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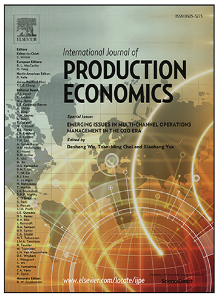

A sustainable inventory model for deteriorating items with power demand and full backlogging under a carbon emission tax

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A sustainable inventory model for deteriorating items with power demand and full backlogging under a carbon emission tax

Luis A. San-José[*] Joaquín Sicilia[†] Leopoldo Eduardo Cárdenas-Barrón[‡]

Manuel González-de-la-Rosa[§]

Second Revised version August 22, 2023

Abstract

Environmental degradation due to carbon emissions occurring in the production, storage and marketing of products has increased notably in recent years. To maintain a sustainable development, it is necessary to penalize commercial activities that generate high carbon emissions. This paper develops and analyzes a sustainable inventory system for a product whose demand follows a power pattern with respect to time. It is considered that the stock items have an estimated life period, after which a percentage of these items begins to deteriorate over time. The inventory system allows shortages which are fully backordered. Several sources of carbon emissions are considered in this article: transportation, stock holding and deterioration. The main objective is to determine the sustainable inventory policy that maximizes the benefit per unit of time, which is given by the difference between the income obtained from sales and the costs associated with inventory management and carbon emissions. Two scenarios are analyzed. In the first, the optimal inventory policy for a system without deterioration is derived. In the second, an algorithm to determine

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the optimal policy for an inventory system with non-instantaneous deterioration is proposed. Thus, our findings serve to determine the best inventory policy that helps decision-makers to obtain the lot size and the reorder point that maximize the profit per unit time under carbon emission taxes in transporting, storage and deteriorating of items. Some numerical examples are solved in order to illustrate the theoretical results previously obtained. Finally, a sensitive analysis of the optimal inventory policy with respect to some input parameters of the system is presented and interesting managerial insights from the numerical examples are proposed.

Keywords: Sustainable inventory; Power Demand Pattern; Non-instantaneous deteriorating; Backlogging; Carbon emissions

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1 Introduction

1.1 Motivation

Over the last few decades, it has been possible to appreciate how the habitability conditions on our planet have gradually been impoverished. Little by little, the air we breathe has been degrading due to the atmospheric pollution from industry, transport and the inadequate conservation of waste generated by humans. At the present time, it is well-known that global warming is a substantial threat to the world and carbon emissions are a major source. Therefore, customers have a greater environmental awareness than before. Many governments and non-governmental organizations have raised their voices indicating the need to seek formulas that allow the sustainable growth of the planet. A good strategy is to impose higher taxes on those companies or factories where the manufacturing or production process is highly polluting, or that carry out poor maintenance of the items in stock needed in the production and supply chain. There should also be a high tax penalty for firms that produce excess contamination in the transportation and distribution of products. From this perspective, it is necessary for organizations around the world to develop and apply inventory models that take into account sustainability issues in order to decrease the carbon emissions associated with their operations. It is imperative to formulate new solutions aimed at adopting responsible practices that mitigate their negative impacts on local and global ecosystems. The purpose is to help promote long-term sustainable development that will preserve the environment for future generations.

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In this work, we address a sustainable inventory management system that considers costs derived from carbon emissions into the atmosphere produced in the transport or shipment of goods, in the incorrect maintenance of products, as well as the disposal of deteriorated items or waste. The main novelty of this study lies in its comprehensive and applied approach. It combines a set of assumptions, including environmental constraints, that affect the current practices associated with inventory activities, providing a more accurate assessment of their effects. In addition, it suggests concrete guidelines for companies to implement optimal and responsible inventory management policies.

1.2 Literature review

In recent years, several articles have been published on sustainable inventory models. Andriolo et al. (2013) raised the convenience of approaching the economic order quantity (EOQ model) from the perspective of sustainability. They argued that there is a necessity for a sustainable inventory management framework. In a subsequent paper, Andriolo et al. (2014) proposed that academics and researchers should build inventory models which include environmental issues, so as to reflect the impact of the amount of greenhouse gases that are emitted to the atmosphere due to the activities carried out during the process of production, delivery and storage. Along these lines, Battini et al. (2014) introduced a sustainable EOQ inventory model which incorporates several sustainability aspects that affect the environment. Hovelaque and Bironneau (2015) stated that, if some adjustments are made to the lot sizes, then the carbon emissions decrease. They developed an EOQ inventory model linking the inventory and carbon emissions policies. They modeled the demand as price and carbon emissions dependent. Taleizadeh et al. (2017) addressed and examined four sustainable inventory models: without shortages, lost sales, partial and full backordering. Battini et al. (2018) developed a bi-objective EOQ model in which costs and emissions are kept separate and analyzed using a Pareto frontier subject to a Cap and Trade mitigation policy. Liao and Deng (2018) formulated a carbon-constrained EOQ model considering that the demand is uncertain. They noted that inbcreasing the carbon tax decreases profits and changes the optimal ordering decisions. Tiwari et al. (2018c) built an inventory model for deteriorating products when some of them are of imperfect quality under carbon emissions. They stated that the inventory model effectively reduces both the costs and the carbon emissions. Wang and Ye (2018) incorporated carbon emissions into the two basic

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inventory models: JIT and EOQ. They mentioned that considering carbon emissions in both the JIT and EOQ inventory models decreases the amount of carbon emissions compared to the case without considering carbon emissions. Yu et al. (2020) derived an inventory model which involves deteriorating items when the processes of ordering and storing of perishable products cause carbon emissions. Mishra et al. (2020) presented an economic production quantity (EPQ) inventory model with carbon tax when the carbon emissions rate can be controlled through investment in green technology. Mishra et al. (2021) examined an EOQ inventory model with shortages and carbon emissions. Ruidas et al. (2021) investigated an imperfect production inventory model, considering that the parameters related to carbon emissions may vary within a certain interval. Mandal et al. (2021) proposed a sustainable stock-dependent inventory model with advertising demand and two progressive periods for delay-in-payments. Taleizadeh et al. (2022) studied an EOQ model by incorporating environmental issues under partial trade credit and partial backordering, in which the demand rate is sensitive to the selling price and to carbon emissions. Kumar et al. (2022) developed an inventory model with a single manufacturer and retailer by assuming that goods that have been remanufactured are as excellent as new items and the cost of carbon emissions is incorporated into the manufacturer's and supplier's holding and degrading costs. An overview of the scientific literature on sustainable inventory management models in the supply chain context up to the year 2021 can be seen in Becerra et al. (2022). Jani et al. (2023) studied a perishable inventory model, with credit predefined duration and shortages from the retailer's perspective, in which demand is determined by the perishable product's quality.

It is well-known that a majority of the stored items available for sale suffer some deterioration over time. Consequently, some of these products cannot be sold due to the fact that they are damaged and this generates an economic loss for the company. One of the first studies of inventory management for articles with a deterioration process is attributed to Ghare and Schrader (1963), who developed an inventory model with known and constant demand and decay rate. Later, Misra (1975) proposed an economic production quantity (EPQ) inventory model considering that goods could be damaged due to a process of deterioration. Then, Shah and Jaiswal (1977) presented an inventory level model for items with a constant rate of deterioration. Subsequently, Aggarwal (1978) corrected the average cost of inventory and modified the proposed policy in the research work of Shah and Jaiswal (1977). A few years later, Dave and Patel (1981) derived an economic order quantity (EOQ)

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inventory model without shortages, with deterioration and time proportional demand. After that, Hollier and Mak (1983) formulated two inventory models with exponentially decreasing demand in which units deteriorate at a constant rate. Raafat (1991) presented a comprehensive review of the inventory models for items with deterioration. In the same direction, and completing the review of Raafat (1991), Goyal and Giri (2001) published a detailed review of the literature related to inventories with deterioration. Then, Lin et al. (2006) optimized the period of the production cycle for an inventory model when articles deteriorate. Li et al. (2010) compiled an interesting review of works on inventory models with product deterioration. Widyadana et al. (2011) introduced an economic order quantity inventory model for items with deterioration and planned shortages; they presented an approximate solution to the inventory problem. More recently, Janssen et al. (2016) presented a review and classification of more than three hundred deteriorated inventory models published between 2012 and 2015. Srivastava and Singh (2017) developed an inventory model for deteriorating items with linear demand, partial backlogging and variable deterioration rate, where the rate of backlogging is variable and dependent on the waiting time for the next replenishment. Sen and Saha (2018) developed an inventory model for deteriorating items with time-dependent holding cost and shortages under permissible delay in payment. Tiwari et al. (2018a) studied a two-echelon inventory model for deteriorating items in which the retailer's warehouse has a limited capacity of display for the products and the demand rate depends on the retailer's selling price and displayed stock level. Tiwari et al. (2018b) analyzed a supplier-retailer-customer supply chain for deteriorating items, assuming a two-level partial trade credit and allowed shortages. This paper considers a non-decreasing deterioration rate over time and the item is fully deteriorated close to its expiry date.

Most of the classic inventory models consider that demand is known and constant. However, constant demand is not usually used today because customer demand is influenced by several factors such as time, price, inventory level and quality, among other reasons. Naddor (1966) was one of the first researchers who suggested the power demand pattern as a good and practical function to adapt consumer demand according to the reality of their behavior in the purchase of items. The power demand pattern allows the demand behavior of different products to be represented and helps to determine the evolution of the inventory level over time. With this type of demand, it is possible to model the following: i) products whose demand remains traditionally constant throughout the whole management period; ii) real-life situations in which the items are highly consumed at the

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beginning of the inventory cycle period and then the inventory decays more smoothly. Products in this category are, for example, cooked foods such as cakes and breads, among others, due to the fact that buyers want these products prepared freshly; and iii) situations in which the products are sold in large quantities at the end of the cycle because they become scarce. For example, basic necessities, such as diesel, gasoline, sugar, water and flour, among others, are products whose demand increases considerably as the stocks start to decrease.

There exist some works that deal with inventory models which assume that demand follows a power demand pattern. For example, Goel and Aggarwal (1981) built an inventory model with power demand pattern, taking into account the fact that products worsen over time with a constant rate of deterioration. Later, Datta and Pal (1988) introduced an inventory system with a power demand pattern in which the items have a variable rate of deterioration. Then, Lee and Wu (2002) formulated an EOQ inventory model for an item that deteriorates with a Weibull distribution rate, considering a power demand pattern. Dye (2004) revisited and extended the inventory model of Lee and Wu (2002), modeling the rate of shortages as proportional to time, with the main idea of having a more complete and applicable inventory model in practice. Singh et al. (2009) constructed an EOQ inventory model with perishable products, partial backordering and a power demand pattern. Singh and Sehgal (2011) studied an EOQ inventory model with a two parameter Weibull deterioration rate, a power demand pattern and shortages. Rajeswari and Vanjikkodi (2011) derived an inventory model for a product that deteriorates, considering the power demand pattern and partial backordering, where the rate of shortage is considered inversely proportional to the waiting time to the next replacement. In a subsequent paper, Rajeswari and Vanjikkodi (2012) examined an inventory model with a power demand pattern which depends on time and shortage, assuming that the products deteriorate with a two-parameter Weibull distribution. Sicilia et al. (2012) provided a detailed study of inventory models for the case where the demand follows a power pattern. They analyzed both the optimal policy when no shortage is allowed and the policy that must be implemented when shortage is permitted. In this last situation, they discussed the inventory models with complete backordering and the situation in which the shortages turn into lost sales. In a subsequent article, Sicilia et al. (2013) introduced an EOQ inventory model to study the optimal replenishment policy, allowing items to deteriorate and in which demand depends on time following a power pattern. They assumed that shortages are not allowed and that the replenishment cycle is not fixed and given; this is, however, a decision variable of the inventory

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problem. Mishra and Singh (2013) presented an EOQ inventory model for perishable products with a quadratic deterioration rate, considering a power demand pattern and shortages. Mandal and Islam (2013) developed a fuzzy inventory model for products that do not deteriorate, assuming a power demand pattern, shortages and inflation. Sicilia et al. (2014a) derived a deterministic inventory model for items with a constant rate of deterioration and permitting shortages. Their inventory model assumes that demand varies over time and follows a power pattern. In the same year, Sicilia et al. (2014b) formulated an EPQ inventory model with a power demand pattern, assuming that the production rate is proportional to the demand rate. In a later article, Sicilia et al. (2015) established the optimal inventory policies for an inventory system without shortages, in which the replenishment rate is uniform and the demand follows a power pattern. Rajeswari et al. (2015) introduced a fuzzy inventory model for items that deteriorate constantly with a power demand pattern, in which shortages are permitted. The rate of shortages is in accordance with a decreasing exponential function of the waiting time. Recently, San-José et al. (2017) addressed an inventory problem with a power demand pattern, permitting shortages. Only one part of the demand that is pending within the time of shortage is covered and the rest of the demand is taken as lost sales. They developed a solution procedure to determine both the optimal batch size and the duration of the inventory cycle. Other more recent works that consider a power demand pattern are the papers of San-José et al. (2018), San-José et al. (2019), San-José et al. (2020) and Khan et al. (2023a,b,c,d), which assume that demand also depends on the selling price of the product.

In the research works previously mentioned on inventory models for products susceptible to deterioration, it is generally assumed that the deterioration process begins from the moment in which the items are stored in the inventory. This hypothesis is usually not true in practice, since the products have a period of life where they remain perfect and therefore do not suffer any deterioration. This type of evolution is known as items with a non-instantaneous deterioration process. Several researchers have developed inventory models for products with this deterioration pattern. Thus, Wu et al. (2006) analyzed an inventory model for non-instantaneous deteriorating items with stock-dependent demand and partial backlogging. Ouyang et al. (2006) developed the optimal replenishment policy for non-instantaneous deteriorating items with permissible delay in payments. Sugapriya and Jeyaraman (2008) studied an economic production quantity model for non-instantaneous deteriorating items in which the holding cost varies with time. Chang and Lin (2010) developed a partial backlogging inventory

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model for non-instantaneous deteriorating items with stock-dependent demand rate and inflation over a finite planning horizon. Soni (2013) studied an inventory system for non-instantaneous deteriorating items with price-and-stock-dependent demand considering permissible delay in payment. Kaur et al. (2013) proposed the optimal replenishing policy for a two-warehouse inventory model of non-instantaneous deteriorating items under stock-dependent demand where no shortage is allowed. Tat et al. (2013) analyzed the optimal inventory policy for non-instantaneous deteriorating products in vendor-managed inventory systems. Maihami and Karimi (2014) analyzed the best pricing and optimal ordering policy for non-instantaneous deteriorating items with stochastic demand, considering promotional efforts. Wu et al. (2014) developed an inventory model for non-instantaneous deteriorating items with price and stock-sensitive demand under permissible delay in payment. Vandana and Sharma (2016) proposed the inventory policy for non-instantaneous deteriorating items over a quadratic demand rate with permissible delay in payments and time-dependent deterioration rate. Rangarajan and Karthikeyan (2017) developed an inventory model for non-instantaneous deteriorating items with cubic demand rate and cubic deterioration rate, where shortages are partially backlogged for the next replenishment cycle. Pal and Samanta (2018) studied the optimal inventory policy for non-instantaneous deteriorating items with a random pre-deterioration period, where no shortages are allowed and demand occurs uniformly, but at different rates during pre- and post-deterioration periods. Shah and Naik (2018) developed an inventory model with time and price-sensitive demand for non-instantaneous deteriorating items, including the learning effect on various costs and the preservation technology investment to reduce the deterioration rate.

Table 1 displays a list of selected papers that have been published since 2015. In this table, we show the differences between this paper and the related literature, reflecting the gap with respect to previous research. The papers are categorized considering demand pattern type, whether backorders are allowed or not, if there is a deterioration process of the items, and if there exist costs for carbon emissions due to transport, storage or deterioration.

Following this research line, this work develops a sustainable inventory model for products that meet the characteristic of non-instantaneous deterioration, that is, items begin to deteriorate after a certain period of time in the inventory. It is also assumed that the demand follows a power demand pattern and that shortages, which are met with the arrival of the next replenishment of products, are allowed. Several sources of carbon emissions

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Table 1. Summary of selected literature from the year 2015

Authors	Demand pattern	Backlogging	Deterioration	Carbon emissions in transporting	Carbon emissions in stocking	Carbon emissions due to deterioration
Battini et al. (2018)	Constant	No	No	Yes	Yes	—
Hovelaque and Bironneau (2015)	Price-and-CO2-dependent	No	No	Yes	Yes	—
Jani et al. (2023)	Quality-dependent	Yes	Yes	No	No	No
Khan et al. (2023a)	Price-and-time-dependent	No	No	No	No	—
Khan et al. (2023b)	Price-and-time-dependent	Yes	No	No	No	—
Khan et al. (2023c)	Price-and-time-dependent	Yes	No	No	No	—
Khan et al. (2023d)	Price-and-time-dependent	No	No	Yes	Yes	—
Kumar et al. (2022)	Advertisement-dependent	No	Yes	—	Yes	Yes
Liao and Deng (2018)	Stochastic	No	No	Yes	Yes	—
Mandal et al. (2021)	Stock-dependent	No	Yes	No	No	No
Mishra et al. (2020)	Constant	Yes	No	—	Yes	—
Mishra et al. (2021)	Price-dependent	Yes	Yes	Yes	Yes	Yes
Pal and Samanta (2018)	Constant	No	Yes	No	No	No
Rajeswari et al. (2015)	Power-time-dependent	Yes	Yes	No	No	No
Rangarajan and Karthikeyan (2017)	Time-dependent	Yes	Yes	No	No	No
Ruidas et al. (2021)	Price-dependent	No	No	Yes	Yes	—
San-José et al. (2017)	Power-time-dependent	Yes	No	No	No	—
San-José et al. (2018)	Price-and-time-dependent	Yes	No	No	No	—
San-José et al. (2019)	Power and price-dependent	No	No	No	No	—
San-José et al. (2020)	Power and price-dependent	Yes	No	No	No	—
Sen and Saha (2018)	Time-dependent	Yes	Yes	No	No	No
Shah and Naik (2018)	Price-and-time-dependent	Yes	Yes	No	No	No
Sicilia et al. (2015)	Power-time-dependent	No	No	No	No	—
Srivastava and Singh (2017)	Time-dependent	Yes	Yes	No	No	No
Taleizadeh et al. (2017)	Constant	Yes	No	Yes	Yes	—
Taleizadeh et al. (2022)	Price-and-emissions-dependent	Yes	No	Yes	Yes	—
Tiwari et al. (2018a)	Price-and-stock-dependent	Yes	Yes	No	No	No
Tiwari et al. (2018b)	Price-dependent	Yes	Yes	No	No	No
Tiwari et al. (2018c)	Constant	No	Yes	Yes	Yes	Yes
Vandana and Sharma (2016)	Time-dependent	Yes	Yes	No	No	No
Wang and Ye (2018)	Constant	No	No	Yes	Yes	—
Yu et al. (2020)	Price-and-stock-dependent	Yes	Yes	Yes	Yes	No
This paper	Power-time-dependent	Yes	Yes	Yes	Yes	Yes

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are considered in the paper. Thus, transportation, stock holding and deterioration can produce environmental degradation and the related costs must be incorporated into the formulation of the inventory model.

1.3 Contribution of this study

The sustainable approach to our inventory management model of deteriorating items that follow a power demand pattern can help reduce the environmental impact and improve profitability and efficiency in the supply chain. This can be potentially beneficial for businesses, consumers, and the planet. In addition, its results may be relevant and valuable for researchers and practitioners interested in inventory systems.

Following the research lines previously commented, the main contributions of this work to the literature on inventory models are:

(a) It describes the displayed stock behavior when demand is power dependent on time, which represents real-life situations. This power demand pattern includes the inventory system with a constant demand rate, as well as other consumer behaviors in which the requested quantity starts off low and increases over time; or conversely, where the demand is initially high and gradually tapers off during the inventory cycle.

(b) It considers a process of deterioration of the elements to reduce the amount of waste generated. It also offers solutions for non-instantaneous deterioration of the stock. Thus, it helps companies to minimize the risk of inventory management due to obsolescence, facilitating the sale of items before they lose their quality and value, since they have a limited useful life and can spoil or expire if they are not sold or consumed by a certain date. This in turn can improve customer satisfaction and loyalty.

(c) Shortages are allowed and backlogged, satisfying all customer demands, although the requests of some clients can be fulfilled with delay.

(d) It incorporates environmental constraints that allow inventory management in a sustainable way. Thus, it considers several sources of carbon emissions: transportation, stock holding, and deterioration.

  - It includes taxes on transportation, maintenance and deterioration processes that are highly polluting.
  - It contributes to generating a positive impact on the environment by promoting the reduction of greenhouse gas emissions, or the carbon footprint associated with product maintenance and transportation.

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- It takes into account the effect of carbon taxes, reflecting the growing concern over the environmental impact of companies and their social responsibility.

(e) It presents an algorithmic procedure to determine the optimal inventory policy that maximizes the profit per unit time of the retailer, that is, the difference between the income obtained from sales and the costs associated with inventory management and carbon emissions.

(f) The results obtained are applicable to various economic sectors, such as the food, pharmaceutical, electronics, and chemical industries, among others, which trade in a wide range of perishable products.

(g) It is in line with the Sustainable Development Goal 12 of the UN 2030 Agenda (United Nations, 2015), which aims to achieve sustainable consumption and production patterns.

To the best of our knowledge, there is no published model that determines the best policy for an inventory system with a power demand pattern, full backlogging and non-instantaneous deterioration, considering three sources of carbon emissions.

The rest of this paper is as follows. Section 2 provides the properties and assumptions that characterize the sustainable inventory system. Section 3 formulates the mathematical model of the inventory system. Section 4 presents the necessary conditions that must be satisfied to obtain the optimal inventory policy and proposes an algorithm to determine the best inventory policy. Section 5 solves some numerical examples. Section 6 presents a sensitivity analysis of the optimal inventory policy with respect to several input parameters of the system and derives some useful managerial insights. Finally, Section 7 gives relevant conclusions and potential future research work.

2 Notation

Table 2 shows the notation used for the development of the inventory model.

3 Assumptions

The inventory model has the following assumptions:

1. The inventory system considers a single product.

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Table 2 Notation

Parameters	
k_0	fixed shipment cost
k_1	shipment cost per transported unit
τ	life period of the item
θ	deterioration rate
α_0	fixed carbon emissions in transporting
α_1	variable carbon emissions in transporting
β_0	fixed carbon emissions in holding
β_1	carbon emission per unit held in stock and per unit of time
γ	carbon kilogram emission per deteriorated unit
c	unit purchasing cost
p	unit selling price
h	unit holding cost per time unit
ω	unit backordering cost
A	ordering cost
v	unit deteriorating cost
μ_1	tax charged on carbon emissions in transporting (\$/per carbon kilogram emission)
μ_2	tax charged on carbon emissions in storage (\$/per carbon kilogram emission)
μ_3	tax charged on carbon emissions in deteriorating (\$/per carbon kilogram emission)
r	average demand per cycle
n	index of demand pattern
Variables and functions	
T	length of inventory cycle (*decision variable*) (> 0)
S	initial inventory level
t_1	instant in which the inventory runs out of stock (*decision variable*) $(0 \leq t_1 \leq T)$
B	maximum number of backorders. Thus, the order level or replacement level is $-B$
Q	replenishment or lot size
U	number of units deteriorated during an inventory cycle
$D(t)$	demand rate at time t
$I(t)$	inventory level at time t
$C(t_1, T)$	total cost per unit time
$C_i(t_1, T)$	total cost per unit time for scenario i, with $i = 1, 2$
$C_3(T)$	total cost per unit time for scenario 2 when $t_1 = T$
$C_4(T)$	total cost per unit time for scenario 2 when $t_1 = \tau$
$\pi(t_1, T)$	benefit function per unit time

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2. The inventory cycle or planning period T is a decision variable.
3. The behavior of the inventory level during a period is repeated later in successive periods.
4. At the beginning of the inventory cycle, the stock of the product is replenished up to the level of S units.
5. The time that occurs from when the order is placed until it is delivered to the inventory is insignificant.
6. The instant t_1, in which the inventory runs out of stock, is a decision variable.
7. Shortages are allowed and these are fully backordered at the beginning of the next period.
8. When the total shortages reach the amount of B units, the inventory must be replenished.
9. Replenishment time is considered instantaneous.
10. The size of the replenishment or lot size Q is constant, but it is not known and must be determined by the inventory model.
11. The cost of shipment includes a fixed cost k_0 and a variable cost $k_1 Q$.
12. It is considered that there is a period of time τ in which the articles do not suffer any deterioration.
13. After the period $[0, \tau]$, a fraction θ of the units in stock starts to deteriorate.
14. Deteriorated units cannot be repaired.
15. The carbon emissions for transporting an order of Q units is the sum of a fixed amount α_0 plus a variable amount $\alpha_1 Q$.
16. The carbon emissions in the inventory are represented by a fixed part β_0 plus an amount β_1 multiplied by the average amount held in stock.
17. The removal of deteriorated items or waste disposal is assumed to be a source of air carbon emissions.
18. There are taxes charged on carbon emissions, depending on how they are generated. Thus, it is considered that μ_1 is the tax charged on carbon emissions in transport, μ_2 is the tax charged on carbon emissions in stock holding, and μ_3 is the tax charged on carbon emissions in the deterioration process of the item.
19. Demand of the product is deterministic at a ratio of r units per inventory cycle T, but the way in which the units are taken from the inventory in order to satisfy the demand of the clients depends on the time when these are requested. Thus, let $f(t)$ be the demand function of the product until time t $(0 \leq t \leq T)$. This demand varies with time and it is assumed that it has the following mathematical expression

$$f(t) = rT \left(\frac{t}{T} \right)^{1/n} \tag{1}$$

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where n is the index of the demand pattern, with $n > 0$. Thus, the demand ratio $D(t)$ at time t $(0 < t < T)$ is given by

$$D(t) = \frac{rt^{1/n-1}}{nT^{1/n-1}} \tag{2}$$

This form of demand is known as a power demand pattern (see Naddor, 1966; Datta and Pal, 1988; Lee and Wu, 2002; Sicilia et al., 2012, 2013, 2014b; and San-José et al., 2017).

There is a large group of products that follow the assumptions supposed in this inventory system. For example, products such as pastries, cakes, sweets, breads, and freshly prepared meals, among others, reflect the common characteristic that they have a greater demand at the beginning of the period, since they are fresh products attracting a greater number of customers. Fish, vegetables, fruit, and yoghurts, among others, are also part of this group, as they can deteriorate over time and their sales decrease considerably when the expiry date approaches. This situation is represented in the power demand pattern with a demand pattern index $n > 1$.

However, there are items whose demand increases as the end of the inventory cycle. Thus, for example, basic household products, such as bottled water, coffee, milk, flour, and sugar, among others, increase their demand at the end of the inventory cycle. They are products of first necessity, and when customers notice that there are few products displayed for sale, demand rises and the stock level decreases considerably. In addition, petroleum products such as gasoline or diesel fuel increase their demand when the stored product begins to be scarce. This situation can be modeled by the power demand pattern with an index $n < 1$.

Finally, there are also other types of product where demand remains more or less uniform throughout the whole inventory cycle (which is represented in the power demand pattern with an index of $n = 1$). For example, items such as construction materials, furniture, decoration products, electrical components, cleaning products, and kitchen utensils, among others, usually have a stable demand. These products are not basic or fundamental, so their demand does not change excessively over time.

4 Development of the inventory model

Note that, if items have a life period τ longer than or equal to the length of the inventory cycle T, then the deterioration process of products does not affect the stored products. Thus, in this situation, the inventory

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system to be analyzed does not consider the deterioration of products. Therefore, in the rest of this section, we consider that the life period τ of the items is less than the inventory cycle T of the system.

The behavior of the inventory level is described as follows. Denote by $I(t)$ the level of net stock in inventory at time t, with $0 \leq t \leq T$. The inventory level begins with $I(0) = S$ units in stock. Then, two different scenarios or situations can occur, depending on whether the period of time τ is greater than or equal to the period t_1 in which there is stock in the inventory system, or whether τ is less than t_1.

Case 1. Suppose that $\tau \geq t_1$, this means that the demand is able to absorb all the stock stored in the inventory system before the products start to deteriorate. In that case, there is no deterioration in the articles and the differential equation that governs the behavior of the inventory level of the system is given by

$$\frac{dI(t)}{dt} = -\frac{rt^{1/n-1}}{nT^{1/n-1}},\ 0 < t < T \tag{3}$$

Solving the above equation, the function that describes the evolution of the inventory level is obtained:

$$I(t) = S - \frac{rt^{1/n}}{T^{1/n-1}},\ 0 \leq t \leq T \tag{4}$$

In this scenario, the maximum number of backorders is $B = -I(T) = -S + rT$ and the lot size is $Q = S + B = rT$.

In this case, the holding cost HC_1 per unit of time is calculated as follows

$$HC_1 = \frac{h}{T}\int_0^{t_1} I(t)dt = \frac{h}{T}\int_0^{t_1}\left(S - \frac{rt^{1/n}}{T^{1/n-1}}\right)dt = \frac{h}{T}\left(St_1 - \frac{nrt_1^{1/n+1}}{(n+1)T^{1/n-1}}\right) \tag{5}$$

The order cost OC_1 per unit of time is computed with the quotient A/T. The shipping cost SC_1 per unit of time is $k_0/T + k_1 r$. The backlogging cost BC_1 per unit of time is calculated by the following formula

$$BC_1 = \frac{\omega}{T}\int_{t_1}^{T}\left[-I(t)\right]dt = \frac{\omega}{T}\int_{t_1}^{T}\left(\frac{rt^{1/n}}{T^{1/n-1}} - S\right)dt = \frac{\omega}{T}\left[S(t_1 - T) + \frac{nr\left(T^{1/n+1} - t_1^{1/n+1}\right)}{(n+1)T^{1/n-1}}\right] \tag{6}$$

Since $I(t_1) = 0$, and from equation (4), we obtain that the initial stock level is

$$S = \frac{rt_1^{1/n}}{T^{1/n-1}},\ \text{if } t_1 \leq \tau \tag{7}$$

Substituting S into equations (5) and (6), it follows that the holding cost per unit of time is given by

$$HC_1 = \frac{h}{T}\int_0^{t_1} I(t)dt = \frac{hrt_1^{1/n+1}}{(n+1)T^{1/n}} \tag{8}$$

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and the shortage or backlogging cost per unit of time is computed as follows

$$\begin{aligned}BC_1 &= \frac{\omega}{T}\int_{t_1}^{T}[-I(t)]\,dt = \frac{\omega}{T}\left[\frac{rt_1^{1/n}}{T^{1/n-1}}(t_1-T)+\frac{nr\left(T^{1/n+1}-t_1^{1/n+1}\right)}{(n+1)T^{1/n-1}}\right]\\ &= \frac{\omega r}{T^{1/n}}\left[-t_1^{1/n}T+\frac{nT^{1/n+1}}{n+1}+\frac{t_1^{1/n+1}}{n+1}\right] \qquad (9)\end{aligned}$$

In this case, the carbon emissions occur in transport and inventory holding. Hence, the carbon emissions in transporting Q units are $\alpha_0+\alpha_1 Q=\alpha_0+\alpha_1 rT$ and the carbon emissions in the holding of stock are

$$\beta_0+\beta_1\int_0^{t_1} I(t)dt=\beta_0+\frac{\beta_1 r t_1^{1/n+1}}{(n+1)T^{1/n-1}}$$

Then, the total carbon emissions cost EC_1 per unit of time is

$$EC_1=\mu_1\left(\frac{\alpha_0}{T}+\alpha_1 r\right)+\mu_2\left(\frac{\beta_0}{T}+\frac{\beta_1 r t_1^{1/n+1}}{(n+1)T^{1/n}}\right) \qquad (10)$$

The total cost per unit of time is determined as the sum of the shipping cost, holding cost, ordering cost, backordering cost and carbon emissions costs. Thus, the total cost of the inventory system per unit of time is expressed as

$$\begin{aligned}C_1(t_1,T) &= \frac{(h+\mu_2\beta_1)\,rt_1^{1/n+1}}{(n+1)T^{1/n}}+\frac{A+k_0+\mu_1\alpha_0+\mu_2\beta_0}{T}+\frac{\omega r}{T^{1/n}}\left[-t_1^{1/n}T+\frac{nT^{1/n+1}}{n+1}+\frac{t_1^{1/n+1}}{n+1}\right]+(k_1+\mu_1\alpha_1)\,r\\ &= \frac{(h+\mu_2\beta_1+\omega)rt_1^{1/n+1}}{(n+1)T^{1/n}}+\frac{\delta_0}{T}+\omega\left(\frac{nrT}{n+1}-\frac{rt_1^{1/n}}{T^{1/n-1}}\right)+(k_1+\mu_1\alpha_1)\,r \qquad (11)\end{aligned}$$

where

$$\delta_0=A+k_0+\mu_1\alpha_0+\mu_2\beta_0. \qquad (12)$$

Case 2. Now, let us analyze the other possible situation, when $\tau<t_1$. At the beginning of the inventory cycle, the inventory level decreases over time due to customer demand until the time period τ is completed. Then, the inventory level continues to decrease, not only due to demand, but also due to the effect of the deterioration of the items, until that stock level is zero at time $t=t_1$. Later, when there is no stock of the product, shortages appear during the period (t_1,T). When a total of shortages of B units is reached, then the inventory is replenished with a sufficient quantity to meet the shortages and leave some units in stock to satisfy customer demand from the next inventory cycle. Taking this scenario into account, the differential equations that describe the inventory level $I(t)$ during the period $[0,T]$ are given below

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$$
\begin{aligned}
&\frac{dI(t)}{dt} = -\frac{rt^{1/n-1}}{nT^{1/n-1}} && \text{if } 0<t<\tau \\
&\frac{dI(t)}{dt} + \theta I(t) = -\frac{rt^{1/n-1}}{nT^{1/n-1}} && \text{if } \tau<t<t_1 \\
&\frac{dI(t)}{dt} = -\frac{rt^{1/n-1}}{nT^{1/n-1}} && \text{if } t_1<t<T
\end{aligned}
\tag{13}
$$

with the boundary conditions $I(0) = S$, $I(t_1) = 0$ and $I(T) = B$. The solutions to the differential equations expressed in (13) are

$$
\begin{aligned}
&I(t) = S - \frac{r}{T^{1/n-1}} t^{1/n} && \text{if } 0 \le t \le \tau \\
&I(t) = e^{-\theta(t-\tau)} \left(S - \frac{r}{T^{1/n-1}} \tau^{1/n} - \int_\tau^t e^{\theta(z-\tau)} \frac{r}{nT^{1/n-1}} z^{1/n-1} dz \right) && \text{if } \tau < t \le t_1 \\
&I(t) = \frac{r}{T^{1/n-1}} \left(t_1^{1/n} - t^{1/n} \right) && \text{if } t_1 < t < T
\end{aligned}
\tag{14}
$$

Since the inventory level at t_1 is zero, $I(t_1) = 0$, then for this scenario the initial stock level is

$$
S = \frac{r}{T^{1/n-1}} \tau^{1/n} + \frac{r}{nT^{1/n-1}} \int_\tau^{t_1} e^{\theta(z-\tau)} z^{1/n-1} dz \tag{15}
$$

So, replacing S in equation (14), we have

$$
\begin{aligned}
&I(t) = \frac{r}{T^{1/n-1}} \left(\tau^{1/n} - t^{1/n} \right) + \frac{r}{nT^{1/n-1}} e^{-\theta\tau} \int_\tau^{t_1} e^{\theta z} z^{1/n-1} dz && \text{if } 0 \le t \le \tau \\
&I(t) = \frac{r}{nT^{1/n-1}} e^{-\theta t} \int_t^{t_1} e^{\theta z} z^{1/n-1} dz && \text{if } \tau < t \le t_1 \\
&I(t) = \frac{r}{T^{1/n-1}} \left(t_1^{1/n} - t^{1/n} \right) && \text{if } t_1 < t < T
\end{aligned}
\tag{16}
$$

By calculating the value of the function $I(t)$ at $t = T$, the maximum number B of shortages is obtained by the following formula

$$
B = -I(T) = \frac{-r}{T^{1/n-1}} \left(t_1^{1/n} - T^{1/n} \right) = rT \left[1 - \left(\frac{t_1}{T} \right)^{1/n} \right] \tag{17}
$$

The quantity to be ordered, or lot size Q, is equal to $S + B$. Therefore, from equations (15) and (17), it can be deduced that the lot size is computed with the expression

$$
Q = \frac{r}{nT^{1/n-1}} e^{-\theta\tau} \int_\tau^{t_1} e^{\theta z} z^{1/n-1} dz + \frac{r}{T^{1/n-1}} \left(\tau^{1/n} + T^{1/n} - t_1^{1/n} \right) \tag{18}
$$

The number of deteriorated units U is obtained as the difference between the size of the replenishment Q and the units demanded, rT, throughout the inventory cycle. So, this amount is

$$
U = Q - rT = \frac{r}{nT^{1/n-1}} e^{-\theta\tau} \int_\tau^{t_1} e^{\theta z} z^{1/n-1} dz + \frac{r}{T^{1/n-1}} \left(\tau^{1/n} - t_1^{1/n} \right) \tag{19}
$$

The amount carried in the inventory is

$$\begin{aligned}\int_0^{t_1} I(t)dt &= \int_0^{\tau} I(t)dt + \int_{\tau}^{t_1} I(t)dt \\ &= \frac{r\tau^{1/n+1}}{(n+1)T^{1/n-1}} + \frac{r\tau}{nT^{1/n-1}}e^{-\theta\tau}\int_{\tau}^{t_1} e^{\theta z}z^{1/n-1}dz + \frac{r}{nT^{1/n-1}}\int_{\tau}^{t_1} e^{-\theta t}\left(\int_t^{t_1} e^{\theta z}z^{1/n-1}dz\right)dt \\ &= \frac{r\tau^{1/n+1}}{(n+1)T^{1/n-1}} + \frac{r\tau}{nT^{1/n-1}}e^{-\theta\tau}\int_{\tau}^{t_1} e^{\theta z}z^{1/n-1}dz + \frac{r}{nT^{1/n-1}}\int_{\tau}^{t_1} e^{\theta z}z^{1/n-1}\left(\int_{\tau}^{z} e^{-\theta t}dt\right)dz \\ &= \frac{r\tau^{1/n+1}}{(n+1)T^{1/n-1}} + \frac{r}{nT^{1/n-1}}\left(\tau+\frac{1}{\theta}\right)e^{-\theta\tau}\int_{\tau}^{t_1} e^{\theta z}z^{1/n-1}dz + \frac{r}{\theta T^{1/n-1}}\left(\tau^{1/n}-t_1^{1/n}\right) \end{aligned} \tag{20}$$

Now, the carbon emissions occur in transport, inventory holding and disposal of deteriorated items. Thus, the carbon emissions are

$$\alpha_0 + \alpha_1 Q + \beta_0 + \beta_1\int_0^{t_1} I(t)dt + \gamma U \tag{21}$$

From (18), (19) and (20), the total carbon emissions are

$$\begin{aligned}&\alpha_0 + \frac{\alpha_1 r}{nT^{1/n-1}}e^{-\theta\tau}\int_{\tau}^{t_1} e^{\theta z}z^{1/n-1}dz + \frac{\alpha_1 r}{T^{1/n-1}}\left(\tau^{1/n}+T^{1/n}-t_1^{1/n}\right) + \beta_0 + \frac{\beta_1 r\tau^{1/n+1}}{(n+1)T^{1/n-1}} \\ &+\frac{\beta_1 r}{nT^{1/n-1}}\left(\tau+\frac{1}{\theta}\right)e^{-\theta\tau}\int_{\tau}^{t_1} e^{\theta z}z^{1/n-1}dz + \frac{\beta_1 r}{\theta T^{1/n-1}}\left(\tau^{1/n}-t_1^{1/n}\right) \\ &+\frac{\gamma r}{nT^{1/n-1}}e^{-\theta\tau}\int_{\tau}^{t_1} e^{\theta z}z^{1/n-1}dz + \frac{\gamma r}{T^{1/n-1}}\left(\tau^{1/n}-t_1^{1/n}\right)\end{aligned}$$

In this scenario, the total cost during the inventory cycle is the sum of the holding, ordering, shipping, shortage and deterioration costs, plus the total carbon emissions cost.

From (20), the holding cost per unit of time is

$$HC_2 = \frac{h}{T}\int_0^{t_1} I(t)dt = \frac{hr\tau^{1/n+1}}{(n+1)T^{1/n}} + \frac{hr}{nT^{1/n}}\left(\tau+\frac{1}{\theta}\right)e^{-\theta\tau}\int_{\tau}^{t_1} e^{\theta z}z^{1/n-1}dz + \frac{hr}{\theta T^{1/n}}\left(\tau^{1/n}-t_1^{1/n}\right) \tag{22}$$

The cost of placing an order per unit of time is $OC_2 = A/T$. The shipping cost per unit of time is $SC_2 = k_0/T + k_1 r$. The cost of the deteriorated units per unit of time is

$$DC_2 = \frac{vU}{T} = \frac{vr}{nT^{1/n}}e^{-\theta\tau}\int_{\tau}^{t_1} e^{\theta z}z^{1/n-1}dz + \frac{vr}{T^{1/n}}\left(\tau^{1/n}-t_1^{1/n}\right) \tag{23}$$

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The shortage cost per unit of time is

$$\begin{aligned} BC_2 &= \frac{\omega}{T}\int_{t_1}^{T}[-I(t)]\,dt = \frac{\omega}{T}\int_{t_1}^{T}\tfrac{-r}{T^{1/n-1}}\left(t_1^{1/n}-t^{1/n}\right)dt = \frac{\omega r}{T^{1/n}}\int_{t_1}^{T}\left(t^{1/n}-t_1^{1/n}\right)dt \\ &= \frac{\omega rn}{n+1}T+\frac{\omega r}{(n+1)\,T^{1/n}}t_1^{1/n+1}-\frac{\omega r}{T^{1/n-1}}t_1^{1/n} \end{aligned} \tag{24}$$

In this case, the total carbon emissions cost, EC_2, per unit of time is

$$\begin{aligned} EC_2 &= \mu_1\frac{\alpha_0}{T}+\frac{\mu_1\alpha_1 r}{nT^{1/n}}e^{-\theta\tau}\int_{\tau}^{t_1}e^{\theta z}z^{1/n-1}dz+\frac{\mu_1\alpha_1 r}{T^{1/n}}\left(\tau^{1/n}+T^{1/n}-t_1^{1/n}\right)+\mu_2\frac{\beta_0}{T}+\frac{\mu_2\beta_1 r\tau^{1/n+1}}{(n+1)T^{1/n}} \\ &+\frac{\mu_2\beta_1 r}{nT^{1/n}}\left(\tau+\frac{1}{\theta}\right)e^{-\theta\tau}\int_{\tau}^{t_1}e^{\theta z}z^{1/n-1}dz+\frac{\mu_2\beta_1 r}{\theta T^{1/n}}\left(\tau^{1/n}-t_1^{1/n}\right)+\frac{\mu_3\gamma r}{nT^{1/n}}e^{-\theta\tau}\int_{\tau}^{t_1}e^{\theta z}z^{1/n-1}dz \\ &+\frac{\mu_3\gamma r}{T^{1/n}}\left(\tau^{1/n}-t_1^{1/n}\right) \end{aligned} \tag{25}$$

As a result, the total cost per unit of time is expressed as follows

$$\begin{aligned} C_2(t_1,T) &= \frac{k_0}{T}+k_1 r+\frac{hr\tau^{1/n+1}}{(n+1)T^{1/n}}+\frac{hr}{nT^{1/n}}\left(\tau+\frac{1}{\theta}\right)e^{-\theta\tau}\int_{\tau}^{t_1}e^{\theta z}z^{1/n-1}dz+\frac{hr}{\theta T^{1/n}}\left(\tau^{1/n}-t_1^{1/n}\right) \\ &+\frac{vr}{nT^{1/n}}e^{-\theta\tau}\int_{\tau}^{t_1}e^{\theta z}z^{1/n-1}dz+\frac{vr}{T^{1/n}}\left(\tau^{1/n}-t_1^{1/n}\right)+\frac{A}{T}+\frac{\omega rn}{n+1}T+\frac{\omega r}{(n+1)\,T^{1/n}}t_1^{1/n+1} \\ &-\frac{\omega r}{T^{1/n-1}}t_1^{1/n}+\mu_1\frac{\alpha_0}{T}+\frac{\mu_1\alpha_1 r}{nT^{1/n}}e^{-\theta\tau}\int_{\tau}^{t_1}e^{\theta z}z^{1/n-1}dz+\frac{\mu_1\alpha_1 r}{T^{1/n}}\left(\tau^{1/n}+T^{1/n}-t_1^{1/n}\right)+\mu_2\frac{\beta_0}{T} \\ &+\frac{\mu_2\beta_1 r\tau^{1/n+1}}{(n+1)T^{1/n}}+\frac{\mu_2\beta_1 r}{nT^{1/n}}\left(\tau+\frac{1}{\theta}\right)e^{-\theta\tau}\int_{\tau}^{t_1}e^{\theta z}z^{1/n-1}dz+\frac{\mu_2\beta_1 r}{\theta T^{1/n}}\left(\tau^{1/n}-t_1^{1/n}\right) \\ &+\frac{\mu_3\gamma r}{nT^{1/n}}e^{-\theta\tau}\int_{\tau}^{t_1}e^{\theta z}z^{1/n-1}dz+\frac{\mu_3\gamma r}{T^{1/n}}\left(\tau^{1/n}-t_1^{1/n}\right) \\ &= \frac{(h+\mu_2\beta_1)\,r\tau^{1/n+1}}{(n+1)T^{1/n}}+\frac{\delta_1 r}{nT^{1/n}}e^{-\theta\tau}\int_{\tau}^{t_1}e^{\theta z}z^{1/n-1}dz+\frac{\delta_2 r}{T^{1/n}}\left(\tau^{1/n}-t_1^{1/n}\right)+\frac{\delta_0}{T}+\frac{\omega rn}{n+1}T \\ &+\frac{\omega r}{(n+1)\,T^{1/n}}t_1^{1/n+1}-\frac{\omega r}{T^{1/n-1}}t_1^{1/n}+(k_1+\mu_1\alpha_1)\,r \end{aligned} \tag{26}$$

where δ_0 is given by (12),

$$\delta_1 = (h+\mu_2\beta_1)\left(\tau+\frac{1}{\theta}\right)+v+\mu_1\alpha_1+\mu_3\gamma \tag{27}$$

and

$$\delta_2 = \frac{h+\mu_2\beta_1}{\theta}+v+\mu_1\alpha_1+\mu_3\gamma. \tag{28}$$

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Since the total demand throughout the whole inventory cycle is rT, the total revenue obtained from the sales of the product is calculated with the expression $(p-c)rT$. Let $\pi(t_1,T)$ be the benefit per unit of time obtained by the company, in other words, the income minus the expenses per unit of time. Considering the costs of the two scenarios given by equations (11) and (26), the benefit function is determined by

$$\pi(t_1,T) = \begin{cases} (p-c)r - C_1(t_1,T) & \text{if } T < \tau \text{ or } 0 < t_1 \le \tau \le T \\ (p-c)r - C_2(t_1,T) & \text{if } \tau < t_1 \le T \end{cases} \tag{29}$$

The main objective is to determine the optimal policy (t_1^*,T^*) for the inventory system that maximizes the benefit given by equation (29). This is equivalent to minimizing the cost function $C(t_1,T)$ defined by

$$C(t_1,T) = \begin{cases} C_1(t_1,T) & \text{if } T < \tau \text{ or } 0 < t_1 \le \tau \le T \\ C_2(t_1,T) & \text{if } \tau < t_1 \le T \end{cases} \tag{30}$$

subject to the constraints $0 < t_1 \le T$ and $T > 0$. Notice that, if $t_1 = \tau$, then both cost functions are equal, that is, $C_1(t_1,T) = C_2(t_1,T)$.

5 Necessary conditions to determine the optimal policy

In this section, we present the approaches to find the optimal inventory policies for the two scenarios.

Case 1. To find the optimal policy for the first scenario (if $T < \tau$ or $0 < t_1 \le \tau \le T$), we have to calculate the partial derivatives of the function $C_1(t_1,T)$ given in (11), that is,

$$\frac{\partial C_1}{\partial t_1} = (h+\omega+\mu_2\beta_1)\,\frac{rt_1^{1/n}}{nT^{1/n}} - \frac{\omega r t_1^{1/n-1}}{nT^{1/n-1}} \tag{31}$$

$$\frac{\partial C_1}{\partial T} = -\frac{(h+\omega+\mu_2\beta_1)\,rt_1^{1/n+1}}{n\,(n+1)\,T^{1/n+1}} + \frac{(1-n)\omega r t_1^{1/n}}{nT^{1/n}} + \frac{\omega r n}{n+1} - \frac{\delta_0}{T^2} \tag{32}$$

Equalizing the partial derivative (31) to zero, the first condition is obtained. Thus, we have

$$t_1 = \frac{\omega T}{h+\omega+\mu_2\beta_1} \tag{33}$$

Substituting the above t_1 in equation (32) and equating to zero, we obtain

$$\frac{\omega r n}{n+1}\left[1-\left(\frac{\omega}{h+\omega+\mu_2\beta_1}\right)^{1/n}\right] - \frac{\delta_0}{T^2} = 0$$

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Thus, the optimal cycle period is given by

$$T^0 = \sqrt{\frac{(n+1)\delta_0}{\omega n r \left[1 - \left(\omega/(h+\omega+\mu_2\beta_1)\right)^{1/n}\right]}} \tag{34}$$

and the time in which the inventory is zero is

$$t_1^0 = T^0 \frac{\omega}{h+\omega+\mu_2\beta_1} = \frac{\omega}{h+\omega+\mu_2\beta_1} \sqrt{\frac{(n+1)\delta_0}{\omega n r \left[1 - \left(\omega/(h+\omega+\mu_2\beta_1)\right)^{1/n}\right]}} \tag{35}$$

In addition, in this case, the corresponding initial stock level is

$$S^0 = rT^0 \left(\frac{\omega}{h+\omega+\mu_2\beta_1}\right)^{1/n} = \sqrt{\frac{(n+1)r\delta_0}{\omega n \left[1 - \left(\omega/(h+\omega+\mu_2\beta_1)\right)^{1/n}\right]}} \left(\frac{\omega}{h+\omega+\mu_2\beta_1}\right)^{1/n} \tag{36}$$

Next, we prove that the policy (t_1^0, T^0), given by (34) and (35), is the optimal inventory policy for this scenario. To do so, we first determine the second partial derivatives and then we calculate the Hessian at the point (t_1^0, T^0). Thus, we have

$$\begin{aligned}
\frac{\partial^2 C_1}{\partial t_1^2} &= \frac{r t_1^{1/n-2}}{n^2 T^{1/n}} \left(t_1 \left(h+\omega+\beta_1\mu_2\right) + (n-1)\,\omega T\right) \\
\frac{\partial^2 C_1}{\partial t_1 \partial T} &= -\frac{r t_1^{1/n-1}}{n^2 T^{1/n+1}} \left(t_1 \left(h+\omega+\beta_1\mu_2\right) + (n-1)\,\omega T\right) \\
\frac{\partial^2 C_1}{\partial T^2} &= \frac{r t_1^{1/n}}{n^2 T^{1/n+2}} \left(t_1 \left(h+\omega+\beta_1\mu_2\right) + (n-1)\,\omega T\right) + \frac{2\delta_0}{T^3}
\end{aligned}$$

and the Hessian determinant is

$$H(t_1, T) = \frac{\partial^2 C_1}{\partial t_1^2}\frac{\partial^2 C_1}{\partial T^2} - \left(\frac{\partial^2 C_1}{\partial t_1 \partial T}\right)^2 = \frac{2\delta_0 r t_1^{1/n-2}}{n^2 T^{1/n+3}} \left(t_1 \left(h+\omega+\beta_1\mu_2\right) + (n-1)\,\omega T\right)$$

Taking into account that $t_1^0 \left(h+\omega+\beta_1\mu_2\right) + (n-1)\,\omega T^0 = n\omega T^0 > 0$, it is clear that the second partial derivative with respect to T_1 at the point (t_1^0, T^0) is positive and the Hessian determinant at the point (t_1^0, T^0) is positive. Therefore, (t_1^0, T^0) is the policy that minimizes the total cost of the inventory system under this scenario, that is, when products are sold before they begin to deteriorate.

Thus, the minimum cost per unit of time is

$$C^0 = \sqrt{\frac{4\omega n r \delta_0 \left[1 - \left(\omega/(h+\omega+\mu_2\beta_1)\right)^{1/n}\right]}{n+1}} + \left(k_1 + \mu_1\alpha_1\right) r \tag{37}$$

and the maximum benefit per unit of time is

$$\pi^0 = \pi(t_1^0, T^0) = (p-c)r - C^0$$

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Remark 1 *This policy extends the optimal inventory policies for some systems studied by other authors. Thus, the inventory system without deterioration, with power demand and full backlogging proposed by Sicilia et al. (2012) is a particular case of the model analyzed here. So, if the parameters related to the carbon emissions are zero, then we obtain the same policy given by these authors. Also, the policy given by equations (34) to (36) extends the environmental economic order quantity model analyzed by Bonney and Jaber (2011). In addition, the inventory model proposed here extends the model developed by Hua et al. (2011) when the firm has no carbon emission quotas per unit of time and neither buys nor sells any carbon credit.*

Case 2. Now, the optimal policy for the case $\tau < t_1 \leq T$ is analyzed in detail. First, the solution of the inventory problem expressed in (30) is analyzed at the upper bound of the feasible region determined by the constraints. Thus, if it is assumed that $t_1 = T$, then the cost function $C_2(t_1, T)$ given in equation (26) is reduced to the following univariable function

$$C_3(T) = \frac{(h+\mu_2\beta_1)\, r\tau^{1/n+1}}{(n+1)T^{1/n}} + \frac{\delta_1 r}{nT^{1/n}} e^{-\theta\tau} \int_\tau^T e^{\theta z} z^{1/n-1} dz + \frac{\delta_2 r}{T^{1/n}} \tau^{1/n} - \delta_2 r + \frac{\delta_0}{T} + (k_1 + \mu_1\alpha_1) r \tag{38}$$

By calculating the derivative of the function (38), the necessary condition to determine the scheduling period T_1 that minimizes the cost function $C_3(T)$ can be obtained. Thus, we have

$$\frac{-\,(h+\mu_2\beta_1)\, r\tau^{1/n+1}}{n\,(n+1)\,T^{1/n+1}} + \frac{\delta_1 r}{nT} e^{\theta(T-\tau)} - \frac{\delta_1 r}{n^2 T^{1/n+1}} e^{-\theta\tau} \int_\tau^T e^{\theta z} z^{1/n-1} dz - \frac{\delta_2 r\tau^{1/n}}{nT^{1/n+1}} - \frac{\delta_0}{T^2} = 0$$

This condition is equivalent to

$$\frac{-r\tau^{1/n}}{n}\left(\frac{(h+\mu_2\beta_1)\,\tau}{n+1} + \delta_2\right) + \frac{\delta_1 r e^{\theta(T-\tau)}}{n} T^{1/n} - \frac{\delta_1 r e^{-\theta\tau}}{n^2} \int_\tau^T e^{\theta z} z^{1/n-1} dz - \delta_0 T^{1/n-1} = 0 \tag{39}$$

Solving equation (39), the value of T_1 is obtained. Replacing this value T_1 in the function $C_3(T)$ given in equation (38), the minimum cost $C_3 = C_3(T_1)$ is determined.

Now let us analyze the other lower bound of the feasible region, that is, when $t_1 = \tau$. In this case, the cost function given in (26) is reduced to

$$C_4(T) = \frac{(h+\omega+\mu_2\beta_1) r\tau^{1/n+1}}{(n+1)T^{1/n}} + \frac{\delta_0}{T} + \frac{\omega r n}{n+1} T - \frac{\omega r}{T^{1/n-1}} \tau^{1/n} + (k_1 + \mu_1\alpha_1)\, r \tag{40}$$

Deriving and equating to zero, the following equation is determined

$$-\frac{(h+\omega+\mu_2\beta_1) r\tau^{1/n+1}}{n(n+1)T^{1/n+1}} - \frac{\delta_0}{T^2} + \frac{\omega r n}{n+1} + \frac{(1-n)\omega r}{nT^{1/n}} \tau^{1/n} = 0 \tag{41}$$

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Now, by solving Equation (41), the value of T_2 that minimizes the cost function $C_4(T)$ is obtained. If T_2 is substituted in the function $C_4(T)$ given in equation (40), the minimum cost $C_4 = C_4(T_2)$ is calculated.

In the following paragraphs, the optimality necessary conditions of the problem given in (30) are established when it is assumed that $\tau < t_1 < T$. To find these conditions, the partial derivatives of the cost function $C_2(t_1, T)$ given in equation (26) with respect to t_1 and T must be calculated. So,

$$\frac{\partial C_2}{\partial t_1} = \frac{\delta_1 r}{nT^{1/n}} e^{\theta(t_1-\tau)} t_1^{1/n-1} + (\omega t_1 - \omega T - \delta_2) \frac{r t_1^{1/n-1}}{nT^{1/n}} \tag{42}$$

$$\begin{aligned}\frac{\partial C_2}{\partial T} = &-\frac{r}{nT^{1/n+1}} \left[\frac{(h+\mu_2\beta_1)\,\tau^{1/n+1}}{n+1} + \delta_1 \frac{e^{-\theta\tau}}{n} \int_\tau^{t_1} e^{\theta z} z^{1/n-1} dz \right] \\ &+ \left[\delta_2 \left(t_1^{1/n} - \tau^{1/n} \right) - \frac{\omega t_1^{1/n+1}}{n+1} + (1-n)\omega T t_1^{1/n} \right] \frac{r}{nT^{1/n+1}} + \frac{\omega r n}{n+1} - \frac{\delta_0}{T^2}\end{aligned} \tag{43}$$

Equalizing the partial derivative (42) to zero, the first condition is obtained

$$T = t_1 + \frac{\delta_1}{\omega} e^{\theta(t_1-\tau)} - \frac{\delta_2}{\omega} \tag{44}$$

Now, if the other partial derivative (43) is equal to zero, the second condition is determined

$$\begin{aligned}&-\frac{r}{nT^{1/n+1}} \left[\frac{(h+\mu_2\beta_1)\,\tau^{1/n+1}}{n+1} + \frac{\delta_1 e^{-\theta\tau}}{n} \int_\tau^{t_1} e^{\theta z} z^{1/n-1} dz \right] \\ &\quad + \left[\delta_2 \left(t_1^{1/n} - \tau^{1/n} \right) - \frac{\omega t_1^{1/n+1}}{n+1} + (1-n)\omega T t_1^{1/n} \right] \frac{r}{nT^{1/n+1}} + \frac{\omega r n}{n+1} - \frac{\delta_0}{T^2} = 0\end{aligned}$$

This last equation is equivalent to

$$\begin{aligned}&-\left[\frac{(h+\mu_2\beta_1)\,\tau^{1/n+1}}{n+1} + \frac{\delta_1 e^{-\theta\tau}}{n} \int_\tau^{t_1} e^{\theta z} z^{1/n-1} dz \right] + \delta_2 \left(t_1^{1/n} - \tau^{1/n} \right) - \frac{\omega t_1^{1/n+1}}{n+1} + (1-n)\omega T t_1^{1/n} \\ &\quad + \frac{\omega n^2}{n+1} T^{1/n+1} - \frac{n\delta_0}{r} T^{1/n-1} = 0\end{aligned} \tag{45}$$

Substituting the value of T given by expression (44) in equation (45), the following nonlinear equation with a single variable t_1 is stated.

$$\begin{aligned}&-\left[\frac{(h+\mu_2\beta_1)\,\tau^{1/n+1}}{n+1} + \frac{\delta_1 e^{-\theta\tau}}{n} \int_\tau^{t_1} e^{\theta z} z^{1/n-1} dz \right] + \delta_2 \left(t_1^{1/n} - \tau^{1/n} \right) - \frac{\omega t_1^{1/n+1}}{n+1} \\ &+ (1-n)\omega t_1^{1/n} \left[t_1 + \frac{\delta_1}{\omega} e^{\theta(t_1-\tau)} - \frac{\delta_2}{\omega} \right] + \frac{\omega n^2}{n+1} \left[t_1 + \frac{\delta_1}{\omega} e^{\theta(t_1-\tau)} - \frac{\delta_2}{\omega} \right]^{1/n+1} \\ &- \frac{n\delta_0}{r} \left[t_1 + \frac{\delta_1}{\omega} e^{\theta(t_1-\tau)} - \frac{\delta_2}{\omega} \right]^{1/n-1} = 0\end{aligned} \tag{46}$$

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It is important to remark that the above equation could be solved by a numerical method to obtain the period of time $\widetilde{t}_1$. Then, substituting $\widetilde{t}_1$ into equation (44), the length of the inventory cycle $\widetilde{T}$ can be found.

The second partial derivatives of $C_2(t_1, T)$ are as follows

$$\begin{aligned}\frac{\partial^2 C_2}{\partial t_1^2} &= \delta_1 \frac{r}{nT^{1/n}} e^{\theta(t_1-\tau)} \left(\theta t_1^{1/n-1} + \frac{1-n}{n} t_1^{1/n-2}\right) + \frac{\omega r t_1^{1/n-1}}{n^2 T^{1/n}} - \frac{(1-n)\omega r t_1^{1/n-2}}{n^2 T^{1/n-1}} \\ &\quad -\delta_2 \frac{(1-n) r t_1^{1/n-2}}{n^2 T^{1/n}} \end{aligned} \tag{47}$$

$$\frac{\partial^2 C_2}{\partial t_1 \partial T} = -\frac{r t_1^{1/n-1}}{n^2 T^{1/n+1}} \left(\delta_1 e^{\theta(t_1-\tau)} - \delta_2\right) + \frac{(1-n)\omega r t_1^{1/n-1}}{n^2 T^{1/n}} - \frac{\omega r t_1^{1/n}}{n^2 T^{1/n+1}} \tag{48}$$

$$\begin{aligned}\frac{\partial^2 C_2}{\partial T^2} &= \frac{(n+1)r}{n^2 T^{1/n+2}} \left[\frac{(h+\mu_2\beta_1)\,\tau^{1/n+1}}{n+1} + \frac{\delta_1 e^{-\theta\tau}}{n} \int_{\tau}^{t_1} e^{\theta z} z^{1/n-1} dz\right] + \delta_2 \frac{(n+1)r}{n^2 T^{1/n+2}} \left(\tau^{1/n} - t_1^{1/n}\right) \\ &\quad + \frac{\omega r t_1^{1/n+1}}{n^2 T^{1/n+2}} + \frac{2\delta_0}{T^3} - \frac{\omega(1-n) r t_1^{1/n}}{n^2 T^{1/n+1}} \end{aligned} \tag{49}$$

The solution $(\widetilde{t}_1, \widetilde{T})$ of equations (44) and (46) must satisfy the Hessian matrix for a positive definite in order to be a minimum. The second partial derivatives at that point $(\widetilde{t}_1, \widetilde{T})$ are

$$\frac{\partial^2 C_2(\widetilde{t}_1, \widetilde{T})}{\partial t_1^2} = \frac{r(\widetilde{t}_1)^{1/n-1}}{n\left(\widetilde{T}\right)^{1/n}} \left[\theta\delta_2 + \theta\omega\left(\widetilde{T} - \widetilde{t}_1\right) + \omega\right] \tag{50}$$

$$\frac{\partial^2 C_2(\widetilde{t}_1, \widetilde{T})}{\partial t_1 \partial T} = -\frac{\omega r(\widetilde{t}_1)^{1/n-1}}{n\left(\widetilde{T}\right)^{1/n}} \tag{51}$$

$$\frac{\partial^2 C_2(\widetilde{t}_1, \widetilde{T})}{\partial T^2} = \frac{\omega r(1-n)(\widetilde{t}_1)^{1/n}}{n\left(\widetilde{T}\right)^{1/n+1}} + \frac{(n-1)\delta_0}{n\left(\widetilde{T}\right)^3} + \frac{\omega r}{\widetilde{T}} \tag{52}$$

From equation (50), it is observed that the second partial derivative with respect to t_1 is positive at the point $(\widetilde{t}_1, \widetilde{T})$. Therefore, it is sufficient to check that the Hessian at that point $(\widetilde{t}_1, \widetilde{T})$ is positive, i.e.

$$H(\widetilde{t}_1, \widetilde{T}) = \frac{\partial^2 C_2(\widetilde{t}_1, \widetilde{T})}{\partial t_1^2} \frac{\partial^2 C_2(\widetilde{t}_1, \widetilde{T})}{\partial T^2} - \left(\frac{\partial^2 C_2(\widetilde{t}_1, \widetilde{T})}{\partial t_1 \partial T}\right)^2 > 0 \tag{53}$$

where the second partial derivatives of $C_2(t_1, T)$ at the point $(\widetilde{t}_1, \widetilde{T})$ are given by the expressions (50), (51) and (52).

Considering the theoretical results obtained above, the following algorithm is developed to determine the optimal policy for an inventory system where the products in stock are kept in perfect conditions for a period of time τ; after which a process of deterioration begins for the stored items, so the products deteriorate with a constant deterioration rate of θ units per unit of time.

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Algorithm 1

Step 1 *Determine the values of* T^0 *and* t_1^0 *by using equations (34) and (35), respectively. If* $T^0 \leq \tau$*, then go to Step 3. Otherwise, go to Step 2.*

Step 2 *If the value of* t_1^0 *is greater than* τ*, then go to Step 5. Otherwise, go to Step 3.*

Step 3 *The optimal policy is* $(t_1^*, T^*) = (t_1^0, T^0)$*. Go to Step 4.*

Step 4 *From (37), calculate the minimum cost* $C^* = C^0 = C_1(t_1^0, T^0)$*. Obtain the optimal benefit* $\pi^* = \pi^0 = \pi(t_1^0, T^0)$ *by using equation (29), that is,* $\pi^0 = (p - c)r - C^0$*. Stop.*

Step 5 *Generate the set* Ω *of solutions of equation (46) that are greater than* τ *by using some numerical method. Select a solution* t_1 *of the set* Ω*.*

Step 6 *From (44), obtain the value of* T *associated with* t_1*. Compute the Hessian value* $H(t_1, T)$ *given by equation (53).*

Step 7 *If* $H(t_1, T) > 0$ *then include the pair* (t_1, T) *in the set* P *of candidate inventory policies, determine the cost* $C_2(t_1, T)$ *with equation (26) and go to Step 8. Otherwise, go directly to Step 8.*

Step 8 *Set* $\Omega = \Omega - \{t_1\}$*. If Card* $(\Omega) = 0$*, then go to Step 9. Otherwise, select a new positive solution for* t_1 *of the set* Ω*. Go to Step 6.*

Step 9 *Compute the value of* T_1*, solving equation (39) by using a numerical method. If* $T_1 > \tau$*, then include the pair* (T_1, T_1) *in the set* P *of candidate inventory policies, calculate the cost* $C_3 = C_3(T_1)$ *with equation (38) and go to Step 10. Otherwise, go to Step 10 directly.*

Step 10 *Calculate the value of* T_2*, solving equation (41) by using a numerical method. If* $T_2 > \tau$*, then include the pair* (τ, T_2) *in the set* P *of candidate inventory policies, compute the cost* $C_4 = C_4(T_2)$ *with equation (40) and go to Step 11. Otherwise, go to Step 11 directly.*

Step 11 *Determine the policy* (t_1^*, T^*) *such that its cost* $C^* = C(t_1^*, T^*)$ *is the lowest cost of the inventory policies included in the set* P *of candidate inventory policies. Obtain the optimal benefit* $\pi^* = \pi(t_1^*, T^*)$ *by equation (29), which is* $\pi^* = (p - c)r - C^*$*. Stop.*

Note that the algorithm described previously also determines the optimal inventory policy for a system with a constant process of deterioration from the beginning of the inventory cycle. For this, it is sufficient to consider only *Step 5* through to *Step 11*, and set $\tau = 0$ in all the equations where τ appears.

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Remark 2 *The inventory model analyzed for this scenario extends the basic sustainable EOQ model and the sustainable EOQ model with full backordering proposed by Taleizadeh et al. (2017). Moreover, the inventory policy proposed here extends the optimal policy for an inventory system with deteriorated items and power demand developed by Sicilia et al. (2013) and the optimal policy for an inventory system with a power demand pattern, deterioration and full backlogging analyzed by Sicilia et al. (2014a).*

6 Numerical examples

This section presents some numerical examples to help to understand the steps of the algorithm proposed above to find the optimal inventory policy.

Example 1 Consider an inventory system for a certain type of cake or pie, which may deteriorate over time. The estimated period without deterioration of these cakes is $\tau = 3$ days. Assume that the system has the characteristics described in this paper and consider the following parameters: average demand $r = 100$ kilograms of cake per week, index of the power demand pattern $n = 2$, deterioration rate $\theta = 0.1$, order cost $A = \$20$, unit holding cost $h = \$1.5$ per kilogram and week, unit deterioration cost $v = \$13$ per kilogram and week, and unit backlogging cost $\omega = \$10$ per kilogram and week. The purchase cost of a unit of the product is $c = \$20$ and the sale price is $p = \$40$. The cost of transporting a batch of Q units has a fixed cost $k_0 = \$20$ per shipment and a variable cost $k_1 = \$0.5$ per unit. The carbon emissions for transporting an order of Q units are a fixed amount of $\alpha_0 = 200$ kilograms plus $\alpha_1 = 0.8$ kilograms per unit ordered. The carbon emissions in the inventory are the sum of a fixed part of $\beta_0 = 100$ kilograms plus an amount $\beta_1 = 1$ kilogram per unit held in stock and per week. The carbon emissions in disposing of deteriorated items are $\gamma = 1.2$ kilograms per deteriorated unit. Finally, the taxes charged on carbon emissions are $\mu_1 = \$0.5$ per carbon emission kilogram in transportation, $\mu_2 = \$0.3$ per carbon emission kilogram in storage and $\mu_3 = \$0.4$ per carbon emission kilogram for deterioration. Applying the algorithm to determine the optimal inventory policy, the following results are obtained:

Step 1 $T^0 = 1.79180$ weeks, $t_1^0 = 1.51848$ weeks, $\tau = 3/7 = 0.428571$ weeks.

Step 2 $t_1^0 > \tau$.

Step 5 $\Omega = \{1.12408\}$, $t_1 = 1.12408$.

Step 6 $T = 1.43639$, $H(t_1, T) = 66651.3$.

Step 7 $P = \{(t_1 = 1.12408, T = 1.43639)\}$ and $C_2(t_1, T) = 303.086$.

Step 9 $T_1 = 1.28115$, $C_3 = 325.039$.

Step 10 $T_2 = 0.843384$, $C_4 = 372.784$.

Step 11 The optimal policy is $t_1^* = 1.12408$ weeks, $T^* = 1.43639$ weeks, $C^* = \$303.086$ per week and $\pi^* = \$1696.91$ per week.

From (18), the optimal lot size is $Q^* = 145.232$ and, from (17), the order level is $-B^* = -16.5719$.

Example 2 Consider the same parameters as in the previous example, but changing the without-deterioration period τ to $\tau = 13$ days.

Step 1 $T^0 = 1.79180$ weeks, $t_1^0 = 1.51848$ weeks, $\tau = 13/7 = 1.85714$ weeks. Then $T^0 < \tau$.

Step 3 The optimal policy is $(t_1^*, T^*) = (t_1^0, T^0)$, with $t_1^0 = 1.51848$ weeks and $T^0 = 1.79180$ weeks.

Step 4 The optimal cost is $C^* = C^0 = \$279.753$ per week and the maximum profit is $\pi^* = \pi^0 = \$1720.25$ per week.

Now, the optimal lot size is $Q^* = rT^* = 179.180$ and, from (36), the initial inventory level is $S^* = 164.949$. The maximum number of backorders is $B^* = 14.2315$.

Example 3 Consider the same parameters as in Example 1, but changing τ to $\tau = 0$ days, that is, the deterioration process of products starts from the beginning of the inventory cycle.

Step 1 $T^0 = 1.79180$ weeks, $t_1^0 = 1.51848$ weeks, $\tau = 0$.

Step 2 $t_1^0 > \tau$.

Step 5 $\Omega = \{1.02566\}$, $t_1 = 1.02566$.

Step 6 $T = 1.37000$, $H(t_1, T) = 77342.55$.

Step 7 $P = \{(t_1 = 1.02566, T = 1.37000)\}$ and $C_2(t_1, T) = 335.127$.

Step 9 $T_1 = 1.21908$, $C_3 = 363.876$.

Step 10 $T_2 = 0.504975$, $C_4 = 763.300$.

Step 11 The optimal policy is $t_1^* = 1.02566$ weeks, $T^* = 1.37000$ weeks, $C^* = \$335.127$ per week and $\pi^* = \$1664.87$ per week.

In this case, the optimal lot size is $Q^* = 141.181$ and the replacement level is $-B^* = -18.4607$.

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Example 4 We keep the same parameters as in the previous example, but changing the index of the demand pattern n to $n = 0.5$.

Step 1 $T^0 = 1.34525$ weeks, $t_1^0 = 1.14004$ weeks, $\tau = 3/7 = 0.428571$ weeks.

Step 2 $t_1^0 > \tau$.

Step 5 $\Omega = \{0.868656\}$, $t_1 = 0.868656$.

Step 6 $T = 1.09270$, $H(t_1, T) = 543179$.

Step 7 $P = \{(t_1 = 0.868656, T = 1.09270)\}$ and $C_2(t_1, T) = 368.865$.

Step 9 $T_1 = 0.894321$, $C_3 = 417.858$.

Step 10 $T_2 = 0.699888$, $C_4 = 430.176$.

Step 11 The optimal policy is $t_1^* = 0.868656$ weeks, $T^* = 1.09270$ weeks, $C^* = \$368.865$ per week and $\pi^* = \$1631.13$ per week.

Thus, the optimal lot size is $Q^* = 110.570$ and the order level is $-B^* = -40.2153$.

Example 5 Consider the same parameters as in Example 4, but changing the without-deterioration period τ to $\tau = 8$ days.

Step 1 $T^0 = 1.34525$ weeks, $t_1^0 = 1.14004$ weeks, $\tau = 8/7 = 1.14286$ weeks.

Step 2 $t_1^0 \leq \tau$.

Step 3 The optimal policy is $(t_1^*, T^*) = (t_1^0, T^0)$, with $t_1^0 = 1.14004$ weeks and $T^0 = 1.34525$ weeks.

Step 4 The optimal cost is $C^* = C^0 = \$342.741$ per week and the maximum profit is $\pi^* = \pi^0 = \$1657.26$ per week.

In this case, the optimal lot size is $Q^* = 134.525$ and the replacement level is $-B^* = -37.4334$.

Example 6 Consider the same parameters as in Example 4, but changing τ to $\tau = 0$ days, that is, the deterioration process of products starts from the beginning of the inventory cycle.

Step 1 $T^0 = 1.34525$ weeks, $t_1^0 = 1.14004$ weeks, $\tau = 0$.

Step 2 $t_1^0 > \tau$.

Step 5 $\Omega = \{0.806380\}$, $t_1 = 0.806380$.

Step 6 $T = 1.07410$, $H(t_1, T) = 526049$.

Step 7 $P = \{(t_1 = 0.806380, T = 1.07410)\}$ and $C_2(t_1, T) = 403.494$.

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Step 9 $T_1 = 0.865728$, $C_3 = 476.476$.

Step 10 $T_2 = 0.714143$, $C_4 = 566.095$.

Step 11 The optimal policy is $t_1^* = 0.806380$ weeks, $T^* = 1.07410$ weeks, $C^* = \$403.494$ per week and $\pi^* = \$1596.51$ per week.

Now, the optimal lot size is $Q^* = 110.765$ and the order level is $-B^* = -46.8715$.

Example 7 Consider the same parameters as in Example 1, but changing the index of the demand pattern n to $n = 1$.

Step 1 $T^0 = 1.49295$ weeks, $t_1^0 = 1.26521$ weeks, $\tau = 3/7 = 0.428571$ weeks.

Step 2 $t_1^0 > \tau$.

Step 5 $\Omega = \{0.952116\}$, $t_1 = 0.952116$.

Step 6 $T = 1.20476$, $H(t_1, T) = 237050$.

Step 7 $P = \{(t_1 = 0.952116, T = 1.20476)\}$ and $C_2(t_1, T) = 342.642$.

Step 9 $T_1 = 1.04459$, $C_3 = 374.606$.

Step 10 $T_2 = 0.746147$, $C_4 = 407.575$.

Step 11 The optimal policy is $t_1^* = 0.952116$ weeks, $T^* = 1.20476$ weeks, $C^* = \$342.642$ per week and $\pi^* = \$1657.36$ per week.

Therefore, the optimal lot size is $Q^* = 121.871$ and the order level is $-B^* = -25.2642$.

6.1 The effect of the sustainable costs in the inventory system

In this section, the optimal inventory policy obtained from the proposed model is compared with the one obtained from a model where the carbon emissions costs are not taken into consideration.

Let us denote by C^* the optimal objective function value of the model with carbon emissions costs. That is, C^* is the value of the cost associated to the optimal policy proposed in this paper. Also, we can work with the cost of the optimal policy for the inventory model without considering the sustainable costs, and this is denoted by $\widehat{C}$. To make the comparison of both inventory policies, it is necessary to calculate this last cost $\widehat{C}$. To do so, we first need the objective function $\overline{C}(t_1, T)$ to be optimized in the model that does not consider carbon

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emissions costs. This function is given by

$$\overline{C}(t_1,T)=\begin{cases}\overline{C}_1(t_1,T) & \text{if } T<\tau \text{ or } 0<t_1\le\tau\le T\\ \overline{C}_2(t_1,T) & \text{if } \tau<t_1\le T\end{cases}$$

Thus, for case 1, the function $\overline{C}_1(t_1,T)$ is deduced from (11), considering $\mu_i=0$ for $i=1,2,3$. Therefore, we have

$$\overline{C}_1(t_1,T)=\frac{(h+\omega)rt_1^{1/n+1}}{(n+1)T^{1/n}}+\frac{A+k_0}{T}+\omega\left(\frac{nrT}{n+1}-\frac{rt_1^{1/n}}{T^{1/n-1}}\right)+k_1r$$

Similarly, for case 2, the cost function $\overline{C}_2(t_1,T)$ follows from (26), taking $\mu_i=0$ for $i=1,2,3$. That is,

$$\overline{C}_2(t_1,T)=\frac{hr\tau^{1/n+1}}{(n+1)T^{1/n}}+\frac{\left(h\left(\tau+\frac{1}{\theta}\right)+v\right)r}{nT^{1/n}}e^{-\theta\tau}\int_{\tau}^{t_1}e^{\theta z}z^{1/n-1}dz+\frac{\left(\frac{h}{\theta}+v\right)r}{T^{1/n}}\left(\tau^{1/n}-t_1^{1/n}\right)+\frac{A+k_0}{T}$$
$$+\frac{\omega rn}{n+1}T+\frac{\omega r}{(n+1)\,T^{1/n}}t_1^{1/n+1}-\frac{\omega r}{T^{1/n-1}}t_1^{1/n}+k_1r$$

Now, considering the function objective $\overline{C}(t_1,T)$ and applying Algorithm 1, we obtain the optimal policy $(\widehat{t}_1,\widehat{T})$ for the inventory model without considering the sustainable costs. Then, the cost $\widehat{C}$ associated to that policy $(\widehat{t}_1,\widehat{T})$ is obtained from $\widehat{C}=C(\widehat{t}_1,\widehat{T})$, where $C(t_1,T)$ is given by (30).

Afterwards, the relative gap RG as a percentage of the two solutions can be calculated as the difference between $\widehat{C}$ and C^* divided by C^*, as indicated in the following equation:

$$RG(\%)=100\frac{\widehat{C}-C^*}{C^*}$$

The values of the measures $RG(\%)$ for the results obtained in the numerical examples are presented in Table 3.

Table 3. Comparison of the optimal policies of numerical examples

Example	t_1^*	T^*	Q^*	$\widehat{t}_1$	$\widehat{T}$	$\widehat{Q}$	C^*	$\widehat{C}$	$RG(\%)$
1	1.12408	1.43639	145.232	0.666242	0.799419	80.1086	303.086	348.119	14.8582
2	1.51848	1.79180	179.180	0.819863	0.942843	94.2843	279.753	320.376	14.5210
3	1.02566	1.37000	141.181	0.550538	0.709011	72.0668	335.127	391.799	16.9105
4	0.868656	1.09270	110.570	0.521123	0.61204	61.2729	368.865	426.917	15.7380
5	1.14004	1.34525	134.525	0.609994	0.701493	70.1493	342.741	398.586	16.2936
6	0.806380	1.07410	110.765	0.426820	0.548916	55.8513	403.494	478.566	18.6057
7	0.952116	1.20476	121.871	0.571665	0.677232	67.8261	342.642	394.918	15.2569

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Note that, for Example 1, the inventory cycle for the model without considering the sustainable costs is $\widehat{T} = 0.799419$, which is 44.3455% lower than the optimal inventory cycle T^*. The optimal lot size for this model is $\widehat{Q} = 80.1086$, which is 44.8409% lower than the optimal lot size Q^*. Moreover, the relative gap RG as a percentage is around 15%. This means that, to apply the optimal policy obtained for the inventory model without considering sustainable costs, leads to an additional cost over the minimum cost corresponding to the optimal solution deduced considering sustainable costs.

7 Sensitivity analysis

7.1 Impact of some parameters

In this section, we include an analysis of the behavior of the optimal inventory policy and the maximum profit when the index of the demand pattern or the parameters of the deterioration rate vary.

We assume the following parameters of the inventory system: $c = 30$, $p = 50$, $A = 30$, $k_0 = 4$, $k_1 = 3$, $\alpha_0 = 5$, $\alpha_1 = 0.8$, $h = 2.5$, $\beta_0 = 7$, $\beta_1 = 1$, $v = 19$, $\gamma = 1.2$, $\mu_1 = \mu_2 = \mu_3 = 0.5$, $\omega = 10$ and $r = 100$. Table 4 shows some computational results when the parameters τ, θ and n vary, that is, $\tau \in \{0, 1/7, 2/7, 3/7\}$, $\theta \in \{0.04, 0.06, 0.08, 0.10, 0.12\}$ and $n \in \{0.5, 1, 2, 4\}$.

7.2 Discussion on numerical results

These results provide certain insights into the inventory model developed here. Some issues are the following:

1. With fixed τ and θ, if the value of the index of the demand pattern n is increasing, then the optimal length of the inventory cycle where the net stock is positive t_1^*, the optimal inventory cycle T^* and the maximum profit π^* are all also increasing.
2. In general, assuming fixed τ and θ, the optimal profit π^* is more sensitive to variations in the parameter n when the value of τ is short. However, the optimal inventory cycle T^* is more sensitive to n when the value τ is large.
3. With fixed θ and n, the optimal inventory cycle T^* and the optimal positive inventory cycle t_1^* both increase as the parameter τ increases. However, the maximum profit π^* is first decreasing and later increasing.

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Table 4. Effects of the parameters θ, τ and n on the optimal policy and the maximum profit

τ	θ	$n = 0.5$			$n = 1$			$n = 2$			$n = 4$		
		t_1^*	T^*	π^*	t_1^*	T^*	π^*	t_1^*	T^*	π^*	t_1^*	T^*	π^*
0	0.04	0.362331	0.501019	1500.59	0.388287	0.536988	1511.30	0.456104	0.631014	1534.32	0.477209	0.660290	1635.80
	0.06	0.341037	0.485749	1495.67	0.363203	0.517422	1505.78	0.425349	0.606294	1529.04	0.469794	0.669915	1611.64
	0.08	0.322578	0.472894	1491.28	0.341509	0.500769	1500.74	0.398663	0.585004	1524.22	0.452083	0.663847	1597.59
	0.10	0.306373	0.461931	1487.34	0.322518	0.486406	1496.11	0.375292	0.566503	1519.78	0.432092	0.652874	1587.77
	0.12	0.292000	0.452475	1483.78	0.305723	0.473879	1491.84	0.354635	0.550272	1515.65	0.412250	0.640463	1580.18
1/7	0.04	0.371939	0.502642	1506.50	0.400592	0.542339	1518.25	0.472134	0.641512	1540.02	0.594880	0.811850	1564.92
	0.06	0.354913	0.487951	1504.11	0.380839	0.524982	1515.86	0.447322	0.620019	1537.75	0.561304	0.783223	1562.85
	0.08	0.340414	0.475552	1502.01	0.364028	0.510294	1513.73	0.426125	0.601756	1535.73	0.532531	0.758819	1560.98
	0.10	0.327898	0.464938	1500.14	0.349527	0.497689	1511.84	0.407784	0.586025	1533.90	0.507568	0.737738	1559.28
	0.12	0.316971	0.455744	1498.47	0.336879	0.486746	1510.13	0.391741	0.572321	1532.25	0.485682	0.719325	1557.73
2/7	0.04	0.389261	0.514761	1510.95	0.420481	0.558009	1522.47	0.492644	0.658030	1543.45	0.615615	0.828661	1567.47
	0.06	0.379324	0.504948	1510.39	0.408619	0.546778	1521.84	0.475887	0.642916	1542.65	0.590160	0.806497	1566.50
	0.08	0.371112	0.496831	1509.93	0.398696	0.537395	1521.30	0.461696	0.630149	1541.95	0.568445	0.787659	1565.64
	0.10	0.364213	0.490007	1509.54	0.390269	0.529436	1520.83	0.449512	0.619216	1541.33	0.549685	0.771435	1564.88
	0.12	0.358338	0.484190	1509.20	0.383022	0.522597	1520.42	0.438933	0.609741	1540.79	0.533303	0.757307	1564.19
3/7	0.04	0.417029	0.542137	1512.44	0.447610	0.583517	1524.09	0.516849	0.679480	1545.13	0.638349	0.848055	1568.96
	0.06	0.417029	0.542137	1512.44	0.445741	0.581660	1524.08	0.509263	0.672431	1545.00	0.621627	0.833251	1568.60
	0.08	0.417029	0.542137	1512.44	0.444206	0.580135	1524.07	0.502887	0.666512	1544.88	0.607414	0.820698	1568.30
	0.10	0.417029	0.542137	1512.44	0.442923	0.578860	1524.06	0.497451	0.661470	1544.79	0.595178	0.809911	1568.03
	0.12	0.417029	0.542137	1512.44	0.441835	0.577779	1524.06	0.492761	0.657123	1544.70	0.584529	0.800539	1567.79

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4. In general, given θ and n, the optimal inventory cycle T^* and the maximum profit π^* are more sensitive to variations in the parameter τ when the value of n is large.
5. Supposing that τ and n are fixed, if the value of the deterioration rate θ increases, then the maximum profit π^* and the optimal positive inventory cycle t_1^* are decreasing. However, the optimal inventory cycle T^* is first increasing and later decreasing.
6. In general, given τ and n, the maximum profit π^* is more sensitive to changes in the parameter θ when the value of n is large or when the value of τ is short.

8 Managerial insights and policy implications

Next, managerial implications based on the sensitivity analysis of the parameters are set out. Some suggestions are provided to inventory managers that could help them to improve the efficiency of their inventory control.

From the computational results, and the above comments described in the paper, we can deduce the following managerial insights:

1. The largest increase in profit per unit of time is obtained when the power demand pattern index n increases. When this index n is greater than one, a larger portion of the demand occurs towards the beginning of the inventory cycle and the remaining demand decreases along the scheduling period. Then, the practitioners and inventory managers should encourage customers to purchase products preferably towards the first half of the inventory cycle. To do so, managers can increment demand by increasing advertising or marketing campaigns (for example, increasing advertisements about the goodness of the product in the press, radio, television or social networks), or by giving incentives to customers to increase the purchase of the product (for example, considering a discount in the sale price, or offering an additional free unit of the product for the purchase of several items of that product).
2. Another way to increase the profit per unit of time is to reduce the deterioration rate θ. If this reduction is small, then the increase in profit is minimal. To achieve a higher profit, the deterioration rate should be noticeably reduced, which is not easy to achieve; since a reduction in the deterioration rate requires a considerable economic investment to improve the infrastructures and hygienic and environmental conditions of the warehouses where the products are stored.

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3. Another alternative to increase the benefit per unit of time would be to decrease the replenishment cost A per order. To do this, the fixed cost of the transportation of products should be reduced. This transportation cost includes the cost of the conveyances used, insurance, taxes, and also the cost of the machinery and labor used in loading and unloading the products. Any reduction achieved in the price of transporting the merchandise or products will suppose an increase in the benefit.
4. Furthermore, the reduction of the unit holding cost h per unit of time, or the reduction of the unit backlogging cost ω per unit of time, leads to an increase in the benefit per unit of time. Decreasing the unit shortage cost is difficult for practitioners, but it is possible to reduce the unit holding cost by acting on the fixed costs related to the warehouse where the stock of products is held. Thus, for example, an increment in the profit could be obtained by reducing costs for insurance, cleaning, electricity, water, heating or cooling items.
5. In general, from the computational results, it can be deduced that, when the power demand pattern index n is less than one, an increase in the time period τ in which the product does not suffer any deterioration leads to an increment in the benefit per unit of time. From the point of view of practitioners or inventory managers, it is not possible to act directly on the time period τ, since it is an intrinsic characteristic of the product. However, it can be done indirectly, by improving the conservation and maintenance conditions of stored products.

Additionally, the inventory model considers that the carbon emission taxes applied to transport (μ_1), maintenance (μ_2) and the item deterioration process (μ_3) increase the inventory management costs. Incorporating the new total carbon emissions cost into the inventory system forces companies to modify their stock management policies, making them more sustainable.

The enterprises could use the results of this study to improve their business models. It encourages compliance with increasingly demanding environmental and health regulations by promoting the sustainable storage and transport of products. It also makes it possible to improve the company's image and brand reputation by demonstrating its commitment to the environment and sustainability. Moreover, it could strengthen relationships with supply chain actors by minimizing risks and supporting sustainable maintenance and transport of stocks. In parallel, the paper provides several direct and indirect policy implications. Its underlying analysis

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could suggest that authorities could generate fiscal incentives to stimulate companies to implement strategies aimed at increasing customer demand in specific periods; promote economic support programs to invest in sustainable infrastructure; establish regulations that require storage standards and environmental conditions that help reduce the rate of deterioration of items; as well as encourage efficient policies in the fields of logistics and transportation. Also, decisions could focus on driving energy efficiency practices to reduce costs; favoring the adoption of cleaner technologies in warehouses; or recognizing companies that adopt socially responsible inventory management practices.

Finally, a sustainable approach to the inventory management model for deteriorated items with a power demand pattern can offer several significant benefits, including reducing waste from product maintenance and transport, enhancing item quality guarantees, reducing greenhouse gas emissions, improving risk management, and strengthening the image and reputation of the company, among others. By adopting responsible and sustainable inventory management practices, companies can increase their profitability, efficiency, and competitiveness while contributing to the reduction of the environmental and social impact of their operations.

9 Conclusions

This paper studies an economic order quantity model for a sustainable inventory system with power demand pattern and backlogged shortages, considering a carbon emissions tax. In this inventory system, it is assumed that there is a period where the items are kept in the inventory in perfect conditions but, after that time, a deterioration process starts in the stored items that causes a percentage of these products to deteriorate and thus they cannot be sold.

Two scenarios are presented and studied. In the first, it is considered that the lifetime of the articles is greater than the time-period when these items are stored in the inventory. In this case, the optimal inventory policy is derived for an inventory system without deterioration, with power demand pattern and shortages completely backlogged, assuming a carbon emissions tax. This model extends to the inventory system analyzed by Sicilia et al. (2012), who developed the optimal inventory policy for items with a power demand pattern and backlogged shortages, but without analyzing the effect of a carbon tax on the inventory system. In addition, the inventory model developed in this paper extends the environmental EOQ model proposed by Bonney and

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Jaber (2011) and the inventory model analyzed by Hua et al. (2011) when the firm has no carbon emissions quotas and neither buys nor sells any carbon credit.

In the second scenario, it is assumed that the lifetime of the articles is less than the period that these articles stay in the inventory. Thus, the behavior pattern of the inventory level is as follows: the inventory starts with an initial stock level and then that level is gradually decreasing to meet the demand of the customers. From a certain point, the inventory level not only decreases to satisfy the orders of customers, but also decreases due to the loss of products because of the deterioration process. When the inventory runs out of stock, shortages appear which are covered with the arrival of the next inventory replenishment. In this case, the inventory model with a carbon emissions tax proposed in this paper extends the sustainable EOQ models without shortages and with full backlogging studied by Taleizadeh et al. (2017). Furthermore, the inventory policy proposed here extends the optimal policy for deteriorated items with a power demand pattern analyzed by Sicilia et al. (2013) and the optimal policy for an inventory system with a power demand pattern, deterioration and full backlogging developed by Sicilia et al. (2014a).

Considering the different evolution of the net level of inventory for each scenario throughout the planning period, the costs involved in inventory management are determined in each situation, and the general problem of profit maximization per inventory cycle is formulated. Subsequently, the necessary optimality conditions that the best inventory policy should satisfy are developed and an algorithm that allows us to determine the optimal inventory policy is proposed. In all the cases analyzed, through numerical examples, it is confirmed that the solutions obtained reflect substantial reductions in inventory management costs with respect to models in which carbon emissions rates are not taken into account. Furthermore, optimal policies that incorporate economic, operational, as well as environmental parameters (taxes applied to transport, maintenance, and the item deterioration process) in inventory management provide a more holistic view that raises awareness of the importance of implementing business activities that reduce pollution, protect the environment, and increase the well-being of society in general.

This paper makes a significant contribution to business knowledge and practice by helping to promote sustainable and efficient practices in the inventory management and commercial distribution of perishable items within the supply chain. It can help managers in organizations to make more informed decisions that allow

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them to generate a positive impact on the profitability of the business and, at the same time, reduce their carbon emissions. It also offers suggestions to political actors to implement actions that contribute to reducing deterioration and waste generation by organizations, satisfying the growing demands of customers who value sustainability, and fulfilling SDG 12 of the UN Agenda 2030.

As future works of research, it would be interesting to study the effect of a carbon-tax on some of the inventory systems characterized by the following hypothesis: (i) a system with deterioration and power demand pattern considering partial backlogging; (ii) a system for items with deterioration assuming power demand pattern and loss of sales; (iii) a system for items with a process of deterioration, power demand pattern and backlogging when the replenishment is not instantaneous and a finite rate of replenishment of the products is considered.

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