of $S_J$ is represented by the transversal section

$$\eta|_{\Delta_\varepsilon}=\textstyle\sum_{j\in J} b_j|_{\Delta_\varepsilon} dx_j/x_j.$$

Let us consider now the cases (a), (b) and (c) in the statement.

Case (a). If $J\subset J_1$, the coefficients $b_j$ are units in $P=Q_0$ and we have that $b_j(Q_0)=\lambda_j\neq 0$, for $j\in J$. Recalling that $\varepsilon$ is small enough, we also have that $b_j(Q_\varepsilon)\neq 0$, for any $j\in J$. Moreover, since $S_J$ is an axis for that points, the transversal type is the same one for each $\varepsilon$. In view of Proposition 5.3.22 and taking a natural number $j_0\in J$, the quotients

$$\lambda^{\varepsilon}_{j,j_0}=(b_j/b_{j_0})|_{\Delta_\varepsilon}(Q_\varepsilon)=(b_j/b_{j_0})(Q_\varepsilon),\quad j\in J,$$

are independent of $\varepsilon$. Then $\lambda^{\varepsilon}_{j,j_0}=(b_j/b_{j_0})(Q_0)=\lambda_j/\lambda_{j_0}$ and we are done.

Case (b). If $J\subset J_2$, considering the structure of the coefficients given in Proposition 5.3.13, we have that $b_j(Q_\varepsilon)=\mu_j F_j(\varepsilon)$, where

$$0\neq F_j(\varepsilon)=\mathbf{m}(Q_\varepsilon)\left(\phi(Q_\varepsilon)+g_j(Q_\varepsilon)\right),\quad j\in J.$$

Reasoning as before, the quotients $\mu_j F_j(\varepsilon)/\mu_{j_0}F_{j_0}(\varepsilon)$ are constant functions, that is, their values do not depend on $\varepsilon$. Recalling that $g_j(Q_\varepsilon)\to 0$ when $\varepsilon\to 0$, we conclude that each quotient $\mu_j F_j(\varepsilon)/\mu_{j_0}F_{j_0}(\varepsilon)$ is equal to $\mu_j/\mu_{j_0}$.

Case (c). Assume now that $J\cap J_1\neq\emptyset\neq J\cap J_2$. Let us write

$$\mathbf{m}=\mathbf{m}_{J\cap J_1}\mathbf{m}_{J_1\setminus J},\quad \mathbf{m}_{J\cap J_1}=\textstyle\prod_{k\in J\cap J_1} x_k^{q_k}.$$

We have that $b_k(Q_\varepsilon)\neq 0$ and $b_k(Q_\varepsilon)\to\lambda_k$, when $\varepsilon\to 0$, for $k\in J\cap J_1$. Moreover, for $j\in J\cap J_2$, we have that

$$b_j|_{\Delta_\varepsilon}=\left[\mu_j\mathbf{m}_{J_1\setminus J}(Q_\varepsilon)\right]\mathbf{m}_{J\cap J_1}(\phi|_{\Delta_\varepsilon}+g_j|_{\Delta_\varepsilon})+x_j f_j|_{\Delta_\varepsilon}.$$

Then, a primary residual vector is $(\lambda^\varepsilon_i)_{i\in J}$, where $\lambda^\varepsilon_j=b_j(Q_\varepsilon)=0=\lambda_j$, for $j\in J\cap J_2$ and $\lambda^\varepsilon_k=b_k(Q_\varepsilon)\neq 0$, for $k\in J_1$. Moreover, we have a secondary residual vector $(\mu^\varepsilon_j)_{j\in J\cap J_2}$, where

$$\mu^\varepsilon_j=\mu_j\mathbf{m}_{J_1\setminus J}(Q_\varepsilon)(\phi(Q_\varepsilon)+g_j(Q_\varepsilon)).$$

Applying Proposition 5.3.22 as before, the quotients $\mu^\varepsilon_j/\mu^\varepsilon_{j_0}$ and $\lambda^\varepsilon_k/\lambda^\varepsilon_{k_0}$ are independent of $\varepsilon$. We conclude that $(\lambda_i)_{i\in J}$ and $(\mu_j)_{j\in J\cap J_2}$ are, respectively, primary and secondary residual vectors for the transversal type of $\mathcal{M}$, hence for $\mathcal{M}$, at $Q$.

*Trace Case* We assume that $P$ is a pre-simple trace point and that it is not a binary-resonant tangent trace. In particular, we know that $P$ is a complete trace point. The local expression of $\eta$ at $Q_\varepsilon$ is given by

$$\eta = b_n dx^*_n + \sum_{j\in J\setminus\{n\}} b_j dx_j/x_j + \sum_{k\in J^c\setminus\{n\}} b_k dx^*_k/(x^*_k + \varepsilon_k).$$

Let us recall the structure of the coefficients $b_i$ given in Eqs. (5.17) and (5.18). We distinguish the cases $n \in J^c$ and $n \in J$.

- Suppose first that $n \in J^c$. Let us see that $Q_\varepsilon$ is a corner. If there is a natural number $j_0 \in J \cap J_1$, we have that

$$b_{j_0}(Q_\varepsilon) = \varepsilon_n \lambda_{j_0} u_{j_0}(Q_\varepsilon) \neq 0.$$

  Otherwise, if $J \subset J_2$, then we have that $b_j(Q_\varepsilon) = \varepsilon_n \mu_j \mathbf{m}(Q_\varepsilon) \neq 0$, for any $j \in J$. In both cases, applying Proposition 5.3.7, we see that $Q_\varepsilon$ is a pre-simple corner. We deduce as above that $\#J$ is the dimensional type of $\mathcal{F}$ at $Q_\varepsilon = Q$ and that $S_J$ is the axis of $\mathcal{F}$ at $Q$. Primary and secondary residual vectors are obtained as in the precedent corner type case.
- Suppose now that $n \in J$. Let us see that $Q_\varepsilon$ is not a corner, then it is a pre-simple trace point and then the dimensional type is equal to $\#J$. Looking at the first and third lines in Eq. (5.17), we have that $b_j(Q_\varepsilon) = 0$, for any $j \in J \setminus \{n\}$. Hence $Q_\varepsilon$ is a pre-simple trace point. By Part b) of Remark 5.2.6 applied to $\bar{S}_{J\setminus\{n\}}$, we have that $S_J$ is the axis of $\mathcal{F}$ at $Q_\varepsilon$. The transversal type of $\mathcal{M}$ at the points of $S_J$ is represented by the transversal section

$$\eta|_{\Delta_\varepsilon} = b_n|_{\Delta_\varepsilon} dx_n + \sum_{j\in J\setminus\{n\}} b_j|_{\Delta_\varepsilon} dx_j/x_j.$$

Let us consider now the cases (a), (b) and (c) in the statement.

If $J \subset J_1$, a primary residual vector is $(\lambda^\varepsilon_j)_{j\in J}$, where $\lambda^\varepsilon_j = \lambda_j u_j(Q_\varepsilon)$. The quotients are constant and independent of $\varepsilon$ and thus $(\lambda_j)_{j\in J}$ is a primary residual vector. If $J \subset J_2$, a primary residual vector is $(\mu^\varepsilon_j)_{j\in J}$, where

$$\mu^\varepsilon_j = \mu_j \mathbf{m}(Q_\varepsilon) u_j(Q_\varepsilon).$$

As above, the quotients are constant and independent of $\varepsilon$ and thus $(\mu_j)_{j\in J}$ is a primary residual vector.

Assume now that $J \cap J_1 \neq \emptyset \neq J \cap J_2$. We distinguish the two possibilities $J \cap J_2 \neq \{n\}$ and $J \cap J_2 = \{n\}$.

– Suppose that $J \cap J_2 \neq \{n\}$. Write $\mathbf{m} = \mathbf{m}_{J\cap J_1}\mathbf{m}_{J_1\setminus J}$ as in the corner case. Looking at Equations (5.17) and (5.18), we have a primary residual vector given by $(\lambda_j^\varepsilon)_{j\in J}$, where $\lambda_j^\varepsilon = 0$, if $j \in J_2 \cap J$, and $\lambda_j^\varepsilon = \lambda_j u_j(Q_\varepsilon)$, if $j \in J_1 \cap J$. As before, we see that $(\lambda_j)_{j\in J}$ is a primary residual vector for $\mathcal{F}$ at $Q_\varepsilon$. Let us separate the cases $n \in J_1$ and $n \in J_2$.

  Assume that $n \in J_1$. We have that $Q_\varepsilon$ is not a binary-resonant tangent trace, then, we are working with a complete trace type pre-simple point. In particular, Eqs. (5.17) and (5.18) apply to the transversal section and we can see the secondary residual vector from it. Note that

$$b_j|_{\Delta_\varepsilon} = x_n(\mu_j \mathbf{m}_{J_1\setminus J}(Q_\varepsilon))\mathbf{m}_{J_1\cap J} u_j|_{\Delta_\varepsilon} + x_j g_j|_{\Delta_\varepsilon}, \quad j \in J \cap J_2.$$

  Hence, a secondary residual vector is

$$(\mu_j \mathbf{m}_{J_1\setminus J}(Q_\varepsilon) u_j(Q_\varepsilon))_{j\in J\cap J_2}.$$

  As before, by analytic triviality and taking limits, we see that it is proportional to $(\mu_j)_{j\in J\cap J_2}$. Then $(\mu_j)_{j\in J\cap J_2}$ is a secondary residual vector for the transversal type, as wanted.

  Assume that $n \in J_2 \cap J$. We see that $Q_\varepsilon$ is a transversal saddle-node trace for the transversal section and hence Equations (5.17) and (5.18) apply to the transversal type. We detect the secondary residual vectors as before.

– Suppose that $J \cap J_2 = \{n\}$. If we can assure that $b_n(Q_\varepsilon) = 0$, we detect a primary residual vector by analytic triviality as before and the secondary residual vector is automatically equal to $(1)$. To assure that, take $j_0 \in J \cap J_1$ and note that the quotients

$$b_n(Q_\varepsilon)/\lambda_{j_0} u_{j_0}(Q_\varepsilon),$$

  does not depend on $\varepsilon$. Recalling that $\lambda_n = b_n(P) = 0$ and taking limits when $\varepsilon \to 0$, we obtain that $b_n(Q_\varepsilon)/\lambda_{j_0} u_{j_0} = 0/\lambda_{j_0}$. Then $b_n(Q_\varepsilon) = 0$ and we are done.

□

**Corollary 5.4.8** *Let $P$ be a pre-simple point for a foliated space $\mathcal{M} = (M, E, \mathcal{F})$ that is not a binary-resonant tangent trace point. Let $\tau$ be the dimensional type of $\mathcal{F}$ at $P$, denote by $\lambda = (\lambda_i)_{i=1}^{\tau}$ a primary residual vector. Put $J_2 = \{j;\ \lambda_j = 0\}$ and let $\mu = (\mu_j)_{j\in J_2}$ be a secondary residual vector, when $J_2$ is not empty. Then $P$ is a simple point if, and only if, one of the following statements holds:*

*(a) $J_2 = \emptyset$ and $\lambda$ is not resonant.*
*(b) $J_2 \neq \emptyset$ and $\mu$ is not resonant.*

***Proof*** Assume first that $J_2 = \emptyset$ and that $P$ is a simple point. Let us consider a subset $J \subset \{1, 2, \ldots, \tau\}$, with $\#J \geq 2$. The transversal type of $\mathcal{M}$ at the stratum $S_J$ is simple and it has $\lambda_J$ as primary residual vector. Hence $\lambda_J$ is not strictly resonant.

In view of Remark 4.1, we see that $\lambda$ is not resonant. The converse statement goes similarly.

Assume now that $J_2 \neq \emptyset$. Let us put $J_1 = \{1, 2, \ldots, \tau\} \setminus J_2$, as usual. The vector $\lambda_{J_1}$ is proportional to a vector with strictly positive integer entries, hence it is not resonant. The rest of the proof follows similarly to the previous case. □

**Remark 5.4.9** Note that if $J_2 \neq \emptyset$, we know that $(\lambda_k)_{k \in J_1} = \lambda_{J_1}$ is automatically not resonant. Hence, we can state the above result by saying that the point is simple if, and only if, both $\lambda_{J_1}$ and $\mu$ are not resonant.

As we have seen, we have two possibilities for a simple point $P$ in a foliated space: there is a secondary residual vector or not. The first case is related with the existence of *saddle-nodes* in bidimensional sections. When there is no secondary residual vector, and hence all the coefficients of the primary residual vector are non-null, we say that $P$ is *complex hyperbolic*.

### *5.4.3 Invariant Hypersurfaces Through Simple Points*

We show that a simple corner has only the invariant components of the divisor as invariant hypersurfaces. Up to add a formal invariant hypersurface, a simple trace point is a simple corner and hence we find a single new formal invariant hypersurface, added to the invariant components of the divisor. The technique for obtaining these results is to show the stability of simple corners under blowing-ups with center at the axis.

**Proposition 5.4.10** *Let $P$ be a simple corner for a foliated space $\mathcal{M} = (M, E, \mathcal{F})$. Let $\pi : \mathcal{M}' \to \mathcal{M}$ be the blowing-up of $\mathcal{M}$ with center $A$, where $A$ is the axis of $\mathcal{F}$ at $P$. Any point $P'$ belonging to $\pi^{-1}(A)$ is a simple corner for $\mathcal{M}'$.*

***Proof*** It is enough to work in the case that the dimensional type is equal to $n = \dim M$. In this situation $A = \{P\}$. We label $E_1, E_2, \ldots, E_n$ the (invariant) components of $E$ through $P$. Take local coordinates $x_1, x_2, \ldots, x_n$, with $E_i = (x_i = 0)$, for any $i = 1, 2, \ldots, n$. Consider a local logarithmic generator $\eta$ for $\mathcal{M}$ at $P$ that we write as follows:

$$\eta = \sum_{i=1}^{n} b_i dx_i / x_i.$$

The primary residual vector attached to $\eta$ is $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_i = b_i(P)$, for any $i = 1, 2, \ldots, n$. We define, as usual, the subsets $J_1 = \{i;\ \lambda_i \neq 0\}$ and $J_2 = \{i;\ \lambda_i = 0\}$. Recall that, when $J_2 \neq \emptyset$, there is a secondary residual vector $\mu = (\mu_j)_{j \in J_2}$. By Corollary 5.4.8, we know that $\lambda_{J_1}$ and $\mu$ are non-resonant vectors.

Let $E'_\infty = \pi^{-1}(A)$ and write $E'_i$ to denote the strict transform of $E_i$ by $\pi$, for each $i = 1, 2, \ldots, n$. Up to a reordering, we can take local coordinates $x'_1, x'_2, \ldots, x'_n$ at $P'$ such that:

1. The blowing-up $\pi$ is given by $x_1 = x'_1$ and $x_i = (x'_i + \varepsilon_i)x'_1$, for $i = 2, 3, \ldots, n$.
2. There is a natural number $s$, with $1 \leq s \leq n$, such that $\varepsilon_i \neq 0$, for all $2 \leq i \leq s$ and $\varepsilon_i = 0$, for all $s + 1 \leq i \leq n$.

Note that $E'_i = (x'_i = 0)$, for all $s + 1 \leq i \leq n$ and $E'_\infty = (x'_1 = 0)$. A local logarithmic generator for $\mathcal{M}'$ at $P'$ is given by

$$\eta' = b'_1 dx'_1/x'_1 + \textstyle\sum_{i=2}^{s} b'_i dx'_i + \sum_{i=s+1}^{n} b_i dx'_i/x'_i,$$

where $b'_1 = \sum_{i=1}^{n} b_i$ and $(x'_i + \varepsilon_i)b'_i = b_i$, for $i = 2, 3, \ldots, s$. Note that

$$b'_1(P') = \textstyle\sum_{i=1}^{n} b_i(P) = \sum_{i=1}^{n} \lambda_i = \lambda'_1$$

and $\lambda'_1 \neq 0$, otherwise we would find a resonance for $\lambda_{J_1}$. Hence $b'_1$ is a unit. Then $E'_\infty$ is invariant and, in view of Proposition 5.3.7, we have that $P'$ is a pre-simple corner. In particular, the dimensional type of $\mathcal{F}'$ at $P'$ is $n - s + 1$.

We look now for primary and secondary residual vectors at $P'$ in order to conclude that $P'$ is simple, thanks to Corollary 5.4.8.

Consider first the case $s = 1$. The primary residual vector for $\mathcal{M}'$ at $P'$ is $\lambda' = (\lambda'_1, \lambda_2, \ldots, \lambda_n)$ and the new sets $J'_1$ and $J'_2$ are given by

$$J'_1 = J_1 \cup \{1\}, \quad J'_2 = J_2 \setminus \{1\}.$$

Given a resonance for $\lambda'_{J'_1}$, we obtain in a direct way a resonance for $\lambda_{J_1}$, thus $\lambda'_{J'_1}$ is not resonant. Moreover, the new secondary residual vector is $\mu_{J'_2}$ and hence it is not resonant (when $J'_2 \neq \emptyset$).

In the general case $1 \leq s \leq n$, we note that

$$B = E'_\infty \cap E'_{s+1} \cap E'_{s+2} \cap \cdots \cap E'_n$$

is the axis of $\mathcal{F}'$ at $P'$. Since the residual vectors are invariant by isomorphisms (see Proposition 5.3.22), we can look for the residual vectors at $P'$ at a point close enough to $O_1 = E'_\infty \cap E'_2 \cap E'_3 \cap \cdots \cap E'_n$. Applying the case $s = 1$ and Proposition 4.3, the new sets $\tilde{J}_1$ and $\tilde{J}_2$ are given by

$$\tilde{J}_1 = J'_1 \cap \{1, s+1, s+2, \ldots, n\}, \quad \tilde{J}_2 = J'_2 \cap \{s+1, s+2, \ldots, n\}$$

and the residual vectors are $\lambda'_{\{1,s+1,s+2,\ldots,n\}}$ and $\mu'_{\tilde{J}_2} = \mu_{\tilde{J}_2}$. In this way we get the non-resonance properties. □

**Corollary 5.4.11** *Let $P$ be a simple corner for $\mathcal{M} = (M, E, \mathcal{F})$. The only formal invariant hypersurfaces through $P$ are the components of $E_{inv}$ through $P$.*

***Proof*** Assume the existence of another formal invariant hypersurface $\hat{H}$ at $P$. Then, there is a formal invariant curve $\Gamma$ not contained in any of the invariant components of $E$. Let us see that this is not possible. We do it by induction on the dimensional type and by applying repeatedly blowing-ups with center the axis at the infinitely near points of $\Gamma$. In this situation, we can reduce ourselves to the case where the dimensional type is $n = \dim M$ and after the blowing-up at the point $P$ the curve $\Gamma$ passes through a point of dimensional type equal to $n$, and so on. This is not possible. Indeed, in view of the classical reduction of singularities of curves, the infinitely near points of $\Gamma$ pass asymptotically through points with only one component of the exceptional divisor. Thus, we reduce the problem to the case of a one dimensional corner, where the result becomes evident. □

Consider the case that $P$ is a simple trace point for $\mathcal{M}$. Since it is complete, there is a formal irreducible invariant hypersurface $\hat{H}$ at $P$ different from the irreducible components of $E_{\text{inv}}$. We know that it is unique under the additional conditions that $\hat{H}$ is non-singular and that it has normal crossings with $E$. Recall that $P$ is a simple corner for $(\hat{M}_P, E \cup \hat{H}, \mathcal{F})$. Hence, the invariant hypersurface $\hat{H}$ is actually unique, with the only condition that it is nor contained in $E$. Moreover, any formal invariant curve at $P$ not contained in $E$ is necessarily contained in $\hat{H}$.

## 5.5 Formal Transversality of Invariant Hypersurfaces

Let us consider a simple trace singular point $P$ for a foliated space $\mathcal{M} = (M, E, \mathcal{F})$. In view of Propositions 5.3.17 and 5.4.5, we know that there is a unique irreducible formal invariant hypersurface $\hat{H}$ at $P$ not contained in $E_{\text{inv}}$. Moreover, we have that $E_{\text{inv}} \cup \hat{H}$ is a formal normal crossings divisor. For any invariant component $D$ of $E$ through $P$, we know that $D \cap \hat{H}$ is an irreducible component of the singular locus $\text{Sing}\mathcal{F}$. In particular $D \cap \hat{H}$ is convergent, despite the fact that $\hat{H}$ may be only of formal nature. In this section we will see that $\hat{H}$ actually defines a formal invariant hypersurface at the points of $D \cap \hat{H}$ near $P$. This property is called *formal transversality*. In [1] the reader can found more details about the ringed spaces built with the transversely formal functions, also called *formal Zariski functions*.

Let us give a quick presentation of the transversely formal functions. This concept does not need of the foliation. Hence, we consider a point $P \in M$ and we choose local coordinates $x_1, x_2, \ldots, x_n$ at $P$ such that $E = (\prod_{\alpha=1}^{e} x_\alpha = 0)$. Recall that

$$\mathcal{O}_{M,P} = \mathbb{C}\{x_1, x_2, \ldots, x_n\} \subset \mathbb{C}[[x_1, x_2, \ldots, x_n]] = \hat{\mathcal{O}}_{M,P}.$$

Given a formal function $\hat{f} \in \hat{O}_{M,P}$ and a natural number $1 \leq \alpha \leq e$, we can write

$$\hat{f} = \sum_{k=0}^{\infty} g_k^{\alpha} x_{\alpha}^k, \quad g_k^{\alpha} \in \mathbb{C}[[x_1, x_2, \ldots, x_{\alpha-1}, x_{\alpha+1}, \ldots, x_n]],$$

We say that $\hat{f}$ is *transversely formal with respect to $E$* if the formal series $g_k^{\alpha}$ are convergent and they have a common ray of convergence for all $k = 0, 1, 2, \ldots$ and $1 \leq \alpha \leq e$. Take a point $Q \in (x_{\alpha} = 0)$, close to $P$. Put $\varepsilon_i = x_i(Q)$, for any $i = 1, 2, \ldots, n$, and take local coordinates $y_i = x_i - \varepsilon_i$ at $Q$. If $\hat{f}$ is transversely formal with respect to $x_{\alpha} = 0$, we get a well-defined formal series

$$\hat{f}_Q = \sum_{k=0}^{\infty} g_k^{\alpha,Q} y_{\alpha}^k,$$

where $g_k^{\alpha,Q}$ is obtained from $g_k^{\alpha}$ by substituting $x_i$ by $y_i + \varepsilon_i$, for any natural number $i \in \{1, 2, \ldots, n\}$ different from $\alpha$. Note that $\hat{f}_Q \in \mathbb{C}[[y_1, y_2, \ldots, y_n]] = \hat{O}_{M,Q}$. We select here one of the main properties of transversely formal functions, without an explicit proof:

If $\hat{f}_Q$ converges for any $Q \in E$ close to $P$, with $Q \neq P$, then $\hat{f}$ converges.

Many classical results as the ones on proper direct images (cf. [20, 30]) apply to the ringed spaces built by using transversely formal functions as fibers of the structural sheaf [1]. We see in next propositions that all the formal invariant hypersurfaces that occur after desingularization are transversely formal with respect to the non-dicritical components of the divisor. In the non-dicritical case (and in certain dicritical situations), this property allows us to project them by the morphism of reduction of singularities.

**Proposition 5.5.1** *If $P$ is a simple transversal saddle-node, then $\hat{H}$ is transversely formal with respect to $E$.*

***Proof*** We work, for simplicity, in the case $\tau_P \mathcal{F} = n = \dim M$. Take a local coordinate system $x_1, x_2, \ldots, x_n$ adapted to the singular locus Sing$\mathcal{F}$. Consider also a local logarithmic generator

$$\eta = b_n dx_n + \sum_{i=1}^{n-1} b_i dx_i / x_i, \quad \text{ord}_P(b_1, b_2, \ldots, b_n) = 1$$

for $\mathcal{M}$ at $P$. The formal hypersurface $\hat{H}$ is of the form $x_n = f(x_1, x_2, \ldots, x_{n-1})$. Up to a reordering of coordinates, let us show that $f$ is transversely formal with respect to $x_1 = 0$. That is, if we write

$$f = \sum_{k=1}^{\infty} \phi_k(x_2, x_3, \ldots, x_{n-1}) x_1^k,$$

we have to show that the series $\phi_k$ are convergent with a common ray of convergence.

We recall that $f$ is the unique solution to the system of equations

$$b_i(x_1, x_2, \ldots, x_{n-1}, f) = -b_n(x_1, x_2, \ldots, x_{n-1}, f)x_i \partial f/\partial x_i, \tag{5.21}$$

for $i = 1, 2, \ldots, n-1$. Since the coordinate system has been chosen adapted to the singular locus, we note that $x_1$ divides $f$ and $b_1|_{x_n=0}$. We must consider the cases $1 \in J_1$ and $1 \in J_2$, where $J_1$ and $J_2$ are defined in the usual way.

Assume first that $1 \in J_1$, that is $\partial b_1/\partial x_n(P) \neq 0$. Up to multiplying $\eta$ by a convergent unit, we can suppose that $b_1 = x_n - g$, where $g$ is a convergent series $g \in \mathbb{C}\{x_1, x_2, \ldots, x_{n-1}\}$ and $x_1$ divides $g = b_1|_{x_n=0}$. Let us write

$$g = \textstyle\sum_{k=1}^{\infty} g_k(x_2, x_3, \ldots, x_{n-1})x_1^k,$$

$$b_n = \textstyle\sum_{j,\ell} \psi_{j\ell}(x_2, x_3, \ldots, x_{n-1})x_1^j x_n^\ell.$$

Let $\hat{z} = x_n - f$. Applying Lemma 5.3.12 to $\eta/\hat{z}$, we conclude that $x_1$ divides $b_n|_{x_n=f}$. As a consequence, we have that $\psi_{00} = 0$. Then, the coefficient of $x_1^k$ in Eq. (5.21) for $i = 1$, is given by

$$\phi_k = g_k + \mathcal{P}_k(\{\psi_{j\ell}\}_{j+\ell<k}, \{\phi_r\}_{r=1}^{k-1}),$$

where the $\mathcal{P}_k$ are polynomials with integer coefficients. The functions $g_k$ and $\psi_{j\ell}$ have a common ray of convergence, for all the natural numbers $k, j, \ell$. This ray of convergence is also a ray of convergence for all the $\phi_k$.

Assume now that $1 \in J_2$. Taking into account Eqs. (5.17) and (5.18), we have that $b_1 = x_n\mu_1\mathbf{m}u_1 + x_1 f_1$ and $b_n = \mu_n\mathbf{m}u_n + x_1x_2\cdots x_{n-1}h + x_n f_n$. In view of Corollary 5.4.8, we know that $\mu_1/\mu_n \notin \mathbb{Q}_{<0}$. Let us decompose $b_1$ and $b_n$ as follows:

$$b_1 = \textstyle\sum_{j,\ell} \xi_{j\ell}(x_2, x_3, \ldots, x_{n-1})x_1^j x_n^\ell,$$

$$b_n = \textstyle\sum_{j,\ell} \psi_{j\ell}(x_2, x_3, \ldots, x_{n-1})x_1^j x_n^\ell,$$

Looking to the coefficient of $x_1^k$ in Eq. (5.21) for $i = 1$, we obtain that

$$(\xi_{01} - k\psi_{00})\phi_k = \mathcal{Q}_k(\{\xi_{j\ell}\}_{j+\ell<k}, \{\psi_{j\ell}\}_{j+\ell<k}, \{\phi_r\}_{r=1}^{k-1}),$$

where the $\mathcal{Q}_k$ are polynomials with integer coefficients. Noting that

$$\xi_{01} = \mu_1\mathbf{m}u_1|_{x_n=0}, \quad \psi_{00} = \mu_n\mathbf{m}u_n|_{x_1=0}, \quad \mu_1 - k\mu_n \neq 0,$$

we conclude that the common ray of convergence for the $\phi_{j\ell}$ and $\psi_{j\ell}$ is also a ray of convergence for the $\phi_k$. □

**Remark 5.5.2** Asking $P$ to be simple is essential in order to have formal transversality. More precisely, we need to assure $\mu_1/\mu_n \notin \mathbb{Z}_{<0}$ in the case $1 \in J_2$ above. The following example illustrates the reason. Consider $(\mathbb{C}^3, 0)$ and $E = E_{\text{inv}} = (xy = 0)$. The foliation generated by

$$(z - xy)dx/x + xzdy/y - xdz$$

is pre-simple but not simple, since $(1, -1)$ is a secondary residual vector. The unique irreducible formal invariant surface having normal crossings with $E$ is $z = \varphi(x)y$, where $\varphi(x) = \sum_{k=0}^{\infty} k!x^{k+1}$. But $\varphi(x)y$ is not formally transversal along $y = 0$.

**Proposition 5.5.3** *Assume that $P$ is a simple tangent trace point. The irreducible formal invariant hypersurface $\hat{H}$ is convergent.*

***Proof*** Again, we work in the case $\tau_P\mathcal{F} = n = \dim M$. If $n = 2$ the result is a direct consequence of Briot-Bouquet's Theorem; for more details, the reader can see [9]. Assume now that $n \geq 3$. Take local coordinates $x_1, x_2, \ldots, x_n$ adapted to the singular locus and let

$$\eta = dx_n + \textstyle\sum_{i=1}^{n-1} b_i dx_i/x_i$$

be a logarithmic generator for $\mathcal{M}$ at $P$. We are going to use the following "blowing-up criteria" appearing, for example, at [8, 25]:

> Let $\pi : M' \to M$ be the blowing-up of $M$ centered at $P$. A formal power series $g \in \hat{O}_{M,P}$ is convergent if, and only if, the pull-back $(\pi^* g)_{P'}$ is convergent at a given point $P' \in \pi^{-1}(P)$.

Let us write $\hat{H}$ as $x_n = f(x_1, x_2, \ldots, x_{n-1})$, where we know that the monomial $x_1 x_2 \cdots x_{n-1}$ divides $f$. We have to prove that $f$ is convergent.

Let us perform the blowing-up $\pi : \mathcal{M}' \to \mathcal{M}$ centered at $P$. Take a point $P'$ belonging to $\pi^{-1}(P)$ and in the strict transform of $\hat{H}$, but not in the strict transform of any irreducible component of $E$. We have local coordinates $x'$ at $P'$ given by the relations $x_1 = x'_1$, $x_n = x'_n x'_1$ and

$$x_i = (x'_i + \varepsilon_i)x'_1, \;\; \varepsilon_i \neq 0, \quad i = 2, 3, \ldots, n-1.$$

A logarithmic generator for the foliated space $\mathcal{M}'$ at the point $P'$ is $\eta' = \pi^*\eta/x'_1$. We write $\eta'$ as follows:

$$\eta' = dx'_n + b'_1 dx'_1/x'_1 + \textstyle\sum_{i=2}^{n-1} b'_i dx'_i,$$

where $x'_1 b'_1 = \sum_{j=1}^{n-1} b_j + x'_1 x'_n$ and $x'_1 b'_i = b_i/(x'_i + \varepsilon_i)$. Note that $x'_n$ divides $b'_1$. Then, in view of Proposition 5.3.7, we see that $P'$ is a tangent trace type singularity with $\tau_{P'}\mathcal{F}' = 2$. Moreover, the strict transform $\hat{H}'$ of $\hat{H}$ is a formal invariant hypersurface at $P'$. Applying the result for the case $n = 2$, we know that $\hat{H}'$ is a convergent hypersurface given by

$$x'_n = (\pi^* f)_{P'}/x'_1.$$

Thus, the series $(\pi^* f)_{P'}$ is convergent and hence $f$ converges. □

## Part II: The Existence of Invariant Hypersurfaces

In the second part of this text, we give, in an expository way, the main lines for the proofs of the known results of existence of invariant hypersurface for a germ of singular codimension one foliation on $(\mathbb{C}^n, 0)$. We stand out the idea of *partial separatrix* that is useful when the dimension of the ambient space is greater or equal than three.

## 5.6 Partial Separatrices

In this section we describe foliated spaces having only simple points, paying special attention to the extension of convergent or formal invariant hypersurfaces along the "trace components" of the singular locus.

We consider a foliated space $\mathcal{M} = (M, E, \mathcal{F})$, where the ambient space is a germ $(M, K)$ along a compact analytic subset $K \subset E$. We suppose that all the points $P \in K$ are simple points for $\mathcal{M}$. Moreover, we assume that $K$ is a union of irreducible components of $E$. We say that a foliated space with these properties is a *simple foliated space*.

**Remark 5.6.1** Simple foliated spaces are the expected objects obtained by reduction of singularities of a germ of foliation over $(\mathbb{C}^n, 0)$. When the first blowing-up of the reduction of singularities morphism is centered at the origin, the condition that $K$ is a union of components of $E$ is reached. This condition is not strictly necessary, but we keep it for simplicity of the exposition.

We say that an irreducible component $D$ of $E$ is a *compact component* when $D \subset K$. We denote by $E^c_{\mathrm{inv}}$ the union of the compact invariant components of $E$ and by $E^c_{\mathrm{dic}}$ the union of the compact dicritical components. Thus, we have that $K = E^c_{\mathrm{inv}} \cup E^c_{\mathrm{dic}}$.

### *5.6.1 The Singular Trace Set*

The set $\mathrm{Tr}\,\mathcal{M} \subset \mathrm{Sing}\,\mathcal{F}$ of trace type singular points is a finite union of non-singular codimension two irreducible closed analytic subsets of $(M, K)$ having normal crossings with the divisor $E$ (including dicritical components). This fact is a consequence of the local description of the singular locus given in Part I. More precisely, around a trace type singular point $P \in K$ of dimensional type $\tau$, there is a local system of coordinates $x_1, x_2, \ldots, x_n$ such that $E_{\mathrm{inv}} = (\prod_{i=1}^{\tau-1} x_i = 0)$, with

$$\mathrm{Tr}\,\mathcal{M} = \bigcup_{i=1}^{\tau-1} (x_i = x_\tau = 0) \subset E_{\mathrm{inv}}.$$

Moreover $E_{\mathrm{dic}} = (\prod_{i=k}^{n} x_i = 0)$, for certain natural number $k$, with $\tau + 1 \leq k \leq n$.

Let us denote by $\mathrm{Tr}^c\,\mathcal{M}$ the union of the irreducible components of $\mathrm{Tr}\,\mathcal{M}$ that are contained in $K$. We call them *compact components of* $\mathrm{Tr}\,\mathcal{M}$. That is, the compact components of $\mathrm{Tr}\,\mathcal{M}$ are exactly the irreducible components of $\mathrm{Tr}^c\,\mathcal{M}$.

**Definition 5.6.2** A *partial separatrix* $S$ *for* $\mathcal{M}$ is a maximal connected union of compact components of $\mathrm{Tr}\,\mathcal{M}$. In other words, partial separatrices are the connected components of $\mathrm{Tr}^c\,\mathcal{M}$.

We take a partial separatrix $S$ and a point $P \in S$. Since $P$ is a simple trace point, there is a unique (formal) invariant hypersurface $\hat{H}$ for $\mathcal{M}$ at $P$ which is not an invariant component of $E$. We know that $\hat{H}$ is transversely formal in the sense of [1]. This allows us to build a formal space $(\hat{H}_S, S)$, where the germ of $\hat{H}_S$ at any point $Q \in S$ is the corresponding invariant formal hypersurface.

**Remark 5.6.3** The formal space $(\hat{H}_S, S)$ converges if, and only if, there is a point $Q \in S$ such that the germ of $(\hat{H}_S, S)$ at $Q$ converges. This property follows by formal transversality and taking into account the structure of the foliation around a simple point.

Let us note that $S \subset \hat{H}_S \cap K$. The immersion $(\hat{H}_S, S) \mapsto (M, K)$ is said to be *closed* if we have the equality $\hat{H}_S \cap K = S$. As a consequence of the local description of $\mathrm{Tr}\,\mathcal{M}$ and thanks to the fact that $K$ is a union of components of $E$, the following two statements are equivalent:

(a) $\hat{H}_S \cap K = S$, that is, the immersion $(\hat{H}_S, S) \mapsto (M, K)$ is closed.
(b) The intersection between $S$ and $E^c_{\mathrm{dic}}$ is empty.

This equivalence justifies next definition:

**Definition 5.6.4** A partial separatrix $S$ is *complete* if, and only if, the equivalent statements (a) and (b) below hold.

The Direct Image Theorems in [1] and [8] give us the following result:

**Proposition 5.6.5** *Assume that there is a morphism*

$$\pi : \mathcal{M} = ((M, K), E, \mathcal{F}) \to ((\mathbb{C}^n, 0), \emptyset, \mathcal{F}_0)$$

*obtained by composition of a finite sequence of blowing-ups with invariant centers. If* $S$ *is a complete partial separatrix for* $\mathcal{M}$*, then its image* $\pi(\hat{H}_S)$ *is a formal invariant hypersurface of* $\mathcal{F}_0$.

### 5.6.2 Extended Partial Separatrices

Here we extend the concept of partial separatrix to take into account the influence of compact dicritical components of $\mathcal{M}$ in order to assure the existence of invariant hypersurfaces supported by them.

The irreducible components of $\operatorname{Tr}\mathcal{M}$ are some of the irreducible components of $\operatorname{Sing}\mathcal{F}$ and hence they have codimension two and are non-singular. We need to extend this class of objects with some subspaces not contained in the singular locus. More precisely, we introduce the following concept:

**Definition 5.6.6** Let $Z \subset K$ be a closed irreducible subspace of codimension two. We say that $Z$ is a *compact trace subspace $Z$ for $\mathcal{M}$* if it is invariant for $\mathcal{F}$ and all the points in $Z$ are of trace type.

Let us note that any compact component of $\operatorname{Tr}\mathcal{M}$ is a compact trace subspace (with all the points being singular), but we can have more compact trace subspaces. More precisely, a compact trace subspace $Z$ is generically contained in a single compact component $D$ of $E$. When $D$ is invariant, we have that $Z \subset \operatorname{Tr}^c\mathcal{M}$. When $D$ is dicritical, the generic points of $Z$ are non-singular points of $\mathcal{F}$ (its dimensional type is one).

**Definition 5.6.7** An *extended partial separatrix* $\mathcal{E}$ for $\mathcal{M}$ is a maximal finite connected union of compact trace subspaces.

The maximality required in the definition of extended partial separatrix implies that there is at most one of them containing a given trace subspace, however the existence is not assured. Anyway, we have the following result:

**Proposition 5.6.8** *A complete partial separatrix $S$ for $\mathcal{M}$ is also an extended partial separatrix for $\mathcal{M}$.*

Let $\mathcal{E}$ be an extended partial separatrix for $\mathcal{M}$. There is a unique germ $(\hat{H}_{\mathcal{E}}, \mathcal{E})$ of formal invariant hypersurface along $\mathcal{E}$ such that $\hat{H}_{\mathcal{E}} \not\subset E$. Let us recall that $\hat{H}_{\mathcal{E}}$ is transversely formal. Moreover, if there is a point $P \in \mathcal{E}$ such that the germ of $\hat{H}_{\mathcal{E}}$ at $P$ converges, then the formal hypersurface $\hat{H}_{\mathcal{E}}$ converges at any point of $\mathcal{E}$ (see Remark 5.6.3). In particular, if the extended partial separatrix $\mathcal{E}$ contains at least one compact trace subspace not contained in $\operatorname{Tr}^c\mathcal{M}$, then $(\hat{H}_{\mathcal{E}}, \mathcal{E})$ is a convergent invariant hypersurface.

The immersion $(\hat{H}_{\mathcal{E}}, \mathcal{E}) \to (M, K)$ is closed if, and only if, $\mathcal{E} = \hat{H}_{\mathcal{E}} \cap K$. This property may be read only with the datum of the extended partial separatrix. More precisely, the following statements are equivalent:

(a) The immersion $(\hat{H}_{\mathcal{E}}, \mathcal{E}) \to (M, K)$ is closed.
(b) For any point $P \in \mathcal{E}$ and any compact component $D$ of $E$ with $P \in D$, there is a compact trace subspace $Z \subset \mathcal{E}$, with $Z \subset D$ and $P \in Z$.

**Definition 5.6.9** An extended partial separatrix $\mathcal{E}$ for $\mathcal{M}$ is said to be *complete* if, and only if, the above equivalent statements (a) and (b) hold.

There is a bijection $\mathcal{E} \mapsto (\hat{H}_{\mathcal{E}}, \mathcal{E})$ between the set of complete extended partial separatrices for $\mathcal{M}$ and the set of closed formal hypersurfaces

$$(\hat{H}, \hat{H} \cap K) \subset (M, K)$$

that are not contained in $E$ and that are invariant for $\mathcal{F}$. The complete extended partial separatrix associated to $(\hat{H}, \hat{H} \cap K)$ is given by $\mathcal{E}_{\hat{H}} = \hat{H} \cap K$. Applying again the Direct Image Theorems, if there is a morphism

$$\pi : ((M, K), E, \mathcal{F}) \to ((\mathbb{C}^n, 0), \emptyset, \mathcal{F}_0)$$

obtained by a finite sequence of blowing-ups with invariant centers, there is a bijection between the set of complete extended partial separatrices and the formal invariant hypersurfaces of $\mathcal{F}_0$.

**Remark 5.6.10** If $S$ is a complete partial separatrix for $\mathcal{M}$, then it is also complete considered as an extended partial separatrix. On the other hand, given a complete extended separatrix $\mathcal{E}$, either $\mathcal{E} \subset \mathrm{Tr}^c\, \mathcal{M}$ or $\mathcal{E} \not\subset \mathrm{Tr}^c\, \mathcal{M}$. If $\mathcal{E} \subset \mathrm{Tr}^c\, \mathcal{M}$, we have that $\mathcal{E}$ is a complete partial separatrix. If $\mathcal{E} \not\subset \mathrm{Tr}^c\, \mathcal{M}$, the connected components of $\mathcal{E} \cap \mathrm{Tr}^c\, \mathcal{M}$ are non-complete partial separatrices.

### *5.6.3 Invariant Hypersurfaces and Reduction of Singularities*

Let us consider a foliated space $\mathcal{M}_0 = ((M_0, K_0), E_0, \mathcal{F}_0)$. A *reduction of singularities of* $\mathcal{M}_0$ is a morphism

$$\pi : \mathcal{M} \to \mathcal{M}_0, \quad \mathcal{M} = ((M, K), E, \mathcal{F}),$$

obtained by the composition of a finite sequence of blowing-ups with invariant centers, where $\mathcal{M}$ is a simple foliated space.

There are results of existence of reduction of singularities when the ambient space dimension is equal to 2 (see [32]), or equal to 3 (see [7]). There are other special cases in higher dimension having reduction of singularities (for instance, the Newton non-degenerate cases [26]). Anyway, the existence of reduction of singularities for ambient dimension bigger or equal than 4 is an open problem.

Let us focus our attention in the case that the ambient space is $(\mathbb{C}^n, 0)$ and assume that we have a reduction of singularities

$$\pi : \mathcal{M} = ((M, K), E, \mathcal{F}) \to ((\mathbb{C}^n, 0), \emptyset, \mathcal{F}_0).$$

In this situation, the problem of finding an invariant hypersurface for $\mathcal{F}_0$ is reduced to detect a complete extended partial separatrix for $\mathcal{M}$, such that the corresponding hypersurface is convergent.

## 5.7 Invariant Curves in Dimension Two

Let us consider a two-dimensional foliated space $\mathcal{M}_0 = ((\mathbb{C}^2, 0), \emptyset, \mathcal{F}_0)$. The classical result of Seidenberg [32], implies the existence of a reduction of singularities

$$\pi : \mathcal{M} = ((M, K), E, \mathcal{F}) \to ((\mathbb{C}^2, 0), \emptyset, \mathcal{F}_0), \tag{5.22}$$

where $\pi$ is a finite composition of blowing-ups centered at points. The exceptional divisor $E = \pi^{-1}(0)$ is a union of compact components, each one isomorphic to a complex projective line. Two components meet at most at a single point.

If there is some dicritical component, say $D$, we get infinitely many convergent invariant curves just by projecting the transverse curves to $D$. Indeed, any given point $P$ in $D$ is necessarily a non-singular point of $\mathcal{F}$ and then, the invariant curve through $P$ is transverse to $D$. Moreover, since $\mathcal{F}$ has normal crossings at the points of $D$, the dicritical component $D$ does not meet any other dicritical component.

Hence, in dimension two, the problem of existence of convergent invariant curve concentrates the difficulty in the non-dicritical case: when $E = E_{\text{inv}}$. This problem was solved by C. Camacho and P. Sad in [4]. The result may be stated as follows:

**Theorem 5.7.1 (Camacho-Sad)** *Assume that the reduction of singularities $\pi$ of Eq. (5.22) does not have any dicritical component. Then, there is at least one simple tangent trace point $P \in E$.*

Recall, in view of Proposition 5.5.3, that the invariant curve obtained at a tangent trace point is convergent. Then, as a consequence of the previous theorem, we can project the invariant curve at the tangent trace point in order to obtain the desired invariant curve for $\mathcal{F}_0$.

The proof of Camacho-Sad theorem is based on a control of the behaviour under blowing-ups of an invariant called *Camacho-Sad Index*. For a short proof, the reader may see [9, 16].

There is a refined version of Camacho-Sad theorem established in [3, 28], that implies the following statement:

**Proposition 5.7.2** *There is at least one tangent trace point in each (non-empty) connected component of $E_{inv}$.*

In [11], the authors find similar results on the distribution of invariant hypersurfaces in higher dimension, concerning the non-dicritical case.

## 5.8 Dicriticalness and Jouanolou's Examples

As we have seen, when the ambient space has dimension two, there are always invariant curves for a germ of foliation on $(\mathbb{C}^2, 0)$. Recall that we find essentially two different situations:

(a) The dicritical case. That is, there is a generically transversal irreducible component of the exceptional divisor after a reduction of singularities. Here the argument is obvious, by taking one of the infinitely many transversal invariant curves to the dicritical component.
(b) The non-dicritical case. That is, all the irreducible components of the exceptional divisor after a reduction of singularities are invariant for the transformed foliation. In this situation, we need to apply the argument of Camacho-Sad to find a simple tangent trace point defining a convergent invariant curve transversal to the exceptional divisor. This curve is projected over an invariant curve of the original germ of foliation.

In higher ambient dimension, the "dicritical" situation may give an obstruction to the existence of invariant hypersurface. This fact is visible in the example given by Jouanolou in [21], which we present here. Consider the "conic" foliation $\mathcal{F}_0$ on $(\mathbb{C}^3, 0)$ defined by the differential 1-form

$$\omega = (zx - y^2)dx + (xy - z^2)dy + (yz - x^2)dz.$$

By conic we mean that it is tangent to the radial vector field $x\partial/\partial x + y\partial/\partial y + z\partial/\partial z$. Let us perform the blowing-up of the origin

$$\sigma : (\widetilde{(\mathbb{C}^3, 0)}, E_0) \to (\mathbb{C}^3, 0), \quad E_0 = \sigma^{-1}(0) = \mathbb{P}^2_{\mathbb{C}}.$$

The exceptional divisor $E_0$ is dicritical, then, the restriction $\mathcal{G}$ of the transform of $\mathcal{F}_0$ to $E_0$ gives a foliation on the complex projective plane. This foliation $\mathcal{G}$ has no algebraic invariant curve, hence Jouanolou's foliation has not a germ of invariant surface (nor a formal one). Indeed, the tangent cone of a potential invariant surface should define an algebraic invariant curve for $\mathcal{G}$. Let us consider a reduction of singularities

$$\pi : ((M, K), E, \mathcal{F}) \to ((\mathbb{C}^3, 0), \emptyset, \mathcal{F}_0),$$

obtained by performing seven additional blowing-ups centered at the seven lines of singularities transverse to $E_0$. Denote by $E_i$ each one of the respectively exceptional divisors, and let $\tilde{E}_0$ be the strict transform of $E_0$ by $\pi$. The new divisor $E$ is then given by the union

$$E = \tilde{E}_0 \cup \bigcup_{i=1}^{7} E_i$$

and $K = \pi^{-1}(0) = \tilde{E}_0 = E_{\text{dic}}$. There are no compact invariant curves contained in the only compact component $\tilde{E}_0$ of $E$. Then, there are no extended partial separatrices for the foliated space $((M, K), E, \mathcal{F})$.

### 5.8.1 Dicritical Foliations

The concept of dicritical foliation, that we introduce below, is given by making a two-dimensional test.

**Definition 5.8.1** Let $\mathcal{F}$ be a foliation on a complex analytic space $M$ and $P$ a point in $M$. We say that $\mathcal{F}$ is *dicritical at* $P$ if there is a map $\phi : (\mathbb{C}^2, 0) \to (M, P)$, such that $\phi^{-1}\mathcal{F} = (dx = 0)$ and the subspace $\phi(y = 0)$ is invariant for $\mathcal{F}$.

In ambient dimensions two and three, this definition is equivalent to the fact that there is a generically transversal component of the exceptional divisor after a morphism of reduction of singularities.

Let us note at this point that any germ of foliation having a holomorphic first integral is non-dicritical. Indeed, consider the foliation $\mathcal{F}$ given by $df = 0$ and assume that it is dicritical, that is, there is a map $\phi : (\mathbb{C}^2, 0) \to (M, P)$ as in the previous definition. The pull-back of $\mathcal{F}$ by $\phi$ is given by $d(f \circ \phi) = 0$. Since it is the foliation $dx = 0$, it means that the function $f \circ \phi$ is of the form $f \circ \phi = \psi(x)$. Now the fact that $\phi(y = 0)$ is invariant means that $\psi$ is a constant function, contradiction.

## 5.9 Invariant Hypersurfaces for Non-dicritical Foliations

The general positive answer to Thom's question about the existence of invariant hypersurfaces in the non-dicritical case is stated as follows:

**Theorem 5.9.1** *Let $\mathcal{F}_0$ be a germ of non-dicritical foliation on $(\mathbb{C}^n, 0)$. Then, there is a germ $(H, 0) \subset (\mathbb{C}^n, 0)$ of analytic hypersurface invariant for $\mathcal{F}_0$.*

The proof of this theorem is given by Camacho-Sad, when $n = 2$ (see [4]), as we have seen. It is due to Cano-Cerveau, for the case $n = 3$ (see [8]) and to Cano-Mattei, for the case $n \geq 4$ (see [10]).

Let us point out the main ideas in the proof, for the case $n = 3$. The most important tool is the existence of a reduction of singularities. In [6] and [8], the authors prove the existence of reduction of singularities for non-dicritical germs of foliation on $(\mathbb{C}^3, 0)$. Consider a non-dicritical germ of foliation $\mathcal{F}_0$ over $(\mathbb{C}^3, 0)$ and take a reduction of singularities

$$\pi : \mathcal{M} = ((M, K), E, \mathcal{F}) \to ((\mathbb{C}^3, 0), \emptyset, \mathcal{F}_0), \quad K = \pi^{-1}(0).$$

Since $\mathcal{F}_0$ is non-dicritical, we have that $E = E_{\text{inv}}$, that is, there are no dicritical components in $E$. In this situation, the extended partial separatrices coincide with the partial separatrices. Moreover, all the partial separatrices are complete. Then, the only thing we have to do is to detect a partial separatrix $S$ such that $H_S$ is convergent. Recall that the convergence can be read at any generic point of $S$ and that a generic point of $S$ has dimensional type equal to two. In order to detect this kind of partial separatrices, we apply Camacho-Sad Theorem to a two-dimensional transversal section of $\mathcal{F}_0$. Then, we show that the convergent curve given by this result lifts to a convergent curve in a generic point of a partial separatrix. In this way, we obtain the convergent invariant hypersurface by taking the projection $\pi(H_S)$.

The case $n \geq 4$, proved in [10], is reduced to the three-dimension case as follows. By taking a three-dimensional transversal section $\Delta$, we obtain a surface $H_\Delta \subset \Delta$ through the origin, invariant for $\mathcal{F}|_\Delta$. By means of equidesingularization arguments, we see that $H_\Delta$ extends locally to hypersurfaces $H_P$, invariant for $\mathcal{F}$, locally at the points $P$ not belonging to a given subset of codimension three. These local extensions allow us to build a hypersurface $\tilde{H}$, invariant for $\mathcal{F}$, in the restriction to a poly-annulus of $\mathbb{C}^n$, whose intersection with $\Delta$ is contained in $H_\Delta$. By cohomological triviality, the hypersurface $\tilde{H}$ extends to a hypersurface $H$, invariant for $\mathcal{F}$, of the whole ambient space. We see that $H$ passes through the origin, since it is "guided" by $H_\Delta$.

## 5.10 Extensions to Dicritical Situations (Dimension Three)

We restrict ourselves in this section to three-dimensional ambient spaces. We are going to describe some situations where the existence of invariant hypersurface is assured for a germ of foliation $\mathcal{F}_0$ in $(\mathbb{C}^3, 0)$.

We present here two criteria of completeness for extended partial separatrices and we apply them to find invariant hypersurfaces.

### *5.10.1 Rational First Integrals in Dicritical Components*

Let us consider a simple foliated space $\mathcal{M} = ((M, K), E, \mathcal{F})$ of dimension three. The following criterion is essentially the same one used in [29]:

**Proposition 5.10.1** *Assume that the restriction $\mathcal{F}|_D$ has a rational first integral, for each compact dicritical component $D$ of $E$. Then every extended partial separatrix is complete.*

***Proof*** Let $\mathcal{E}$ be an extended partial separatrix and take a point $P$ in $\mathcal{E}$ contained in a compact irreducible component $D^*$ of $E$. Assume that $D^*$ is an invariant component. Then, there is a trace compact curve $Y$ in $\mathrm{Tr}^c\,\mathcal{M}$ with $P \in Y \subset D^*$, thus $Y \subset \mathcal{E}$. Assume now that $D^*$ is a dicritical component. Then, there is a unique invariant branch $(\Gamma, P) \subset (D^*, P)$ invariant for $\mathcal{F}|_{D^*}$ and not contained in any other component of $E$. Since $\mathcal{F}|_{D^*}$ has a rational first integral, the branch $(\Gamma, P)$ extends to a compact trace curve $Y \subset D^*$. Hence $Y \subset \mathcal{E}$ and we are done. □

Note that the existence of extended partial separatrix can fail when two compact dicritical components meet and it appears an infinite chain of invariant trace curves of these two components. This phenomenon breaks the finiteness needed in the definition of extended partial separatrix. To avoid this situation, we can ask that two dicritical components never meet. Then, we can state the following result:

**Proposition 5.10.2** *Assume that the restriction $\mathcal{F}|_D$ has a rational first integral, for each compact dicritical component $D$ of $E$. Suppose also that two compact dicritical components never meet. Then, each trace point in $E$ is contained in a complete extended partial separatrix.*

**Corollary 5.10.3** *Assume that $\pi : \mathcal{M} \to ((\mathbb{C}^3, 0), \emptyset, \mathcal{F}_0)$ is a reduction of singularities and that $\mathcal{M}$ is a simple foliated space satisfying the hypothesis of the proposition. Then $\mathcal{F}_0$ has a convergent invariant surface.*

***Proof*** Similarly to the non-dicritical case, it is enough to make a two-dimensional transversal section for $\mathcal{F}_0$ and to lift a convergent invariant curve. In this way, we detect a complete extended partial separatrix $\mathcal{E}$ such that $H_{\mathcal{E}}$ converges. □

### *5.10.2 Prolongation of Isolated Branches*

The property of prolongation for isolated branches has been studied in [27]. In order to introduce it, we need to develop first the concept of isolated branches for two-dimensional foliated spaces. This concept can be implicitly found in [2].

Let $\mathcal{X} = (X, D, \mathcal{G})$ be a foliated surface. Consider a (maybe formal) branch of curve $(\Gamma, P)$ in $(X, P)$, not contained in $D$. We say that $(\Gamma, P)$ is an *isolated branch for $\mathcal{X}$* when the following statement holds:

> Given a composition $\sigma : \mathcal{X}' = (X', D', \mathcal{G}') \to \mathcal{X} = (X, D, \mathcal{G})$ of a finite sequence of blowing-ups, the strict transform of the branch $(\Gamma, P)$ passes through a non-simple regular point for $\mathcal{X}'$ or a singular point of $\mathcal{G}'$, that is, passes through a point belonging to $\mathrm{Sing}(\mathcal{G}', D')$.

**Remark 5.10.4** An isolated branch for $\mathcal{X}$ is, in particular, an invariant branch of $\mathcal{G}$.

**Definition 5.10.5** The foliated surface $\mathcal{X}$ has the property of *prolongation for isolated branches* if, for each isolated branch $(\Gamma, P)$, the following properties hold:

1. There is a (unique) closed irreducible curve $Y \subset X$ extending $(\Gamma, P)$.
2. Moreover, all the branches $(\Upsilon, Q) \subset (Y, Q)$ are isolated, for each $Q \in Y \cap D$.

If all the points in $X$ are simple points for $\mathcal{X}$, there is a bijection between trace type singularities and isolated branches. Indeed, we associate to each singular trace point $P$ the only invariant branch $(\Gamma_P, P)$ through it not contained in $D$. Under this assumption, the following statements are equivalent:

(a) The prolongation property for isolated branches holds for $\mathcal{X}$.
(b) For each trace type (simple) singularity $P$ in $\mathcal{X}$, there is a (unique) closed irreducible curve $Y \subset X$ extending $(\Gamma_P, P)$. Moreover $Y$ does not meet the dicritical part $D_{\mathrm{dic}}$ of $D$.

We take now a foliated space $\mathcal{M} = (M, E, \mathcal{F})$ of dimension three. Given a dicritical component $X$ of $E$, it makes sense to consider the foliated surface $\mathcal{M}|_X$, obtained by restriction of $\mathcal{M}$ to $X$. We say that $\mathcal{M}$ has the *prolongation property for isolated branches* if the property holds in the restriction $\mathcal{M}|_X$, for each dicritical component $X$ of the divisor $E$. This prolongation property for isolated branches assures the completeness of some extended partial separatrices, as we see in the following proposition:

**Proposition 5.10.6 (See [12])** *Let $\mathcal{M} = (M, E, \mathcal{F})$ be a simple three-dimensional foliated space satisfying the prolongation property for isolated branches. Then, any extended partial separatrix containing singularities of $\mathcal{F}$ is complete.*

### *5.10.3 Invariant Surfaces for Toric Type Foliations*

Here we present the class of toric type foliations, introduced in [5, 12]. We show the existence of invariant hypersurface for this family of foliations.

Let $\mathcal{F}$ be a foliation defined on a complex analytic space $M$. We say that $\mathcal{F}$ is a *toric type foliation* if there are a normal crossings divisor $E$ on $M$ and a reduction of singularities morphism

$$\pi : (M', E', \mathcal{F}') \to (M, E, \mathcal{F})$$

given by composition of a finite sequence of combinatorial blowing-ups. Roughly speaking, by combinatorial blowing-up we mean that it is centered at an intersection of components of the corresponding divisor.

We say that the toric type foliation $\mathcal{F}$ is *complex hyperbolic* when the reduction of singularities $\pi$ satisfies that all the points in $(M', E', \mathcal{F}')$ are complex hyperbolic. Under this assumption, we have that any formal invariant surface converges.

**Theorem 5.10.7 (See [12])** *Every complex hyperbolic toric type foliation on $(\mathbb{C}^3, 0)$ has an invariant surface.*

Let us give some ideas on the proof. Take a germ of complex hyperbolic foliation $\mathcal{F}_0$ on $(\mathbb{C}^3, 0)$. Assume that $\mathcal{F}_0$ is of toric type with respect to a normal crossings divisor $E_0$ and fix a combinatorial reduction of singularities

$$\pi : \mathcal{M} = ((M, \sigma^{-1}(0)), E, \mathcal{F}) \to ((\mathbb{C}^3, 0), E_0, \mathcal{F}_0). \tag{5.23}$$

Note that, if $E_0$ has some invariant component, we are trivially done. Thus, we assume that $E_0$ has only dicritical components. We remark also that $E_0$ must have three components, in order to be allowed to blow-up the origin. Recall that, in this situation, we assure that $\pi^{-1}(0)$ is a union of components of $E$. In Proposition 5.10.8, the existence of a complete extended partial separatrix $\mathcal{E}$ is assured. The associated hypersurface $H_{\mathcal{E}}$ converges in view of the complex hyperbolicity of $\mathcal{F}_0$. We finish by taking the hypersurface $\pi(H_{\mathcal{E}})$, that is invariant for $\mathcal{F}_0$.

**Proposition 5.10.8** *The next statements hold for the foliated space* $\mathcal{M} = (M, E; \mathcal{F})$.

*(a)* $\mathcal{M}$ *has the prolongation property for isolated branches.*
*(b) There is at least one trace type singularity for* $\mathcal{M}$.

Part (a) follows by seeing that every compact component $X$ of $E$ is a non-singular projective toric surface. The restriction $E|_X$ is recovered in a natural way as the union of non-dense orbits of the torus action on $X$. The study done in [27] for foliated spaces defined in toric surfaces is applied to assure that $\mathcal{M}$ has the prolongation property for isolated branches. The main tool to prove part (b) is the refined version of Camacho-Sad theorem stated as in Proposition 5.7.2.

**Remark 5.10.9** In the definition of simple foliated space given in this text, we have asked the germification compact $K$ to be a union of irreducible components of the divisor $E$. This property is not strictly necessary in many of the results presented here, for instance the ones that can be found in the paper [12].

## 5.11 Local Brunella's Alternative

A natural question is to ask for a property of the germs of foliation in $(\mathbb{C}^3, 0)$ without invariant surface that represents in some sense the "limit" to the transcendence of solutions imposed by the analyticity of the foliation. A similar phenomenon has been described by M. Brunella for foliations of $\mathbb{P}^3_{\mathbb{C}}$ (see [17]). A local way for presenting this question is as follows:

> A germ of foliation $\mathcal{F}$ in $(\mathbb{C}^3, 0)$ without invariant hypersurface satisfies that all the "leaves" contain a germ of analytic curve at the origin.

This statement has a positive answer for certain classes of foliations [13, 14], but the general question is open.

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