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An inventory model with price- and stock-dependent demand and time- and stock quantity-dependent holding cost under profitability maximization

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ABSTRACT

This paper focuses on inventory models with a broad framework for the storage cost and the demand rate. The cumulative storage cost is modelled with a power function, depending on both time and stock quantity, by using two elasticity coefficients. Similarly, the demand rate has an isoelastic dependence on sale price and stock quantity, modelled with another two elasticity coefficients. These four elasticity coefficients allow many real practical situations to be modelled. A reference price is used to measure the effect of the sale price on the demand rate. The goal is to maximize the income expense ratio (IER), and the sale price, the order level and the reorder point are the decision variables. The operating expense ratio (OER) of the system, defined as the quotient cost/income, is used to solve the problem. The optimum values are obtained with explicit expressions, which is an interesting result for inventory managers. Under the optimum policy, the reorder point is always equal to zero and the order quantity depends on the replenishing cost, the purchase price and the four elasticity coefficients. However, the optimum ordering policy does not depend on the scale parameters of the storage cost and the demand rate. A complete sensitivity analysis for most of the model parameters is performed. A numerical example is used to compare the optimum policies for the maximum income expense ratio and the maximum profit per unit time. Finally, some managerial insights derived from the results are given.

1. Introduction

Nowadays, the intense competition and plurality of the markets require an adequate diversification of the available resources. Investors prefer to invest their money in products with high profitability. In this way, a larger overall profit per unit of time would be achieved. Indeed, if the inventory manager diversifies the available resources into the most profitable items, instead of using them all in just one of the items, the sum of the obtained profits per unit time could increase. For this reason, the number of inventory models focused on profitability maximization, instead of minimum cost or maximum profit, is greatly increasing. Recent papers in this research line are the works of Abdeltawab and Mohamed (2022) and Hussein (2022).

In inventory theory, different profitability measures have been defined, with some differences between them. The residual income (RI) was proposed by Morse and Scheiner (1979). Later, Otake et al. (1999) used the return on investment (ROI) in setup operation policies of inventories. Pando et al. (2021a) introduced the name return on inventory management expense (ROIME). The term return on assets (ROA) was considered by Bradley and Arntzen (1999) and Patel and Tsionas (2022). More recently, San-José et al. (2022) have defined the term return on inventory investment (ROII). Usually, the terms profitability index (PI) or income expense ratio (IER) are used in many areas of economic theory (see, for example, Arnold (2008)). Furthermore, Lubbe et al. (1995) proposed the operating expense ratio (OER) as a profitability measure for life insurance companies.

The role of the sale price of goods in inventory models has been evolving based on the pursued objectives. When the goal is to maximize profit or profitability, it is common to assume a price-dependent demand and to use the sale price as a decision variable. High sale prices lead to higher revenues but lower sales because demand decreases. Thus, the inventory manager needs to find the optimum sale price that maximizes either profit or profitability because both policies are

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different. Although many inventory models consider a price-dependent demand, some of them do not use the sale price as a decision variable. Avinadav et al. (2014), Cárdenas-Barrón et al. (2021), San-José et al. (2021), Das et al. (2022) and Rahman et al. (2022) consider profit maximization. Instead, Jaber et al. (2019) and Pando et al. (2021b) use profitability maximization.

A power function with a negative exponent is the most commonly used to model the price-dependent demand. It is known as isoelastic demand because the quotient between the relative changes in the demand rate and the sale price is a fixed constant named the elasticity coefficient. Thus, Agrawal and Ferguson (2007), Chang et al. (2010) and Duary et al. (2022) use this approach. In another way, Khan et al. (2020) consider a multiplicative dependence of demand concerning the sale price and the frequency of advertising. Also, Alfares and Ghaithan (2016) analyse an inventory system with a demand linearly decreasing with price, time-varying storage cost and quantity discounts. Other ways to represent the price-dependent demand rate were used, for example, by San-José et al. (2018, 2020). Macías-López et al. (2021) define up to six different price-dependent demand functions. Furthermore, Alfares and Ghaithan (2022) also consider a generalized production–inventory model with price-dependent demand, variable production and cost rates.

Another factor that influences demand in inventory models is the stock quantity. It is a common fact in commercial activity that high stock quantities lead to higher demand. In these models, the reorder point when the stock is replenished is also a decision variable, because restocking increases sales, helps to have an adequate stock of products, and avoids falling into shortage. There is extensive literature on this topic. Three recent papers are Barron (2022), Li and Mizuno (2022) and Shah et al. (2023). Mostly, stock-dependent demand is modelled by a power function of the inventory level with a positive exponent in the interval $[0, 1)$. Then, the dependence is also of an isoelastic type, because the quotient between the relative changes in the demand rate and the stock quantity is always the fixed exponent. Thus, Cárdenas-Barrón et al. (2020) consider this dependence type for the demand rate, with trade credit and non-linear stock-dependent storage cost.

Nevertheless, in many real practical situations, the stock level and the sale price influence the demand rate simultaneously. A higher stock quantity and a lower sale price together lead to greater customer demand. This occurs, for example, with sales of style merchandise, such as women's dresses or sports clothes. Therefore, it seems very useful to consider this issue in inventory models. Thus, Pal et al. (2014) proposed a model where the demand rate depends on both the stock quantity and the sale price, and the type of this dependence is multiplicative and isoelastic. Later, Onal et al. (2016) and Feng et al. (2017) also used this approach. It will also be used in this paper.

The modelling of the storage cost is another topic widely discussed in inventory theory. The stock level and the storage time are the two most commonly used factors. Naddor (1982) introduced a model where the storage cost has a multiplicative and isoelastic dependence of time and quantity of items (the general system). Since then, Pando et al. (2013, 2018, 2019) have also considered this approach. It was included by Alfares and Ghaithan (2018) in their review and classification of EOQ and EPQ inventory models formulated under the assumption of variable storage cost.

Specifically, the inventory model to be analysed combines the following features: (i) the demand rate has a multiplicative and isoelastic dependence of the sale price and the stock quantity; (ii) the storage cost depends on both the stock quantity and the stocking time, also with a multiplicative and isoelastic dependence; (iii) the sale price, the initial stock quantity and the reorder point are the decision variables; and (iv) the purpose is to maximize profitability, instead of to maximize profit per unit time. As far as we know, there are no inventory models in the inventory theory that consider all these properties together, simultaneously.

To highlight the contribution of this work, Table 1 collects the most related papers on EOQ models cited in this introduction, classified by the demand rate type, the storage cost type, the objective function and the sale price role (parameter or decision variable). Note that this paper is the only one that considers profitability maximization with the sale price as a decision variable, a demand rate dependent on both the sale price and the stock level, and a storage cost rate dependent on both time and stock level. This is the gap that this paper aims to fill.

The structure of the paper is as follows. Assumptions and notations of the model are included in Section 2, along with the expressions for the demand rate and the storage cost functions. The formulation of the model, with all involved functions, is introduced in Section 3. The core theorem of the paper with the solution of the model is given in Section 4, along with the managerial insights addressed to the inventory managers. Section 5 includes a sensitivity analysis of the parameters of the system, based on the partial derivatives of the optimum values. Computational results are shown in Section 6 and, finally, Section 7 provides the conclusions and future research lines.

2. Assumptions and notation

The following assumptions are considered for the inventory system: (i) it refers to a single item and is continuously reviewed, (ii) the replenishment is instantaneous without lead time, (iii) inventory never falls into shortage, (iv) the planning horizon is infinite, (v) the unit purchase price $c > 0$ is fixed, and (vi) the replenishing cost $K > 0$ is also fixed and does not depend on the order quantity.

Regarding the storage cost, a broad framework is considered. Basic models suppose that the storage cost rate per unit per time is a constant $h > 0$, and therefore the cumulative storage cost for x units during the time t is $H(t, x) = htx$. Nevertheless, in many situations, this cost is not linear as concerns time or quantity, because the storage cost speeds up as the quantity of items increases or time extends. Thus, the model proposed in this paper expresses the cumulative storage cost as a function of the number x of units stored during the time t

$$H(t, x) = ht^{\gamma_1}x^{\gamma_2} \tag{1}$$

with $\gamma_1 \geq 1$ and $\gamma_2 \geq 1$. As a consequence, it can be non-linear regarding time and stock quantity if $\gamma_1 > 1$ and $\gamma_2 > 1$. The conditions $\gamma_1 \geq 1$ and $\gamma_2 \geq 1$ punish inventories with a long scheduling period or a large order quantity. Moreover, the parameters γ_1 and γ_2 satisfy that $\gamma_1 = \frac{\partial H(t,x)/\partial t}{H(t,x)/t}$ and $\gamma_2 = \frac{\partial H(t,x)/\partial x}{H(t,x)/x}$. Therefore, γ_1 depicts the relative change in the cumulative storage cost regarding the relative change in time, and γ_2 depicts the relative change in the cumulative storage cost regarding the relative change in the stock quantity. As a consequence, if, for example, $\gamma_1 = 2.5$ and $\gamma_2 = 1.5$, then the storage cost increases by 2.5% if time in stock increases by 1% with a fixed quantity of items, and this storage cost increases 1.5% if the quantity of items increases by 1% with a fixed time. The parameter $h = H(1, 1)$ depicts the storage cost for one single item during a unit of time. Furthermore, with this function, the storage cost rate per unit per time to calculate the total storage cost in a scheduling period is $\frac{\partial^2 H(t,x)}{\partial t \partial x} = h\gamma_1\gamma_2 t^{\gamma_1 - 1}x^{\gamma_2 - 1}$. Note that the three parameters h, γ_1 and γ_2 let us model a lot of real practical situations for the storage cost in inventory models, and the basic models are achieved when $\gamma_1 = \gamma_2 = 1$.

To model the behaviour of the customers, the demand is also considered with a very broad framework that allows the resulting model to be adapted to many real situations. Thus, the demand rate depends on the sale price p (> 0) and the stock quantity x (> 0) according to the following function

$$D(p, x) = \lambda (p/\eta)^{-\alpha} x^{\beta} \tag{2}$$

with the following conditions for the four parameters:

(i) β is the elasticity coefficient of the demand rate with respect to the number of items in stock, with $0 \leq \beta < 1$ and $\beta \leq \gamma_2/\gamma_1$. It

Table 1
Summary of most related literature cited in this paper.

Paper	Sale price role:		Demand rate type:			Storage cost rate type:		Objective function:		
	P	V	P	S	T	S	T	C	G	R
Baker and Urban (1988)	✓			✓					✓	
Otake et al. (1999)	✓									✓
Chang et al. (2010)		✓	✓	✓					✓	
Pando et al. (2013)	✓			✓		✓	✓		✓	
Pal et al. (2014)	✓		✓	✓				✓		
Onal et al. (2016)		✓	✓	✓					✓	
Alfares and Ghaithan (2016)		✓	✓				✓		✓	
Feng et al. (2017)		✓	✓	✓					✓	
San-José et al. (2018)		✓	✓		✓		✓		✓	
Pando et al. (2019)	✓			✓		✓	✓			✓
San-José et al. (2020)	✓		✓		✓				✓	
Cárdenas-Barrón et al. (2020)	✓			✓		✓			✓	
Pando et al. (2020)	✓			✓		✓				✓
Khan et al. (2020)		✓	✓				✓		✓	
Cárdenas-Barrón et al. (2021)		✓	✓		✓		✓		✓	
Macías-López et al. (2021)		✓	✓	✓			✓		✓	
Pando et al. (2021a)	✓		✓	✓						✓
San-José et al. (2021)	✓		✓		✓				✓	
Pando et al. (2021b)	✓		✓	✓						✓
Duary et al. (2022)		✓	✓						✓	
San-José et al. (2022)	✓				✓					✓
This paper		✓	✓	✓		✓	✓			✓

Sale price role: P = parameter, V = decision variable.
Demand rate type: P = price-dependent, S = stock-dependent, T = time-dependent.
Storage cost rate type: S = stock-dependent, T = time-dependent.
Objective function: C = minimum cost, G = maximum gain or profit, R = maximum profitability.

satisfies that $\beta = \frac{\partial D(p,x)/\partial x}{D(p,x)/x}$ and, therefore, depicts the ratio between the relative change in the demand rate and the relative change in the stock quantity. Then, as $\beta \geq 0$, a high stock quantity leads to a greater demand rate. For example, if $\beta = 0.2$, the demand rate increases by 0.2% as the inventory level increases by 1%. The condition $\beta < 1$ ensures that the demand does not speed up by increasing the stock quantity, and the condition $\beta \leq \gamma_2/\gamma_1$ is necessary for the model, as will be justified later.

(ii) α is the elasticity coefficient of the demand rate with respect to the sale price, with $\alpha > 0$. By the negative exponent $(-\alpha)$, the demand rate is greater with a lower sale price, as seems logical. This coefficient satisfies that $\alpha = -\frac{\partial D(p,x)/\partial p}{D(p,x)/p}$ and, therefore, it depicts the ratio between the relative change in the demand rate and the relative change in the sale price. Then, for example, if $\alpha = 3$, the demand rate decreases by 3% as the sale price increases by 1%.

(iii) η is a reference price used to quantify the elasticity of the demand rate with respect to the sale price p, with $0 < \eta \leq c$, where c is the purchase price. It can be understood as the production cost of the item, and below such a value the item cannot be sold, or it can be understood as the factory price. It could perhaps be difficult for the inventory manager to determine this parameter without information from the producer. Then, as a starting point, it could be useful to suppose that $\eta = c$.

(iv) λ is the scale parameter of the demand rate, with $\lambda = D(\eta, 1)$. It can be viewed as the population size of potential customers. This is because, if only one unit of the item were available, and its sale price was the lowest possible, all potential customers would be willing to buy it.

Note that these four parameters for the demand rate let the inventory model include a lot of real practical situations with different sale prices or stock quantities.

The time spent in the scheduling period is denoted by t, and the inventory level curve is depicted by $I(t)$. Finally, the decision variables of the model are: the sale price p (> 0), the order level S (> 0), and the reorder point r where the inventory level is restored to the initial level, with $0 \leq r < S$. Although the usual case is that $p > c$ to obtain a profit, the most general case with $p > 0$ is considered in the mathematical problem. With these decision variables, the order quantity is $q = S - r$, and the length of the scheduling period T is determined by the three decision variables, $T = T(p, S, r) > 0$.

The notation used throughout the paper is collected in Table 2.

Table 2
Inventory model notation.

c	Unit purchase price ($c > 0$)
K	Replenishing cost per order ($K > 0$)
h	Storage cost for a single item during a unit time period ($h > 0$)
γ_1	Elasticity coefficient of the storage cost regarding time ($\gamma_1 \geq 1$)
γ_2	Elasticity coefficient of the storage cost regarding the stock quantity ($\gamma_2 \geq 1$)
λ	Population size of the potential customers per unit time ($\lambda > 0$)
η	Reference price or production cost of the item ($0 < \eta \leq c$)
β	Elasticity coefficient of the demand rate regarding the stock quantity ($0 \leq \beta < 1$ and $\beta \leq \gamma_2/\gamma_1$)
α	Elasticity coefficient of the demand rate regarding the sale price ($\alpha > 0$)
T	Length of the scheduling period ($T > 0$)
t	Elapsed time in the scheduling period ($0 \leq t \leq T$)
$I(t)$	Inventory level at time t
x	Quantity of items in stock at time t, that is, $x = I(t)$
p	Unit sale price, *decision variable* ($p > 0$)
S	Order level, *decision variable* ($S > 0$)
r	Reorder point, *decision variable* ($0 \leq r < S$)
q	Order quantity, that is, $q = S - r$
$H(t,x)$	Cumulative storage cost for x items during time t, with $H(t,x) = ht^{\gamma_1}x^{\gamma_2} \geq 0$
$D(p,x)$	Demand rate with sale price p and x items in stock, with $D(p,x) = \lambda (p/\eta)^{-\alpha} x^{\beta}$
ξ	Auxiliary parameter, with $\xi = (1-\beta)\gamma_1 + \gamma_2 > 1$

3. Model formulation

With the previous assumptions and notation, the differential equation that defines the inventory level function $I(t)$ is

$$\frac{d}{dt} I(t) = -\lambda (p/\eta)^{-\alpha} (I(t))^{\beta} \tag{3}$$

with the conditions given by the decision variables S and r, namely, $I(0) = S$ and $I(T) = r$. Solving the differential Eq. (3) and using the first boundary condition, the inventory level function is

$$I(t) = \left(S^{1-\beta} - (1-\beta)\lambda\eta^{\alpha}p^{-\alpha}t\right)^{1/(1-\beta)} \tag{4}$$

Moreover, by the second boundary condition $I(T) = r$, the length of the scheduling period T is

$$T = T(p,S,r) = \frac{S^{1-\beta} - r^{1-\beta}}{(1-\beta)\lambda\eta^{\alpha}p^{-\alpha}} \tag{5}$$

The storage cost during a scheduling period is evaluated in the next lemma.

Lemma 1. *Consider an inventory system with the cumulative storage cost given by (1), the inventory level curve given by (4), and the scheduling period given by (5). Then, the storage cost during an inventory cycle depends on the sale price p, the order level S and the reorder point r, and it is given by*

$$HC(p,S,r) = \left(\frac{h\gamma_1}{(1-\beta)^{\gamma_1-1}(\lambda\eta^{\alpha}p^{-\alpha})^{\gamma_1}}\right)\int_r^S \left(S^{1-\beta} - x^{1-\beta}\right)^{\gamma_1-1} x^{\gamma_2-\beta}dx \tag{6}$$

Proof. Please see the proof in Appendix A. □

For each scheduling period, the total expense is the sum of the purchase cost $cq = c(S-r)$, the replenishing cost K, and the storage cost $HC(p,S,r)$, that is,

$$TC(p,S,r) = c(S-r) + K + HC(p,S,r) \tag{7}$$

The sales income in each scheduling period is

$$IN(p,S,r) = pq = p(S-r) \tag{8}$$

and the profit obtained during each inventory cycle is $IN(p,S,r) - TC(p,S,r)$.

The profitability of the inventory system can be measured in three ways:

(i) *The return on inventory management expense (ROIME or ROI)*. This is defined as the ratio profit/cost, that is, $W(p,S,r) = \frac{IN(p,S,r)-TC(p,S,r)}{TC(p,S,r)}$.

(ii) *The profitability index (PI) or income expense ratio (IER)*. This is defined as the ratio income/expense, that is, $R(p,S,r) = \frac{IN(p,S,r)}{TC(p,S,r)}$.

(iii) *The operating expense ratio (OER)*. This is defined as the ratio expense/income, that is, $O(p,S,r) = \frac{TC(p,S,r)}{IN(p,S,r)}$. This index is an efficiency ratio of the inventory management expense because it quantifies the necessary expense to enter a currency unit. The lower the *OER*, the higher the efficiency, and, as a consequence, the higher the profitability. Maximizing the income expense ratio is equivalent to minimizing the operating expense ratio. For example, if $O(p,S,r) = 0.8$ then the inventory system needs to expend 0.8 currency units for each currency unit entered, and the income expense ratio is $1/0.8 = 1.25$, that is, the profitability of the inventory system is 25%.

The relation between these three profitability measurements is

$$W(p,S,r) = R(p,S,r) - 1 = \frac{1}{O(p,S,r)} - 1 \tag{9}$$

and it is clear that maximizing $W(p,S,r)$ is equivalent to maximizing $R(p,S,r)$, which is equivalent to minimizing $O(p,S,r)$.

Note that, for any fixed sale price p, we can consider the parameter $\Lambda = \lambda(p/\eta)^{-\alpha}$, so that the demand rate only depends on the stock quantity $D(t) = \Lambda(I(t))^{\beta}$. Then the inventory model exactly matches the system studied by Pando et al. (2019) with $\lambda = \Lambda$. That paper, using the function $W(p,S,r)$ with a fixed p, showed that the reorder point for the maximum profitability is $r^* = 0$ as long as $\beta \le \gamma_2/\gamma_1$ (please see Theorem 2 in that paper). Furthermore, it showed that, if $\beta > \gamma_2/\gamma_1$, the maximum profitability ratio $W(p,S,r)$ is $p/c - 1$ (please, see Theorem 1 in that paper). Then, if p is now a decision variable of the inventory system and $\beta > \gamma_2/\gamma_1$, the maximum profitability ratio will be ∞ and it is obtained with $p \to \infty$. This case is not a real practical situation, because there are no inventories with infinite income expense ratio. This is why the problem is limited to the case $\beta \le \gamma_2/\gamma_1$, and this assumption has been considered in this section.

As a consequence, if $\beta \le \gamma_2/\gamma_1$, for any fixed p the optimum value for the reorder point is $r^* = 0$ and the problem that uses p as a decision variable can be restricted to the case $r = 0$. Then the order level S equals the order quantity q and the maximizing profitability problem can be reduced to maximizing the income expense ratio where the decision variables are now p and q. To solve this problem, we use the minimization of the operating expense ratio with $r = 0$. Thus, we formulate the mathematical problem as

$$\min_{(p,q)\in\Omega} O(p,q) \tag{10}$$

where

$$O(p,q) = \frac{TC(p,q,0)}{IN(p,q,0)} = \frac{cq + K + HC(p,q,0)}{pq} \tag{11}$$

and $\Omega = \left\{(p,q) \in \mathbb{R}^2 / p > 0, q > 0\right\}$.

By Lemma 1, and using the change of variable $u = (x/q)^{1-\beta}$, the function $HC(p,q,0)$ can be evaluated as follows:

$$\begin{aligned} HC(p,q,0) &= \left(\frac{h\gamma_1}{(1-\beta)^{\gamma_1-1}(\lambda\eta^{\alpha}p^{-\alpha})^{\gamma_1}}\right)\int_0^q \left(q^{1-\beta} - x^{1-\beta}\right)^{\gamma_1-1} x^{\gamma_2-\beta}dx \\ &= \left(\frac{h\gamma_1 q^{(1-\beta)\gamma_1+\gamma_2}}{(1-\beta)^{\gamma_1}(\lambda\eta^{\alpha}p^{-\alpha})^{\gamma_1}}\right)\int_0^1 (1-u)^{\gamma_1-1} u^{\frac{\gamma_2}{1-\beta}}du = Ap^{\alpha\gamma_1}q^{\xi} \end{aligned} \tag{12}$$

where

$$A = \frac{\gamma_1 B\left(\gamma_1, 1 + \gamma_2/(1-\beta)\right) h}{(1-\beta)^{\gamma_1}\lambda^{\gamma_1}\eta^{\alpha\gamma_1}} > 0, \tag{13}$$

$$\xi = (1-\beta)\gamma_1 + \gamma_2 > 1, \tag{14}$$

and $B(a,b) = \int_0^1 (1-u)^{a-1}u^{b-1}du$ is the beta function with $a \ge 1$ and $b \ge 1$.

As a consequence, the objective function to minimize is

$$O(p,q) = \frac{c}{p} + \frac{K}{pq} + Ap^{\alpha\gamma_1-1}q^{\xi-1} \tag{15}$$

subject to $(p,q) \in \Omega$.

4. Solution of the model

Three possible scenarios can be analysed, depending on the relation between the values $\alpha\gamma_1$ and ξ: (i) $\alpha\gamma_1 < \xi$, (ii) $\alpha\gamma_1 = \xi$, and (iii) $\alpha\gamma_1 > \xi$.

4.1. Scenario $\alpha\gamma_1 < \xi$ (or equivalently $\alpha < 1 - \beta + \gamma_2/\gamma_1$)

In this scenario, we can consider a value δ with $\frac{\alpha\gamma_1}{\xi} < \delta < 1$ and $q = p^{-\delta}$ to obtain

$$\lim_{p\to\infty, q=p^{-\delta}} O(p,q) = \lim_{p\to\infty}\left(\frac{c}{p} + \frac{K}{p^{1-\delta}} + Ap^{\alpha\gamma_1-1-\delta(\xi-1)}\right) = 0$$

Then, the maximal income expense ratio $R(p,q,0) = 1/O(p,q)$ would be ∞, obtained with p infinitely large and q infinitely small. This case is not a real practical situation, because there are no inventories with zero order quantity and zero operating cost ratio, that is, with infinite income expense ratio.

4.2. Scenario $\alpha\gamma_1 = \xi$ (or equivalently $\alpha = 1 - \beta + \gamma_2/\gamma_1$)

In this case the function to minimize is

$$O(p,q) = \frac{c}{p} + \frac{K}{pq} + Ap^{\xi-1}q^{\xi-1}$$

For each fixed p, the minimum value of the function

$$f(q) = \frac{c}{p} + \frac{K}{pq} + Ap^{\xi-1}q^{\xi-1}$$

is obtained at the unique positive solution of the equation

$$f'(q) = -\frac{K}{pq^2} + (\xi - 1)\, A p^{\xi-1} q^{\xi-2} = 0$$

which is given by

$$q_p^* = \frac{\left(\frac{K}{(\xi-1)A}\right)^{1/\xi}}{p}$$

This is due to the fact that $\lim_{q\to 0^+} f(q) = \lim_{q\to\infty} f(q) = \infty$ and

$$f''(q_p^*) = \frac{2K + (\xi-1)(\xi-2)\, A p^{\xi} \left(q_p^*\right)^{\xi}}{p\left(q_p^*\right)^3} = \frac{\xi K}{p\left(q_p^*\right)^3} > 0$$

As a consequence, for fixed p, the minimum value is given by

$$O\left(p, q_p^*\right) = f\left(q_p^*\right) = \frac{c}{p} + \frac{K}{\left(\frac{K}{(\xi-1)A}\right)^{1/\xi}} + A\left(\frac{K}{(\xi-1)\,A}\right)^{(\xi-1)/\xi}$$
$$= \frac{c}{p} + \frac{\xi K}{(\xi-1)\left(\frac{K}{(\xi-1)A}\right)^{1/\xi}}$$

Then, as the function $O\left(p, q_p^*\right)$ is decreasing on p, the minimum value for $O(p,q)$ is obtained when $p \to \infty$, and it is given by

$$\lim_{p\to\infty}\left(\frac{c}{p} + \frac{\xi K}{(\xi-1)\left(\frac{K}{(\xi-1)A}\right)^{1/\xi}}\right) = \frac{\xi K}{(\xi-1)\left(\frac{K}{(\xi-1)A}\right)^{1/\xi}} = \frac{\xi A^{1/\xi} K^{1-1/\xi}}{(\xi-1)^{1-1/\xi}} > 0$$

Therefore, the minimum value for the function $O(p,q)$ is obtained with p infinitely large and q infinitely small, although the minimum value is not 0 and, therefore, the maximal income expense ratio is finite. In addition, this scenario is not a real solution, because there are no inventories with p infinitely large and q infinitely small.

4.3. Scenario $\alpha\gamma_1 > \xi$ (or equivalently $\alpha > 1 - \beta + \gamma_2/\gamma_1$)

Note that, in this case, necessarily $\alpha > 1$ because $\beta \le \gamma_2/\gamma_1$. In this scenario, the solution of the model can be obtained by closed expressions for the sale price, the order quantity, the minimum operating expense ratio, and the maximum income expense ratio. The following theorem gives the solution and provides the essential results that we wish to disclose in this paper.

Theorem 1. *Consider an inventory system with the cumulative storage cost function given by (1), the demand rate function given by (2), and the auxiliary parameters $\xi = (1-\beta)\gamma_1 + \gamma_2$ and*

$$A = \frac{\gamma_1 B\left(\gamma_1, 1 + \gamma_2/(1-\beta)\right) h}{(1-\beta)^{\gamma_1} \lambda^{\gamma_1} \eta^{\alpha\gamma_1}}$$

where $B(a,b) = \int_0^1 (1-u)^{a-1} u^{b-1} du$ is the beta function with $a \ge 1$ and $b \ge 1$. Suppose that the four elasticity coefficients of the system satisfy the conditions $\beta < 1$, $\beta \le \gamma_2/\gamma_1$ and $\alpha\gamma_1 > \xi$ and that the goal is to maximize profitability. Then, the following statements are true:

(i) The optimum reorder point is $r^* = 0$.

(ii) The optimum order quantity is equal to the order level, and is given by

$$q^* = S^* = \frac{\left(\alpha\gamma_1 - \xi\right) K}{(\xi-1)\, c} \tag{16}$$

(iii) The optimum sale price is

$$p^* = \left(\frac{K\left(q^*\right)^{-\xi}}{(\xi-1)\,A}\right)^{1/(\alpha\gamma_1)}$$
$$= \eta\left(\frac{(\xi-1)^{\xi-1}(1-\beta)^{\gamma_1}\lambda^{\gamma_1}c^{\xi}}{\left(\alpha\gamma_1-\xi\right)^{\xi}\gamma_1 B\left(\gamma_1, 1+\gamma_2/(1-\beta)\right) K^{\xi-1} h}\right)^{1/(\alpha\gamma_1)} \tag{17}$$

(iv) The minimum operating expense ratio is

$$O^* = O\left(p^*, q^*\right) = \left(\frac{\alpha\gamma_1}{\alpha\gamma_1 - \xi}\right)\left(\frac{c}{p^*}\right) \tag{18}$$

(v) The maximum income expense ratio is

$$R^* = \left(1 - \frac{\xi}{\alpha\gamma_1}\right)\left(\frac{p^*}{c}\right)$$
$$= \eta\left(\frac{\left(\alpha\gamma_1-\xi\right)^{\alpha\gamma_1-\xi}(\xi-1)^{\xi-1}(1-\beta)^{\gamma_1}\lambda^{\gamma_1}}{\left(\alpha\gamma_1\right)^{\alpha\gamma_1}\gamma_1 B\left(\gamma_1, 1+\gamma_2/(1-\beta)\right) h K^{\xi-1} c^{\alpha\gamma_1-\xi}}\right)^{1/(\alpha\gamma_1)} \tag{19}$$

(vi) The optimum scheduling period is

$$T^* = \frac{\left(q^*\right)^{1-\beta}\left(p^*\right)^{\alpha}}{(1-\beta)\lambda\eta^{\alpha}} = \left(\frac{(\xi-1)^{\gamma_2-1} c^{\gamma_2}}{\left(\alpha\gamma_1-\xi\right)^{\gamma_2}\gamma_1 B\left(\gamma_1, 1+\gamma_2/(1-\beta)\right) K^{\gamma_2-1} h}\right)^{1/\gamma_1} \tag{20}$$

(vii) For each inventory cycle, the optimum storage cost is equal to $\frac{K}{\xi-1}$ and the optimum total expense is $\frac{\alpha\gamma_1 K}{\xi-1}$.

(viii) The optimum costs follow a proportional distribution for the storage cost, the replenishing cost, and the purchase cost, and these proportions are, respectively, $\frac{1}{\alpha\gamma_1}$, $\frac{\xi-1}{\alpha\gamma_1}$, and $1 - \frac{\xi}{\alpha\gamma_1}$.

Proof. Please see the proof in Appendix A. □

The following managerial insights for the inventory managers are deduced from the optimum solution of the inventory problem:

(i) To maximize profitability, replenishment should be done when stock is depleted, that is, the reorder point is equal to zero.

(ii) The optimum order quantity q^* does not depend on the population size of potential customers λ, the reference price of the item η, or the scale parameter of the storage cost h. Only the elasticity coefficients of the demand rate and the storage cost influence the optimum order quantity, along with the unit purchase price c and the replenishing cost K. Even more, the optimum order quantity is proportional to K and inversely proportional to c.

(iii) The optimum scheduling period T^* does not depend on the population size of potential customers λ or the reference price of the item η. Furthermore, if the storage cost rate per unit per time is fixed ($\gamma_1 = \gamma_2 = 1$) then the optimum length of the scheduling period T^* is proportional to the purchase price c, inversely proportional to the storage cost parameter h, and does not depend on the replenishing cost K.

(iv) The total demand per unit time served with this optimum policy is

$$q^*/T^* = (1-\beta)\,\lambda\left(p^*/\eta\right)^{-\alpha}\left(q^*\right)^{\beta} = (1-\beta)\, D(p^*, q^*)$$

and, therefore, the ratio between the total sales per unit time and the initial demand rate is equal to $1-\beta$.

(v) The optimum sale price p^* and the maximum income expense ratio R^* are proportional to the reference price η.

(vi) As the maximum income expense ratio is given by (19) and p^*/c would be the income expense ratio if there are no costs in the inventory, then the ratio $\xi/(\alpha\gamma_1)$ represents the relative loss of profitability due to the total expense in the inventory system. As a particular case, in the basic model where the storage cost rate per unit per time is fixed ($\gamma_1 = \gamma_2 = 1$) and the demand rate does not depend on the stock quantity ($\beta = 0$), this value is $2/\alpha$.

(vii) If the auxiliary parameter ξ is equal to 2, then the storage cost per scheduling period is equal to the replenishing cost, as in Harris' rule of the basic EOQ model. For example, this occurs if the storage cost rate per unit per time is fixed ($\gamma_1 = \gamma_2 = 1$) and the demand rate does not depend on the stock quantity ($\beta = 0$).

Table 3
Profitability thresholds for the parameters K, h, c, λ, and η of the inventory system.

Parameter:	$K <$	$h <$	$c <$	$\eta >$	$\lambda >$
Threshold:	$\left(\frac{\Delta\lambda^{\gamma_1}\eta^{\alpha\gamma_1}}{c^{\alpha\gamma_1-\xi}h}\right)^{1/(\xi-1)}$	$\frac{\Delta\lambda^{\gamma_1}\eta^{\alpha\gamma_1}}{K^{\xi-1}c^{\alpha\gamma_1-\xi}}$	$\left(\frac{\Delta\lambda^{\gamma_1}\eta^{\alpha\gamma_1}}{K^{\xi-1}h}\right)^{1/(\alpha\gamma_1-\xi)}$	$\left(\frac{K^{\xi-1}c^{\alpha\gamma_1-\xi}h}{\Delta\lambda^{\gamma_1}}\right)^{1/(\alpha\gamma_1)}$	$\left(\frac{K^{\xi-1}c^{\alpha\gamma_1-\xi}h}{\Delta\eta^{\alpha\gamma_1}}\right)^{1/\gamma_1}$

From these results, the following corollary characterizes, a priori, profitability as a function of only the initial parameters of the system.

Corollary 1. *Consider an inventory system with all the conditions given in Theorem 1. Then the following statements are true:*

(i) The inventory system is profitable ($R^* > 1$) if, and only if, the parameters satisfy the condition

$$\frac{K^{\xi-1}c^{\alpha\gamma_1-\xi}h}{\lambda^{\gamma_1}\eta^{\alpha\gamma_1}} < \Delta \tag{21}$$

where

$$\Delta = \frac{(\xi-1)^{\xi-1}(1-\beta)^{\gamma_1}\left(\alpha\gamma_1-\xi\right)^{\alpha\gamma_1-\xi}}{\gamma_1\left(\alpha\gamma_1\right)^{\alpha\gamma_1}B\left(\gamma_1, 1+\gamma_2/(1-\beta)\right)} \tag{22}$$

is an auxiliary parameter obtained from the four elasticity coefficients.

(ii) If the condition (21) is true, then the optimum sale price p^* is greater than the unit purchase price c.

(iii) If the purchase price c satisfies that

$$c \geq \left(\frac{(\xi-1)^{\xi-1}}{\left(\alpha\gamma_1-\xi\right)^{\xi}AK^{\xi-1}}\right)^{1/(\alpha\gamma_1-\xi)} \tag{23}$$

then $p^* \leq c$ and the inventory system never makes a profit ($R^* < 1$).

Proof. Please see the proof in Appendix A. □

From (21) and (22), the inventory manager can evaluate the profitability threshold for each parameter of the system, keeping all the others fixed. Indeed, clearing each parameter in (21), the profitability thresholds given in Table 3 are obtained, with upper bounds for the cost parameters K, h, and c, and lower bounds for the demand parameters λ and η.

The inventory manager can also evaluate the total expense and the total profit per unit time for the solution with maximum profitability. Indeed, the total cost per unit time $C(p,S,r)$ is given by

$$C(p,S,r) = \frac{TC(p,S,r)}{T(p,S,r)} = \frac{c(S-r)+K+HC(p,S,r)}{T(p,S,r)} \tag{24}$$

and, for the maximum profitability solution, by statements (vi) and (vii) of Theorem 1, we have

$$C\left(p^*,q^*,0\right) = \frac{\frac{\alpha\gamma_1 K}{\xi-1}}{T^*}$$
$$= \left(\frac{\alpha^{\gamma_1}\left(\alpha\gamma_1-\xi\right)^{\gamma_2}\left(\gamma_1\right)^{\gamma_1+1}B\left(\gamma_1,1+\gamma_2/(1-\beta)\right)K^{\gamma_1+\gamma_2-1}h}{(\xi-1)^{\gamma_1+\gamma_2-1}c^{\gamma_2}}\right)^{1/\gamma_1} \tag{25}$$

In a similar way, the profit per unit time $G(p,S,r)$ is given by

$$\begin{aligned} G(p,S,r) &= \frac{IN(p,S,r)-TC(p,S,r)}{T(p,S,r)} \\ &= \left(\frac{IN(p,S,r)-TC(p,S,r)}{TC(p,S,r)}\right)\left(\frac{TC(p,S,r)}{T(p,S,r)}\right) \\ &= (R(p,S,r)-1)\,C(p,S,r) \end{aligned}$$

and, for the maximum profitability solution, we have

$$G\left(p^*,q^*,0\right) = \left(R^*-1\right)C\left(p^*,q^*,0\right) \tag{26}$$

where R^* is given by (19) and $C(p^*,q^*,0)$ is given by (25).

A special case of the inventory model solved with Theorem 1 is obtained when $\beta = 0$ and $\eta = \gamma_1 = \gamma_2 = 1$. With these values, the storage cost rate per unit per time is a fixed constant h which does not depend on time or stock quantity. In addition, the demand rate only depends on the sale price through the function $D(p) = \lambda p^{-\alpha}$ and does not depend on the stock quantity. This demand function has already been used in inventory theory and is known as isoelastic price-dependent demand (see, for example, Arcelus and Srinivasan (1987)). For this case, $\xi = 2$, $B\left(\gamma_1, 1+\gamma_2/(1-\beta)\right) = B(1,2) = 1/2$, and, if $\alpha > 2$, the optimum solutions given in Theorem 1 are simpler, as shown in Table 4.

This model also makes it possible to find the optimum solution in the event that the sale price is predetermined in advance, and cannot be chosen by the inventory manager. Then, the sale price is not a decision variable in the model, but a fixed parameter that influences the demand rate. In this case, the operating cost ratio only depends on the order quantity q through the function

$$g(q) = O(p,q) = \frac{c}{p} + \frac{K}{pq} + Ap^{\alpha\gamma_1-1}q^{\xi-1}$$

whose derivative is

$$g'(q) = \frac{-K+(\xi-1)Ap^{\alpha\gamma_1}q^{\xi}}{pq^2}$$

Then it is easy to prove that the absolute minimum of this function is obtained at the unique positive solution of the equation $g'(q) = 0$, which is

$$q_p^* = \left(\frac{K}{(\xi-1)Ap^{\alpha\gamma_1}}\right)^{1/\xi}$$

with

$$g''(q_p^*) = \frac{2K+(\xi-1)(\xi-2)Ap^{\alpha\gamma_1}\left(q_p^*\right)^{\xi}}{p\left(q_p^*\right)^3} = \frac{\xi K}{p\left(q_p^*\right)^3} > 0$$

and $\lim_{q\to 0^+} g(q) = \lim_{q\to\infty} g(q) = \infty$. Then, replacing the value of A given by (13), we have

$$q_p^* = \left(\frac{(1-\beta)^{\gamma_1}\Lambda^{\gamma_1}K}{(\xi-1)\gamma_1 B\left(\gamma_1,1+\gamma_2/(1-\beta)\right)h}\right)^{1/\xi} \tag{27}$$

where $\Lambda = \lambda(p/\eta)^{-\alpha}$. As expected, this expression for the order quantity agrees with the other one given by Pando et al. (2019) (please see Theorem 2 in that paper) with Λ instead of λ. Also, evaluating the minimum operating expense ratio $O_p^* = O\left(p,q_p^*\right) = g\left(q_p^*\right)$, we obtain

$$O_p^* = g\left(q_p^*\right) = \frac{c}{p} + \frac{K+Ap^{\alpha\gamma_1}\left(q_p^*\right)^{\xi}}{pq_p^*} = \frac{c}{p} + \frac{\xi K}{(\xi-1)pq_p^*}$$

and, as a consequence, the maximum income expense ratio for each fixed p is

$$R_p^* = \frac{1}{O_p^*} = \frac{p}{c+\frac{\xi K}{(\xi-1)q_p^*}} \tag{28}$$

These expressions can be used to evaluate the difference between the solutions that consider the sale price as a fixed parameter or a decision variable of the system. Note that the expression for the optimum order quantity is simpler when the sale price is a decision variable in the model. Furthermore, if the sale price is a fixed parameter, the optimum order quantity depends on the reference price of the item η, but not on the purchase price c. Instead, if the sale price is a decision variable in the model, the optimum order quantity depends on the purchase price c, but not on the reference price η.

Table 4
Optimum solution for the model with $\beta = 0$ and $\eta = \gamma_1 = \gamma_2 = 1$.

Sale price	Order quantity	Income expense ratio	Scheduling period
$p^* = \left(\frac{2\lambda c^2}{(\alpha-2)^2 hK}\right)^{1/\alpha}$	$q^* = \frac{(\alpha-2)K}{c}$	$R^* = \left(\frac{2(\alpha-2)^{\alpha-2}\lambda}{\alpha^\alpha K h c^{\alpha-2}}\right)^{1/\alpha}$	$T^* = \frac{2c}{(\alpha-2)h}$

Table 5
Partial derivatives of p^*, q^*, R^* and T^* regarding the parameters K, h, λ, c, η, and α.

	$\partial p^*/\partial z$	$\partial q^*/\partial z$	$\partial R^*/\partial z$	$\partial T^*/\partial z$
$z = K$	$\left(\frac{\xi-1}{\alpha\gamma_1}\right)\left(\frac{-p^*}{K}\right)$	$\frac{q^*}{K}$	$\left(\frac{\xi-1}{\alpha\gamma_1}\right)\left(\frac{-R^*}{K}\right)$	$\left(\frac{\gamma_2-1}{\gamma_1}\right)\left(\frac{-T^*}{K}\right)$
$z = h$	$\left(\frac{1}{\alpha\gamma_1}\right)\left(\frac{-p^*}{h}\right)$	0	$\left(\frac{1}{\alpha\gamma_1}\right)\left(\frac{-R^*}{h}\right)$	$\left(\frac{1}{\gamma_1}\right)\left(\frac{-T^*}{h}\right)$
$z = c$	$\left(\frac{\xi}{\alpha\gamma_1}\right)\left(\frac{p^*}{c}\right)$	$\frac{-q^*}{c}$	$\left(1-\frac{\xi}{\alpha\gamma_1}\right)\left(\frac{-R^*}{c}\right)$	$\left(\frac{\gamma_2}{\gamma_1}\right)\left(\frac{T^*}{c}\right)$
$z = \eta$	$\frac{p^*}{\eta}$	0	$\frac{R^*}{\eta}$	0
$z = \lambda$	$\left(\frac{1}{\alpha}\right)\left(\frac{p^*}{\lambda}\right)$	0	$\left(\frac{1}{\alpha}\right)\left(\frac{R^*}{\lambda}\right)$	0
$z = \alpha$	$\left(\ln\left(\frac{p^*}{\eta}\right)+\frac{\xi}{\alpha\gamma_1-\xi}\right)\left(\frac{-p^*}{\alpha}\right)$	$\frac{\gamma_1 q^*}{\alpha\gamma_1-\xi}$	$\left(\frac{-R^*}{\alpha}\right)\ln\left(\frac{p^*}{\eta}\right)$	$\frac{-\gamma_2 T^*}{\alpha\gamma_1-\xi}$

5. Sensitivity analysis

In this section, the closed expressions obtained in Theorem 1 for the optimum values of the sale price p^*, the order quantity q^*, the income expense ratio R^*, and the scheduling period T^* are used to develop a sensitivity analysis by calculating the partial derivatives regarding the parameters K, h, λ, c, η, and α of the model. The parameters β, γ_1 and γ_2 have been excluded because the obtained derivatives do not ensure the behaviour of the optimum values (they can increase or decrease depending on the values of the other parameters). Furthermore, the analysis is limited to the case where the inventory is profitable because only in this case would the model be used in real practical situations. So, we suppose that the profitability condition established by the expression (21) is satisfied.

The partial derivatives obtained for the selected parameters are shown in Table 5, and all the proofs are given in Appendix B.

The sensitivity analysis for the maximum income expense ratio R^* is included in the next proposition.

Proposition 1. *Suppose that the inventory system satisfies the conditions $\beta < 1$, $\beta \le \gamma_2/\gamma_1$, $\alpha\gamma_1 > \xi$, and the profitability condition (21). Then, the maximum income expense ratio R^* given by (19) satisfies that:*

(i) R^* decreases as the replenishing cost K, the scale parameter of the storage cost h, the unit purchase price c, or the price elasticity coefficient of the demand rate α increase.
(ii) R^* increases as the population size of potential customers λ, or the reference price of the item η increase.

Proof. Taking into account that $\alpha\gamma_1 > \xi > 1$ and $p^* > c \ge \eta$ because the inventory is profitable, the signs of the partial derivatives in the fourth column of Table 5 prove the proposition. □

In order to analyse which parameters are more influential in the maximum income expense ratio, it is interesting to evaluate the ratio, in absolute value, between the relative change on the maximum income expense ratio and the relative change on each of the parameters, that is, $\left|\frac{\partial R^*/\partial z}{R^*/z}\right|$ for each parameter z. Then, from the fourth column in Table 5, it is easily seen that

$$\left(\frac{\alpha\gamma_1}{\xi-1}\right)\left|\frac{\partial R^*/\partial K}{R^*/K}\right| = \alpha\gamma_1\left|\frac{\partial R^*/\partial h}{R^*/h}\right| = \left(\frac{\alpha\gamma_1}{\alpha\gamma_1-\xi}\right)\left|\frac{\partial R^*/\partial c}{R^*/c}\right| = \left(\frac{\partial R^*/\partial\eta}{R^*/\eta}\right) = \alpha\left(\frac{\partial R^*/\partial\lambda}{R^*/\lambda}\right) = \frac{\left|\frac{\partial R^*/\partial\alpha}{R^*/\alpha}\right|}{\ln(p^*/\eta)} = 1$$

The following assertions are deduced from the previous equalities:

(i) As R^* is linear on η and $\partial R^*/\partial\eta = R^*/\eta$, a relative change on parameter η leads to an equal relative change on the maximum income expense ratio. That is, an $m\%$ -increase in parameter η leads to an $m\%$-increase in R^*.

(ii) The relative effect of parameter η is greater than that of λ, and this is greater than or equal to that of h, because $\alpha\gamma_1 \ge \alpha > 1$.

(iii) The relative effect of parameter η is greater than that of c, and also greater than that of K, because $\frac{\alpha\gamma_1}{\alpha\gamma_1-\xi} > 1$ and $\frac{\alpha\gamma_1}{\xi-1} > 1$.

(iv) As a particular case, if the storage cost rate per unit per time is fixed ($\gamma_1 = \gamma_2 = 1$), then the relative effect of parameter η is greater than that of λ or h (which are equal except for the sign), and these are greater than the parameter K. Indeed, if $\gamma_1 = \gamma_2 = 1$, then $\xi = 2-\beta$ and $1 < \alpha < \frac{\alpha}{1-\beta}$.

In a similar way, the sensitivity analysis for the optimum sale price p^* is given in the next proposition.

Proposition 2. *Suppose that the inventory system satisfies the conditions $\beta < 1$, $\beta \le \gamma_2/\gamma_1$, $\alpha\gamma_1 > \xi$, and the profitability condition (21). Then, the optimum sale price p^* given by (17) satisfies that:*

(i) p^* decreases as the replenishing cost K, the scale parameter of the storage cost h, or the price elasticity coefficient of the demand rate α increase.
(ii) p^* increases as the population size of potential customers λ, the unit purchase price c, or the reference price of the item η increase.

Proof. Taking into account that $\xi > 1$ and $p^* > c \ge \eta$ because the inventory is profitable, the signs of the partial derivatives in the second column of Table 5 prove the proposition. □

For the relative changes in absolute values we have

$$\left(\frac{\alpha\gamma_1}{\xi-1}\right)\left|\frac{\partial p^*/\partial K}{p^*/K}\right| = \alpha\gamma_1\left|\frac{\partial p^*/\partial h}{p^*/h}\right| = \left(\frac{\alpha\gamma_1}{\xi}\right)\left(\frac{\partial p^*/\partial c}{p^*/c}\right) = \left(\frac{\partial p^*/\partial\eta}{p^*/\eta}\right) = \alpha\left(\frac{\partial p^*/\partial\lambda}{p^*/\lambda}\right) = \frac{\left|\frac{\partial p^*/\partial\alpha}{p^*/\alpha}\right|}{\ln(p^*/\eta)+\frac{\xi}{\alpha\gamma_1-\xi}} = 1$$

As a consequence, the following assertions are deduced:

(i) As p^* is linear on η and $\partial p^*/\partial\eta = p^*/\eta$, a relative change on parameter η leads to an equal relative change on the optimum sale price. That is, an $m\%$-increase on parameter η leads to an $m\%$-increase on p^*.

(ii) The relative effect of parameter η is greater than that of λ, and this is greater than or equal to that of h, because $\alpha\gamma_1 \ge \alpha > 1$.

(iii) The relative effect of parameter α is greater than that of c, and this is greater than that of K. Indeed, as $\alpha\gamma_1 > \xi$ and $p^* > \eta$, then $\frac{\alpha\gamma_1}{\xi-1} > \frac{\alpha\gamma_1}{\xi} > \frac{\alpha\gamma_1-\xi}{\xi} > \frac{1}{\ln(p^*/\eta)+\frac{\xi}{\alpha\gamma_1-\xi}}$.

(iv) The relative effect of parameter c is greater than that of h, and it is lower than that of η, because $\alpha\gamma_1 > \frac{\alpha\gamma_1}{\xi} > 1$.

(v) As a particular case, if the storage cost rate per unit per time is fixed ($\gamma_1 = \gamma_2 = 1$) and parameter α is excluded, then parameter η has the greatest relative effect on the sale price, then parameter c, then parameters λ and h (which are equal except for the sign), and finally parameter K. Indeed, if $\gamma_1 = \gamma_2 = 1$, then $\xi = 2 - \beta$ and $1 < \frac{\alpha}{2-\beta} < \alpha < \frac{\alpha}{1-\beta}$.

For the optimum order quantity q^*, the following proposition provides the sensitivity analysis regarding the parameters K, h, λ, c, η, and α.

Proposition 3. *Suppose that the inventory system satisfies the conditions $\beta < 1$, $\beta \leq \gamma_2/\gamma_1$, $\alpha\gamma_1 > \xi$, and the profitability condition (21). Then, the optimum order quantity q^* given by (17) satisfies that:*

(i) q^* does not depend on the scale parameter of the demand rate h, or the population size of potential customers λ, or the reference price of the item η.
(ii) q^* increases as the replenishing cost K or the price elasticity coefficient α increase.
(iii) q^* decreases as the unit purchase price c increases.

Proof. Taking into account that $\alpha\gamma_1 > \xi$, the signs of the partial derivatives in the third column of Table 5 prove the proposition. □

The relative changes in absolute values for the three parameters K, c, and α satisfy the equalities

$$\frac{\partial q^*/\partial K}{q^*/K} = \left|\frac{\partial q^*/\partial c}{q^*/c}\right| = \left(1 - \frac{\xi}{\alpha\gamma_1}\right)\left(\frac{\partial q^*/\partial \alpha}{q^*/\alpha}\right) = 1$$

Then the following assertions are deduced:

(i) As q^* is linear on K and $\partial q^*/\partial K = q^*/K$, a relative change on parameter K leads to an equal relative change on the optimum order quantity q^*. That is, an $m\%$ -increase on parameter K leads to an $m\%$-increase in q^*.

(ii) As $\partial q^*/\partial c = -q^*/c$, a small relative increase on parameter c leads to an equal relative decrease in the optimum order quantity q^*.

(iii) The relative effect of parameter α is greater than that of parameters K or c (which are equal except for the sign), because $\alpha\gamma_1 > \xi$.

Finally, the sensitivity analysis for the scheduling period T^* is provided in the next proposition.

Proposition 4. *Suppose that the inventory system satisfies the conditions $\beta < 1$, $\beta \leq \gamma_2/\gamma_1$, $\alpha\gamma_1 > \xi$, and the profitability condition (21). Then, the optimum scheduling period T^* given by (20) satisfies that:*

(i) T^* does not depend on the population size of potential customers λ, or the reference price of the item η.
(ii) T^* decreases as the replenishing cost K, or the scale parameter of the storage cost h, or the price elasticity coefficient α increase.
(iii) T^* increases as the unit purchase price c increases.

Proof. As $\alpha\gamma_1 > \xi$ and $\gamma_2 \geq 1$, the signs of the partial derivatives in the last column of Table 5 prove the proposition. □

Note that, if $\gamma_2 = 1$, then $\partial T^*/\partial K = 0$ because the optimum scheduling period T^* does not depend on K.

For the relative changes in absolute values we have that, if $\gamma_2 > 1$,

$$\left(\frac{\gamma_1}{\gamma_2 - 1}\right)\left|\frac{\partial T^*/\partial K}{T^*/K}\right| = \gamma_1\left|\frac{\partial T^*/\partial h}{T^*/h}\right| = \left(\frac{\gamma_1}{\gamma_2}\right)\left(\frac{\partial T^*/\partial c}{T^*/c}\right)$$
$$= \left(\frac{\gamma_1 - \xi/\alpha}{\gamma_2}\right)\left|\frac{\partial T^*/\partial \alpha}{T^*/\alpha}\right| = 1$$

Then the following assertions are deduced:

(i) As $\frac{\gamma_1-\xi/\alpha}{\gamma_2} < \frac{\gamma_1}{\gamma_2} \leq \min\left(\gamma_1, \frac{\gamma_1}{\gamma_2-1}\right)$, the parameter α has the greatest relative effect, then the parameter c, which is greater than or equal to that of h, and, if $\gamma_2 > 1$, than that of K.

(ii) If $\gamma_2 = 1$, then the relative effect of the parameters h and c are equal except for the sign.

(iii) If $\gamma_1 = \gamma_2$, then $\frac{\partial q^*/\partial\alpha}{q^*/\alpha} = -\left(\frac{\partial T^*/\partial\alpha}{T^*/\alpha}\right) = \frac{\alpha\gamma_1}{\alpha\gamma_1-\xi}$ and, promptly, a relative increase on parameter α leads to a relative increase in the optimum order quantity q^*, which is equal to the relative decrease in the optimum scheduling period T^*.

6. Numerical results

This section illustrates all the obtained results with a double objective: (i) solving a numerical example that includes a sensitivity analysis regarding all the initial parameters, and (ii) comparing the maximum profitabililty policy with the maximum profit per unit time policy. This comparison is an interesting question taking into account the fact that both solutions are not always equal, because profit and profitability are different objectives. The income expense ratio $R(p,S,s)$ used in this paper is given by

$$R(p,S,s) = \frac{IN(p,S,r)}{TC(p,S,r)} \tag{29}$$

while the profit per unit time is given by

$$G(p,S,r) = \frac{IN(p,S,r) - TC(p,S,r)}{T(p,S,r)} \tag{30}$$

where $T(p,S,r)$, $TC(p,S,r)$, and $IN(p,S,r)$ are those given by (5), (7) and (8), respectively.

The maximization of the function $G(p,S,r)$ has already been studied in the inventory theory, and it is known that the optimum solution can be obtained with $r \neq 0$, both when p is a parameter (see, for example, Baker and Urban (1988)), and when it is a decision variable (see, for example, Chang et al. (2010)).

As the solution of the model can be obtained directly for any values of the initial parameters, they will be chosen simply so that they can be reasonable for a hypothetical situation. Thus, using a week as the unit time and a euro (€) as the currency unit, let us suppose that the replenishing cost, the reference price and the purchase price are, respectively, $K = 500$€, $\eta = 18$€ and $c = 20$€. Consider that the storage cost of one item during a week is $h = 3$€, and the population size of potential consumers for a week is $\lambda = 800$. The elasticity coefficients for the storage cost are $\gamma_1 = 1.2$ and $\gamma_2 = 1.5$, that is, the cumulative storage cost increases 1.2% if the time in stock increases 1%, and it increases 1.5% if the stock level increases 1%. Regarding the demand rate, consider that the elasticity coefficients are $\alpha = 4$ and $\beta = 0.2$, that is, a 1% increase in the sale price leads to a 4% decrease in the demand rate, and a 1% increase in the stock quantity leads to a 0.2% increase in the demand rate. Finally, let us suppose that the inventory manager needs to obtain the inventory policy with the maximum income expense ratio.

With these input data, $B\left(\gamma_1, 1 + \gamma_2/(1-\beta)\right) = B(1.2, 2.875) = 0.2488$ and the values for the parameters A, ξ and Δ given by (13), (14) and (22) are, respectively, $A = 3.6257 \cdot 10^{-10}$, $\xi = 2.46$ and $\Delta = 0.0175$. Then, the conditions $\beta < 1$, $\beta \leq \gamma_2/\gamma_1$, and $\alpha\gamma_1 > \xi$ considered in Section 4.3 are true, and Theorem 1 can be used to obtain the optimum policy. Then, the optimum order quantity given by (16) is $q^* = 40.07$ items, the optimum sale price given by (17) is $p^* = 47.14$€, and the reorder point is, as we know, $r^* = 0$. The minimum operating expense ratio given by (18) is $O^* = 0.8703$, that is, the inventory system needs 0.8703€ for each entered euro. Furthermore, the operating cost of the system is 87.03% of the income. As a consequence, the maximum income expense ratio is $R^* = 1/O^* = 1.1490$, that is, the profitability of the inventory system is 14.9%. The optimum scheduling period, given by (20), is $T^* = 1.41$ weeks (about 10 days).

Table 6
Profitability thresholds for the parameters of the system.

	K	h	c	η	λ	α	β	γ_1	γ_2
Actual value	500	3	20	18	800	4	0.2	1.2	1.5
Profitability threshold	< 789.5	< 5.8	< 26.5	> 15.7	> 458.9	< 4.8	> 0.02	≥ 1	< 1.8

Table 7
Absolute and relative changes to p^*, q^*, R^*, T^* regarding K, h, c, λ, η, α.

Actual value	$K = 500$	$h = 3$	$c = 20$	$\eta = 18$	$\lambda = 800$	$\alpha = 4$
	$z = K$	$z = h$	$z = c$	$z = \eta$	$z = \lambda$	$z = \alpha$
$\partial p^*/\partial z$	−0.03	−3.27	1.21	2.62	0.01	−23.74
$(\partial p^*/\partial z)\,/\,(p^*/z)$	−0.30	−0.21	0.51	1	0.25	−2.01
$\partial q^*/\partial z$	0.08	0	−2.00	0	0	20.55
$(\partial q^*/\partial z)\,/\,(q^*/z)$	1	0	−1	0	0	2.05
$\partial R^*/\partial z$	−0.0007	−0.08	−0.03	0.06	0.0004	−0.28
$(\partial R^*/\partial z)\,/\,(R^*/z)$	−0.30	−0.21	−0.49	1	0.25	−0.96
$\partial T^*/\partial x$	−0.001	−0.39	0.09	0	0	−0.90
$(\partial T^*/\partial z)\,/\,(T^*/z)$	−0.42	−0.83	1.25	0	0	−2.56

For this optimum policy, the storage cost in a scheduling period is $K/(\xi-1) = 342.47$€, and the total expense in a scheduling period is $\alpha\gamma_1 K/(\xi-1) = 1643.85$€. Then, the costs distribution is $342.47/1643.85 = 20.83\%$ for the storage cost, $500/1643.85 = 30.42\%$ for the replenishing cost, and the other 48.75% for the purchase cost. The income in a cycle, given by (8), is $IN(47.14, 40.07, 0) = 1888.89$€, and the total expense in a scheduling period, given by (7), is $TC(47.14, 40.07, 0) = 1643.85$€, which coincides with the previous value given by $\alpha\gamma_1 K/(\xi-1)$. The quotient between these two quantities is the optimum income expense ratio, that is, $R(47.14, 40.07, 0) = 1.1490$, which coincides with the value given by expression (19). Also, for this optimum solution, the profit per unit time, given by (26) or (30), is $G(47.14, 40.1, 0) = 174.05$€ per week, and the total cost per unit time, given by (24), is $C(47.14, 40.07, 0) = 1167.55$ € per week.

Using the expressions given in Table 3 with $\Delta = 0.0175$, the profitability thresholds for each of the parameters K, h, c, λ, and η, keeping all the others fixed, have been calculated and included in Table 6. In addition, the profitability thresholds for the elasticity coefficients of the model have numerically been evaluated for this example, taking into account that this inventory system is profitable because condition (21) is satisfied.

To illustrate the sensitivity analysis, the partial derivatives of the optimum values for p^*, q^*, R^* and T^* have been evaluated, in absolute and relative values, and are included in Table 7.

The optimum sale price p^* decreases with parameters K, h and α, and it increases with parameters c, η and λ. Parameter α has the highest relative effect, then parameter η, then parameter c, and parameters K, λ and h have a lower effect. The optimum order quantity q^* does not depend on parameters h, η and λ, it increases with K and α, and it decreases with parameter c. Parameter α also has the highest relative effect, and parameters K and c have an equal relative effect, but with the opposite sign. Regarding the maximum income expense ratio R^*, it decreases with parameters K, h, c and α, and increases with parameters η and λ. The highest relative effect is obtained with parameter η, then parameter α, then parameter c, while parameters K, h and λ have a lower effect. Finally, the optimum scheduling period T^* does not depend on parameters η and λ, it increases with c, and decreases with parameters K, h and α. Also now, the highest relative effect is obtained for parameter α, then parameter c, while parameters K and h have a lower effect. All these remarks agree with the results obtained in Section 5.

The sensitivity analysis has been completed solving the model with percentage changes between −50% and 50% in each of the parameters, keeping all the others fixed. Then, the percentage changes in the sale price p^*, the order quantity q^*, the maximum income expense ratio R^*, and the optimum scheduling period T^* have been evaluated and plotted in Fig. 1.

The second objective in this section is the comparison of the optimum solutions to the problems of the maximum income expense ratio and the maximum profit per unit time for this numerical example. To do that, for each fixed $p \in (20, 60)$ with a step of one-hundredth, the optimum solution for the maximum income expense ratio problem has been evaluated by using the expressions (27) for the optimum order quantity q_p^*, and (28) for the maximum income expense ratio R_p^*. Furthermore, the optimum scheduling period T_p^* and the profit per unit time $G_p^R = G\left(p, q_p^*, 0\right)$ for this optimum solution were evaluated with the expressions (5) and (30), respectively. For the maximum profit per unit time problem, the algorithm included in Pando Fernández (2014) (Algoritmo 2, p. 64) has been used to obtain the optimum order level S_p^G, the optimum reorder point r_p^G, the optimum order quantity $q_p^G = S_p^G - r_p^G$, the optimum scheduling period $T_p^G = T\left(p, S_p^G, r_p^G\right)$, and the maximum profit per unit time $G_p^* = G\left(p, S_p^G, r_p^G\right)$. The income expense ratio for this solution $R_p^G = R\left(p, S_p^G, r_p^G\right)$ was also evaluated with the expression (29). Then, the point $\left(p, S_p^G, r_p^G\right)$ with the greatest value of G_p^* provides the optimum sale price p^G, the optimum order level S^G, the optimum reorder point r^G and the maximum profit per unit time G^*. The order quantity $q^G = S^G - r^G$, the scheduling period $T^G = T\left(p^G, S^G, r^G\right)$, and the income expense ratio $R^G = R\left(p^G, S^G, r^G\right)$ for this optimum solution were evaluated to compare them with the optimum values q^*, T^* and R^*. The profit per unit time for the maximum income expense ratio solution was denoted by $G^R = G_{p^*}^R = G(p^*, q^*, 0) = 174.05$. All these quantities are included in Table 8.

Both optimum solutions are only near the reorder point r, which is 0 in the maximum profitability problem, and roughly 0 in the maximum profit per unit time problem. In fact, in this example, for every $p \in (20, 60)$ the reorder point r_p^G is roughly 0, although this is not true in all cases (see for example Pando et al., 2019). The maximum profit per unit time is obtained with a sale price of $p^G = 33.82$, which is 28% lower than the optimum sale price with a maximum income expense ratio, which is $p^* = 47.14$. In addition, the optimum order quantity for the solution with a maximum profit per unit time is $q^G = 79.30$, which is 98% higher than the optimum order quantity $q^* = 40.07$ for the maximum income expense ratio solution. The scheduling period with a maximum profit per unit time is $T^G = 0.64$ weeks, which is 55% lower than the optimum scheduling period with the maximum income expense ratio, which is $T^* = 1.41$ weeks. The maximum profit per unit time is $G^* = 346.34$ €, while the profit per unit time for the solution with maximum income expense ratio is $G^R = 174.05$€, which is roughly 50% lower. Note that the total sales with the maximum profit solution are $q^G/T^G = 123.9$ items per week, while the total sales with the maximum income expense ratio policy are $q^*/T^* = 28.9$ items per week, considerably lower.

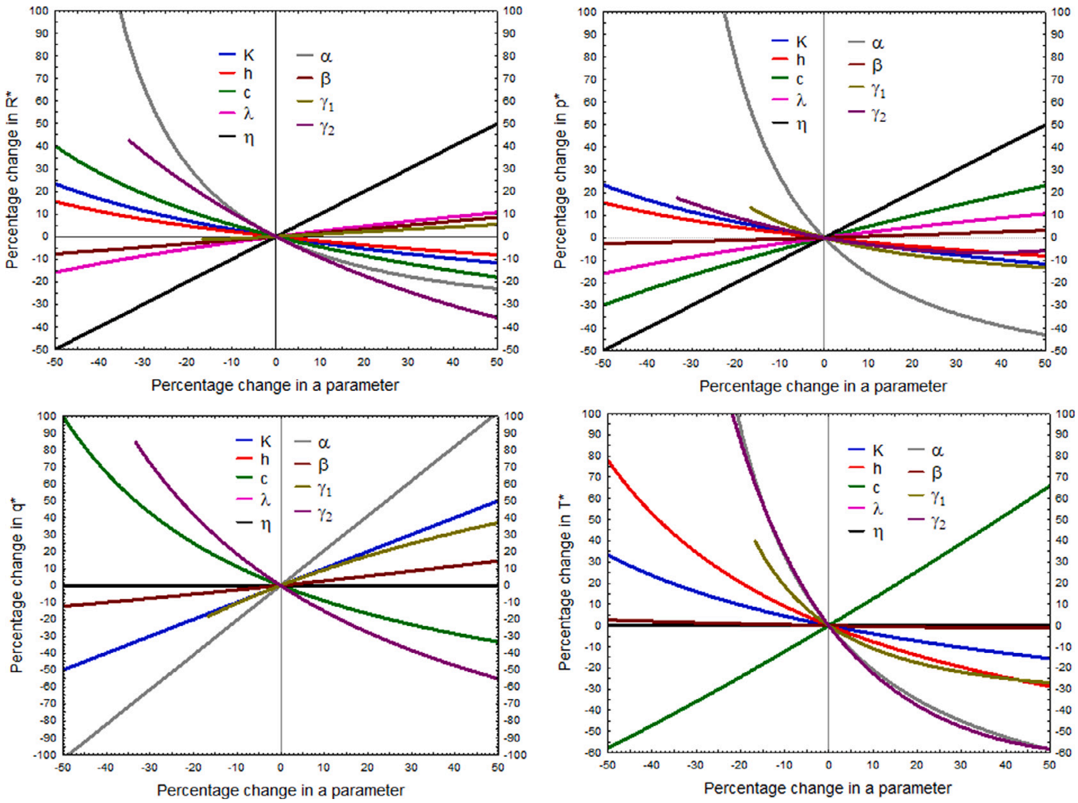

Fig. 1. Percentage changes in p^*, q^*, R^* and T^* versus percentage changes in each parameter.

Table 8
Optimum solutions for the problems of maximum profitability and maximum profit per week.

Problem	p	S	r	q	T	G	R
$R(p,S,r)$	$p^* = 47.14$	$S^* = 40.07$	$r^* = 0$	$q^* = 40.07$	$T^* = 1.41$	$G^R = 174.05$	$R^* = 1.1490$
$G(p,S,r)$	$p^G = 33.82$	$S^G = 79.31$	$r^G = 0.01$	$q^G = 79.30$	$T^G = 0.64$	$G^* = 346.34$	$R^G = 1.0906$

Furthermore, the income expense ratio for the maximum profit per unit time solution is $R^G = 1.0906$ (9.1% profitability), while the maximum income expense ratio is $R^* = 1.1490$ (14.9% profitability). Then, the solution with a maximum income expense ratio yields a profit per unit time 50% lower, but 5.8% more profitability. Consequently, the optimum policy proposed in this paper provides higher profitability with lower sales but less profit per week. Evaluating the total expense for the maximum profit per unit time solution we obtain $C(33.82, 79.30, 0.01) = 3821.06$€ per week, with a profit $G^* = 346.34$€ per week. On the other hand, as we said before, the maximum income expense ratio solution yields a profit of $G(47.14, 40.1, 0) = 174.05$€ per week with a total expense of $C(47.14, 40.07, 0) = 1167.55$€ per week. Then, the difference in total expense, $3821.06 - 1167.55 = 2653.51$€ could be invested in other items with profitability greater than 9.1%, which yields the maximum profit per unit time solution. For example, if the inventory manager markets other items with 14.9% profitability, the profit per week would reach the value of $(3821.06)(0.149) = 569.34$€, which is 64% higher than $G^* = 346.34$ €. Consequently, the inventory manager might prefer to diversify the monetary resources in the most profitable items, instead of using all of them in the solution with the maximum profit per unit time for one of them. In this way, the inventory manager would increase the total profit per unit with the sum of the profits from the other items. For this numerical example the increase could reach 64%, which is a significant amount. However, if there are no other investment options, the maximum profit per unit time policy might be a better option. Therefore, both policies may be necessary depending on each particular case.

Besides that, this numerical example suggests that, regarding the solution of maximum profit per unit time, the solution with the maximum income expense ratio requires higher values for the sale price and the scheduling period, and smaller values for the order quantity and the total cost per unit time in the inventory system. Similar results were obtained by Pando et al. (2020) when the demand rate depends exponentially on the sale price, but the storage cost rate per unit per time is constant.

Finally, for each $p \in (25, 60)$ with increments of one-hundredth, Fig. 2 plots the G_p^*-quantities in the left vertical axis, and the R_p^*-quantities in the right vertical axis. This plot shows that everything is right.

7. Conclusions and managerial insights

This paper analyses an inventory system focused on profitability maximization with two significant characteristics: (i) the demand rate depends simultaneously and isoelastically on the sale price and the stock quantity; and (ii) the storage cost is simultaneously non-linear in time and stock quantity, also with an isoelastic dependence on both variables. The obtained results lead to the following list of conclusions which are also managerial implications for the inventory manager:

(a) The optimum reorder point is always equal to zero, that is, the replenishment should be done when the stock is depleted.
(b) The optimum order quantity is given in a closed form, and it only depends on the purchase price, the replenishing cost and the four elasticity coefficients of the model.

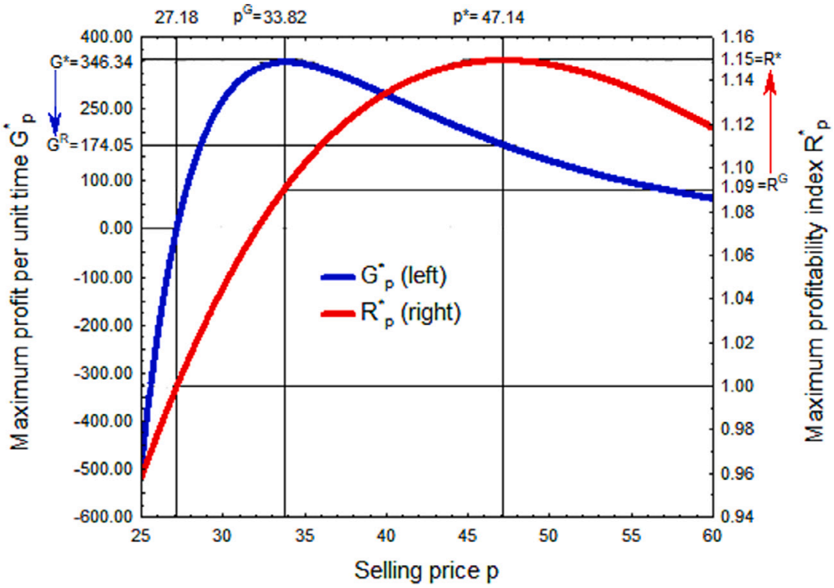

Fig. 2. Maximum profit per unit time and maximum income expense ratio for each p.

(c) The optimum sale price, the maximum income expense ratio, and the optimum scheduling period are also obtained with explicit expressions regarding the initial parameters.
(d) The previous statements make the applicability of the model in a real-life engineering setting does not have any obstacles or limitations. The inventory manager only needs to evaluate the closed expressions for the decision variables and the maximum profitability. However, to implement this solution in practice, the inventory manager should estimate the four elasticity coefficients of the model as best as possible. It could be done using data previously obtained during inventory management.
(e) Excluding the elasticity coefficients of the model, we obtain profitability thresholds for each of the parameters.
(f) The optimum order quantity is proportional to the replenishing cost and inversely proportional to the unit purchase price.
(g) The optimum sale price increases with the unit purchase price, the reference price of the item, and the population size of potential customers. Nevertheless, it decreases with the replenishing cost, the price elasticity of the demand rate and the scale parameter of the storage cost.
(h) The maximum income expense ratio decreases with the replenishing cost, the unit purchase price, the price elasticity of the demand rate, and the scale parameter of the storage cost. On the other hand, it increases with the reference price of the item and the population size of potential customers.
(i) The optimum scheduling period increases with the unit purchase price, and decreases with the replenishing cost, the price elasticity of the demand rate, and the scale parameter of the storage cost. Furthermore, it does not depend on the reference price of the item or the population size of potential customers.
(j) Numerical results suggest that, regarding the maximum profit per unit time policy, the optimum policy for profitability maximization leads to higher values for the sale price and the scheduling period, and lower values for the order quantity and the total sales per unit time.
(k) If the inventory manager diversifies the monetary resources in items with greater profitability, the total profit per unit time could be increased with respect to investing all of them in the maximum profit per unit time policy for one of the items.

Future research topics extending this model could be: (i) to consider perishable or deteriorating items; (ii) to include a unit purchase price or a replenishing cost depending on the order quantity; and (iii) to study inventory systems with multiple items.

CRediT authorship contribution statement

Valentín Pando: Conceptualization, Methodology, Writing – review & editing. **Luis A. San-José:** Conceptualization, Methodology, Writing – review & editing. **Joaquín Sicilia:** Conceptualization, Methodology, Writing – review & editing. **David Alcaide-López-de-Pablo:** Conceptualization, Methodology, Writing – review & editing.

Data availability

No data was used for the research described in the article.

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Appendix A

In this appendix, all the proofs for the results given in this paper are included.

Proof of Lemma 1. As the storage cost rate per unit per time is given by

$$\frac{\partial^2}{\partial t \partial x} H(t,x) = h\gamma_1\gamma_2 t^{\gamma_1-1} x^{\gamma_2-1}$$

the storage cost for the region of the (t,x)-plane under the inventory level curve $x = I(t)$, with $0 \le t \le T(p,S,r)$, can be evaluated as

$$HC(p,S,r) = \int_0^{T(p,S,r)} \left(\int_0^{I(t)} h\gamma_1\gamma_2 t^{\gamma_1-1} x^{\gamma_2-1} dx \right) dt$$
$$= \int_0^{T(p,S,r)} h\gamma_1 t^{\gamma_1-1} \left(I(t)\right)^{\gamma_2} dt$$

With the change of variable $x = I(t)$ we have $t = \frac{S^{1-\beta} - x^{1-\beta}}{(1-\beta)\lambda\eta^\alpha p^{-\alpha}}$ and, using (3), $dx = -\lambda\eta^\alpha p^{-\alpha} \left(I(t)\right)^\beta dt$. Then, the storage cost in a scheduling period is

$$HC(p,S,r) = \left(\frac{h\gamma_1}{(1-\beta)^{\gamma_1-1} \left(\lambda\eta^\alpha p^{-\alpha}\right)^{\gamma_1}} \right) \int_r^S \left(S^{1-\beta} - x^{1-\beta}\right)^{\gamma_1-1} x^{\gamma_2-\beta} dx$$

and the proof is completed. □

Proof of Theorem 1. For any fixed sale price p, if $\Lambda = \lambda(p/\eta)^{-\alpha}$, the inventory model leads to the system studied by Pando et al. (2019) with $\lambda = \Lambda$. Then, using the assumption $\beta \le \gamma_2/\gamma_1$, the first statement is proved by Theorem 2 in that paper.

The function $O(p,q)$ given by (15) is differentiability class C^2 and satisfies that $\lim_{p\to 0^+} O(p,q) = \lim_{p\to\infty} O(p,q) = \infty$ for any $q > 0$, and also $\lim_{q\to 0^+} O(p,q) = \lim_{q\to\infty} O(p,q) = \infty$ for any $p > 0$. Then the minimum value of the function $O(p,q)$ in the feasible region Ω is necessarily obtained at a stationary point inside the Ω set.

The first partial derivatives of the function $O(p,q)$ are given by

$$\frac{\partial O}{\partial p} = -\frac{c}{p^2} - \frac{K}{p^2 q} + \left(\alpha\gamma_1 - 1\right) A p^{\alpha\gamma_1-2} q^{\xi-1} = \frac{-cq - K + \left(\alpha\gamma_1 - 1\right) A p^{\alpha\gamma_1} q^{\xi}}{p^2 q}$$

$$\frac{\partial O}{\partial q} = -\frac{K}{p q^2} + (\xi - 1) A p^{\alpha\gamma_1-1} q^{\xi-2} = \frac{-K + (\xi - 1) A p^{\alpha\gamma_1} q^{\xi}}{p q^2}$$

Then, the stationary points are given by the solutions of the system

$$\left.\begin{array}{r} -cq - K + \left(\alpha\gamma_1 - 1\right) A p^{\alpha\gamma_1} q^{\xi} = 0 \\ -K + (\xi - 1) A p^{\alpha\gamma_1} q^{\xi} = 0 \end{array}\right\}$$

with $p > 0$ and $q > 0$.

By the second equation, we have $A p^{\alpha\gamma_1} q^{\xi} = K/(\xi - 1)$, and substituting in the first equation, we have

$$-cq - K + \frac{K\left(\alpha\gamma_1 - 1\right)}{\xi - 1} = 0$$

Then the unique solution for q is

$$q^* = \frac{\left(\alpha\gamma_1 - \xi\right) K}{(\xi - 1)\, c}$$

and, using the second equation, the unique positive solution for p is

$$p^* = \left(\frac{K\,(q^*)^{-\xi}}{(\xi - 1)\, A}\right)^{1/(\alpha\gamma_1)} \tag{31}$$

As a consequence, (p^*, q^*) is the unique stationary point for the function $O(p,q)$. To prove that it is the minimum we need to calculate the second-order derivatives, as follows:

$$\frac{\partial^2 O}{\partial p^2} = \frac{2c}{p^3} + \frac{2K}{p^3 q} + \left(\alpha\gamma_1 - 1\right)\left(\alpha\gamma_1 - 2\right) A p^{\alpha\gamma_1-3} q^{\xi-1}$$

$$\frac{\partial^2 O}{\partial q^2} = \frac{2K}{p q^3} + (\xi - 1)(\xi - 2) A p^{\alpha\gamma_1-1} q^{\xi-3}$$

$$\frac{\partial^2 O}{\partial p \partial q} = \frac{K}{p^2 q^2} + \left(\alpha\gamma_1 - 1\right)(\xi - 1) A p^{\alpha\gamma_1-2} q^{\xi-2}$$

Then, if $\frac{\partial O}{\partial p} = 0$, we have $\frac{c}{p^2} + \frac{K}{p^2 q} = \left(\alpha\gamma_1 - 1\right) A p^{\alpha\gamma_1-2} q^{\xi-1}$, and also, if $\frac{\partial O}{\partial q} = 0$, we have $\frac{K}{p q^2} = (\xi - 1) A p^{\alpha\gamma_1-1} q^{\xi-2}$. Therefore, if $\frac{\partial O}{\partial p} = \frac{\partial O}{\partial q} = 0$, the second partial derivatives are:

$$\begin{aligned}\frac{\partial^2 O}{\partial p^2} &= \frac{2\left(\alpha\gamma_1 - 1\right) A p^{\alpha\gamma_1-2} q^{\xi-1}}{p} + \left(\alpha\gamma_1 - 1\right)\left(\alpha\gamma_1 - 2\right) A p^{\alpha\gamma_1-3} q^{\xi-1} \\ &= \alpha\gamma_1\left(\alpha\gamma_1 - 1\right) A p^{\alpha\gamma_1-3} q^{\xi-1}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 O}{\partial q^2} &= \frac{2(\xi - 1) A p^{\alpha\gamma_1-1} q^{\xi-2}}{q} + (\xi - 1)(\xi - 2) A p^{\alpha\gamma_1-1} q^{\xi-3} \\ &= \xi(\xi - 1) A p^{\alpha\gamma_1-1} q^{\xi-3}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 O}{\partial p \partial q} &= \frac{(\xi - 1) A p^{\alpha\gamma_1-1} q^{\xi-2}}{p} + \left(\alpha\gamma_1 - 1\right)(\xi - 1) A p^{\alpha\gamma_1-2} q^{\xi-2} \\ &= \alpha\gamma_1(\xi - 1) A p^{\alpha\gamma_1-2} q^{\xi-2}\end{aligned}$$

As a consequence, if $\frac{\partial O}{\partial p} = 0$ and $\frac{\partial O}{\partial q} = 0$, all these second partial derivatives are strictly positive because $\alpha\gamma_1 > \xi > 1$. Moreover, the Hessian of the function $O(p,q)$ is

$$\begin{aligned}\left(\frac{\partial^2 O}{\partial p^2}\right)\left(\frac{\partial^2 O}{\partial q^2}\right) - \left(\frac{\partial^2 O}{\partial p \partial q}\right)^2 &= \alpha\gamma_1(\xi - 1) A^2 p^{2\alpha\gamma_1-4} q^{2\xi-4} \\ &\quad \times \left(\left(\alpha\gamma_1 - 1\right)\xi - \alpha\gamma_1(\xi - 1)\right) \\ &= \alpha\gamma_1(\xi - 1)\left(\alpha\gamma_1 - \xi\right) A^2 p^{2\alpha\gamma_1-4} q^{2\xi-4}\end{aligned}$$

Therefore, the Hessian matrix of the function $O(p,q)$ at the point (p^*, q^*) is positive definite, because $\alpha\gamma_1 > \xi > 1$. This ensures that the minimum of the function $O(p,q)$ is obtained at the point (p^*, q^*). Then, the order quantity q^* is given by the expression (16) in statement (ii). Moreover, substituting the values for A and q^* in the expression of p^*, we obtain the expression (17) in statement (iii). To prove statement (iv), we evaluate the function $O(p,q)$ at the point (p^*, q^*) taking into account the fact that $\frac{K}{p^*(q^*)^2} = (\xi - 1) A (p^*)^{\alpha\gamma_1-1} (q^*)^{\xi-2}$. Thus, we get

$$\begin{aligned} O^* &= O\left(p^*, q^*\right) = \frac{c}{p^*} + \frac{K}{p^* q^*} + A\left(p^*\right)^{\alpha\gamma_1-1}\left(q^*\right)^{\xi-1} \\ &= \frac{c}{p^*} + \frac{K}{p^* q^*} + \frac{K}{(\xi - 1) p^* q^*} \\ &= \left(c + \frac{\xi K}{(\xi - 1) q^*}\right)\left(\frac{1}{p^*}\right) = \left(\frac{\alpha\gamma_1}{\alpha\gamma_1 - \xi}\right)\left(\frac{c}{p^*}\right)\end{aligned}$$

Then, statement (iv) is proved. Moreover, using the expression (9), the maximum income expense ratio will be

$$R^* = \frac{1}{O\left(p^*, q^*\right)} = \left(1 - \frac{\xi}{\alpha\gamma_1}\right)\left(\frac{p^*}{c}\right)$$

and, using the expression (17) for p^*, we obtain the final expression in (19) for R^*. Then, statement (v) is also proved.

Furthermore, using the expression (5), the optimum scheduling period is

$$T^* = T\left(p^*, q^*, 0\right) = \frac{(q^*)^{1-\beta}\,(p^*)^{\alpha}}{(1 - \beta)\lambda\eta^{\alpha}}$$

and substituting the values for q^* and p^*, the expression (20) in the statement (vi) is obtained.

Now, as the storage cost in each cycle is given by (12) and $\frac{\partial O}{\partial q}(p^*, q^*) = 0$, then

$$HC\left(p^*, q^*, 0\right) = A\left(p^*\right)^{\alpha\gamma_1}\left(q^*\right)^{\xi} = \frac{K}{\xi - 1}$$

and the total expense in each scheduling period is the sum of the storage cost, the replenishing cost and the purchase cost, that is

$$\frac{K}{\xi - 1} + K + cq^* = \frac{K}{\xi - 1} + K + \frac{\left(\alpha\gamma_1 - \xi\right) K}{\xi - 1} = \frac{\alpha\gamma_1 K}{\xi - 1}$$

Therefore, statement (vii) is proved. From the previous expression, the cost proportion of the storage cost on the total expense is $\frac{1}{\alpha\gamma_1}$, the cost proportion for the replenishing cost is $\frac{\xi-1}{\alpha\gamma_1}$, and the remaining proportion for the purchase cost is $1 - \frac{\xi}{\alpha\gamma_1}$. As a consequence, statement (viii) is also proved and the proof of Theorem 1 is completed. □

Proof of Corollary 1. For statement (i), the inventory system is profitable if, and only if, the income expense ratio R^* is strictly greater than 1, that is

$$R^* > 1 \Leftrightarrow \left(1 - \frac{\xi}{\alpha\gamma_1}\right)\left(\frac{p^*}{c}\right) > 1 \Leftrightarrow p^* > \left(\frac{\alpha\gamma_1}{\alpha\gamma_1 - \xi}\right) c$$

Then, using the formula (17) in this expression, the inequality is equivalent to

$$\frac{(\xi - 1)^{\xi-1}(1 - \beta)^{\gamma_1}\lambda^{\gamma_1} c^{\xi}}{\gamma_1 B\left(\gamma_1, 1 + \gamma_2/(1 - \beta)\right) K^{\xi-1} h} > \frac{\left(\alpha\gamma_1\right)^{\alpha\gamma_1} c^{\alpha\gamma_1}}{\left(\alpha\gamma_1 - \xi\right)^{\alpha\gamma_1-\xi}\eta^{\alpha\gamma_1}}$$

that is,

$$\frac{K^{\xi-1} c^{\alpha\gamma_1-\xi} h}{\lambda^{\gamma_1}\eta^{\alpha\gamma_1}} < \frac{(\xi - 1)^{\xi-1}(1 - \beta)^{\gamma_1}\left(\alpha\gamma_1 - \xi\right)^{\alpha\gamma_1-\xi}}{\gamma_1\left(\alpha\gamma_1\right)^{\alpha\gamma_1} B\left(\gamma_1, 1 + \gamma_2/(1 - \beta)\right)} = \Delta$$

as statement (i) establishes. Even more, if $R^* > 1$, as $\alpha\gamma_1 > \xi$, necessarily $p^* > c$ and statement (ii) is proved. Finally, for statement (iii), as

$\alpha\gamma_1 > \xi$, if condition (23) is satisfied then

$$p^* = \left(\frac{K(q^*)^{-\xi}}{(\xi-1)A}\right)^{1/(\alpha\gamma_1)} = \left(\frac{K\left(\frac{(\alpha\gamma_1-\xi)K}{(\xi-1)c}\right)^{-\xi}}{(\xi-1)A}\right)^{1/(\alpha\gamma_1)}$$
$$= \left(\frac{(\xi-1)^{\xi-1}c^{\xi}}{(\alpha\gamma_1-\xi)^{\xi}AK^{\xi-1}}\right)^{1/(\alpha\gamma_1)} \le \left(c^{\alpha\gamma_1-\xi}c^{\xi}\right)^{1/(\alpha\gamma_1)} = c$$

and the inventory system never makes a profit, because

$$R^* = \left(1-\frac{\xi}{\alpha\gamma_1}\right)\left(\frac{p^*}{c}\right) < 1$$

Then, the proof of the corollary is completed. □

Appendix B

The partial derivatives of the optimum order quantity q^* given by expression (16) are easily obtained as:

$$\frac{\partial q^*}{\partial K} = \frac{\alpha\gamma_1-\xi}{(\xi-1)c} = \frac{q^*}{K} > 0$$

$$\frac{\partial q^*}{\partial h} = \frac{\partial q^*}{\partial \lambda} = \frac{\partial q^*}{\partial \eta} = 0$$

$$\frac{\partial q^*}{\partial c} = -\frac{(\alpha\gamma_1-\xi)K}{(\xi-1)c^2} = \frac{-q^*}{c} < 0$$

and

$$\frac{\partial q^*}{\partial \alpha} = \frac{\gamma_1 K}{(\xi-1)c} = \frac{\gamma_1 q^*}{\alpha\gamma_1-\xi} > 0$$

because $\alpha\gamma_1 > \xi$.

Now, the derivatives of the optimum sale price p^*, given by expression (17), will be evaluated. For the parameter K, using logarithmic differentiation, we have

$$\frac{\partial p^*}{\partial K} = p^*\left(\frac{\partial \ln p^*}{\partial K}\right) = p^*\left(\frac{\partial\left(\frac{-(\xi-1)\ln K}{\alpha\gamma_1}\right)}{\partial K}\right) = \left(\frac{\xi-1}{\alpha\gamma_1}\right)\left(\frac{-p^*}{K}\right) < 0$$

In a similar way, for the parameters h, λ, c, and η, we obtain:

$$\frac{\partial p^*}{\partial h} = p^*\left(\frac{\partial \ln p^*}{\partial h}\right) = p^*\left(\frac{\partial\left(\frac{-\ln h}{\alpha\gamma_1}\right)}{\partial h}\right) = \left(\frac{1}{\alpha\gamma_1}\right)\left(\frac{-p^*}{h}\right) < 0$$

$$\frac{\partial p^*}{\partial \lambda} = p^*\left(\frac{\partial \ln p^*}{\partial \lambda}\right) = p^*\left(\frac{\partial\left(\frac{\gamma_1\ln \lambda}{\alpha\gamma_1}\right)}{\partial \lambda}\right) = \left(\frac{1}{\alpha}\right)\left(\frac{p^*}{\lambda}\right) > 0$$

$$\frac{\partial p^*}{\partial c} = p^*\left(\frac{\partial \ln p^*}{\partial c}\right) = p^*\left(\frac{\partial\left(\frac{\xi\ln c}{\alpha\gamma_1}\right)}{\partial c}\right) = \left(\frac{\xi}{\alpha\gamma_1}\right)\left(\frac{p^*}{c}\right) > 0$$

$$\frac{\partial p^*}{\partial \eta} = p^*\left(\frac{\partial \ln p^*}{\partial \eta}\right) = \frac{p^*}{\eta} > 0$$

For the parameter α, we have

$$\ln p^* = \ln\eta + \left(\frac{1}{\alpha\gamma_1}\right)\ln\left(\frac{(\xi-1)^{\xi-1}(1-\beta)^{\gamma_1}\lambda^{\gamma_1}c^{\xi}}{(\alpha\gamma_1-\xi)^{\xi}\gamma_1 B(\gamma_1, 1+\gamma_2/(1-\beta))K^{\xi-1}h}\right)$$

and

$$\frac{\partial \ln p^*}{\partial \alpha} = \left(\frac{-1}{\alpha^2\gamma_1}\right)\ln\left(\frac{(\xi-1)^{\xi-1}(1-\beta)^{\gamma_1}\lambda^{\gamma_1}c^{\xi}}{(\alpha\gamma_1-\xi)^{\xi}\gamma_1 B(\gamma_1, 1+\gamma_2/(1-\beta))K^{\xi-1}h}\right)$$
$$+\left(\frac{1}{\alpha\gamma_1}\right)\left(\frac{-\gamma_1\xi}{\alpha\gamma_1-\xi}\right)$$
$$= \frac{-\ln(p^*/\eta)}{\alpha} - \left(\frac{1}{\alpha}\right)\left(\frac{\xi}{\alpha\gamma_1-\xi}\right)$$

Then, we obtain

$$\frac{\partial p^*}{\partial \alpha} = p^*\left(\frac{\partial \ln p^*}{\partial \alpha}\right) = \left(\ln(p^*/\eta) + \frac{\xi}{\alpha\gamma_1-\xi}\right)\left(\frac{-p^*}{\alpha}\right) < 0$$

because $\alpha\gamma_1 > \xi$ and, as the inventory is profitable, $p^* > c \ge \eta$.

For the maximum income expense ratio R^*, given by (19), the partial derivatives are:

$$\frac{\partial R^*}{\partial K} = \left(1-\frac{\xi}{\alpha\gamma_1}\right)\left(\frac{\partial p^*/\partial K}{c}\right) = \left(\frac{\left(1-\frac{\xi}{\alpha\gamma_1}\right)(\xi-1)}{\alpha\gamma_1 c}\right)\left(\frac{-p^*}{K}\right)$$
$$= \left(\frac{\xi-1}{\alpha\gamma_1}\right)\left(\frac{-R^*}{K}\right) < 0$$

$$\frac{\partial R^*}{\partial h} = \left(1-\frac{\xi}{\alpha\gamma_1}\right)\left(\frac{\partial p^*/\partial h}{c}\right) = \left(\frac{\left(1-\frac{\xi}{\alpha\gamma_1}\right)}{\alpha\gamma_1 c}\right)\left(\frac{-p^*}{h}\right)$$
$$= \left(\frac{1}{\alpha\gamma_1}\right)\left(\frac{-R^*}{h}\right) < 0$$

$$\frac{\partial R^*}{\partial \lambda} = \left(1-\frac{\xi}{\alpha\gamma_1}\right)\left(\frac{\partial p^*/\partial \lambda}{c}\right) = \left(\frac{1-\frac{\xi}{\alpha\gamma_1}}{c}\right)\left(\frac{p^*}{\alpha\lambda}\right) = \left(\frac{1}{\alpha}\right)\left(\frac{R^*}{\lambda}\right) > 0$$

$$\frac{\partial R^*}{\partial c} = \left(1-\frac{\xi}{\alpha\gamma_1}\right)\left(\frac{c(\partial p^*/\partial c) - p^*}{c^2}\right) = \left(1-\frac{\xi}{\alpha\gamma_1}\right)\left(\frac{\frac{\xi p^*}{\alpha\gamma_1} - p^*}{c^2}\right)$$
$$= \left(1-\frac{\xi}{\alpha\gamma_1}\right)\left(\frac{-R^*}{c}\right) < 0$$

$$\frac{\partial R^*}{\partial \eta} = \left(1-\frac{\xi}{\alpha\gamma_1}\right)\left(\frac{\partial p^*/\partial \eta}{c}\right) = \left(1-\frac{\xi}{\alpha\gamma_1}\right)\left(\frac{p^*}{\eta c}\right) = \frac{R^*}{\eta} > 0$$

and

$$\frac{\partial R^*}{\partial \alpha} = \left(\frac{\xi}{\alpha^2\gamma_1}\right)\left(\frac{p^*}{c}\right) + \left(1-\frac{\xi}{\alpha\gamma_1}\right)\left(\frac{\partial p^*/\partial \alpha}{c}\right) =$$
$$= \left(\frac{\xi}{\alpha^2\gamma_1}\right)\left(\frac{p^*}{c}\right) - \left(\frac{\alpha\gamma_1-\xi}{\alpha\gamma_1}\right)\left(\ln(p^*/\eta) + \frac{\xi}{\alpha\gamma_1-\xi}\right)\left(\frac{p^*}{\alpha c}\right)$$
$$= \left(1-\frac{\xi}{\alpha\gamma_1}\right)\left(\frac{-p^*\ln(p^*/\eta)}{\alpha c}\right) = \left(\frac{-R^*}{\alpha}\right)\ln(p^*/\eta) < 0$$

because, as the inventory is profitable, $p^* > c \ge \eta$.

Finally, for the optimum scheduling period T^* given by (20), we use similar reasonings as for p^* to obtain the following results:

$$\frac{\partial T^*}{\partial K} = T^*\left(\frac{\partial \ln T^*}{\partial K}\right) = T^*\left(\frac{\partial\left(\frac{-(\gamma_2-1)\ln K}{\gamma_1}\right)}{\partial K}\right) = \left(\frac{\gamma_2-1}{\gamma_1}\right)\left(\frac{-T^*}{K}\right) \le 0$$

$$\frac{\partial T^*}{\partial h} = T^*\left(\frac{\partial \ln T^*}{\partial h}\right) = T^*\left(\frac{\partial\left(\frac{-\ln h}{\gamma_1}\right)}{\partial h}\right) = \left(\frac{1}{\gamma_1}\right)\left(\frac{-T^*}{h}\right) < 0$$

$$\frac{\partial T^*}{\partial \lambda} = \frac{\partial T^*}{\partial \eta} = 0$$

$$\frac{\partial T^*}{\partial c} = T^*\left(\frac{\partial \ln T^*}{\partial c}\right) = T^*\left(\frac{\partial\left(\frac{\gamma_2\ln c}{\gamma_1}\right)}{\partial c}\right) = \left(\frac{\gamma_2}{\gamma_1}\right)\left(\frac{T^*}{c}\right) > 0$$

$$\frac{\partial T^*}{\partial \alpha} = T^*\left(\frac{\partial \ln T^*}{\partial \alpha}\right) = T^*\left(\frac{\partial\left(\frac{-\gamma_2\ln(\alpha\gamma_1-\xi)}{\gamma_1}\right)}{\partial \alpha}\right) = \frac{-\gamma_2 T^*}{\alpha\gamma_1-\xi} < 0$$

as in Table 4. □

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