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A new approach to maximize the profit/cost ratio in a stock-dependent demand inventory model

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ABSTRACT

This work analyzes an inventory system with stock-dependent demand and non-linear holding cost. It presents a new approach to maximize the return on investment, that is, the profit/cost ratio. When an inventory manager can invest in different projects and the resources are limited, it seems sensible to select those projects that provide a higher return on investment. Thus, the goal of the manager will be to find the inventory policy that gives a major return on investment. Note that the solution for the maximum profit per unit time does not necessarily match the solution of the maximum profit/cost ratio. Consequently, a new procedure to obtain the inventory policy that maximizes the return on investment should be proposed. In this paper, it is proved that maximizing the profit/cost ratio is equivalent to minimizing the inventory cost per unit of an item, instead of minimizing the inventory cost per unit time. The optimal policy can be obtained in a closed form and the replacement should be done when the stock is depleted. Thus, the inventory manager does not need to process a new order while there are items available in stock. This optimal solution is different from the other policies proposed for the problems of minimum cost or maximum profit per unit time. Finally, numerical examples are solved to illustrate the theoretical results and the solution methodology proposed in the work.

1. Introduction

This paper presents an economic order quantity model (EOQ) where the demand rate and the holding cost rate both depend on the inventory level. Since the 1980s, many models have appeared in the inventory literature that use one of these two assumptions. Some consider the aim of minimizing the inventory cost per unit time, while others focus on maximizing the profit per unit time. The two approaches may be appropriate depending on the ultimate goal of the inventory manager. We have cited the ones that inspired this work in Table 1 and we have classified them according to which of the two assumptions they assume and which objective function is considered.

Specifically, in this paper, we assume that the demand rate is a concave power function on the stock level, and that the holding cost rate per unit time is a convex power function on the quantity of items held in stock. The Model B of Goh (1994) already considered these two assumptions simultaneously in an inventory model focusing on the minimization of the inventory cost per unit time. However, it is known that, when the demand rate depends on the stock level, the revenues can be increased by maintaining a higher level of inventory, even though the inventory costs increase. Therefore, if the aim is to maximize the profit per unit time, the best inventory policy can be different from the other one with the minimum inventory cost per unit time.

Pando et al. (2012b) solved an inventory model from the maximum profit per unit time approach, proving that the best solution was different from the one for the minimization of the inventory cost per unit time. The comparison of these two models leads to the observation that the solution of the minimum inventory cost per unit time carries a low profit per unit time, while the maximum profit per unit time solution carries high purchasing and inventory costs per unit time. This issue could be expected, as each model only takes into account one of the two targets, forgetting the other. However, in many real situations, the inventory man-

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Table 1
Inventory models with stock-dependent demand or stock-dependent holding cost

Model	Objective: Minimum Cost	Objective: Maximum Profit	Demand rate dependent on the stock level	Holding cost rate dependent on the stock level
Muhlemann and Valtis-Spanopoulos (1980)	x			x
Baker and Urban (1988)		x	x	
Goh (1992)	x		x	
Goh (1994)	x		x	x
Padmanabhan and Vrat (1995)		x	x	
Giri and Chaudhuri (1998)	x		x	x
Chung et al. (2000)		x	x	
Chang (2004)		x	x	x
Dye and Ouyang (2005)		x	x	
Teng and Chang (2005)		x	x	
Berman and Perry (2006)		x	x	
Wu et al. (2006)	x		x	
Alfares (2007)	x		x	
Sana and Chaudhuri (2008)		x	x	
Urban (2008)		x	x	
Chang et al. (2010)		x	x	
Yang et al. (2010)		x	x	
Lee and Dye (2012)		x	x	
Pando et al. (2012b)		x	x	x
Pando et al., 2013		x	x	x
Yang (2014)		x	x	x
Alfares (2015)		x	x	
Choudhury et al. (2015)		x	x	x
Pervin et al. (2017)	x		x	x
Pando et al. (2018)		x	x	x

ager may prefer a solution that provides a high profit per unit time without greatly increasing the total cost invested in the inventory. That is, perhaps the manager is more interested in maximizing the return on the investment, which is given by the profit/cost ratio of the money handled in the inventory. In this case, the formulation of the model should aim to maximize the profit/cost ratio.

This alternative approach has been less used in inventory theory. As a starting point, Schroeder and Krishnan (1976) considered the return on investment (ROI) as a criterion for inventory models. Arcelus and Srinivasan (1985) adapted the EOQ model to maximizing ROI and they obtained the optimal solution in an inventory model with price-dependent demand rate, where the selling price is set as a markup of the unit purchasing cost. Also, Trietsch (1995) developed the company-wide ROI maximizing order quantity, proving that it does not necessarily follow the square root of the demand level and that it is bounded from above by EOQ. Other papers focused on maximizing the return on investment are Otake et al. (1999), Chen (2001), Otake and Min (2001) and Li et al. (2008). More recently, Chen and Liao (2014) also considered the return on investment maximization in an inventory model for deteriorating items.

As can be seen in Table 1, none of the models with stock-dependent demand rate and/or a stock-dependent holding cost are focused on the maximization of the profit/cost ratio, that is, the profitability of the inventory system. So, this is the gap that we aim to fill with this paper, building an inventory model with demand rate and holding cost both dependent on the stock level whose objective is the maximization of the profit/cost ratio in the inventory system. In Section 2, the notation and assumptions are established, and the formulation of the model is presented to obtain the mathematical formulation of the problem. Section 3 includes the theoretical results leading to the resolution of the problem and the choice of the optimal policy for the inventory system. A sensitivity analysis of the optimal solution with respect to the main parameters of the model is included in Section 4. Section 5 illustrates the use of the model with some numerical examples. Finally, the conclusions and future research lines on this topic are set out in Section 6.

2. Mathematical model

This paper considers an inventory system for a single item over an infinite planning horizon. The inventory control assumes a continuous review with instantaneous replenishment. We suppose that shortages are not allowed. Also, the unit purchasing cost, the unit selling price, and the ordering cost are known constants.

Moreover, we assume that the demand rate at time t is a known function $D(t)$, which depends on the inventory level $I(t)$ at each time. As in Baker and Urban (1988), we suppose that this function is given by $D(t) = \lambda[I(t)]^{\beta}$, with $\lambda > 0$ and $0 \leq \beta < 1$. Then, the demand rate is a concave power function of the inventory level. With this functional form, as time goes by, the inventory level decreases and so does the demand rate. The coefficient λ is the scale parameter, and the exponent β is known as the elasticity of the demand rate with respect to the inventory level, which is a measure of the responsiveness of the demand rate to changes in the inventory level. Obviously, the basic case with constant demand rate is obtained if $\beta = 0$.

With this assumption for the demand rate, it could be interesting to set a new order before the stock is depleted, thus leading to an increase in the demand rate. Therefore, the profit is improved and this could offset the higher ordering and holding costs. In fact, Baker and Urban (1988) found that the maximum profit per unit time in an inventory model with this demand rate function is obtained with a non-zero order point, which implies that the optimal length of the inventory cycle is strictly shorter than the length of the cycle τ that would be necessary to deplete the inventory. Now, we want to explore whether this statement is also true when the aim is to maximize the return on investment of the inventory system, which is calculated as the ratio between the profit and the total cost invested to obtain it.

In addition, the holding cost rate per unit time for x items held in stock is assumed as the convex power function $H(x) = hx^{\gamma}$,

Table 2
Notation for the model

τ	Time period over which there is stock in the inventory, *decision variable* ($\tau > 0$)
T	Length of the inventory cycle, *decision variable*($0 < T \leq \tau$)
t	Elapsed time in the inventory ($0 \leq t \leq T$)
x	Quantity of items in stock ($x \geq 0$)
$I(t)$	Inventory level at time t ($I(t) \geq 0$)
q	Lot size, that is, $q = I(0) - I(T)$ ($q > 0$)
p	Unit purchasing cost ($p > 0$)
v	Unit selling price ($v > p$)
K	Ordering cost per order ($K > 0$)
h	Scale parameter of the holding cost ($h > 0$)
γ	Elasticity parameter of the holding cost ($\gamma \geq 1$)
$H(t)$	Holding cost rate per unit time, $H(t) = h[I(t)]^{\gamma}$
λ	Scale parameter of the demand rate ($\lambda > 0$)
β	Elasticity parameter of the demand rate with respect to the stock level ($0 \leq \beta < 1$)
$D(t)$	Demand rate at time t, $D(t) = \lambda[I(t)]^{\beta}$

where $h > 0$ is the scale parameter and $\gamma \geq 1$ is the elasticity parameter. Thus, the holding cost rate is non-linear with respect to the quantity of stored items when $\gamma > 1$. In the inventory system, the number x of items in stock at time t is given by the inventory level $I(t)$. Therefore, the holding cost rate per unit time is $h[I(t)]^{\gamma}$, just as in Goh (1994), Giri and Chaudhuri (1998), Chang (2004) and Pando et al. (2012b).

Two decision variables are considered in the model: the length of the inventory cycle T, and the time τ where the inventory level would be depleted (that is, $I(\tau) = 0$). As shortages are not allowed, the condition $T \leq \tau$ must be considered. Table 2 includes the notation used in the paper.

For $0 \leq t < \tau$, the inventory level curve $I(t)$ is obtained by solving the differential equation

$$\frac{dI(t)}{dt} = -\lambda[I(t)]^{\beta} \tag{1}$$

with the condition $I(\tau) = 0$. The solution can be written as

$$I(t) = [(1-\beta)\lambda(\tau - t)]^{1/(1-\beta)} \tag{2}$$

Then, the holding cost in an inventory cycle, $HC(\tau, T)$, can be evaluated as

$$\begin{aligned} HC(\tau, T) &= \int_0^T h[I(t)]^{\gamma} dt \\ &= h[(1-\beta)\lambda]^{\gamma/(1-\beta)} \int_0^T (\tau - t)^{\gamma/(1-\beta)} dt = \\ &= A\left[\tau^{1+\gamma/(1-\beta)} - (\tau - T)^{1+\gamma/(1-\beta)}\right] \end{aligned} \tag{3}$$

with

$$A = \frac{h[(1-\beta)\lambda]^{\gamma/(1-\beta)}}{1 + \gamma/(1-\beta)} \tag{4}$$

As $I(0) = [(1-\beta)\lambda\tau]^{1/(1-\beta)}$ and $I(T) = [(1-\beta)\lambda(\tau - T)]^{1/(1-\beta)}$, the lot size is

$$q = I(0) - I(T) = B\left[\tau^{1/(1-\beta)} - (\tau - T)^{1/(1-\beta)}\right] \tag{5}$$

with

$$B = [(1-\beta)\lambda]^{1/(1-\beta)} \tag{6}$$

The income obtained in each inventory cycle is vq. The total cost is the sum of the purchasing cost pq, the ordering cost K, and the holding cost $HC(\tau, T)$. Then, the total cost per unit time $TC(\tau, T)$ is

$$TC(\tau, T) = \frac{pq + K + HC(\tau, T)}{T} \tag{7}$$

The profit per unit time $G(\tau, T)$ is given by

$$G(\tau, T) = \frac{(v-p)q - K - HC(\tau, T)}{T} \tag{8}$$

and the profit/cost ratio $R(\tau, T)$ for the inventory model is given by

$$R(\tau, T) = \frac{G(\tau, T)}{TC(\tau, T)} = \frac{v}{p + r(\tau, T)} - 1, \tag{9}$$

where

$$r(\tau, T) = \frac{K + HC(\tau, T)}{q} = \frac{K + A\left[\tau^{1+\gamma/(1-\beta)} - (\tau - T)^{1+\gamma/(1-\beta)}\right]}{B\left[\tau^{1/(1-\beta)} - (\tau - T)^{1/(1-\beta)}\right]} \tag{10}$$

Note that, if we define the inventory cost to be the sum of the ordering cost and the holding cost, then the function $r(\tau, T)$ evaluates the average inventory cost per unit of an item. Moreover, as $r(\tau, T) > 0$, it is clear that $-1 < R(\tau, T) < v/p - 1$.

The aim of the model is to maximize the profit/cost ratio $R(\tau, T)$. Therefore, the mathematical problem is

$$\max_{\substack{\tau > 0 \\ 0 < T \leq \tau}} R(\tau, T) \tag{11}$$

which is equivalent to

$$\min_{\substack{\tau > 0 \\ 0 < T \leq \tau}} r(\tau, T) \tag{12}$$

From (3), (5) and (8), it follows that the profit per unit time $G(\tau, T)$ can be evaluated as

$$G(\tau, T) = \frac{(v-p)B\left[\tau^{1/(1-\beta)} - (\tau - T)^{1/(1-\beta)}\right] - K - A\left[\tau^{1+\gamma/(1-\beta)} - (\tau - T)^{1+\gamma/(1-\beta)}\right]}{T} \tag{13}$$

In a similar way, the inventory cost per unit time (that is, the sum of the ordering cost and the holding cost per unit time) can be expressed as

$$C(\tau, T) = \frac{K + HC(\tau, T)}{T} = \frac{K + A\left[\tau^{1+\gamma/(1-\beta)} - (\tau - T)^{1+\gamma/(1-\beta)}\right]}{T} \tag{14}$$

Note that, if the demand rate is constant (that is, $\beta = 0$), then $G(\tau, T) = \lambda(v - p) - \lambda r(\tau, T)$ and $C(\tau, T) = \lambda r(\tau, T)$. As a consequence, if $\beta = 0$, then the solution of the problem (12) is equal to the solution for the problem of the minimization of the inventory cost per unit time, and is also the same solution for the problem of the maximum profit per unit time. But this is not true if $\beta > 0$, that is, if the demand rate depends on the inventory level. Then, the purpose of this paper is to obtain the optimal solution for the problem of the maximum profit/cost ratio.

3. Problem solution

In order to solve the problem (12), we begin by proposing a first result which will be used in the search for the optimal inventory policy.

Lemma 1. *Let $K > 0$, $\tau > 0$ and $T > 0$ with $T < \tau$. Then the following inequalities are held:*

(i) $\dfrac{K}{\tau^{1/(1-\beta)} - (\tau - T)^{1/(1-\beta)}} > \dfrac{K}{\tau^{1/(1-\beta)}}$

(ii) $\dfrac{\tau^{1+\gamma/(1-\beta)} - (\tau - T)^{1+\gamma/(1-\beta)}}{\tau^{1/(1-\beta)} - (\tau - T)^{1/(1-\beta)}} > \tau^{1+(\gamma-1)/(1-\beta)}$

Proof. The inequality (i) is obvious and (ii) follows from

$$\begin{aligned}
&\frac{\tau^{1+\gamma/(1-\beta)} - (\tau - T)^{1+\gamma/(1-\beta)}}{\tau^{1/(1-\beta)} - (\tau - T)^{1/(1-\beta)}} \\
&> \frac{\tau^{1+\gamma/(1-\beta)} - \tau^{1+(\gamma-1)/(1-\beta)}(\tau - T)^{1/(1-\beta)}}{\tau^{1/(1-\beta)} - (\tau - T)^{1/(1-\beta)}} \\
&= \frac{\tau^{1+(\gamma-1)/(1-\beta)}\left[\tau^{1/(1-\beta)} - (\tau - T)^{1/(1-\beta)}\right]}{\tau^{1/(1-\beta)} - (\tau - T)^{1/(1-\beta)}} \\
&= \tau^{1+(\gamma-1)/(1-\beta)} \quad \square
\end{aligned}$$

Therefore, from Lemma 1, the function $r(\tau, T)$ satisfies that

$$r(\tau, T) > \frac{K + A\tau^{1+\gamma/(1-\beta)}}{B\tau^{1/(1-\beta)}} = r(\tau, \tau) \tag{15}$$

for every (τ, T), with $0 < T < \tau$.

As a consequence, the minimum of the function $r(\tau, T)$ is obtained when $T = \tau$. Thus, considering the function

$$\Phi(\tau) = r(\tau,\tau) = \left(\frac{1}{B}\right)\left[\frac{K}{\tau^{1/(1-\beta)}} + A\tau^{1+(\gamma-1)/(1-\beta)}\right] \tag{16}$$

we only need to solve the problem

$$\min_{\tau>0} \ \Phi(\tau) \tag{17}$$

to obtain the optimal solution of the problem (12). $\square$

The following proposition gives the solution for the problem (17):

Proposition 1. *Consider the function $\Phi(\tau)$ given by (16) with $\tau > 0$ and the auxiliary parameters A and B given by (4) and (6) respectively. Then, the minimum value for the function $\Phi(\tau)$ is obtained at the point*

$$\tau^* = \left(\frac{K}{A(\gamma - \beta)}\right)^{(1-\beta)/(\gamma+1-\beta)} \tag{18}$$

and the minimum value is

$$\Phi^* = \Phi(\tau^*) = \left(\frac{K(\gamma + 1 - \beta)}{B(\gamma - \beta)}\right)\left(\frac{K}{A(\gamma - \beta)}\right)^{-1/(\gamma+1-\beta)} \tag{19}$$

Proof. Taking into account that $\beta \in [0, 1)$ and $\gamma \geq 1$, it is easily seen that $\lim_{\tau\to 0^+} \Phi(\tau) = \lim_{\tau\to\infty} \Phi(\tau) = \infty$. Moreover, the first two derivatives of this function are:

$$\begin{aligned}
\Phi'(\tau) &= \left(\frac{1}{(1-\beta)B}\right)\left[-\frac{K}{\tau^{1+1/(1-\beta)}} + A(\gamma - \beta)\tau^{(\gamma-1)/(1-\beta)}\right] \\
&= \frac{-K + A(\gamma - \beta)\tau^{1+\gamma/(1-\beta)}}{(1-\beta)B\tau^{1+1/(1-\beta)}}
\end{aligned}$$

and

$$\Phi''(\tau) = \frac{(2-\beta)K + (\gamma - \beta)(\gamma - 1)A\tau^{1+\gamma/(1-\beta)}}{B(1-\beta)^2\tau^{2+1/(1-\beta)}} > 0$$

Therefore, Φ is a strictly convex function and the only root of the equation $\Phi'(\tau) = 0$ is

$$\tau^* = \left(\frac{K}{A(\gamma - \beta)}\right)^{(1-\beta)/(\gamma+1-\beta)}$$

Then, the minimum of the function $\Phi(\tau)$ is obtained at τ^*, and the minimum value is

$$\begin{aligned}
\Phi^* = \Phi(\tau^*) &= \frac{K + A(\tau^*)^{1+\gamma/(1-\beta)}}{B(\tau^*)^{1/(1-\beta)}} = \left(\frac{K(\gamma + 1 - \beta)}{B(\gamma - \beta)}\right)(\tau^*)^{-1/(1-\beta)} \\
&= \left(\frac{K(\gamma + 1 - \beta)}{B(\gamma - \beta)}\right)\left(\frac{K}{A(\gamma - \beta)}\right)^{-1/(\gamma+1-\beta)}
\end{aligned}$$

which proves the proposition. $\square$

From this last result, the solution of the equivalent problems (11) and (12) are given in the next theorem, which establishes the optimal policy that maximizes the profit/cost ratio $R(\tau, T)$ or, equivalently, minimizes the average inventory cost per unit of an item $r(\tau, T)$.

Theorem 1. *Consider the functions $R(\tau, T)$ and $r(\tau, T)$ given by (9) and (10) respectively. Then, the following statements are satisfied for the equivalent problems (11) and (12):*

(i) *The optimal solution is reached at the point (τ^*, T^*), where*

$$T^* = \tau^* = \left(\frac{K(\gamma + 1 - \beta)}{h(1-\beta)(\gamma - \beta)[(1-\beta)\lambda]^{\gamma/(1-\beta)}}\right)^{(1-\beta)/(\gamma+1-\beta)} \tag{20}$$

(ii) *The minimum inventory cost per unit of an item r^* is given by*

$$r^* = r(\tau^*, \tau^*) = \left(\frac{K(\gamma + 1 - \beta)}{\gamma - \beta}\right)\left(\frac{\lambda K(\gamma + 1 - \beta)}{h(\gamma - \beta)}\right)^{-1/(\gamma+1-\beta)} \tag{21}$$

(iii) *The maximum profit/cost ratio R^* is given by*

$$R^* = R(\tau^*, \tau^*) = \frac{v}{p + r^*} - 1 \tag{22}$$

(iv) *The optimal lot size q^* is given by*

$$q^* = \left(\frac{\lambda K(\gamma + 1 - \beta)}{h(\gamma - \beta)}\right)^{1/(\gamma+1-\beta)} \tag{23}$$

Proof. If we take into account the expression (16) for $\Phi(\tau)$, the statement (i) follows directly from Proposition 1 using the expressions (18) and (4). Similarly, from (19), (4) and (6), the statement (ii) is easily seen. The statement (iii) is obtained from the expression (9) for the function $R(\tau, T)$. Finally, using the expressions (5), (6) and (20), the optimal lot size is given by (23) and the proof is finished. $\square$

Note that, if $\beta = 0$ and $\gamma = 1$, from (20), the optimal inventory cycle is $T^* = \sqrt{2K/(h\lambda)}$ and, from (23), the economic lot size is $q^* = \sqrt{2K\lambda/h}$, as in the basic *EOQ* model.

It could also be interesting for inventory managers to know the optimal policy with minimum inventory cost per unit time, which can be obtained by solving the problem

$$\min_{\substack{\tau>0 \\ 0<T\leq\tau}} C(\tau, T) \tag{24}$$

where $C(\tau, T)$ is given by (14). The following result provides the optimal solution of the problem (24).

Proposition 2. *The solution for the problem (24) is obtained at the point* $(\tau^\#, T^\#)$, *with*

$$T^\# = \tau^\# = \left(\frac{K(\gamma+1-\beta)}{h\gamma[(1-\beta)\lambda]^{\gamma/(1-\beta)}} \right)^{(1-\beta)/(\gamma+1-\beta)} \tag{25}$$

Moreover, the value for the minimum cost per unit time is

$$C^* = C\left(\tau^\#, T^\#\right) = \frac{K(\gamma+1-\beta)}{\gamma T^\#} \tag{26}$$

Proof. Taking into account the expression (14) of the function $C(\tau, T)$, it is easily seen that, if $0 < T < \tau$, then

$$\begin{aligned} C(\tau, T) &= \frac{K + A\left[\tau^{1+\gamma/(1-\beta)} - (\tau - T)^{1+\gamma/(1-\beta)}\right]}{T} \\ &> \frac{K}{\tau} + \frac{A\left[\tau^{1+\gamma/(1-\beta)} - (\tau - T)\tau^{\gamma/(1-\beta)}\right]}{T} \\ &= \frac{K}{\tau} + A\tau^{\gamma/(1-\beta)} = C(\tau, \tau) \end{aligned}$$

Thus, we only need to solve the problem $\min_{\tau>0} C(\tau, \tau)$. This function satisfies that $\lim_{\tau\to 0^+} C(\tau, \tau) = \lim_{\tau\to\infty} C(\tau, \tau) = \infty$ and

$$\frac{d}{d\tau} C(\tau, \tau) = \frac{-K + \left(\frac{A\gamma}{1-\beta}\right)\tau^{1+\gamma/(1-\beta)}}{\tau^2}$$

As a consequence, the optimal solution is obtained solving the equation $\frac{d}{d\tau} C(\tau, \tau) = 0$, whose only root $T^\#$ is given by (25), if we use the expression (4) for the auxiliary parameter A. Finally, using the values $\tau = \tau^\#$ and $T = T^\#$ in the expression (14), we obtain

$$C^* = C\left(\tau^\#, T^\#\right) = \frac{K + A\left(T^\#\right)^{1+\gamma/(1-\beta)}}{T^\#} = \frac{K + \frac{K(1-\beta)}{\gamma}}{T^\#} = \frac{K(\gamma+1-\beta)}{\gamma T^\#}$$

and the proof is finished. □

Theorem 1 and Proposition 2 prove that the optimal inventory cycle of the maximum profit/cost ratio and the optimal inventory cycle of the minimum cost per unit time are different, and the ratio between the two solutions is given by

$$\frac{T^*}{T^\#} = \left(\frac{\gamma}{(\gamma-\beta)(1-\beta)} \right)^{(1-\beta)/(\gamma+1-\beta)} \tag{27}$$

which verifies that $T^*/T^\# > 1$ if $0 < \beta < 1$. Then, when the demand rate depends on the inventory level with elasticity parameter $\beta > 0$, the optimal length of the inventory cycle is larger for the solution of the maximum profit/cost ratio than the optimal length of the inventory cycle for the minimum cost per unit time solution. However, both solutions are equal if, and only if, $\beta = 0$, that is, if the demand rate is constant. For the two problems, the solution is always obtained for $T = \tau$, that is, the cycle time is extended until the stock is depleted. Moreover, both solutions do not depend on the purchasing cost p or the selling price v. Thus, the inventory manager does not need to change the cycle time when these prices change, if the objective is maximizing the profit/cost ratio or minimizing the inventory cost per unit time.

Nevertheless, if $\gamma = 1$, Baker and Urban (1988) proved that the solution for the problem of the maximum profit per unit time is obtained when a new order is requested before the stock is depleted (that is, $T < \tau$). Furthermore, in that case, the solution depends on the purchasing cost and the selling price.

From (3) and (20) we can ensure that

$$HC(\tau^*, T^*) = A(T^*)^{1+\gamma/(1-\beta)} = \frac{K}{\gamma - \beta} \tag{28}$$

Table 3
Partial derivatives of T^* and R^*

x	$\partial T^*/\partial x$	$\partial R^*/\partial x$
K	$\frac{(1-\beta)T^*}{(\gamma+1-\beta)K} > 0$	$\frac{-vr^*(\gamma-\beta)}{(p+r^*)^2 K(\gamma+1-\beta)} < 0$
h	$\frac{-(1-\beta)T^*}{(\gamma+1-\beta)h} < 0$	$\frac{-vr^*}{(p+r^*)^2 h(\gamma+1-\beta)} < 0$
λ	$\frac{-\gamma T^*}{(\gamma+1-\beta)\lambda} < 0$	$\frac{vr^*}{(p+r^*)^2 \lambda(\gamma+1-\beta)} > 0$
v	0	$\frac{1}{p+r^*} > 0$
p	0	$\frac{-v}{(p+r^*)^2} < 0$

Thus, the optimal policy that maximizes the profit/cost ratio satisfies the equality (28). Therefore, for the optimal solution with the maximum profit/cost ratio, the holding cost is higher than the ordering cost if $\gamma < 1 + \beta$ (expensive storage). On the other hand, the holding cost is less than the ordering cost if $\gamma > 1 + \beta$ (cheap storage). Obviously, if $\gamma = 1 + \beta$, then the ordering cost is equal to the holding cost, as in Harris' rule of the basic *EOQ* model.

Also, from (22), the optimal profit/cost ratio is positive ($R^* > 0$), if and only if the selling price v satisfies that $v > p + r^*$. Taking into account (21), it follows that

$$p + r^* = p + \left(\frac{K(\gamma+1-\beta)}{(\gamma-\beta)} \right) \left(\frac{\lambda K(\gamma+1-\beta)}{h(\gamma-\beta)} \right)^{-1/(\gamma+1-\beta)}$$

Then, we obtain a profitability condition for the inventory model:

$$v > p + \left(\frac{K(\gamma-\beta+1)}{\gamma-\beta} \right)^{(\gamma-\beta)/(\gamma+1-\beta)} \left(\frac{h}{\lambda} \right)^{1/(\gamma+1-\beta)} \tag{29}$$

Thus, the inventory manager can know the minimum selling price that ensures the profitability of the inventory system, solely from the parameters of the model.

The evaluation of the inventory cost per unit time and the profit per unit time for the maximum profitability solution leads to

$$C(\tau^*, T^*) = \frac{r^* q^*}{T^*} = \frac{K(\gamma+1-\beta)}{T^*(\gamma-\beta)} \tag{30}$$

and

$$\begin{aligned} G(\tau^*, T^*) &= \frac{(v-p)q^*}{T^*} - C(\tau^*, T^*) \\ &= (v-p)B(T^*)^{\beta/(1-\beta)} - \frac{K(\gamma+1-\beta)}{T^*(\gamma-\beta)} \end{aligned} \tag{31}$$

These expressions can be used to compare the solution obtained for the maximum profit/cost ratio with the minimum cost per unit time solution or with the maximum profit per unit time solution.

4. Sensitivity analysis

In this section, we use the closed expressions (20) and (22) to develop a sensitivity analysis of the optimal solution, by calculating the partial derivatives of the optimal cycle time T^* and the maximum profit/cost ratio R^* with respect to the main parameters of the inventory system. Specifically, these partial derivatives are obtained in the Appendix of this paper and are included in Table 3.

The sensitivity of the optimal cycle time with respect to these parameters is analyzed in the next theorem, both in absolute and relative terms.

Theorem 2. *Let* T^* *be the optimal inventory cycle given by (20). Then:*

(i) T^* *increases as the ordering cost* K *increases.*

(ii) *T^* decreases as the unit holding cost h or the scale parameter of the demand rate λ increase.*
(iii) *T^* does not change if the unit selling price v or the unit purchasing cost p vary.*
(iv) *The relative changes in T^* with respect to a relative change in the parameters K or h are equal, but with the opposite sign.*
(v) *The absolute value of the relative change in T^* with respect to a relative change in λ is greater than the relative changes with respect to the parameters K or h, except when $\beta = 0$ and $\gamma = 1$, in which case they are equal.*

Proof.

(i)-(ii) The first two assertions follow directly from the sign of the partial derivatives with respect to K, h and λ, given in Table 3.
(iii) The expression (20) for T^* allows the fact that this value does not depend on the values of the parameters v and p to be confirmed.
(iv) The relative change in T^* with respect to the relative change in a parameter x is given by $\frac{\partial T^*/\partial x}{T^*/x}$. Then, by using the partial derivatives included in Table 3, we deduce that

$$\frac{\partial T^*/\partial K}{T^*/K} = -\frac{\partial T^*/\partial h}{T^*/h} = \frac{1-\beta}{\gamma+1-\beta}$$

and they are equal, but with the opposite sign.
(v) In a similar way, from Table 3, we observe that

$$\frac{\partial T^*/\partial \lambda}{T^*/\lambda} = \frac{-\gamma}{\gamma+1-\beta}$$

and, from the proof of the statement (iv), we have

$$\frac{\partial T^*/\partial \lambda}{T^*/\lambda} = \frac{\gamma}{1-\beta}\frac{\partial T^*/\partial h}{T^*/h}$$

Therefore,

$$\left|\frac{\partial T^*/\partial \lambda}{T^*/\lambda}\right| = \frac{\gamma}{1-\beta}\left|\frac{\partial T^*/\partial h}{T^*/h}\right| = \frac{\gamma}{1-\beta}\left|\frac{\partial T^*/\partial K}{T^*/K}\right| = \frac{\gamma}{\gamma+1-\beta}$$

□

As a consequence, if $\beta = 0$ and $\gamma = 1$, then the absolute value of the relative change in T^* with respect to a relative change in λ, K or h is always equal to 1/2. However, when $\beta > 0$ or $\gamma > 1$, we have $\gamma > 1-\beta$ and we deduce that the absolute value of the relative change in T^* with respect to a relative change in λ is greater than the relative change with respect to the parameters K or h.

In a similar way, the next theorem analyzes the changes in the maximum profit/cost ratio with respect to the parameters of the inventory model.

Theorem 3. *Let R^* be the maximum profit/cost ratio, given by (22), and r^* the minimum inventory cost per unit of an item, given by (21). Then:*

(i) *R^* decreases as the ordering cost K, or the unit holding cost h, or the unit purchasing cost p, increase.*
(ii) *R^* increases as the scale parameter of the demand rate λ, or the unit selling price v, increase.*
(iii) *R^* increases linearly with respect to the unit selling price v.*
(iv) *If the inventory is profitable (that is, $R^* > 0$), then the absolute value of the change in R^* with respect to a change in the parameter p is greater than with respect to an equal change in the parameter v. If the inventory is not profitable (that is, $R^* < 0$), the opposite occurs.*
(v) *The relative change in R^* with respect to a relative change in the parameters h or λ are equal, but with the opposite sign.*
(vi) *If $\gamma > 1+\beta$, then the absolute value of the relative variation in R^* with respect to a relative change in the parameter K is greater than with the parameters h or λ. The opposite occurs if $\gamma < 1+\beta$. If $\gamma = 1+\beta$, then the three relative changes are equal in absolute value.*
(vii) *The absolute value of the relative variation in R^* with respect to a relative change in v is greater than with respect to an equal relative change in the parameter p.*

Proof.

(i)-(ii) The first two assertions follow directly from the sign of the partial derivatives given in Table 3.
(iii) As r^* does not depend on v, then $\partial R^*/\partial v = 1/(p+r^*)$ is a positive constant and R^* increases linearly with v.
(iv) From Table 3, we have

$$\frac{\partial R^*}{\partial p} = -\frac{v}{p+r^*}\frac{\partial R^*}{\partial v}$$

Therefore, if $R^* > 0$, then $v > p + r^*$ and

$$\left|\frac{\partial R^*}{\partial p}\right| = \frac{v}{p+r^*}\frac{\partial R^*}{\partial v} > \frac{\partial R^*}{\partial v}$$

and the opposite happens if $R^* < 0$.
(v) Also, from the partial derivatives given in Table 3, we observe that

$$\frac{\partial R^*/\partial h}{R^*/h} = -\frac{\partial R^*/\partial \lambda}{R^*/\lambda}$$

and both relative changes are equal, except for the sign.
(vi) In a similar way, from Table 3, we observe that

$$\frac{\partial R^*/\partial K}{R^*/K} = (\gamma-\beta)\frac{\partial R^*/\partial h}{R^*/h}$$

Then, if $\gamma > 1+\beta$, we have

$$\left|\frac{\partial R^*/\partial K}{R^*/K}\right| > \left|\frac{\partial R^*/\partial h}{R^*/h}\right| = \left|\frac{\partial R^*/\partial \lambda}{R^*/\lambda}\right|$$

However, when $\gamma < 1+\beta$, then

$$\left|\frac{\partial R^*/\partial K}{R^*/K}\right| < \left|\frac{\partial R^*/\partial h}{R^*/h}\right| = \left|\frac{\partial R^*/\partial \lambda}{R^*/\lambda}\right|$$

Otherwise (that is, if $\gamma = 1+\beta$), the three relative changes are equal in absolute value.
(vii) Finally, with respect to the parameters v and p, Table 3 shows that

$$\frac{\partial R^*/\partial v}{R^*/v} = -\left(\frac{p+r^*}{p}\right)\frac{\partial R^*/\partial p}{R^*/p}$$

and, taking into account that $r^* > 0$, we deduce that

$$\left|\frac{\partial R^*/\partial v}{R^*/v}\right| > \left|\frac{\partial R^*/\partial p}{R^*/p}\right| \qquad \square$$

All the statements in the two previous theorems help the inventory manager to know how to improve the optimal profit/cost ratio, or how the changes in the parameters of the inventory system affect the optimal cycle time.

5. Numerical examples

In this section, some numerical examples are used to illustrate the proposed model, the solution methodology and the sensitivity analysis. To compare the results, we begin by considering the same example proposed in Pando et al. (2012b), who solved the problem of maximum profit per unit time using the condition $T = \tau$. They assumed the following parameters for the inventory system: $K = 10$, $p = 50$, $v = 62$, $h = 0.5$, $\lambda = 1$, $\gamma = 1.5$ and $\beta = 0.3$.

Then, from the expressions (4) and (6), the auxiliary parameters are $A = 0.0741$ and $B = 0.6008$. From Theorem 1, the optimal cycle time is $T^* = 4.49$, the minimum cost per unit of an item is $r^* = 3.57$, the lot size is $q^* = 5.14$ and the maximum profit/cost ratio is $R^* = 0.1575$, that is, the maximum profitability of the inventory system is 15.75%. For this optimal solution, the inventory cost per unit time given by (30) leads to $C(\tau^*, T^*) = 4.08$, and the profit per unit time given by (31) is $G(\tau^*, T^*) = 9.65$. Also, by using (3), the holding cost in an inventory cycle is $HC(\tau^*, T^*) = 8.33$, which is less than the ordering cost $K = 10$ because, in this case, $\gamma > 1 + \beta$. Then, in this case, the inventory manager needs to invest fewer resources in storage than in order (note that, if $\gamma < 1 + \beta$, the opposite occurs). Moreover, we observe that $\gamma - \beta = 1.2$ and $HC(\tau^*, T^*) = K/1.2$, as in the expression (28). Also, the total cost per unit time for this optimal solution, given by (7), is $TC(\tau^*, T^*) = 61.28$. In addition, the minimum selling price to obtain a profitable system is $v = p + r^* = 53.57$.

For the problem of the minimum cost per unit time, we have used Proposition 2 to evaluate the optimal solution. From (25), the obtained values are $T^\# = \tau^\# = 3.74$, with a minimum cost per unit time $C^* = C(\tau^\#, T^\#) = 3.92$, which is lower than the inventory cost per unit time for the solution of the maximum profit/cost ratio, $C(\tau^*, T^*) = 4.08$. If we evaluate the average inventory cost per unit of an item with (10), the obtained value is $r(\tau^\#, T^\#) = 3.71$, which is greater that the minimum inventory cost per unit of an item $r^* = 3.57$. The profit/cost ratio for this solution, given by (9), is $R(\tau^\#, T^\#) = 0.1543$, which is less than the maximum profit/cost ratio $R^* = 0.1575$. The difference between the two solutions is 0.32% in absolute terms, and 2.03% in relative terms. Also, we observe that $T^\# = 3.74 < 4.49 = T^*$. The relation between them is $T^*/T^\# = 1.20$, which coincides with the right side of the equality (27), if $\beta = 0.3$ and $\gamma = 1.5$.

For the problem of maximum profit per unit time, if we suppose that $T = \tau$, Pando et al. (2012b) obtained the optimal solution $\widetilde{T} = 6.37$ with a profit per unit time $G(\widetilde{T}, \widetilde{T}) = 10.46$. Now, for the problem with two decision variables given by (13), we have used a numerical search procedure to obtain the maximum value for the function $G(\tau, T)$. The obtained solution was $\widehat{\tau} = 5.63$ and $\widehat{T} = 4.81$, with a total cost per unit time $TC(\widehat{\tau}, \widehat{T}) = 80.40$ and a profit per unit time $G(\widehat{\tau}, \widehat{T}) = 11.12$. As expected, this value is greater than the profit per unit time for the solution with the maximum profit/cost ratio, which is $G(\tau^*, T^*) = 9.65$. Instead, the profit/cost ratio for this new solution, evaluated with the expression (9), is $R(\widehat{\tau}, \widehat{T}) = 0.1383$, which is less than the maximum profit/cost ratio $R^* = 0.1575$. The difference between them is 1.92% in absolute terms, and 12.19% in relative terms.

Therefore, the three solutions are different because the demand rate depends on the inventory level with $\beta = 0.3 > 0$. This does not happen if the demand rate is constant, that is, $\beta = 0$.

The optimal policy obtained maximizing the profit per unit time leads to a higher inventory cost per unit time, while the optimal policy deduced minimizing the inventory cost per unit time yields a lower profit per unit time. This fact is expected because the maximum profit problem is focused on income while the minimum cost problem forgets to consider the income. However, if the goal is to maximize the profit/cost ratio, the optimal solution provides the best balance between profit and cost. This issue should be taken into account by the inventory manager to select the most interesting option for the inventory system.

Note that, for the solution that maximizes the profit per unit time, the total cost per unit time is $TC(\widehat{\tau}, \widehat{T}) = 80.40$, which is greater than the total cost for the maximum profit/cost solution $TC(\tau^*, T^*) = 61.28$. If the inventory manager has other investment alternatives for the company savings, then following the optimal profit/cost ratio policy, he could obtain a profit per unit time of 12.67, which is 13.94% greater than the profit per unit time obtained with the maximum profit solution $G(\widehat{\tau}, \widehat{T}) = 11.12$. This issue should also be taken into account by the inventory manager when choosing between the maximum profit per unit time and the maximum profit/cost ratio policies. If there are no other investment alternatives maybe he/she prefers the maximum profit per unit time solution, but otherwise, he/she probably prefers the maximum profit/cost ratio policy.

Table 4
Changes in the optimal solution (T^*, R^*)

Parameter	Absolute changes		Relative changes	
x	$\partial T^*/\partial x$	$\partial R^*/\partial x$	$\frac{\partial T^*/\partial x}{T^*/x}$	$\frac{\partial R^*/\partial x}{R^*/x}$
K	0.1430	−0.0042	0.3182	−0.2670
h	−2.8598	−0.0701	−0.3182	−0.2225
λ	−3.0640	0.0350	−0.6818	0.2225
v	0	0.0187	0	7.3514
p	0	−0.0216	0	−6.8620

The sensitivity of the optimal solution (T^*, R^*) with respect to the parameters K, h, λ, v and p have been evaluated by using the partial derivatives given in Table 3, both in absolute terms and relative terms. The obtained results are included in Table 4.

All the statements given by Theorems 2 and 3 are verified. Indeed, T^* increases with K, and decreases with h or λ. The relative changes with respect to K and h are equal in absolute value (0.3182), but with the opposite sign. Also, the absolute value of the relative change with respect to λ (0.6818) is greater than with respect to K or h (0.3182). Furthermore, R^* increases with λ or v, and decreases with K, h or p. The relative change with respect to h or λ are equal in absolute value (0.2225), but with the opposite sign. Also, the absolute value of the relative change with respect to K is greater than with respect to h or λ ($0.2670 > 0.2225$), as expected in this case, because $\gamma = 1.5 > 1.3 = 1 + \beta$. Finally, as the inventory system is profitable ($R^* > 0$), a decrease of 1$ in the purchasing cost improves the optimal profit/cost ratio more than an increase of 1$ in the selling price ($0.0216 > 0.0187$). Instead, an increase of 1% in the selling price improves the optimal profit/cost ratio more than a decrease of 1% in the purchasing cost ($7.3514 > 6.8620$). Also, we observe that R^* is much more sensitive to relative changes in v and p than to the changes in the parameters K, h or λ.

In order to confirm all the previous results, we have analyzed the inventory model, allowing changes between −50% and +50% in each of these parameters, keeping all the other ones fixed. For each system, the optimal cycle time T^* and the maximum profit/cost ratio R^* were obtained by using Theorem 1, while the relative changes were evaluated as percentages. The results are shown in Figs. 1 and 2, respectively. As the optimal cycle time does not depend on v or p, in Fig. 1 we only plot the percentage changes with respect to the parameters K, h and λ. In Fig. 2, as the relative changes with respect to v or p are much larger than with respect to K, h or λ, we have used the left vertical axis for K, h or λ and the right vertical axis for v or p. These graphics endorse all the comments mentioned above.

To quantify the changes of the optimal solution in absolute terms, we have evaluated the optimal quantities T^*, R^* and q^* assuming different values in each of the parameters while keeping all the others fixed. Specifically, for each parameter, we have chosen size drops of −15%, −10% and −5%, and increments by 5%, 10% and 15%. The obtained results are shown in Table 5. The calculated values support the relevance of the model and allow obtaining more representative issues. The optimal profit/cost ratio is much more sensitive to changes in purchasing or selling prices than to changes in the other parameters, which seems to be logical. How-

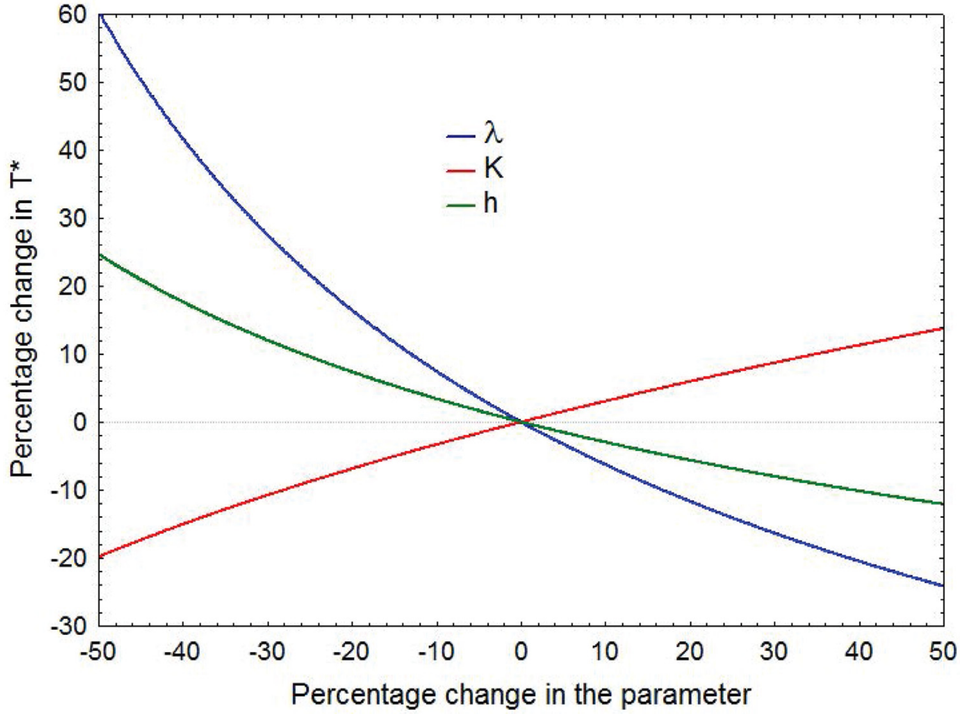

Fig. 1. Percentage changes in T^*

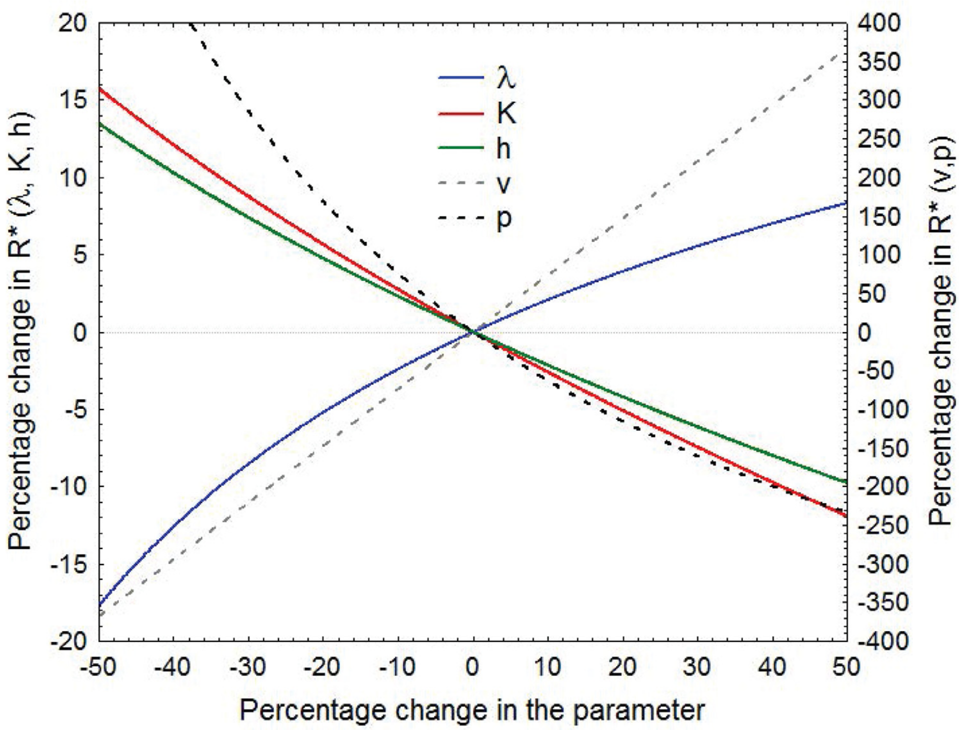

Fig. 2. Percentage changes in R^*

ever, the optimal values for the cycle time and the lot size do not change when these prices vary. Also, the elasticity parameter γ of the holding cost is more influential on the cycle time and the lot size than the other parameters.

It is also interesting to check how the inventory model behaves when the initial parameters are random variables. To analyze this topic, we have calculated the optimal solution of the model when one of the parameters has a uniform probability distribution and all others remain fixed. Table 6 contains the obtained values for the mean, the standard deviation and the variation coefficient of the optimal values for the cycle time, the profit/cost ratio and the lot size, using a random sample with size 100. The mean values are always similar to those obtained in the model with fixed parameters. Only the elasticity parameter γ of the holding cost and the

Table 5
Optimal policy of the model for different values of the initial parameters

x		$\Delta x = -15\%$	$\Delta x = -10\%$	$\Delta x = -5\%$	$\Delta x = 5\%$	$\Delta x = 10\%$	$\Delta x = 15\%$
	T^*	4.27	4.36	4.42	4.56	4.63	4.70
$K = 10$	R^*	0.1640	0.1618	0.1596	0.1554	0.1533	0.1514
	q^*	4.77	4.90	5.02	5.26	5.37	5.48
	T^*	4.73	4.65	4.57	4.42	4.36	4.30
$h = 0.5$	R^*	0.1630	0.1611	0.1592	0.1557	0.1540	0.1524
	q^*	5.53	5.39	5.26	5.03	4.92	4.82
	T^*	5.30	4.99	4.73	4.29	4.11	3.95
$\gamma = 1.5$	R^*	0.1674	0.1639	0.1606	0.1544	0.1515	0.1487
	q^*	6.52	5.98	5.52	4.81	4.53	4.28
	T^*	5.02	4.83	4.65	4.35	4.21	4.09
$\lambda = 1$	R^*	0.1516	0.1537	0.1556	0.1591	0.1607	0.1622
	q^*	4.77	4.90	5.02	5.26	5.37	5.48
	T^*	4.41	4.44	4.47	4.52	4.55	4.59
$\beta = 0.3$	R^*	0.1556	0.1562	0.1568	0.1581	0.1587	0.1593
	q^*	4.94	5.00	5.07	5.21	5.29	5.36
	T^*	4.49	4.49	4.49	4.49	4.49	4.49
$v = 62$	R^*	−0.0162	0.0417	0.0996	0.2153	0.2732	0.3311
	q^*	5.14	5.14	5.14	5.14	5.14	5.14
	T^*	4.49	4.49	4.49	4.49	4.49	4.49
$p = 50$	R^*	0.3459	0.2766	0.2141	0.1058	0.0586	0.0153
	q^*	5.14	5.14	5.14	5.14	5.14	5.14

Table 6
Descriptives statistics for the solutions of the model with random parameters

	T^*			R^*			q^*		
	Mean	Std.	CV (%)	Mean	Std.	CV (%)	Mean	Std.	CV (%)
$K \sim U[5, 15]$	4.43	0.39	8.75	0.1591	0.0112	7.05	5.05	0.63	12.39
$h \sim U[0.25, 0.75]$	4.56	0.43	9.39	0.1580	0.0098	6.22	5.26	0.71	13.53
$\gamma \sim U[0.75, 2.25]$	5.30	1.84	34.77	0.1617	0.0183	11.32	6.74	3.44	51.03
$\lambda \sim U[0.5, 1.5]$	4.80	0.99	20.65	0.1545	0.0109	7.03	5.03	0.67	13.26
$\beta \sim U[0.15, 0.45]$	4.54	0.19	4.20	0.1580	0.0039	2.46	5.24	0.45	8.64
$v \sim U[57, 67]$	4.49	0	0	0.1675	0.0557	33.25	5.14	0	0
$p \sim U[45, 55]$	4.49	0	0	0.1562	0.0622	39.80	5.14	0	0

Table 7
Profitability threshold for the main parameters

	K	h	λ	v	p
Actual value	10	0.5	1	62	50
Threshold	< 92.5	< 7.22	> 0.06	> 53.6	< 58.5

selling price v seem to lead to slightly greater changes. The variation coefficient of the optimal profit/cost ratio for purchasing and selling prices are above 30%, which again confirms the influence of these parameters on the profitability of the inventory system. For the other parameters, the variation coefficients are below 12% and, therefore, their randomness has a low effect on the profit/cost ratio. As expected, the randomness of the purchasing and selling prices has no effect on the optimal values for the cycle time and the lot size. Also now, the randomness of the elasticity parameter γ of the holding cost leads to greater variability on the optimal values for the cycle time and the lot size.

Finally, we have used the inequality given by (29) to calculate the threshold value that leads to a profitable system ($R^* > 0$) for each of these parameters, keeping all the other ones fixed. The results are included in Table 7. Note that the inventory is profitable for values of K lower than 92.5, values of h lower than 7.22, values of λ greater than 0.06, values of v greater than 53.6, and values of p lower than 58.5.

6. Conclusions

The EOQ model studied in this paper provides some interesting issues that have not been much explored in the inventory literature. Thus, the approach aimed at maximizing the profitability of the inventory system is analyzed instead of minimizing the inventory cost per unit time or maximizing the profit per unit time, which are usually more common targets. Besides, in the inventory model developed in this work, the demand rate and the holding cost rate are both dependent on the stock level, which allows a wider range of real practical situations to be included. Also, the replacement of the inventory before the stock runs out is allowed in the model. Further, the length of the inventory cycle and the time required for stock depletion are used as the decision variables, instead of the lot size and the reorder point, which is more usual in the inventory literature.

The mathematical formulation obtained for the problem makes it possible to check that the maximization of the profit/cost ratio is equivalent to the minimization of the average inventory cost per unit of an item, assuming that the unit purchasing cost and the unit selling price are not dependent on the lot size. Moreover, the average inventory cost per unit of an item is always greater when a new order is set out before the stock is depleted. Then, the optimal length of the cycle time must be equal to the time required for stock depletion, and the known policy of zero stock at the end of an inventory cycle is optimal for maximizing the profitability of the inventory system. As expected, that policy is also optimal for the minimization of the inventory cost per unit time. However, this is not the best policy if the aim is the maximization of the profit per unit time, as shown in the numerical example included in this paper, or as was proven by Baker and Urban (1988) for the simple case with constant holding cost rate.

The optimal length of the inventory cycle, the optimal lot size and the maximum profit/cost ratio are determined in a closed form, and the best solution for the other problem of minimum inventory cost per unit time is also obtained. The comparison of both solutions allows us to conclude that, if the demand rate depends on the inventory level, the optimal length of the cycle time is longer for maximizing the profit/cost ratio than for minimizing the inventory cost per unit time. Moreover, in both cases, they do not depend on the unit purchasing cost or the unit selling price and, therefore, the inventory manager does not need to change the warehouse order policy if these prices change. However, this is not true if the goal is the maximization of the profit per unit time.

The optimal policy which maximizes the profitability of the system can be identified by a rule that relates the holding cost and the ordering cost. The holding cost can be greater or lower than the ordering cost, depending on the values for the elasticity parameters of the demand rate and the holding cost rate. If the difference between them is equal to one, then the holding cost is equal to the ordering cost, as in Harris' rule of the basic EOQ model.

Another interesting condition is proposed to assure the profitability of the inventory from the initial parameters of the model. It can be used to establish the minimum selling price that should be fixed to obtain a profit in the inventory system or to evaluate the profitability threshold for each of the parameters, while keeping all the other ones fixed.

The sensitivity analysis of the optimal solution of the model shows that the optimal length of the inventory cycle decreases as the scale parameter of the holding cost or the scale parameter of the demand rate increase. Instead, the inventory cycle increases as the ordering cost increases. As we said before, it does not change if the unit purchasing cost or the unit selling price change. In addition, the absolute value of the relative change in the inventory cycle with respect to a relative change in the ordering cost is equal to that with an equal relative change in the scale parameter of the holding cost. Furthermore, this common value can be greater or lower than with an equal relative change in the scale parameter of the demand rate, depending on the elasticity parameters of the model.

Regarding the maximum profit/cost ratio, this decreases as the ordering cost, the purchasing cost or the scale parameter of the holding cost increase. However, it increases if the scale parameter of the demand rate or the unit selling price increase. Moreover, the maximum profit/cost ratio increases linearly with respect to the selling price. Also, the absolute value of the relative change in the maximum profit/cost ratio with respect to a relative change in the scale parameter of the demand rate is equal to that with an equal relative change in the scale parameter of the holding cost. This common value can be greater or lower than with an equal relative change in the ordering cost, depending on the elasticity parameters of the model. Also, the maximum profit/cost ratio is more sensitive to relative changes in the unit purchasing cost, or the unit selling price, than to relative changes in the ordering cost, or the scale parameters of the demand rate and the holding cost. Finally, the absolute value of the relative change in the maximum profit/cost ratio with respect to a relative change in the unit purchasing cost is lower than to a relative change in the unit selling price.

Finally, several future works on this research line could be proposed. Thus, this model could be extended by considering other mathematical functions for the demand rate or the holding cost. It would be interesting to study the inventory system with a unit purchasing cost dependent on the lot size, or a time-dependent holding cost. Furthermore, it could be interesting to develop the inventory model with a price-dependent demand rate, while still being dependent on the stock level.

CRediT authorship contribution statement

Valentín Pando: Conceptualization, Methodology, Writing - review & editing. **Luis A. San-José:** Conceptualization, Methodology, Writing - review & editing. **Joaquín Sicilia:** Conceptualization, Methodology, Writing - review & editing.

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Appendix A

In this appendix, we calculate the partial derivatives of T^* and R^* with respect to the parameters K, h, λ, v and p.

First of all, from (20), we have

$$T^* = \left(K^{1-\beta}h^{-(1-\beta)}\lambda^{-\gamma}\right)^{1/(\gamma+1-\beta)} \times (1-\beta)^{-1}\left(\frac{\gamma+1-\beta}{\gamma-\beta}\right)^{(1-\beta)/(\gamma+1-\beta)}$$

and therefore

$$\ln T^* = \frac{(1-\beta)\ln K - (1-\beta)\ln h - \gamma\ln\lambda}{\gamma+1-\beta} - \ln(1-\beta) + \left(\frac{1-\beta}{\gamma+1-\beta}\right)\ln\left(\frac{\gamma+1-\beta}{\gamma-\beta}\right) \tag{32}$$

Also, from (21), we can observe that

$$r^* = \left(K^{\gamma-\beta}h\lambda^{-1}\right)^{1/(\gamma+1-\beta)}\left(\frac{\gamma+1-\beta}{\gamma-\beta}\right)^{(\gamma-\beta)/(\gamma+1-\beta)}$$

and therefore

$$\ln r^* = \frac{(\gamma-\beta)\ln K + \ln h - \ln\lambda}{\gamma+1-\beta} + \left(\frac{\gamma-\beta}{\gamma+1-\beta}\right)\ln\left(\frac{\gamma+1-\beta}{\gamma-\beta}\right) \tag{33}$$

Now, from (32), by derivation with respect to the parameter K, we have

$$\frac{\partial T^*}{\partial K} = T^*\frac{\partial(\ln T^*)}{\partial K} = \frac{(1-\beta)T^*}{(\gamma+1-\beta)K} > 0 \tag{34}$$

and, from (22) and (33), we obtain

$$\frac{\partial R^*}{\partial K} = \frac{-v}{(p+r^*)^2}\frac{\partial r^*}{\partial K} = \frac{-vr^*}{(p+r^*)^2}\frac{\partial \ln r^*}{\partial K} = \frac{-(\gamma-\beta)vr^*}{(\gamma+1-\beta)K(p+r^*)^2} < 0 \tag{35}$$

In a similar way, by derivation with respect to the parameter h, we obtain

$$\frac{\partial T^*}{\partial h} = T^*\frac{\partial(\ln T^*)}{\partial h} = \frac{-(1-\beta)T^*}{(\gamma+1-\beta)h} < 0 \tag{36}$$

and, from (22) and (33), we have

$$\frac{\partial R^*}{\partial h} = \frac{-vr^*}{(p+r^*)^2}\frac{\partial \ln r^*}{\partial h} = \frac{-vr^*}{(\gamma+1-\beta)h(p+r^*)^2} < 0 \tag{37}$$

Likewise, by derivation with respect to the parameter λ, we have

$$\frac{\partial T^*}{\partial \lambda} = T^* \frac{\partial (\ln T^*)}{\partial \lambda} = \frac{-\gamma T^*}{(\gamma + 1 - \beta)\lambda} < 0 \quad (38)$$

and

$$\frac{\partial R^*}{\partial \lambda} = \frac{-v r^*}{(p + r^*)^2} \frac{\partial \ln r^*}{\partial \lambda} = \frac{v r^*}{(\gamma + 1 - \beta)\lambda (p + r^*)^2} > 0 \quad (39)$$

These partial derivatives allow us to evaluate the absolute instant changes in T^* and R^* with respect to the parameters K, h and λ. Furthermore, they show the following relations for the relative instant changes in T^* and R^*:

$$\frac{\partial T^*/\partial \lambda}{T^*/\lambda} = \frac{-\gamma}{\gamma + 1 - \beta} = \left(\frac{\gamma}{1 - \beta}\right) \frac{\partial T^*/\partial h}{T^*/h} = \left(\frac{-\gamma}{1 - \beta}\right) \frac{\partial T^*/\partial K}{T^*/K}$$

and

$$\frac{\partial R^*/\partial K}{R^*/K} = \frac{-(\gamma - \beta) v r^*}{(\gamma + 1 - \beta)(p + r^*)^2 R^*} = (\gamma - \beta) \frac{\partial R^*/\partial h}{R^*/h} = (\beta - \gamma) \frac{\partial R^*/\partial \lambda}{R^*/\lambda}$$

Therefore,

$$\left|\frac{\partial R^*/\partial K}{R^*/K}\right| = (\gamma - \beta)\left|\frac{\partial R^*/\partial h}{R^*/h}\right| = (\gamma - \beta)\left|\frac{\partial R^*/\partial \lambda}{R^*/\lambda}\right| = \frac{(\gamma - \beta) v r^*}{(\gamma + 1 - \beta)(p + r^*)^2 |R^*|}$$

As T^* and r^* do not depend on the parameters v or p, it is sure that $\partial T^*/\partial v = \partial T^*/\partial p = 0$ and $\partial r^*/\partial v = \partial r^*/\partial p = 0$. As a consequence, deriving the expression (22) with respect to v, we obtain

$$\frac{\partial R^*}{\partial v} = \frac{1}{p + r^*} > 0$$

and, deriving with respect to p, we have

$$\frac{\partial R^*}{\partial p} = \frac{-v}{(p + r^*)^2} < 0$$

Then, for the relative instant changes, we get

$$\frac{\partial R^*/\partial v}{R^*/v} = \frac{v}{(p + r^*) R^*} = -\left(\frac{p + r^*}{p}\right) \frac{\partial R^*/\partial p}{R^*/p}$$

and, in consequence

$$\left|\frac{\partial R^*/\partial v}{R^*/v}\right| = \left(\frac{p + r^*}{p}\right)\left|\frac{\partial R^*/\partial p}{R^*/p}\right| = \frac{v}{(p + r^*)|R^*|} = \frac{1 + R^*}{|R^*|}$$

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