Applied Mathematical Modelling 132 (2024) 57–72

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Pricing decision in a newsvendor model with partial backorders under normal probability distribution for the demand

Valentín Pando [a,b,*], Luis A. San-José [c,d], Joaquín Sicilia [e], David Alcaide-López-de-Pablo [e]

[a] *Departamento de Estadística e Investigación Operativa, Universidad de Valladolid, Spain*
[b] *Instituto Universitario Investigación Gestión Forestal Sostenible (IUFOR), Universidad de Valladolid, Spain*
[c] *Instituto de Investigación en Matemáticas (IMUVA), Universidad de Valladolid, Spain*
[d] *Departamento de Matemática Aplicada, Universidad de Valladolid, Spain*
[e] *Departamento de Matemáticas, Estadística e Investigación Operativa, Universidad de La Laguna, Spain*

ARTICLE INFO

Keywords:
Stochastic inventory
Newsvendor
Emergency order
Price-dependent normal distribution for the demand
Partial backordering

ABSTRACT

This paper presents a newsvendor model with backorders for customers who are willing to wait to be served. Demand follows a normal probability distribution, with the particularity that the expected value depends on the sale price and the variation coefficient is fixed. Three parameters are considered to characterize this dependence on the expected demand and the sale price: the population size of potential customers, the unit production cost of the item and an elasticity parameter with an isoelastic type. Backorders and lost sales are combined with a fixed proportion for the backorders. The quantities to be determined are the sale price and the order quantity. The goal is the maximization of the expected profit. The optimal solution is obtained in a closed form if there are no lost sales. In the case of a mixture of partial backordering and lost sales, a methodological proposal based on a numerical algorithm is given. The study reveals that the maximum expected profit and the optimal quantities to be determined are highly influenced by the unit purchasing cost and the degree of dependence of the demand concerning the sale price. Other parameters, such as the proportion of backorders and the variation coefficient of demand, are less influential. Numerical examples are used to illustrate the model, and a sensitivity analysis of the optimal solution regarding the nine initial parameters is presented. Some managerial insights deduced from the obtained results are also proposed.

1. Introduction

The newsvendor problem (see, for example, Hillier & Lieberman [1]) is a basic single-period stochastic inventory model that has been the subject of several papers throughout the history of the inventory literature. In brief, the original formulation of the newsvendor problem (also known as the newsboy problem) can be described as follows. The problem arises when a vendor has to order goods or items before the sales period starts and there is no possibility of reordering in case of need. It is also assumed that customer demand is described by a random variable that follows a known probability distribution and that the excess stock at the

* Corresponding author at: Departamento de Estadística e Investigación Operativa, Escuela Técnica Superior de Ingenierías Agrarias, Avenida de Madrid 57, 34004-Palencia (Spain).

E-mail addresses: vpando@uva.es (V. Pando), luisaugusto.san-jose@uva.es (L.A. San-José), jsicilia@ull.es (J. Sicilia), dalcaide@ull.es (D. Alcaide-López-de-Pablo).

https://doi.org/10.1016/j.apm.2024.04.037
Received 2 October 2023; Received in revised form 1 March 2024; Accepted 16 April 2024

Available online 23 April 2024
0307-904X/

end of the selling period cannot be sold. The objective is to determine the quantity to be ordered to maximize the expected profit. In commercial activity, this situation is common for seasonal products, fashion items or promotional launches.

Since this first formulation, the newsboy model has been extended in several ways and remains a current area of research within the Inventory Theory. This paper considers a further extension that combines some features not previously considered. First of all, it assumes a normal probability distribution for the stochastic demand. Although the Poisson distribution seems to be the most logical for customer demand, its discrete nature has meant that it is not often used when the population size of potential customers is high. Considering that the Poisson distribution can be approximated by a normal distribution with continuity correction when its expected value is large, we propose its use in the newsboy problem. Furthermore, this paper assumes that the stochastic demand depends on the sale price of the item. Then the retailer can increase demand by lowering the sale price. This assumption seems to be a more realistic scenario because, for a large majority of products, the demand is always higher when the sale price is lower. But in this case, the sale price is a decision variable along with the order size. The pricing decision is also essential for the vendor. In addition, this paper assumes that, in the event of a shortage, a certain proportion of customers will be served with backorders and the others will be lost. The vendor incurs an additional cost for the backorders and weighs a goodwill cost for the lost sales. Considering the sale price as an additional decision variable, we did not find any paper with all these features.

Then, the purpose of this paper is to establish a general solution methodology for the newsvendor problem with backorders and lost sales, considering a price-dependent normal probability distribution function for the demand and using the sale price as an additional decision variable. To this end, Section 2 presents a review of the more relevant literature on these topics. Section 3 includes the assumptions and notation, together with the calculation of the expected profit to be maximized. The formulation of the model is given in Section 4. The optimal solution to the inventory problem is proposed in Section 5, with a subsection for the case of total backorders and another subsection for the case of partial backorders with lost sales. Numerical examples with sensitivity analysis are given in Section 6. Finally, Section 7 contains a further discussion of the managerial insights derived from the obtained results and Section 8 includes the conclusions and some future research lines.

2. Literature review

There is an extensive literature on the newsboy problem. Proof of this is that many reviews have been published on the subject. Already in the 1990s, three review papers were published: Gallego & Moon [2], Khouja [3], Petruzzi & Dada [4]. Another three can be cited in the current century: Qin et al. [5], Sharma & Nandi [6], and Mu et al. [7]. The first extensions of the newsboy problem focused on the consideration of backorders to fill shortages during the sale period. Gallego & Moon [2] and Khouja [8] were the first authors to consider this possibility. In these papers, a fixed fraction of the unsatisfied demand was considered for backorders. Some years later, Lodree [9], Lodree et al. [10] and Lee & Lodree [11] extended this assumption by considering that the cited fraction depends on the extent of the shortage. Wee & Wang [12] developed a newsboy problem in the coordination of the supply chain with an option contract and partial backorders. Also, Li & Ou [13] considered a model where the unsatisfied demand due to shortages is partially backordered via an emergency channel with a relatively higher unit cost. Zhang et al. [14] proposed a model combining long-term contract procurement and spot replenishment.

All these extensions considered the sale price of the item as a fixed parameter of the system. However, the assumption that the expected demand for the item is not constant and depends on the sale price certainly seems to be a more realistic scenario. Therefore, recent enhancements have made the logical assumption that the demand is higher when the sale price is lower. In this way, the inventory manager can choose the appropriate sale price to maximize the expected profit. Thus, there are two variables to be determined in the newsvendor problem: the order quantity and the sale price. As in Petruzzi & Dada [4] or Hrabec et al. [15], two types of stochastic dependence have mostly been used: the additive case and the multiplicative case. The former considered that the sale price only affects the expected value of demand, assuming that it is the sum of a price-dependent deterministic function and a random variable. The second supposed that the sale price also influences the variance of demand, assuming it to be the product of a deterministic price-dependent function and a random variable.

Price-dependent demand has been widely used in both production-inventory models and newsboy models. For instance, Ahmadi & Shavandi [16] studied dynamic pricing in a production system with a single product demanded by several customer classes. Similarly, Singer & Khmelnitsky [17] considered a stochastic production-inventory problem with price-sensitive demand. Additionally, Yu et al. [18] proposed a newsvendor model with fuzzy price-dependent demand. Other papers that have studied newsboy models with price-dependent demand include Lau & Lau [19], Sana [20] and Zhang et al. [21].

In this paper, we consider an isoelastic function for the price-dependent demand, which has been widely used in different forms, both in deterministic and stochastic inventories. This approach has been studied by Agrawal & Ferguson [22], Chang et al. [23] and Duary et al. [24]. This paper employs the formulation introduced by Pando et al. [25] to describe price-dependent demand using three parameters: the population size of potential customers, price elasticity, and the production cost of the item. The production cost is the minimum sale price below which the item cannot be sold. Together, these parameters provide an adequate representation of the expected demand for the item. Yao et al. [26] compiled up to six other types of dependence used by other authors as possible extensions of the newsvendor problem with pricing. Raza et al. [27] studied a stochastic inventory model with pricing decisions and a multi-objective approach. Ullah et al. [28] introduced a multi-period newsvendor under stochastic price-dependent demand with dynamic pricing. Ma et al. [29] also studied the joint decision of the order quantity and sale prices for a stochastic inventory model with multiple discounts.

The probability distribution of demand is another issue that needs to be addressed in the newsvendor problem. Although the basic model can be solved for any probability distribution, this is not true for some extensions, such as the case of price-dependent

Table 1
Comparison table of the most related literature cited in this paper.

Paper	Demand distribution	Sale price	Backlogging type	Price-dependent demand
Lau & Lau [19]	Normal	Variable	No	Yes
Khouja [8]	Free	Parameter	Fixed	No
Shore [30]	Poisson	Parameter	No	No
Lau [36]	Uniform, Exponential, Normal	Parameter	No	No
Yao et al. [26]	Free	Variable	No	Yes
Lodree [9]	Free	Parameter	Variable	No
Lodree et al. [10]	Exponential, Normal	Parameter	Variable	No
Lee & Lodree [11]	Normal	Parameter	Variable	No
Halkos & Kevork [37]	Truncated normal	Parameter	No	No
Su & Pearn [42]	Normal	Parameter	No	No
Sana [20]	Free	Variable	No	Yes
Pando et al. [34]	Exponential	Parameter	Variable	No
Saidane et al. [35]	Gamma	Parameter	No	No
Yu et al. [18]	Fuzzy	Variable	No	Yes
Pando et al. [33]	Uniform	Parameter	Variable	No
Rossi et al. [31]	Binomial, Poisson, Exponential	Parameter	No	No
Raza et al. [27]	Normal	Variable	No	Yes
Chen [32]	Poisson	Parameter	No	No
Ullah et al. [28]	Free	Variable	No	Yes
Ma et al. [29]	Normal	Variable	No	Yes
This paper	Normal	Variable	Fixed	Yes

stochastic demand or when the size of the backorders depends on the extent of the shortage. A particular probability distribution must then be specified for the demand. Although the Poisson distribution may seem the most logical for customer demand, its discrete nature has meant that it is not widely used. Some papers on the newsvendor problem with Poisson distribution demand are: Shore [30], Rossi et al. [31] and Chen [32]. However, continuous probability distributions have been used more often. For example, Pando et al. [33] considered uniform distribution, Pando et al. [34] worked with exponential distribution, and Saidane et al. [35] assumed gamma distribution.

The normal probability distribution has also been used, perhaps because the Poisson distribution can be approximated by a normal distribution with continuity correction when its expected value is large. In the basic newsvendor problem, when demand is normally distributed, the standard critical fractile formula solves the problem only when the variation coefficient is sufficiently small. Note that this is true when a Poisson distribution with an expected value greater than thirty is approximated by a normal probability distribution. In the basic newsvendor problem, Lau [36] provided a simple formula for computing the optimal order quantity and the maximum expected profit when demand is normal with a variation coefficient of less than 0.3. Halkos & Kevork [37] used a normal distribution singly truncated at point zero. Perakis & Roels [38] stated that a normal distribution with a small coefficient of variation is robust and also maximizes the entropy when only mean and variance are known. When the variation coefficient of the demand is large, Gallego et al. [39] suggested the fit of the empirical distribution to non-negative random variables such as gamma or lognormal. Other papers that used normal probability distribution in the newsvendor problem are Khouja [40], Ouyang et al. [41] and Su & Pearn [42].

Table 1 presents a collection of papers on newsvendor models cited in this literature review that are most relevant to the model being presented. The papers are classified based on the probability distribution for demand, the role of the sale price (parameter or decision variable), the backlogging type (no backorder, fixed fraction or variable fraction), and the dependence or independence of demand on the sale price. They are arranged chronologically.

3. Assumptions and notation

In this paper, we consider a newsvendor problem with the following assumptions:

- The inventory manager has the ability to place an extraordinary order to satisfy the demand of those customers who could not be served with the initial order and are willing to wait for the arrival of a new order. The fraction of customers backlogged is denoted by β, with $0 \leq \beta \leq 1$, and the fraction of lost sales is $1-\beta$.
- If p is the sale price of the item, the random demand X has a normal distribution with positive expected value μ_p and variation coefficient ν, where the dependence of μ_p concerning the sale price p is isoelastic. In addition, the variation coefficient ν is fixed for any p and the standard deviation is $\nu\mu_p$. That is, we suppose that $X \rightsquigarrow N\left(\mu_p, \nu\mu_p\right)$ and, therefore, $X=\mu_p\varepsilon$, where ε follows a normal probability distribution with expected value 1 and standard deviation ν. Then, it is the multiplicative demand case considered by Petruzzi & Dada [4], but assuming a normal probability distribution for demand with expected value μ_p and variation coefficient ν.
- The decision variables are the initial order quantity q and the sale price p.
- The aim is to maximize the expected profit during the sales period.

Table 2
Notation of the model.

c	unit purchasing cost before the sale period (> 0)
η	unit production cost ($0 < \eta \leq c$)
o	unit overstocking cost excluding the purchasing cost ($> -c$)
ω	unit extra cost of the backorders (> 0)
γ	unit goodwill cost of lost sale (> 0)
β	intensity of backorder ($0 \leq \beta \leq 1$)
s	unit shortage cost, that is, $s = \beta(c+\omega) + (1-\beta)\gamma$
q	order quantity (*decision variable*) (≥ 0)
p	sale price (*decision variable*) ($\geq c$)
X	random demand in the sale period
x	a specific value for demand
$f(x)$	density function of the random variable X
ν	variation coefficient of demand ($\nu > 0$)
λ	expected demand if $p = \eta$ ($\lambda > 0$)
α	elasticity of the expected demand regarding the sale price (> 2)
μ_p	expected value for demand if the sale price is p, that is, $\mu_p = E(X) = \lambda (p/\eta)^{-\alpha}$
z	standardized order quantity (*auxiliary decision variable*), that is, $z = \left(q - \mu_p\right) / \left(\nu \mu_p\right)$
$\varphi(z)$	standard normal density function, that is, $\varphi(z) = \exp\left(-z^2/2\right) / \sqrt{2\pi}$
$\Phi(z)$	standard normal distribution function, that is, $\Phi(z) = \int_{-\infty}^{z} \varphi(t)\, dt$
$L(z)$	standard normal loss function, that is, $L(z) = \int_{z}^{\infty} (t-z)\varphi(t)\, dt$

- The expected demand μ_p is defined by a negative power function of the sale price p, as is usual for inventory models with isoelastic price-dependent demand.
- To characterize the isoelastic dependence of demand regarding the sale price, we suppose that there exists a minimum reference price η below which the item can not be sold. This value η can be seen as the unit production cost or the factory price.
- The unit purchasing cost before the sale period is a value c, which has to be at least the unit production cost η.
- The maximum possible value λ for the expected demand μ_p would be obtained when $p = \eta$. Thus, the value λ can be understood as the population size of potential customers that can only be achieved if the sale price p is η and, therefore, without profit on the sale because it is below the purchasing cost c.
- The function that defines the expected demand μ_p regarding the sale price p is

$$\mu_p = \lambda \left(\frac{p}{\eta}\right)^{-\alpha} \tag{1}$$

  with $\alpha > 2$. The parameter α reflects the dependence degree of the expected demand concerning the sale price. Note that it satisfies

$$\alpha = -\frac{\partial \mu_p / \partial p}{\mu_p / p}$$

  and, therefore, it is the ratio between the relative decrease in the expected demand and the relative increase in the sale price. Jeuland & Shugan [43] introduced the condition $\alpha > 2$ for this type of price-dependent demand with a negative exponent of the sale price. In this way, the rational conjectural behavior and the Nash equilibrium for the demand functions are achieved, which is highly valued in economic theory. This assumption is logical in highly competitive markets where demand is very sensitive to the sales price.
- The leftover items at the end of the sale period have a unit additional cost o, which can be negative if they can be sold below the purchasing cost c. Therefore, we have $o > -c$.
- The unit purchasing cost for the items in the extraordinary order has a surplus cost $\omega > 0$ and, as a consequence, they are purchased at a unit cost $c + \omega$.
- The inventory manager considers that each lost sale has a positive unit goodwill cost γ. This is an intangible value that takes into account the loss of customer confidence or the company's brand reputation. This cost can influence the choice of the sale price or the optimal order quantity, even though it is not included in the warehouse accounting.

Table 2 collects the notation used in the stochastic inventory model to be developed.

4. Formulation of the model

Let p, q and x be the sale price, the order quantity and the observed value for the demand, respectively. If $x \leq q$ the income is px, and the total cost is the sum of the purchasing cost cq and the overstocking cost $o(q-x)$. Then the obtained profit is $B(p,q,x) = px - cq - o(q-x)$.

On the other hand, if $x > q$ the income from the initial order and the backordered items are pq and $\beta p(x-q)$ respectively. The total cost is the sum of the initial purchasing cost cq, the purchasing cost of the backordered items $\beta(c+\omega)(x-q)$, and the goodwill cost for the lost sales $(1-\beta)\gamma(x-q)$. Then the unit shortage cost is $s=\beta(c+\omega)+(1-\beta)\gamma$ and the obtained profit is $B(p,q,x)=pq+\beta p(x-q)-cq-s(x-q)$.

As a consequence, the mathematical function for the profit is

$$B(p,q,x)=\begin{cases} px-cq-o(q-x) & \text{if } x\le q \\ pq+\beta p(x-q)-cq-s(x-q) & \text{if } x>q \end{cases} \tag{2}$$

Thus, if $f(x)$ is the density function of the random demand, the expected profit $E(B(p,q,x))$ is

$$G(p,q)=E(B(p,q,x))=\int_{-\infty}^{q}(px-cq-o(q-x))\,f(x)\,dx+\int_{q}^{\infty}(pq+\beta p(x-q)-cq-s(x-q))\,f(x)\,dx$$

The first integral can be evaluated as

$$\begin{aligned}\int_{-\infty}^{q}(px-cq-o(q-x))\,f(x)\,dx&=(p+o)\int_{-\infty}^{\infty}xf(x)dx-(c+o)\int_{-\infty}^{\infty}qf(x)\,dx-\int_{q}^{\infty}(px-cq-o(x-q))\,f(x)\,dx\\&=(p+o)\,\mu_p-(c+o)\,q-\int_{q}^{\infty}(px-cq-o(x-q))\,f(x)\,dx\end{aligned}$$

and, adding the second integral, the function $G(p,q)$ can be expressed as

$$\begin{aligned}G(p,q)&=(p+o)\,\mu_p-(c+o)\,q+\int_{q}^{\infty}(-p(x-q)+\beta p(x-q)-(s+o)(x-q))\,f(x)\,dx\\&=(p+o)\,\mu_p-(c+o)\,q-((1-\beta)\,p+s+o)\int_{q}^{\infty}(x-q)f(x)dx\end{aligned} \tag{3}$$

To maximize the expected profit, we can reformulate the problem by using the standardized initial order quantity $z=\left(q-\mu_p\right)/\left(\nu\mu_p\right)$ as an auxiliary decision variable. Thus, $q=\mu_p+\nu\mu_p z$ and, using the change of variable $t=\left(x-\mu_p\right)/\left(\nu\mu_p\right)$ in the integral given in (3), we have $x=\mu_p+\nu\mu_p t$, $dx=\nu\mu_p dt$ and

$$\int_{q}^{\infty}(x-q)f(x)dx=\nu\mu_p\int_{z}^{\infty}(t-z)f(\mu_p+\nu\mu_p t)\nu\mu_p dt$$

Moreover, as $X\rightsquigarrow N\left(\mu_p,\nu\mu_p\right)$, we have

$$f(\mu_p+\nu\mu_p t)=\frac{\exp\left(-0.5\left(\frac{\mu_p+\nu\mu_p t-\mu_p}{\nu\mu_p}\right)^2\right)}{\sqrt{2\pi}\nu\mu_p}=\frac{\exp(-0.5t^2)}{\sqrt{2\pi}\nu\mu_p}$$

As a consequence, the expected profit can be formulated as a function of p and z as follows:

$$\begin{aligned}\Lambda(p,z)&=(p+o)\,\mu_p-(c+o)\left(\mu_p+\nu\mu_p z\right)-((1-\beta)\,p+s+o)\nu\mu_p\int_{z}^{\infty}(t-z)f(\mu_p+\nu\mu_p t)\nu\mu_p dt\\&=(p-c)\,\mu_p-(c+o)\,\nu\mu_p z-((1-\beta)\,p+s+o)\nu\mu_p\int_{z}^{\infty}(t-z)\frac{\exp(-0.5t^2)}{\sqrt{2\pi}}dt\\&=\mu_p\left((p-c)-(c+o)\,\nu z-((1-\beta)\,p+s+o)\nu L(z)\right)\end{aligned} \tag{4}$$

where $\varphi(t)=\frac{\exp(-0.5t^2)}{\sqrt{2\pi}}$ is the standard normal density function and $L(z)=\int_z^{\infty}(t-z)\varphi(t)\,dt$ is the standard normal loss function.

5. Solution of the model

First of all, the next lemma allows us to obtain the optimal value of z for each fixed $p\in[c,\infty)$.

Lemma 1. *For each fixed sale price p, the maximum value of the expected profit $\Lambda(p,z)$ given by (4) is obtained when the standardized order quantity is*

$$z_p^* = \Phi^{-1}\left(1 - \frac{c+o}{(1-\beta)\,p + s + o}\right) \tag{5}$$

where Φ^{-1} is the inverse cumulative standard normal distribution function.

Proof. Please see the proof in the Appendix. □

Note that, as $p \geq c$, $o > -c$, $\omega > 0$, $\gamma > 0$ and $s = \beta(c+\omega) + (1-\beta)\gamma$, then we can ensure that

$$(1-\beta)\,p + s + o \geq (1-\beta)\,c + s + o = c + \beta\omega + (1-\beta)\gamma + o > c + o > 0$$

As a consequence, we have

$$0 < \frac{c+o}{(1-\beta)\,p + s + o} < 1$$

and the value z_p^* is always well defined.

From the previous lemma, the problem of the expected profit maximization consists in the maximization of the price-dependent function

$$g(p) = \Lambda\left(p, z_p^*\right) = \mu_p\left(p - c - \nu\,(c+o)\,z_p^* - \nu((1-\beta)\,p + s + o)L(z_p^*)\right) \tag{6}$$

with $p \geq c$, $\mu_p = \lambda\,(p/\eta)^{-\alpha}$ and $z_p^* = \Phi^{-1}\left(1 - \frac{c+o}{(1-\beta)p+s+o}\right)$.

Now, taking into account that $L(z) = \varphi(z) - z\,(1-\Phi(z))$ and $1 - \Phi\left(z_p^*\right) = \frac{c+o}{(1-\beta)p+s+o}$, we have

$$\begin{aligned}((1-\beta)\,p + s + o)L(z_p^*) &= ((1-\beta)\,p + s + o)\varphi\left(z_p^*\right) - ((1-\beta)\,p + s + o)z_p^*\left(1 - \Phi\left(z_p^*\right)\right)\\ &= ((1-\beta)\,p + s + o)\varphi\left(z_p^*\right) - (c+o)\,z_p^*\end{aligned}$$

and therefore the function $g(p)$ can also be written as

$$g(p) = \Lambda\left(p, z_p^*\right) = \mu_p \xi\,(p) \tag{7}$$

where $\xi\,(p)$ is the function

$$\xi\,(p) = p - c - \nu((1-\beta)\,p + s + o)\varphi\left(z_p^*\right) \tag{8}$$

Note that, for each sale price p, the function $\xi\,(p)$ is the ratio between the expected profit $g\,(p)$ and the expected demand μ_p if the order quantity is optimally chosen for that p. Then, it can be seen as a measure of the average profit per demanded unit in the inventory.

Hence, the problem of expected profit maximization is

$$\max_{p\in[c,\infty)} g\,(p) = \max_{p\in[c,\infty)} \mu_p \xi\,(p) \tag{9}$$

To solve the problem, we first need to calculate the derivatives of z_p^* and $\xi\,(p)$. Taking into account that $\Phi'\,(z) = \varphi\,(z)$, $\varphi'\,(z) = -z\varphi\,(z)$ and (5), the derivative of z_p^* regarding p is

$$\frac{dz_p^*}{dp} = \frac{(1-\beta)\,(c+o)}{((1-\beta)\,p + s + o)^2\,\varphi\left(z_p^*\right)} = \frac{(1-\beta)\left(1 - \Phi\left(z_p^*\right)\right)}{((1-\beta)\,p + s + o)\,\varphi\left(z_p^*\right)} \tag{10}$$

As a consequence, the first derivative of $\xi\,(p)$ is

$$\begin{aligned}\xi'\,(p) &= 1 - \nu\,(1-\beta)\,\varphi\left(z_p^*\right) + \nu((1-\beta)\,p + s + o)z_p^*\varphi\left(z_p^*\right)\left(\frac{dz_p^*}{dp}\right)\\ &= 1 - \nu\,(1-\beta)\left(\varphi\left(z_p^*\right) - z_p^*\left(1 - \Phi\left(z_p^*\right)\right)\right) = 1 - \nu\,(1-\beta)\,L\left(z_p^*\right)\end{aligned} \tag{11}$$

Then, from (7), the first derivative of the function $g\,(p)$ is

$$g'\,(p) = \lambda\eta^{\alpha}\left(\frac{\xi_1\,(p)}{p^{\alpha+1}}\right) \tag{12}$$

with

$$\xi_1\,(p) = p\xi'\,(p) - \alpha\xi\,(p) \tag{13}$$

Note that, as the signs of the functions $g'(p)$ and $\xi_1(p)$ are equal for any sale price p, the critical values of the function $g(p)$ are the roots of the equation $\xi_1(p)=0$.

Depending on the intensity of backorder (that is, the value of the parameter β), two cases will be considered in the resolution of the problem.

5.1. Case $\beta = 1$ (total backordering, there are no lost sales)

In this scenario, when there is no stock, all the customers are willing to wait for the arrival of the extraordinary order to receive the item. In this case, as $\beta = 1$, from (5) it follows that z_p^* is constant and does not depend on p because

$$z_p^* = z^* = \Phi^{-1}\left(1 - \frac{c+o}{s+o}\right) \tag{14}$$

for any p. Then, as $\beta = 1$, the objective function given by (7) is

$$g(p) = \lambda\eta^{\alpha}\left(\frac{p - c - v(s+o)\varphi(z^*)}{p^{\alpha}}\right)$$

From (11), as $\beta = 1$, the function $\xi'(p)$ is also constant with $\xi'(p) = 1$ for any $p \geq c$, and $\xi_1(p)$ is:

$$\xi_1(p) = (1-\alpha)p + \alpha\left(c + v(s+o)\varphi\left(z^*\right)\right)$$

Then the unique solution of $\xi_1(p) = 0$ is

$$p^* = \frac{\alpha\left(c + v(s+o)\varphi(z^*)\right)}{\alpha - 1} \tag{15}$$

Note that $\xi_1(p) > 0$ if $p \in [c, p^*)$ and $\xi_1(p) < 0$ if $p \in (p^*, \infty)$. As a consequence, $g(p)$ increases on $[c, p^*)$ and decreases on (p^*, ∞), which means that the point $(p^*, g(p^*))$ is the global maximum of $g(p)$ in $[c, \infty)$. Moreover, the value of $g(p^*)$ is

$$g(p^*) = \lambda\eta^{\alpha}\left(\frac{p^* - c - v(s+o)\varphi(z^*)}{(p^*)^{\alpha}}\right) = \lambda\eta^{\alpha}\left(\frac{c + v(s+o)\varphi(z^*)}{(\alpha-1)(p^*)^{\alpha}}\right)$$

Finally, as $q = \mu_p + v\mu_p z = \mu_p(1 + vz)$, from (1) the optimal order quantity is

$$q^* = \lambda\eta^{\alpha}\left(\frac{1 + vz^*}{(p^*)^{\alpha}}\right) \tag{16}$$

with maximum profit

$$G^* = G\left(p^*, q^*\right) = \Lambda\left(p^*, z^*\right) = g(p^*) = \lambda\eta^{\alpha}\left(\frac{c + v(s+o)\varphi(z^*)}{(\alpha-1)(p^*)^{\alpha}}\right) \tag{17}$$

5.2. Case $0 \leq \beta < 1$ (partial backordering, there are lost sales)

In this case, the following lemma provides a lower bound for the optimal sale price.

Lemma 2. *Let $\xi(p)$ be the function given by (8), with $0 \leq \beta < 1$, and z_p^* given by (5). Then, the following is satisfied:*

(i) The function $\xi(p)$ is strictly convex on the interval (c, ∞).
(ii) There is a unique solution p_l of the equation $\xi(p) = 0$ in the interval (c, ∞).
(iii) $\xi(p) < 0$ if $p \in \left[c, p_l\right)$, $\xi(p) > 0$ if $p \in \left(p_l, \infty\right)$ and $\xi'(p) > 0$ if $p \geq p_l$.

Proof. Please see the proof in the Appendix. □

Corollary 1. *Under the hypotheses of Lemma 2, it follows that $\xi(p)$ is positive and strictly convex on the interval $\left(p_l, \infty\right)$.*

Proof. Obvious by Lemma 2 and because $c < p_l$. □

As $\mu_p > 0$, and from (7), it follows that, for each value $p \geq c$, the functions $g(p)$ and $\xi(p)$ have the same sign. Therefore, by (7), Lemma 2 and Corollary 1, we can ensure that $g(p) < 0$ if $p \in \left[c, p_l\right)$, $g(p) > 0$ if $p \in \left(p_l, \infty\right)$ and the problem of the expected profit maximization can be reduced to

$$\max_{p \in \left(p_l, \infty\right)} g(p) \tag{18}$$

where the function $g(p)$ is positive on the interval $\left(p_l, \infty\right)$.

Now, Lemma 3 provides an upper bound for the optimal sale price.

Lemma 3. *Let $\xi_1(p)$ be the function given by (13), with $0 \le \beta < 1$, and z_p^* given by (5). Let p_l be the value in the interval (c, ∞) such that $\xi\left(p_l\right) = 0$, and let $u(p)$ be the function given by*

$$u(p) = 1 - \frac{\alpha \xi(p)}{p} \tag{19}$$

Then, the following statements are true:

(i) The function $u(p)$ is strictly decreasing on the interval $\left(p_l, \infty\right)$.
(ii) There is a unique solution p_u of the equation $u(p) = 0$ in the interval $\left(p_l, \infty\right)$.
(iii) $\xi_1(p) < 0$ for $p \in \left(p_u, \infty\right)$.

Proof. Please see the proof in the Appendix. □

Note that the equation $u(p) = 0$ is equivalent to $\xi(p) = p/\alpha$ and therefore $\xi\left(p_u\right) = p_u/\alpha$.

From (12) it follows that, for each value $p \ge c$, the functions $g'(p)$ and $\xi_1(p)$ have the same sign. In addition, from Lemma 3, we can ensure that $g'(p) < 0$ for any $p \ge p_u$ and the problem of the expected profit maximization can be reduced from (18) to

$$\max_{p \in \left(p_l, p_u\right)} g(p) \tag{20}$$

Once the search interval for the optimal sale price has been limited to the interval $\left(p_l, p_u\right)$, the following lemma allows us to find the roots of equation $\xi_1(p) = 0$ in such an interval $\left(p_l, p_u\right)$ and, therefore, the critical values of the function $g(p)$.

Lemma 4. *Let $\xi(p)$ and $\xi_1(p)$ be the functions given by (8) and (13), respectively, with $0 \le \beta < 1$, and z_p^* given by (5). Let p_l and p_u be the values given in Lemmas 2 and 3, respectively. Then, the following statements are true:*

(i) $\xi_1(p)$ is a strictly concave function on the interval $\left(p_l, p_u\right)$.
(ii) There is a unique solution p^* of the equation $\xi_1(p) = 0$ in the interval $\left(p_l, p_u\right)$.
(iii) $\xi_1(p) > 0$ for $p \in \left(p_l, p^*\right)$, $\xi_1(p) < 0$ for $p \in \left(p^*, p_u\right)$, and $\xi_1'(p) < 0$ for $p \in \left(p^*, p_u\right)$.

Proof. Please see the proof in the Appendix. □

Now, the solution of the problem (20) is proposed in the next theorem.

Theorem 1. *Let $g(p)$ be the function given by (7), with $0 \le \beta < 1$ and z_p^* given by (5). Let p_l, p_u and p^* be the sale prices given by Lemmas 2, 3 and 4, respectively. Then, the following is satisfied:*

(i) p^* is the unique local maximum of $g(p)$ in the interval $\left(p_l, \infty\right)$.
(ii) p^* is the global maximum of $g(p)$ in the interval $[c, \infty)$.
(iii) The function $g(p)$ is strictly pseudoconcave on the interval $\left(p_l, \infty\right)$.

Proof. Please see the proof in the Appendix. □

Once this optimal sale price p^* has been obtained, we can do the following

$$z^* = z_{p^*}^* = \Phi^{-1}\left(1 - \frac{c+o}{(1-\beta)p^* + s + o}\right)$$
$$\mu^* = \mu_{p^*}^* = \lambda\left(p^*/\eta\right)^{-\alpha}$$
$$\xi^* = \xi\left(p^*\right) = p^* - c - \nu((1-\beta)p^* + s + o)\varphi\left(z^*\right)$$

and

$$g\left(p^*\right) = \mu^* \xi\left(p^*\right) = \mu^* \xi^*$$

Then the maximum value of $G(p, q)$ is reached for the point (p^*, q^*) with

$$q^* = \mu^*\left(1 + \nu z^*\right)$$

and the maximum expected profit is

$$G^* = G\left(p^*, q^*\right) = \Lambda\left(p^*, z^*\right) = g\left(p^*\right) = \mu^* \xi\left(p^*\right) = \mu^* \xi^*$$

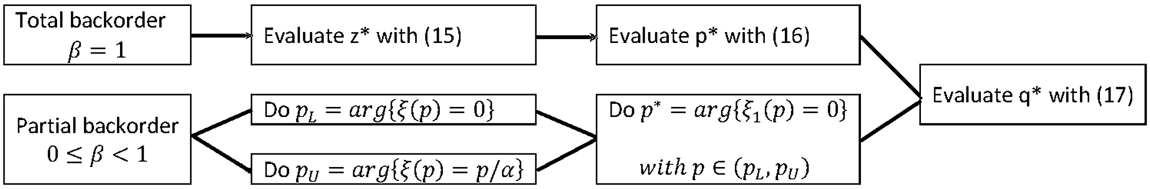

Fig. 1. Solution procedure flow chart.

Note that the equation $\xi_1(p) = 0$ does not depend on the parameters λ or η of the expected demand. Therefore the optimal sale price p^* does not depend on these parameters. Nor does z^* depend on λ or η. As a consequence, the optimal selling price p^* does not change if the population size of the potential customers λ or the reference price η move.

If $0 \leq \beta < 1$, the calculation process to obtain the optimal solution is summarized by the next algorithm.

Algorithm 1. Optimal solution for the inventory model when $0 \leq \beta < 1$:

1. Program the functions

$$\xi(p) = p - c - \nu((1-\beta)p + s + o)\varphi\left(\Phi^{-1}\left(1 - \frac{c+o}{(1-\beta)p + s + o}\right)\right)$$

   and

$$\xi'(p) = 1 - \nu(1-\beta)L\left(\Phi^{-1}\left(1 - \frac{c+o}{(1-\beta)p + s + o}\right)\right)$$

2. Solve the equation $\xi(p) = 0$, with $p > c$, to obtain the unique root p_l.
3. Program the function $u(p) = 1 - \alpha\xi(p)/p$.
4. Solve the equation $u(p) = 0$, with $p > p_l$, to obtain the unique root p_u.
5. Program the function $\xi_1(p) = p\xi'(p) - \alpha\xi(p)$.
6. Solve the equation $\xi_1(p) = 0$, with $p \in (p_l, p_u)$, to obtain the unique root p^*, which is the optimal sale price.
7. Calculate the optimal expected demand $\mu^* = \lambda(p^*/\eta)^{-\alpha}$.
8. Calculate the optimal standardized order quantity

$$z^* = \Phi^{-1}\left(1 - \frac{c+o}{(1-\beta)p^* + s + o}\right)$$

9. Calculate the optimal order quantity $q^* = \mu^*(1 + \nu z^*)$.
10. Calculate the maximum expected profit $G^* = \mu^*\xi^*$, with $\xi^* = p^* - c - \nu((1-\beta)p^* + s + o)\varphi(z^*)$.

A flow chart of the optimal solution search procedure is shown in Fig. 1.

6. Numerical examples with sensitivity analysis

In this section, the proposed model is illustrated with numerical examples where the optimal inventory policy is obtained. A sensitivity analysis is also included to analyse the effect of the parameters of the model on the maximum expected profit and the optimal quantities to be determined.

Example 1. Let us suppose that a fashion clothing store needs to place an order of swimsuits to sell during the upcoming summer season. The unit production cost of each swimsuit is $\eta = \$18$ and the retailer purchases it from a producer at a price $c = \$30$. The leftover swimsuits at the end of the summer are stored for the following season at a holding cost of \$5 per item. Then, the overstocking cost is $o = \$5$. In case of shortage, the retailer can order the swimsuit from the manufacturer for \$38 as long as the customer is willing to wait for the product to arrive. Then, the extra cost of the backorders is $\omega = \$8$. It is estimated that only 70% of customers would be willing to wait for backordered items, that is, the intensity of the backorder is $\beta = 0.7$. Also, the retailer considers a unit goodwill cost $\gamma = \$4$ for the lost sales due to the loss of customer confidence and the company's brand reputation. The expected demand at a sale price equal to the production cost is $\lambda = 8000$ and the dependence degree of the expected demand concerning the sale price is $\alpha = 3$. It is also assumed that the probability distribution of demand is a normal distribution with a variation coefficient $\nu = 0.25$, that is, 25% of the expected demand. With these assumptions, the retailer wants to know how many swimsuits should be ordered from the manufacturer and what the sale price should be to maximize the expected profit during the summer season.

With these values, the unit shortage cost is $s = \beta(c + \omega) + (1-\beta)\gamma = \$27.8$. In this case, the solution of the equation $\xi(p) = 0$ is $p_l = 32.79$. Thus, the expected profit is positive if the sale price is larger than \$32.79 and the order quantity is optimally chosen. Furthermore, if the sale price is lower than this value, the expected profit is negative for any order quantity. Now, to find an upper bound for the sale price, we need to solve the equation $u(p) = 0$. In this example, the solution is $p_u = 50.99$, that is, if the sale price is greater than \$50.99, then the expected profit can no longer be improved. So, the optimal sale price must belong to the interval

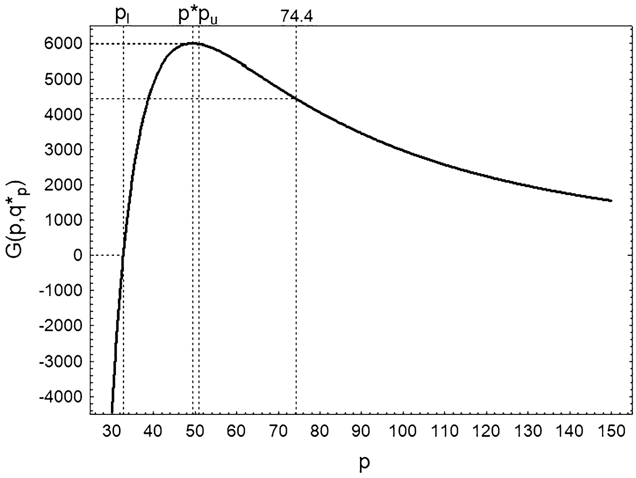

Fig. 2. Graphic view of the function $g(p) = G\left(p, q_p^*\right)$ for Example 1.

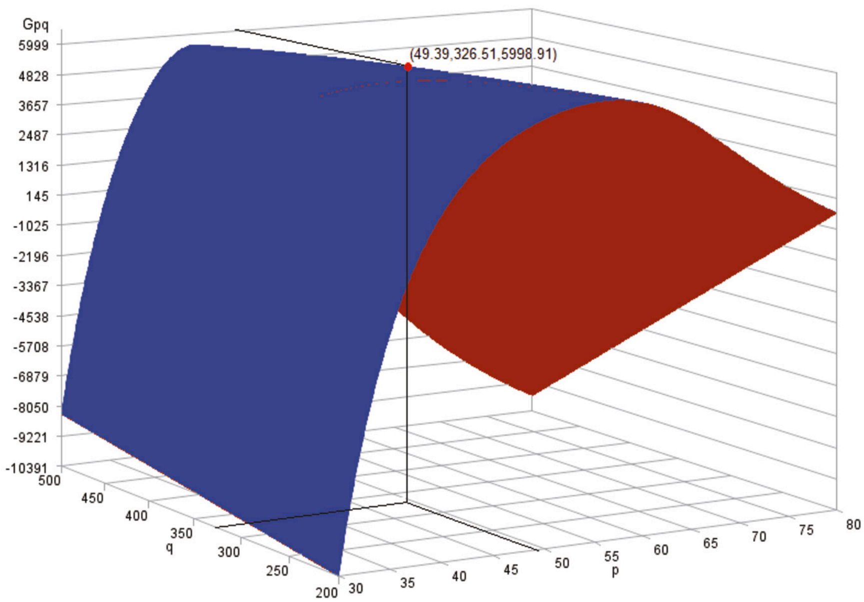

Fig. 3. Graphic view of the surface $G(p,q)$ for Example 1.

$(32.79, 50.99)$. Solving the equation $\xi_1(p) = 0$ in this interval we obtained $p^* = 49.39$ and, therefore, if the sale price is \$49.39, the maximum expected profit is reached. As a result, the expected demand for this sale price, given by (1), is $\mu^* = \lambda(p^*/\eta)^{-\alpha} = 387.33$. The optimal value z_p^* can be evaluated with the expression (5) to obtain $z^* = -0.6282$, which leads to an optimal order quantity $q^* = \mu^*(1 + \nu z^*) = 326.51$. Note that, in this case, the order quantity is below the expected demand. Now, by using the expression (8), we have $\xi^* = \xi(p^*) = 15.4877$, that is, the optimal measure of the average profit per unit in the inventory is \$15.4877. Finally, the maximum expected profit is $G^* = \mu^*\xi^* = \$5998.91$. If the retailer rounds up to 327 swimsuits and a price of \$50, the expected profit would be \$5984.72.

Fig. 2 plots the function $g(p) = G\left(p, q_p^*\right)$ with the points p_l, p^* and p_u used to solve the problem. Note that this function has an inflection point at $p = 74.4$ and is convex on the interval $(74.4, \infty)$, but it is a pseudoconcave function on the interval $(30, \infty)$.

Furthermore, the surface defined by the function $G(p,q)$ for $p \in [30, 80]$ and $q \in [200, 500]$ is plotted in Fig. 3 to show that the global maximum is obtained for $p^* = 49.39$ and $q^* = 326.51$ with a maximum expected profit of $G^* = \$5998.91$.

To illustrate the use of the model, we performed a simulation study considering this optimal policy and using a sample of size $n = 1000$ drawn from a normal distribution with expected value 387.33 and variation coefficient 0.25. Then we evaluated the profit for each case by using the expression (2) to obtain a sampling distribution of the profit. The obtained results show an average sample profit of \$5972.53, a sample median of \$6793.99, and a variation coefficient of 38.58%. The sampling distribution of the profit is highly skewed to the left, and the optimal value $G^* = \$5998.91$ turns out to be, approximately, the first quartile of the sample profit.

Table 3
Effects of the parameters on the optimal policy and the maximum expected profit.

	Δ	−40%	−20%	−10%	10%	20%	40%
c	$\Delta p^*(\%)$	−37.8499	−18.8776	−9.4293	9.4136	18.8140	37.5828
	$\Delta q^*(\%)$	334.8704	90.8278	35.7733	−24.2421	−41.2470	−62.6091
	$\Delta G^*(\%)$	161.0632	52.5110	22.1149	−16.5950	−29.3681	−47.4501
o	$\Delta p^*(\%)$	−0.1686	−0.0831	−0.0411	0.0407	0.0808	0.1593
	$\Delta q^*(\%)$	1.5182	0.7456	0.3695	−0.3632	−0.7202	−1.4165
	$\Delta G^*(\%)$	0.5359	0.2639	0.1310	−0.1291	−0.2563	−0.5052
ω	$\Delta p^*(\%)$	−1.9113	−0.9190	−0.4511	0.4356	0.8567	1.6598
	$\Delta q^*(\%)$	1.8374	0.9166	0.4568	−0.4524	−0.8994	−1.7745
	$\Delta G^*(\%)$	3.1133	1.4882	0.7283	−0.6993	−1.3715	−2.6424
γ	$\Delta p^*(\%)$	−0.3856	−0.1913	−0.0953	0.0946	0.1885	0.3742
	$\Delta q^*(\%)$	0.3913	0.1953	0.0975	−0.0973	−0.1945	−0.3881
	$\Delta G^*(\%)$	0.6224	0.3085	0.1535	−0.1522	−0.3031	−0.6011
β	$\Delta p^*(\%)$	0.5951	0.2550	0.1157	−0.0890	−0.1474	−0.1505
	$\Delta q^*(\%)$	3.5705	2.0846	1.1265	−1.3190	−2.8612	−6.7900
	$\Delta G^*(\%)$	−5.0194	−2.6207	−1.3403	1.4059	2.8841	6.0915
ν	$p^*(\%)$	−3.7302	−1.8962	−0.9559	0.9718	1.9598	3.9854
	$q^*(\%)$	19.9024	9.5207	4.6563	−4.4553	−8.7166	−16.6837
	$G^*(\%)$	10.5137	5.1448	2.5447	−2.4901	−4.9264	−9.6403
λ	$p^*(\%)$	0	0	0	0	0	0
	$q^*(\%)$	−40	−20	−10	10	20	40
	$G^*(\%)$	−40	−20	−10	10	20	40
η	$p^*(\%)$	0	0	0	0	0	0
	$q^*(\%)$	−78.4	−48.8	−27.1	33.1	72.8	174.4
	$G^*(\%)$	−78.4	−48.8	−27.1	33.1	72.8	174.4
α	$p^*(\%)$		14.6618	6.0184	−4.4314	−7.8304	−12.7015
	$q^*(\%)$		35.6699	17.0003	−15.0120	−28.0439	−48.7940
	$G^*(\%)$		90.1462	36.5010	−25.6007	−43.9706	−67.3287

So, for this numerical example, more than 75% of the time, the inventory manager will make a profit greater than the optimal expected profit.

We also evaluated the percentage changes in the optimal policy and the maximum expected profit when each individual initial parameter of the inventory system is moved, while keeping all the others fixed. The change values for each parameter were ±10%, ±20% and ±40%. The obtained results are included in Table 3.

Some findings obtained from this sensitivity analysis are now listed:

(i) The purchasing cost c is the most influential parameter in the sale price p^*, with changes in percentage terms roughly resembling the changes in c. As expected, this effect is always positive, that is, when c is increased, the value p^* is also increased. The elasticity parameter α of the expected demand with respect to the sale price is also very influential in the optimal sale price, with changes between −12% and 14% for changes in α between −20% and 40%. Note that changes in α of −40% have not been evaluated because in that case $\alpha = 1.8 < 2$. The effect of this parameter is always negative, that is, the optimal sale price p^* decreases if parameter α increases. The remaining parameters are much less influential, with changes in p^* of less than 4% for the variation coefficient ν and less than 2% for the others when the parameters change between −40% and 40%. For parameter β, the effect is negative, that is, a larger intensity of backorder leads to a smaller sale price p^*. Instead, the effect is positive for the cost parameters o, ω or γ. Finally, the optimal sale price p^* does not change if either the population size of the potential customers λ or the production cost of the item η changes.

(ii) The purchasing cost c is also very influential in the order quantity q^*, with changes between 334% and −62% for changes in c between −40% and 40%. The effect is now negative, that is, there is a decrease in the order quantity q^* when c increases. The variations in the optimal order quantity q^* are also notable for the parameter α (between 35% and −48% for changes in c between −20% and 40%). It also has an effect with a negative sign, that is, there is a decrease in the order quantity q^* when α increases. Regarding the parameters λ and η of the expected demand, the influence is positive, that is, there is an increase in the order quantity q^* when any of the parameters λ or η increases. The relative changes in q^* are equal to the relative changes in parameter λ, and equal to the power of order α in parameter η. The effect of parameter β on the order quantity q^* goes from 3.5% to −6.8% and has a negative effect, that is, the optimal order quantity q^* decreases when the intensity of backorder β increases. Finally, the effect of the cost parameters o, ω and γ are much less influential, with relative changes of less than ±2%, always with a positive effect, that is, there is an increase in the order quantity q^* when any of the parameters o, ω or γ increases.

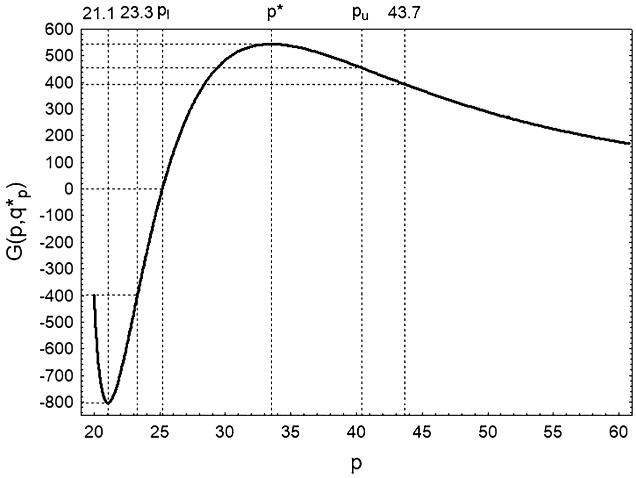

Fig. 4. Graphic view of the function $g(p) = G\left(p, q_p^*\right)$ for Example 3.

(iii) There is a decrease in the maximum expected profit G^* when any of the cost parameters c, o, ω or γ increases, with large relative changes for the purchasing cost c and small ones for o, ω or γ. The effect of parameter v in the maximum expected profit is also negative, that is, G^* decreases if the variation coefficient v increases, with relative changes between -10% and 10% for relative changes in v between -40% and 40%. On the other hand, the effect of parameter β is positive, that is, the maximum expected profit increases if the intensity of backorder β increases, with relative changes between -5% and 6% for relative changes in β between -40% and 40%. The effects of the expected demand parameters λ and η in the maximum expected profit G^* and the optimal order quantity q^* are exactly equal, and with the same sign, that is, an increase in either of the parameters λ or η leads to an increase in the maximum expected profit G^*. Finally, the elasticity parameter α is now the most influential in the maximum expected profit G^*, with relative changes between 90% and -67% for relative changes in α between -20% and 40%. This effect is now negative, that is, an increase in the elasticity parameter α leads to a decrease in the maximum expected profit G^*. Note that, the larger relative changes in the maximum expected profit are stronger if either of the parameters c or α decreases, than when such a parameter increases.

Example 2. To illustrate the case of full backordering (without lost sales) we consider the same parameters as in Example 1 with $\beta = 1$ instead of $\beta = 0.7$. As $\beta = 1$, from (14) and (15), we have $z^* = -0.8926$ and $p^* = 49.32$. Then, from (1), the expected demand is $\mu^* = 388.92$ and, from (16), the optimal order quantity is $q^* = 302.13$. Finally, from (17), the maximum expected profit is $G^* = 6393.69$. As expected, we obtain a lower sale price, a lower order quantity and a higher maximum expected profit than in Example 1 with $\beta = 0.7$.

Example 3. Consider now an inventory system with the following values for the parameters: $c = \$20$, $o = \$7$, $\omega = \$0.1$, $\gamma = \$0.1$, $\eta = \$15$, $\beta = 0.1$, $v = 0.7$, $\lambda = 8000$ and $\alpha = 5$. In this case, the unit shortage cost is $s = \$2.1$ and the unique solution of $\xi(p) = 0$ is $p_l = 25.19$. Next, we solve the equation $u(p) = 0$ to obtain $p_u = 40.45$ and, finally, we solve the equation $\xi_1(p) = 0$ in the interval $\left(p_l, p_u\right)$ to obtain $p^* = 33.52$. Then, the optimal sale price that gives us the maximum expected profit is now \$33.52. For this sale price, the expected demand given by (1), is $\mu^* = 143.62$ and the optimal value for z_p^*, given in (5), is $z^* = -0.4891$. As a consequence, the optimal order quantity is $q^* = \mu^*(1 + vz^*) = 94.45$, and the optimal value for the function $\xi(p)$ given by (8) is $\xi^* = 3.7881$, that is, the optimal measure of the average profit per unit in the inventory is \$3.7881. Finally, the maximum expected profit is $G^* = \mu^* \xi^* = \$544.06$. Fig. 4 plots the function $g(p) = G\left(p, q_p^*\right)$ with the points p_l, p^* and p_u obtained in this case. Note that now this function is not pseudoconcave on the interval (c, ∞) because it has a local minimum at $p = 21.1$. Moreover, it has two inflection points at $p = 23.3$ and $p = 43.7$. However, it is pseudoconcave on the interval $\left(p_l, \infty\right)$, as Theorem 1 establishes.

In this case, Fig. 5 plots the surface defined by the function $G(p, q)$ for $p \in [20, 60]$ and $q \in [50, 150]$. It shows that the global maximum is obtained for $p^* = 33.52$ and $q^* = 94.45$ with a maximum expected profit of $G^* = \$544.06$.

7. Managerial insights

In this section, some managerial insights are given to assist the inventory manager in deciding the optimal policy. From the results obtained in sections 5 and 6, the following statements can be highlighted:

(i) The dependence degree of the expected demand concerning the sale price is the most relevant parameter on the optimal policy and the maximum expected profit. Then, the inventory manager needs to estimate this parameter as precisely as possible before choosing the order quantity and the sale price.

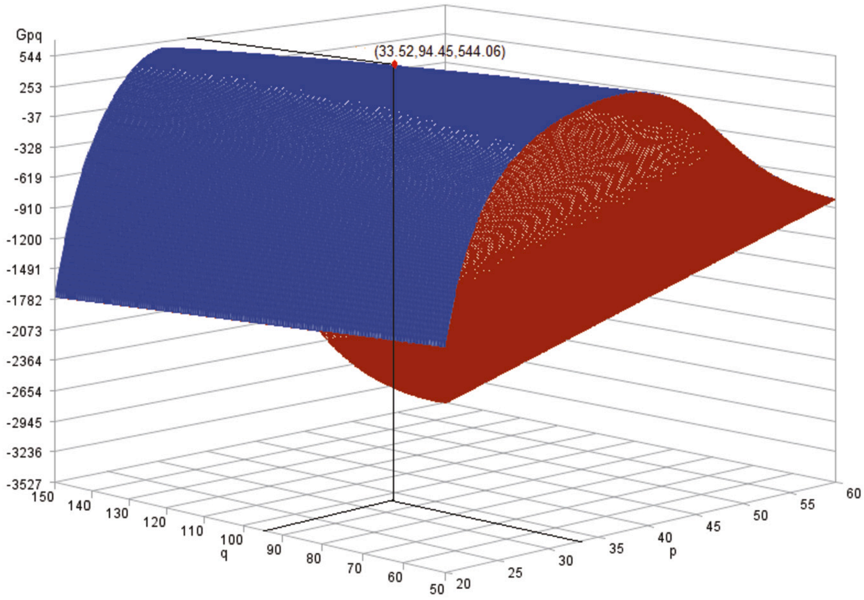

Fig. 5. Graphic view of the surface $G(p, q)$ for Example 3.

(ii) The unit purchasing cost of the item also has a great influence on the optimal policy and the maximum expected profit. A smaller order quantity, a higher sale price and larger maximum profit are obtained when the unit purchasing cost is close to the unit production cost of the item, which is the real market value for customers.
(iii) The optimal sale price does not depend on the unit production cost of the item, but only on the unit purchasing cost. However, the optimal order quantity and the maximum expected profit are deeply influenced by the unit production cost. Usually, a higher unit production cost (i.e., a more expensive item) leads to a higher sale price and a higher expected profit, despite reduced demand.
(iv) The optimal sale price does not change when the expected size of the potential customers changes. Furthermore, the relative changes in the optimal order quantity and the maximum expected profit are equal to the relative changes in this parameter.
(v) The proportion of customers who are willing to wait for the extraordinary order (i.e., the intensity backorder parameter) has little influence on the optimal solution. The optimal values for the order quantity and the sale price decrease if this parameter increases. Otherwise, the maximum expected profit increases if this parameter increases. Nevertheless, the degree of influence is low.
(vi) When the variation coefficient of the demand increases, the maximum expected profit and the optimal order quantity decrease. However, the optimal sale price increases.
(vii) The optimal policy and the maximum expected profit hardly change if the unit overstocking cost, the unit extra cost of the backorders or the unit goodwill cost change.

8. Conclusions and future research

This paper presents a methodological proposal for calculating the optimal solution for the newsvendor problem with a normal probability distribution for demand. It is assumed that the expected demand is a function of the sale price with three parameters: the population size of potential customers, the price elasticity, and the production cost of the item below which it can not be sold. However, the variation coefficient of demand does not change with the sale price. Furthermore, a combination of partial backorders and lost sales is considered for the customers not served with the initial order. A portion of them are willing to wait for an extraordinary order. The values to be determined are the initial order quantity and the sale price. The goal is the maximization of the expected profit in the sales period.

Given a sale price, the optimal order quantity is obtained in a closed form. Then, the optimal expected profit is evaluated, and the problem is reduced to an optimization problem to obtain the optimal sale price. Lower and upper bounds are obtained by solving two equations that ensure the optimal solution is always between them. Then, the maximization problem is solved in this interval. The function defined by the expected profit for each optimal order quantity turns out to be pseudoconcave in this interval. The unique critical value is calculated in a closed form in the case of total backorder without lost sales, and numerically in the case of a mixture of partial backorders and lost sales. An algorithmic procedure is proposed in this latter situation.

Numerical examples are used to illustrate the relevance of the proposed model. A sensitivity analysis of the optimal policy concerning the nine initial parameters is also developed.

Note that the unit production cost and the elasticity parameter of the expected demand regarding the sale price are decisive parameters for the optimal solution. Then, they must be well estimated from previous data on the market, especially the demand elasticity. If the inventory manager does not know the unit production cost from the producer, it may be useful to assume that it is equal to the unit purchasing cost as a starting point. Perhaps, these issues could be considered as a limitation for this model.

Possible extensions to the newsvendor model under consideration could be the use of different probability distributions for demand. Furthermore, an interesting work would be to develop the inventory model considering a different type of demand dependence on the selling price. Finally, another study could be to analyze the inventory problem assuming that the fraction of backorders is not fixed and that it depends on the extent of the shortage.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Funding

This work is supported by Grant [PID2021-125042NB-I00] funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe". Also, we wish to thank the editors and the anonymous referees for their valuable suggestions and comments to improve the paper.

Appendix A

This appendix includes the proofs of the four lemmas and the theorem included in the paper.

Proof of Lemma 1. Taking into account that the derivative of the function $L(z)$ is $L'(z) = \Phi(z) - 1$, for each fixed $p \geq c$ we have

$$\frac{\partial \Lambda(p,z)}{\partial z} = v\mu_p\left(-(c+o) + ((1-\beta)p + s + o)(1-\Phi(z))\right)$$

and

$$\frac{\partial^2 \Lambda(p,z)}{\partial z^2} = -v\mu_p((1-\beta)p + s + o)\varphi(z) < 0$$

Then, the unique solution of the equation $\frac{\partial \Lambda(p,z)}{\partial z} = 0$ is the value z_p^* given by (5). Moreover, as $\frac{\partial^2 \Lambda(p,z)}{\partial z^2} < 0$ for any z, the maximum value of $\Lambda(p,z)$ is $\Lambda\left(p, z_p^*\right)$ for each fixed sale price p. □

Proof of Lemma 2. The second derivative of the function $\xi(p)$ is

$$\xi''(p) = v(1-\beta)\left(1-\Phi\left(z_p^*\right)\right)\left(\frac{dz_p^*}{dp}\right) = \frac{v(1-\beta)^2(c+o)^2}{((1-\beta)p+s+o)^3\,\varphi\left(z_p^*\right)}$$

As $0 \leq \beta < 1$, we can ensure that $\xi''(p) > 0$ for any $p \in [c,\infty)$ and, therefore, $\xi(p)$ is strictly convex on this interval, which proves (*i*). On the other hand, as $\lim_{p\to\infty} z_p^* = \infty$ and $\lim_{p\to\infty} \varphi\left(z_p^*\right) = 0$, then, from (8),

$$\lim_{p\to\infty} \xi(p) = \lim_{p\to\infty} p\left(1 - \frac{c + v((1-\beta)p+s+o)\varphi\left(z_p^*\right)}{p}\right) = \lim_{p\to\infty} p = \infty$$

Moreover, $\xi(c) < 0$ and, therefore, the equation $\xi(p) = 0$ has at least one root p_l in (c,∞). Furthermore, since $\xi(p)$ is strictly convex, p_l is the unique root of the equation $\xi(p) = 0$ in the interval (c,∞) and (*ii*) is proved. In addition, it has to be $\xi(p) < 0$ if $p \in \left[c, p_l\right)$ and $\xi(p) > 0$ if $p \in \left(p_l, \infty\right)$. Also, $\xi'\left(p_l\right) > 0$ because, if $\xi'\left(p_l\right) = 0$, then p_l would be an inflection point of $\xi(p)$ and this is impossible because $\xi(p)$ is a strictly convex function on the interval (c,∞). Similarly, $\xi'(p) > 0$ if $p \in \left(p_l, \infty\right)$ because, as $\xi(p)$ is a strictly convex function on this interval, $\xi'(p)$ is strictly increasing on the interval $\left(p_l, \infty\right)$. Then, the proof is finished. □

Proof of Lemma 3. The derivative of the function $u(p)$ is

$$u'(p) = -\frac{\alpha\xi_2(p)}{p^2}$$

with $\xi_2(p) = p\xi'(p) - \xi(p)$, whose derivative is $\xi_2'(p) = p\xi''(p)$. Then, the function $\xi_2(p)$ is an increasing function on the interval (p_l, ∞) because $\xi''(p) > 0$ for any $p \in (p_l, \infty)$. Moreover, from Lemma 2, $\xi(p_l) = 0$ and $\xi'(p_l) > 0$, then we have $\xi_2(p_l) = p_l\xi'(p_l) > 0$. Consequently, as $\xi_2(p)$ increases on (p_l, ∞), we have $\xi_2(p) > 0$ for any $p \in (p_l, \infty)$ and the function $u(p)$ is strictly decreasing on the interval (p_l, ∞), which proves (*i*). Furthermore, as $\xi(p_l) = 0$ we have $u(p_l) = 1 > 0$ and, from (8)

$$\lim_{p\to\infty} u(p) = \lim_{p\to\infty}\left(1 - \frac{\alpha\xi(p)}{p}\right) = 1 - \alpha + \alpha \lim_{p\to\infty}\left(\frac{c + v((1-\beta)p + s + o)\varphi\left(z_p^*\right)}{p}\right) = 1 - \alpha < 0$$

because $\lim_{p\to\infty} z_p^* = \infty$ and $\lim_{p\to\infty}\varphi\left(z_p^*\right) = 0$. As a consequence, in the interval (p_l, ∞), there is a unique root p_u. Then (*ii*) is proved. Moreover, if $p \in (p_u, \infty)$, we have $u(p) < 0$, $\xi'(p) < 1$ and $\xi_1(p) = p\xi'(p) - p + pu(p) < 0$. Then the proof is finished. □

Proof of Lemma 4. As $\xi_1(p) = p\xi'(p) - \alpha\xi(p)$, the two first derivatives of the function $\xi_1(p)$ are $\xi_1'(p) = p\xi''(p) - (\alpha-1)\xi'(p)$ and $\xi_1''(p) = p\xi'''(p) - (\alpha-2)\xi''(p)$. Moreover, as $\xi''(p) > 0$ in the interval (p_l, p_u), taking (5) and (10) into account, the third derivative of the function $\xi(p)$ is

$$\begin{aligned}
\xi'''(p) &= \xi''(p)\left(\frac{d\left(\ln \xi''(p)\right)}{dp}\right) = -\xi''(p)\left(\frac{3(1-\beta)}{(1-\beta)p+s+o} - z_p^*\left(\frac{dz_p^*}{dp}\right)\right) \\
&= \frac{-(1-\beta)\xi''(p)}{(1-\beta)p+s+o}\left(2 + \frac{\varphi\left(z_p^*\right) - z_p^*\left(1-\Phi\left(z_p^*\right)\right)}{\varphi\left(z_p^*\right)}\right) \\
&= \left(\frac{-(1-\beta)\xi''(p)}{(1-\beta)p+s+o}\right)\left(2 + \frac{L\left(z_p^*\right)}{\varphi\left(z_p^*\right)}\right) < 0
\end{aligned}$$

Therefore, as $0 \le \beta < 1$ and $\alpha > 2$, we have $\xi_1''(p) < 0$ and the function $\xi_1(p)$ is strictly concave on the interval (p_l, p_u), which proves statement *(i)*.

Now, to prove statement (*ii*), from Lemmas 2 and 3, we have $\xi(p_l) = 0$, $\xi'(p_l) > 0$, $u(p_u) = 0$, $\xi(p_u) = p_u/\alpha$, and taking (11) into account, we observe that

$$\begin{aligned}
&\xi_1(p_l) = p_l\xi'(p_l) - \alpha\xi(p_l) > 0 \\
&\xi_1(p_u) = p_u\xi'(p_u) - \alpha\xi(p_u) = p_u\left(\xi'(p_u) - 1\right) = -v(1-\beta)p_u L\left(z_{p_u}^*\right) < 0
\end{aligned}$$

As a consequence, the equation $\xi_1(p) = 0$ has at least one root p^* in the interval (p_l, p_u). Moreover, it is the unique root because, in any other case, $\xi_1(p)$ would have to have a local minimum, and this is not possible because $\xi_1(p)$ is a strictly concave function in the interval (p_l, p_u). Then, p^* is the unique root of the equation $\xi_1(p) = 0$ in the interval (p_l, p_u) and statement (*ii*) is proved.

From statement (*ii*), $\xi_1(p)$ is necessarily strictly positive on the interval (p_l, p^*) and $\xi_1(p)$ is strictly negative on the interval (p^*, p_u). Moreover, $\xi_1'(p) < 0$ if $p \in (p^*, p_u)$ because, otherwise, the function $\xi_1(p)$ would have to have a local minimum, and this is impossible because $\xi_1(p)$ is a strictly concave function on the interval (p_l, p_u). Then, statement (*iii*) is proved and the proof is finished. □

Proof of Theorem 1. From (12), $g'(p)$ and $\xi_1(p)$ have the same sign. Then, the previous lemmas ensure that, on the interval (p_l, p_u), the value p^* is the unique critical value of $g(p)$. Moreover, $g(p)$ is strictly increasing on the interval (p_l, p^*) and strictly decreasing on the interval (p^*, p_u). As a consequence, p^* is the global maximum of $g(p)$ on the interval (p_l, p_u). Furthermore, from Lemma 3, $\xi_1(p) < 0$ if $p \in (p_u, \infty)$ and, in the interval (p_u, ∞), $g(p)$ is strictly decreasing. Therefore, the unique local maximum of the function $g(p)$ in the interval (p_l, ∞) is attained at p^*, which proves statement (*i*). In addition, from (7), $g(p^*) > 0$ because $p^* > p_l$ and $\xi(p) > 0$ for any $p > p_l$. However, $g(p) < 0$ if $p \in (c, p_l)$ and $g(p)$ is strictly decreasing if $p \ge p_u$ because, from Lemma 3, $\xi_1(p) < 0$ if $p \in [p_u, \infty)$. What is more, from (7) and (8), and considering that $\lim_{p\to\infty}\varphi\left(z_p^*\right) = 0$, we have

$$\lim_{p\to\infty} g(p) = \lambda\eta^\alpha \lim_{p\to\infty}\frac{\xi(p)}{p^\alpha} = \lambda\eta^\alpha \lim_{p\to\infty}\frac{1 - \frac{c+v((1-\beta)p+s+o)\varphi\left(z_p^*\right)}{p}}{p^{\alpha-1}} = 0$$

Finally, as $g(p)$ has the global maximum on the interval (p_l, p_u) at the point p^*, and $g(p)$ is a strictly decreasing function if $p \ge p_u$, we can ensure that p^* is also the global maximum of $g(p)$ on the interval (p_l, ∞). Moreover, as p^* is the unique point in the interval (p_l, p_u) with $g'(p^*) = 0$, the function $g(p)$ is strictly pseudoconcave on the interval (p_l, ∞) (see for example Cambini & Martein [44], Theorem 3.2.7, p. 45). Then, the proof is complete. □

References

[1] F.S. Hillier, G.J. Lieberman, Introduction to Operations Research, seventh ed., McGraw-Hill, Boston, MA, USA, 2001.
[2] G. Gallego, I. Moon, The distribution free newsboy problem: review and extensions, J. Oper. Res. Soc. 44 (1993) 825–834.
[3] M.A. Khouja, The single period (news-vendor) problem: literature review and suggestions for future research, Omega 27 (1999) 537–553.
[4] H.C. Petruzzi, M. Dada, Pricing and the newsvendor problem: a review with extensions, Oper. Res. 47 (2) (1999) 183–194.
[5] Y. Qin, R. Wang, A.J. Vakharia, Y. Chen, M. Seref, The newsvendor problem: review and directions for future research, Eur. J. Oper. Res. 213 (2) (2011) 361–374.
[6] A. Sharma, S. Nandi, A review of behavioral decision making in the newsvendor problem, Oper. Suppl. Chain Manag., Int. J. 11 (4) (2018) 200–213.
[7] M.T. Mu, J.L. Chen, Y. Yang, J. Guo, The multi-product newsvendor problem: review and extensions, in: 2019 6th International Conference on Behavioral, Economic and Socio-Cultural Computing (BSC), Beijing, China, 2019, pp. 1–4.
[8] M.A. Khouja, A Note on the newsboy problem with an emergency supply option, J. Oper. Res. Soc. 47 (1996) 1530–1534.
[9] E.J. Lodree, Advanced supply chain planning with mixtures of backorders, lost sales, and lost contract, Eur. J. Oper. Res. 181 (2007) 168–183.
[10] E. Lodree, Y. Kim, W. Jang, Time and quantity dependent waiting costs in a newsvendor problem with backlogged shortages, Math. Comput. Model. 47 (2008) 60–71.
[11] H. Lee, E.J. Lodree, Modeling customer impatience in a newsboy problem with time-sensitive shortages, Eur. J. Oper. Res. 205 (2010) 595–603.
[12] H.M. Wee, W.T. Wang, Supply chain coordination for short-life-cycle products with option contract and partial backorders, Eur. J. Ind. Eng. 7 (1) (2013) 78–99.
[13] Y.H. Lee, J.W. Ou, Optimal ordering policy for complementary components with partial backordering and emergency replenishment under spectral risk measure, Eur. J. Oper. Res. 284 (2) (2020) 538–549.
[14] J. Zhang, F. Chan, X. Xu, Data-driven analysis on optimal purchasing decisions in combined procurement, Int. J. Prod. Res. 61 (13) (2023) 4265–4278.
[15] D. Hrabec, J. Kučera, P. Martinek, Marketing effort within the newsvendor problem framework: a systematic review and extensions of demand-effort and cost-effort formulations, Int. J. Prod. Econ. 257 (2023) 108754.
[16] M. Ahmadi, H. Shavandi, Dynamic pricing in a production system with multiple demand classes, Appl. Math. Model. 39 (2015) 2332–2344.
[17] G. Singer, E. Khmelnitsky, A production-inventory problem with price-sensitive demand, Appl. Math. Model. 89 (2021) 688–699.
[18] Y. Yu, J. Zhu, C. Wang, A newsvendor model with fuzzy price-dependent demand, Appl. Math. Model. 37 (2013) 2644–2661.
[19] A.H.L. Lau, H.S. Lau, The Newsboy problem with price-dependent demand distribution, IIE Trans. 20 (2) (1988) 168–175.
[20] S.S. Sana, Price sensitive demand with random sales price - a newsboy problem, Int. J. Syst. Sci. 43 (3) (2012) 491–498.
[21] W.S. Zhang, J.L. Li, R. Zhang, Y.H. Chen, Impact of emergency order in price-dependent newsvendor problems, Asia-Pac. J. Oper. Res. 34 (2) (2017) 1750001.
[22] V. Agrawal, M. Ferguson, Bid-response models for customized pricing, J. Revenue Pricing Manag. 6 (3) (2007) 212–228.
[23] C.T. Chang, Y.J. Chen, T.R. Tsai, S.J. Wu, Inventory models with stock- and price-dependent demand for deteriorating items based on limited shelf space, Yugosl. J. Oper. Res. 20 (1) (2010) 55–69.
[24] A. Duary, M.A. Khan, S. Pani, A.A. Shaik, L.M. Hezam, A.F. Alraheedi, J. Gwak, Inventory model with nonlinear price-dependent demand for non- instantaneous decaying items via advance payment and installment facility, AIMS Math. 7 (11) (2022) 19794–19821.
[25] V. Pando, L.A. San-José, J. Sicilia, D. Alcaide-López-de-Pablo, An inventory model with price and stock-dependent demand and time- and stock quantity-dependent holding cost under profitability maximization, Comput. Oper. Res. 164 (2024) 106520.
[26] L. Yao, Y.F. Chen, H. Yan, The newsvendor problem with pricing: extensions, Int. J. Manag. Sci. Eng. Manag. 1 (2006) 3–16.
[27] S.A. Raza, F.C. Abdullakutty, S. Rathinam, Joint determination of process mean, price differentiation, and production decisions with demand leakage: a multi-objective approach, Appl. Math. Model. 40 (2016) 8446–8463.
[28] M. Ullah, I. Khan, B. Sarkar, Dynamic pricing in a multi-period newsvendor under stochastic price-dependent demand, Mathematics 7 (2019) 520, https://doi.org/10.3390/math7060520.
[29] S. Ma, Z. Jemai, Q. Bai, Optimal pricing and ordering decisions for a retailer using multiple discounts, Eur. J. Oper. Res. 299 (2022) 1177–1192.
[30] H. Shore, A general solution for the newsboy model with random order size and possibly a cutoff transaction size, J. Oper. Res. Soc. 55 (11) (2004) 1218–1228.
[31] R. Rossi, S. Prestwich, S.A. Tarim, B. Hnich, Confidence-based optimisation for the newsvendor problem under binomial, Poisson and exponential demand, Eur. J. Oper. Res. 239 (3) (2014) 674–684.
[32] Y.C. Chen, A newsboy problem with lost sale depends on degree of freshness, J. Inf. Optim. Sci. 7 (4) (2016) 633–640.
[33] V. Pando, L.A. San-José, J. García-Laguna, J. Sicilia, A newsvendor inventory model with an emergency order to supply a non-increasing fraction of shortage, Appl. Math. Comput. 237 (2014) 560–570.
[34] V. Pando, L.A. San-José, J. García-Laguna, J. Sicilia, A newsboy problem with an emergency order under a general backorder rate function, Omega 41 (2013) 1020–1028.
[35] S. Saidane, M.Z. Babai, M.S. Aguir, O. Korbaa, On the performance of the base-stock inventory system under a compound Erlang demand distribution, Comput. Ind. Eng. 6 (3) (2013) 548–554.
[36] H. Lau, Simple formulas for the expected costs in the newsboy problem: an educational note, Eur. J. Oper. Res. 100 (1997) 557–561.
[37] G.E. Halkos, I.S. Kevork, Non negative demand in newsvendor models: the case of singly truncated normal samples, MPRA Paper No. 31842, online at http://mpra.ub.uni-muenchen.de/31842/, 2011.
[38] G. Perakis, G. Roels, Regret in the newsvendor model with partial information, Oper. Res. 56 (2008) 188–203.
[39] G. Gallego, K. Katircioglu, B. Ramachandran, Inventory management under highly uncertain demand, Oper. Res. Lett. 35 (2007) 281–289.
[40] M. Khouja, S.S. Robbins, Linking advertising and quantity decisions in the single-period inventory model, Int. J. Prod. Econ. 86 (2) (2003) 93–105.
[41] L.Y. Ouyang, B.R. Chuang, Y.J. Lin, Effective investment to reduce lost-sales rate in a periodic review inventory model, OR Spektrum 29 (4) (2007) 681–697.
[42] R.H. Su, W.L. Pearn, Product selection for newsboy-type products with normal demands and unequalcosts, Int. J. Prod. Econ. 132 (2011) 214–222.
[43] A.P. Jeuland, S.M. Shugan, Channel of distribution profits when channel members form conjectures, Mark. Sci. 7 (2) (1988) 202–210.
[44] A. Cambini, L. Martein, Generalized Convexity and Optimization, Springer, 2009.