Commun Nonlinear Sci Numer Simulat 137 (2024) 108130

Contents lists available at ScienceDirect

Communications in Nonlinear Science and Numerical Simulation

journal homepage: www.elsevier.com/locate/cnsns

Research paper

# On solitary-wave solutions of Rosenau-type equations

Angel Durán [a,*], Gulcin M. Muslu [b,c]

[a] Applied Mathematics Department, University of Valladolid, P/ Belen 15, 47011, Valladolid, Spain
[b] Istanbul Technical University, Department of Mathematics, Maslak 34469, Istanbul, Turkey
[c] Istanbul Medipol University, School of Engineering and Natural Sciences, Beykoz 34810, Istanbul, Turkey

ARTICLE INFO

*MSC:*
35C07
76B25

*Keywords:*
Rosenau-type equations
Solitary waves
Normal form theory
Concentration-Compactness theory
Petviashvili' iterative method

ABSTRACT

The present paper is concerned with the existence of solitary wave solutions of Rosenau-type equations. By using two standard theories, Normal Form Theory and Concentration-Compactness Theory, some results of existence of solitary waves of three different forms are derived. The results depend on some conditions on the speed of the waves with respect to the parameters of the equations. They are discussed for several families of Rosenau equations present in the literature. The analysis is illustrated with a numerical study of generation of approximate solitary-wave profiles from a numerical procedure based on the Petviashvili iteration.

## 1. Introduction

In this study, we consider Rosenau-type equations of the form

$$u_t + \epsilon u_x + \alpha u_{xxt} + \eta u_{xxx} + \beta u_{xxxxt} + \gamma u_{xxxxx} + (g(u))_x = 0, \tag{1.1}$$

where $\alpha, \beta, \gamma, \epsilon, \eta \in \mathbb{R}$ and $g$ is a smooth nonlinear function satisfying one of these two conditions:

(P1) $g = g(u, u', u'', u''')$ and its Taylor series vanishes at the origin along with its first derivatives, and the dependence of $g$ on $u'$ and $u'''$ occurs as sums of even-order products of these two variables, [1].
(P2) $g = g(u)$ is a pure function of $u$, with no dependence on the derivatives.

Some particular cases of (1.1), present in the literature and that will be analyzed below, are:

- The Rosenau equation ([2–4] and references therein) corresponds to $\epsilon = \beta = 1, \alpha = \eta = \gamma = 0$, and the cases of $g$:
  - Classical: $g(u) = u^2/2$.
  - Single power: $g(u) = \frac{u^{p+1}}{p+1}$, $p \geq 1$.
  - Cubic-quintic, [5]: $g(u) = \frac{u^3}{3} + \frac{r}{5}u^5$, with $r$ constant.
- The Rosenau-RLW equation ([6,7] and references therein) corresponds to $\eta = \gamma = 0, \epsilon, \beta > 0, \alpha \in \mathbb{R}$ and $g$ quadratic, say $g(u) = u^2/2$. It may be generalized to a single power $g(u) = \frac{u^{p+1}}{p+1}$, $p \geq 1$.
- The Rosenau-KdV equation ([7–9] and references therein) corresponds to taking $\alpha = \gamma = 0, \epsilon, \beta > 0, \eta \in \mathbb{R}$ and $g$ quadratic or, in general, $g(u) = \frac{u^{p+1}}{p+1}$, $p \geq 1$.

* Corresponding author.
*E-mail addresses:* angeldm@uva.es (A. Durán), gulcin@itu.edu.tr (G.M. Muslu).

https://doi.org/10.1016/j.cnsns.2024.108130
Received 12 March 2024; Accepted 4 June 2024
Available online 7 June 2024
1007-5704/© 2024 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY-NC license (http://creativecommons.org/licenses/by-nc/4.0/).

- The Rosenau–Kawahara equation ([8,10] and references therein) corresponds to $\alpha = 0, \epsilon, \beta > 0, \gamma \in \mathbb{R}$ (with $\gamma = -1$ in [8]) and $g(u) = \frac{u^{p+1}}{p+1}$, $p \geq 1$, [10] (with $p = 1$ in [8]).
- The Rosenau-RLW-Kawahara equation ([11,12] and references therein) corresponds to $\epsilon = 2k > 0, \alpha = -\mu, \eta \in \mathbb{R}, \beta > 0$, and, [12]

$$g(u) = A\frac{u^2}{2} + B\frac{u^{m+1}}{m+1} + s\left(\frac{u_x^2}{2} + uu_{xx}\right), \ A, B, s \in \mathbb{R}.$$

  (Other cases in [11] may not satisfy (P1).)

The literature on the previous families of Rosenau-type equations, sketched above, is now described in more detail. The Rosenau equation was originally derived in [2] to study the dynamics of dense discrete lattices. Some mathematical properties of the equation with different types of nonlinearities are studied in [3–5]. They concern well-posedness of the initial-value problem (ivp), Lie symmetries, exact periodic traveling wave solutions (of cnoidal type), as well as the numerical generation of solitary wave solutions and computational study of their dynamics. In [13], the Concentration-Compactness theory of [14] is applied to study the existence of solitary wave solutions of equations of BBM type which include the Rosenau equation. This is mentioned by the authors in [5] as starting point of their numerical investigation of the waves with the Petviashvili iteration, [15]. The orbital stability, [16] of solitary wave solutions of the Rosenau-RLW equation (whose existence is assumed as critical points of the energy subject to constant charge) is investigated in [7]. Some references are concerned with the derivation of exact solitary wave solutions for particular values of the speed. This is the case of [8,9] for the Rosenau-RLW, Rosenau-KdV equations and generalized versions, [8,10] for the Rosenau–Kawahara equation, and [11] for the Rosenau-RLW-Kawahara equation, among others. The numerical treatment of the equations is mainly based on finite differences, [10,11,17], finite element methods, ([18] and references therein), and Fourier pseudospectral discretizations in space with an explicit, fourth-order Runge–Kutta time integrator for the Rosenau equation in [4].

The main contributions of the present paper are concerned with the existence of solitary wave solutions of (1.1), focused on three types: Classical Solitary Waves (CSW), with monotone and nonmonotone decay, and Generalized Solitary Waves (GSW). To this end, some mathematical properties of (1.1) are described in Section 2. They include well-posedness of the ivp, conserved quantities and Hamiltonian formulation. This section is completed with the description of the problem of searching for solitary wave solutions. The problem is then analized in the following two sections using two classical theories. In Section 3 we study the existence with the Normal Form Theory (NFT), [19,20], which is used to identify solitary-wave structures as homoclinic orbits of the solitary wave problem considered as a dynamical system. Our study will be based on different approaches, [1,21,22], and will be applied to the families of Rosenau-type equations mentioned above. This includes a discussion, for each family, on the range of values of the speed and depending on the parameters of the equations, required for the existence results. The existence of the three types of waves will be illustrated here with a computational study of generation of approximate profiles. This will be done by solving numerically the fourth-order differential equation satisfied by the solitary-wave profiles. The numerical procedure consists of discretizing the periodic, ivp on a long enough interval with a Fourier collocation method and solving iteratively the resulting algebraic system for the discrete Fourier coefficients of the approximation with Petviashvili-type methods, [15,23–25]. From the numerical experiments some additional properties of the waves, such as the asymptotic decay and the speed–amplitude relation, are explored.

A different point of view is taken in Section 4, where the existence of CSW's is analyzed from the Concentration-Compactness theory (CCT) of Lions, [14]. In this case, the CSW's are found as minimizers of a constrained variational problem. Compared to the study performed in [13] for BBM-type equations, our approach consists of the analysis of different minimization problems, more related to other applications of the theory, cf. *e.g.* [26–28]. We will make a general assumption on a key property of coercivity of the functional to be minimized, that will be characterized in terms of the speed of the wave. The study of this condition for each of the families of Rosenau equations introduced above will lead us to the corresponding, specific conditions on the speed in order to ensure the existence. We will also prove that the results obtained are extensions of some derived from the NFT, in the sense that the conditions for the speed are similar but do not impose restrictions on the proximity to limiting values, that are required by the local character of the NFT. The numerical procedure will be used again to illustrate the generation of some approximate profiles in this case; all were shown to be nonmonotone. The paper will be closed with a section of concluding remarks. The interested reader is referred to the extended version [29] for more detailed explanations and computations.

Throughout the paper, $C$ will denote a generic constant, that may depend on the corresponding parameters involved in the context.

## 2. Some mathematical properties of the rosenau equations

In this section, some mathematical properties of (1.1), that will be used in the paper, are collected. In order to delimit the 'admissible' Eq. (1.1) from a modeling point of view, a first step would be the study of linear well-posedness of the ivp in $L^2$, meaning, as usual, existence and uniqueness of solutions and continuous dependence on the initial data. If

$$\widehat{f}(k) = \int_{\mathbb{R}} f(x)e^{-ikx}dx, \quad k \in \mathbb{R},$$

denotes the Fourier transform of $f \in L^2$, we consider the ivp of the linearized equation

$$u_t + \epsilon u_x + \alpha u_{xxt} + \eta u_{xxx} + \beta u_{xxxxt} + \gamma u_{xxxxx} = 0, \tag{2.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, t > 0,$$

in Fourier space (with respect to $x$) to have

$$\frac{d}{dt}\widehat{u}(k,t) + ikl(k)\widehat{u}(k,t) = 0, \quad k \in \mathbb{R}, t > 0, \tag{2.2}$$
$$\widehat{u}(k,0) = \widehat{u_0}(k),$$

where

$$l(k) = \frac{\epsilon - \eta k^2 + \gamma k^4}{1 - \alpha k^2 + \beta k^4}.$$

The solution of (2.2) is

$$\widehat{u}(k,t) = e^{-ikl(k)t}\widehat{u_0}(k), \quad k \in \mathbb{R}, \quad t > 0.$$

Then, well-posedness of (2.1) holds if $m(k,t) = e^{-ikl(k)t}$ is bounded in finite intervals of $t$. It looks clear that this happens when the denominator of $l(k)$ has no zeros on $k \in \mathbb{R}$. If we consider the polynomial

$$P(x) = 1 - \alpha x + \beta x^2, \quad x \geq 0,$$

then linear well-posedness requires

$$\alpha^2 < 4\beta. \tag{2.3}$$

Condition (2.3) will be assumed throughout the paper. In particular, (2.3) implies $\beta > 0$. (Note that the case $\alpha = \beta = 0$ corresponds to the Kawahara or fifth-order KdV equation, [27,30,31].) We will also assume $\epsilon > 0$, as in all the literature on Rosenau-type equations that we are aware of.

On the other hand, local well-posedness results of the full nonlinear problem (1.1) with suitable nonlinear terms can be derived from several approaches, [32,33]. In what follows, for $s \in \mathbb{R}$, $H^s = H^s(\mathbb{R})$ will be the $L^2$-based Sobolev space over $\mathbb{R}$, with norm denoted by $\|\cdot\|_s$, and for $T > 0$, $X_s = C(0,T,H^s)$ will stand for the space of continuous functions $u : [0,T] \to H^s$ with norm

$$||u|_{X^s} = \max_{0\leq t\leq T} \|u(t)\|_s.$$

**Theorem 2.1.** *Let $s \geq 0$. Assume that $\epsilon > 0$, that (2.3) holds, and that $g$ in (1.1) is locally Lipschitz in $H^s$ with $g(0) = 0$. Let $u_0 \in H^s$. Then there exists $T > 0$ and a unique solution $u \in X^s$ of (1.1) with initial condition $u_0$.*

**Proof.** We write (1.1) in the form

$$\frac{du}{dt} + \mathcal{L}u = \mathcal{G}(u), \tag{2.4}$$

where $\mathcal{L}$ is the linear operator with Fourier symbol $ikl(k), k \in \mathbb{R}$, and

$$\mathcal{G}(u) = -M^{-1}\partial_x g(u),$$

where $M$ is given by

$$M = 1 + \alpha\partial_x^2 + \beta\partial_x^4, \tag{2.5}$$

which, from (2.3), is invertible. From Duhamel formula, the solution of (2.4) can be written as

$$u = \mathcal{S}(t)u_0 + \int_0^t \mathcal{S}(t-\tau)\mathcal{G}(u)d\tau,$$

where $\mathcal{S}(t)$ denotes the group generated by $\mathcal{L}$. We consider the mapping $\widetilde{u} \mapsto u$ with

$$u = \mathcal{S}(t)u_0 + \int_0^t \mathcal{S}(t-\tau)\mathcal{G}(\widetilde{u})d\tau. \tag{2.6}$$

Note that due (2.3), $\mathcal{S}(t)$ is a unitary group on $H^s$. On the other hand, let $T, R > 0$ and $\widetilde{u}_1, \widetilde{u}_2$ be in a closed ball of radius $R$ centered at 0 in $X^s$. Since $g$ is locally Lipschitz and from the arguments used in *e.g.* [33], we may find $C = C(R) > 0$ such that

$$\|\mathcal{G}(\widetilde{u}_1)(\tau) - \mathcal{G}(\widetilde{u}_2)(\tau)\|_{X^s} \leq C\|\widetilde{u}_1(\tau) - \widetilde{u}_2(\tau)\|_{X^s}, \tag{2.7}$$

for any $0 \leq \tau \leq T$. Then, (2.7) and the hypothesis $g(0) = 0$ imply the existence of some small enough $T > 0$ such that (2.6) is a contraction of the closed ball into itself. The result follows from the application of the Contraction Mapping Theorem. □

Some additional properties of (1.1) with $g$ satisfying (P1) or (P2), can also be mentioned. Note first that in the case of (P2) and if $G'(u) = g(u)$, then (1.1) admits

$$V(u) = \frac{1}{2}\int_{\mathbb{R}} (u^2 - \alpha u_x^2 + \beta u_{xx}^2)dx,$$

as an invariant in time quantity by smooth enough solutions $u$ which decay, along with its derivatives in space up to second order, to zero as $|x| \to \infty$. Furthermore, if (2.3) holds, then (1.1) admits a Hamiltonian structure

$$u_t = \mathcal{J}\delta H(u),$$

with $\mathcal{J} = -\partial_x M^{-1}$, $M$ as in (2.5), and the Hamiltonian function given by

$$H(u) = \int_{\mathbb{R}} \left( \frac{1}{2} \left( \epsilon u^2 - \eta u_x^2 + \gamma u_{xx}^2 \right) + G(u) \right) dx.$$

The present paper is focused on the existence of solitary wave solutions. Solitary wave solutions of Eq. (1.1) are smooth functions $u$ of the form $u(x,t) = \varphi(X)$ where $X = x - c_s t$, $c_s \neq 0$ and where $\varphi$ satisfies

$$(\gamma - \beta c_s)\varphi''''' + (\eta - \alpha c_s)\varphi''' + (\epsilon - c_s)\varphi' + [g(\varphi)]' = 0. \tag{2.8}$$

Here $'$ denotes the derivative with respect to $X$. Integrating Eq. (2.8) and assuming that $\varphi$ and derivatives vanish as $|X| \to \infty$ then

$$(\gamma - \beta c_s)\varphi'''' + (\eta - \alpha c_s)\varphi'' + (\epsilon - c_s)\varphi + g(\varphi) = 0. \tag{2.9}$$

Assuming that $\gamma - \beta c_s \neq 0$, then (2.9) can be alternatively written as

$$\varphi'''' - b\varphi'' + a\varphi = f(\varphi), \tag{2.10}$$

where $f = g/(\beta c_s - \gamma)$ and

$$a = a(c_s) = \frac{c_s - \epsilon}{\beta c_s - \gamma}, \quad b = b(c_s) = \frac{\alpha c_s - \eta}{\gamma - \beta c_s}. \tag{2.11}$$

In what follows we will make a general study of existence of solutions of (2.9). Under the assumption that $\varphi \to 0$ as $|X| \to \infty$, the corresponding solution will be called Classical Solitary Wave (CSW). However, we may consider other solutions of (2.9), with a different asymptotic behavior, and our study will also consider two types of them: Generalized Solitary Waves (GSW), [34–36], and Periodic Traveling Waves (PTW), [37].

## 3. Existence of solitary wave solutions via normal form theory

One of the approaches, considered here, to study the existence of the solutions for (2.10), [1,38–40], is based on the application of the Normal Form Theory to the equivalent first-order differential system for $U = (U_1, U_2, U_3, U_4)^T := (\varphi, \varphi', \varphi'', \varphi''')$, given by

$$U' = V(U, c_s) := L(c_s)U + R(U, c_s), \tag{3.1}$$

where

$$L(c_s) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\epsilon - c_s}{\beta c_s - \gamma} & 0 & \frac{\alpha c_s - \eta}{\gamma - \beta c_s} & 0 \end{pmatrix}, \qquad R(U, c_s) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{g(U)}{\beta c_s - \gamma} \end{pmatrix}, \tag{3.2}$$

The system (3.1), (3.2) admits the solution $U = 0$ if $g(0) = 0$. We also assume that $g'(U) = 0$ so that $\partial_U R(0) = 0$. Furthermore, note that if $g$ satisfies (P1) then (2.10) is reversible under the transformation

$$\begin{aligned} &t \mapsto -t \\ &(\varphi, \varphi', \varphi'', \varphi''')^T \mapsto (\varphi, -\varphi', \varphi'', -\varphi''')^T, \end{aligned} \tag{3.3}$$

Then, for $g$ satisfying (P1) or (P2), the vector field $V$ is reversible, in the sense that for all $U, c_s$

$$SV(U, c_s) = -V(SU, c_s), \quad S = \text{diag}(1, -1, 1, -1). \tag{3.4}$$

In order to study the homoclinic solutions of (2.10), we first consider the linearization of (3.1), (3.2) at the equilibrium point $U = 0$ yields to the linear system $U' = LU$. The characteristic equation is indeed

$$\lambda^4 - b\lambda^2 + a = 0, \tag{3.5}$$

where $a = a(c_s), b = b(c_s)$ are given by (2.11). The distribution of the roots of (3.5) in the $(b, a)$-plane, given in Fig. 1, reproduces the arguments of [1]. Thus the linearized dynamics is determined by the four regions 1 to 4, separated by the bifurcation curves

$$\begin{aligned} C_0(c_s) &= \{(b,a)|a = 0, b > 0\} = \{c_s = \epsilon, \frac{\alpha c_s - \eta}{\gamma - \beta c_s} > 0\}, \\ C_1(c_s) &= \{(b,a)|a = 0, b < 0\} = \{c_s = \epsilon, \frac{\alpha c_s - \eta}{\gamma - \beta c_s} < 0\}, \\ C_2(c_s) &= \{(b,a)|a > 0, b = -2\sqrt{a}\} \\ &= \{\frac{c_s - \epsilon}{\beta c_s - \gamma} > 0, \frac{\alpha c_s - \eta}{\gamma - \beta c_s} = -2\sqrt{\frac{c_s - \epsilon}{\beta c_s - \gamma}}\}, \end{aligned}$$

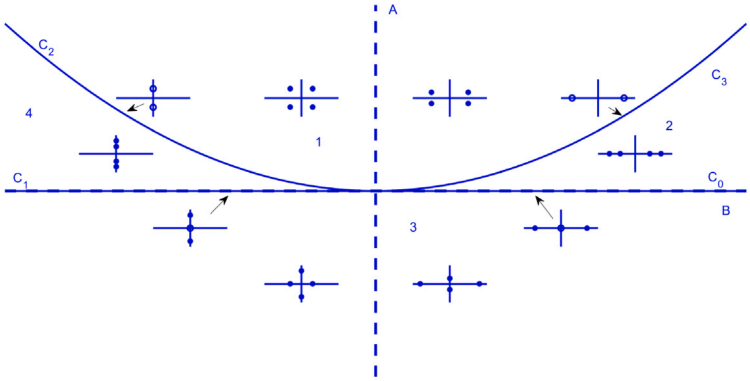


**Fig. 1.** Linearization at the origin of (3.1) (as in Figure 1 of [1]): Regions 1 to 4 in the $(b,a)$-plane, delimited by the bifurcation curves $\mathbb{C}_0$ to $\mathbb{C}_3$, and schematic representation of the position in the complex plane of the roots of (3.5) for each curve and region. (Dot: simple root, larger dot: double root.).

$$
\begin{aligned}
C_3(c_s) &= \{(b,a)|a>0, b=2\sqrt{a}\}\\
&= \{\frac{c_s-\epsilon}{\beta c_s-\gamma}>0, \frac{\alpha c_s-\eta}{\gamma-\beta c_s}=2\sqrt{\frac{c_s-\epsilon}{\beta c_s-\gamma}}\}.
\end{aligned}
$$

In terms of the spectrum of $L(c_s)$, $\mathbb{C}_0$ is characterized by the presence of two zero eigenvalues and other two, nonzero and real; in $\mathbb{C}_1$, there are two zero eigenvalues and other two, imaginary; in $\mathbb{C}_2$, the spectrum consists of a double complex conjugate pair of imaginary eigenvalues $\pm i\sqrt{b/2}$, while in $\mathbb{C}_3$ we have two double real eigenvalues $\pm\sqrt{b/2}$. The four curves meet at the origin $(b,a)=(0,0)$, where $L(c_s)$ has only the zero eigenvalue with geometric multiplicity equals one, cf. Fig. 1.

The theory of normal forms consists of finding a change of variables to transform, locally near the equilibrium, the system into a simpler one (the normal form) up to some fixed order and from which the dynamics of the original is better analyzed. The local transformation is polynomial and can be computed from the resolution of a sequence of linear problems. The dynamics of the resulting normal form is determined by the linear part. The theory, first introduced by Poincaré and Birkhoff, was developed by Arnold, [41], and has many applications, see *e.g.* [19] and references therein. Its particular use to determine the existence of solitary wave solutions is also widely considered in the literature (cf. *e.g.* [1,28,38–40] and references therein). On the other hand, the search for solitary waves as homoclinic trajectories sometimes requires the introduction of bifurcation parameters in order to analyze the emergence of structures via the bifurcation theory and the application of the corresponding version of the normal form theorem. In addition, of particular relevance is the influence of symmetries and reversible transformations, which are inherited by the normal form, see [19] for details.

One of the applications of the normal form concerns the analysis of the systems obtained from a center manifold reduction. For the case at hand, we will consider the approach developed in [22], based on determining a normal form of the reversible vector field given by (3.1), (3.2) in a general way and then discussing its principal part near the bifurcation curves $C_j, j=0,1,2$, with specific bifurcation parameter $\mu$ for each curve. The case of $C_3$ will be explored computationally will the help of the references, cf. [1,39,42,43]. The conclusions will be illustrated by their application to several types of Rosenau equations.

The corresponding normal form theorem (cf. *e.g.* [19]) establishes, in the case of (3.1), (3.2) and for any positive integer $m\geq 2$, the existence of neighborhoods $V_1$ of $U=0$ and $V_2$ of $\mu=0$ such that for any $\mu\in V_2$ there is a polynomial $\Phi(\mu,\cdot):\mathbb{R}^4\rightarrow\mathbb{R}^4$ of degree $m$ satisfying:

(i) The coefficients of $\Phi$ are smooth functions of $\mu$ and

$$\Phi(0,0)=0,\quad \partial_U\Phi(0,0)=0.$$

(ii) For $V\in V_1$, the change of variable

$$U=V+\Phi(\mu,V),$$

transforms (3.1) into the normal form

$$V'=LV+N(\mu,V)+\rho(\mu,V),$$

where

(a) For any $\mu\in V_2$, $N(\mu,\cdot):\mathbb{R}^4\rightarrow\mathbb{R}^4$ is a polynomial of degree $m$, with coefficients which are smooth functions of $\mu$ and $N(0,0)=\partial_V N(0,0)=0$.

(b) The equality

$$N(\mu,e^{tL^*}V)=e^{tL^*}N(\mu,V),$$

holds for all $(t, V) \in \mathbb{R}\times\mathbb{R}^4$ and $\mu \in V_2$, and where $L^*$ denotes the adjoint of $L$. This is equivalent, [19], to the condition

$$\partial_V N(\mu, V)L^*V = L^*N(\mu, V), \quad V \in \mathbb{R}^4, \mu \in V_2.$$

(c) $\rho$ is smooth in $(\mu, V) \in V_2 \times V_1$ and

$$\rho(\mu, V) = o(\|V\|^m),$$

for all $\mu \in V_2$ and where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^4$.

(d) The polynomials $\Phi(\mu, \cdot), N(\mu, \cdot)$ satisfy, for all $\mu \in V_2$

$$S\Phi(\mu, V) = \Phi(\mu, SV), \quad SN(\mu, V) = -N(\mu, SV), \quad V \in \mathbb{R}^4,$$

where $S$ is given in (3.4).

We note that (3.1), (3.2) can alternatively be written as

$$U' = L_0 U + \widetilde{R}(U, c_s), \tag{3.6}$$

where $\widetilde{R}(U, c_s) = R_{11}(U, c_s) + R(U, c_s)$,

$$L_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad R_{11}(U, c_s) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -a(c_s)U_1 + b(c_s)U_3 \end{pmatrix}, \tag{3.7}$$

and observe that the approach made in [22] can be used here to compute a general expression of a reversible normal form for (3.6), (3.7). We notice that the approach in [22] initially assumes a regular, nonlinear function $\widetilde{R}$ at least quadratic near the origin. However, the treatment of the linear term $R_{11}$ in (3.7) (which can be considered quadratic as $a$ and $b$ will involve some bifurcation parameters) can be included in the construction of the normal form from the arguments used in [21], based on the preservation of the eigenvalue structure. All this leads to a normal form for (3.6) as, [22]

$$\begin{aligned} U' &= L_0 U + P_4(U_1, V_2, V_4)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + P_2(U_1, V_2, V_4)\begin{pmatrix} 0 \\ U_1 \\ U_2 \\ U_3 \end{pmatrix} \\ &\quad + P_0(V_2, V_4)\begin{pmatrix} 0 \\ V_2 \\ W_2 \\ X_2 \end{pmatrix} + P_1(U_1, V_2, V_4)\begin{pmatrix} V_3 \\ W_3 \\ X_3 \\ Y_3 \end{pmatrix} \\ &\quad + P_3(U_1, V_2, V_4)\begin{pmatrix} 0 \\ 0 \\ V_3 \\ W_3 \end{pmatrix}, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} &V_2 = U_2^2 - 2U_1U_3, \quad V_3 = U_2^3 - 3U_1U_2U_3 + 3U_1^2U_4, \\ &V_4 = 3U_2^2U_3^2 - 6U_2^3U_4 - 8U_1U_3^3 + 18U_1U_2U_3U_4 - 9U_1^2U_4^2, \\ &W_2 = -3U_1U_4 + U_2U_3, \quad W_3 = 3U_1U_2U_4 - 4U_1U_3^2 + U_2^2U_3, \\ &X_2 = -3U_2U_4 + 2U_3^2, \quad X_3 = -3U_1U_3U_4 + 3U_2^2U_1 - U_2U_3^2, \\ &Y_3 = 3U_2U_3U_4 - \frac{4}{3}U_3^3 - 3U_1U_4^2, \end{aligned}$$

and $P_j, j = 0, 1, 2, 3, 4$, are polynomials in their arguments with

$$\begin{aligned} P_4(U_1, V_2, V_4) &= \mu_2 U_1 + \nu_1 U_1^2 + \nu_2 V_2 + O(\|U\|^3), \\ P_2(U_1, V_2, V_4) &= \mu_1 + \nu_3 U_1 + (\|U\|^2), \\ P_0(V_2, V_4) &= \nu_4 + O(\|U\|^2), \end{aligned}$$

where $\nu_j, j = 1, 2, 3, 4$, are constants and

$$\mu_1 = \frac{b}{3}, \quad \mu_2 = \left(\frac{b}{3}\right)^2 - a = \mu_1^2 - a, \tag{3.9}$$

with $a, b$ given by (2.11). In order to apply here the discussion, established in [22], on the normal form near each bifurcation curve $C_j, j = 0, 1, 2$, it is also convenient to write the bifurcation curves in the $(\mu_1, \mu_2)$ plane, as

$$\begin{aligned} C_0 &= \{(\mu_1, \mu_2) | \mu_1 > 0, \mu_2 = \mu_1^2\}, \\ C_1 &= \{(\mu_1, \mu_2) | \mu_1 < 0, \mu_2 = \mu_1^2\}, \\ C_2 &= \{(\mu_1, \mu_2) | \mu_1 < 0, \mu_2 = -\frac{5}{4}\mu_1^2\}, \\ C_3 &= \{(\mu_1, \mu_2) | \mu_1 > 0, \mu_2 = -\frac{5}{4}\mu_1^2\}. \end{aligned}$$

(Cf. Figure 1 of [22].)

Note that the linearization of (3.8) at the origin leads to the system

$$U' = L(\mu_1, \mu_2)U,$$

where

$$L(\mu_1, \mu_2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \mu_1 & 0 & 1 & 0 \\ 0 & \mu_1 & 0 & 1 \\ \mu_2 & 0 & \mu_1 & 0 \end{pmatrix}, \tag{3.10}$$

which, using (3.9), has the same characteristic polynomial as $L$ in (3.2), preserving then the eigenvalue structure, cf. [21].

Following the discussion in [22], the principal part of the normal form (3.8) will be described near the curves $C_j, j = 0, 1, 2$, in order to show the existence of different homoclinic structures leading to different types of solutions of (2.10). The results will be specified for typical choices of the nonlinear term $g$ and illustrated numerically.

### 3.1. Near $C_0$

We consider $\mu = -a$ as bifurcation parameter, in such a way that $\mu_2 = \mu_1^2 + \mu$, $C_0$ is characterized by the conditions $\mu = 0, \mu_1 > 0$, and the points near $C_0$ depend on the (small) values of $\mu$. If $L$ denotes the matrix (3.10) at $\mu = 0$, then a basis of the generalized eigenspace $\operatorname{Ker} L^2 = \operatorname{span}(\xi_0, \xi_1)$ with $L\xi_1 = \xi_0$ is given explicitly by, [21,22]

$$\xi_0 = (1, 0, -\mu_1, 0)^T, \quad \xi_1 = (0, 1, 0, -2\mu_1)^T,$$

with, in addition, $S\xi_0 = \xi_0, S\xi_1 = -\xi_1$. Using the Center Manifold Theorem, [19,44], the dynamics of bounded solutions near $C_0$ can be studied on the two-dimensional center manifold. From the change of variables

$$U = A\xi_0 + B\xi_1 + \Phi(\mu, A, B),$$

with suitable $\Phi$, the principal (linear and quadratic) part of the normal form (3.8) takes the form, [19,22]

$$\begin{aligned} \frac{dA}{dX} &= B, \\ \frac{dB}{dX} &= -\frac{1}{3\mu_1}\left(\mu A + (\nu_1 + 2\mu_1)(\nu_2 - \nu_3)A^2\right). \end{aligned} \tag{3.11}$$

(The variables $A, B, X$ can be rescaled in order to reformulate (3.11) as a nonsingular system as $\mu_1 \to 0$, [22].) When $g$ in (3.2) is quadratic in the components of $U$ then (3.11) can be reduced to, [21]

$$\begin{aligned} \frac{dA}{dX} &= B, \\ \frac{dB}{dX} &= -\frac{\mu}{3\mu_1}A - \frac{2}{3}A^2, \end{aligned}$$

which admits a homoclinic to zero solution, in the form of a Classical Solitary Wave

$$A(X) = \frac{9a}{4b}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{a}{b}}X\right),$$

when $\mu < 0$ (that is $a > 0$, region 2 in Fig. 1). The persistence of the homoclinic solution in the original system (3.1), which corresponds to a CSW solution of (1.1) can be proved following [1,40]. These conclusions are applied to different families of equations of Rosenau-type in Tables 1–4, where the range of speed $c_s > 0$ ensuring the existence of CSW's is specified in each case when $g$ is quadratic in the components of $U$. (The existence of CSW solutions of the Rosenau equation will be considered in Section 4, as well as the generalized case $g(u) = \frac{u^{p+1}}{p+1}, p \geq 1$.) The justification of Tables 1–4 is in the extended version [29], where the curves $C_j, j = 0, 1, 2, 3$, and the regions of Fig. 1 are described for each equation of Rosenau type considered in this paper. The generation of some of the CSW profiles are illustrated in Fig. 2. (All the figures are generated from the experiments performed using the Petviashvili method, [15]; the numerical procedure is also described in [29].)

Two properties of the profiles can be observed from Fig. 2. Note first that the wave is smooth, even, and decreases (if it is of elevation) or increases (if it is of depression) fast in both directions away from its maximum (or minimum) point at $X = 0$.

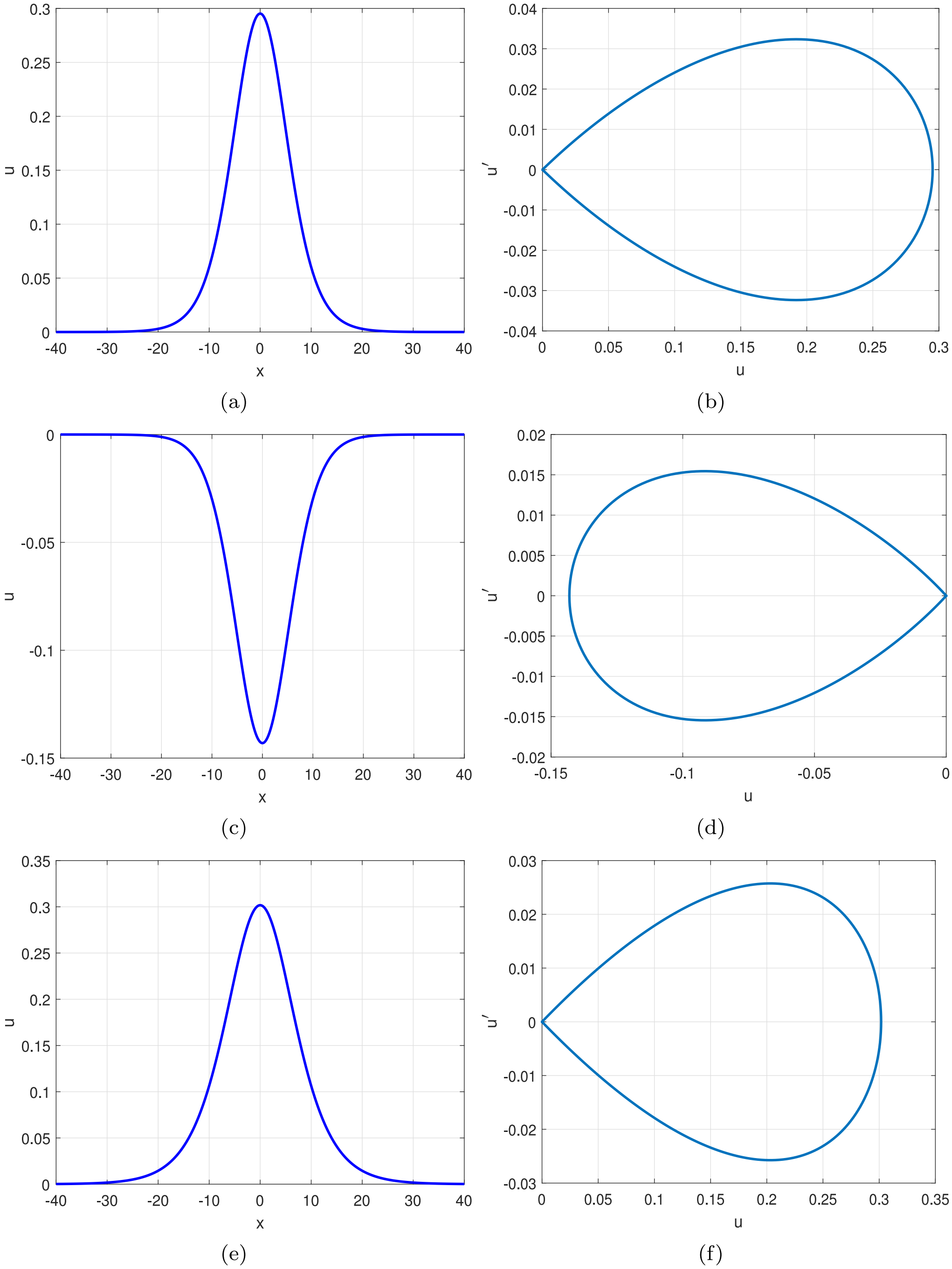

**Fig. 2.** CSW generation, $u$ profiles and phase portraits. (a), (b) Rosenau-RLW equation with $\alpha=-1, \epsilon=\beta=1, g(u)=u^2/2; c_s=1.1$; (c), (d) Rosenau–Kawahara equation with $\eta=-1/2, \epsilon=\beta=1, \gamma=2, g(u)=u^2/2, c_s=y_-+\epsilon+0.01\approx 0.951$; (e), (f) Rosenau-RLW-Kawahara equation with $\alpha=\gamma=-1, \epsilon=\beta=1, \eta=1, g(u)=u^2/2, c_s=z_++\epsilon-0.565\approx 1.102$. The values of $y_-$ and $z_+$ are given in Tables 3 and 4 resp.

Furthermore, the corresponding phase portrait and the way how the homoclinic orbit approaches zero at $\pm\infty$ suggest that the profile goes to zero exponentially as $|X|\rightarrow\infty$. This behavior can be theoretically checked from the application of some standard results (cf. *e.g.* [45]) in the case of nonlinear terms of the form $g(u)=\frac{u^{p+1}}{p+1}, p\geq 1$.

**Remark 3.1.** When $\mu>0$ is small (region 3 of Fig. 1, close to $C_0$) NFT predicts the existence of periodic solutions of the reduced system, as in [40]. They correspond to periodic traveling wave (PTW) solutions of (1.1): The description of the region for several families of Rosenau equations and the computation of some approximate PTW's are can be checked in [29].

**Table 1**
Admissible parameters and type of solitary wave for the Rosenau-RLW equation, according to the Normal Form Theory (cf. [29]). NMCSW=Nonmonotone CSW.

| Rosenau-RLW ($\gamma=\eta=0,\epsilon,\beta>0$) | |
|---|---|
| Admissible parameters | Type of solitary wave |
| $\alpha<0, 0<c_s-\epsilon<\frac{\epsilon\alpha^2}{4\beta-\alpha^2}$ <br> $c_s-\epsilon$ small | CSW |
| $\alpha>0, c_s-\epsilon<0$ <br> $\lvert c_s-\epsilon\rvert$ small | GSW |
| $\alpha>0, c_s-\epsilon-\frac{\epsilon\alpha^2}{4\beta-\alpha^2}>0$ small | NMCSW |

**Table 2**
Admissible parameters and type of solitary wave for the Rosenau-KdV equation, according to the Normal Form Theory, where $x_+=\frac{1}{2}\left(-\epsilon+\sqrt{\epsilon^2+\frac{\eta^2}{\beta}}\right)$ (cf. [29]).

| Rosenau-KdV ($\alpha=\gamma=0,\epsilon,\beta>0$) | |
|---|---|
| Admissible parameters | Type of solitary wave |
| $\eta>0, \epsilon<c_s<x_+$ <br> $c_s-\epsilon$ small | CSW |
| $\eta<0, c_s-\epsilon<0$ <br> $\lvert c_s-\epsilon\rvert$ small | GSW |
| $\eta<0, c_s-\epsilon-x_+>0$ small | NMCSW |

### 3.2. Near $C_1$

We consider the same parameters defined in the previous section. In $C_1$ (where $\mu_1<0$) the spectrum of $L$ consists of two imaginary simple eigenvalues $\pm i\sqrt{-3\mu_1}$ and 0, which is double. A basis of the generalized eigenspace may consist of the vectors $\xi_0,\xi_1$, defined above, along with $\zeta_0,\overline{\zeta_0}$, where

$$\zeta_0=(1,i\sqrt{-3\mu_1},2\mu_1,i\mu_1\sqrt{-3\mu_1})^T.$$

For $\mu>0$, the double eigenvalue splits into two real eigenvalues (region 3 of Fig. 1), [22]. The change of variables

$$U=A\xi_0+B\xi_1+C\zeta_0+\overline{C\zeta_0}+\Phi(\mu,A,B,C,\overline{C}),$$

for some polynomial $\Phi$ of degree at least two, can be chosen to write (3.8) in a normal form, [40]

$$\begin{aligned}\frac{dA}{dX}&=B,\\ \frac{dB}{dX}&=-\frac{1}{3\mu_1}\left(\mu A+(\nu_1+2\mu_1)(\nu_2-\nu_3)A^2+\nu_5|C|^2\right)+\cdots,\\ \frac{dC}{dX}&=i\sqrt{-3\mu_1}C\left(1+\frac{\mu}{18\mu_1^2}+\nu_6A+\cdots\right),\end{aligned}\tag{3.12}$$

where

$$\begin{aligned}\nu_5&=2\nu_1+2\mu_1(\nu_3-7\nu_2)+12\mu_1^2\nu_4,\\ \nu_6&=\frac{1}{9\mu_1^2}(\nu_1+\mu_1(4\nu_3-\nu_2)-12\mu_1^2\nu_4),\end{aligned}$$

(which, as before, can be reformulated in order to be nonsingular as $\mu_1\to0$). For a quadratic nonlinearity $g$, (3.12) has the form, [21]

$$\begin{aligned}\frac{dA}{dX}&=B,\\ \frac{dB}{dX}&=-\frac{\mu}{3\mu_1}A-\frac{2}{3}A^2-\frac{4}{3}|C|^2,\\ \frac{dC}{dX}&=i\sqrt{-3\mu_1}C\left(1+\frac{\mu}{18\mu_1^2}-\frac{1}{9\mu_1}\right).\end{aligned}$$

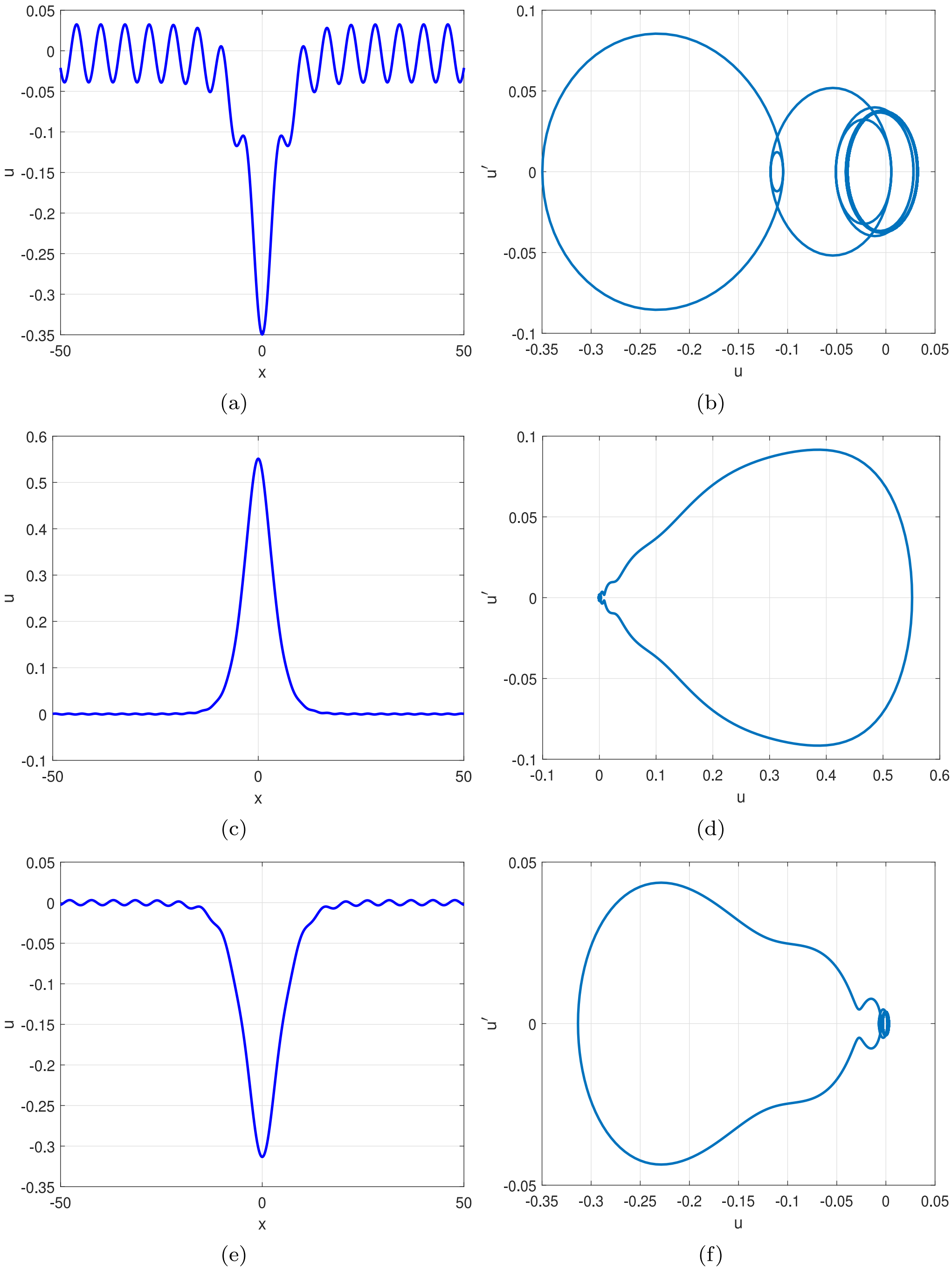

**Fig. 3.** GSW generation, $u$ profiles and phase portraits. (a), (b) Rosenau-RLW equation with $\alpha = 1, \epsilon = \beta = 1, g(u) = u^2/2; c_s = 0.9$; (c), (d) Rosenau–Kawahara equation with $\eta = 1, \epsilon = 0.25, \beta = 4, \gamma = 2, g(u) = u^2/2, c_s = 0.43$; (e), (f) Rosenau–Kawahara equation with $\eta = -1, \epsilon = 1, \beta = 2, \gamma = 1, g(u) = u^2/2, c_s = 0.9$.

As mentioned in *e.g.* [21,34–36,40], in region 3 close to $C_1$, there are orbits which are homoclinic to periodic orbits as $|X| \to \infty$, leading to smooth Generalized Solitary Wave solutions of (1.1). Some of the ripples may have exponentially small amplitude and on isolated curves the oscillations may vanish and an orbit homoclinic to zero (CSW, also called embedded soliton) is formed. For the families of Rosenau equations mentioned in the introduction, the range of speed and parameters ensuring the emergence of GSW's is specified in Tables 1–4 and some of approximate profiles are shown in Fig. 3. In addition, close to $C_1$, but in region 4, [40], PTW's are also formed, cf. [29].

### 3.3. Near $C_2$

In this case, according to the definition of $C_2$ in the $(\mu_1, \mu_2)$ plane, we consider a bifurcation parameter $\mu$ such that

$$\mu_2 = -\frac{5}{4}\mu_1^2 - \mu, \quad \mu_1 < 0.$$

Recall that the spectrum of $L$ in $C_2$ consists of two double imaginary eigenvalues

$$\lambda_{\pm} = \pm i\sqrt{\frac{-3}{2}\mu_1}.$$

When $\mu > 0$ is small (region 1 of Fig. 1) they become two complex simple pairs of eigenvalues, symmetric with respect to the axis, while if $\mu < 0$ (region 4) they become two pairs of imaginary eigenvalues which are simple. The generalized eigenspace when $\mu = 0$ can be generated by the eigenvalues $\zeta_0, \overline{\zeta_0}$ and the generalized eigenvectors $\zeta_1, \overline{\zeta_1}$, where, [22]

$$\zeta_0 = (1, \lambda_+, \frac{\mu_1}{2}, -\mu_1\frac{\lambda_+}{2})^T, \quad \zeta_1 = (0, 1, 2\lambda_+, \frac{5}{2}\mu_1)^T.$$

Following [1,20,22,46] we can change the variable

$$U = A\zeta_0 + B\zeta_1 + \overline{A\zeta_0} + \overline{B\zeta_1} + \Phi(\mu, A, B, \overline{A}, \overline{B}),$$

in such a way that the following normal form for (3.8) can be derived

$$\begin{aligned}
\frac{dA}{dX} &= \lambda_+ A + B - \frac{i\mu A}{6\mu_1\sqrt{-6\mu_1}} + \cdots, \\
\frac{dB}{dX} &= \lambda_+ B - \frac{i\mu B}{6\mu_1\sqrt{-6\mu_1}} - \frac{\mu}{6\mu_1}A + \frac{76\nu_1^2}{243\mu_1^3}A|A|^2 + \cdots
\end{aligned}$$

After rescaling, the study made in [46] reveals the existence of homoclinic to zero orbits when $\mu > 0$ is small (region 1 in Fig. 1) corresponding to CSW but with nonmonotone decay; they are different from the ones obtained close to $C_0$, (which are positive or negative) as well as the possible formation of orbits which are homoclinic to periodic orbits (corresponding to GSW's) and periodic orbits (corresponding to PTW's) when $\mu < 0$ (region 4). The description given in [29] gives the range of speeds and parameters corresponding to these two situations for each Rosenau-type equation considered in the present paper. The information corresponding to nonmonotone CSW's is displayed in Tables 1–4 and some approximate profiles are shown in Fig. 4. The form of both the approximate profiles and phase portraits reveals the nonmonotone behavior of the waves and their oscillatory (exponential) asymptotic decay to zero as $|X| \to \infty$.

### 3.4. Near $C_3$

In this case, the numerical experiments with $g(u) = u^2/2$ suggest that the bifurcation from region 2 to region 1 in Fig. 1 generates new homoclinic orbits in different ways: multimodal homoclinic orbits, [39], CSW's and CSW's with nonmonotone decay, and PTW's, cf. [28,42,43]. The bifurcation from CSW's to nonmonotone CSW's case is illustrated by some examples in Figs. 5 and 6.

## 4. Existence of classical solitary wave solutions with Concentration-Compactness theory

For the case $g(u) = G'(u), g(0) = 0$, with $G(u)$ homogeneous of some degree $q > 2$ and such that

$$\int_{\mathbb{R}} G(u)dx > 0, \tag{4.1}$$

for some $u$, the existence of CSW solutions of (1.1) can be analyzed using the theory of Concentration-Compactness, developed by Lions in [14] and which is a classical procedure to study solitary-wave solutions in nonlinear dispersive equations, [37,47]. In the present case we look for solitary waves of (1.1) as solutions of minimization problems of the form

$$I_\lambda = \inf\{E(u) : u \in H^2/F(u) = \lambda\}, \tag{4.2}$$

for $\lambda > 0$ where,

$$E(u) = \int_{-\mathbb{R}} \left((c_s - \epsilon)u^2 - (c_s\alpha - \eta)u_x^2 + (c_s\beta - \gamma)u_{xx}^2\right) dx, \tag{4.3}$$

$$F(u) = \int_{\mathbb{R}} G(u)dx. \tag{4.4}$$

The main hypothesis we assume here is the existence of positive constants $C_1, C_2$ such that for all $u \in H^2$

$$C_1\|u\|_2^2 \le E(u) \le C_2\|u\|_2^2. \tag{4.5}$$

Note that the continuity condition (second inequality in (4.5)) easily follows from the definition of $E$. The coercivity condition (first inequality in (4.5)) can be characterized as follows.

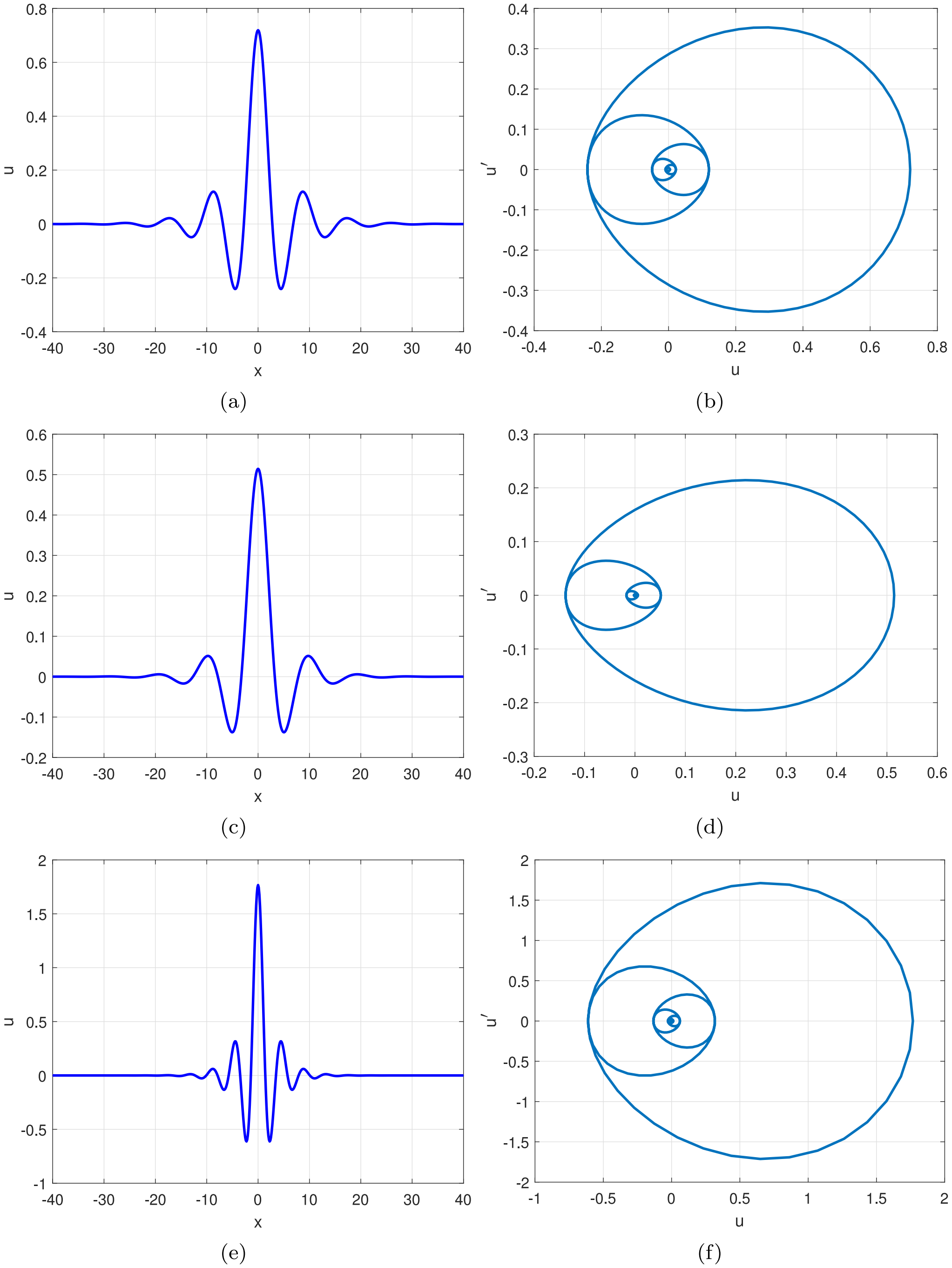


**Fig. 4.** Generation of nonmonotone CSW's, $u$ profiles and phase portraits. (a), (b) Rosenau-RLW equation with $\alpha = 1, \epsilon = \beta = 1, g(u) = u^2/2; c_s = 1.5$; (c), (d) Rosenau-KdV equation with $\eta = -1, \epsilon = \beta = 1, g(u) = u^2/2, c_s \approx 1.3071$; (e), (f) Rosenau–Kawahara equation with $\gamma = 2, \epsilon = \beta = 1, \eta = -1, g(u) = u^2/2, c_s \approx 2.2590$.

**Proposition 4.1.** *Let $\rho = \beta\epsilon - \gamma, \delta = \alpha\epsilon - \eta$, and*

$$x_{\pm} = \frac{1}{4\beta - \alpha^2}\left(\alpha\delta - 2\rho \pm \sqrt{(\alpha\delta - 2\rho)^2 + \delta^2(4\beta - \alpha^2)}\right). \tag{4.6}$$

*If*

$$c_s - \epsilon < x_- \text{ or } c_s - \epsilon > x_+, \tag{4.7}$$

*then (4.5) holds for some positive constants $C_j, j = 1, 2$ and all $u \in H^2$.*

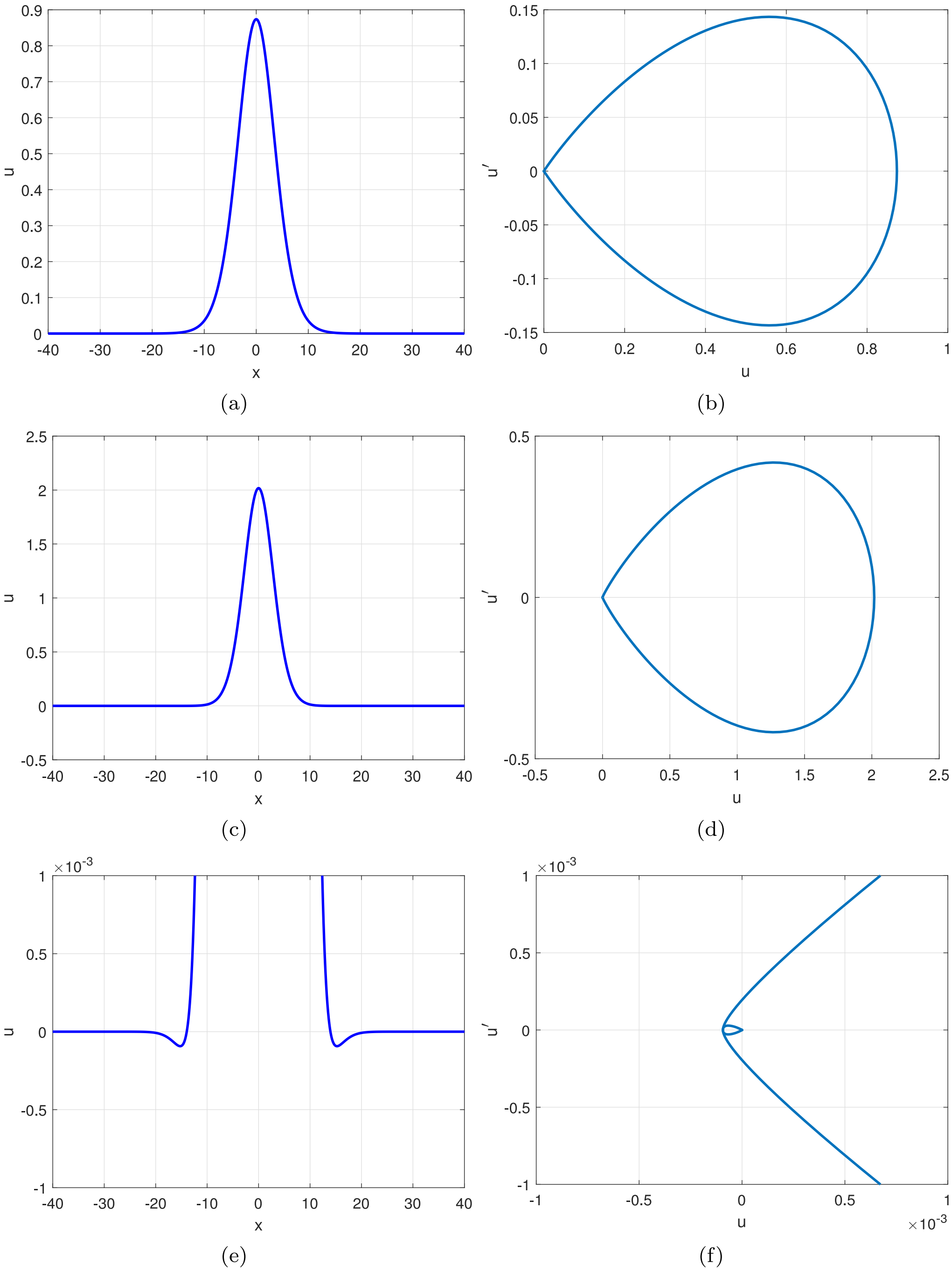

**Fig. 5.** Bifurcation from region 2 to region 1 close to $C_3$, $u$ profiles and phase portraits. (a), (b) Rosenau-RLW equation with $\alpha = -1, \epsilon = \beta = 1, g(u) = u^2/2; c_s = 1.3$; (c), (d) Rosenau-RLW equation with $\alpha = -1, \epsilon = \beta = 1, g(u) = u^2/2; c_s = 1.7$; (e) and (f) are magnifications of (c) and (d) resp.

**Proof.** As mentioned before, we just have to prove the first inequality in (4.5). We can write (4.3) in the form

$$\begin{aligned} E(u) &= \int_{-\mathbb{R}} \left((c_s-\epsilon)u^2 + (c_s\alpha-\eta)uu_{xx} + (c_s\beta-\gamma)u_{xx}^2\right)dx \\ &= \langle A\begin{pmatrix} u \\ u_{xx}\end{pmatrix}, \begin{pmatrix} u \\ u_{xx}\end{pmatrix}\rangle, \end{aligned} \tag{4.8}$$

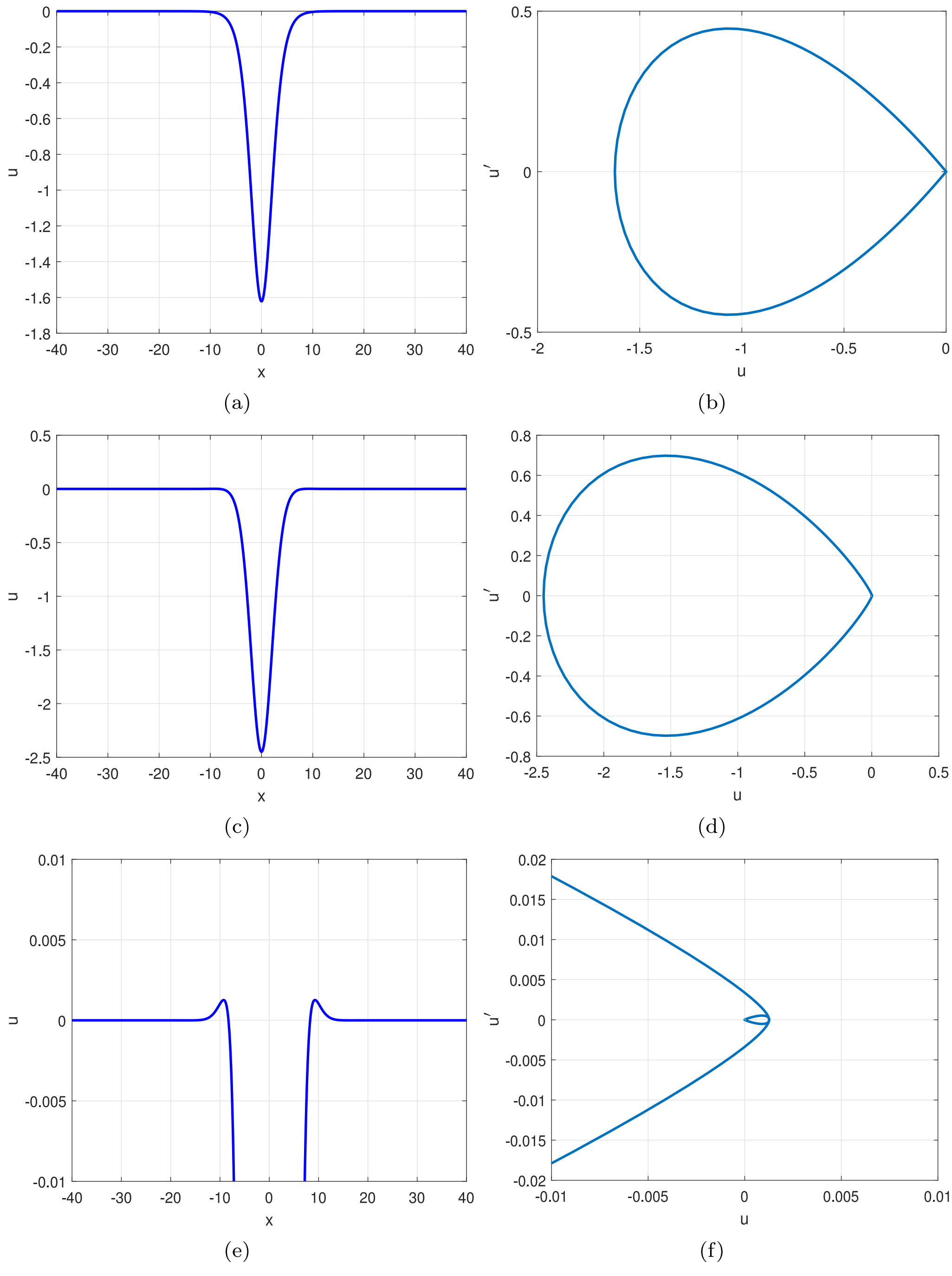


**Fig. 6.** Bifurcation from region 2 to region 1 close to $C_3$, $u$ profiles and phase portraits. (a), (b) Rosenau–Kawahara equation with $\eta = -1, \epsilon = 1, \beta = 2, \gamma = 1, g(u) = u^2/2, c_s \approx 0.4538$; (c), (d) Rosenau–Kawahara equation with $\gamma = 1, \epsilon = 1, \beta = 2, \eta = -1, g(u) = u^2/2, c_s \approx 0.1438$; (e) and (f) are magnifications of (c) and (d) resp.

where $\langle \cdot, \cdot \rangle$ denotes the $L^2$ inner product and

$$A = \begin{pmatrix} c_s - \epsilon & \frac{\alpha(c_s-\epsilon)+\delta}{2} \\ \frac{\alpha(c_s-\epsilon)+\delta}{2} & \beta(c_s - \epsilon) + \rho \end{pmatrix}.$$

The eigenvalues $z_{\pm}$ of $A$ satisfy $z_+ > 0$ while $z_- > 0$ when

$$P(c_s - \epsilon) > 0, \tag{4.9}$$

**Table 3**
Admissible parameters and type of solitary wave for the Rosenau–Kawahara equation, according to the Normal Form Theory, where $y_{\pm} = \frac{1}{2}\left(-\frac{\rho}{\beta} \pm \sqrt{\left(\frac{\rho}{\beta}\right)^2 + \frac{\eta^2}{\beta}}\right)$ (cf. [29]).

| Rosenau–Kawahara ($\alpha = 0, \epsilon, \beta > 0$) | | |
|---|---|---|
| $\rho = \epsilon\beta - \gamma$ | Admissible parameters | Type of solitary wave |
| $\rho < 0$ | $\eta < 0, y_- < c_s - \epsilon < 0$, $\lvert c_s - \epsilon\rvert$ small<br>or<br>$\eta > 0, c_s - \epsilon > -\rho/\beta$ | CSW |
| | $\eta > 0, 0 < c_s - \epsilon < -\rho/\beta$<br>$c_s - \epsilon$ small | GSW |
| | $\eta > 0, y_- - (c_s - \epsilon) > 0$ small<br>or<br>$\eta < 0, c_s - \epsilon - y_+ > 0$ small | NMCSW |
| $\rho > 0$ | $\eta > 0, 0 < c_s - \epsilon < y_+$, $\lvert c_s - \epsilon\rvert$ small<br>or<br>$\eta < 0, c_s - \epsilon < -\rho/\beta$ | CSW |
| | $\eta < 0, -\rho/\beta < c_s - \epsilon < 0$<br>$\lvert c_s - \epsilon\rvert$ small | GSW |
| | $\eta > 0, y_- - (c_s - \epsilon) > 0$ small<br>or<br>$\eta < 0, c_s - \epsilon - y_+ > 0$ small | NMCSW |
| $\rho = 0$ | $\beta$ large and $\eta > 0, 0 < c_s - \epsilon < y_+$<br>or<br>$\beta$ large and $\eta < 0, y_- < c_s - \epsilon < 0$ | CSW |
| | $\eta > 0, y_- - (c_s - \epsilon) > 0$ small<br>or<br>$\eta < 0, (c_s - \epsilon) - y_+ > 0$ small | NMCSW |

**Table 4**
Admissible parameters and type of solitary wave for the Rosenau-RLW-Kawahara equation, according to the Normal Form Theory (cf. [29]), where $z_+ = \frac{1}{2}\left(\frac{2(\eta-3)}{3} + \sqrt{\left(\frac{2(\eta-3)}{3}\right)^2 + 4\frac{(1+\eta)^2}{3}}\right)$.

| Rosenau-RLW-Kawahara ($\alpha = \gamma = -1, \epsilon = \beta = 1, \eta > 0$) | |
|---|---|
| Admissible parameters | Type of solitary wave |
| $0 < c_s - \epsilon < z_+$<br>$c_s - \epsilon$ small | CSW |

with

$$P(x) = x^2 + \frac{2(2\rho - \alpha\delta)}{4\beta - \alpha^2}x - \frac{\delta^2}{4\beta - \alpha^2}.$$

The roots of $P$ are given by (4.6) and they satisfy $x_- \le 0 \le x_+$. Thus, (4.9) holds when one of the conditions (4.7) is satisfied. In such case, then $A$ is shown to be positive definite and (4.5) follows from (4.8). □

The characterization (4.7) will be discussed below for the families of Rosenau equations considered in the present paper. This will give us a range of speeds ensuring the existence of CSW's in each case and that will be related to the corresponding results obtained in Section 3 from the NFT.

The following lemma uses to be the starting point for the application of the Concentration Compactness theory.

**Lemma 4.1.** *Under the hypotheses (4.1) and (4.5), the problems (4.2) satisfy the following properties:*

*(i)* $I_\lambda > 0$ *for* $\lambda > 0$.
*(ii) Every minimizing sequence for* $I_\lambda, \lambda > 0$, *is bounded in* $H^2$.
*(iii) For all* $\theta \in (0, \lambda)$

$$I_\lambda < I_\theta + I_{\lambda-\theta}. \tag{4.10}$$

**Proof.** Note first that since $G$ is homogeneous of degree $q > 2$ then [30]

$$|G(u)| \le C|u|^q,$$

for some constant $C$. Then, if $u$ minimizes (4.2) for $\lambda > 0$, from (4.4) and the Sobolev embedding of $H^2$ into $L^\infty$ it holds that

$$\lambda = \int_{\mathbb{R}} G(u)dx \leq C \int_{\mathbb{R}} |u|^q dx \leq C\|u\|_{L^\infty}^{q-2}\|u\|_{L^2}^2 \tag{4.11}$$

$$\leq C\|u\|_2^q, \tag{4.12}$$

for some constant $C$. Using (4.5) and (4.12) we have

$$I_\lambda = E(u) \geq C\lambda^{2/q} > 0,$$

for some constant $C$ and (i) follows. Property (ii) is also a consequence of (4.12) and coercivity hypothesis (4.5). On the other hand, since $E$ and $F$ are homogeneous of degrees 2 and $q$ respectively, then

$$I_{\tau\lambda} = \tau^{\frac{2}{q}} I_\lambda.$$

Therefore, if $\theta \in (0, \lambda)$, with $\theta = \tau\lambda, \tau \in (0, 1)$, then we have, for $q > 2$

$$\begin{aligned} I_\theta + I_{\lambda-\theta} &= I_{\tau\lambda} + I_{(1-\tau)\lambda} = \tau^{\frac{2}{q}} I_\lambda + (1-\tau)^{\frac{2}{q}} I_\lambda \\ &> (\tau + (1-\tau))I_\lambda, \end{aligned}$$

and (4.10) follows. □

The application of Concentration-Compactness theory in order to prove the existence of CSW's of (1.1) under the assumption (4.5) is as follows. Let $\{u_n\}_n$ be a minimizing sequence for (4.2) and consider the sequence of nonnegative functions

$$\rho_n(x) = |u_n(x)|^2 + |u_n'(x)|^2 + |u_n''(x)|^2.$$

Then $\rho_n \in L^1$ and its $L^1$ norm satisfies $\lambda_n = \|\rho_n\|_{L^1} = \|u_n\|_2^2$. From Lemma 4.1, $\lambda_n$ is bounded and from (4.5), (4.12)

$$\lambda_n > C\lambda^{\frac{2}{q}},$$

for some constant $C$. Let $\sigma = \lim_{n\to\infty} \lambda_n > 0$. Normalizing $\rho_n$ as $\widetilde{\rho}_n(x) = \sigma\rho_n(\lambda_n x)$, to have

$$\widetilde{\lambda}_n = \|\widetilde{\rho}_n\|_{L^1} = \sigma,$$

(and dropping tildes from now on), then from Lemma 1.1 of [14] there is a subsequence $\{\rho_{n_k}\}_{k\geq 1}$ satisfying one of the following three possibilities:

(1) (Compactness.) There are $y_k \in \mathbb{R}$ such that $\rho_{n_k}(\cdot + y_k)$ satisfies that for any $\widetilde{\epsilon} > 0$ there exists $R = R(\widetilde{\epsilon}) > 0$ large enough such that

$$\int_{|x-y_k|\leq R} \rho_{n_k}(x)dx \geq \sigma - \widetilde{\epsilon}.$$

(2) (Vanishing.) For any $R > 0$

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{|x-y|\leq R} \rho_{n_k}(x)dx = 0. \tag{4.13}$$

(3) (Dichotomy.) There is $\theta_0 \in (0, \sigma)$ such that for any $\widetilde{\epsilon} > 0$ there exists $k_0 \geq 1$ and $\rho_{k,1}, \rho_{k,2} \in L^1$ with $\rho_{k_1}, \rho_{k_2} \geq 0$ such that for $k \geq k_0$

$$\int_{\mathbb{R}} |\rho_{n_k} - (\rho_{k,1} + \rho_{k,2})|dx \leq \widetilde{\epsilon}, \tag{4.14}$$

$$\left|\int_{\mathbb{R}} \rho_{k,1}dx - \theta_0\right| \leq \widetilde{\epsilon}, \quad \left|\int_{\mathbb{R}} \rho_{k,2}dx - (\sigma - \theta_0)\right| \leq \widetilde{\epsilon},$$

with

$$\begin{aligned} &\mathrm{supp}\rho_{k,1} \cap \mathrm{supp}\rho_{k,2} = \emptyset, \\ &\mathrm{dist}\left(\mathrm{supp}\rho_{k,1}, \mathrm{supp}\rho_{k,2}\right) \to +\infty, \; k \to \infty. \end{aligned}$$

Since the supports of $\rho_{k,1}$ and $\rho_{k,2}$ are disjoint, we may assume the existence of $R_0 > 0$ and sequences $\{y_k\}_k$ and $R_k \to \infty$ as $k \to \infty$ such that, [26]

$$\mathrm{supp}\rho_{k,1} \subset (y_k - R_0, y_k + R_0),$$

$$\mathrm{supp}\rho_{k,2} \subset (-\infty, y_k - 2R_k) \cup (y_k + 2R_k, \infty). \tag{4.15}$$

Our goal is to prove that compactness holds by ruling out the other two possibilities. Note first that if vanishing property (4.13) hods, then

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{y-R}^{y+R} \left(|u_{n_k}(x)|^2\right) dx = 0.$$

Since $\{u_n\}_n$ is a bounded sequence in $H^2$ and from (4.11) we have

$$F(u_{n_k}) \leq C\|u_{n_k}\|_{L^\infty}^{q-2}\|u_{n_k}\|_{L^2}^2 \leq C\|u_{n_k}\|_{L^2}^2,$$

for some constant $C$. This implies that

$$\lim_{k\to\infty} F(u_{n_k}) = 0,$$

which contradicts the fact that $\lambda > 0$ and vanishing is not possible.

We now assume that dichotomy holds and consider cutoff functions $\varphi, \phi \in C^\infty(\mathbb{R}), 0 \leq \varphi, \phi \leq 1$, with

$$\phi(x) = 1, \quad |x| \leq 1, \quad \phi(x) = 0, \quad |x| \geq 2,$$
$$\varphi(x) = 1, \quad |x| \geq 2, \quad \varphi(x) = 0, \quad |x| \leq 1.$$

Let $R > R_0$ and define, for $x \in \mathbb{R}$,

$$\phi_k(x) = \phi\left(\frac{x-y_k}{R}\right), \quad u_{k,1} = \phi_k(x)u_{n_k}(x),$$
$$\varphi_k(x) = \varphi\left(\frac{x-y_k}{R_k}\right), \quad u_{k,2} = \varphi_k(x)u_{n_k}(x),$$
$$w_k(x) = u_{n_k}(x) - u_{k,1}(x) - u_{k,2}(x).$$

In the following lemma some auxiliary results are collected, [26].

**Lemma 4.2.** *Let $\widetilde{\epsilon} > 0$. If dichotomy holds, then:*

*(1) For $R$ large enough*

$$\int_{|x-y_k|\leq R_0} |\rho_{n_k}(x) - \rho_{k,1}(x)|dx \leq \widetilde{\epsilon} \tag{4.16}$$
$$\int_{|x-y_k|\geq 2R_k} |\rho_{n_k}(x) - \rho_{k,2}(x)|dx \leq \widetilde{\epsilon}, \tag{4.17}$$
$$\int_{R_0\leq|x-y_k|\leq 2R_k} \rho_{n_k}(x)dx \leq \widetilde{\epsilon}. \tag{4.18}$$

*(2) For $R, R_k$ large enough*

$$\left|\|u_{k,1}\|_2^2 - \int_{\mathbb{R}} \rho_{k,1}dx\right| = O(\widetilde{\epsilon}), \tag{4.19}$$
$$\left|\|u_{k,2}\|_2^2 - \int_{\mathbb{R}} \rho_{k,2}dx\right| = O(\widetilde{\epsilon}). \tag{4.20}$$

*(3) For $R, R_k$ large enough*

$$\|w_k\|_2 = O(\widetilde{\epsilon}). \tag{4.21}$$

**Proof.** Using (4.14) and (4.15) we have

$$\begin{aligned} O(\widetilde{\epsilon}) &= \int_{\mathbb{R}} |\rho_{n_k} - (\rho_{k,1} + \rho_{k,2})|dx \\ &= \int_{|x-y_k|\leq R_0} |\rho_{n_k}(x) - \rho_{k,1}(x)|dx + \int_{|x-y_k|\geq 2R_k} |\rho_{n_k}(x) - \rho_{k,2}(x)|dx \\ &\quad + \int_{R_0\leq|x-y_k|\leq 2R_k} \rho_{n_k}(x)dx, \end{aligned}$$

which implies (4.16)–(4.18). Note now that

$$\phi_k'(x) = \frac{1}{R}\phi'\left(\frac{x-y_k}{R}\right), \quad \phi_k''(x) = \frac{1}{R^2}\phi''\left(\frac{x-y_k}{R}\right),$$
$$\varphi_k'(x) = \frac{1}{R_k}\varphi'\left(\frac{x-y_k}{R_k}\right), \quad \varphi_k''(x) = \frac{1}{R_k^2}\varphi''\left(\frac{x-y_k}{R_k}\right).$$

Then, for some constant $C$ and $R$ large enough

$$\begin{aligned} \|u_{k,1}\|_2^2 &= \int_{\mathbb{R}} \left(|u_{k,1}(x)|^2 + |u_{k,1}'(x)|^2 + |u_{k,1}''(x)|^2\right) dx \\ &\leq \int_{|x-y_k|\leq 2R} \left(|u_{n_k}(x)|^2 + |u_{n_k}'(x)|^2 + |u_{n_k}''(x)|^2\right) dx \\ &\quad + \frac{C}{R}\int_{|x-y_k|\leq 2R} |u_{n_k}(x)|^2 dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{|x-y_k|\le R_0} \rho_{n_k}(x)dx - \int_{\mathbb{R}} \rho_{k,1}(x)dx + \int_{\mathbb{R}} \rho_{k,1}(x)dx \\
&\quad + \frac{C}{R}\int_{|x-y_k|\le 2R} |u_{n_k}(x)|^2 dx + \int_{R_0\le|x-y_k|\le 2R} \rho_{n_k}(x)dx.
\end{aligned}
$$

Therefore, using (4.15), we have

$$
\begin{aligned}
\left| \|u_{k,1}\|_2^2 - \int_{\mathbb{R}} \rho_{k,1}(x)dx \right| \le & \underbrace{\int_{|x-y_k|\le R_0} |\rho_{n_k}(x) - \rho_{k,1}(x)|dx}_{I_1} \\
& + \underbrace{\frac{C}{R}\int_{|x-y_k|\le 2R} |u_{n_k}(x)|^2 dx}_{I_2} \\
& + \underbrace{\int_{R_0\le|x-y_k|\le 2R} \rho_{n_k}(x)dx}_{I_3}.
\end{aligned}
$$

For $R$ large enough, (4.16), and (4.18), the integrals $I_j, j = 1,2,3$, are $O(\widetilde{\epsilon})$, leading to (4.19). Similarly,

$$
\begin{aligned}
\left| \|u_{k,2}\|_2^2 - \int_{\mathbb{R}} \rho_{k,2}(x)dx \right| \le & \underbrace{\int_{|x-y_k|\ge R_k} |\rho_{n_k}(x) - \rho_{k,2}(x)|dx}_{J_1} \\
& + \underbrace{\frac{C}{R_k}\int_{|x-y_k|\le 2R_k} |u_{n_k}(x)|^2 dx}_{J_2} \\
& + \underbrace{\int_{R_k\le|x-y_k|\le 2R_k} \rho_{n_k}(x)dx}_{J_3},
\end{aligned}
$$

and for $R$ large enough, (4.17), and (4.18), the integrals $J_j, j = 1,2,3$, are $O(\widetilde{\epsilon})$ and (4.20) holds. Note finally that

$$
\begin{aligned}
& w_k = \chi_k u_{n_k}, \quad \chi_k = 1 - \phi_k - \varphi_k, \\
& \operatorname{supp}\chi_k \subset \{R \le |x-y_k| \le 2R_k\}.
\end{aligned}
$$

Then

$$
\|w_k\|_2^2 \le C\left(\|\chi_k\|_{L^\infty}^2 + \|\chi_k'\|_{L^\infty}^2 + \|\chi_k''\|_{L^\infty}^2\right)\int_{R\le|x-y_k|\le 2R_k} \rho_{n_k}(x)dx, \tag{4.22}
$$

for some constant $C$. The parenthesis on the right-hand side of (4.22) is bounded while the integral is $O(\widetilde{\epsilon})$ because of (4.18), implying (4.21). □

**Lemma 4.3.** *Let $\widetilde{\epsilon} > 0$. If dichotomy holds and $R, R_k$ are large enough, then there are subsequences of $\{u_{k,1}\}_k, \{u_{k,2}\}_k$, denoted again in the same way, such that*

$$
E(u_{n_k}) = E(u_{k,1}) + E(u_{k,2}) + O(\widetilde{\epsilon}), \tag{4.23}
$$
$$
F(u_{n_k}) = F(u_{k,1}) + F(u_{k,2}) + O(\widetilde{\epsilon}). \tag{4.24}
$$

**Proof.** Since $u_{n_k} = w_k + u_{k,1} + u_{k,2}$ then we can write

$$
E(u_{n_k}) = E(w_k) + E(u_{k,1}) + E(u_{k,2}) + N,
$$

where

$$
\begin{aligned}
N &= 2A_1\int_{\mathbb{R}} w_k(u_{k,1}+u_{k,2})dx + 2A_2\int_{\mathbb{R}} w_k'(u_{k,1}'+u_{k,2}')dx \\
&\quad + 2A_2\int_{\mathbb{R}} w_k''(u_{k,1}''+u_{k,2}'')dx,
\end{aligned}
$$

where

$$
A_1 = c_s - \epsilon, \quad A_2 = -(c_s\alpha - \eta), \quad A_3 = c_s\beta - \gamma.
$$

Therefore, from Cauchy–Schwarz inequality we have

$$
|N| \le C\|w_k\|_2\left(\|u_{k,1}\|_2 + \|u_{k,2}\|_2\right),
$$

for some constant $C$. Now, (4.19), (4.20), (4.14), and (4.21) imply that $N = O(\widetilde{\epsilon})$ and (4.23) holds. On the other hand, since $\{u_{n_k}\}_k$ is a minimizing sequence, using Lemma 4.1(i) and (iii) along with the property $0 \le \phi \le 1$, then $F(u_{k,1})$ is bounded, and there is a subsequence (denoted again by $u_{k,1}$) such that $F(u_{k,1})$ converges to some $\theta \in \mathbb{R}$. Thus, there is some $k_0 > 0$ such that for $k \ge k_0$

$$|F(u_{k,1}) - \theta| \le \widetilde{\epsilon}.$$

Now, since $F(u_{n_k}) = \lambda$, $\phi_k\varphi_k = 0$, and $G$ is homogeneous of degree $q$, from (4.12) we have

$$\begin{aligned} |F(u_{k,2}) - (\lambda-\theta)| &\le \left|\int_{\mathbb{R}} \left(G(\varphi_k u_{n_k}) - G(u_{n_k}) + G(\phi_k u_{n_k})\right) dx\right| + \widetilde{\epsilon} \\ &= \left|\int_{\mathbb{R}} \left(\varphi_k^q + \phi_k^q - 1\right) G(u_{n_k}) dx\right| + \widetilde{\epsilon} \\ &\le \int_{\mathbb{R}} \chi_k^q G(u_{n_k}) dx + \widetilde{\epsilon} = F(w_k) + \widetilde{\epsilon} \\ &\le C\|w_k\|_2^q + \widetilde{\epsilon} = O(\widetilde{\epsilon}), \end{aligned} \tag{4.25}$$

where in the last equality (4.21) was used; then (4.24) follows. □

As a consequence of (4.23) we have

**Corollary 4.1.** *Under the conditions of Lemma 4.3*

$$\begin{aligned} I_\lambda &\ge \lim_{k\to\infty} \inf E(u_{n_k}) \\ &\ge \lim_{k\to\infty} \inf E(u_{k,1}) + \lim_{k\to\infty} \inf E(u_{k,2}) + \widetilde{\epsilon}. \end{aligned} \tag{4.26}$$

In order to rule out dichotomy, we discuss the existence of the limit obtained in the proof of Lemma 4.3

$$\theta = \lim_{k\to\infty} F(u_{k,1}).$$

We have the following possibilities, [26]:

(1) $\theta = 0$. Then, using (4.25) we have, for $k$ large enough and $\widetilde{\epsilon} < \lambda/2$

$$F(u_{k,2}) > \lambda - \widetilde{\epsilon} > \frac{\lambda}{2} > 0. \tag{4.27}$$

We consider $d_k > 0$ such that $\lambda = F(d_k h_{k,2})$; explicitly

$$d_k = \left(\frac{\lambda}{F(u_{k,2})}\right)^{\frac{1}{q}},$$

Then

$$|d_k - 1| < \left|\left(\frac{\lambda}{F(u_{k,2})}\right)^{\frac{1}{q}} - 1\right| < \frac{1}{F(u_{k,2})^{1/q}}\left|\lambda^{\frac{1}{q}} - F(h_{k,2})^{\frac{1}{q}}\right|.$$

For $x > 0$ let $r(x) = x^{\frac{1}{q}}$. Then

$$\begin{aligned} \left|\lambda^{\frac{1}{q}} - F(u_{k,2})^{\frac{1}{q}}\right| &= |r(\lambda) - r(F(h_{k,2}))| \\ &= |r'(\xi_k)(\lambda - F(u_{k,2}))|, \end{aligned}$$

for some $\xi_k$ between $F(u_{k,2})$ and $\lambda$ and consequently, by (4.27), between $\lambda/2$ and $\lambda$. Therefore

$$r'(\xi_k) = \frac{1}{q}\left(\frac{1}{\xi_k}\right)^{\frac{1}{q}-1} < \frac{1}{q}\left(\frac{2}{\lambda}\right)^{\frac{1}{q}-1}.$$

This yields, for some constant $C(\lambda)$

$$|d_k - 1| < \left(\frac{2}{\lambda}\right)^{1/q} |\lambda - f(u_{k,2})| = O(\widetilde{\epsilon}).$$

Then $\lim_{k\to\infty} d_k = 1$ and

$$I_\lambda \le E(d_k u_{k,2}) = d_k^2 E(u_{k,2}) = E(u_{k,2}) + O(\widetilde{\epsilon}). \tag{4.28}$$

Therefore, from coercivity property of $E$, dichotomy, and (4.19)

$$\begin{aligned} \lim_{k\to\infty} \inf E(u_{k,1}) &\ge C \lim_{k\to\infty} \inf \|u_{k,1}\|_2^2 \\ &\ge C \lim_{k\to\infty} \inf \|\rho_{k,1}\|_{L^1} + O(\widetilde{\epsilon}) \ge C\theta_0 + O(\widetilde{\epsilon}). \end{aligned}$$

Therefore, (4.26) and (4.28) imply, for $\widetilde{\epsilon} > 0$ arbitrarily small

$$I_\lambda \geq C\theta_0 + I_\lambda + O(\widetilde{\epsilon}),$$

that taking $\widetilde{\epsilon} \to 0$ leads to $I_\lambda \geq C\theta_0 + I_\lambda$, which is the contradiction with $\theta_0 \leq 0$.

(2) $\lambda > \theta > 0$. Then, from (4.23)

$$\begin{aligned} E(u_{n_k}) &= E(u_{k,1}) + E(u_{k,2}) + O(\widetilde{\epsilon}) \\ &\geq I_{F(u_{k,1})} + I_{F(u_{k,2})} + O(\widetilde{\epsilon}) \\ &= \left( F(u_{k,1})^{\frac{2}{q}} + F(u_{k,2})^{\frac{2}{q}} \right) I_1 + O(\widetilde{\epsilon}). \end{aligned}$$

If $k \to \infty$ then

$$I_\lambda \geq \left( \theta^{\frac{2}{q}} + (\lambda - \theta)^{\frac{2}{q}} \right) I_1 + O(\widetilde{\epsilon}) = I_\theta + I_{\lambda-\theta} + O(\widetilde{\epsilon}),$$

for $\widetilde{\epsilon} > 0$ arbitrarily small. This contradicts Lemma 4.1(ii).

(3) $\theta < 0$. Then, from (4.25)

$$\lim_{k\to\infty} F(h_{k,2}) = \lambda - \theta > \frac{\lambda}{2},$$

and we apply the analysis of item (1) leading to contradiction.

(4) $\theta = \lambda$. Then, as in item (1), we may take $\widetilde{\epsilon}$ small enough so that $F(h_{k,1}) > \lambda/2$ for $k$ large. The same arguments apply to have

$$I_\lambda \leq E(u_{k,1}) + O(\widetilde{\epsilon}),$$

and

$$\begin{aligned} \lim_{k\to\infty} \inf E(u_{k,2}) &\geq C \lim_{k\to\infty} \inf \|u_{k,2}\|_2^2 \\ &\geq C \lim_{k\to\infty} \inf \|\rho_{k,2}\|_{L^1} + O(\widetilde{\epsilon}) \geq C(\sigma - \theta_0) + O(\widetilde{\epsilon}). \end{aligned}$$

This leads, for $k \to \infty, \widetilde{\epsilon} \to 0$

$$I_\lambda \geq C(\sigma - \theta_0) + I_\lambda,$$

implying the contradiction $\sigma - \theta_0 \leq 0$.

(5) $\theta > \lambda$. Then $F(h_{k,1}) > 0$ for $k$ large enough. We consider $e_k$ such that $F(e_k u_{k,1}) = \lambda$. Explicitly

$$e_k = \left( \frac{\lambda}{F(u_{k,1})} \right)^{\frac{2}{q}}.$$

Then

$$\lim_{k\to\infty} e_k = \left( \frac{\lambda}{\theta} \right)^{\frac{2}{q}}.$$

Therefore, for $k$ large enough, we have

$$I_\lambda \leq E(e_k u_{k,1}) = e_k^2 E(u_{k,1}) < E(u_{k,1}). \tag{4.29}$$

From coercivity property of $E$, dichotomy, and (4.20)

$$\begin{aligned} \lim_{k\to\infty} \inf E(u_{k,2}) &\geq C \lim_{k\to\infty} \inf \|u_{k,2}\|_2^2 \geq C \lim_{k\to\infty} \inf \|\rho_{k,2}\|_{L^1} + O(\widetilde{\epsilon}) \\ &\geq C(\sigma - \theta_0) + O(\widetilde{\epsilon}). \end{aligned} \tag{4.30}$$

Thus, (4.26), (4.29), and (4.30) imply

$$I_\lambda \geq I_\lambda + C(\sigma - \theta_0) + O(\widetilde{\epsilon}).$$

When $\widetilde{\epsilon} \to 0$, we have the contradiction $\sigma - \theta_0 \leq 0$.

Since vanishing and dichotomy do not hold, then Lemma 1.1 of [14] implies that necessarily compactness is satisfied. This means that there exists $\{y_k\}_k \subset \mathbb{R}$ such that for any $\widetilde{\epsilon} > 0$ there is $R = R(\widetilde{\epsilon}) > 0$ large and $k_0 >$ such that for $k \geq k_0$

$$\int_{|x-y_k|\leq R} \rho_{n_k} dx \geq \sigma - \widetilde{\epsilon}, \quad \int_{|x-y_k|\geq R} \rho_{n_k} dx = O(\widetilde{\epsilon}). \tag{4.31}$$

From the arguments of Lemma 4.1(i), applied to $P_k = \{|x - y_k| \geq R\}$, we have

$$\int_{P_k} G(u_{n_k}) dx \leq C \|u_{n_k}\|_{L^\infty}^{q-2} \int_{P_k} u_{n_k}^2 dx$$

$$\leq C\|u_{n_k}\|_2^{q-2}\int_{P_k}\rho_{n_k}dx = O(\widetilde{\epsilon}),$$

where in the last inequality Lemma 4.1(iii) and (4.31) were used. Therefore

$$\left|\int_{|x-y_k|\leq R} G(u_{n_k})dx - \lambda\right| \leq \widetilde{\epsilon}. \tag{4.32}$$

Let $\widetilde{u}_{n_k}(x) = u_{n_k}(x-y_k)$. Then $\widetilde{u}_{n_k}$ is bounded in $H^2$ and therefore there exists a subsequence $\widetilde{u}_{n_k}$ which converges weakly in $H^2$ to some $\widetilde{u}$. From (4.32)

$$\lambda \geq \int_{-R}^{R} G(\widetilde{u}_{n_k})dx \geq \lambda - \widetilde{\epsilon}.$$

The compact embedding of $H^2(-R,R)$ in $W^{1,p}(-R,R)$ $\forall p\geq 1$ and Lemma 2.2 of [30] imply

$$\lambda \geq \int_{-R}^{R} G(\widetilde{u})dx \geq \lambda - \widetilde{\epsilon}.$$

Taking $\widetilde{\epsilon}\to 0$ then $R = R(\widetilde{\epsilon})\to\infty$ and therefore $F(\widetilde{u}) = \lambda$. Furthermore, from lower semicontinuity and invariance by translations of $E$ we have

$$I_\lambda = \lim_{k\to\infty}\inf E(\widetilde{u}_{n_k}) \geq E(\widetilde{u}) \geq I_\lambda.$$

Therefore $\widetilde{u}$ satisfies the variational problem

$$\delta E(\widetilde{u}) = \kappa\delta F(\widetilde{u}),$$

with $\kappa$ a Lagrange multiplier. This means

$$(\gamma - \beta c_s)\widetilde{u}'''' + (\eta - \alpha c_s)\widetilde{u}'' + (\epsilon - c_s)u + \kappa g(\widetilde{u}) = 0. \tag{4.33}$$

Using the Euler theorem to the homogeneous function $G$, then

$$G'(\widetilde{u})\widetilde{u} = qG(\widetilde{u}),$$

and multiplying (4.33) by $\widetilde{u}$ and integrating yield

$$q\kappa\lambda = E(\widetilde{u}) = I_\lambda \Rightarrow \kappa = \frac{I_\lambda}{q\lambda} > 0.$$

Defining

$$u = \kappa^{\frac{1}{q-1}}\widetilde{u},$$

then $u$ is a solution of (2.9). This completes the proof of the first part of the following theorem.

**Theorem 4.1.** *Under the assumptions (2.3), (4.1), (4.7), there is a solution $u \in H^u$ of (2.9). Furthermore if $g(u)\in H^{s-2}$ when $u\in H^s, s>0$, then $u\in H^\infty$.*

**Proof.** The last statement is proved from writing (2.9) in the form

$$Q\varphi = g(\varphi), \tag{4.34}$$

where $Q$ is the linear operator with Fourier symbol

$$\widehat{Qu}(k) = \left((\beta c_s - \gamma)k^4 - (\alpha c_s - \eta)k^2 + (c_s - \epsilon)\right)\widehat{u}(k), \quad k\in\mathbb{R}.$$

Note that from Proposition 4.1, the determinant of the matrix $A$ in (4.8) is positive, leading to

$$(\alpha c_s - \eta)^2 < 4(\beta c_s - \gamma)(c_s - \epsilon).$$

This implies that $Q$ is invertible. The hypothesis on $g$ and (4.34) yield $u\in H^4$. A bootstrap argument applied to (4.34) proves the result. □

### *4.1. Some examples*

In this section Theorem 4.1 will be applied in order to prove the existence of CSW solutions of several families of Rosenau-type Eq. (1.1). The application mainly depends on the coercivity property (4.5) of the corresponding functional (4.3). We will consider nonlinear terms of the form $g(u) = \frac{u^{p+1}}{p+1}, p\geq 1$, for which

$$G(u) = \frac{u^{p+2}}{(p+2)(p+1)},$$

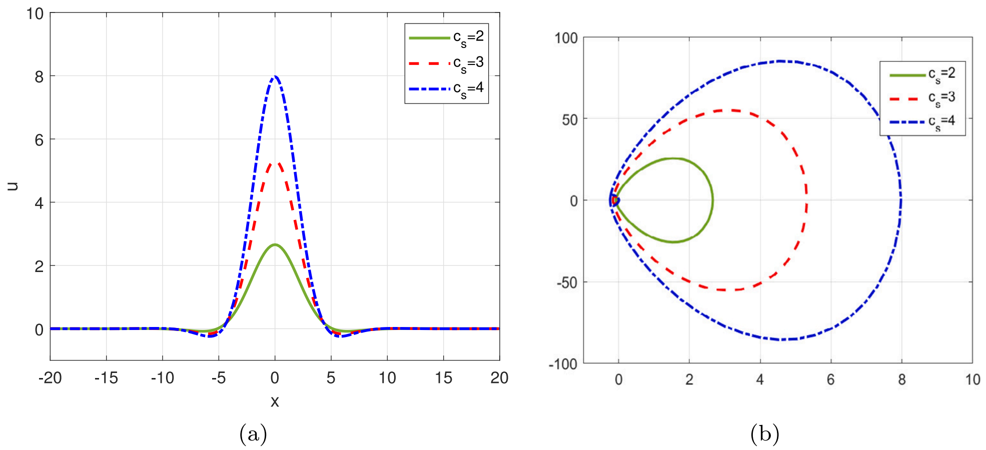


**Fig. 7.** Generation of nonmonotone CSW's, $u$ profiles and phase portraits. Rosenau equation with $\epsilon = \beta = 1, g(u) = u^2/2$ and for several speeds $c_s > \epsilon$.

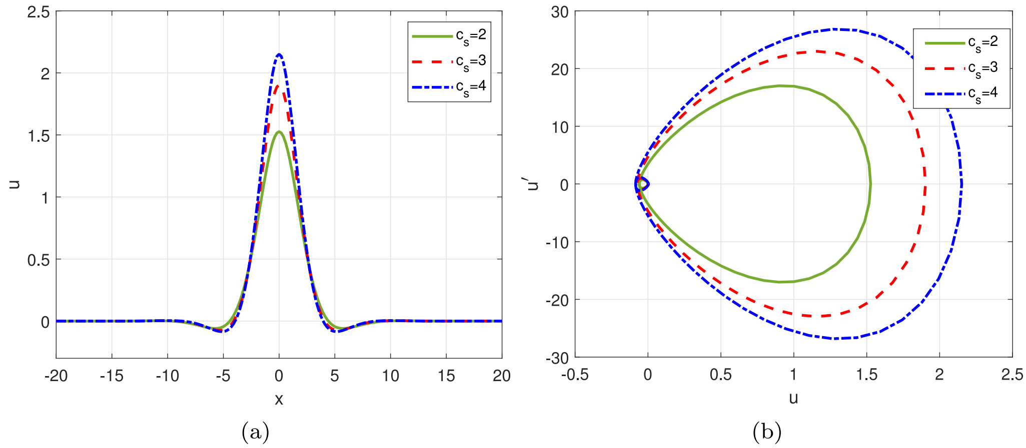


**Fig. 8.** Generation of nonmonotone CSW's, $u$ profiles and phase portraits. Rosenau equation with $\epsilon = \beta = 1, g(u) = u^3/3 + u^5/5$ and for several speeds $c_s > \epsilon$.

and consequently $G$ is homogeneous of degree $q = p + 2$. Thus

$$F(u) = \int_{\mathbb{R}} \frac{u^{p+2}}{(p+2)(p+1)} dx.$$

The corresponding conditions on the speed $c_s$ for coercivity property will be derived in each case from the characterization given in Proposition 4.1. In all the cases, the CSW's predicted from CCT are nonmonotone and were partially anticipated by NFT.

#### 4.1.1. Rosenau equation

Rosenau equation corresponds to (1.1) with $\alpha = \eta = \gamma = 0, \epsilon, \beta > 0$. Then

$$E(u) = \int_{\mathbb{R}} \left((c_s - \epsilon)u^2 + (c_s\beta)u_{xx}^2\right) dx.$$

In this case the roots in (4.6) are $x_+ = 0, x_- = -\epsilon$. Therefore, (4.7) is satisfied for $c_s > 0$ only when $c_s - \epsilon > 0$. In this case the application of NFT is special, since from (2.11), $a = a(c_s) = \frac{c_s - \epsilon}{\beta c_s}$, $b = 0$, and the eigenvalues of the linearization are the roots of $\lambda^4 + a = 0$. The existence of (nonmonotone) CSW's for speeds $c_s > \epsilon$ with $c_s - \epsilon$ small can be established (region 1 close to $C_3$ in Fig. 1). The existence result obtained from Concentration-Compactness theory is valid for $c_s > \epsilon$ with no restriction on the size of $c_s - \epsilon$. The numerical generation of some of the profiles is illustrated in Fig. 7. For $c_s < \epsilon$ (region 3 with $b = 0$) PTW's were found experimentally.

It may be worth mentioning that when $g(u)$ is a sum of homogeneous functions, the application of Concentration-Compactness theory to (1.1) can follow the steps developed in [27]. For the Rosenau equation, the corresponding result would ensure the existence of solitary waves under the same restriction $c_s > \epsilon$ on the speeds. This is illustrated in Fig. 8 for $g(u) = u^3/3 + u^5/5$.

In both Figs. 7 and 8, we observe that the amplitude of the waves is an increasing function of the speed and, from the phase portraits, the profiles decay to zero exponentially as $|X| \to \infty$ in an oscillatory way.

*4.1.2. Rosenau-RLW equation*

Rosenau-RLW equation corresponds to (1.1) with $\eta = \gamma = 0, \epsilon, \beta > 0$. Then

$$E(u) = \int_{\mathbb{R}} \left( (c_s - \epsilon)u^2 - \alpha c_s u_x^2 + (c_s\beta)u_{xx}^2 \right) dx.$$

In this case

$$x_+ = \frac{\alpha^2\epsilon}{4\beta - \alpha^2}, \quad x_- = -\epsilon,$$

and for speeds $c_s > 0$(4.7) holds when

$$c_s - \epsilon > \frac{\alpha^2\epsilon}{4\beta - \alpha^2},$$

which was partially anticipated by NFT (cf. Table 1) but now $\alpha \in \mathbb{R}$ and the proximity between the two values $c_s - \epsilon$ and $\frac{\alpha^2\epsilon}{4\beta-\alpha^2}$ is not required.

*4.1.3. Rosenau-KdV equation*

Rosenau-KdV equation corresponds to (1.1) with $\alpha = \gamma = 0, \epsilon, \beta > 0$. Then

$$E(u) = \int_{\mathbb{R}} \left( (c_s - \epsilon)u^2 + \eta u_x^2 + (c_s\beta)u_{xx}^2 \right) dx.$$

In this case

$$x_\pm = \frac{1}{2}\left( -\epsilon \pm \sqrt{\epsilon^2 + \frac{\eta^2}{\beta}} \right),$$

and (4.7) holds when $c_s - \epsilon > x_+$ or $c_s - \epsilon < x_-$. Note that $x_- < -\epsilon$, so for $c_s > 0$ only the first condition is possible, and this is the same as that in Table 2 without the restrictions on the sign of $\eta$ and on the size of the difference between $c_s - \epsilon$ and $x_+$.

*4.1.4. Rosenau-Kawahara equation*

Rosenau–Kawahara equation corresponds to (1.1) with $\alpha = 0, \epsilon, \beta > 0$. Then

$$E(u) = \int_{\mathbb{R}} \left( (c_s - \epsilon)u^2 + \eta u_x^2 + (c_s\beta - \gamma)u_{xx}^2 \right) dx.$$

We may write $E$ in the form

$$E(u) = \int_{\mathbb{R}} \left( (c_s - \epsilon)2u^2 + \eta u_x^2 + ((c_s - \epsilon)\beta + \rho)u_{xx}^2 \right) dx,$$

with $\rho = \epsilon\beta - \gamma$. In this case

$$x_\pm = \frac{1}{2}\left( -\frac{\lambda}{\beta} + \sqrt{\left(\frac{\lambda}{\beta}\right)^2 + \frac{\eta^2}{\beta}} \right).$$

Hence, as before, CCT gives the same conditions on the speed $c_s$ to ensure the existence of CSW as some given by NFT, cf. Table 3, for the case of nonmonotone CSW's, now independently of the sign of $\eta$ and $\rho$, and of the difference between $c_s - \epsilon$ and $x_\pm$.

*4.1.5. Rosenau-RLW-Kawahara equation*

Rosenau-RLW-Kawahara equation corresponds to (1.1) with $\alpha = \gamma = -1, \epsilon = \beta = 1, \eta > 0$. Then

$$E(u) = \int_{\mathbb{R}} \left( (c_s - \epsilon)u^2 + (\eta + c_s)u_x^2 + (c_s\beta + 1)u_{xx}^2 \right) dx.$$

Now

$$x_\pm = \frac{1}{2}\left( \frac{2(\eta-3)}{3} \pm \sqrt{\left(\frac{2(\eta-3)}{3}\right)^2 + \frac{4(1+\eta)^2}{3}} \right).$$

Since $x_- < -\epsilon$ then, for $c_s > 0$(4.7) holds only when $c_s - \epsilon > x_+$, and the existence of nonmonotone CSW's can be ensured in the corresponding region 1 (right) of Fig. 1 but not necessarily close to the curve $C_3$. By way of illustration, approximations of some of these profiles are shown in Fig. 9.

We note that for nonlinear terms $g$ of the form

$$g(u) = g(u, u_x, u_{xx}),$$

the Concentration-Compactness theory for (1.1) can also be analyzed using the approach in [27]. Some of the approximate profiles for this case are given in Fig. 10.

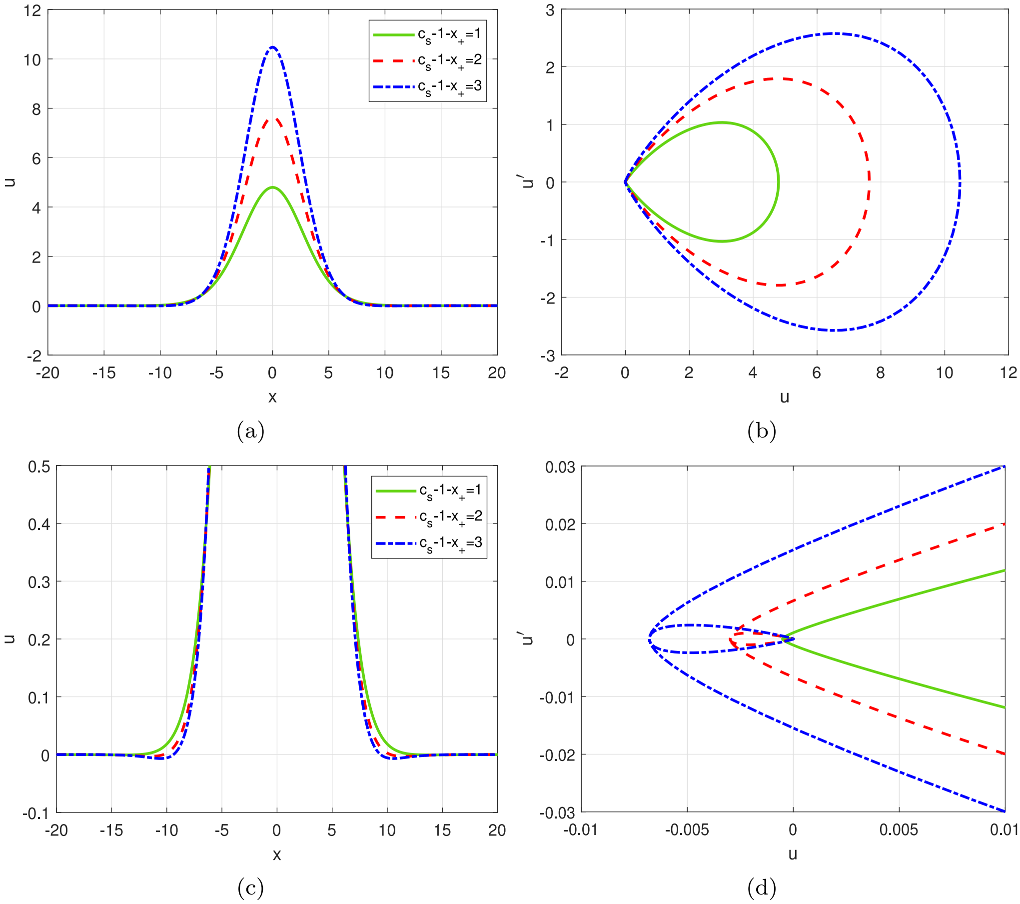


**Fig. 9.** Generation of NMCSW's, $u$ profiles and phase portraits. Rosenau-RLW-Kawahara equation with $\eta = 1, g(u) = u^2/2$ and several speeds $c_s$; (c) and (d) are magnifications of (a) and (b) resp.

## 5. Concluding remarks

The present paper is focused on the existence of solitary-wave solutions of equations of Rosenau type of the form (1.1), which involve different generalizations of the Rosenau equation derived in [2]. After the study, in Section 2, of some mathematical properties of the corresponding ivp, such as well-posedness, conserved quantities and Hamiltonian formulation, we apply two standard theories in order to obtain existence results of solitary-wave solutions. In Section 3, for two types of nonlinearities, the equation for the solitary waves (2.9) is written as a reversible first-order system (3.6) and where the Normal Form Theory (NFT) is applied. The derivation and analysis of a normal form close to suitable bifurcation curves reveal the existence of classical solitary waves (CSW's) of two types (positive/negative and with nonmonotone decay) as well as generalized solitary waves (GSW's). The general conclusions are then applied to several families of Rosenau-type equations, with the purpose of identifying the range of speeds which, depending on the parameters of the equation under study, ensures the existence of each type of solitary wave. A more detailed description is made in the extended version [29].

Section 4 is devoted to the use of Concentration-Compactness Theory (CCT) with the aim at obtaining additional results on the existence of CSW's and which may extend those from NFT. The corresponding theorem establishes the formation of such waves under a general condition on the speed and ensuring the coercivity property of the minimized functional, a key feature for the successful application of CCT. The resulting restriction on the speed is analyzed for the above mentioned families of Rosenau equations. The main conclusions are that the corresponding results of existence do extend those obtained with NFT for CSW's with nonmonotone decay, in the sense of allowing a wider range of values of velocities which in the case of NFT is somehow restricted because of the local character of this theory.

In both Sections 3 and 4, the existence results are illustrated with the numerical generation of approximate CSW and GSW solutions, as well as some additional periodic traveling wave (PTW) solutions. The numerical procedure, described and checked in [29], consists of discretizing the fourth-order Eq. (2.9) for the solitary-wave profiles on a long enough interval and periodic boundary conditions with a Fourier collocation scheme, and solving the algebraic system for the discrete Fourier coefficients of the approximation (in the Fourier space) using the Petviashvili iteration. The performance of the method, checked with several

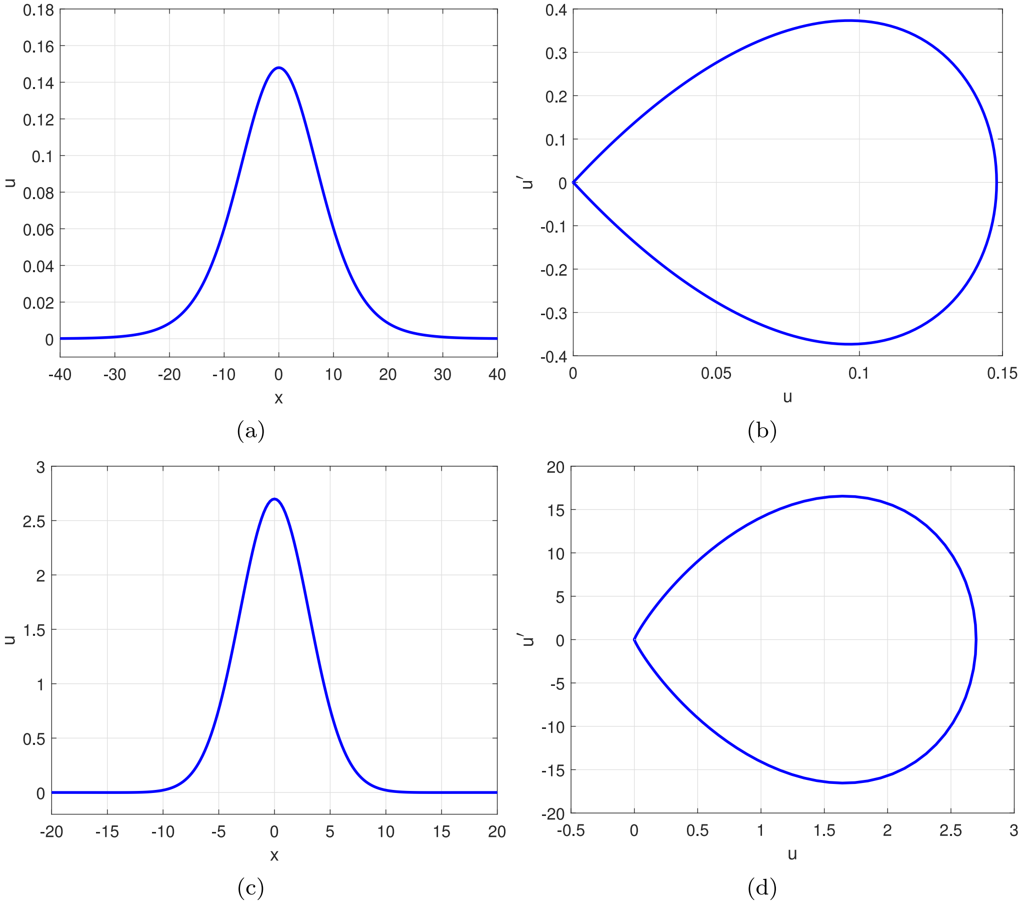


**Fig. 10.** Generation of CSW's, $u$ profiles and phase portraits. Rosenau-RLW-Kawahara equation with $\eta = 1, g(u) = u^2 + u_x^2 + uu_{xx}$ and speeds (a), (b) $c_s = 1.1$; (c), (d) $c_s = 2.9$.

examples in [29], guarantees an accuracy of the computations which enables to suggest some additional conclusions on the behavior of the waves. The main properties concern the CSW's, that seem to decay to zero exponentially (in a monotone or oscillating way, depending on the type of CSW) and whose amplitude seems to be an increasing function of the speed.

We believe that the study performed in the present paper gives a complete enough picture about the existence of solitary-wave solutions of Rosenau-type equations. This will also serve as motivation for a second part, developed in a future research, concerning the dynamics of the waves. It is our purpose to design efficient discretizations of the equations that may suggest, from a computational point of view, some conclusions on different stability issues.

## CRediT authorship contribution statement

**Angel Durán:** Conceptualization, Formal analysis, Investigation, Methodology, Writing – original draft. **Gulcin M. Muslu:** Conceptualization, Formal analysis, Investigation, Methodology, Writing – original draft.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

## References


[1] Champneys AR. Homoclinic orbits in reversible systems and their applications in mechanics, fluids and optics. Phys D 1998;112:158–86.
[2] Rosenau P. Dynamics of dense discrete systems: High order effects. Progr Theoret Phys 1988;79:1028–42.
[3] Park MI. On the Rosenau equation in multidimensional space. Nonlinear Anal.: Theory, Methods and Applications 1993;21:77–85.
[4] Demirci A, Hasanoğlu Y, Muslu GM, Özemir C. On the Rosenau equation: Lie symmetries, periodic solutions and solitary wave dynamics. Wave Motion 2022;109:102848.
[5] Erbay HA, Erbay S, Erkip A. Numerical computation of solitary wave solutions of the Rosenau equation. Wave Motion 2020;98:102618.
[6] Zuo JM, Zhang YM, Zhang TD. A new conservative difference scheme for the general Rosenau-RLW equation. Bound Value Probl 2010;13:13.
[7] Esfahani A, Pourgholi R. Dynamics of solitary waves of the Rosenau-RLW equation. Differ Equ Dyn Syst 2014;22(1):93–111.
[8] Zuo J-M. Solitons and periodic solutions for the Rosenau-KdV and Rosenau-Kawahara equations. Appl Math Comput 2006;215:150–64.
[9] Esfahani A. Solitary wave solutions for generalized Rosenau-KdV equations. Commun Theor Phys (Beijing) 2011;55:396–8.
[10] He D. New solitary solutions and a conservative numerical method for the Rosenau-Kawahara equation with power law nonlinearity. Nonlinear Dynam 2015;82:1177–90.
[11] He D. Exact solitary solution and a three-level linearly implicit conservative finite difference method for the generalized Rosenau-Kawahara-RLW equation with generalized Novikov type perturbation. Nonlinear Dynam 2016;85:479–98.
[12] Sukantamala N, Nanta S. On solitary wave solutions for the Camassa–Holm and the Rosenau-RLW-Kawahara equations with the dual-power law nonlinearities. Abstr Appl Anal 2021;ID6649285. http://dx.doi.org/10.1155/2021/6649285.
[13] Zeng L. Existence and stability of solitary-wave solutions of equations of Benjamin–Bona–Mahony type. J Differential Equations 2003;188:1–32.
[14] Lions PL. The concentration-compactness principle in the calculus of variations. The locally compact case. Part I and part ii.. Ann Inst Henri Poincaré Sect A (N.S.) 1984;1. pp. 109–145 and pp. 223–283.
[15] Petviashvili VI. Equation of an extraordinary soliton. Soviet J Plasma Phys 1976;2:257–8.
[16] Grillakis M, Shatah J, Strauss WA. Stability of solitary waves in the presence of symmetry: I. J Funct Anal 1987;74:170–97.
[17] He D, Pan K. A linearly implicit conservative difference scheme for the generalized Rosenau-Kawahara-RLW equation. Appl Math Comput 2015;271:323–36.
[18] Shi D, Jia X. Convergence analysis of the Galerkin finite element method for the fourth-order Rosenau equation. Appl Math Lett 2023;135:108432.
[19] Haragus M, Iooss G. Local bifurcations, center manifolds, and normal forms in infinite-dimensional dynamical systems. Heidelberg New York: Springer London Dordrecht; 2011.
[20] Iooss G, Adelmeyer M. Topics in bifurcation theory and applications. 2nd ed.. Singapore: World Scientific; 1999.
[21] Roy Choudhury S. Solitary-wave families of the Ostrovsky equation: An approach via reversible systems theory and normal forms. Chaos Solitons Fractals 2007;33:1468–79.
[22] Iooss G. A codimension 2 bifurcation for reversible vector fields. Fields Inst Commun 1995;4:201–17.
[23] Pelinovsky DE, Stepanyants YA. Convergence of petviashvili's iteration method for numerical approximation of stationary solutions of nonlinear wave equations. SIAM J Numer Anal 2004;42:1110–27.
[24] Álvarez J, Durán A. An extended petviashvili method for the numerical generation of traveling and localized waves. Comm Nonlinear Sci Numer Simul 2014;19:2272–83.
[25] Álvarez J, Durán A. Petviashvili type methods for traveling wave computations: II. Acceleration with vector extrapolation methods. Math Comput Simulation 2016;123:19–36.
[26] Angulo-Pava J, Saut J-C. Existence of solitary wave solutions for internal waves in two-layer systems. Quart Appl Math 2020;78:75–105.
[27] Esfahani A, Levandosky S. Existence and stability of traveling waves of the fifth-order KdV. Phys D 2021;421:132872, equation.
[28] Dougalis VA, Durán A, Saridaki L. On solitary-wave solutions of Boussinesq/Boussinesq systems for internal waves. Phys D 2021;428:133051.
[29] Durán A, Muslu GM. Notes on solitary-wave solutions of Rosenau-type equations, Preprint available at http://arxiv.org/abs/2403.06958.
[30] Levandosky S. A stability analysis of fifth-order water wave models. Phys D 1999;125:222–40.
[31] Kabakouala A, Molinet L. On the stability of the solitary waves to the (generalized) Kawahara equation. J Math Anal Appl 2018;457:478–97.
[32] Albert JP, Bona JL. Comparisons between model equations for long waves. J Nonlinear Sci 1991;1:345–74.
[33] Bona JL, Chen M, Saut J-C. Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media: II. The nonlinear theory. Nonlinearity 2004;17:925–52.
[34] Lombardi E. Homoclinic orbits to small periodic orbits for a class of reversible systems. Proc Roy Soc Edinburgh A 1996;126:1035–54.
[35] Lombardi E. Homoclinic orbits to exponentially small periodic orbits for a class of reversible systems: Application to water waves. Arch Ration Mech Anal 1997;137:227–304.
[36] Lombardi E. Oscillatory integrals and phenomena beyond all algebraic orders. Berlin-Heidelberg: Springer-Verlag; 2000.
[37] Angulo-Pava J. Nonlinear dispersive equations. existence and stability of solitary and periodic travelling wave solutions. Providence, Rhode Island: Amer. Math. Soc.; 2009.
[38] Champneys AR, Spence A. Hunting for homoclinic orbits in reversible systems: A shooting technique. Adv Comput Math 1993;1:81–108.
[39] Champneys AR, Toland JF. Bifurcation of a plethora of multi-modal homoclinic orbits for autonomous Hamiltonian systems. Nonlinearity 1993;6:665–772.
[40] Iooss G, Kirchgässner K. Water waves for small surface tension: an approach via normal form. Proc. Roy. Soc. Edinburgh A 1992;112:200–67.
[41] Arnold V. Geometrical methods in the theory of ordinary differential equations. Springer; 1983.
[42] Devaney RL. Homoclinlc orbits in Hamiltonian systems. J Differential Equations 1976;21:431–8.
[43] Belyakov LA. Bifurcation of systems with homoclinic curve of a saddle-focus with saddle quantity zero. Mat Zametki 1984;36:838–43.
[44] Wiggins S. Introduction to applied nonlinear dynamical systems and chaos. Springer; 1990.
[45] Bona JL, Li YA. Decay and analyticity of solitary waves. J Math Pures Appl 1997;76:377–430.
[46] Iooss G, Peroueme MC. Perturbed homoclinic solutions in reversible 1 : 1 resonance vector fields. J Differential Equations 1993;102:62–88.
[47] Weinstein MI. Existence and dynamic stability of solitary wave solutions of equations arising in long wave propagation. Comm Partial Differential Equations 1987;12:1133–73.