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Optimizing price, order quantity, and backordering level using a nonlinear holding cost and a power demand pattern

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ABSTRACT

It is well-known that the demand rate for some products depends on several factors, such as price, time, and stock, among others. Moreover, the holding cost can vary over time. More specifically, it increases with time since a long period of storage requires more expensive warehouse facilities. This research introduces an inventory model with shortages for a single product where the demand rate depends simultaneously on both the selling price and time according to a power pattern. Shortages are completely backordered. Demand for the product jointly combines the impact of the selling price and a time power function, which is performed as an addition. Furthermore, the holding cost is a power of the time that the product is held in storage. The main objective is to derive the optimal inventory policy such that the total profit per unit of time is maximized. For optimizing the inventory problem, some theoretical results are derived first to prove that the total profit function is strictly pseudo concave with respect to the decision variables. Next, an efficient algorithm that obtains the optimal solution is provided. The proposed inventory model is a generalmodel because it contains several published inventory models as special cases. Some numerical examples are presented and solved to illustrate and validate the proposed inventory model. Also, a sensitivity analysis is conducted in order to highlight and generate significant insights.

1. Introduction

In any economic sector, product inventories are critical for all firms. Therefore, firms must develop robust inventory models to determine optimal product inventory policies (e.g., what to order, when to order, and in which quantities), and in some cases specify the degree of shortages that can be permitted in order to minimize costs or maximize profits. To accomplish this, firms usually have a department responsible for managing the inventory which proposes effective and efficient methodologies for controlling product stocks with the aim of always having items available to satisfy customer demands. It is for this reason that researchers across the world have been developing inventory models that perfectly fit and solve inventory issues. For instance, (Akan et al., 2021), (Fang et al., 2021), (Feng et al., 2021) and (Hemmati et al., 2021) have studied the joint pricing-inventory management.

In many production and inventory models, the rate of demand is considered to be constant and known. However, in real life situations, the rate of demand actually depends on many factors such as price, time, and stock, to name a few. This paper is strongly related to four topics in inventory management: price-dependent demand, power demand pattern, nonlinear holding cost, and shortages. Within the inventory theory literature, large efforts are still dedicated to building inventory models that incorporate these topics in isolation, or by perhaps considering the effects of two topics jointly. However, no previous study has considered the combined impact of all four topics. In the next section, a literature review of the research works most closely associated with the mentioned topics is presented.

1.1. Literature review

1.1.1. Power demand pattern

In the research area of inventory models, one stream of studies investigates the inventory problems associated with demand that is dependent on time (i.e., products are sold at the beginning of the period, withdrawn at the end of the cycle, or consumed uniformly during the

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period). These distinctive manners in which the demand happens within a time period are referred to as power patterns.

There exists a vast amount of research literature that models the demand by using the power demand pattern function. In this direction, Naddor (1966) proposed a power demand pattern that depends on both the time and the length of the cycle time. Since then, several other researchers have developed inventory models that model the demand using the power pattern as a function of time. For example, Aggarwal and Goel (1982) developed an inventory model using the power demand pattern for the case when a constant portion of the on–hand inventory deteriorates over time. Afterwards, Datta and Pal (1988) introduced an inventory system that used a power demand pattern for items with a variable rate of deterioration. Girlich (1990)solved the EOQ inventory model that used a power demand pattern. Later, Lee and Wu (2002) examined an EOQ inventory model with permissible shortages and a power demand pattern when the products deteriorate according to a Weibull distributed rate. Dye (2004) revisited and extended the research work of Lee and Wu (2002) by including a general class time-proportional backlogging rate and a power demand pattern. Jung et al. (2008) identified some questionable results in the inventory model proposed by Dye (2004) and improved it. Abdul-Jalbar et al. (2009) formulated an inventory model as a mixed nonlinear programming problem and analyzed the implications of utilizing a power demand pattern and backordering in a scenario with one-warehouse and N-retailers. Singh et al. (2009) formulated an EOQ inventory model in which shortages are permitted and these are partially backordered when the demand of deteriorating products follows a power demand pattern. Their inventory model considers the backordering rate as being inversely proportional to the waiting time of the subsequent replenishment. Tripathy and Pradhan (2010) built an EOQ inventory model when the items deteriorate with a two-parameter Weibull distributed rate and assumed a power demand pattern with partial backlogging. Kumar and Singh (2011) modeled an inventory system by considering that the product deteriorates after a fixed time period referred to as the *life time*) and by taking into account an incremental holding cost and the impact of partial backlogging. Rajeswari and Vanjikkodi (2011) presented an inventory model in which the products have a constant deterioration rate and demand follows a power pattern. Shortages are permissible and these are partially backordered. Sarbjit and Shivraj (2011) proposed deterministic and probabilistic EOQ inventory models with shortages and a power demand pattern for products having a variable rate of deterioration. Moreover, the impacts of inflation and a permissible delay in payment are studied and analyzed. Singh and Sehgal (2011) constructed an EOQ inventory model for articles that deteriorate with a two-parameter Weibull rate by considering a power demand pattern when shortages are permissible and are completely backordered. Krishnaraj and Ramasamy (2012) dealt with an inventory system without shortages for a power demand pattern including a two-parameter Weibull distribution to model the deterioration rate. Mishra et al. (2012) investigated the effects of both the time value of money and inflation in an inventory system with shortages for perishable products with a power demand pattern when a two-parameter Weibull distribution is used to account for the deterioration rate by taking into consideration that deterioration begins after a fixed time period. Rajeswari and Vanjikkodi (2012) considered an inventory model with a time-dependent power demand pattern when deterioration follows a two-parameter Weibull distribution. Their inventory model includes three different situations: complete, partial, and no backlogging. Sicilia et al. (2012) developed inventory systems with a power demand pattern for cases without and with shortages. In addition, complete backordering and fully lost sales inventory models are derived. Sicilia et al. (2014aa) developed a production-inventory system with a power demand pattern, a production rate proportional to the demand rate and full backlogging. San-José et al. (2017) determined the optimal inventory policy for an inventory system with a power demand pattern and fixed partial backlogging. San-José et al. (2018a) studied an economic order quantity inventory model with shortages fully backlogged and where the demand rate was the product of a price-logit function and a power-time function. San-José et al. (2019) analyzed an inventory model without shortages for a single item where the demand rate was the sum of a linear function with respect to the unit selling price and of a power-time function. Other inventory models with a power demand pattern have been proposed by Sicilia et al. (2013), Sicilia et al. (2014b), Sicilia et al. (2015), Rajeswari et al. (2017), Tripathi et al. (2017), San-José et al. (2018b) and San-José et al. (2020). Table 1 presents a list of selected inventory models with a power demand pattern that have been developed since 2000.

1.1.2. Holding cost

The majority of the production and inventory models are derived assuming a constant holding cost. However, in the real world, the holding cost varies. Therefore, another research area of interest is the development of inventory models that consider a variable holding cost. In this field, there are different types of models such as those that consider stock-dependent holding cost, time-dependent holding cost, or include multiple-dependence holding cost or any other holding cost variability. In this direction, Alfares and Ghaithan (2019) presented an excellent and comprehensive state-of-the-art review on inventory models that consider variable holding cost. For the case of time-dependent holding cost, normally the authors use linear or nonlinear time functions. Weiss (1982) introduced deterministic and stochastic EOQ inventory models by assuming that the unit holding cost is non-linearly dependent on the duration of time in storage. Goh (1994) considered two types of variations of the holding cost: a nonlinear function for the duration of time the products are maintained in storage, and a nonlinear function with respect to the amount of on-hand inventory. Giri and Chaudhuri (1998) revisited and extended the inventory models of Goh (1994) by taking into account that the products are perishable. Chang (2004) also improved the inventory models of Giri and Chaudhuri (1998) by optimizing the maximum profit and relaxing the constraint of a zero-ending inventory. They showed that the profits are significantly larger than those obtained by Giri and Chaudhuri (1998)'s inventory model. Ferguson et al. (2007) extended Weiss (1982)'s inventory model and pointed out that it is an approximation of the optimal order quantity for the case of perishable items. They also incorporated surcharges for infrequent ordering and discounts. On the one hand, Goh (1994) treated the variation of the holding cost over time as a continuous nonlinear function, whereas Alfares (2007) proposed two kinds of discontinuous step functions for the variable holding cost in which the storage time of the items is separated into several periods and the holding cost increases continuously. Additionally, as the storage time of the items approaches the subsequent time period, the holding cost can be charged either retroactively to all storage cycles or incrementally to the subsequent storage cycle only. It is important to remark that Alfares (2007) imposed the constraint that the inventory level at the end of the cycle must be equal to zero. Conversely, Urban (2008) revisited and generalized the Alfares (2007)'s inventory model by permitting that the ending inventory level be a non-zero value. Mahata and Goswami (2009) studied fuzzy inventory models for perishable goods by assuming that the holding cost varies according to a nonlinear function of the time the product is held in storage and the deterioration rate occurs according to a triangular fuzzy number. Mao and Xiao (2009) formulated and solved an inventory model for non-instantaneous deteriorating goods by taking into consideration that shortages occur and that these shortages are completely backordered. A generalized function of the on-hand inventory was used in this study to represent the holding cost. By considering the holding cost as a nonlin-

Table 1
Selected inventory models related to power demand pattern from 2000.

Authors	Price dependent demand	Power demand pattern	Holding cost	Allowed shortages	Type of backlogging	Objective function
Lee and Wu (2002)	No	Yes	Constant	Yes	Full	Min. cost
Dye (2004)	No	Yes	Constant	Yes	Partial	Min. cost
Jung et al. (2008)	No	Yes	Constant	Yes	Partial	Min. cost
Abdul-Jalbar et al. (2009)	No	Yes	Constant	Yes	Full	Min. cost
Singh et al. (2009)	No	Yes	Constant	Yes	Partial	Min. cost
Tripathy and Pradhan (2010)	No	Yes	Constant	Yes	Partial	Min. cost
Kumar and Singh (2011)	No	Yes	Constant	Yes	Partial	Min. cost
Rajeswari and Vanjikkodi (2011)	No	Yes	Constant	Yes	Partial	Min. cost
Sarbjit and Shivraj (2011)	No	Yes	Constant	Yes	Full	Min. cost
Singh and Sehgal (2011)	No	Yes	Constant	Yes	Full	Min. cost
Krishnaraj and Ramasamy (2012)	No	Yes	Constant	No		Min. cost
Mishra et al. (2012)	No	Yes	Constant	Yes	Partial	Min. cost
Rajeswari and Vanjikkodi (2012)	No	Yes	Constant	Yes	Full/Partial	Min. cost
Sicilia et al. (2012)	No	Yes	Constant	Yes	Full	Min. cost
Sicilia et al. (2013)	No	Yes	Constant	No		Min. cost
Sicilia et al. (2014aa)	No	Yes	Constant	Yes	Full	Min. cost
Sicilia et al. (2014b)	No	Yes	Constant	Yes	Full	Min. cost
Sicilia et al. (2015)	No	Yes	Constant	No		Min. cost
Rajeswari et al. (2017)	No	Yes	Linear	Yes	Partial	Min. cost
San-José et al. (2017)	No	Yes	Constant	Yes	Partial	Max. profit
Tripathi et al. (2017)	No	Yes	Constant	No		Min. cost
San-José et al. (2018a)	Yes	Yes	Linear	Yes	Full	Max. profit
San-José et al. (2018b)	Yes	Yes	Non-linear	Yes	Partial	Max. profit
San-José et al. (2019)	Yes	Yes	Non-linear	No		Max. profit
San-José et al. (2020)	Yes	Yes	Constant	Yes	Full	Max. profit
This paper	Yes	Yes	Non-linear	Yes	Full	Max. profit

ear function of time and taking into account inflation, Valliathal and Uthayakumar (2011) formulated a production inventory model assuming shortages with partial backordering. Pando et al. (2012) examined an inventory system without stockouts when both the holding cost and demand are nonlinear functions with respect to the time in storage. Sazvar et al. (2012) determined the optimal (r,Q) policy for a three-echelon supply chain with a nonlinear holding cost when the lead time for the purchaser is uncertain. Pando et al. (2013) analyzed an economic lot size inventory model for when the demand depends on inventory level, and the holding cost is a nonlinear function for both the amount of units in inventory and the time that these units are held in storage. Prasher and Pundir (2013) studied the nonlinearity of the holding cost with respect to the amount of on–hand inventory. Sazvar et al. (2013a) dealt with a continuous review inventory system, which assumes that the lead time is stochastic and the demand rate is constant and known during the course of an infinite planning horizon. Their inventory model uses the time dependent nonlinear holding cost function of Weiss (1982), and also allows for shortages that are fully backordered. Sazvar et al. (2013b) proposed a new way to determine the inventory up to a level policy for perishable goods with a normally distributed demand and lead time by including service level requirements. Other inventory models with a nonlinear holding cost have been derived by San-José et al. (2015), Khalilpourazari and Pasandideh (2017), Paknejad et al. (2018), Pando et al. (2018), San-José et al. (2018b), Edalatpour and Al-e-Hashem (2019), Pando et al. (2019), San-José et al. (2019), Tripathi (2019) and (Cárdenas-Barrón et al., 2020).

1.1.3. Price-dependent demand

Given that the demand of several products is influenced by price, an increase in price induces clients to buy fewer products. Alternatively, a low price motivates clients to buy more items. In this line of research, a large variety of inventory models have been developed to account for demand that is dependent on a linear price when analyzing inventory policies. The most recent works in this area include those by Jadidi et al. (2017), Panda et al. (2017), Rubio-Herrero and Baykal-Gursoy (2018), Marand et al. (2019) and San-José et al. (2019).

In the business world, the demand is affected by several factors. To model this complexity, the demand is defined as a function that depends simultaneously on some factors in additive form. In this context, Herbon and Khmelnitsky (2017) built an inventory model for deriving the optimal ordering and pricing policies for a perishable good by considering that the demand is influenced by time and price in an additive way. In the same direction, San-José et al. (2019) derived an inventory model for goods whose demand depends on both price and time. However, their inventory model did not consider the case of shortages and this modeling aspect is the focus of the present research.

1.2. Our contribution

The research work reported here develops and studies an inventory model for a product whose demand rate jointly combines the impact of the selling price and a time power function in an additive way. More specifically, the demand rate changes linearly at the same time with respect to selling price, and nonlinearly with respect to both time and the length of cycle time. The inventory model allows shortages that are completely backordered. Furthermore, the holding cost is a power function of the time period in storage. This means that the holding cost is nonlinear as reported by Weiss (1982).

The main aim of this research work is to simultaneously determine the selling price, order quantity, and backordering level in order to maximize the total profit per unit of time. To optimize the total profit, some theoretical results are derived first. Then, the theoretical results are used to develop an effective and efficient algorithm to obtain the optimal selling price, the optimal time at which the inventory level reaches zero, and the optimal cycle time that collectively maximize the total profit per unit of time. With these three optimal values, the optimal maximum inventory, the optimal order quantity, and the optimal backordering level are then calculated.

As can be seen in Table 1, and to the best of our knowledge, this is the first research paper that simultaneously considers the following characteristics in the inventory system: (i) the demand rate additively combines the effects of a time-power pattern and a selling price-linear function, (ii) a non-linear holding cost and (iii) shortages are allowed and completely backordered. Thus, for example, the differences between this paper and that developed by San-José et al. (2018b) is that, there the demand rate is ramp-type and, although there the demand also depends on the selling price, this price is fixed and, therefore, it is not a decision variable of the optimization problem.

The rest of the manuscript is organized as follows. Section 2 introduces the notation, assumptions and the mathematical formulation of the inventory model with a nonlinear holding cost, a power demand pattern, and full backordering. Section 3 presents the optimal inventory policy when the inventory level at the beginning of the scheduling period is zero. Section 4 derives the theoretical results and develops an efficient algorithm to determine the optimal solution for the inventory model when the inventory level attains zero at positive time. Section 5 solves five numerical examples. Section 6 presents a sensitivity analysis. Finally, Section 7 provides some conclusions and suggests some future lines of research.

2. The inventory problem

In this section, the notation and assumptions of the inventory system to be studied are introduced first. Then, a mathematical formulation of the problem is presented.

2.1. Notation

The nomenclature utilized for the development of the inventory model is shown below.

Parameters:

K = Ordering cost per order (>0).
p = Unit purchasing cost (>0).
w = Unit backordering cost per unit of time (>0).
h = Scale parameter for the holding cost (>0).
δ = Elasticity parameter for the holding cost ($\geqslant 1$).
α = Scale parameter for the part of the price-dependent demand (>0).
β = Sensitivity parameter for the demand with respect to price (>0).
γ = Scale parameter for the part of the time-dependent demand (>0).
n = Demand pattern index (>0).

Decision variables:

s = Unit selling price ($s \geqslant p$).
τ = Time period at which the inventory level is greater than or equal to zero (>0).
T = Cycle time (>0).

Dependent decision variables:

Q = Order quantity (>0).
b = Backordering level (>0).
I_m = Maximum inventory level (>0).

Functions:

$H(t)$ = Cumulative holding cost per unit maintained in storage during t units of time.
$D(s,t)$ = Demand rate at time t for a selling price s, with $0<t<T$.
$I(s,t)$ = Stock level at time t for a selling price s, with $0 \leqslant t<T$.
$TP(s,\tau,T)$ = Total profit per cycle per unit of time.

2.2. Assumptions

The inventory model in based on the following assumptions:

1. The inventory system is for a unique product.
2. The planning horizon is infinite.
3. The lot size Q is the order quantity to replenish the inventory.
4. The replenishment is instantaneous and the product is restocked in each inventory cycle T.
5. The purchasing cost p is known and fixed.
6. The selling price s is a value that must be obtained.
7. Shortages are permitted and these are completely backordered.
8. The ordering cost K is known and fixed. It is independent of the order quantity.
9. The demand rate $D(s,t)$ is a function, at the same time, of both the unit selling price and the time that the inventory is maintained in storage. It is assumed that $D(s,t) = D_1(s) + D_2(t)$, where $D_1(s)$ is the linear price-demand which is expressed as

$$D_1(s) = \alpha - \beta s, \text{ with } \alpha>0,\ \beta>0 \text{ and } p \leqslant s \leqslant \alpha/\beta$$

and $D_2(t)$ is the power-time demand which is denoted as

$$D_2(t) = \left(\frac{\gamma}{n}\right)\left(\frac{t}{T}\right)^{(1-n)/n}, \text{ with } \gamma>0 \text{ and } n>0.$$

Here, α represents the scale parameter in the linear price-demand, β is the coefficient of the selling price sensitivity, γ is the scale parameter of the time-dependent demand, and n is the index of the power time demand pattern. The index n represents the form in which the products are withdrawn from the stock in order to cover the customer demand. Thus, the demand rate jointly combines the impacts of the selling price and a time-power function. In order to illustrate the effect of the parameter n on the evolution of the net inventory level, we have depicted the function $I(s,t)$ for different demand pattern indexes in Figs. 1–3 A justification of the practical utility of the function $D_2(t)$ to represent the demand for certain items can be seen in San-José et al. (2017). We assume, as is done extensively in the literature, that the demand varies linearly with the selling price. This is wholly justified for some products in which demands are lost due to price sensitivity (see, e.g., Panda et al., 2017). Therefore, the function $D(s,t)$ allows us to describe the behavior of customer demand for a wide variety of products.

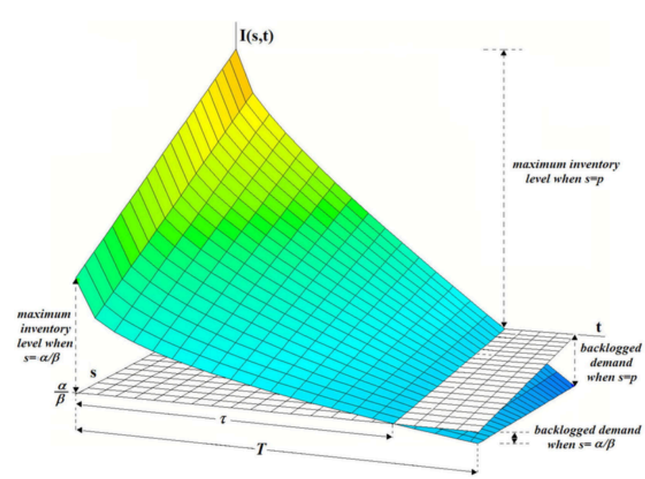

Fig. 1. Net stock level when $n>1$.

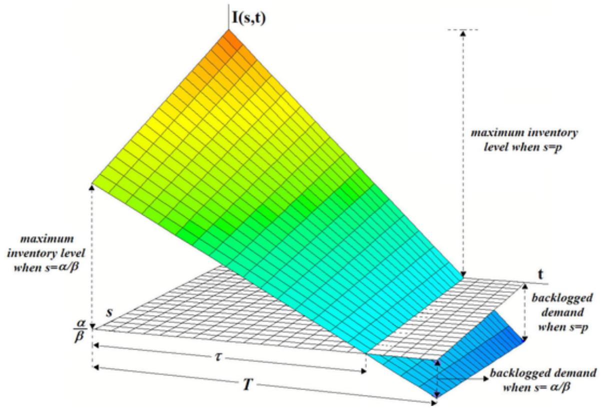

Fig. 2. Net stock level when $n = 1$.

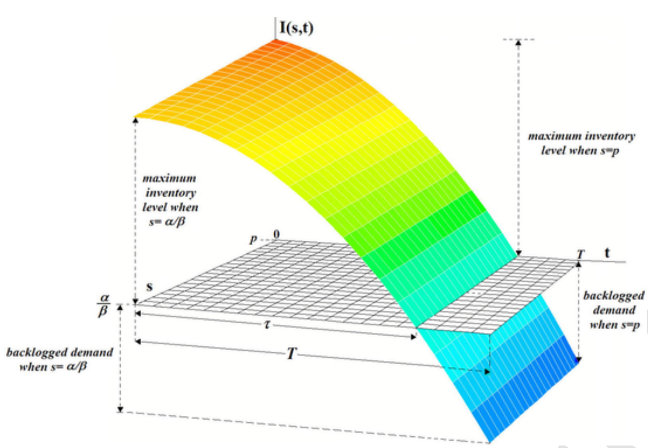

Fig. 3. Net stock level when $n<1$.

10. The cumulative holding cost $H(t)$ for a unit maintained in storage during t units of time is a power function of the time in stock. Hence, it is considered that $H(t) = ht^{\delta}$, here $h>0$ is the scale parameter of the holding cost and $\delta \geqslant 1$ is the elasticity parameter of the holding cost. That is, δrepresents the relative change in the holding cost related to the corresponding relative change over time, i.e., $\delta = (\partial H/\partial t)/(H/t)$. Therefore, in the linear case ($\delta = 1$), the holding cost per unit increases at a constant rate h, while in the general case ($\delta>1$), the unit holding cost increases slowly initially and subsequently grows faster.

2.3. Formulation of the inventory model

At the beginning of the inventory cycle (i.e., at $t = 0$), Q units are received and this quantity immediately decreases to I_m units in the stock due to the covering of shortages from the previous cycle. During the period $(0, \tau)$, the on-hand inventory level $I(s,t)$ decreases due to demand and eventually reaches zero at $t = \tau$, and then the occurrence of shortages starts which are accumulated until the maximum backordering level of b units is reached. The quantity I_m is determined with

$$I_m = \int_0^{\tau} D(s,u)\,du = (\alpha - \beta s)\,\tau + \gamma\tau^{\left(\frac{1}{n}\right)}T^{\left(\frac{n-1}{n}\right)}$$

For all $t \in [0, T)$, the inventory level at time t is computed as follows:

$$I(s,t) = I_m - \int_0^{t} D(s,u)du = I_m - \left[(\alpha - \beta s)\,t + \gamma t^{\left(\frac{1}{n}\right)}T^{\left(\frac{n-1}{n}\right)}\right]$$
$$= (\alpha - \beta s)(\tau - t) + \gamma T^{\left(\frac{n-1}{n}\right)}\left(\tau^{\left(\frac{1}{n}\right)} - t^{\left(\frac{1}{n}\right)}\right)$$

The backordering level is calculated with:

$$b = \int_{\tau}^{T} D(s,u)\,du$$
$$= (\alpha - \beta s + \gamma)\,T - I_m$$
$$= (\alpha - \beta s + \gamma)\,T - (\alpha - \beta s)\,\tau - \gamma\tau^{\left(\frac{1}{n}\right)}T^{\left(\frac{n-1}{n}\right)}$$

The order quantity is equal to the total demand during the cycle length, that is,

$$Q = \int_0^{\tau} D(s,t)\,dt + \int_{\tau}^{T} D(s,u)\,du = (\alpha - \beta s + \gamma)\,T.$$

Thus, it follows that $Q = I_m + b$.

By considering the above assumptions, the revenue and the inventory costs at each cycle are obtained below.

Revenue: sQ

Purchase cost: pQ

Ordering cost: K

Holding cost:

$$\int_0^{\tau} H(t)\,D(s,t)\,dt = h\left[\frac{(\alpha - \beta s)\,\tau^{(\delta+1)}}{(\delta+1)} + \frac{\gamma}{(n\delta+1)}\tau^{\left(\frac{1+n\delta}{n}\right)}T^{\left(\frac{n-1}{n}\right)}\right]$$

Backordering cost:

$$\int_{\tau}^{T} w\,[-I(s,t)]\,dt = w\left[\frac{(\alpha - \beta s)\left(T^2 - \tau^2\right)}{2} - (\alpha - \beta s)\left(T\tau - \tau^2\right) + \frac{n\gamma T^2}{(n+1)} - \frac{n\gamma}{(n+1)}\tau^{\left(\frac{n+1}{n}\right)}T^{\left(\frac{n-1}{n}\right)} - \gamma\tau^{\left(\frac{1}{n}\right)}T^{\left(\frac{n-1}{n}\right)}(T - \tau)\right]$$

The total profit per unit of time $TP(s,\tau,T)$ is calculated as the difference between the revenue per inventory cycle and the sum of the purchasing cost, the ordering cost, the holding cost, and the backordering cost per cycle. This difference is then divided by the cycle length T. Mathematically speaking,

$$TP(s,\tau,T) = \frac{1}{T}\left[(s - p)\,Q - K - \int_0^{\tau} H(t)\,D(s,t)dt - \int_{\tau}^{T} w[-I(s,t)]\,dt\right]$$
$$= \frac{1}{T}\Big\{(s - p)\,T\gamma + (s - p)\,(\alpha - \beta s)\,T - K$$
$$-h\left[\frac{(\alpha-\beta s)\tau^{(\delta+1)}}{(\delta+1)} + \frac{\gamma}{(1+n\delta)}\tau^{\left(\frac{1+n\delta}{n}\right)}T^{\left(\frac{n-1}{n}\right)}\right]$$
$$-w\left[\frac{(\alpha-\beta s)(T-\tau)^2}{2} + \frac{n\gamma T^2}{(n+1)} - \frac{n\gamma}{(n+1)}\tau^{\left(\frac{n+1}{n}\right)}T^{\left(\frac{n-1}{n}\right)} - \gamma\tau^{\left(\frac{1}{n}\right)}T^{\left(\frac{n-1}{n}\right.}\right.$$

Then, the objective is to maximize the total profit per unit of time $TP(s,\tau,T)$. Therefore, the optimization problem is formulated as below.

$$\max_{(s,\tau,T)\in\Omega} TP(s,\tau,T), \tag{2}$$

where $\Omega = \{(s,\tau,T) : T>0, 0 \leqslant \tau \leqslant T \text{ and } p \leqslant s \leqslant \alpha/\beta\}$.

Notice that the above problem is a nonlinear optimization problem. In the next section, we begin determining the optimal solution of the inventory problem for the case in which the stocking period $\tau$is equal to zero. Then, in Section 4, we will develop a procedure to search the solution of the inventory problem for the scenario $0<\tau \leqslant T$. In that case, the solution to the optimization problem can be obtained with an algorithm and by applying a numerical method to solve the nonlinear equations. In general, it is not possible to determine the closed form expressions for the decision variables.

3. Analysis of the optimal inventory policy when $\tau = 0$

In this section, the function $TP(s, \tau, T)$ is studied when $\tau = 0$. Mathematically speaking,

$$TP(s, 0, T) = (\alpha - \beta s + \gamma)(s - p) - \frac{K}{T} - \omega\left(\frac{\alpha - \beta s}{2} + \frac{n\gamma}{n+1}\right)T$$

This case can be interpreted as the analysis of make-to-order production. In this situation, there are fully backordered stockouts during the inventory cycle, and backorders are supplied with the arrival of the new replenishment of items.

Let us consider T as a given value. Then the first derivative of $TP(s, \tau, T)$ with respect to s is

$$\frac{d}{ds}TP(s, 0, T) = \alpha + \beta(p - 2s) + \gamma + \frac{\omega\beta T}{2}$$

This derivative is, evidently, a decreasing function in s, and at the point $s = p$ takes the value of $\alpha - \beta p + \gamma + \frac{\omega\beta T}{2}>0$. Additionally, this derivative has one root at the point

$$s_0(T) = \frac{\alpha + \beta p + \gamma}{2\beta} + \frac{\omega}{4}T$$

For a fixed value of T, since the second derivative $\frac{d^2}{ds^2}TP(s, 0, T) = -2\beta<0$, the function $TP(s, 0, T)$ is strictly concave in the interval $(0, \infty)$. Thus, the maximum value of $TP(s, 0, T)$ is attained at the point

$$s_0^*(T) = \begin{cases} \frac{\alpha+\beta p+\gamma}{2\beta} + \frac{\omega}{4}T & \text{if } T<T_0 \\ \frac{\alpha}{\beta} & \text{si } T \geqslant T_0 \end{cases} \quad (3)$$

where

$$T_0 = \frac{2}{\omega\beta}(\alpha - \beta p - \gamma). \quad (4)$$

Furthermore, it easy to see that $T<T_0$ if and only if $s_0(T)<\alpha/\beta$.

Notice that always $s_0(T)>p$ because, if $s_0(T) \leqslant p$, then $T \leqslant -2(\alpha - \beta p + \gamma)/\beta\omega<0$, and it is absurd, since T must be always positive.

Evidently, two possible cases must be analyzed:

1. If $\alpha - \beta p>\gamma$, then $T_0>0$ and the value of the function at the point $s_0^*(T)$ is,

$$TP_0(T) = TP\left(s_0^*(T), 0, T\right) = \begin{cases} P_1(T) & \text{if } T<T_0 \\ P_2(T) & \text{if } T \geqslant T_0 \end{cases} \quad (5)$$

where

$$P_1(T) = \frac{(\alpha - \beta p + \gamma)^2}{4\beta} - \frac{K}{T} - \frac{\omega}{4}\left(\alpha - \beta p + \frac{3n-1}{n+1}\gamma - \frac{\omega\beta}{4}T\right)T \quad (6)$$

and

$$P_2(T) = \frac{(\alpha - \beta p)\gamma}{\beta} - \frac{K}{T} - \frac{\omega n\gamma}{n+1}T \quad (7)$$

2. If $\alpha - \beta p \leqslant \gamma$, then $T_0 \leqslant 0$ and $TP_0(T) = P_2(T)$, due to always $T>T_0$.

Some properties of the function $TP_0(T)$ given in (5) are:

1. $\lim_{T\to0^+}TP_0(T) = \lim_{T\to\infty}TP_0(T) = -\infty$.
2. It is continuous, since

$$P_1(T_0) = -\frac{K\omega\beta}{2(\alpha-\beta p-\gamma)} + \frac{\gamma}{(n+1)\beta}[(\alpha - \beta p)(1 - n) + 2\gamma n] = P_2(T_0)$$

3. It is differentiable and its derivative is continuous:

$$P_1'(T) = \frac{K}{T^2} - \frac{\omega}{4}\left(\alpha - \beta p + \frac{3n-1}{n+1}\gamma\right) + \frac{\omega^2\beta}{8}T = \frac{1}{T^2}f_0(T),$$

where

$$f_0(T) = K - \frac{\omega}{4}\left(\alpha - \beta p + \frac{3n-1}{n+1}\gamma\right)T^2 + \frac{\omega^2\beta}{8}T^3. \quad (8)$$

Also, we have

$$P_2'(T) = \frac{K}{T^2} - \frac{\omega n\gamma}{n+1}$$

Then

$$P_1'(T_0) = \frac{K\omega^2\beta^2}{4(\alpha - \beta p - \gamma)^2} - \frac{\omega n\gamma}{n+1} = P_2'(T_0)$$

4. The function $P_2'(T)$ has a unique positive root at point

$$\hat{T}_2 = \sqrt{\frac{(n+1)K}{\omega n\gamma}} \quad (9)$$

which corresponds to the maximum of the function $P_2(T)$, due to $P_2''(T) = -2K/T^3<0$ (the function $P_2(T)$ is strictly concave).

Analysis of the function $f_0(T)$

1. $\lim_{T\to0^+}f_0(T) = K$
2. $\lim_{T\to T_0^-}f_0(T) = K - \frac{4\gamma n(\alpha-\beta p-\gamma)^2}{\beta^2\omega(n+1)}$ and $\lim_{T\to\infty}f_0(T) = \infty$.
3. $f_0'(T) = \left[-\frac{\omega}{2}\left(\alpha - \beta p + \frac{3n-1}{n+1}\gamma\right) + \frac{3\omega^2\beta}{8}T\right]T$ and, thus, $f_0'(T)$ has one root in the point

$$T_1 = \frac{4}{3\omega\beta}\left(\alpha - \beta p + \frac{3n-1}{n+1}\gamma\right). \quad (10)$$

4. $f_0''(T) = \frac{3\omega^2\beta}{4}T - \frac{\omega}{2}\left(\alpha - \beta p + \frac{3n-1}{n+1}\gamma\right) = \frac{3\omega^2\beta}{4}\left(T - \frac{T_1}{2}\right)$ y
$f_0''(T_1) = \frac{3\omega^2\beta}{8}T_1 = \frac{\omega}{2}\left(\alpha - \beta p + \frac{3n-1}{n+1}\gamma\right)$.

Determination of the optimal value of T

The following theoretical result permits the determination of the optimal inventory policy when $\tau = 0$.

Theorem 1 *Given* $T_0 = 2(\alpha - \beta p - \gamma)/\omega\beta, T_1 = 4\left[\alpha - \beta p + (3n-1)\gamma/(n+1)\right]/3\omega\beta, f_0(T)$ *the function given by (8),* $\widehat{T}_1 = \arg\{f_0(T) = 0 : T \in (0, T_0)\}, \widetilde{T}_1 = \arg\{f_0(T) = 0 : T \in (0, T_1)\}$ *and* $\widehat{T}_2 = \sqrt{(n+1)K/\omega n\gamma}$. *The maximum value of the function* $TP(s, 0, T)$ *is attained at the point* (s^*, T^*) *given by* Table 2, *where* $s_0(T) = (\alpha + \beta p + \gamma)/2\beta + \omega T/4$, *and* $P_1(T)$ *y* $P_2(T)$ *are given by (6) y (7), respectively.*

Proof See Appendix A. ■

The following corollary states the optimal benefit when $\tau = 0$.

Corollary 1 *If* $T^* < T_0$*, then the maximum profit per unit time is*

$$P^* = TP(s^*, 0, T^*) = \frac{(\alpha - \beta p + \gamma)^2}{4\beta} - \frac{\beta\omega^2}{16}(T^*)^2 - \frac{2K}{T^*}$$

Otherwise, the maximum benefit per unit time is

$$P^* = TP\left(s^*, 0, \widehat{T}_2\right) = \frac{(\alpha - \beta p)\gamma}{\beta} - 2\sqrt{\frac{nK\omega\gamma}{n+1}}$$

Proof If $T^* < T_0$, then from (6), $P^* = P_1(T^*)$. Now, as $f_0(T^*) = 0$, from (8), $\frac{K}{T} = \frac{\omega}{4}\left(\alpha - \beta p + \frac{3n-1}{n+1}\gamma\right)T - \frac{\omega^2\beta}{8}T^2$. Substituting the right side of equality into (6), we obtain the expression proposed.Otherwise, $T^* = \widehat{T}_2$ and $P^* = P_2(T^*)$. From (9), substituting $\widehat{T}_2$ into (7), the desired expression is obtained. ■

Corollary 2 *If* $s^* = \alpha/\beta$*, then* $T^* = \widehat{T}_2$*.Otherwise:*

1. *If* $T_0 \leqslant T_1$*, then* $T^* = \widehat{T}_1$.
2. *If* $T_0 > T_1$*, then* $T^* = \widetilde{T}_1$.

Proof It is easily deduced from Table 2. ■

4. Theoretical results and optimal solution for the inventory model when $0 < \tau \leqslant T$

In this section, we analyze the inventory problem proposed in (2) when $0 < \tau \leqslant T$. We start presenting the optimal policy when $\tau = T$.

A) If $\tau = T$, Eq. (1) is reduced to

$$TP(s, \tau = T, T) = \frac{1}{T}\left((s-p)Q - K - \int_0^\tau H(t)D(s,t)\,dt\right)$$
$$= \frac{1}{T}\left[(s-p)T\gamma + (s-p)(\alpha - \beta s)T - K - h\left(\frac{(\alpha-\beta s)}{(\delta+1)} + \frac{\gamma}{(1+\right.\right.$$

Now let,

$$b(s) = \left\{\frac{(\alpha - \beta s)}{(\delta + 1)} + \frac{\gamma}{(1 + n\delta)}\right\}$$

Then,

$$TP(s, \tau = T, T) = \frac{1}{T}\left[(s-p)T\gamma + (s-p)(\alpha - \beta s)T - K - hb(s)T^{(\delta+1)}\right]$$
$$= (s-p)(\alpha - \beta s + \gamma) - \left(\frac{K}{T} + hb(s)T^{\delta}\right)$$

This is the profit function of the inventory model without shortages of San-José et al. (2019). Therefore, in this case, the optimal solution of the inventory problem can be obtained applying the algorithm proposed by those authors.

B) In the following paragraphs, we search for the optimal inventory policy for the case $0 < \tau < T$.

Given selling price s and time τ at which the inventory reaches zero, obtaining the first-order derivative of $TP(s, \tau, T)$ with respect to T and setting this equal to zero, then a necessary condition for the optimal inventory cycle time T^* is

$$K + \frac{h(\alpha - \beta s)\tau^{(\delta+1)}}{\delta + 1} + \frac{h\gamma}{n(1+n\delta)}\tau^{\left(\frac{1+n\delta}{n}\right)}T^{\left(\frac{n-1}{n}\right)} + \frac{w(\alpha - \beta s)(\tau^2 - T^2)}{2} - \frac{nw\gamma T^2}{(n+1)} + \frac{w\gamma}{n(n+1)}\left[(n^2-1)T + \tau\right]\tau^{\left(\frac{1}{n}\right)}T^{\left(\frac{n-1}{n}\right)} = 0 \tag{11}$$

For a detailed derivation of this result see Appendix B.

To maximize the total profit $TP(s, \tau, T)$ subject to $0 < \tau < T$, some theoretical results in concave fractional programming are utilized. Cambini and Martein (2009) state that a real function $f(x)$ defined on an open convex set and represented by $f(x) = \frac{y(x)}{g(x)}$ is (strictly) pseudo concave if $y(x)$ is non-negative, differentiable and (strictly) concave, and $g(x)$ is positive, differentiable and convex. For simplicity, the following value is defined

$$J = \left[\frac{w(n-1)\gamma}{n(n+1)}\tau^{\left(\frac{n+1}{n}\right)} + \frac{w(n-1)\gamma}{n^2}\tau^{\left(\frac{1}{n}\right)}(T - \tau) - \frac{h(n-1)\gamma}{n^2(1+n\delta)}\tau^{\left(\frac{1+n\delta}{n}\right)}\right]T^{-\left(\frac{n+1}{n}\right)} + w\left[(\alpha - \beta s) + \frac{2n\gamma}{(n+1)} - \frac{2(n-1)\gamma}{n}\tau^{\left(\frac{1}{n}\right)}T^{\left(\frac{-1}{n}\right)}\right]. \tag{12}$$

For a fixed and given s and τ, applying the theoretical result from Cambini and Martein (2009), it is easy to prove that total profit $TP(s, \tau, T)$ is strictly pseudo-concave with respect to T if $J > 0$. Therefore, in this case, there exists a unique global optimal solution T^* such that $TP(s, \tau, T)$ is maximized.

Table 2
Optimal inventory policy (s^*, T^*) when $\tau = 0$.

		$f_0(T_0) < 0$	$f_0(T_0) = 0$	$f_0(T_0) > 0$
$0 < T_0 \leqslant T_1$		$\left(s_0(\widehat{T}_1), \widehat{T}_1\right)$	$\left(\alpha/\beta, \widehat{T}_2\right)$	$\left(\alpha/\beta, \widehat{T}_2\right)$
$0 < T_1 < T_0$	$f_0(T_1) < 0$	$\left(s_0(\widetilde{T}_1), \widetilde{T}_1\right)$	$\left(s_0(\widetilde{T}_1), \widetilde{T}_1\right)$	$\begin{cases}\left(s_0(\widetilde{T}_1), \widetilde{T}_1\right) & \text{if } P_1(\widetilde{T}_1) \geqslant P_2(\widehat{T}_2) \\ \left(\alpha/\beta, \widehat{T}_2\right) & \text{if } P_1(\widetilde{T}_1) < P_2(\widehat{T}_2)\end{cases}$
	$f_0(T_1) \geqslant 0$	–	–	$\left(\alpha/\beta, \widehat{T}_2\right)$
$T_0 \leqslant 0$		$\left(\alpha/\beta, \widehat{T}_2\right)$	$\left(\alpha/\beta, \widehat{T}_2\right)\left(\alpha/\beta, \widehat{T}_2\right)$	

Note. The symbol "–" means that this situation cannot occur.

Theorem 2 *Given selling price s and time τ at which the inventory reaches zero, if $J>0$ then $TP(s,\tau,T)$ given by (1) is a strictly pseudo-concave function with respect to T, and there exists a unique maximum solution for T^*.*

Proof See Appendix C. ■

Likewise, for any given T, the function $TP(s,\tau,T)$ given by (1) is a strictly concave function in both s and τ if some conditions are hold. For simplicity, define

$$M = h\beta\tau^{\delta} + w\beta(\tau - T)$$

$$N = -\left\{ h\left[(\alpha-\beta s)\,\delta\tau^{(\delta-1)} + \frac{\gamma(1+n\delta-n)}{n^2}\tau^{\left(\frac{1+n\delta-2n}{n}\right)}T^{\left(\frac{n-1}{n}\right)}\right] + w\left[(\alpha-\beta s) + \frac{\gamma(n-1)}{n^2}\tau^{\left(\frac{1-2n}{n}\right)}T^{\left(\frac{n-1}{n}\right)}(T-\tau) + \frac{\gamma}{n}\tau^{\left(\frac{1-n}{n}\right)}T^{\left(\frac{n-1}{n}\right)}\right]\right\}$$

$$L = -2\beta T$$

Note that $L<0$ always. By assuming that $N<0$ and $M^2<LN$, then the following theorem is stated.

Theorem 3 *For any given cycle time T, if $N<0$ and $LN-M^2>0$, then $TP(s,\tau,T)$ given by (1) is a strictly concave function in both s and τ, and, consequently, there exists a unique maximum solution for s^* and τ^*.*

Proof See Appendix D. ■

For any given cycle time T, by obtaining the first order partial derivative of $TP(s,\tau,T)$ with respect to s, and setting this equal to zero, then a necessary condition for the optimal selling price s is determined. Thus, the condition is

$$(\alpha - 2\beta s + \beta p + \gamma)\,T + \frac{h\beta\tau^{(\delta+1)}}{(\delta+1)} + \frac{w\beta}{2}(T-\tau)^2 = 0 \tag{13}$$

Similarly, we conclude that a necessary condition for the optimal time τ at which the inventory level attains to zero is

$$h\left\{(\alpha-\beta s)\,\tau^{\delta} + \frac{\gamma}{n}\tau^{\left(\frac{1+n\delta-n}{n}\right)}T^{\left(\frac{n-1}{n}\right)}\right\} - w\left\{(\alpha-\beta s)(T-\tau) + \frac{\gamma}{n}\tau^{\left(\frac{1-n}{n}\right)}T^{\left(\frac{n-1}{n}\right)}(T-\tau)\right\} = 0 \tag{14}$$

For a detailed derivation of these two conditions, see Appendix E.

Eq. (14) can be reduced to

$$\left[h\tau^{\delta} - w(T-\tau)\right]\left[\alpha - \beta s + \frac{\gamma}{n}\tau^{\left(\frac{1-n}{n}\right)}T^{\left(\frac{n-1}{n}\right)}\right] = 0$$

Therefore, as $0<\tau<T$ and $s \leqslant \alpha/\beta$, it is easy to show that the optimal cycle time T is given by

$$T = \tau + \frac{h\tau^{\delta}}{w} \tag{15}$$

Substituting (15) into Eq. (13), it is straightforward to prove that the selling price is given by

$$s = \frac{\alpha + \beta p + \gamma}{2\beta} + \frac{h\tau^{\delta}}{4}\left[1 - \frac{(\delta-1)\,w}{(\delta+1)\left(w + h\tau^{\delta-1}\right)}\right] \tag{16}$$

Notice that the selling price has the following constraint $s \leqslant \alpha/\beta$. Therefore, any value of s given by (16) such that $s \geqslant \alpha/\beta$ implies that it is not allowed. Thus, in this case, the solution for the selling price must be $s^* = \alpha/\beta$. From (16), the condition $s \geqslant \alpha/\beta$ is equivalent to

$$\frac{\alpha - \beta p - \gamma}{2\beta} - \frac{h\tau^{\delta}}{4}\left[1 - \frac{(\delta-1)\,w}{(\delta+1)\left(w + h\tau^{\delta-1}\right)}\right] \leqslant 0.$$

Since the last term of the above expression is negative, it leads to $s \geqslant \alpha/\beta$ when $\alpha - \beta p - \gamma \leqslant 0$. Consequently, if $\alpha - \beta p \leqslant \gamma$, then the optimal selling price is $s^* = \alpha/\beta$.

Let (s_1, τ_1) be the point obtained by equating to zero the first partial derivatives with respect to s and τ. Evaluating M and N at this point, we have $M = 0$ and

$$N = -\left[\alpha - \beta s + \frac{\gamma}{n}\left(1 + \frac{h}{w}\tau^{\delta-1}\right)^{1-1/n}\right]\left(w + \delta h\tau^{\delta-1}\right) < 0$$

Thus, taking into account Eqs. (15), (16), we conclude that for any given T, there exists a unique optimal solution (s_1, τ_1).

Considering the theoretical results derived above, the following algorithm is constructed. The procedure uses the sets Σ, S and S_1. The set Σ contains all the potential solutions obtained along the algorithm. The set S collects the positive points τ that are solutions of Eq. (11) when the selling price is determined by Eq. (16). The set S_1 includes the points τ obtained solving Eq. (11) when the selling price is $s = \alpha/\beta$.

Algorithm

Step 0. Input the inventory parameters.

Step 1. By using Theorem 1, obtain the optimal inventory policy (s_1^*, T_1^*) when $\tau = 0$ and calculate its profit $TP(s_1^*, 0, T_1^*)$. Go to Step 2.

Step 2. By using the algorithm proposed by San-José et al., 2019, obtain the optimal inventory policy (s_2^*, T_2^*) when $\tau = T$ and calculate $TP(s_2^*, T_2^*, T_2^*)$. Go to Step 3.

Step 3. Set $\Sigma = \{(s_1^*, 0, T_1^*), (s_2^*, T_2^*, T_2^*)\}$ and $k = 0$. Go to Step 4.

Step 4. If $\alpha - \beta p \leqslant \gamma$ then go to Step 7. Otherwise, go to Step 5.

Step 5. Using T given in Eq. (15) and s given in Eq. (16), determine the set S of real positive points τ that solve Eq. (11). Go to Step 6.

Step 6. While $|S|>0$ do

$k = k + 1$.
Obtain $\tau_k = \min\{\tau : \tau \in S\}$ and set $S = S \setminus \{\tau_k\}$.
Calculate s_k with τ_k and Eq. (16).
If $s_k<\alpha/\beta$, then:
(a) Determine T_k with τ_k and Equation (15).
(b) From Eq. (12), obtain $J(s_k, \tau_k, T_k)$.
(c) If $J(s_k, \tau_k, T_k)>0$, then put $\Sigma = \Sigma \cup \{(s_k, \tau_k, T_k)\}$ and calculate $TP(s_k, \tau_k, T_k)$.
End_While.

Step 7. Using T given in Eq. (15) and $s = \alpha/\beta$, determine the set S_1 of positive points τ that solve Eq. (11). Go to Step 8.

Step 8. Set $j = 0$.

While $|S_1|>0$ do.
$j = j + 1$.
$\tau_j = \min\{\tau : \tau \in S_1\}$. Put $S_1 = S_1 \setminus \{\tau_j\}$ and $s_j = \alpha/\beta$.
Calculate T_j with τ_j and Eq. (15). From Eq. (12), obtain $J(s_j, \tau_j, T_j)$.
If $J(s_j, \tau_j, T_j)>0$, then put $\Sigma = \Sigma \cup \{(s_j, \tau_j, T_j)\}$ and calculate $TP(s_j, \tau_j, T_j)$.
End_While.

Step 9. Set (s^*, τ^*, T^*) the inventory policy such that its profit $TP(s^*, \tau^*, T^*)$ is the greatest profit per unit time of the inventory policies belonging to the set Σ. Go to Step 10.

Step 10. Report the optimal solution for $s^*, \tau^*, T^*, I_m^*, Q^*, b^*$ and $TP^*(s^*, \tau^*, T^*)$.

Step 11. Stop.

Remark

In the inventory model analyzed here, customer demand is additively affected by both price and time. For that, it is likely that the managerial decisions depend on the parameters associated with both demand components. Thus, $\alpha - \beta p$ represents the price-dependent average demand when the product is sold at the purchasing price (note that it also is the maximum price-dependent demand rate), while γ is the average demand during the inventory cycle that is dependent on time. Hence, $\alpha - \beta p - \gamma$ is the gap between the average demand due to the selling price if the item was sold at the purchasing price and the average demand due to the variation of customer orders over time.

Taking into account the above paragraph, in Step 4 of the algorithm, if the condition $\alpha - \beta p \leqslant \gamma$ is satisfied, then the optimal selling price must always be the maximum possible value, that is, $s^* = \alpha/\beta$. However, if $\alpha - \beta p > \gamma$, then the selling price will depend on the solutions obtained by solving Eq. (16).

Special cases

It is important to highlight that the inventory model developed in this research is a general model which contains several previously published inventory models as special cases. The conditions that make it possible to reduce the model analyzed here to the other inventory models are outlined in Table 3.

5. Numerical examples

With the aim to illustrate the proposed inventory model and accompanying algorithm, this section presents and solves five numerical examples. The data for each example are taken from San-José et al. (2019), adding a backordering cost w.

Example 1 Consider an inventory system with the following parameters: $\alpha = 120, \beta = 1, \gamma = 10, n = 0.5, K = 200, p = 40, h = 1.05, \delta = 1.5$ and $w = 0.25$. By applying the algorithm, the optimal solution is as follows: time at which the inventory level reaches zero is $\tau^* = 1.192677$, the selling price is $s^* = 85.32967$, the inventory cycle time is $T^* = 6.663257$ and the maximum profit per unit of time is $TP^* = 1966.683$. Additionally, the optimal values for the dependent variables are: order quantity is $Q^* = 297.6499, I_m^* = 43.4853$, and the backordering level is $b^* = 254.1646$. In order to ensure that solution to the inventory problem is optimal, it is needed to satisfy the following conditions: $J>0, L<0, N<0, LN - M^2>0$. For this example, all conditions are satisfied: $J = 10.536>0, L = -13.32651<0, N = -75.354920<0, LN - M^2 = 1004.218>0$

Example 2 Consider the following input parameters: $\alpha = 120, \beta = 1, \gamma = 60, n = 25, K = 1600, p = 35, h = 1.5, \delta = 2$ and $w = 0.75$. By using the algorithm, the optimal inventory policy is as follows: $\tau^* = 2.570826, s^* = 109.8439, T^* = 15.78912, TP^* = 5064.396, Q^* = 1107.703, I_m^* = 907.1134$ and $b^* = 200.5896$. The conditions are satisfied: $J = 14.13317>0, L = -31.578250<0, N = -201.94690<0, LN - M^2 = 6377.129>0$

Example 3 Take into consideration the same data as in Example 2, but modify the values of K, p, γ and w to $K = 1000, p = 55, \gamma = 40$ and $w = 1.5$, respectively. By employing the algorithm, then the optimal inventory policy is given by $\tau^* = 2.790789, s^* = 110.1639, T^* = 10.57929, TP^* = 2578.897, Q^* = 527.2310, I_m^* = 428.6566$ and $b^* = 98.57446$. The conditions are satisfied: $J = 21.64756>0, L = -21.158580<0, N = -153.87630<0, LN - M^2 = 3255.805>0$

Example 4 Take into account the same parameters as in Example 2, but change the values of n, γ and w to $n = 2, \gamma = 80$ and $w = 1.75$, respectively. By applying the algorithm, the following optimal inventory policy is obtained: $\tau^* = 2.158918, s^* = 119.0435$, $T^* = 6.153996$ $TP^* = 6350.918, Q^* = 498.2062, I_m^* = 293.6645$ and $b^* = 204.5417$. The conditions are satisfied: $J = 110.9445>0, L = -12.30799<0, N = -563.4523<0, LN - M^2 = 6934.966>0$

Example 5 Consider the same data as in Example 1, but modify the values of β, and w to $\beta = 2.8$ and $w = 2$ respectively. By utilizing the algorithm, the optimal inventory policy is given by $\tau^* = 3.156389, s^* = 42.85714, T^* = 6.100438, TP^* = 0.000000, Q^* = 61.00438, I_m^* = 16.33125$ and $b^* = 44.67305$. The conditions are satisfied: $J = 23.39244>0, L = -34.162410<0, N = -49.65189<0, LN - M^2 = 1696.23>0$

Notice that the total profit is zero. Therefore, the inventory system is non-profitable for any unit selling price.

Table 3
Special cases.

Conditions	Inventory models
$w \to \infty$ and $\tau \to T$	San-José et al. (2019)
$w \to \infty$, $\tau \to T$, $n = 1$ and $\alpha, \beta \to 0$	Weiss (1982) and Ferguson et al. (2007)
$w \to \infty$, $\tau \to T$, $\delta = 1$ and $\alpha, \beta \to 0$	Sicilia et al. (2012) (inventory system without shortages)
$w \to \infty$, $\tau \to T$, $n = 1$, $\delta = 1$ and $\gamma \to 0$	Kunreuther and Richard (1971) and Smith et al. (2007), considering a linear demand
$w \to \infty$, $\tau \to T$, $n = 1$, $\delta = 1$ and $\gamma \to 0$	Kabirian (2012) considering a constant production cost, a linear demand rate and the production rate tends to infinity
$\delta = 1$, $\alpha, \beta \to 0$	Sicilia et al. (2012) (inventory system with full back-ordering)
$n = 1$, $\delta = 1$ and $\beta, \gamma \to 0$	Hadley and Whitin (1963) (inventory system with full backordering)
$w \to \infty$, $\tau \to T$, $n = 1$, $\delta = 1$ and $\beta, \gamma \to 0$	Harris (1913)

6. Sensitivity analysis

This section presents a sensitivity analysis. For this, several examples are considered in which some of the input parameters are fixed, while the rest of the parameters are allowed to vary across a range of values. The computational runs were performed on a computer HP Elite 8300 *Intel*® Core™ i5-3470M CPU @ 3.30 GHz, 8.00 GB RAM, and 64-bit Windows 7 operating system). It is important to remark that the run times required to solve the numerical instances are insignificant, because the computer solves them instantaneously.

6.1. Effects of the parameters p, n and δ on the optimal policy

Consider an inventory model with the following input parameters: $\alpha = 120, \beta = 1, \gamma = 10, K = 200, h = 1.5$ and $w = 0.25$. To examine the impact of the unit purchasing cost p, the demand pattern index n and the holding cost elasticity δ on the optimal inventory policy, a sensitivity analysis is done when $p \in \{45, 50, 55, 65\}, n \in \{0.5, 1, 2, 4\}$ and $\delta \in \{1, 1.5, 2\}$. Table 4 shows the impact of the parameters p, n and δ on the optimal inventory policy: $s^*, \tau^*, T^*, Q^*, I_m^*, b^*$ and TP^*. The information presented in Table Table 2 reveals the following relevant insights about the inventory model:

1. By considering n and δ as fixed, the optimal unit selling price s^*, the optimal time τ^* at which inventory level reaches zero, and the optimal inventory cycle T^* increment as the unit purchasing cost p increases. But, the optimal maximum inventory level I_m^*, the optimal order quantity Q^*, the optimal backordering level b^*, and the optimal maximum profit per unit of time TP^* decrement as p increases.
2. By fixing n and p, the optimal unit selling price s^*, the optimal time τ^* at which inventory level reaches zero, the optimal inventory cycle T^*, the optimal maximum inventory level I_m^*, and the optimal order quantity Q^* decrease as the unit holding cost elasticity δ increments. In contrast, the optimal maximum profit per unit of time TP^* increases as δ increments.
3. By placing p and δ as fixed, the optimal unit selling price s^*, the optimal time τ^* at which inventory level reaches zero, the optimal inventory cycle T^*, the optimal order quantity Q^*, and the optimal maximum profit per unit of time TP^* decrement as the power demand index n increases for values of $n \leqslant 1$; and increment as the power demand index n increases when $n \geqslant 1$. However, the optimal maximum inventory level I_m^* increases and optimal backordering level b^* decrements when the power demand index n increases.

6.2. Effects of the parameters h, w and γ on the optimal policy

Now, consider an inventory system with the following data: $\alpha = 120, \beta = 1, n = 0.25, K = 1000, p = 40$ and $\delta = 2$. To study the effects of the scale parameter of the holding cost h, the backordering cost w, and the scale parameter for the part of the time-dependent demand γ on the optimal inventory policy, a sensitivity analysis is carried out when $\gamma \in \{30, 40, 50\}, w \in \{1.5, 2, 2.5, 3\}$ and $h \in \{0.75, 1, 1.5, 1.75\}$. Table 5 shows the effects of the parameters h, w, and γ on the optimal inventory policy: $s^*, \tau^*, T^*, Q^*, I_m^*, b^*$ and TP^*. From the information shown in Table 5, the following significant insights about the inventory model are observed.

(i) When w and γ are fixed, the optimal time τ^* at which the inventory level reaches zero, the optimal inventory cycle T^*, the optimal maximum inventory level I_m^*, the optimal order quantity Q^*, and the optimal maximum profit per unit of time TP^* decrease as the scale parameter of the holding cost h increments. Conversely, the optimal unit selling price s^* and the optimal backordering level b^* increase as h increments.

(ii) If w and h are fixed, the optimal time τ^* at which inventory level reaches zero, the optimal inventory cycle T^* and the optimal maximum inventory level I_m^* decrement as the scale parameter of the holding cost γ increases. In contrast, the optimal unit selling price s^*, the optimal order quantity Q^*, the optimal backordering level b^*, and the optimal maximum profit per unit of time TP^* increment as γ increases.

(iii) If h and γ are fixed, the optimal unit selling price s^*, the optimal time τ^* at which inventory level reaches zero, and the optimal maximum inventory level I_m^* increase as the backordering cost w increments. However, the optimal inventory cycle T^*, the optimal order quantity Q^*, the optimal backordering level b^*, and the optimal maximum profit per unit of time TP^* decrement when w increases.

6.3. Effects of the parameters K, h and w on the optimal policy

Now assume an inventory system with: $\alpha = 120, \beta = 1, \gamma = 40, n = 0.25, p = 40$ and $\delta = 2$. To investigate the effects of the ordering cost K, scale parameter of the holding cost h, and the backordering cost w on the optimal policy, a sensitivity analysis is performed when $K \in \{1000, 1500, 2000, 2500\}, w \in \{1.5, 2, 2.5, 2.75\}$ and $h \in \{1, 1.5, 1.75\}$. Table 6 presents the impact of the parameters K, h and w on the optimal inventory policy: $s^*, \tau^*, T^*, Q^*, I_m^*, b^*$ and TP^*. From the information displayed in Table 6, the following significant insights are obtained.

(a) When K and w are fixed, the optimal time τ^* at which the inventory level reaches zero, the optimal inventory cycle T^*, the optimal maximum inventory level I_m^*, the optimal order quantity Q^*, and the optimal maximum profit per unit of time TP^* decrease as the scale parameter of the holding cost h increases. However, the optimal unit selling price s^* and the optimal backordering level b^* increase as h increases.

(b) If w and h are fixed, the optimal unit selling price s^*, the optimal time τ^* at which inventory level reaches zero, the optimal inventory cycle T^*, the optimal order quantity Q^*, the optimal maximum inventory level I_m^*, and the optimal backordering level b^* increase as the ordering cost K increases. Conversely, the optimal maximum profit per unit of time TP^* decreases as the ordering cost K increments.

(c) With K and h as fixed values, the optimal unit selling price s^*, the optimal time τ^* at which inventory level reaches zero, and the optimal maximum inventory level I_m^* increase as the backordering cost w increases. However, the optimal inventory cycle T^*, the optimal order quantity Q^*, the optimal backordering level b^*, and the optimal maximum profit per unit of time TP^* decrease when w increases.

(d) From Figs. 4–6, it is deduced that when h is fixed and w increases then the optimal order quantity Q^*, the optimal backordering level b^*, the optimal inventory cycle T^*, and the optimal maximum profit per unit of time TP^* decrease. In contrast, the optimal maximum inventory level I_m^*, the optimal unit selling price s^*, and the optimal time τ^* at which inventory level reaches zero increase.

(e) From Figs. 4–6, it is observed that when h is fixed and K increases then the optimal order quantity Q^*, the optimal backordering level b^*, the optimal inventory cycle T^*, the optimal maximum inventory level I_m^*, the optimal unit selling price s^* and the optimal time τ^* at which inventory level reaches zero increase, while the optimal maximum profit per unit of time TP^* decreases.

Table 4
Impacts of the parameters p, n and δ on the optimal inventory policy.

n	p	s	τ	T	τ/T	I_m	Q	b	TP
$\delta = 1$									
0.5	45	87.86710	0.978940	6.852582	0.142857	32.85467	288.7191	255.8645	1747.743
	50	90.37928	1.011421	7.079946	0.142857	31.40390	280.5125	249.1086	1543.359
	55	92.89277	1.047384	7.331689	0.142857	29.88795	272.0587	242.1707	1351.538
	65	97.92475	1.132657	7.928602	0.142857	26.62178	254.3119	227.6901	1005.619
1	45	87.85653	0.950759	6.655310	0.142857	40.06826	280.4778	240.4096	1746.020
	50	90.36766	0.980416	6.862913	0.142857	38.85619	271.9933	233.1371	1541.581
	55	92.87989	1.013050	7.091348	0.142857	37.60451	263.2316	225.6271	1349.699
	65	97.90858	1.089535	7.626742	0.142857	34.96472	244.7530	209.7883	1003.636
2	45	87.85792	0.954449	6.681145	0.142857	55.93034	281.5574	225.6270	1746.252
	50	90.36917	0.984467	6.891266	0.142857	55.21709	273.1065	217.8895	1541.819
	55	92.88157	1.017522	7.122656	0.142857	54.51471	264.3818	209.8671	1349.946
	65	97.91067	1.095110	7.665770	0.142857	53.16414	245.9895	192.8253	1003.901
4	45	87.86911	0.984284	6.889987	0.142857	73.98474	290.2813	216.2966	1748.059
	50	90.38150	1.017320	7.121242	0.142857	73.91206	282.1330	208.2209	1543.684
	55	92.89523	1.053944	7.377608	0.142857	73.92357	273.7445	199.8209	1351.876
	65	97.92787	1.140980	7.986857	0.142857	74.28610	256.1555	181.8694	1005.985
$\delta = 1.5$									
0.5	45	87.85168	0.976979	6.770984	0.144289	32.81790	285.3856	252.5677	1748.481
	50	90.36387	0.999239	6.992388	0.142904	31.04152	277.1512	246.1097	1544.019
	55	92.87739	1.023639	7.237638	0.141433	29.21151	268.6800	239.4685	1352.122
	65	97.90956	1.080530	7.819697	0.138181	25.36246	250.9375	225.5751	1006.050
1	45	87.84061	0.956547	6.569751	0.145599	40.32744	276.9767	236.6493	1746.972
	50	90.35164	0.976908	6.770287	0.144293	38.73281	268.4308	229.6980	1542.452
	55	92.86378	0.999084	6.990837	0.142913	37.10218	259.6133	222.5111	1350.490
	65	97.89228	1.050171	7.507331	0.139886	33.71861	241.0433	207.3247	1004.265
2	45	87.84193	0.958996	6.593767	0.145440	55.98582	277.9805	221.9947	1747.459
	50	90.35303	0.979459	6.795545	0.144133	54.83715	269.4228	214.5857	1542.946
	55	92.86525	1.001743	7.017440	0.142750	53.69559	260.5909	206.8953	1350.991
	65	97.89392	1.053067	7.536948	0.139721	51.45170	241.9819	190.5302	1004.782
4	45	87.85328	0.979930	6.800210	0.144103	73.39927	286.6065	213.2073	1749.292
	50	90.36551	1.002224	7.022250	0.142721	72.86208	278.3233	205.4612	1544.838
	55	92.87906	1.026640	7.267991	0.141255	72.40038	269.7946	197.3943	1352.949
	65	97.91124	1.083469	7.850152	0.138019	71.78035	251.9016	180.1213	1006.894
$\delta = 2$									
0.5	45	87.84131	0.978055	6.717606	0.145596	32.87698	283.2055	250.3285	1748.980
	50	90.35358	0.995096	6.936397	0.143460	30.92861	275.0033	244.0747	1544.467
	55	92.86719	1.013672	7.178860	0.141202	28.93511	266.5713	237.6362	1352.518
	65	97.89963	1.056583	7.754795	0.136249	24.79048	248.9318	224.1413	1006.346
1	45	87.82990	0.961915	6.513603	0.147678	40.56407	274.6793	234.1152	1747.612
	50	90.34094	0.977526	6.710868	0.145663	38.76777	266.1468	227.3790	1543.039
	55	92.85309	0.994433	6.927821	0.143542	36.94013	257.3471	220.4070	1351.024
	65	97.88165	1.033027	7.435889	0.138924	33.17911	238.8285	205.6494	1004.692
2	45	87.83129	0.963902	6.538548	0.147418	56.11231	275.7221	219.6098	1748.238
	50	90.34237	0.979540	6.736529	0.145407	54.73876	267.1548	212.4160	1543.672
	55	92.85457	0.996467	6.954149	0.143291	53.37363	258.3148	204.9412	1351.663
	65	97.88319	1.035063	7.463198	0.138689	50.68597	239.6942	189.0082	1005.347
4	45	87.84283	0.980186	6.744778	0.145325	73.16406	284.3407	211.1767	1750.061
	50	90.35505	0.997129	6.962729	0.143210	72.39225	276.0370	203.6448	1545.555
	55	92.86859	1.015565	7.203803	0.140976	71.69535	267.4874	195.7920	1353.614
	65	97.90072	1.058004	7.774243	0.136091	70.59999	249.5476	178.9476	1007.454

6.4. Effects of the parameters α, β and δ on the optimal policy

Now, consider an inventory system with: $\gamma = 40, p = 40, h = 1, \delta = 2$ and $w = 1.5$. To investigate the impacts of α, β and n on the optimal policy a sensitivity analysis is done when $\alpha \in \{1600, 3200, 6400, 12800\}$, $\beta \in \{4, 8, 16, 32\}$ and $n \in \{0.25, 0.5, 1, 2, 4\}$. Figs. 7–11, shows the effects of the parameters α, β and n on the optimal inventory policy: $s^*, \tau^*, T^*, Q^*, I_m^*, b^*$ and TP^*. From the information displayed in Figs. 7–11, the following significant insights are found.

(I) It is deduced that when α increases then the optimal time τ^* at which inventory level reaches zero and the optimal inventory cycle T^* decrease. In contrast, the optimal order quantity Q^*, the optimal backordering level b^*, the optimal maximum inventory level I_m^*, the optimal unit selling price s^* and the optimal maximum profit per unit of time TP^* increase.

Table 5
Effects of the parameters h, w and γ on the optimal inventory policy.

w	h	s	τ	T	τ/T	I_m	Q	b	TP
$\gamma = 30$									
1.5	0.75	96.31020	2.846459	6.897621	0.412673	73.43327	370.3318	296.8986	2756.655
	1	96.40321	2.531313	6.803009	0.372087	63.64288	364.6195	300.9766	2750.253
	1.5	96.52684	2.134502	6.690595	0.319030	52.18281	357.7673	305.5844	2742.217
	1.75	96.57117	1.997660	6.653412	0.300246	48.42492	355.4840	307.0591	2739.444
2	0.75	96.34915	2.924978	6.133290	0.476902	78.69592	329.0563	250.3603	2727.853
	1	96.46206	2.614552	6.032493	0.433412	67.92701	322.9672	255.0401	2719.394
	1.5	96.61609	2.219410	5.913747	0.375297	55.41801	315.6990	260.2810	2708.739
	1.75	96.67232	2.081943	5.874617	0.354396	51.34697	313.2797	261.9327	2705.052
2.5	0.75	96.36314	2.970791	5.618472	0.528754	83.39532	301.3572	217.9619	2701.097
	1	96.49109	2.666714	5.511260	0.483866	71.75459	294.9015	223.1469	2694.644
	1.5	96.66978	2.276609	5.386379	0.422660	58.27065	287.2568	228.9862	2681.421
	1.75	96.73613	2.139948	5.345511	0.400326	53.90222	284.7226	230.8204	2676.835
3	0.75	96.36400	2.997843	5.244608	0.571605	87.65344	281.2997	193.6463	2686.541
	1	96.50305	2.700239	5.130669	0.526294	75.25625	274.4752	199.2189	2674.171
	1.5	96.70145	2.316450	4.999419	0.463344	60.88273	266.4618	205.5791	2658.442
	1.75	96.77630	2.181293	4.956816	0.440059	56.23430	263.8201	207.5858	2652.970
$\gamma = 40$									
1.5	0.75	101.2888	2.824154	6.812079	0.414580	60.89290	399.9453	339.0524	3324.923
	1	101.3874	2.517594	6.743112	0.373358	52.10013	395.2316	343.1314	3319.442
	1.5	101.5185	2.128893	6.661078	0.319602	42.12510	389.5498	347.4247	3312.719
	1.75	101.5655	1.994196	6.633819	0.300611	38.92888	387.6438	348.7149	3310.435
2	0.75	101.3147	2.889141	6.019317	0.479978	66.76341	353.2455	286.4821	3293.584
	1	101.4339	2.590396	5.945472	0.435692	56.66323	348.2031	291.5398	3286.094
	1.5	101.5971	2.206865	5.859554	0.376627	45.32875	342.2152	296.8864	3276.924
	1.75	101.6567	2.072603	5.831327	0.355426	41.74077	340.2189	298.4781	3273.817
2.5	0.75	101.3169	2.922423	5.484590	0.532843	72.28462	321.8526	249.5680	3268.633
	1	101.4512	2.632475	5.404446	0.487094	60.99853	316.4239	255.4253	3259.102
	1.5	101.6398	2.257032	5.313549	0.424769	48.35868	310.0996	261.7409	3247.413
	1.75	101.7101	2.124562	5.284195	0.402060	44.38132	308.0152	263.6338	3243.455
3	0.75	101.3077	2.938247	5.096571	0.576514	77.44325	299.1296	221.6864	3248.202
	1	101.4524	2.656587	5.009071	0.530355	65.12519	293.2688	228.1436	3236.645
	1.5	101.6609	2.289984	4.912002	0.466202	51.27765	286.5617	235.2840	3222.402
	1.75	101.7400	2.159893	4.881224	0.442490	46.92493	284.3803	237.4554	3217.573
$\gamma = 50$									
1.5	0.75	106.2677	2.801972	6.727497	0.416495	48.59948	428.7589	380.1594	3943.238
	1	106.3716	2.503858	6.683393	0.374639	40.70648	425.2536	384.5472	3938.656
	1.5	106.5101	2.123235	6.631362	0.320181	32.12680	421.0244	388.8976	3933.227
	1.75	106.5598	1.990694	6.614036	0.300980	29.46917	419.5957	390.1265	3931.428
2	0.75	106.2813	2.853957	5.908358	0.483037	55.23527	376.4728	321.2376	3909.434
	1	106.4063	2.566468	5.859848	0.437975	45.66872	372.6495	326.9808	3902.867
	1.5	106.5781	2.194311	5.805563	0.377967	35.37596	368.1996	332.8237	3895.139
	1.75	106.6411	2.063228	5.788023	0.356465	32.23514	366.7228	334.4876	3892.600
2.5	0.75	106.2729	2.875545	5.356173	0.536866	61.72058	341.3331	279.6126	3882.386
	1	106.4126	2.598924	5.300687	0.490299	50.62864	337.0567	286.4281	3873.705
	1.5	106.6104	2.237604	5.241728	0.426883	38.66382	332.2710	293.6072	3863.475
	1.75	106.6844	2.109231	5.223431	0.403802	35.02953	330.7247	295.6952	3860.126
3	0.75	106.2549	2.881201	4.956531	0.581294	67.89877	315.9546	248.0558	3860.195
	1	106.4043	2.614267	4.892397	0.534353	55.48653	311.1356	255.6490	3849.354
	1.5	106.6215	2.263941	4.826654	0.469050	41.96938	305.9060	263.9366	3836.491
	1.75	106.7044	2.138732	4.806999	0.444920	37.85401	304.2619	266.4079	3832.275

(II) It is observed that when β increases then the optimal time τ^* at which inventory level reaches zero and the optimal inventory cycle T^* increase. On the contrary, the optimal order quantity Q^*, the optimal backordering level b^*, the optimal maximum inventory level I_m^*, the optimal unit selling price s^* and the optimal maximum profit per unit of time TP^* decrease.

From the computational results given in Tables 4–6 and Figs. 7–11, the behavior of the decision variables and the total profit per unit time can be deduced according with a variation of each of the parameters $p, \gamma, n, K, h, \delta, w, \alpha$ and β. This is shown in Table 7.

6.5. Managerial implications

In this section, some findings obtained from the sensitivity analysis are presented. In addition, some comments or suggestions are proposed

Table 6
Impacts of the parameters K, h and w on the optimal inventory policy.

K	w	s	τ	T	τ/T	I_m	Q	b	TP
$h = 1$									
1000	1.5	101.3874	2.517594	6.743113	0.373358	52.10014	395.2316	343.1315	3319.443
	2	101.4339	2.590396	5.945473	0.435692	56.66324	348.2031	291.5399	3286.094
	2.5	101.4512	2.632475	5.404446	0.487094	60.99853	316.4239	255.4254	3259.102
	2.75	101.4533	2.646253	5.192674	0.509613	63.08850	304.0141	240.9256	3247.387
1500	1.5	101.7901	2.844543	8.238824	0.345261	56.48186	479.5815	423.0997	3252.690
	2	101.8670	2.938271	7.254990	0.405000	61.08733	421.7545	360.6671	3210.334
	2.5	101.9033	2.995685	6.585337	0.454902	65.49194	382.5861	317.0941	3175.693
	2.75	101.9120	3.015573	6.322365	0.476969	67.63460	367.2537	299.6191	3160.541
2000	1.5	102.1422	3.100585	9.509668	0.326046	59.66835	550.2085	490.5402	3196.342
	2	102.2485	3.211422	8.368039	0.383772	64.26831	483.2669	418.9986	3146.321
	2.5	102.3042	3.281638	7.589297	0.432403	68.68362	437.8703	369.1867	3105.139
	2.75	102.3199	3.306720	7.282864	0.454041	70.84387	420.0765	349.2326	3087.034
2500	1.5	102.4611	3.314440	10.63811	0.311563	62.14112	612.1047	549.9635	3146.705
	2	102.5959	3.440012	9.356851	0.367646	66.70811	537.1219	470.4138	3089.898
	2.5	102.6710	3.521391	8.481468	0.415186	71.10313	486.2339	415.1308	3042.910
	2.75	102.6939	3.551053	8.136501	0.436435	73.26296	466.2713	393.0084	3022.176
$h = 1.5$									
1000	1.5	101.5185	2.128893	6.661077	0.319602	42.12509	389.5497	347.4246	3312.719
	2	101.5971	2.206865	5.859554	0.376627	45.32875	342.2152	296.8864	3276.924
	2.5	101.6398	2.257032	5.313549	0.424769	48.35869	310.0996	261.7409	3247.413
	2.75	101.6525	2.275219	5.098830	0.446224	49.83073	297.5040	247.6733	3234.423
1500	1.5	101.9458	2.398577	8.151745	0.294241	45.74842	473.2427	427.4943	3245.201
	2	102.0637	2.495211	7.164767	0.348261	48.97058	415.0998	366.1292	3200.134
	2.5	102.1341	2.559724	6.491037	0.394347	52.01081	375.6097	323.5989	3162.689
	2.75	102.1573	2.583876	6.225556	0.415043	53.49281	360.1031	306.6103	3146.107
2000	1.5	102.3180	2.609667	9.420028	0.277034	48.36346	543.3656	495.0022	3188.284
	2	102.4728	2.721406	8.275945	0.328833	51.56913	476.0915	424.5224	3135.363
	2.5	102.5699	2.797681	7.493891	0.373328	54.58670	430.3751	375.7884	3091.175
	2.75	102.6035	2.826780	7.185334	0.393410	56.06066	412.4126	356.3520	3071.532
2500	1.5	102.6543	2.785949	10.54746	0.264135	50.37780	604.8514	554.4736	3138.198
	2	102.8442	2.910631	9.264458	0.314172	53.54451	529.5174	475.9729	3078.346
	2.5	102.9671	2.997064	8.386500	0.357368	56.52002	478.3061	421.7860	3028.198
	2.75	103.0112	3.030464	8.039764	0.376934	57.97582	458.1766	400.2008	3005.844
$h = 1.75$									
1000	1.5	101.5655	1.994196	6.633820	0.300611	38.92888	387.6438	348.7149	3310.435
	2	101.6567	2.072603	5.831327	0.355426	41.74077	340.2188	298.4781	3273.817
	2.5	101.7101	2.124562	5.284196	0.402060	44.38133	308.0152	263.6339	3243.455
	2.75	101.7274	2.143895	5.068804	0.422959	45.66330	295.3725	249.7092	3230.032
1500	1.5	102.0013	2.244660	8.122904	0.276337	42.29562	471.1179	428.8223	3242.658
	2	102.1350	2.340785	7.135149	0.328064	45.12399	412.8751	367.7511	3196.682
	2.5	102.2192	2.406540	6.460545	0.372498	47.76569	373.2958	325.5301	3158.298
	2.75	102.2485	2.431678	6.194534	0.392552	49.04983	357.7439	308.6940	3141.235
2000	1.5	102.3804	2.440687	9.390461	0.259911	44.71813	541.0748	496.3567	3185.552
	2	102.5537	2.551121	8.245812	0.309384	47.52955	473.6913	426.1618	3131.658
	2.5	102.6672	2.628140	7.463124	0.352150	50.14399	427.8822	377.7382	3086.468
	2.75	102.7082	2.658062	7.154157	0.371541	51.41567	409.8742	358.4585	3066.311
2500	1.5	102.7225	2.604387	10.51769	0.247620	46.57887	602.4266	555.8477	3135.316
	2	102.9334	2.727070	9.234367	0.295317	49.35140	526.9743	477.6229	3074.444
	2.5	103.0751	2.813799	8.356023	0.336739	51.92100	475.6660	423.7450	3023.246
	2.75	103.1277	2.847868	8.009002	0.355583	53.17166	455.4903	402.3186	3000.355

to inventory systems managers that could help in improving the effectiveness and efficiency of the inventory control practices.

The sensitivity analysis reveals that the unit purchasing cost p has the greatest impact on the total profit per unit time among the parameters p, n and δ. Hence, decision makers should negotiate a reduction of the purchasing cost with their suppliers by promising them that, if the price is lowered, then the organization will be able to buy more from them since the quantity to order of the product will increase.

The variation of the scale parameter γ associated with the time-dependent demand has a greater effect on the total profit per unit time in a positive manner, more so than the modification of the cost parameters h and w. Thus, the decision maker should boost the time-dependent demand by implementing policies that augment demand (e.g., by increasing advertising) instead of reducing the inventory costs h and w.

From the results obtained with the sensitivity analysis, it is deduced that the impact of the ordering cost K on the total profit per unit time

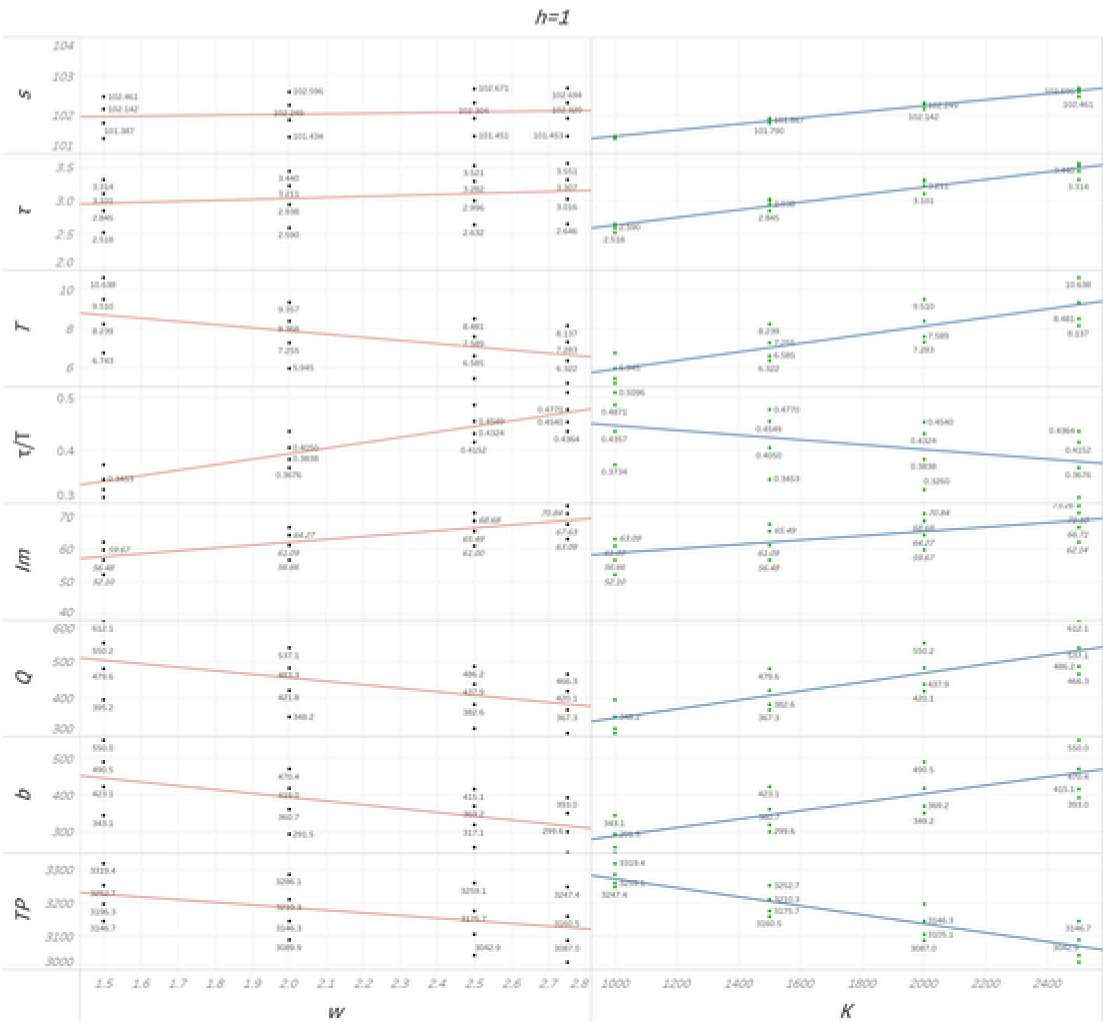

Fig. 4. Effects of changes in w and K on the optimal solution when $h = 1$.

in negative sense, is greater than the effect of the cost parameters h and w. For this reason, the decision maker should try to reduce the ordering cost as much as possible. Finally, the increment of the scale parameter α for the price-dependent demand has a greater effect on the total profit per unit of time in a positive way, more so than the decrease of thesensitivity parameter β for the price-dependent demand. Therefore, the decision maker should boost the price-dependent demand by implementing policies that increase the scale parameter α of the demand (e.g. applying marketing policies such as quantity discount).

7. Conclusions

This study develops and presents an inventory model for a single product in which the demand rate of the product is the addition of a linear function with respect to selling price and of a power time function. Additionally, the holding cost is considered as a power function of the time that the product is held in storage. Furthermore, shortages are permitted and these are backordered. To optimize the total profit per unit of time, an effective and efficient algorithm is proposed. It is important to remark that the algorithm obtains an optimal solution. Based on the assumptions assumed in the inventory system developed in this paper, the results obtained can be useful for the inventory management of items where demand is sensitive to both the selling price and time spent on inventory, the value of the item decreases non-linearly the longer it is held in stock (see Weiss, 1982) and shortages are allowed. Perhaps the main limitation of the inventory model is this last assumption. Sometimes this condition can be restrictive in real practical situations where not all customers facing a shortage are willing to wait until the next order arrives. For this reason, later on we propose as a possible line of research the inclusion of this topic. Another limitation of the model is that the payment of the replenishing quantity is made when the lot size is received. There is no a credit policy for the retailer, where he/she has a time period to pay the ordered quantity. Thus, to analyze the inventory system considering a permissible delay in payments would be another new research line.

Future developments of the current research are certainly required. Principally, the imminent research directions that can be explored in the near future from this research paper are to build an inventory model considering some of the following issues: a) deteriorating products, b) stochastic demand, c) discounts, d) permissible delay in payment (trade credit), e) production rate, f) supply chain environment, g) advertising, h) imperfect quality, i) partial backordering, j) multiple products subject to constrains, and k) sustainable issues like carbon emissions, among others.

CRediT authorship contribution statement

Leopoldo Eduardo Cárdenas-Barrón: Conceptualization, Methodology, Investigation, Validation, Formal analysis, Software, Visualization, Supervision, Writing - original draft, Writing - review & editing. **Buddhadev Mandal:** Conceptualization, Methodology, Investigation, Validation, Formal analysis, Software, Visualization, Writing -

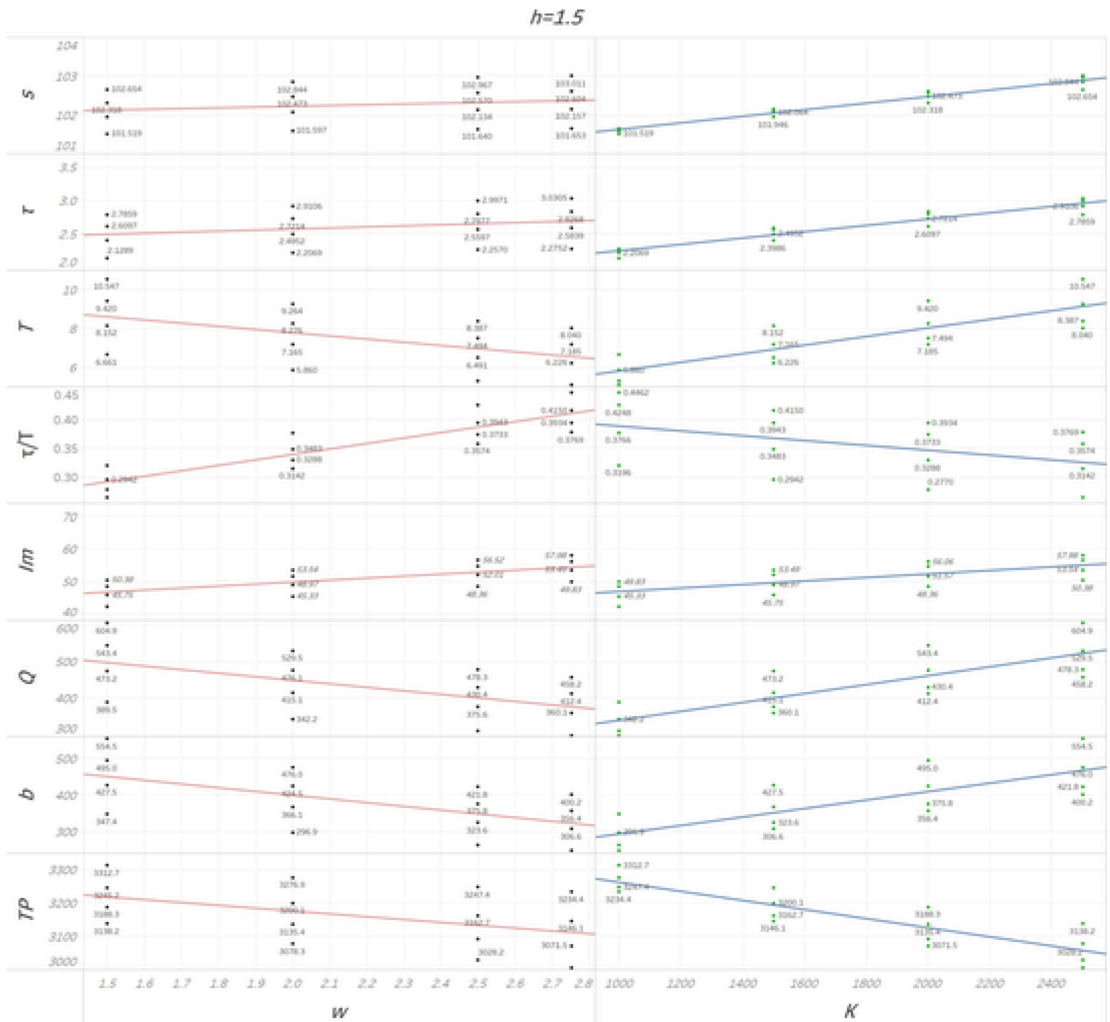

Fig. 5. Effects of changes in w and K on the optimal solution when $h = 1.5$.

original draft. **Joaquín Sicilia:** Conceptualization, Methodology, Investigation, Validation, Formal analysis, Software, Visualization, Writing - original draft, Writing - review & editing. **Luis A. San-José:** Conceptualization, Methodology, Investigation, Validation, Formal analysis, Software, Visualization, Writing- original draft, Writing - review & editing. **Beatriz Abdul-Jalbar:** Conceptualization, Methodology, Investigation, Validation, Formal analysis, Software, Visualization, Writing - review & editing.

Appendix A. Proof of Theorem 1

Consider the following cases:

1. If $T_0 \leqslant 0$, then $TP_0(T) = P_2(T)$ and optimal planning period is $T^* = \widehat{T}_2$, where $\widehat{T}_2$ is given by (9).
2. If $T_0>0$, then $T_1>0$. This is due to if $T_0>0$, then $T_1 = \frac{4}{3\omega\beta}\left(\alpha - \beta p + \frac{3n-1}{n+1}\gamma\right) > \frac{4}{3\omega\beta}\left(\gamma + \frac{3n-1}{n+1}\gamma\right) = \frac{4}{3\omega\beta}\left(\frac{4n}{n+1}\gamma\right) > 0$.Hence, in this case the following two alternatives occur:
   A. If $0<T_1<T_0$, then the function $f_0(T)$ has a minimum in the point $T_1 \in (0, T_0)$.
      a) If $f_0(T_1) \geqslant 0$, or equivalently, $K \geqslant \frac{4}{27\beta^2\omega}\left(\alpha - \beta p + \frac{3n-1}{n+1}\gamma\right)^3$, then $P_1(T)$ is strictly increasing in $(0, T_0)$ and, as $P_2'(T_0) = f_0(T_0)/T_0^2 > 0$, then $P_2(T)$ is increasing en $\left(T_0, \widehat{T}_2\right)$ and decreasing in $\left(\widehat{T}_2, \infty\right)$. Thus, $T^* = \widehat{T}_2$.
      b) If $f_0(T_1) < 0$, then the following point is defined

      $\widetilde{T}_1 = \arg\left\{f_0(T) = 0 : T \in (0, T_1)\right\}$

      Two possibilities can occur:
      i) If $f_0(T_0) \leqslant 0$, then the function $P_1(T)$ is strictly increasing in $\left(0, \widetilde{T}_1\right)$ and strictly decreasing in $\left(\widetilde{T}_1, T_0\right)$. Moreover, since $P_2'(T_0) = P_1'(T_0) = f_0(T_0)/T_0^2 \leqslant 0$, it is concluded that the function $P_2(T)$ is strictly decreasing in (T_0, ∞). Hence, $T^* = \widetilde{T}_1$.
      ii) If $f_0(T_0) > 0$, then the function $f_0(T)$ has two roots in the interval $(0, T_0)$: $\widetilde{T}_1$ and x. Hence, $P_1(T)$ is strictly increasing in $\left(0, \widetilde{T}_1\right)$, strictly decreasing in $\left(\widetilde{T}_1, x\right)$ and strictly increasing in (x, T_0). Additionally, the function $P_2(T)$ is strictly increasing in $\left(T_0, \widehat{T}_2\right)$ and decreasing in $\left(\widehat{T}_2, \infty\right)$. Thus $T^* = \widetilde{T}_1$, if $P_1\left(\widetilde{T}_1\right) \geqslant P_2\left(\widehat{T}_2\right)$ and $T^* = \widehat{T}_2$, if $P_1\left(\widetilde{T}_1\right) < P_2\left(\widehat{T}_2\right)$.
   B. If $T_1 \geqslant T_0$, or equivalently, $\gamma \geqslant (\alpha - \beta p)(n+1)/(9n+1)$, then the function $f_0'(T) = \frac{3\omega^2\beta}{8}(T - T_1)T < 0$ on the interval $(0, T_0)$ and, consequently, the function $f_0(T)$ is strictly decreasing on the

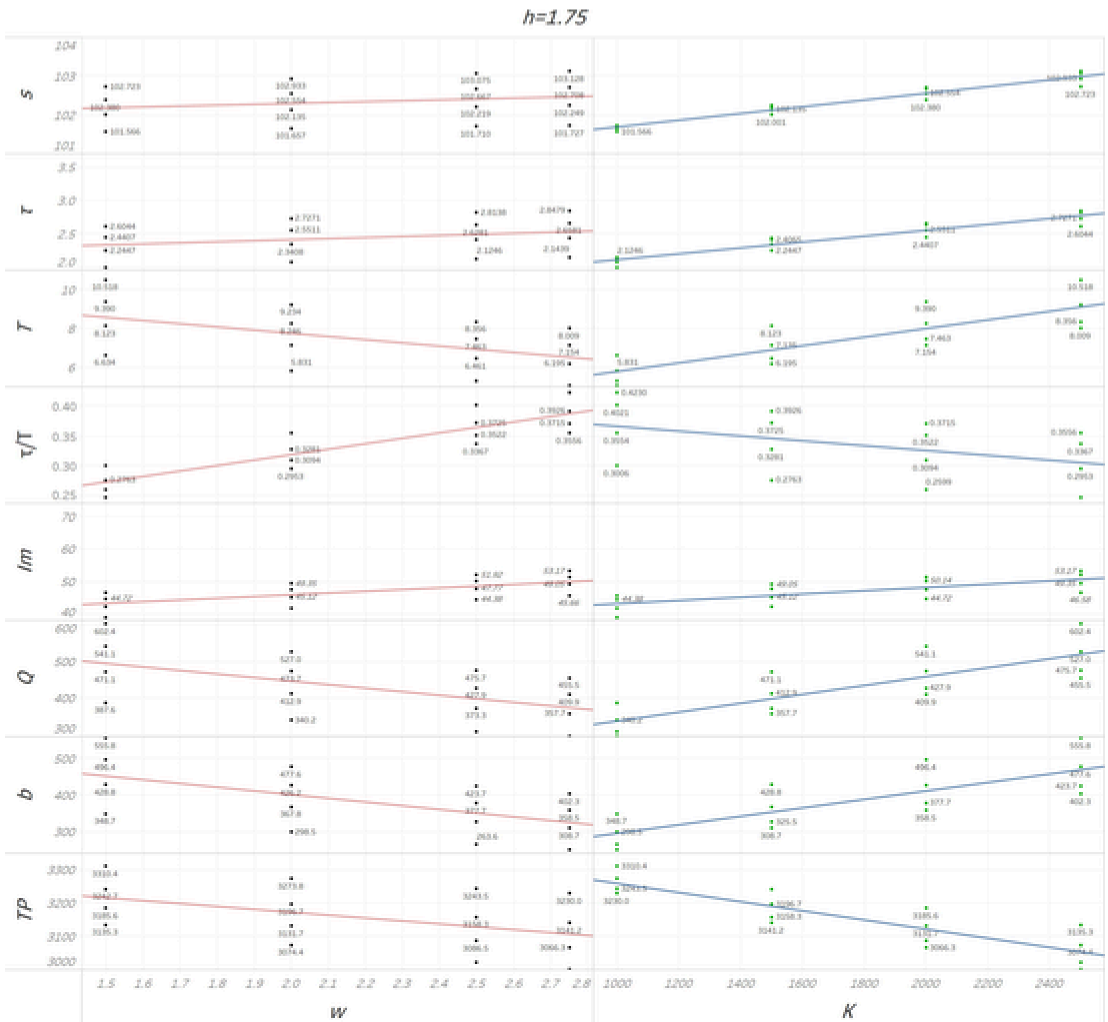

Fig. 6. Effects of changes in w and K on the optimal solution when $h = 1.75$.

mentioned interval. Two cases can occur:

a) If $f_0(T_0) \geqslant 0$, or equivalently, $K \geqslant \frac{4\gamma n(\alpha-\beta p-\gamma)^2}{\beta^2\omega(n+1)}$, then $f_0(T_0)>0$ on $(0.T_0)$ and $P_1(T)$ is strictly increasing on $(0,T_0)$. Furthermore, taking into account that the function $P_2(T)$ is concave on the interval $(0,\infty)$ and the function $TP_0(T)$ is of class C^1, we have $P_2'(T_0) = P_1'(T_0) = f_0(T_0)/T_0^2 \geqslant 0$. From this, it is noted that the function $P_2(T)$ is increasing on $\left(T_0, \widehat{T}_2\right)$ and decreasing on $\left(\widehat{T}_2, \infty\right)$. Hence, $T^* = \widehat{T}_2$.

b) If $f_0(T_0)<0$, or equivalently, $K<\frac{4\gamma n(\alpha-\beta p-\gamma)^2}{\beta^2\omega(n+1)}$, then it is stated the following point

$$\widehat{T}_1 = \arg\left\{f_0(T) = 0 \,:\, T \in (0,T_0)\right\}$$

The function $P_1(T)$ is strictly increasing on $\left(0,\widehat{T}_1\right)$ and decreasing on $\left(\widehat{T}_1, T_0\right)$. Additionally, as $P_2'(T_0) = f_0(T_0)/T_0^2<0$, thus $P_2(T)$ is strictly decreasing on (T_0,∞). Then $T^* = \widehat{T}_1$.

Appendix B. A necessary condition for the optimal cycle time T^*

For given s and τ, the first order derivative of $TP(s,\tau,T)$ with respect to T is

$$\frac{dTP(s,\tau,T)}{dT} = \frac{1}{T^2}\left[K + \frac{h(\alpha-\beta s)\tau^{(\delta+1)}}{(\delta+1)} + \frac{h\gamma}{n(1+n\delta)}\tau^{\left(\frac{1+n\delta}{n}\right)}T^{\left(\frac{n-1}{n}\right)} - \frac{w(\alpha-\beta s)(T^2-\tau^2)}{2} \right.$$
$$\left. - \frac{nw\gamma T^2}{(n+1)} - \frac{w\gamma}{(n+1)}\tau^{\left(\frac{n+1}{n}\right)}T^{\left(\frac{n-1}{n}\right)} - \frac{w\gamma}{n}\tau^{\left(\frac{1}{n}\right)}T^{\left(\frac{n-1}{n}\right)}(T-\tau) + w\gamma\tau\right]$$

Setting this result to zero, then a necessary condition to determine T^* is obtained as follows:

$$K + \frac{h(\alpha-\beta s)\tau^{(\delta+1)}}{(\delta+1)} + \frac{h\gamma}{n(1+n\delta)}\tau^{\left(\frac{1+n\delta}{n}\right)}T^{\left(\frac{n-1}{n}\right)} - \frac{w(\alpha-\beta s)\left(T^2-\tau^2\right)}{2} - \frac{nw\gamma T^2}{(n+1)} - \frac{w\gamma}{(n+1)}\tau^{\left(\frac{n+1}{n}\right)}T^{\left(\frac{n-1}{n}\right)} - \frac{w\gamma}{n}\tau^{\left(\frac{1}{n}\right)}T^{\left(\frac{n-1}{n}\right)}(T-\tau) + w\gamma\tau^{\left(\frac{1}{n}\right)}T^{\left(\frac{2n-1}{n}\right)} = 0. \quad (17)$$

■

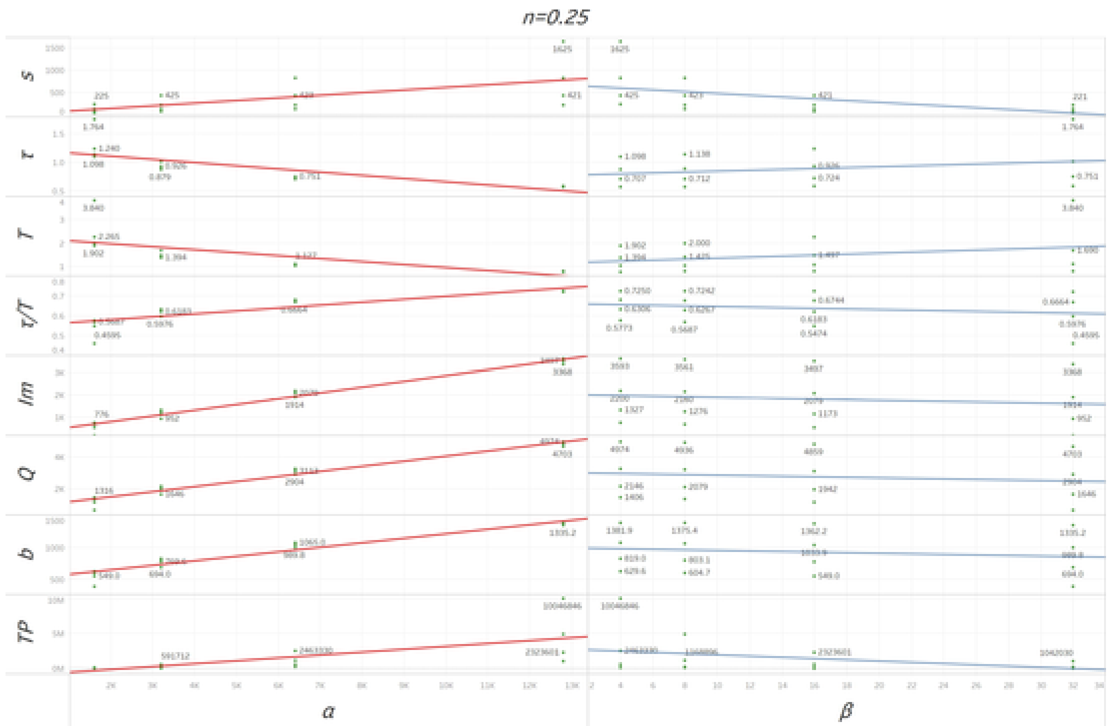

Fig. 7. Effects of changes in α and β on the optimal solution when $n = 0.25$.

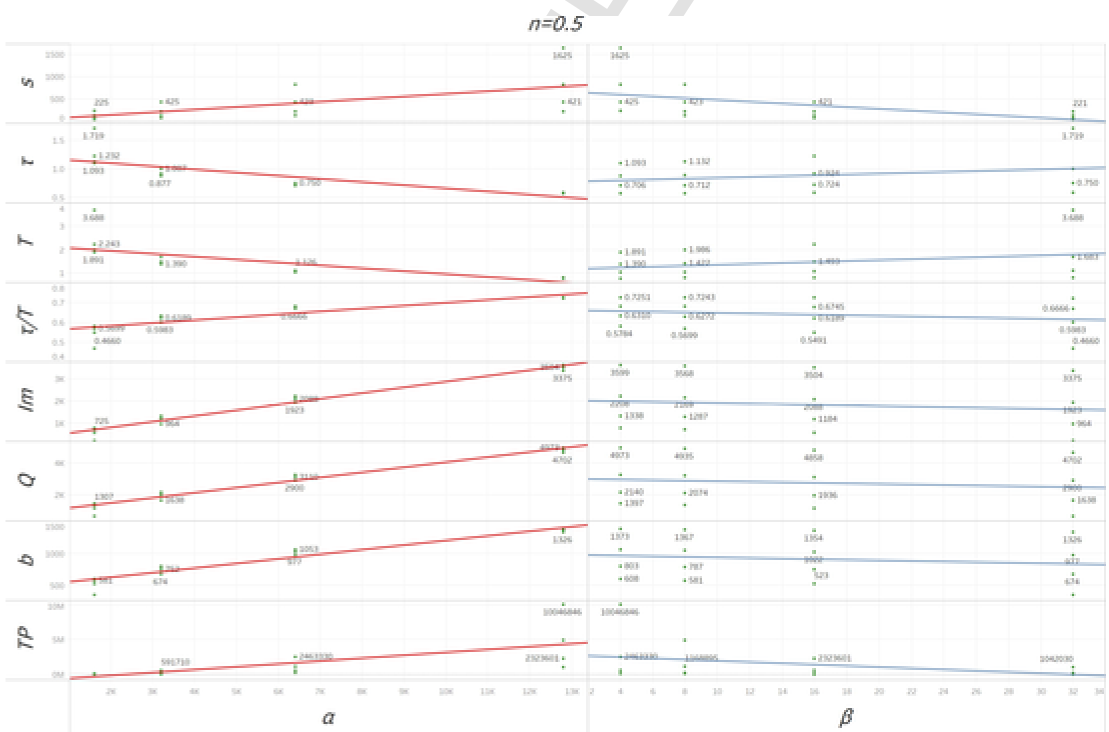

Fig. 8. Effects of changes in α and β on the optimal solution when $n = 0.5$.

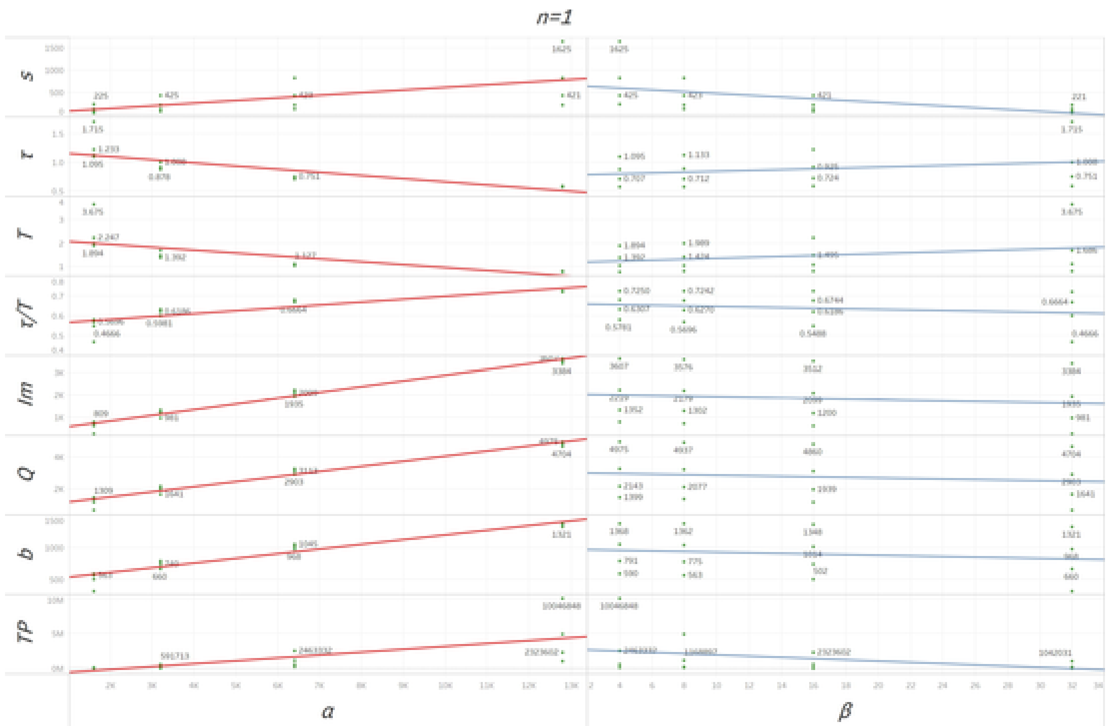

Fig. 9. Effects of changes in α and β on the optimal solution when $n = 1$.

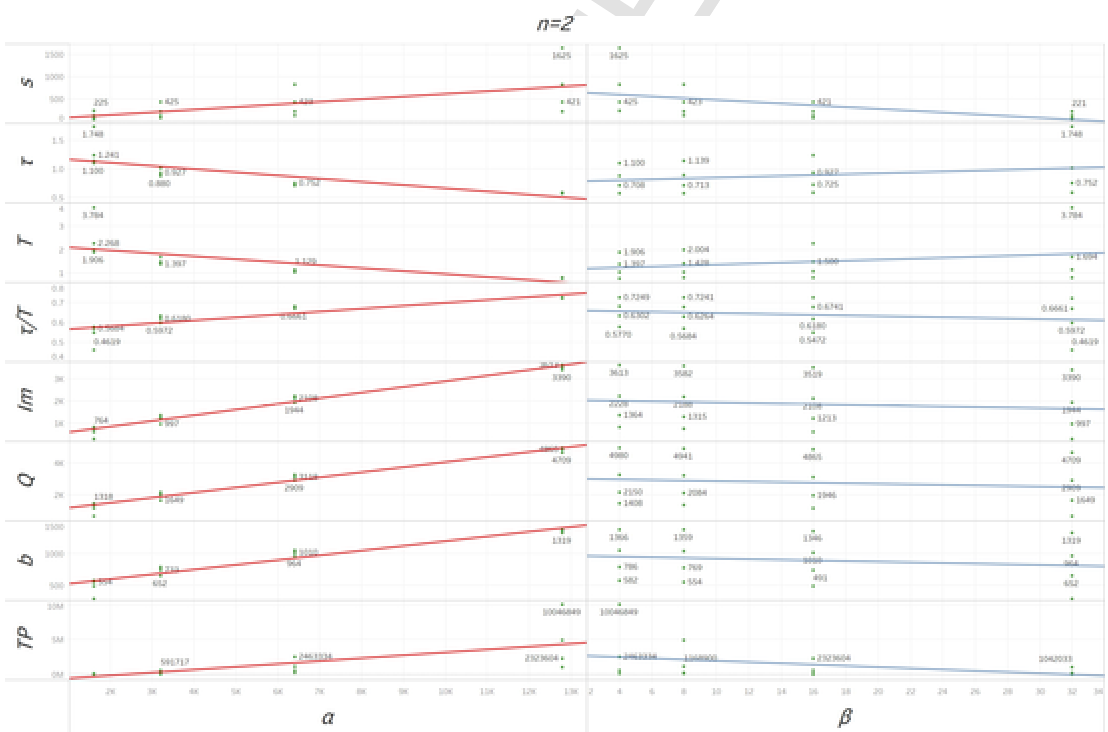

Fig. 10. Effects of changes in α and β on the optimal solution when $n = 2$.

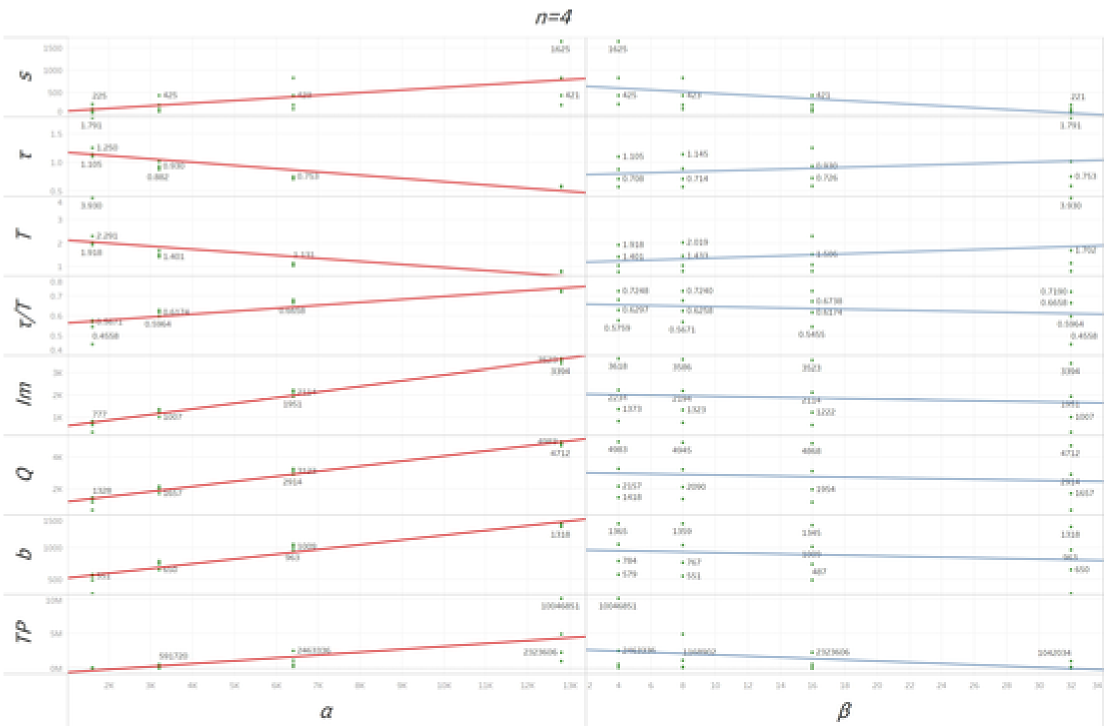

Fig. 11. Effects of changes in α and β on the optimal solution when $n = 4$.

Table 7
Evolution of the optimal policy and the maximum profit as functions of each parameter.

	Decision variables			Other variables			Profit
Parameter	s	τ	T	I_m	Q	b	TP
p↑	↑	↑	↑	↓	↓	↓	↓
α↑	↑	↓	↓	↑	↑	↑	↑
β↑	↓	↑	↑	↓	↓	↓	↓
γ↑	↑	↓	↓	↓	↑	↑	↑
n↑	↘ ≠ *arrow*	↘ ≠ *arrow*	↘ ≠ *arrow*	↑	↘ ≠ *arrow*	↓	↘ ≠ *arrow*
K↑	↑	↑	↑	↑	↑	↑	↓
h↑	↑	↓	↓	↓	↓	↑	↓
δ↑	↓	↘ ≠ *arrow*	↓	↘ ≠ *arrow*	↓	↓	↑
w↑	≠ *arrow*↘	↑	↓	↑	↓	↓	↓

Appendix C. Proof of Theorem 2

From Eq. (1), let

$$y(T) = (s-p)T\gamma + (s-p)(\alpha-\beta s)T - K - h\left[\frac{(\alpha-\beta s)\tau^{(\delta+1)}}{(\delta+1)} + \frac{\gamma}{(1+n\delta)}\tau^{\left(\frac{1+n\delta}{n}\right)} \ldots\right.$$

$$-w\left[\frac{(\alpha-\beta s)(T-\tau)^2}{2} + \frac{n\gamma T^2}{(n+1)} - \frac{n\gamma}{(n+1)}\tau^{\left(\frac{n+1}{n}\right)}T^{\left(\frac{n-1}{n}\right)} - \gamma\tau^{\left(\frac{1}{n}\right)}T^{\left(\frac{n-1}{n}\right)}(T - \ldots\right.$$

and

$$g(T) = T > 0$$

As a result, it is determined that

$$TP(s,\tau,T) = \frac{y(T)}{g(T)}$$

Given s and τ, by taking first and second order derivative of $y(T)$ with respect to T, thus

$$y'(T) = (s-p)\gamma + (s-p)(\alpha-\beta s) - h\left\{\frac{(n-1)\gamma}{n(1+n\delta)}\tau^{\left(\frac{1+n\delta}{n}\right)}T^{\left(\frac{-1}{n}\right)}\right\}$$

$$-w\left\{(\alpha-\beta s)(T-\tau) + \frac{2n\gamma T}{(n+1)} - \frac{(n-1)\gamma}{(n+1)}\tau^{\left(\frac{n+1}{n}\right)}T^{\left(\frac{-1}{n}\right)}\right.$$

$$\left.-\frac{(n-1)\gamma}{n}\tau^{\left(\frac{1}{n}\right)}T^{\left(\frac{-1}{n}\right)}(T-\tau) - \gamma\tau^{\left(\frac{1}{n}\right)}T^{\left(\frac{n-1}{n}\right)}\right\}$$

and

$$y''(T) = -\left[\left\{\frac{w(n-1)\gamma}{n(n+1)}(\tau)^{\left(\frac{n+1}{n}\right)} + \frac{w(n-1)\gamma}{n^2}\tau^{\left(\frac{1}{n}\right)}(T-\tau) - \frac{h(n-1)\gamma}{n^2(1+n\delta)}\tau^{\left(\frac{1+n\delta}{n}\right)}\right\}T^{\prime} \ldots\right.$$

$$\left.+w\left\{(\alpha-\beta s) + \frac{2n\gamma}{(n+1)} - \frac{2(n-1)\gamma}{n}\tau^{\left(\frac{1}{n}\right)}T^{\left(\frac{-1}{n}\right)}\right\}\right] = -J$$

Consequently, if $J>0$ then $y''(T)<0$ and therefore $y(T)$ is nonnegative, differentiable and strictly concave. Hence, if $J>0$ then $TP(s,\tau,T)$ as in Eq. (1) is a strictly pseudo-concave function in T; and there exists a unique optimal solution for T.

Appendix D. Proof of Theorem 3

From Eq. (1), let

$$z(s,\tau) = (s-p)T\gamma + (s-p)(\alpha-\beta s)T - K - h\left\{\frac{(\alpha-\beta s)\tau^{(\delta+1)}}{(\delta+1)} + \frac{\gamma}{1+n\delta}\tau^{\left(\frac{1+n\delta}{n}\right.}\right.$$

$$-w\left\{\frac{(\alpha-\beta s)(T-\tau)^2}{2} + \frac{n\gamma T^2}{(n+1)} - \frac{n\gamma}{(n+1)}\tau^{\left(\frac{n+1}{n}\right)}T^{\left(\frac{n-1}{n}\right)} - \gamma\tau^{\left(\frac{1}{n}\right)}T^{\left(\frac{n-1}{n}\right)}(T\right.$$

Therefore, for any given T, the total profit per unit of time is expressed as follows: $TP(s,\tau,T) = \frac{1}{T}z(s,\tau)$. Taking first order and second order derivatives of $z(s,\tau)$ with respect to s,

$$\frac{\partial z(s,\tau)}{\partial s} = T\gamma + (\alpha-\beta s)T - \beta(s-p)T + \frac{h\beta\tau^{(\delta+1)}}{(\delta+1)} + w\left\{\frac{\beta(T-\tau)^2}{2}\right\}$$

$$\frac{\partial^2 z(s,\tau)}{\partial s^2} = -2\beta T = L$$

Also, taking first order and second order derivatives of $z(s,\tau)$ with respect to τ,

$$\frac{\partial z(s,\tau)}{\partial \tau} = -h\left\{(\alpha-\beta s)\tau^{\delta} + \frac{\gamma}{n}\tau^{\left(\frac{1+n\delta-n}{n}\right)}T^{\left(\frac{n-1}{n}\right)}\right\} + w\left\{(\alpha-\beta s)(T-\tau) + \frac{\gamma}{n}\tau^{\left(\frac{1-n}{n}\right)}T^{\left(\frac{n-1}{n}\right)}(T-\tau)\right\}$$

$$\frac{\partial^2 z(s,\tau)}{\partial \tau^2} = -6pt - h\left\{(\alpha-\beta s)\delta\tau^{(\delta-1)} + \frac{\gamma(1+n\delta-n)}{n^2}\tau^{\left(\frac{1+n\delta-2n}{n}\right)}T^{\left(\frac{n-1}{n}\right)}\right\} - w\left\{(\alpha-\beta s) + \frac{\gamma(n-1)}{n^2}\tau^{\left(\frac{1-2n}{n}\right)}T^{\left(\frac{n-1}{n}\right)}(T-\tau) + \frac{\gamma}{n}\tau^{\left(\frac{1-n}{n}\right)}T^{\left(\frac{n-1}{n}\right)}\right\} = N$$

and

$$\frac{\partial z(s,\tau)}{\partial s\partial \tau} = \frac{\partial z(s,\tau)}{\partial \tau\partial s} = h\beta\tau^{\delta} + w\beta\tau - w\beta T = M$$

It is obvious that $L<0$. Therefore, if $N<0$ and $LN-M^2>0$, then the Hessian matrix associated with $z(s,\tau)$ is negative definite. Hence, for any given T, if $N<0$ and $LN-M^2>0$, then $TP(s,\tau,T)$ in Equation (1) is a strictly concave function in s and τ. Thus, there exists a unique optimal solution. ■

Appendix E. Optimal selling price and time at which the inventory level attains zero

For any given T, setting the first derivative of $z(s,\tau)$ with respect to s to zero. Thus, a necessary condition for s^* is given as follow:

$$T\gamma + (\alpha-\beta s)T - \beta(s-p)T + \frac{h\beta\tau^{(\delta+1)}}{(\delta+1)} + \frac{w\beta(T-\tau)^2}{2} = 0$$

Likewise, a necessary condition for τ^* is given by

$$h\left\{(\alpha-\beta s)\tau^{\delta} + \frac{\gamma}{n}\tau^{\left(\frac{1+n\delta-n}{n}\right)}T^{\left(\frac{n-1}{n}\right)}\right\} - w\left\{(\alpha-\beta s)(T-\tau) + \frac{\gamma}{n}\tau^{\left(\frac{1-n}{n}\right)}T^{\left(\frac{n-1}{n}\right)}(T-\tau)\right\} = 0.$$

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