Optimality conditions for weak solutions of vector optimization problems through quasi interiors and improvement sets*

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Abstract

This work concerns with a vector optimization problem with set-valued mappings. In solving this problem, weakly efficient solutions with respect to the so-called vector criterion are considered. These solutions are defined via the notion of quasi interior, in order to provide useful results to certain problems where the topological and algebraic interior of the ordering cone is empty. Moreover, the domination set that defines the domination structure of the problem is assumed to be free disposal with respect to a convex cone. In this setting, optimality conditions via linear scalarization results and Lagrangian multiplier theorems are stated. Some of them improve several recent ones of the literature, since they are obtained under weaker assumptions.

Key words. Set-valued optimization · Free disposal set · Weak efficiency · Quasi interior · Quasi-relative interior · Linear scalarization · Nearly subconvexlikeness · Lagrangian optimality condition

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1 Introduction

A vector optimization problem is a mathematical programming problem that involves (single or set valued) vector mappings. In particular, the image space of the objective function is usually assumed to be a real Hausdorff topological vector space. Two concepts are really important in this kind of problems. On the one hand, the preference

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order assumed by the decision maker. On the other hand, the notion of solution (see [8, 15, 16, 18, 19]).

In dealing with the above two concepts, a very popular approach in the literature focuses on an ordering in the objective space defined through a convex cone C (see (2.1) with $M = C$), which is assumed to be pointed ($C \cap (-C) = \{0\}$). This binary relation is a partial order. Then, in a problem whose objective function is single valued, a feasible point is said to be a solution if its image is a nondominated point of the image set.

This criterion is extended to problems with set-valued objective mappings by considering as solutions the feasible points whose image sets have some nondominated point with respect to the whole image set of the problem.

In the last years, researchers have been interested in preference orders defined by a free disposal set E, i.e., by a set that coincides with its conical extension with respect to the ordering cone: $E = E + C$ (see, for instance, [4, 11, 13, 20]). By this kind of domination sets, one can deal simultaneously with exact and approximate solutions of the problem (see [4, 13] and the references therein).

On the other hand, it is well known that a vector optimization problem is more tractable from a mathematical point of view if one considers the topological interior of C as ordering set instead of C. Although the nondominated solutions of this problem, called weakly efficient solutions, could be improper from a practical point of view, they deserve to be taken into account since they give valuable information for the first ones. For example, necessary optimality conditions for the second ones are also necessary optimality conditions for the first ones.

Unfortunately, there are a lot of important ordering cones whose topological interior is empty (for example, the positive cones ℓ^p_+ and L^p_+ of the Banach spaces ℓ^p and L^p, for all $p \in [1, +\infty)$). To overcome this problem, the concepts of algebraic interior, relative algebraic interior (also named as intrinsic core), quasi interior and quasi-relative interior of a convex set are considered (see [2, 14, 15, 17, 21]).

The aim of this work is to provide optimality conditions for weakly efficient solutions of vector optimization problems, that are defined through the quasi interior of a free disposal domination set. We study the convex case and the optimality conditions are formulated by linear scalarization results and Lagrangian multiplier theorems. These results improve several ones of the literature since they are stated by weaker assumptions. Moreover, they can be applied to problems where the topological and algebraic interior of the ordering cone is empty. The mathematical approaches and techniques of the paper are strongly motivated by the ones in [7, 22].

This paper is structured as follows. In Section 2, the setting and the basic notations and concepts are introduced. In particular, the vector optimization problem is introduced, the notions of quasi interior and quasi-relative interior and their main properties are recalled, and some basic properties on generalized convexity are stated in connection with the concept of quasi-relative interior.

In Section 3, characterizations of weakly efficient solutions of a vector optimization problem with set-valued mappings are stated via linear scalarization results. It is noticed that they encompass other similar ones of the literature. In section 4, Lagrangian optimality conditions are derived via a scalar Lagrangian set-valued mapping. As in the previous section, it is noticed that the obtained Lagrange multiplier rules improve other similar results of the literature since they are proved by weaker assumptions. Finally, in Section 5, the highlights of the paper are underlined.

2 Preliminaries

Throughout Y, Z denote two real locally convex Hausdorff topological linear spaces and Y^*, Z^* are their topological dual spaces, respectively. We refer to the duality pairing in Y and Z as $\langle \lambda, y\rangle$ and $\langle \mu, z\rangle$, i.e., $\langle \lambda, y\rangle := \lambda(y)$ and $\langle \mu, z\rangle := \mu(z)$, for all $y \in Y$, $z \in Z$, $\lambda \in Y^*$ and $\mu \in Z^*$. Moreover, let us assume that $K \subset Y$ and $D \subset Z$ are two convex cones.

In the sequel, $\operatorname{cl} M$ and $\operatorname{int} M$ denote the closure and the topological interior of a set $M \subset Y$. We say that M is solid if $\operatorname{int} M \neq \emptyset$. Moreover, $\operatorname{co} M$ stands for the convex hull of M. Let us recall that the positive polar cone of a cone $C \subset Y$ is the set

$$C^+ := \{\lambda \in Y^* : \langle \lambda, y\rangle \geq 0, \forall y \in C\},$$

and the positive polar cone of M is the set $M^+ := (\operatorname{cone} M)^+$, where $\operatorname{cone} M$ denotes the generated cone by M, i.e., $\operatorname{cone} M := \cup_{\alpha \geq 0} \alpha M$. We say that a cone $C \subset Y$ is proper if $C \neq Y$. Moreover, we denote $\operatorname{cone}_+ M = \cup_{\alpha > 0} \alpha M$ and $\mathbb{R}_+ := [0, +\infty)$.

In this paper we focus on the following set-valued optimization problems:

$$\operatorname{Min}\{F(x) : x \in S\}, \tag{VOP}$$

$$\operatorname{Min}\{F(x) : x \in Q, G(x) \cap (-D) \neq \emptyset\}, \tag{CVOP}$$

where X is a real linear space, $S, Q \subset X$ and $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$ are set-valued mappings. We assume that the feasible sets of problems (VOP) and (CVOP) are nonempty. Let us notice that problem (CVOP) is a constrained version of problem (VOP). In other words, problem (VOP) encompasses problem (CVOP) by taking the feasible set

$$S = \{x \in X : G(x) \cap (-D) \neq \emptyset\} \cap Q.$$

In this paper, the vector criterion will be considered to solve problem (VOP) (see [15, 16, 18]). This approach is based on the pointwise preferences introduced in the image space Y by a nonempty domination set $M \subset Y$ via the following binary relation:

$$y_1, y_2 \in Y, \quad y_1 \leq_M y_2 :\Longleftrightarrow\ y_2 - y_1 \in M. \tag{2.1}$$

Then, a feasible point $x_0 \in S$ is said to be a solution of problem (VOP) with domination set M, denoted by $x_0 \in \mathrm{Sol}(F, S, M)$, if there exists a point $y_0 \in F(x_0)$ such that y_0 is a nondominated point of the range $F(S)$, i.e., if we have

$$y \in F(S), y \leq_M y_0 \Rightarrow y = y_0.$$

The pair (x_0, y_0) is called M-minimizer of problem (VOP) and it is denoted by $(x_0, y_0) \in \mathrm{Min}(F, S, M)$. Notice that $\mathrm{Min}(F, S, M)$ is a subset of the graph of F:

$$\operatorname{gph} F := \{(x, y) \in X \times Y : y \in F(x)\}.$$

A nonempty set $E \subset Y$ is said to be free disposal with respect to a convex cone C if $E = E + C$. The class of free disposal sets was introduced by Debreu [5]. In the last years, they have been successfully used to model domination sets (see [13]).

Next, the concepts of algebraic interior, relative algebraic interior (also named as intrinsic core), quasi interior and quasi-relative interior of a convex set are recalled (see [2, 14, 15, 17, 21]). The third one is due to Limber and Goodrich [17] and the fourth one was introduced by Borwein and Lewis [2].

Definition 2.1. Let $M \subset Y$ be a convex set. The algebraic interior, the intrinsic core, the quasi interior and the quasi-relative interior of M are, respectively, the following sets:

$$\begin{aligned}
\operatorname{core} M &:= \{y \in M : \operatorname{cone}(M - y) = Y\}, \\
\operatorname{icr} M &:= \{y \in M : \operatorname{cone}(M - y) \text{ is a linear subspace of } Y\}, \\
\operatorname{qi} M &:= \{y \in M : \operatorname{cl} \operatorname{cone}(M - y) = Y\}, \\
\operatorname{qri} M &:= \{y \in M : \operatorname{cl} \operatorname{cone}(M - y) \text{ is a linear subspace of } Y\}.
\end{aligned} \tag{2.2}$$

Remark 2.2. It is clear that

$$\begin{aligned}
\operatorname{int} M \subset \operatorname{core} M \subset \operatorname{icr} M \subset \operatorname{qri} M, \\
\operatorname{core} M \subset \operatorname{qi} M \subset \operatorname{qri} M,
\end{aligned}$$

and $\operatorname{int} M = \operatorname{core} M = \operatorname{icr} M = \operatorname{qi} M = \operatorname{qri} M$ whenever M is solid. On the other hand, let us notice that $\operatorname{icr} \ell^p_+ = \operatorname{icr} L^p_+ = \emptyset$, while $\operatorname{qi} \ell^p_+ \neq \emptyset$ and $\operatorname{qi} L^p_+ \neq \emptyset$, for all $p \in [1, +\infty)$ (see [3, 17]).

The next proposition collects the basic properties of the quasi-relative interior (see [1, 2, 3, 22, 23, 24]). Let $M \subset Y$ be a nonempty convex set. Recall that the normal cone of M at a point $y \in M$ is the set

$$N_M(y) := \{\lambda \in Y^* : \langle \lambda, z - y \rangle \leq 0, \forall z \in M\}.$$

Proposition 2.3. *Let $M, A \subset Y$ be two convex sets. We have that:*

(i) $y \in \operatorname{qri} M$ if and only if $y \in M$ and $N_M(y)$ is a linear subspace of Y^*;

(ii) $\operatorname{qri} M + \operatorname{qri} A \subset \operatorname{qri}(\operatorname{qri} M + \operatorname{qri} A) \subset \operatorname{qri}(M + A)$;

(iii) if $M \cap \operatorname{qri} A \neq \emptyset$, then $\operatorname{qri}(M \cap A) \subset M \cap \operatorname{qri} A$;

(iv) if $\operatorname{qri} M \cap \operatorname{qri} A \neq \emptyset$, then $\operatorname{qri}(M \cap A) \subset \operatorname{qri} M \cap \operatorname{qri} A$;

(v) $\operatorname{qri}(M \times A) = \operatorname{qri} M \times \operatorname{qri} A$;

(vi) $\operatorname{qri}(M + y) = \operatorname{qri} M + y$, for all $y \in Y$;

(vii) $\operatorname{qri}(\alpha M) = \alpha \operatorname{qri} M$, for all $\alpha \in \mathbb{R}$;

(viii) $\beta \operatorname{qri} M + (1 - \beta)M \subset \operatorname{qri} M$, for all $\beta \in (0, 1]$;

(ix) if M is an affine set, then $\operatorname{qri} M = M$;

(x) $\operatorname{qri}(\operatorname{qri} M) = \operatorname{qri} M$;

(xi) $\operatorname{qri} M = M \cap \operatorname{qri} \operatorname{cl} M$;

Moreover, if $\operatorname{qri} M \neq \emptyset$, then:

(xii) $\operatorname{cl}(\operatorname{qri} M) = \operatorname{cl} M$;

(xiii) $\operatorname{cl} \operatorname{cone} \operatorname{qri} M = \operatorname{cl} \operatorname{cone} M$;

On the other hand, if $C \subset Y$ is a convex cone, then $\operatorname{qri} C + C = \operatorname{qri} C$. In addition, if $\operatorname{qri} C \neq \emptyset$, then for each nonempty set $L \subset Y$ it follows that

$$\operatorname{cl} \operatorname{cone}(L + C) = \operatorname{cl} \operatorname{cone}(L + \operatorname{qri} C) = \operatorname{cl}(\operatorname{cone} L + \operatorname{qri} C). \tag{2.3}$$

Remark 2.4. Let $M \subset Y$ be a convex set. If $\operatorname{qi} M \neq \emptyset$, then $\operatorname{qi} M = \operatorname{qri} M$ (see [17]). In particular, all properties of Proposition 2.3 are also true by replacing "qri" by "qi" provided that the quasi interiors of the involved sets are nonempty. Moreover, if $A \subset Y$ is also a convex set, then the following additional properties hold (see [23]):

$$M + \operatorname{qi} A = \operatorname{qi}(M + \operatorname{qi} A) \subset \operatorname{qi}(M + A); \tag{2.4}$$

$$M \subset A \Rightarrow \operatorname{qi} M \subset \operatorname{qi} A. \tag{2.5}$$

To complete the properties above, we state two characterizations of the elements of the quasi interior of a convex cone C through properties formulated in the dual space Y^*. The first one involves the normal cone of C and it was proved in [3, Proposition 2.4]. The second one involves the elements of $C^+\backslash\{0\}$ and it was obtained in [17, Theorem 2.2] in the setting of a normed space.

Theorem 2.5. *Let $C \subset Y$ be a convex cone and $y \in C$. We have that*

$$\begin{aligned} y \in \operatorname{qi} C &\iff N_C(y) = \{0\} && (2.6)\\ &\iff \langle \lambda, y \rangle > 0 \quad \forall \lambda \in C^+\backslash\{0\}. && (2.7) \end{aligned}$$

Proof. Let us only check the second characterization. Consider $y \in \operatorname{qi} C$ and $\lambda \in C^+\backslash\{0\}$. As $y \in C$, we have $\langle \lambda, y \rangle \geq 0$. Reasoning by contradiction, let us suppose that

$\langle\lambda, y\rangle = 0$ and consider an arbitrary point $z \in Y$. By the definition of quasi interior it follows that there exist nets $(\alpha_i) \subset \mathbb{R}_+$ and $(y_i) \subset C$ such that $\alpha_i(y_i - y) \to z$. Thus,

$$\langle\lambda, z\rangle = \lim_i \alpha_i\langle\lambda, y_i - y\rangle = \lim_i \alpha_i\langle\lambda, y_i\rangle \geq 0,$$

since $\lambda \in C^+$. Then, $\langle\lambda, z\rangle \geq 0$, for all $z \in Y$ and so $\lambda = 0$, which is a contradiction. Thus, $\langle\lambda, y\rangle > 0$ and the necessary condition of (2.7) is proved.

Reciprocally, assume that $y \in C$ satisfies $\langle\lambda, y\rangle > 0$, for all $\lambda \in C^+\backslash\{0\}$. Let us prove that $N_C(y) = \{0\}$. Indeed, suppose that $\bar{\lambda} \in N_C(y)$ and $\bar{\lambda} \neq 0$. By definition we have

$$\langle\bar{\lambda}, z - y\rangle \leq 0 \quad \forall z \in C$$

and since C is a cone we see that $\langle\bar{\lambda}, z\rangle \leq 0$, for all $z \in C$. Thus, $-\bar{\lambda} \in C^+\backslash\{0\}$. Moreover, by considering $z = 0$ we obtain $\langle-\bar{\lambda}, y\rangle \leq 0$, that is a contradiction. Therefore, $N_C(y) = \{0\}$ and the result follows by applying the sufficient condition of (2.6). □

It is well-known that optimality conditions for problem (VOP) can be obtained via alternative theorems and linear scalarization processes whenever suitable convexity conditions are satisfied. Next we recall a generalized convexity notion introduced in [12, Definition 2.3] from which the results of the following sections will be derived.

Definition 2.6. Consider a nonempty set $M \subset Y$. A set-valued mapping $H : X \rightrightarrows Y$ is said to be nearly M-subconvexlike on a nonempty set $N \subset X$ if $\operatorname{cl}\operatorname{cone}(H(N) + M)$ is convex.

Next we state some relationships between the notion of nearly M-subconvexlikeness and other generalized convexity concepts of the literature. A previous lemma is needed.

Lemma 2.7. *Consider a nonempty set $L \subset Y$ and a convex set $A \subset Y$ such that $\operatorname{cone}_+ A = A$. It follows that*

$$L + A \textit{ is convex} \Rightarrow \operatorname{cone} L + A \textit{ is convex.} \tag{2.8}$$

Proof. Let $y_1, y_2 \in \operatorname{cone} L$, $a_1, a_2 \in A$ and $\alpha \in (0, 1)$. Let us check that

$$y := \alpha(y_1 + a_1) + (1 - \alpha)(y_2 + a_2) \in \operatorname{cone} L + A. \tag{2.9}$$

Indeed, if $y_1 = 0$ or $y_2 = 0$, then (2.9) follows since A is convex. Then, assume that $y_1 \neq 0$ and $y_2 \neq 0$. There exist $\alpha_1, \alpha_2 > 0$ and $u_1, u_2 \in L$ such that $y_i = \alpha_i u_i$, $i = 1, 2$. By defining $c := \alpha\alpha_1 + (1 - \alpha)\alpha_2$ we have that:

$$y = c\left(\frac{\alpha\alpha_1}{c}(u_1 + (1/\alpha_1)a_1) + \frac{(1-\alpha)\alpha_2}{c}(u_2 + (1/\alpha_2)a_2)\right)$$

and since $\operatorname{cone}_+ A = A$ and $L + A$ is convex we deduce that

$$\begin{aligned} y \in \operatorname{cone}_+ \operatorname{co}(L + \operatorname{cone}_+ A) &= \operatorname{cone}_+ \operatorname{co}(L + A) \\ &= \operatorname{cone}_+(L + A) \\ &\subset \operatorname{cone} L + \operatorname{cone}_+ A \\ &= \operatorname{cone} L + A. \end{aligned}$$

Therefore, statement (2.9) is true and the proof is complete. □

Proposition 2.8. *Consider $H : X \rightrightarrows Y$, $\emptyset \neq N \subset X$ and $\emptyset \neq M \subset Y$. The following statements imply that H is nearly M-subconvexlike on N:*

(i) $H(N) + M$ is convex.

(ii) M is free disposal with respect to a convex cone $C \subset Y$, $\operatorname{qri} C \neq \emptyset$ and either $H(N) + M + \operatorname{qri} C$ is convex or $\operatorname{cone}(H(N) + M) + \operatorname{qri} C$ is convex.

(iii) M is a convex cone such that $\operatorname{qri} M \neq \emptyset$ and either $H(N) + \operatorname{qri} M$ is convex or $\operatorname{cone} H(N) + \operatorname{qri} M$ is convex.

Proof. It is clear that H is nearly M-subconvexlike on N provided that $H(N) + M$ is convex.

Moreover, part (iii) is obtained as a result of part (ii) by taking $C = M$, since by Proposition 2.3 it follows that $C + \operatorname{qri} C = \operatorname{qri} C$ and

$$\begin{aligned} \operatorname{cone}(H(N) + C) + \operatorname{qri} C &= \operatorname{qri} C \cup \operatorname{cone}_+(H(N) + C + \operatorname{qri} C) \\ &= \operatorname{qri} C \cup \operatorname{cone}_+(H(N) + \operatorname{qri} C) \\ &= \operatorname{cone} H(N) + \operatorname{qri} C. \end{aligned}$$

On the other hand, by Lemma 2.7 we see that part (ii) is proved if we only check the case $\operatorname{cone}(H(N) + M) + \operatorname{qri} C$ is a convex set. Indeed, by (2.3) we have that

$$\begin{aligned} \operatorname{cl}(\operatorname{cone}(H(N) + M) + \operatorname{qri} C) &= \operatorname{cl} \operatorname{cone}(H(N) + M + \operatorname{qri} C) \\ &= \operatorname{cl} \operatorname{cone}(H(N) + M + C) \\ &= \operatorname{cl} \operatorname{cone}(H(N) + M) \end{aligned}$$

and then H is nearly M-subconvexlike on N if the second assumption of part (ii) is satisfied. This finishes the proof. □

Remark 2.9. (i) Conditions (i)-(iii) of Proposition 2.8 correspond to different generalize convexity concepts (see [11, Section 2]).

(ii) Proposition 2.8 and (2.8) extend parts (i) and (ii) of [11, Proposition 2.13] to problems whose ordering cone has intrinsic core empty, but its quasi-relative interior is nonempty (see Remark 2.2).

(iii) In [24, Definition 2.3], the concept of generalized C-subconvexlike set-valued mapping is introduced, where C is a convex cone which is assumed to be closed and pointed ($C \cap (-C) = \{0\}$). Specifically, $H : X \rightrightarrows Y$ is said to be generalized C-subconvexlike on $N \subset X$ if $\operatorname{cone} H(N) + \operatorname{qri} C$ is convex. Then, by part (iii) of Proposition 2.8 we see that a set-valued mapping is nearly C-subconvexlike on N whenever it is generalized C-subconvexlike on N.

3 Linear scalarization

Let $E \subset Y$ be a free disposal set. The goal of this section is to characterize a kind of E-minimizers of problem (VOP) by linear scalarization. We derive these optimality conditions as a consequence of the next result. It is denoted $\tau_E : Y^* \to \mathbb{R} \cup \{-\infty\}$,

$$\tau_E(\lambda) := \inf_{e \in E} \langle \lambda, e \rangle \quad \forall \lambda \in Y^*.$$

Notice that mapping τ_E is close to the support function σ_{-E} of the set $-E$: $\tau_E(\lambda) = -\sigma_{-E}(\lambda)$, for all $\lambda \in Y^*$.

Theorem 3.1. *Let $C \subset Y$ be a convex cone and $E \subset Y$ be a free disposal set with respect to C. Consider a nonempty set $M \subset Y$ and the statements:*

(i) $0 \notin \operatorname{qri} \operatorname{co}((M + E) \cup \{0\})$;

(ii) $0 \notin \operatorname{qri} \operatorname{co} \operatorname{cone}(M + E)$;

(iii) $0 \notin \operatorname{qri} \operatorname{cone}(\operatorname{co}(M + E) + C)$;

(iv) $0 \notin \operatorname{qri} \operatorname{cl} \operatorname{co} \operatorname{cone}(M + E)$;

(v) $0 \notin \operatorname{qri}(\operatorname{cone} \operatorname{co}(M + E) + (\operatorname{qri} C \cup \{0\}))$;

(vi) There exist $\lambda \in C^+ \backslash \{0\}$, $y_0 \in M$ and $e_0 \in E$ such that $\langle \lambda, y \rangle + \tau_E(\lambda) \geq 0$, for all $y \in M$, and $\langle \lambda, y_0 \rangle + \langle \lambda, e_0 \rangle > 0$;

(vii) $0 \notin \operatorname{qri} \operatorname{co} \operatorname{cone}(M + E + \operatorname{qri} C)$;

(viii) $0 \notin \operatorname{qri}((\operatorname{cone} \operatorname{co}(M + E) + \operatorname{qri} C) \cup \{0\})$;

(ix) $0 \notin \operatorname{qri} \operatorname{cl} \operatorname{co} \operatorname{cone}(M + E + \operatorname{qri} C)$;

Then, conditions (i)-(vi) are equivalent. If additionally $\operatorname{qri} C \neq \emptyset$*, then all statements above are equivalent.*

Proof. Given two nonempty sets $A, M \subset Y$, the next properties are true:

$\operatorname{co} \operatorname{cone} A = \operatorname{cone} \operatorname{co} A$; $\operatorname{cl} \operatorname{cone} A$ is a cone;

$\operatorname{co}(A + M) = \operatorname{co} A + M$, provided M is convex;

$\operatorname{cone} A = \operatorname{cone}(A \cup \{0\})$;

$\operatorname{cl} \operatorname{cone}(A + M) = \operatorname{cl}(\operatorname{cone} A + M) = \operatorname{cl} \operatorname{cone}(A + M \backslash \{0\})$, provided M is a cone and $M \neq \{0\}$;

The equivalence between statements (i)-(v) is a consequence of Proposition 2.3 and the above relationships, and also the equivalence between (i)-(v) and (vii)-(ix) whenever $\operatorname{qri} C \neq \emptyset$. For example, let us check that parts (ii), (iv) and (v) are equivalent. For this aim, let us denote by $\mathcal{W}$ the family of all linear subspaces of Y. It is clear that $0 \in \operatorname{co}\operatorname{cone}(M+E)$. Thus, by (2.2) and the first of the above properties we have that

$$\begin{aligned} 0 \notin \operatorname{qri}\operatorname{co}\operatorname{cone}(M+E) &\Longleftrightarrow \operatorname{cl}\operatorname{cone}\operatorname{co}\operatorname{cone}(M+E) \notin \mathcal{W} \\ &\Longleftrightarrow \operatorname{cl}\operatorname{co}\operatorname{cone}(M+E) \notin \mathcal{W}. \end{aligned}$$

Analogously, as $0 \in \operatorname{cl}\operatorname{co}\operatorname{cone}(M+E)$, by (2.2) and the two first properties above it follows that

$$\begin{aligned} 0 \notin \operatorname{qri}\operatorname{cl}\operatorname{co}\operatorname{cone}(M+E) &\Longleftrightarrow \operatorname{cl}\operatorname{cone}\operatorname{cl}\operatorname{co}\operatorname{cone}(M+E) \notin \mathcal{W} \\ &\Longleftrightarrow \operatorname{cl}\operatorname{co}\operatorname{cone}(M+E) \notin \mathcal{W}. \end{aligned}$$

For dealing with part (v) assume that $\operatorname{qri} C \neq \emptyset$ (otherwise (v) coincides with (ii)). Moreover, notice that $0 \in \operatorname{cone}\operatorname{co}(M+E) + (\operatorname{qri} C \cup \{0\})$. Therefore, by (2.2), (2.3) and the properties above we deduce that

$$\begin{aligned} &0 \notin \operatorname{qri}(\operatorname{cone}\operatorname{co}(M+E) + (\operatorname{qri} C \cup \{0\})) \\ &\Longleftrightarrow \operatorname{cl}\operatorname{cone}(\operatorname{cone}\operatorname{co}(M+E) + (\operatorname{qri} C \cup \{0\})) \notin \mathcal{W} \\ &\Longleftrightarrow \operatorname{cl}\operatorname{co}\operatorname{cone}(M+E) \notin \mathcal{W}. \end{aligned}$$

Then it follows that parts (ii), (iv) and (v) are equivalent.

Finally, in order to complete the proof let us prove that statements (i) and (vi) are equivalent. Indeed, by [2, Proposition 2.16] we see that condition (i) holds if and only if there exists $\lambda \in Y^* \backslash \{0\}$, $\bar{y} \in \operatorname{co}((M+E) \cup \{0\})$ such that

$$\begin{aligned} \langle \lambda, y+e \rangle \geq 0 \quad \forall y \in M, e \in E, \\ \langle \lambda, \bar{y} \rangle > 0. \end{aligned} \tag{3.1}$$

Since $E = E + C$, inequality (3.1) can be rewritten as follows:

$$\langle \lambda, y+e \rangle + \langle \lambda, d \rangle \geq 0 \quad \forall y \in M, e \in E, d \in C.$$

Since C is a cone, we see that $\langle \lambda, d \rangle \geq 0$, for all $d \in C$, i.e., $\lambda \in C^+$.

It is clear that condition (3.1) is equivalent to the next one:

$$\langle \lambda, y \rangle + \tau_E(\lambda) \geq 0 \quad \forall y \in M.$$

On the other hand, as $\bar{y} \in \operatorname{co}((M+E) \cup \{0\})$, there exist n points $\bar{y}_i \in M+E$ and n real numbers $\alpha_i > 0$, $i = 1, 2, \ldots, n$, such that $\sum_{i=1}^n \alpha_i \leq 1$ and $\bar{y} = \sum_{i=1}^n \alpha_i \bar{y}_i$. Then,

$$\langle \lambda, \bar{y} \rangle = \sum_{i=1}^n \alpha_i \langle \lambda, \bar{y}_i \rangle$$

and so $\langle \lambda, \bar{y}_i \rangle > 0$ for some i.

Thus, conditions (i) and (vi) are equivalent and the proof finishes. □

Theorem 3.1 generalizes [7, Theorem 2.1] to free disposal sets. To be precise, it reduces to [7, Theorem 2.1] by considering the ordering cone as free disposal set.

By dealing with the quasi interior instead of the quasi-relative interior, the following result is obtained.

Theorem 3.2. *Let $C \subset Y$ be a convex cone and $E \subset Y$ be a free disposal set with respect to C. Consider a nonempty set $M \subset Y$ and the statements:*

(i) $0 \notin \operatorname{qi} \operatorname{cone} \operatorname{co}(M + E)$*;*

(ii) $0 \notin \operatorname{qi} \operatorname{cone}(\operatorname{co}(M + E) + C)$*;*

(iii) $0 \notin \operatorname{qi} \operatorname{cl} \operatorname{co} \operatorname{cone}(M + E)$*;*

(iv) There exists $\lambda \in C^+ \backslash \{0\}$ such that $\langle \lambda, y\rangle + \tau_E(\lambda) \geq 0$, for all $y \in M$.

(v) $0 \notin \operatorname{qi} \operatorname{co} \operatorname{cone}(M + E + \operatorname{qi} C)$*;*

(vi) $0 \notin \operatorname{qi}((\operatorname{cone} \operatorname{co}(M + E) + \operatorname{qi} C) \cup \{0\})$*;*

(vii) $0 \notin \operatorname{qi} \operatorname{cl} \operatorname{co} \operatorname{cone}(M + E + \operatorname{qi} C)$*;*

We have that conditions (i)-(iv) are equivalent. If additionally $\operatorname{qi} C \neq \emptyset$, then all statements above are equivalent.

Proof. By the properties of the generated cone and the convex hull (see the proof of Theorem 3.1) it follows that

$$\begin{aligned}
\operatorname{cl} \operatorname{cone} \operatorname{cone}(\operatorname{co}(M + E) + C) &= \operatorname{cl} \operatorname{cone}(\operatorname{co}(M + E) + C) \\
&= \operatorname{cl} \operatorname{cone} \operatorname{cone} \operatorname{co}(M + E) \\
&= \operatorname{cl} \operatorname{co} \operatorname{cone}(M + E) \\
&= \operatorname{cl} \operatorname{cone} \operatorname{cl} \operatorname{co} \operatorname{cone}(M + E).
\end{aligned}$$

Then it is clear that parts (i)-(iii) are equivalent. On the other hand, by (2.6) we have that (i) it is true if and only if there exists $\lambda \in Y^* \backslash \{0\}$ such that

$$\langle \lambda, y\rangle \geq 0 \quad \forall y \in \operatorname{co}(M + E).$$

This condition coincides with part (iv) (see the proof of Theorem 3.1) and so parts (i) and (iv) are also equivalent.

Let us suppose that $\operatorname{qi} C \neq \emptyset$. Then we have that parts (v)-(vii) coincide with statements (vii)-(ix) of Theorem 3.1, respectively. Let us check this fact only for condition (v), since the proofs for the other parts follow the same steps. By properties (2.4) and (2.5) we have that

$$\operatorname{co}(M + E) + \operatorname{qi} C = \operatorname{qi}(\operatorname{co}(M + E) + \operatorname{qi} C) \subset \operatorname{qi} \operatorname{cone}(\operatorname{co}(M + E) + \operatorname{qi} C)$$

and so $\operatorname{qi} \operatorname{cone}(\operatorname{co}(M + E) + \operatorname{qi} C) \neq \emptyset$. Therefore, by Remark 2.4 we deduce that

$$\operatorname{qri} \operatorname{cone}(\operatorname{co}(M + E) + \operatorname{qri} C) = \operatorname{qi} \operatorname{cone}(\operatorname{co}(M + E) + \operatorname{qi} C)$$

and then it is clear that part (v) is a reformulation of Theorem 3.1(vii).

Analogously, condition (iv) coincides with statement (vi) of Theorem 3.1. Indeed, it is obvious that Theorem 3.1(vi) implies part (iv) above. Reciprocally, assume that $\lambda \in C^+\backslash\{0\}$ satisfies $\langle \lambda, y\rangle + \tau_E(\lambda) \geq 0$, for all $y \in M$. Consider three arbitrary points $y_0 \in M$, $e \in E$ and $d \in \operatorname{qi} C$. As E is free disposal we have $e_0 := e + d \in E$. Moreover, by Theorem 2.5 we see that $\langle \lambda, d\rangle > 0$. Therefore,

$$\langle \lambda, y_0\rangle + \langle \lambda, e_0\rangle = \langle \lambda, y_0\rangle + \langle \lambda, e\rangle + \langle \lambda, d\rangle \geq \langle \lambda, y_0\rangle + \tau_E(\lambda) + \langle \lambda, d\rangle \geq \langle \lambda, d\rangle > 0$$

and part (vi) of Theorem 3.1 is satisfied. This finishes the proof. □

From Theorem 3.2 and the generalized convexity notion of Definition 2.6, one can deduce the following Gordan-type alternative theorem.

Theorem 3.3. *Let $C \subset Y$ be a convex cone. Let $E \subset Y$ be a free disposal set with respect to C. Assume that the set-valued mapping $H : X \rightrightarrows Y$ is nearly E-subconvexlike on a nonempty set $N \subset X$. Then, one and only one of the following statements is true:*

(i) $0 \in \operatorname{qi} \operatorname{cl} \operatorname{cone}(H(N) + E)$*;*

(ii) There exists $\lambda \in C^+\backslash\{0\}$ such that $\langle \lambda, y\rangle + \tau_E(\lambda) \geq 0$, for all $y \in H(N)$.

If additionally $\operatorname{qi} C \neq \emptyset$*, then condition (i) can be replaced with*

(i') $0 \in \operatorname{qi} \operatorname{cl} \operatorname{cone}(H(N) + E + \operatorname{qi} C)$;

Proof. Let us notice that

$$\operatorname{cl} \operatorname{co} \operatorname{cone}(H(N) + E) = \operatorname{cl} \operatorname{cone}(H(N) + E).$$

Indeed, as $H : X \rightrightarrows Y$ is nearly E-subconvexlike on N is follows that

$$\begin{aligned}
\operatorname{cl} \operatorname{cone}(H(N) + E) &\subset \operatorname{cl} \operatorname{co} \operatorname{cone}(H(N) + E) \\
&\subset \operatorname{cl} \operatorname{co} \operatorname{cl} \operatorname{cone}(H(N) + E) \\
&= \operatorname{cl} \operatorname{cl} \operatorname{cone}(H(N) + E) \\
&= \operatorname{cl} \operatorname{cone}(H(N) + E).
\end{aligned}$$

Assume that $\operatorname{qi} C \neq \emptyset$. Then, by (2.3), Remark 2.4 and the basic properties collected at the beginning of the proof of Theorem 3.1 we have that

$$\begin{aligned}
\operatorname{cl} \operatorname{co} \operatorname{cone}(H(N) + E + \operatorname{qi} C) &= \operatorname{cl} \operatorname{cone}(\operatorname{co}(H(N) + E) + \operatorname{qi} C) \\
&= \operatorname{cl} \operatorname{cone}(\operatorname{co}(H(N) + E) + C) \\
&= \operatorname{cl} \operatorname{cone}(H(N) + E) \\
&= \operatorname{cl} \operatorname{cone}(H(N) + E + \operatorname{qi} C).
\end{aligned}$$

Therefore, the result follows by applying parts (iii), (iv) and (vii) of Theorem 3.2 and the proof is complete. □

Remark 3.4. (i) In the setting of Theorem 3.3, assume additionally that $\operatorname{qi} C \neq \emptyset$, $0 \notin \operatorname{qi} C$ and $\operatorname{cone}(H(N)+E)$ is closed. Let us prove that

$$H(N) \cap -(E + \operatorname{qi} C) = \emptyset \iff 0 \notin \operatorname{qi}\operatorname{cone}(H(N)+E). \tag{3.2}$$

Indeed, it is clear that $H(N) \cap -(E + \operatorname{qi} C) = \emptyset$ if and only if $0 \notin H(N) + E + \operatorname{qi} C$. Then, let us check that

$$0 \notin H(N) + E + \operatorname{qi} C \iff 0 \notin \operatorname{qi}\operatorname{cone}(H(N)+E). \tag{3.3}$$

To deduce the necessary condition, assume by contradiction that $0 \in \operatorname{qi}\operatorname{cone}(H(N)+E)$. Then,

$$\operatorname{cone}(H(N)+E) = Y, \tag{3.4}$$

since $\operatorname{cone}(H(N)+E)$ is closed. Take an arbitrary point $q \in \operatorname{qi} C$. By the hypothesis we have $q \neq 0$ and from (3.4) we see that there exists $\alpha > 0$ such that $-\alpha q \in H(N)+E$. Therefore, $0 \in H(N)+E+\operatorname{qi} C$, as $\alpha q \in \operatorname{qi} C$ by Proposition 2.3(vii) and Remark 2.4, that is a contradiction.

Reciprocally, by (2.4) it follows that

$$\begin{aligned} H(N)+E+\operatorname{qi} C &\subset \operatorname{cone}(H(N)+E)+\operatorname{qi} C \\ &= \operatorname{qi}(\operatorname{cone}(H(N)+E)+\operatorname{qi} C) \\ &\subset \operatorname{qi}(\operatorname{cone}(H(N)+E)+C). \end{aligned}$$

Moreover, it is clear that

$$0 \in \operatorname{qi}(\operatorname{cone}(H(N)+E)+C) \iff 0 \in \operatorname{qi}\operatorname{cone}(H(N)+E)$$

and the sufficient condition of (3.3) is stated.

By (3.2) it follows that [20, Theorem 4.1] is a particular case of Theorem 3.3. As a result, notice that the assumptions E convex and $0 \notin E$ of [20, Theorem 4.1] are not required in Theorem 3.3.

(ii) The necessary condition of [22, Proposition 28] is obtained by applying Theorem 3.3 to $E := C$. Analogously, Theorem 3.2(iii),(iv) encompasses the sufficient condition of [22, Proposition 28] by considering $E := C$.

Consider problem (VOP) and a nonempty set $E \subset Y$. Next we provide conditions for E-minimizers of problem (VOP) through certain solutions of the following family of associated set-valued scalar optimization problems:

$$\operatorname{Min}\{\langle \lambda, F(x)\rangle : x \in S\} \quad \lambda \in E^+ \backslash \{0\}. \tag{OP(λ)}$$

Notice that $\langle \lambda, F(\cdot)\rangle : X \rightrightarrows \mathbb{R}$ is the set-valued mapping:

$$\langle \lambda, F(x)\rangle := \{\langle \lambda, y\rangle : y \in F(x)\} \quad \forall x \in X.$$

Suppose that $\operatorname{qi} E \neq \emptyset$. Then, a point $x_0 \in S$ is said to be a weak solution of problem (VOP) with domination set E if $x_0 \in \operatorname{WSol}(F, S, E) := \operatorname{Sol}(F, S, \operatorname{qi} E)$. Analogously, the pair $(x_0, y_0) \in \operatorname{gph} F$ is called weak E-minimizer of problem (VOP) if $(x_0, y_0) \in \operatorname{WMin}(F, S, E) := \operatorname{Min}(F, S, \operatorname{qi} E)$.

On the other hand, a pair $(x_0, y_0) \in \operatorname{gph} F$ is a suboptimal solution of problem (OP(λ)) with error $\varepsilon \in \mathbb{R}_+$, and it is denoted by $(x_0, y_0) \in \operatorname{Min}(\langle \lambda, F\rangle, S, \varepsilon)$, if $\langle \lambda, y_0\rangle - \varepsilon \leq \langle \lambda, y\rangle$, for all $y \in F(S)$.

Theorem 3.3 actually characterizes several types of weak solutions of problem (VOP) introduced in the literature. The following result is the bridge for showing this assertion. Notice that (i) and the first part of (ii) work for any nonempty set $E \subset Y$.

Theorem 3.5. *Consider problem* (VOP)*, $x_0 \in S$ and $y_0 \in F(x_0)$. Let E be an arbitrary nonempty set of Y.*

(i) If $\operatorname{cl} \operatorname{cone}(F(S) - y_0 + E)$ *is proper and convex, then*

$$(x_0, y_0) \in \bigcup_{\lambda \in E^+ \backslash \{0\}} \operatorname{Min}(\langle \lambda, F\rangle, S, \tau_E(\lambda)). \tag{3.5}$$

(ii) Reciprocally, if (3.5) *is satisfied, then* $\operatorname{cl} \operatorname{co} \operatorname{cone}(F(S) - y_0 + E)$ *is proper, and if additionally E is free disposal with respect to K and $\operatorname{qi} K \neq \emptyset$, then $(x_0, y_0) \in \operatorname{Min}(F, S, E + \operatorname{qi} K)$.*

Proof. (i) It is obvious that E is free disposal with respect to the convex cone $C := \{0\}$. By the assumptions we have that the set-valued mapping $F - y_0$ is nearly E-subconvexlike on S and $0 \notin \operatorname{qi} \operatorname{cl} \operatorname{cone}(F(S) - y_0 + E)$. By Theorem 3.3 we deduce that there exists $\lambda \in Y^* \backslash \{0\}$ such that

$$\langle \lambda, y\rangle \geq \langle \lambda, y_0\rangle - \tau_E(\lambda) \quad \forall y \in F(S). \tag{3.6}$$

By taking $y = y_0$ in (3.6) it follows that $\tau_E(\lambda) \geq 0$, i.e., $\lambda \in E^+ \backslash \{0\}$. Therefore, $(x_0, y_0) \in \operatorname{Min}(\langle \lambda, F\rangle, S, \tau_E(\lambda))$.

(ii) Consider again the convex cone $C := \{0\}$. We have that $E^+ \subset C^+$. Then, by the hypothesis, statement (iv) of Theorem 3.2 is satisfied with $M = F(S) - y_0$. Therefore, part (i) of Theorem 3.2 is satisfied too, i.e., we obtain that $0 \notin \operatorname{qi} \operatorname{cone} \operatorname{co}(F(S) - y_0 + E)$. Thus, the cone $\operatorname{cl} \operatorname{co} \operatorname{cone}(F(S) - y_0 + E)$ is proper.

Suppose that $\operatorname{qi} K \neq \emptyset$ and $E + K = E$. By (2.4) and (2.5) it follows that

$$\begin{aligned} F(S) - y_0 + E + \operatorname{qi} K &\subset \operatorname{co}(F(S) - y_0 + E) + \operatorname{qi} K \qquad (3.7) \\ &\subset \operatorname{qi}(\operatorname{co}(F(S) - y_0 + E) + K) \\ &\subset \operatorname{qi}(\operatorname{cone}(\operatorname{co}(F(S) - y_0 + E) + K)). \qquad (3.8) \end{aligned}$$

Moreover, we have that

$$0 \in \operatorname{qi} \operatorname{cone}(\operatorname{co}(F(S) - y_0 + E) + K) \iff \operatorname{cl} \operatorname{co} \operatorname{cone}(F(S) - y_0 + E) = Y. \tag{3.9}$$

Therefore, if $(x_0, y_0) \notin \mathrm{Min}(F, S, E + \mathrm{qi}\, K)$, then $(F(S) - y_0) \cap -(E + \mathrm{qi}\, K) \neq \emptyset$, i.e., $0 \in F(S) - y_0 + E + \mathrm{qi}\, K$. By (3.7), (3.8) and (3.9) we deduce that $\mathrm{cl}\,\mathrm{co}\,\mathrm{cone}(F(S) - y_0 + E) = Y$, which is a contradiction. This finishes the proof. $\square$

Remark 3.6. (i) Theorem 3.5 encompasses [20, Theorem 4.2]. Indeed, consider problem (VOP) and a convex free disposal set $E \subset Y\backslash\{0\}$ with respect to K. Assume that $\mathrm{qi}\, K \neq \emptyset$ and $\mathrm{qi}\, E = E + \mathrm{qi}\, K$. Let $x_0 \in S$, $y_0 \in F(S)$ and suppose that $F - y_0$ is nearly E-subconvexlike on S and $\mathrm{cone}(F(S) - y_0 + E)$ is closed. By (3.2) we see that

$$\begin{aligned}(x_0, y_0) \in \mathrm{WMin}(F, S, E) &\iff (F(S) - y_0) \cap -\mathrm{qi}\, E = \emptyset \\ &\iff 0 \notin \mathrm{qi}\,\mathrm{cone}(F(S) - y_0 + E) \\ &\iff \mathrm{cone}(F(S) - y_0 + E) \text{ is proper.}\end{aligned}$$

On the other hand, by the assumptions it follows that $\mathrm{cone}(F(S) - y_0 + E)$ is convex and closed. Thus, by Theorem 3.5 we see that

$$(x_0, y_0) \in \mathrm{WMin}(F, S, E) \iff (x_0, y_0) \in \bigcup_{\lambda \in E^+\backslash\{0\}} \mathrm{Min}(\langle \lambda, F\rangle, S, \tau_E(\lambda))$$

and the conclusion of [20, Theorem 4.2] is obtained. Notice that in this result one can consider $\lambda \in E^+\backslash\{0\}$ instead of $\lambda \in K^+\backslash\{0\}$. Moreover, the assumptions E convex and $0 \notin E$ are not required in Theorem 3.5.

(ii) Theorem 3.5 reduces to [10, Theorem 4.1] and a version of [10, Theorem 4.2] when the objective function F is a vector-valued function $f : X \to Y$ and K is solid and proper. Indeed, let $\emptyset \neq C \subset Y$, $E := C + K$ and suppose that $C \cap -\mathrm{int}\, K = \emptyset$ and f is nearly E-subconvexlike on S. Let $x_0 \in \mathrm{WSol}(f, S, E)$. Then, $x_0 \in S$ and

$$(f(S) - f(x_0)) \cap -(C + \mathrm{int}\, K) = \emptyset, \tag{3.10}$$

since $\mathrm{int}\, E = C + \mathrm{int}\, K$ and $0 \notin \mathrm{int}\, E$. As $K + \mathrm{int}\, K = \mathrm{int}\, K$, condition (3.10) implies

$$\mathrm{cl}\,\mathrm{cone}(f(S) - f(x_0) + E) \cap -\mathrm{int}\, K = \emptyset$$

and so we see that $\mathrm{cl}\,\mathrm{cone}(f(S) - f(x_0) + E)$ is proper and convex. By (3.5) we deduce that there exists $\lambda \in E^+\backslash\{0\}$ such that

$$\langle \lambda, f(x)\rangle \geq \langle \lambda, f(x_0)\rangle - \tau_E(\lambda) \quad \forall x \in S. \tag{3.11}$$

Then, [10, Theorem 4.2] follows since $E^+ = C^+ \cap K^+$ and so $\tau_E(\lambda) = \tau_C(\lambda)$. As a result, [10, Theorem 4.2] is a particular case of Theorem 3.5 whenever C is free disposal with respect to the ordering cone K.

Reciprocally, suppose that condition (3.11) is satisfied and $\lambda \in E^+\backslash\{0\}$. By Theorem 3.5(ii) we have that $x_0 \in \mathrm{Sol}(f, S, E + \mathrm{int}\, K) = \mathrm{Sol}(f, S, C + \mathrm{int}\, K)$. This proves that Theorem 3.5 encompasses [10, Theorem 4.1].

(iii) In [9, Theorems 6 and 8], a characterization for the elements of $\mathrm{WSol}(f, U, K)$ is obtained, where $f : Y \to Y$ is the identity function, $K \subset Y$ is a pointed convex cone such that $\operatorname{qi} K \neq \emptyset$, $U \subset Y$ is nonempty and convex and it is assumed that $U + \operatorname{qi} K = \operatorname{qi}(U + K)$.

Theorem 3.5 encompasses this result. Indeed, if $x_0 \in \mathrm{WSol}(f, U, K)$, then $x_0 \in U$ and $(U - x_0) \cap -\operatorname{qi} K = \emptyset$. By the assumptions, Proposition 2.3(vi) and Remark 2.4 we deduce that

$$0 \notin U - x_0 + \operatorname{qi} K = \operatorname{qi}(U + K) - x_0 = \operatorname{qi}(U + K - x_0).$$

Then $\operatorname{cl}\operatorname{cone}(U - x_0 + K)$ is a proper convex cone and by Theorem 3.5(i) we obtain that there exists $\lambda \in K^+\backslash\{0\}$ such that $\langle\lambda, x\rangle \geq \langle\lambda, x_0\rangle$, for all $x \in U$.

Reciprocally, if this scalarization condition is satisfied, by Theorem 3.5(ii) it follows that $x_0 \in \mathrm{Sol}(f, U, K + \operatorname{qi} K) = \mathrm{WSol}(f, U, K)$ because of Proposition 2.3. Therefore, [9, Theorems 6 and 8] are particular cases of Theorem 3.5. Notice that the pointedness condition on the cone K is superfluous.

4 Lagrangian optimality conditions

In the sequel, we provide a kind of Lagrange multiplier rule for problem (CVOP) from which one can derive different Lagrange optimality conditions for weakly efficient solutions of the problem. We denote $(F \times G)(Q) := \bigcup_{x \in Q} F(x) \times G(x)$.

Theorem 4.1. *Consider problem* (CVOP)*, a nonempty set $E \subset Y$ and $(x_0, y_0) \in \operatorname{gph} F$, where x_0 is a feasible point of* (CVOP)*.*

(i) If the cone $\operatorname{cl}\operatorname{cone}((F \times G)(Q) - (y_0, 0) + (E \times D))$ is convex and proper, then there exists $\lambda \in E^+$, $\mu \in D^+$, $(\lambda, \mu) \neq (0, 0)$, such that

$$\langle\lambda, y_0\rangle - \tau_E(\lambda) \leq \langle\lambda, y\rangle + \langle\mu, z\rangle \quad \forall (y, z) \in (F \times G)(Q), \tag{4.1}$$

$$-\tau_E(\lambda) \leq \inf_{z \in G(x_0)} \{\langle\mu, z\rangle\} \leq 0. \tag{4.2}$$

(ii) Let $(\lambda, \mu) \in (Y^* \times D^+)\backslash\{(0, 0)\}$ and suppose that statement (4.1) *holds. It follows that $\lambda \neq 0$ whenever one of the next two conditions is satisfied:*

(a) There exists $x \in Q$ such that $G(x) \cap -\operatorname{qi} D \neq \emptyset$;

(b) $G(Q) + D$ is convex and $0 \in \operatorname{qi}(G(Q) + D)$.

(iii) If $\lambda \in E^+$, $\mu \in D^+$, $(\lambda, \mu) \neq (0, 0)$ and conditions (4.1) *and* (4.2) *are satisfied, then the cone $\operatorname{cl}\operatorname{co}\operatorname{cone}((F \times G)(Q) - (y_0, 0) + (E \times D))$ is proper. If additionally $\lambda \neq 0$, $\operatorname{qi} K \neq \emptyset$ and E is free disposal with respect to K, then (x_0, y_0) is an $(E + \operatorname{qi} K)$-minimizer of problem* (CVOP)*.*

Proof. (i) Consider the set-valued mapping $H : X \rightrightarrows Y \times Z$ given by $H(x) = F(x) \times G(x)$, for all $x \in X$. As x_0 is a feasible point of problem (CVOP), there exists $z_0 \in$

$G(x_0) \cap (-D)$. Then, by the assumption we see that $\operatorname{cl}\operatorname{cone}(H(Q) - (y_0, z_0) + (E \times (D + z_0)))$ is convex and proper, and so Theorem 3.5 can be applied. As a result, we deduce that there exists $(\lambda, \mu) \in (E \times (D + z_0))^+ \backslash \{(0, 0)\}$ satisfying

$$\langle(\lambda, \mu), (y, z)\rangle \geq \langle(\lambda, \mu), (y_0, z_0)\rangle - \tau_{E\times(D+z_0)}(\lambda, \mu) \quad \forall (y, z) \in H(Q). \tag{4.3}$$

It is clear that

$$\tau_{E\times(D+z_0)}(\lambda, \mu) = \tau_E(\lambda) + \tau_D(\mu) + \langle\mu, z_0\rangle.$$

Therefore, (4.3) is equivalent to the following condition:

$$\langle\lambda, y\rangle + \langle\mu, z\rangle \geq \langle\lambda, y_0\rangle - \tau_E(\lambda) - \tau_D(\mu) \quad \forall (y, z) \in (F \times G)(Q). \tag{4.4}$$

Since D is a cone, we have $\mu \in D^+$ and $\tau_D(\mu) = 0$. Thus, (4.4) becomes to the assertion

$$\langle\lambda, y\rangle + \langle\mu, z\rangle \geq \langle\lambda, y_0\rangle - \tau_E(\lambda) \quad \forall (y, z) \in (F \times G)(Q). \tag{4.5}$$

Moreover, as $(\lambda, \mu) \in (E \times (D + z_0))^+ \backslash \{(0, 0)\}$, we see that

$$\langle\lambda, e\rangle + \langle\mu, d\rangle + \langle\mu, z_0\rangle \geq 0 \quad \forall e \in E, d \in D.$$

Thus, since $\mu \in D^+$ and $z_0 \in -D$, by taking $d = 0$ above it follows that

$$\langle\lambda, e\rangle \geq \langle\lambda, e\rangle + \langle\mu, z_0\rangle \geq 0 \quad \forall e \in E,$$

i.e., $\lambda \in E^+$. On the other hand, by taking $y = y_0$ in (4.5) we have

$$\langle\mu, z\rangle \geq -\tau_E(\lambda) \quad \forall z \in G(x_0).$$

Therefore,

$$0 \geq \inf_{z\in G(x_0)} \{\langle\mu, z\rangle\} \geq -\tau_E(\lambda),$$

since $G(x_0) \cap (-D) \neq \emptyset$.

(ii) Assume that $G(x) \cap -\operatorname{qi} D \neq \emptyset$ for some $x \in Q$, and consider a point $\bar{z} \in G(x) \cap -\operatorname{qi} D$. If $\lambda = 0$, then by (4.1) we obtain $\langle\mu, z\rangle \geq 0$, forall $z \in G(Q)$. In particular, $\langle\mu, \bar{z}\rangle \geq 0$. However, $\mu \in D^+\backslash\{0\}$ and by (2.7) we obtain $\langle\mu, \bar{z}\rangle < 0$, that is a contradiction.

Suppose that $0 \in \operatorname{qi}(G(Q) + D)$. Then $\operatorname{cl}\operatorname{cone}(G(Q) + D) = Y$. If $\lambda = 0$, then by (4.1) we obtain $\langle\mu, z\rangle \geq 0$, forall $z \in G(Q)$. Since $\mu \in D^+$, it follows that $\langle\mu, y\rangle \geq 0$, forall $y \in \operatorname{cl}\operatorname{cone}(G(Q) + D)$. Therefore, $\mu \in Y^+ = \{0\}$, that is a contradiction since $(\lambda, \mu) \neq (0, 0)$.

(iii) Consider $\lambda \in E^+$, $\mu \in D^+$, $(\lambda, \mu) \neq (0, 0)$, and assume that statements (4.1) and (4.2) are satisfied. Take an arbitrary point $z_0 \in G(x_0)$. By (4.1) we have that

$$\langle(\lambda, \mu), (y, z)\rangle \geq \langle(\lambda, \mu), (y_0, z_0)\rangle - \tau_{E\times(D+z_0)}(\lambda, \mu) \quad \forall (y, z) \in (F \times G)(Q).$$

Moreover, $(\lambda, \mu) \in (E \times (D + z_0))^+$, since statement (4.2) implies

$$\langle(\lambda, \mu), (e, d + z_0)\rangle \geq \tau_E(\lambda) + \langle\mu, z_0\rangle \geq \tau_E(\lambda) + \inf_{z \in G(x_0)} \langle\mu, z\rangle \geq 0 \quad \forall e \in E, \forall d \in D.$$

Then, by Theorem 3.5(ii) it follows that $\operatorname{cl}\operatorname{co}\operatorname{cone}((F\times G)(Q)-(y_0, z_0)+(E\times(D+z_0)))$ is proper.

Moreover, if additionally we have $\lambda \neq 0$, $\operatorname{qi} K \neq \emptyset$ and E is free disposal with respect to K, then

$$(F(S) - y_0) \cap -(E + \operatorname{qi} K) = \emptyset. \tag{4.6}$$

Indeed, suppose reasoning by contradiction that there exist $\bar{x} \in S$, $\bar{y} \in F(\bar{x})$, $e \in E$ and $q \in \operatorname{qi} K$ such that $\bar{y} - y_0 = -e - q$. Consider an arbitrary point $\bar{z} \in G(\bar{x}) \cap -D$. As $E + K = E$ we have that $E^+ \subset K^+$. Then, by Theorem 2.5 it follows that

$$\begin{aligned}
\langle\lambda, \bar{y}\rangle + \langle\mu, \bar{z}\rangle &\leq \langle\lambda, y_0\rangle - \langle\lambda, e\rangle - \langle\lambda, q\rangle \\
&\leq \langle\lambda, y_0\rangle - \tau_E(\lambda) - \langle\lambda, q\rangle \\
&< \langle\lambda, y_0\rangle - \tau_E(\lambda),
\end{aligned}$$

that is contrary to (4.1). Thus, statement (4.6) is true and so (x_0, y_0) is a $(E + \operatorname{qi} K)$-minimizer of problem (CVOP) and the proof is completed. □

Remark 4.2. (i) Notice that Theorem 4.1(ii) considers two Slater type constraint qualifications in order to guaranty that the scalarization functional corresponding to the objective function does not vanish.

(ii) Theorem 4.1 reduces to [24, Theorem 3.1] by considering $E = K$. In particular, let us notice that the following assumptions of [24, Theorem 3.1] are superfluous: K, D closed and pointed, $\operatorname{qri} K \neq \emptyset$, $\operatorname{qri} D \neq \emptyset$, $\operatorname{qri}(\operatorname{cone}((F-y_0)\times G)(Q)+(\operatorname{qri} K\times \operatorname{qri} D)) \neq \emptyset$. On the other hand, assumptions (ii) and (iv) of [24, Theorem 3.1] are stronger than the hypothesis of Theorem 4.1(i) (see Proposition 2.8).

(iii) Theorem 4.1(iii) reduces to [24, Theorem 3.2] by considering $E = K$. In order to check this assertion, notice that $\operatorname{qi} K = \operatorname{qri} K$ whenever $\operatorname{cl}(K - K) = Y$ (see [6, Proposition 3.1]).

(iv) Theorem 4.1 encompasses [22, Proposition 23] by considering $E = K$. Notice that [22, Proposition 23] is provided under a stronger convexity assumption (see part (i) of Proposition 2.8).

5 Conclusions

The paper has provided optimality conditions for weakly efficient solutions of vector optimization problems with set-valued mappings. In defining these solutions, the so-called vector criterion has been considered in connection with the concept of quasi interior and an domination set given by a free disposal set with respect to the ordering cone of the problem.

Because of this general approach, the obtained results could be useful in some problems where other similar results of the literature cannot be applied. For example, consider problems (VOP) and (CVOP) given by the following data: the image space is ℓ^p, $p \in [1, +\infty)$, the ordering cone is the positive cone ℓ^p_+, and the domination set is $E = \varepsilon q + \ell^p_+$, where q is a point in $\operatorname{qi} \ell^p_+$ and $\varepsilon > 0$. The set $\mathrm{WSol}(F, S, E)$ is interesting since its elements are good approximations for the set $\mathrm{WSol}(F, S, \ell^p_+)$ of weakly efficient solutions of the problem under mild assumptions. Then, optimality conditions for the solutions in $\mathrm{WSol}(F, S, E)$ can be obtained from Theorems 3.5 and 4.1. However, the results in [10, 11, 20, 22, 24] cannot be applied. Indeed, papers [10, 11] are not useful in this setting since $\operatorname{core} \ell^p_+ = \emptyset$. Work [20] does not deal with constrained vector optimization problems. So, its results do not work in problem (CVOP). Finally, papers [22, 24] cannot be applied because the domination set is not a cone.

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