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Financial boundary conditions in a continuous model with discrete-delay for pricing commodity futures and its application to the gold market

Lourdes Gómez-Valle [a], Miguel Ángel López-Marcos [b], Julia Martínez-Rodríguez [a,*]

[a] Departamento de Economía Aplicada e IMUVA, Facultad de Ciencias Económicas y Empresariales, Universidad de Valladolid, Spain
[b] Departamento de Matemática Aplicada e IMUVA, Facultad de Ciencias, Universidad de Valladolid, Spain

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ABSTRACT

In this work, we approach the solution of a differential problem for pricing commodity futures when the spot price follows a stochastic diffusion process with memory, that is, it depends on two discrete times: the present instant and a delayed one. In this kind of models, a closed-form solution is not feasible to obtain and, in most of the cases, numerical methods should be applied. To this end, it is normal to introduce a bounded domain for the state variable, so suitable boundary conditions have to be established. The conditions based on mathematical reasons often introduce difficulties in the boundary and poor accuracy. Here, we propose new nonstandard boundary conditions based on some financial reasons and then, we face the numerical solution of the problem that arises. Some experiments are presented which show that the drawbacks in the behavior of the solutions are overcome, providing more accurate futures prices. This new procedure is implemented in order to obtain a more precise valuation of gold futures contracts traded on the Commodity Exchange Inc. (US).

1. Introduction

Commodities are a part of our daily life. They include energy sources such as crude oil and natural gas; metals such as gold and silver; agricultural products such as wheat and coffee; and livestock and meat products such as pork and cattle. Furthermore, investing in commodities, as an asset class, is a way to diversify an investment portfolio [1].

Investors can gain exposure to commodities through direct investment in them or through commodity derivatives. On the one hand, a direct investment in a commodity provides exhibition to the performance of its spot price. However, it involves taking the physically holding or delivery of the commodity [2]. On the other hand, commodity derivative contracts allow us to obtain profits from changes in the commodity price without owning it. Options and futures are the most famous commodity derivatives, both having their own particularities. While in a future contract the buyer is obliged to buy the underlying commodity, in an option the buyer has the right but not the obligation. Furthermore, commodity futures markets offer transparency in the price mechanism, low margins, risk management and an organized marketplace which makes them very attractive for investors. Therefore, commodity markets are of fundamental importance and a very decisive part of the global economy. A detailed list and characteristics of the main commodity exchanges markets in the world can be founded in [3].

Valuing commodity derivatives requires a deep knowledge of the behavior of the underlying commodity price, which is different from other financial assets. For example, as documented in [4,5], commodity prices show strong evidence of mean reversion as producers and consumers adapt their production and consumption decisions in the long term. Moreover, many commodity markets are characterized by exceptional abrupt changes in the price because of different factors: weather conditions, unanticipated macroeconomic events, etc. Then, some researchers have considered a jump term, as well as the mean reversion, when modeling the commodity spot price to value commodity futures (see [6–8]). Prior studies [9,10], have reported that commodity prices show a seasonal behavior, specially some agricultural and energetic products (see, for example, [11–14]).

In order to price commodity futures contracts, standard non-arbitrage arguments for valuing financial derivatives allow to obtain the futures price as the expectation under the risk-neutral measure of the commodity spot price at maturity. Nevertheless, instead of calculating this expectation, an alternative procedure arises that consist in solving an equivalent partial differential equation (PDE) subject to a final condition (see [15]).

In most of the cases, the dynamics of the commodity spot price is chosen with the aim of building a tractable and easy-to-implement

* Corresponding author.
E-mail addresses: lourdes.gomez@uva.es (L. Gómez-Valle), malm@uva.es (M.Á. López-Marcos), juliamr@uva.es (J. Martínez-Rodríguez).

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model (see [7,13,16]). These restrictive properties have made this kind of models very popular and standard in commercial applications. However, on the one hand, they fail to incorporate the most realistic features of the commodity prices which could improve the futures valuation. In this paper, we consider the valuation problem in the more general case, where the resulting equation is not so easily tractable. On the other hand, considering that the spot price process depends only on its present value does not adjust to reality. Thus, this study proposes further credibility adding the dependence on its past value.

If a more realistic model is considered, usually a closed-form solution is not known. As a result, numerical methods would be necessary to provide approximated solutions. In previous studies, this problem has usually solved using the Monte Carlo (MC) simulation approach (as in [8,14]), although it is not very efficient due to its low accuracy and high computational cost. An alternative choice is considering discretization techniques for the PDE, such as finite difference methods [17]. In this particular case, one of the main issues is that a bounded domain for the state variables must usually be considered. Hence, suitable boundary conditions for the emerging problems have to be established (see [18,19], for pricing renewable energy certificate derivatives, and [20,21], for mining projects). However, these boundary conditions are sometimes arbitrarily chosen without taking into account the financial characteristics of the derivative to valuate, giving as a result poor approximations. In [22], we face this problem for zero-coupon bond models by assuming boundary conditions specially adapted to this derivative. Nevertheless, the literature on the commodity futures shows a lack of research in this issue.

In this paper, we introduce specific boundary conditions, based on financial arguments, for the numerical pricing of commodity futures contracts with and without memory, when a closed-form solution is not known. The incorporation of these boundary requirements will be the previous step for the numerical treatment of the model. The suitability of these conditions will be exemplified by considering a second order discretization of them, coupled to the well-known Crank–Nicolson discretization technique for the PDE. Our test problem confirms that dealing with this numerical method which incorporates these novel restrictions offers a competitive choice for pricing. In particular, it reveals as a very valuable tool in its application to the gold market.

Other approximation formulas for these financial boundary conditions could be proposed. Furthermore, the problem posed on the bounded interval can be addressed using other PDE discretization techniques, other than the Crank–Nicolson method, which might be more suitable for its numerical integration. Finally, the boundary conditions proposed here, together with their designed discretization, are consistent with other numerical techniques for the PDE problem such as the method of lines.

The remainder of this work is organized as follows. In Section 2, we describe one-factor commodity futures pricing models and their corresponding valuation problems with and without delay in the stochastic process. In Section 3, we deal with the discretization of the non-delayed problem. In Section 3.1 we incorporate some widely used boundary conditions, and show the adverse effects appearing in the approximation of the solution to a test problem based on the Schwartz model. In Section 3.2, we propose novel boundary conditions which take into account the behavior of the commodity futures contracts. We present a discretization of the corresponding problem and validate the effectiveness of this technique by means of some numerical experiments. In Section 4, we go one step further, addressing the problem with delay, by relating its treatment to the previous case without memory. In Section 5, we carry out a valuation of the gold futures contracts, traded on the Commodity Exchange Inc. (COMEX), by means of the new procedure. We compare the results with those obtained with the MC method. Section 6 concludes.

2. Pricing commodity futures

In this section, we present a framework to price commodity futures with a single state variable which follows a stochastic diffusion process (with and without delay).

Firstly, we assume that the commodity spot price S follows the below stochastic differential equation (SDE), which is the usual dynamics in the one-factor case:

$$dS(t) = \hat{\mu}(S(t))\,dt + \sigma(S(t))\,dW(t), \quad t \in (0,T],$$

where $\hat{\mu}$ and σ are deterministic continuous functions which depend on the spot price and are, respectively, the drift and volatility of the process, verifying suitable regularity conditions [23], whereas W is a Wiener process. Here, we consider $(\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}_{t\geq 0})$ a filtered probability space satisfying the usual conditions [23,24].

As we consider that the market is arbitrage-free, there exists a martingale measure $\mathcal{Q}$, equivalent to $\mathcal{P}$, which is known as the risk-neutral measure [15]. Thus, the commodity spot price S, under the $\mathcal{Q}$-measure, follows the SDE

$$dS(t) = \mu(S(t))\,dt + \sigma(S(t))\,dW^{\mathcal{Q}}(t), \quad t \in (0,T], \tag{1}$$

where $W^{\mathcal{Q}}$ is a Wiener process, μ is the new drift, and the volatility does not change with the measure.

Commodity futures are financial derivative contracts that obligate the buyer to purchase, and the seller to sell, a commodity at a predetermined future date (delivery date) and price (delivery price) [25]. Then, the futures price is defined as the delivery price which makes the initial value of the non-arbitrage futures price zero. Therefore, no initial investment is required for the futures contracts.

The price at time t, of a commodity futures maturing at time T (with $t \leq T$), under the above assumptions, will be expressed as $F(S,t;T)$. This futures contract is assumed to have a maturity value equal to the commodity spot price, that is,

$$F(S,T;T) = S. \tag{2}$$

On the one hand, by means of standard non-arbitrage arguments [15], the futures price at time t can be expressed as

$$F(S,t;T) = E^{\mathcal{Q}}\left[S(T)|\mathcal{F}_t\right], \tag{3}$$

where $E^{\mathcal{Q}}$ denotes the conditional expectation under the $\mathcal{Q}$-measure.

On the other hand, the Feynman–Kac theorem [15], allows us to obtain the commodity futures price (3) as the solution of the following PDE

$$F_t(S,t) + \mu(S)\,F_S(S,t) + \frac{1}{2}\sigma^2(S)\,F_{SS}(S,t) = 0, \qquad S>0, \quad 0 \leq t < T. \tag{4}$$

Here, in order to obtain the futures price, we have to solve the final value problem (4), (2). To this end, as usual, a change on the time variable is introduced, $\tau = T - t$, which transforms the original problem into the following initial value problem

$$F_\tau(S,\tau) = \mu(S)\,F_S(S,\tau) + \frac{1}{2}\sigma^2(S)\,F_{SS}(S,\tau), \qquad S>0, \quad 0 < \tau \leq T, \tag{5}$$

$$F(S,0) = S, \qquad S>0. \tag{6}$$

However, the need for improving the valuation of the commodity futures has motivated us to take into account the influence of past events as well as present ones on the current and future prices. Then, it seems natural to consider a commodity spot price process where the drift, the volatility, or both functions depend on the past spot price. That is, we assume that the commodity spot price follows a stochastic delay differential equation (SDDE) as (see [26])

$$dS(t) = \hat{\mu}(S(t), S(t-\delta))\,dt + \sigma(S(t), S(t-\delta))\,dW(t), \quad t \in (0,T], \tag{7}$$

$$S(t) = \phi(t), \quad t \in [-\delta, 0], \tag{8}$$

where δ is a positive constant (a fixed delay) which is incorporated into the drift and volatility terms in order to take into account past events. Again, $\hat{\mu}$ and σ are deterministic continuous functions (the drift and volatility of the process, respectively), satisfying suitable regularity conditions (see [23]). Therefore, these functions depend on the spot price in two instants of time: t and $t-\delta$. Here, W is a Wiener process and $\phi \in C([-\delta,0],\mathbb{R})$, in (8), represents previous known price information. Again, as we assume that the market is arbitrage-free, there exists an equivalent martingale measure (the risk-neutral measure). Then, we consider that the spot price, under this risk-neutral measure Q, follows the process:

$$dS(t) = \mu(S(t), S(t-\delta))\,dt + \sigma(S(t), S(t-\delta))\,dW^Q(t), \quad t \in (0,T], \tag{9}$$

$$S(t) = \phi(t), \quad t \in [-\delta, 0], \tag{10}$$

where W^Q is a Wiener process and μ is the new drift.

Then, as in the previous case, the commodity futures price is given by the conditional expectation (3) or, by means of a riskless strategy (see [26]), we can obtain it as the solution of the following PDE

$$F_t + \mu(S, S(t-\delta))F_S + \frac{1}{2}\sigma^2(S, S(t-\delta))F_{SS} = 0, \qquad S > 0, \quad 0 \le t < T,$$

with the final condition (2). Note that the function $S(t-\delta)$, in the μ and σ coefficients, follows a stochastic process so, following [27], the PDE can be labeled as a random partial differential equation (RPDE).

As we did in (4) to reach (5), we introduce the change on the time variable $\tau = T - t$, and obtain the equivalent RPDE

$$\begin{aligned} F_\tau =& \mu(S, S(T-\tau-\delta))\,F_S + \frac{1}{2}\sigma^2(S, S(T-\tau-\delta))\,F_{SS}, \\ & S > 0, \quad 0 < \tau \le T. \end{aligned} \tag{11}$$

3. Numerical solution of the model without delay

In this section, we deal with the numerical solution of the problem (5)–(6). When numerical methods are used to approximate the solution of the model, it is usually restricted to a suitable bounded domain of the state variable. Thus, the incorporation of appropriate boundary conditions is mandatory. In the commodity derivatives pricing literature, several boundary conditions have been considered. For example, Baamonde-Seoane et al. [18] assume that the first derivative with respect to one of the state variables is zero on the boundary, while Aminrostamkolaee et al. [21] consider that the second derivative with respect to the spot price is canceled. However, these artificial conditions are not based on financial but on mathematical reasons and, as we will show, they are not appropriate for pricing commodity futures.

Then, for the numerical solution of the pricing problem by means of a finite difference method, a suitable bounded interval $[S_{min}, S_{max}]$ for the spot price variable S must be introduced. To adequately represent the dynamics of the problem, this interval should contain a sufficiently large number of market observations to provide an accurate estimate of the model functions, but adjusting its size to avoid biased estimates. Once this interval has been set, appropriate constrains on the solution at the boundary should be detailed.

3.1. Artificial boundary conditions

As used in [21] for other derivatives, we impose that $F_{SS} = 0$ on the boundary (note that the initial data in (6) verifies it). Then, we consider the initial boundary value problem consisting of (5)–(6) on (S_{min}, S_{max}), together with

$$F_{SS}(S_{min}, \tau) = 0, \quad 0 \le \tau \le T, \tag{12}$$

$$F_{SS}(S_{max}, \tau) = 0, \quad 0 \le \tau \le T. \tag{13}$$

In order to discretize this problem, firstly we introduce a uniform grid on the state variable interval $[S_{min}, S_{max}]$: given a positive integer J, we use the constant step $h = (S_{max} - S_{min})/J$ to define the discrete prices $S_j = S_{min} + j\,h$, $j = 0, 1, \dots, J$. Secondly, we introduce a uniform grid on the time variable interval $[0, T]$: given a positive integer N, we define the time step $k = T/N$, and the discrete time levels $\tau_n = n\,k$, $n = 0, 1, \dots, N$. Finally, F_j^n, represents an approximation to $F(S_j, \tau_n)$, $j = 0, 1, \dots, J$, $n = 0, 1, \dots, N$.

A suitable approximation to the initial condition of the problem must be taken as initial data of the numerical method. For the numerical simulation, we choose the grid restriction of (6), that is,

$$F_j^0 = S_j, \qquad j = 0, 1, \dots, J. \tag{14}$$

For $n = 1, \dots, N$, at the inner grid points of the interval $[S_{min}, S_{max}]$, we consider the discretization of the PDE (5) based on the second-order Crank–Nicolson scheme: denoting $\mu_j = \mu(S_j)$ and $\sigma_j = \sigma(S_j)$, then

$$\begin{aligned} \frac{F_j^n - F_j^{n-1}}{k} &= \frac{1}{2}\mu_j \left[\frac{F_{j+1}^n - F_{j-1}^n}{2h} + \frac{F_{j+1}^{n-1} - F_{j-1}^{n-1}}{2h} \right] \\ &+ \frac{1}{4}\sigma_j^2 \left[\frac{F_{j+1}^n - 2F_j^n + F_{j-1}^n}{h^2} + \frac{F_{j+1}^{n-1} - 2F_j^{n-1} + F_{j-1}^{n-1}}{h^2} \right], \\ j &= 1, 2, \dots, J-1. \end{aligned} \tag{15}$$

Now, we discretize the boundary conditions (12) and (13) by means of second order approximations to F_{SS}: forward finite differences on S_{min}, and backward on S_{max}. In this way, we obtain, for $n = 1, \dots, N$,

$$2F_0^n + 5F_1^n + 4F_2^n - F_3^n = 0, \tag{16}$$

$$2F_J^n + 5F_{J-1}^n + 4F_{J-2}^n - F_{J-3}^n = 0. \tag{17}$$

Denoting as $\mathbf{F}^n = [F_0^n, \dots, F_J^n]^T$, the $(J+1)$-dimensional vector that collects the approximations to the solution over the spot price grid at time τ_n, $n = 0, \dots, N$, we can write the numerical method described by (14)–(17), in vectorial form: starting from the initial approximation $\mathbf{F}^0 = \mathbf{S} = [S_0, \dots, S_J]^T$, we have to solve the following linear system to obtain $\mathbf{F}^n$ from $\mathbf{F}^{n-1}$:

$$M\mathbf{F}^n = P\mathbf{F}^{n-1}, \quad n = 1, \dots, N, \tag{18}$$

where M and P are tridiagonal matrices, except the first and the last rows which represent the discretizations of the boundary conditions. Denoting $c = k/h$ and $d = k/h^2$, the matrices M and P are given in Box I.

Note that the numerical method represented by (18) describes a one-step linearly-implicit method. The matrix of the system that must be solved at each time step is always the same, which allows us to implement the method in an efficient way. However, this simple and cheap procedure exhibits failures in the numerical approach of the futures price.

In order to illustrate that the obtained approximation is not satisfactory, we consider a test problem with an affine stochastic process for the spot dynamics, which provides an explicitly solvable model. As a particular case, we look at the very well-known Schwartz model [16], where the spot price follows the diffusion process (1) with

$$\mu(S) = \alpha(\mu - \ln S)S, \tag{19}$$

$$\sigma(S) = \sigma S, \tag{20}$$

where μ, α and σ are constant. In this case, the solution to the equivalent problem (5)–(6) is

$$F(S,\tau) = \exp\left[e^{-\alpha\tau} \ln S + \left(\mu - \frac{\sigma^2}{2\alpha} \right)(1 - e^{-\alpha\tau}) + \frac{\sigma^2}{4\alpha}\left(1 - e^{-2\alpha\tau}\right) \right]. \tag{21}$$

Note that the previous assumptions about the functions of the spot price process (19) and (20) can be considered very restrictive and, as a consequence, the model could provide prices very different from those observed in the markets. The performance of these models may be improved by means of more general functions for the spot price (for example, those obtained with nonparametric techniques). Nonetheless, in most of the cases, a feasible expression of the solution of the initial

$$M=\begin{bmatrix} 2 & 5 & 4 & -1 & \cdots & 0\\ \frac{1}{4}(c\mu_1-d\sigma_1^2) & 1+\frac{1}{2}d\sigma_1^2 & -\frac{1}{4}(c\mu_1+d\sigma_1^2) & 0 & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \frac{1}{4}(c\mu_{J-1}-d\sigma_{J-1}^2) & 1+\frac{1}{2}d\sigma_{J-1}^2 & -\frac{1}{4}(c\mu_{J-1}+d\sigma_{J-1}^2)\\ 0 & \cdots & -1 & 4 & 5 & 2 \end{bmatrix},$$

$$P=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0\\ -\frac{1}{4}(c\mu_1-d\sigma_1^2) & 1-\frac{1}{2}d\sigma_1^2 & \frac{1}{4}(c\mu_1+d\sigma_1^2) & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & -\frac{1}{4}(c\mu_{J-1}-d\sigma_{J-1}^2) & 1-\frac{1}{2}d\sigma_{J-1}^2 & \frac{1}{4}(c\mu_{J-1}+d\sigma_{J-1}^2)\\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

Box I.

Table 1
Estimated parameter values in (21), obtained from corn futures prices traded on the CBOT from January 2013 to December 2019.

Corn	μ	α	σ
	6.1568	0.7891	0.0003497

value problem (5)–(6) is not known. Then, we need to use numerical methods for providing approximated solutions.

The Schwartz model will be used as an academic problem along this section with the values of the parameters shown in Table 1. These values have been estimated with data obtained in the Nasdaq Data Link (https://data.nasdaq.com) from corn[1] futures, traded on the CBOT (Chicago Board of Trade) from January 2013 to December 2019, and minimizing the quadratic pricing error (see [28]).

On the one hand, the interval $[S_{min}, S_{max}]$ must be determined by the values of the market observations used to estimate the functions. The estimations obtained outside the interval of observations may have a bias that would be transmitted to the approximation of the computed price. Taking into account the observations of the corn prices in the market, and trying to fit the minimum and maximum values that the spot price reaches, we consider that $S_{min} = 30$ and $S_{max} = 130$ provide a representative interval. However, note that the solution (21) does not satisfies the boundary conditions (12)–(13).

On the other hand, in the experiments we chose $T = 1$ (one year). This is one of the maturities for which we have observations of the futures prices.

For prescribed values of the discretization parameters h and k, we price the futures contracts with the numerical method, and measure the error produced by the approximation with respect to the exact solution, offered by (21), at the grid points. So, for $n = 0, 1, \dots, N$, the vector $\mathbf{e}^n = [e_0^n, e_1^n, \dots, e_J^n]$ recovers the errors along the spot grid at time τ_n, described by

$$e_j^n = |F_j^n - F(S_j, \tau_n)|, \qquad j = 0, 1, \dots, J.$$

In Fig. 1, we show the distribution of the error $\mathbf{e}^n$ along the spot price interval, obtained in the simulation at a very short time $\tau_n = 0.05$, for different values of the step parameters.

On the one hand, for $h = 0.625$ and $k = 0.003125$ (dash-dotted line), we observe that the errors are small, but their highest values are at the boundary (in particular, in the right extreme of the interval). On the other hand, when the discretization parameters are halved ($h = 0.3125$ and $k = 0.0015625$, dashed line), we notice that the errors decrease at the center of the interval as these parameters are reduced (according to the well-known convergence property of the Crank–Nicolson method), but not at the right boundary.

Therefore, the maximum error at time τ_n:

$$\|\mathbf{e}^n\| = \max_{j=0,1,\dots,J} e_j^n, \tag{22}$$

does not decrease as we would expect from a convergent method.

Besides, this malfunction intensifies with time. In Fig. 2, we show the distribution of the error in spot price at several times until the maturity: we see that the problems at the boundary are transmitted towards the center of the interval. So, the maximum error, at the maturity time, is very high, and the price of the commodity is unreliable.

3.2. Financial boundary conditions

To avoid errors on the boundary from being transmitted towards the center, we look for new boundary conditions for this futures pricing problem. To this end, we analyze the financial behavior of the prices.

In the case of certainty, the relation between the spot price S and the futures price F with maturity τ, is given by

$$F(S,\tau) = S\mathrm{e}^{a\tau},$$

where a is a constant known as cost of carry [2]. In general, when stochastic models are considered (see [16]), the futures prices are represented as

$$F(S,\tau) = \mathrm{e}^{A(\tau)\ln S + B(\tau)}. \tag{23}$$

Note that this expression is equivalent to

$$F(S,\tau) = S^{A(\tau)}\mathrm{e}^{B(\tau)},$$

which is quite similar to that presented for the certainty case, with $A(\tau)$ and $B(\tau)$ being more general functions of the maturity, and depending on the particular problem that we want to analyze.

From (23), we conclude that the logarithm of the futures price is linear with respect to the logarithm of the spot price. Thus, we pay attention to the following restriction obtained from (23) to obtain homogeneous boundary conditions that, in addition to retain this linearity property, do not involve the unknown functions $A(\tau)$ and $B(\tau)$:

$$\frac{\partial}{\partial S}\left(S\,\frac{\partial\left(\ln F(S_{min},\tau)\right)}{\partial S}\right) = 0, \qquad 0 \le \tau \le T, \tag{24}$$

$$\frac{\partial}{\partial S}\left(S\,\frac{\partial\left(\ln F(S_{max},\tau)\right)}{\partial S}\right) = 0, \qquad 0 \le \tau \le T. \tag{25}$$

Note that, again, the initial condition (6) satisfies these proposed boundary conditions.

[1] We achieved similar conclusions with other commodity data on the CME Group.

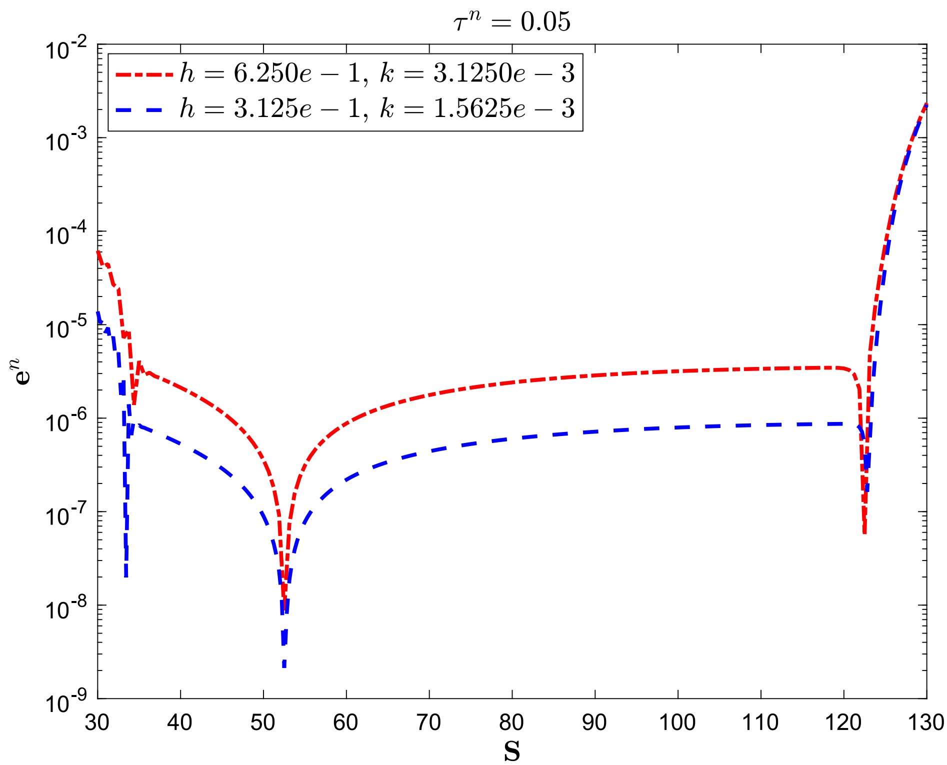

Fig. 1. Distribution of the error along the spot price interval, for different values of the discretization parameters, with the boundary condition $F_{SS} = 0$.

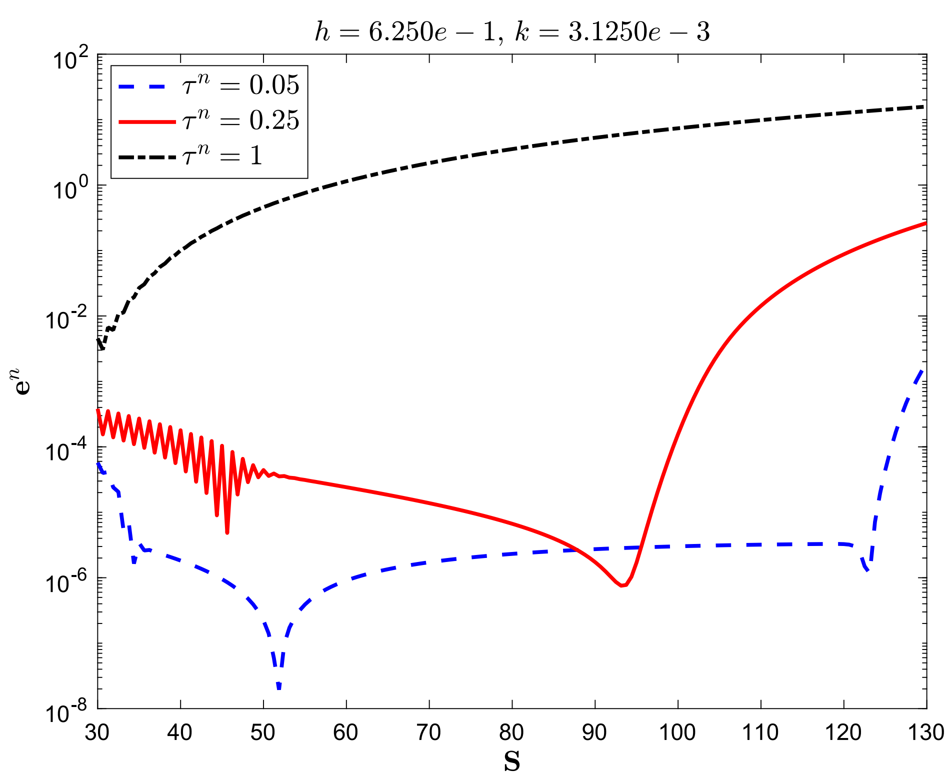

Fig. 2. Distribution of the error along the spot price interval at different times, with the boundary condition $F_{SS} = 0$.

If we expand the left-hand side of (24) and (25), we obtain the following equivalent equalities

$$F(S_{min},\tau)\left(F_S(S_{min},\tau)+S_{min}\,F_{SS}(S_{min},\tau)\right)-S_{min}\,F_S^2(S_{min},\tau)=0,\quad 0\le\tau\le T, \tag{26}$$

$$F(S_{max},\tau)\left(F_S(S_{max},\tau)+S_{max}\,F_{SS}(S_{max},\tau)\right)-S_{max}\,F_S^2(S_{max},\tau)=0,\quad 0\le\tau\le T. \tag{27}$$

Therefore, we propose to solve the problem (5)–(6) on $[S_{min},S_{max}]$ with the boundary conditions (26)–(27). To this end, we design a specific numerical method for the approximation of its solution.

As in the previous subsection, we introduce a uniform meshgrid on $[S_{min},S_{max}]\times[0,T]$, with discrete steps h and k for the spot price and time variables, defined by the discretizations parameters J and N, respectively. Again, we choose (14) as initial approximation (the grid restriction of the initial condition (6)), and the discretization (15) as the approximation to the PDE (5) at the inner grid points. Finally, we have to introduce appropriate discretizations of the boundary conditions (26)–(27). With this goal, given a sufficiently smooth function H of the real variable S, we will build approximation formulas for

$$H(S^*)\left(H'(S^*)+S^*H(S^*)\right)-S^*\left(H'(S^*)\right)^2, \tag{28}$$

at the boundary grid points $S^*=S_{min}$ and $S^*=S_{max}$, based on quadratic values of the function H over the grid; that is H_iH_j, $i,j=0,1,\dots,J$ (again, we denote $H_j=H(S_j)$, $j=0,1,\dots,J$). Different alternatives for these numerical formulas can be chosen, but we take in mind that they will be substituted into (26) and (27) in order to obtain estimations to F_0^n and F_J^n, $n=1,\dots,N$, respectively. On the one hand, we develop numerical formulas that contain few grid values. These grid values should be as close as possible to the point where the approximation is desired, and we look for preserving the order of convergence obtained with the Crank–Nicolson method (15). On the other hand, when we consider, for example, $S^*=S_{min}=S_0$ in order to obtain a simple explicit expression of F_0^n, in the approximation formula of (28) we avoid using H_0^2 (same considerations can be made in the $S^*=S_{max}=S_J$ case, to avoid H_J^2). Therefore, by using Taylor expansions of the function H, it is easy to achieve to the following result:

Lemma 1. *Assuming $H\in C^4([S_{min},S_{max}])$ and, for a positive integer J, let $S_j=S_{min}+j\,h$, $j=0,1,\dots,J$, be the uniform meshgrid with diameter $h=(S_{max}-S_{min})/J$ defined on the interval $[S_{min},S_{max}]$. Then, as $h\to 0$,*

$$\begin{aligned}H(S_0)\left(H'(S_0)+S_0H(S_0)\right)-S_0\left(H'(S_0)\right)^2 =& -\frac{13}{3h}H_0H_1 + \left(\frac{13}{6h}+\frac{3S_0}{h^2}\right)H_0H_2 + \left(\frac{1}{3h}-\frac{S_0}{h^2}\right)H_0H_3 \\ &+ \left(\frac{22}{3h}-\frac{3S_0}{h^2}\right)H_1^2 + \frac{11}{6h}H_2^2+\left(-\frac{22}{h}+\frac{S_0}{h^2}\right)\times H_1H_2+\mathcal{O}(h^2),\end{aligned} \tag{29}$$

$$\begin{aligned}H(S_J)\left(H'(S_J)+S_JH(S_J)\right)-S_J\left(H'(S_J)\right)^2 =& \frac{13}{3h}H_JH_{J-1} - \left(\frac{13}{6h}-\frac{3S_J}{h^2}\right)H_JH_{J-2} - \left(\frac{1}{3h}+\frac{S_J}{h^2}\right)H_JH_{J-3} \\ &- \left(\frac{22}{3h}+\frac{3S_J}{h^2}\right)H_{J-1}^2 - \frac{11}{6h}H_{J-2}^2+\left(\frac{22}{3h}+\frac{3S_J}{h^2}\right)H_{J-1}H_{J-2}+\mathcal{O}(h^2).\end{aligned} \tag{30}$$

Therefore, the discretization of the boundary conditions (26)–(27) by means of the formulas (29)–(30), respectively, produces the following numerical boundary conditions: for $n=1,\dots,N$,

$$-\frac{13}{3h}F_0^nF_1^n+\left(\frac{13}{6h}+\frac{3S_0}{h^2}\right)F_0^nF_2^n+\left(\frac{1}{3h}-\frac{S_0}{h^2}\right)F_0^nF_3^n+\left(\frac{22}{3h}-\frac{3S_0}{h^2}\right)\times(F_1^n)^2+\frac{11}{6h}(F_2^n)^2+\left(-\frac{22}{3h}+\frac{S_0}{h^2}\right)F_1^nF_2^n=0, \tag{31}$$

$$\frac{13}{3h}F_J^nF_{J-1}^n-\left(\frac{13}{6h}-\frac{3S_J}{h^2}\right)F_J^nF_{J-2}^n-\left(\frac{1}{3h}+\frac{S_J}{h^2}\right)F_J^nF_{J-3}^n-\left(\frac{22}{3h}+\frac{3S_J}{h^2}\right)\times(F_{J-1}^n)^2-\frac{11}{6h}(F_{J-2}^n)^2+\left(\frac{22}{3h}+\frac{3S_J}{h^2}\right)F_{J-1}^nF_{J-2}^n=0. \tag{32}$$

Note that the discrete initial data in (14), F_j^0, $j=0,1,\dots,J$, satisfies both numerical boundary conditions (31)–(32).

As a conclusion, the numerical method requires, at each time level $n=1,\dots,N$, the solution of the linearly implicit system of Eqs. (15), coupled with the two nonlinear equations (31) and (32), starting from the initial data (14).

For simplicity, the numerical method will be described in vector form. Again, we denote $\mathbf{F}^n=[F_0,\dots,F_J]^T$ the $(J+1)$-dimensional vector that recovers the approximations to the solution at the time level τ_n, $n=0,1,\dots,N$, but also, we use its reduced $(J-1)$-dimensional version $\widetilde{\mathbf{F}}^n=[F_1,\dots,F_{J-1}]^T$, that only incorporates the approximations at the inner values of the spot price. Then, starting from $\mathbf{F}^0$, with components described by (14), we obtain $\mathbf{F}^n$ from $\mathbf{F}^{n-1}$, $n=1,\dots,N$, solving the following nonlinear system of $J+1$ equations:

$$L\,\widetilde{\mathbf{F}}^n=R\,\widetilde{\mathbf{F}}^{n-1}+\mathbf{b}\,(F_0^{n-1},F_J^{n-1})+\mathbf{b}\,(F_0^n,F_J^n), \tag{33}$$

$$F_0^n=\frac{6S_0F_1^n\left(F_2^n-3F_1^n\right)+11h\left(2F_1^n-F_2^n\right)^2}{6S_0\left(F_3^n-3F_2^n\right)+h\left[13\left(2F_1^n-F_2^n\right)-2F_3^n\right]}, \tag{34}$$

$$F_J^n=\frac{-6S_JF_{J-1}^n\left(F_{J-2}^n-3F_{J-1}^n\right)+11h\left(2F_{J-1}^n-F_{J-2}^n\right)^2}{-6S_J\left(F_{J-3}^n-3F_{J-2}^n\right)+h\left[13\left(2F_{J-1}^n-F_{J-2}^n\right)-2F_{J-3}^n\right]}, \tag{35}$$

where, denoting $c=k/h$, and $d=k/h^2$ as in the previous subsection, then we obtain the matrices as shown in Box II.

Note that, at each time step $n=1,2,\dots,N$, the nonlinear system (33)–(35) must be solved, unlike in the previous subsection where the corresponding system (18) was linear. To this end, an iterative procedure is used: starting from a suitable initial guess of the solution at time τ_n (that we denote $\mathbf{F}^{n,0}$), we compute a new iterant $\mathbf{F}^{n,\nu+1}$ in terms of the previous one $\mathbf{F}^{n,\nu}$, $\nu=0,1,\dots$. In this case, taken into account the special structure of the nonlinear system, the following iteration, which involves a linear system at each iteration, is of practical interest

$$F_0^{n,\nu+1}=\frac{6S_0F_1^{n,\nu}\left(F_2^{n,\nu}-3F_1^{n,\nu}\right)+11h\left(2F_1^{n,\nu}-F_2^{n,\nu}\right)^2}{6S_0\left(F_3^{n,\nu}-3F_2^{n,\nu}\right)+h\left[13\left(2F_1^{n,\nu}-F_2^{n,\nu}\right)-2F_3^{n,\nu}\right]}, \tag{36}$$

$$F_J^{n,\nu+1}=\frac{-6S_JF_{J-1}^{n,\nu}\left(F_{J-2}^{n,\nu}-3F_{J-1}^{n,\nu}\right)+11h\left(2F_{J-1}^{n,\nu}-F_{J-2}^{n,\nu}\right)^2}{-6S_J\left(F_{J-3}^{n,\nu}-3F_{J-2}^{n,\nu}\right)+h\left[13\left(2F_{J-1}^{n,\nu}-F_{J-2}^{n,\nu}\right)-2F_{J-3}^{n,\nu}\right]}, \tag{37}$$

$$L\,\widetilde{\mathbf{F}}^{n,\nu+1}=R\,\widetilde{\mathbf{F}}^{n-1}+\mathbf{b}\,(F_0^{n-1},F_J^{n-1})+\mathbf{b}\,(F_0^{n,\nu+1},F_J^{n,\nu+1}). \tag{38}$$

But the linear system (38) can be solved in an efficient way: the matrices L and R of this linear system do not change with iteration over time, so they are always the same, and only the right-hand side of the system must be updated.

With respect to the initial iterant, note than only $F_1^{n,0}$, $F_2^{n,0}$, $F_3^{n,0}$ are required to produce $F_0^{n,1}$ in (36) and, respectively, $F_{J-3}^{n,0}$, $F_{J-2}^{n,0}$, $F_{J-1}^{n,0}$ to produce $F_J^{n,1}$ in (37). Thus, observing (38) at $\nu=0$, we conclude that only these six values of the $\mathbf{F}^{n,0}$ are needed to calculate the full vector $\mathbf{F}^{n,1}$. Here, we propose to compute these few components of $\mathbf{F}^{n,0}$ using the explicit Euler method, that is:

$$F_j^{n,0}=-\frac{1}{2}(2c\mu_j-d\sigma_j^2)F_{j-1}^{n-1}+(1-d\sigma_j^2)F_j^{n-1}+\frac{1}{2}(2c\mu_j+d\sigma_j^2)F_{j+1}^{n-1},\quad j=1,2,3,J-3,J-2,J-1.$$

$$L = \begin{bmatrix} 1+\frac{1}{2}d\sigma_1^2 & -\frac{1}{4}(c\mu_1+d\sigma_1^2) & 0 & \cdots & 0 \\ \frac{1}{4}(c\mu_2-d\sigma_2^2) & 1+\frac{1}{2}d\sigma_2^2 & -\frac{1}{4}(c\mu_2+d\sigma_2^2) & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \frac{1}{4}(c\mu_{J-2}-d\sigma_{J-2}^2) & 1+\frac{1}{2}d\sigma_{J-2}^2 & -\frac{1}{4}(c\mu_{J-2}+d\sigma_{J-2}^2) \\ 0 & \cdots & 0 & \frac{1}{4}(c\mu_{J-1}-d\sigma_{J-1}^2) & 1+\frac{1}{2}d\sigma_{J-1}^2 \end{bmatrix},$$

$$R = \begin{bmatrix} 1-\frac{1}{2}d\sigma_1^2 & \frac{1}{4}(c\mu_1+d\sigma_1^2) & 0 & \cdots & 0 \\ -\frac{1}{4}(c\mu_2-d\sigma_2^2) & 1-\frac{1}{2}d\sigma_2^2 & \frac{1}{4}(c\mu_2+d\sigma_2^2) & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -\frac{1}{4}(c\mu_{J-2}-d\sigma_{J-2}^2) & 1-\frac{1}{2}d\sigma_{J-2}^2 & \frac{1}{4}(c\mu_{J-2}+d\sigma_{J-2}^2) \\ 0 & \cdots & 0 & -\frac{1}{4}(c\mu_{J-1}-d\sigma_{J-1}^2) & 1-\frac{1}{2}d\sigma_{J-1}^2 \end{bmatrix},$$

$$\mathbf{b}\,(F_0^n, F_J^n) = \begin{bmatrix} -\frac{1}{4}(c\mu_1-d\sigma_1^2)\,F_0^n \\ 0 \\ \vdots \\ 0 \\ \frac{1}{4}(c\mu_{J-1}+d\sigma_{J-1}^2)\,F_J^n \end{bmatrix}, \quad n=0,\dots,N,$$

Box II.

In contrast to the technique presented in the previous subsection, this new approach corrects the shortcomings at the boundary. We show it considering the same test problem (the Schwartz model) with the same parameters values (Table 1). Remember that the solution (21) satisfies the boundary conditions (26)–(27), regardless of the S_{min} and S_{max} values chosen. In practice, the iterative procedure is performed until convergence, the iteration will be stopped when the distance between two consecutive iterants is less than a very small prescribed tolerance.

Taking the behavior observed in Fig. 1 as a guide, we compare the distribution of the errors along the spot price interval at $\tau_n = 0.05$, obtained with both procedures. For example, for $h = 0.625$ and $k = 0.003125$, in Fig. 3 we plot again the errors obtained with the technique presented in Section 3.1 (dashed line), and the errors that the new method proposed in this subsection causes (dash-dotted line). We observe that the inaccuracies on the boundary disappear.

This good performance is maintained even over time. In Fig. 4, we show the distribution of the error for the times $\tau_n = 0.05$, 0.25 and 1 (the maturity of the futures contract priced) with the same discretization steps as in the previous experiments. Although the maximum error increases over time, the approximation at the boundary does not distort the values at the center of the interval, offering a very accurate solution at maturity time.

Finally, note that the accuracy of the approximation obtained with this method can be improved by refinement of the discretization parameters. In Fig. 5 we present a dash-dotted line, the maximum error defined in (22) at the maturity time $T = 1$, for the values of the step values $h = 5$, 2.5, 1.25, 0.625, 0.3125 and 0.15625, and the associated $k = ch$, with $c = 0.0025$. We observe the second order of convergence of the new procedure, confirmed by the solid line plotted in the lower right-hand corner which describes the quadratic behavior.

4. Numerical solution of the model with delay

In this section, we consider the futures pricing model with delay. Therefore, we have to deal with the numerical approximation of the RPDE (11) with the initial condition (6). To this end, we adapt a procedure that has already been used for pricing futures and other derivatives (see, [26,27,29]). Note that, unlike the case without delay, a new source of difficulty arises: the commodity spot price not also appears in the coefficients of (11) as an independent variable S, but also as a function that follows a stochastic process evaluated at different times involving the discrete delay. As a consequence, we are impelled to use some procedure to estimate $\mu(S, S(T-\tau-\delta))$ and $\sigma(S, S(T-\tau-\delta))$.

From a theoretical point of view, if the values of a specific trajectory of the stochastic process (7)–(8) were known, we could substitute them in the coefficients of Eq. (11) to price the futures along the prescribed trajectory. Thus, as these coefficients, which depend on the trajectory, would now be deterministic: the initial RPDE would be understood as a deterministic PDE. By taking a large number of trajectories $S^{(i)}$, $i = 1,\dots,M$, and denoting as $F_{(i)}(S,\tau)$ the corresponding solution of the PDE along the ith trajectory, $i = 1,\dots,M$, we can consider

$$\frac{1}{M}\sum_{i=1}^{M} F_{(i)}(S,\tau),$$

a good estimation of $F(S,\tau)$, solution of the problem (11), (6), assuming M is large enough.

For the numerical procedure that we propose, we reproduce this schedule with numerical approximations. In such a case, taking into account that the RPDE (11) is basically a PDE equation like (5) over a trajectory of (7)–(8), we will adapt the advantageous technique presented in Section 3.2 for the no-delay scenario.

Again, we introduce a uniform meshgrid on $[S_{min}, S_{max}]\times[0,T]$, with discrete steps h and k for the spot price and time variables, respectively, associated to the discretizations parameters J and N: $S_j = S_{min} + j\,h$, $j = 0,1,\dots,J$; $\tau_n = n\,k$, $n = 0,1,\dots,N$. Taking a sufficiently large value of M (the number of considered trajectories of (7)–(8) to be used), for $i = 1,\dots,M$, we propose:

- Computing numerical approximations to $S(T-\tau_n-\delta)$, $n = 0,1,\dots,N$, by means of the Euler discretization (see [30]) of the stochastic delay process (7)–(8) with constant step k. Let us denote them $\psi^n \approx S(T-\tau_n-\delta)$, $n = 0,1,\dots,N$.
- Obtaining, from the previous computed ψ^n, $n = 0,1,\dots,N$, estimates of the coefficients $\mu(S_j, S(T-\tau_n-\delta))$ and $\sigma(S_j, S(T-\tau_n-\delta))$ (denoted by μ_j^n and σ_j^n, respectively), $j = 0,1,\dots,J$, $n = 0,1,\dots,N$. For parametric expressions of these functions, we can use $\mu_j^n = \mu(S_j,\psi^n)$ and $\sigma_j^n = \sigma(S_j,\psi^n)$, $j = 0,1,\dots,J$, $n = 0,1,\dots,N$.
- Using the discretization (33)–(35), which incorporates the financial boundary conditions (26)–(27), to approximate the solution of the problem (11) and (6), along the computed trajectory, by means of the approximations μ_j^n and σ_j^n, $j = 0,1,\dots,J$, $n = 0,1,\dots,N$. Note that now, the coefficients associated to μ and σ depend on τ_n, $n = 0,1,\dots,N$. Therefore, now we have the matrices L^n and R^n, and vectors $\mathbf{b^n}$ in (33), that change with each

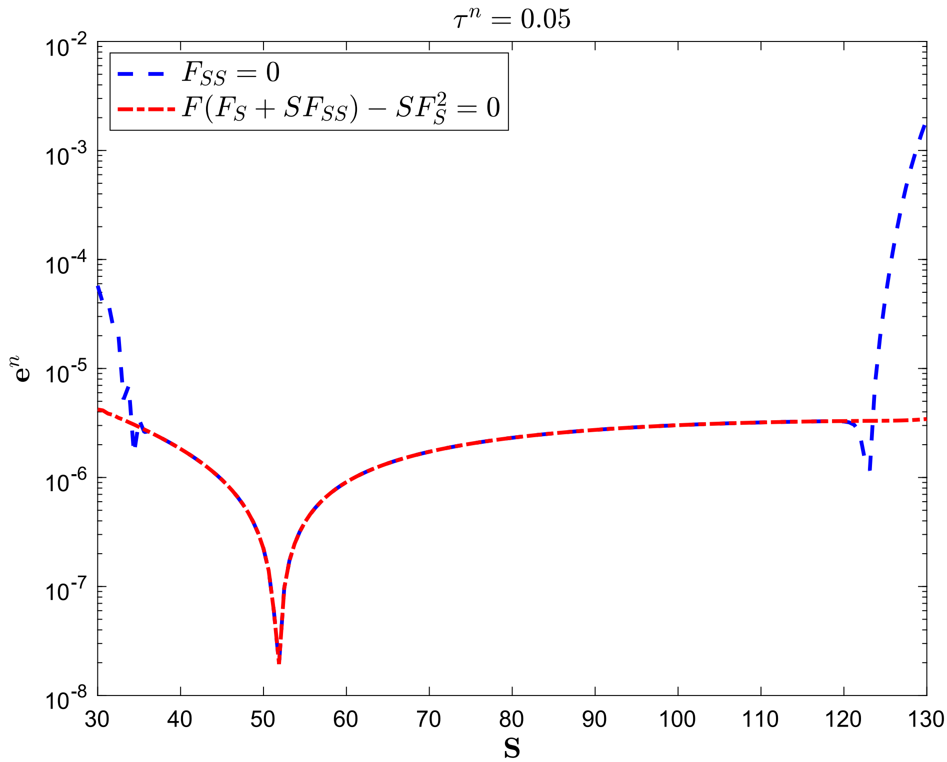

Fig. 3. Distribution of the error along the spot price interval, for $h = 6.250e - 1$ and $k = 3.1250e - 3$, with the boundary conditions $F_{SS} = 0$ and $F(F_S + SF_{SS}) - SF_S^2 = 0$.

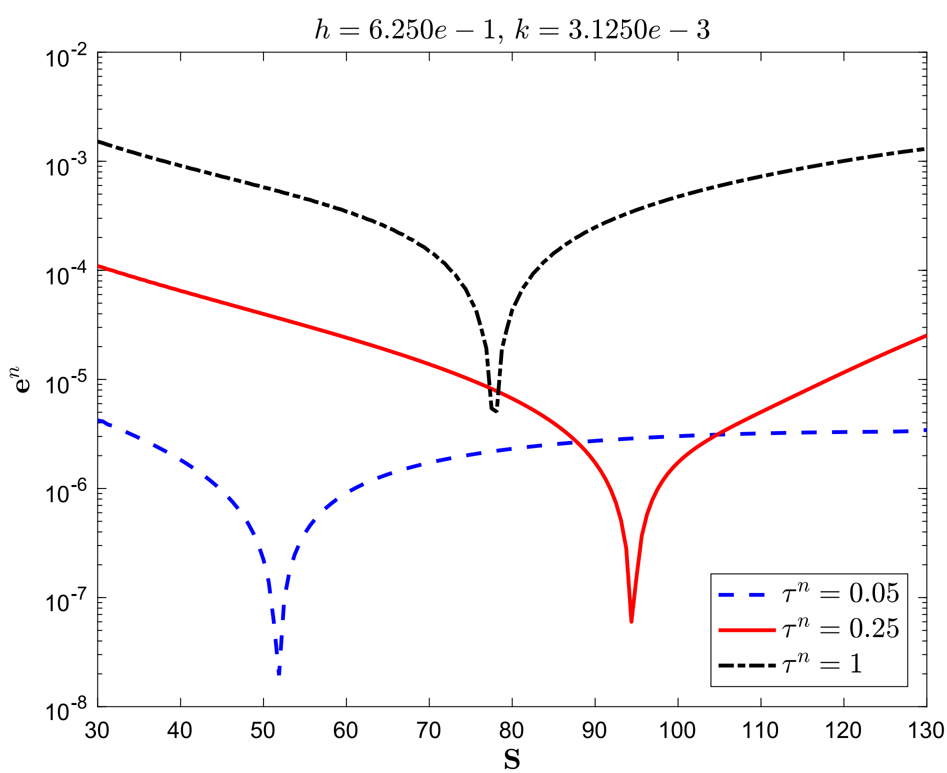

Fig. 4. Distribution of the error along the spot price interval at different times, with the boundary condition $F(F_S + SF_{SS}) - SF_S^2 = 0$.

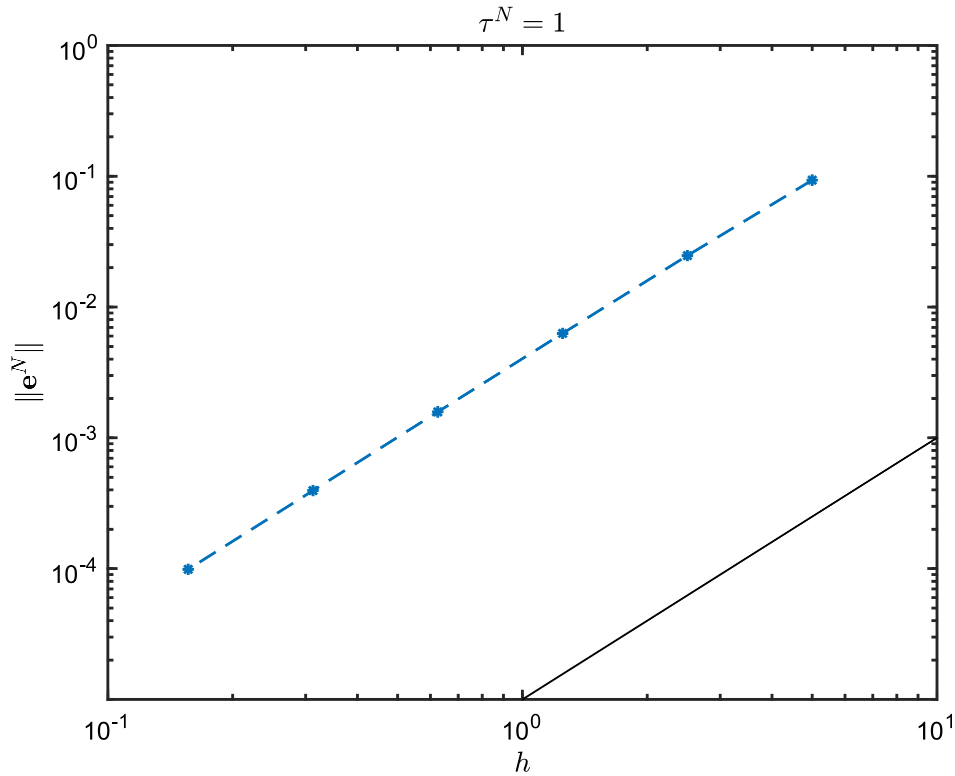

Fig. 5. Maximum error at the maturity time, for different refinements of the discretization parameters h, and $k = ch$ with the boundary condition $F(F_S + SF_{SS}) - SF_S^2 = 0$. The line plotted in the lower right-hand corner represents quadratic slope.

time step. Let us denote $\mathbf{F}^n_{(i)}$, $n = 0, 1, \dots, N$, the vectors with the computed numerical approximations, at each time level.

Finally, we obtain a numerical approximation to the solution of the RPDE problem (11) and (6) by

$$\mathbf{F}^n = \frac{1}{M}\sum_{i=1}^{M}\mathbf{F}^n_{(i)}, \qquad n = 0, 1, \dots, N.$$

5. Empirical application: pricing gold futures

Finally, we apply the different models and the new numerical methods to price gold futures. On the one hand, we will evaluate the interest of introducing memory in the process. On the other hand, we will corroborate the usefulness of the numerical approximations designed for each of the situations: with and without delay. For this purpose, we will compare the prices obtained in this problem with the approximations provided by the MC simulation method, which is commonly used by researchers and partitioners in the markets.

From the high variety of commodities traded on the markets we have chosen the gold because of its relevance in the global word. Gold is one of the most malleable, ductile, brilliant and beautiful metals, but it is also one of the oldest way to store wealth. In fact, it is also used as an investment asset, especially in periods of crisis or high inflation, and also serves as a reserve for central banks [31]. Gold is mainly traded on 7 different markets and the most important are the London Over-the-Counter (OTC) market and the COMEX in New York.

For the present empirical application and comparison of the techniques, we use daily gold futures prices traded on the COMEX obtained from Nasdaq Data Link, with maturities from 1 to 6 months. More precisely, we use data from October 2012 to March 2021 to estimate the risk-neutral functions. We keep data from April 2021 to June 2021 to make an out-of-sample analysis of the futures prices. As usual in the literature (see [32,33]), we use the front-month futures prices as a proxy to the spot price. Fig. 6 shows the spot prices (top) and their first differences (bottom) along the time interval. Table 2 reports its main descriptive statistics.

Table 2
Summary of the main statistics of the gold spot price and their first differences, from October 2012 to June 2021.

Gold	Variable	N	Mean	Std. dev.	Max	Min
	$S(t)$	2191	1429.0596	212.768	2097	1120.1
	$S(t+1) - S(t)$	2190	0.0032	14.6021	108.7	−140.4

In order to avoid imposing arbitrary restrictions to the model, we estimate the functions of the spot risk-neutral process μ and σ (in (1) for the non-delayed case, and in (9) for the delayed one) by means of nonparametric techniques and futures prices traded in the COMEX (see [26]). More precisely, we use the Nadaraya–Watson estimator [34] with a Gaussian kernel (for commodity futures see [8,26]). Unlike the case without delay (in which the functions only depend on one variable S), in the case with delay we use a nonparametric estimation procedure with two independent variables (S, η). From the prices in the market we obtain the values of the functions μ and σ on a discrete grid over a rectangle. Here, we use the gold price vector and its delayed (represented by η). Finally, we compute the values in each $(S_j, \psi(\tau_n))$, $j = 0, 1, \dots, J$, $n = 0, 1, \dots, N$, by means of interpolation.

With respect to the classic Monte Carlo approach, we approximate the futures prices in the out-of-sample period by means of (3). In the experiments we compute 5.000 trajectories with the Euler method applied to the associated stochastic equation (that is, (1) in the case without delay, and (9) in the case with delay), with a daily time step (1/250), and the antithetic variate as a variance reduction technique [35].

Regarding the new techniques that we propose (in Section 3.2 for the case without delay, and in Section 4 for the case with delay), we choose $S_{min} = 1000$ and $S_{max} = 2200$ (see the top picture of Fig. 6). For the numerical experiments, we provide the results obtained with the discretization parameters $h = 3$ ($J = 400$), and $k = c\,h$ with $c = 0.0025$ ($N = 12096$).

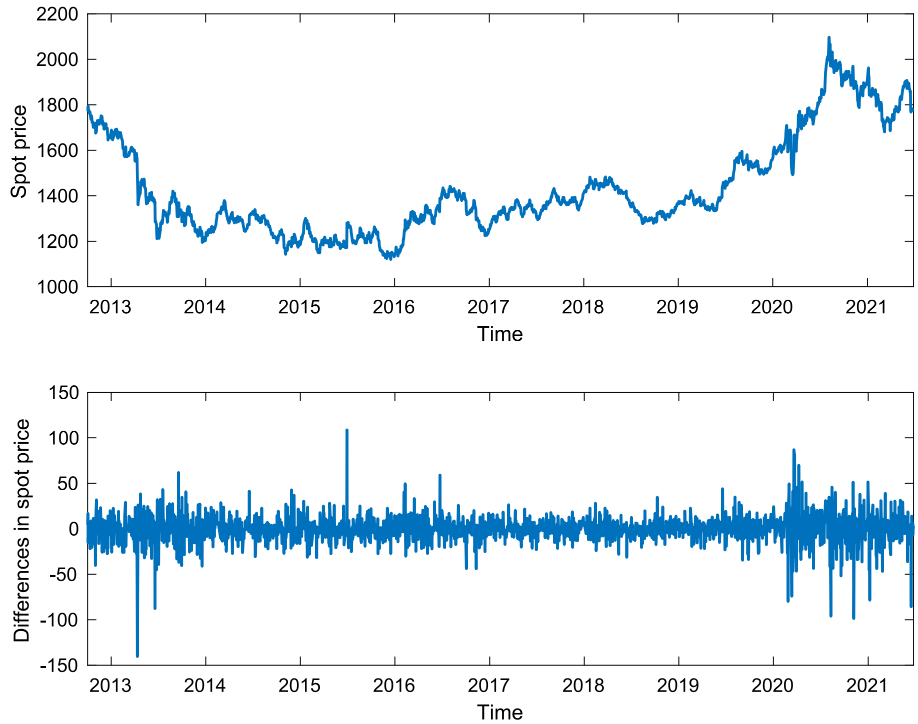

Fig. 6. Daily gold spot prices (top) and their differences (bottom): from October 2012 to June 2021.

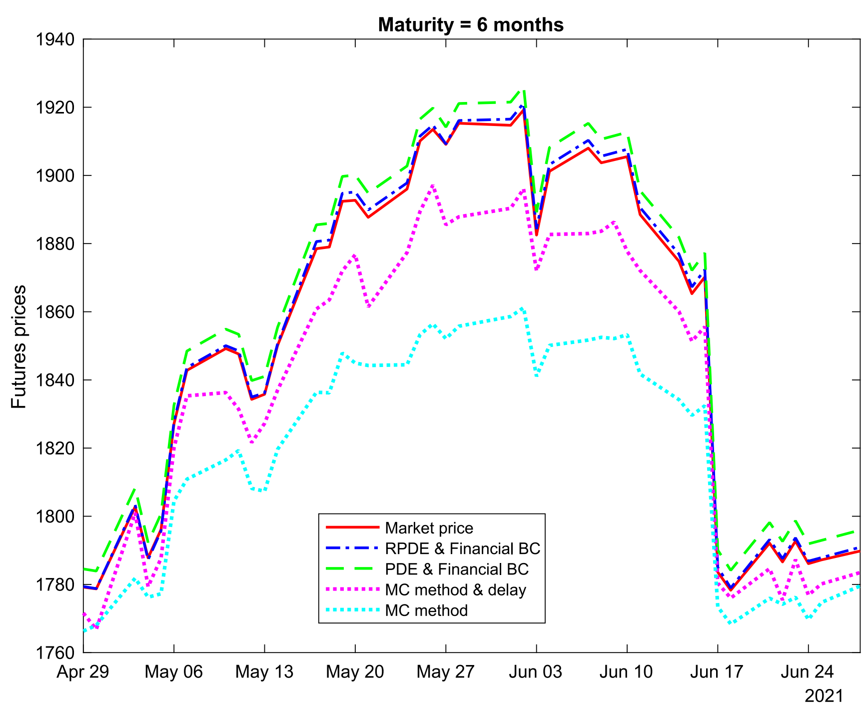

Fig. 7. Futures prices in the gold market and their numerical approximations, for 6-months maturity. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 7 presents the results obtained in the out-sample period. The futures prices for a maturity equal to 6 months[2] observed in the COMEX are represented with a solid line; their approximations obtained by the MC technique for the model without delay, with a light-colored dotted line; the approximations with the MC technique for the model with delay, with a dark dotted line; the numerical procedure proposed in Section 3.2, with a dashed line; and the numerical procedure in the model with delay described in Section 4, with a dash-dotted line. Firstly, we observe that all methods capture approximately the profile shown by the observed futures prices. Secondly, the incorporation of memory in the model improves the valuation process, and notably in the MC approach. The new numerical approaches we have designed provide much more accurate approximations than the MC method for

[2] Similar conclusions are obtained for other maturities.

the whole prediction period. Note that the approximation with the numerical procedure for the model with delay provides values practically indistinguishable from the observed ones. Finally, we observe that the technique based on PDE discretizations, slightly overestimates the observed prices in the COMEX along the whole out-sample period; but the MC technique underestimates them systematically.

6. Conclusions

In the financial literature, it is very common to be confronted with the task of solving a final value problem, based on PDEs, to price derivatives. In most of the cases, a feasible expression of the solution to this kind of problems does not exist or it is not easy to find. In these cases, numerical methods are necessary to approximate the solution. In general, a bounded region for the state variables must be introduced, and then, appropriate boundary conditions must be incorporated.

In this paper, we focus on approximating commodity futures prices in a single-factor diffusion model (with and without memory in the process) by discretization of the corresponding PDE problem. We conclude that the inclusion of boundary conditions based on financial characteristics of the derivative paves the way for its accurate valuation.

We infer that incorporating a discrete delay in the stochastic process of the continuous model improves the valuation of the futures price. Moreover, the best approximation is obtained when applying the numerical technique we propose, which discretizes the PDE problem with the boundary conditions based on financial reasons, is applied.

Finally, we corroborate the effectiveness of the proposed techniques by performing a very accurate pricing of gold futures contracts traded on the COMEX.

CRediT authorship contribution statement

Lourdes Gómez-Valle: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Miguel Ángel López-Marcos:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Methodology, Investigation, Formal analysis, Conceptualization. **Julia Martínez-Rodríguez:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data are obtained in the Nasdaq Data Link: https://data.nasdaq.com.

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