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On the Existence of Weak Efficient Solutions of Nonconvex Vector Optimization Problems

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Abstract We study vector optimization problems with arbitrary convex ordering cones (in particular, non polyhedral ones) without assuming any convexity or quasiconvexity assumption. We state a Weierstrass-type theorem, coercive and noncoercive existence results for weak efficient solutions to these problems. Our approach is based on a new coercivity notion for vector valued functions, two realizations of the Gerstewitz scalarization function, asymptotic analysis and a regularization of the objective function. We define new boundedness and lower semicontinuity properties for vector valued functions and study their properties. These mathematical tools rely heavily on the solidness of the order cone through the notion of colevel and level sets. As a consequence of this approach, we improve several similar existence results from the literature, since weaker assumptions are required.

Keywords Vector optimization · Weak efficient solution · Noncoercive existence result · Coercive vector valued function · Colevel set · Level set · Nonlinear scalarization

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1 Introduction

The aim of this paper is to obtain existence results for weak non-dominated solutions of a vector optimization problem whose decision space is a normed space, the image space is a (possibly infinite dimensional) Hausdorff real locally convex topological vector space and the preferences of the decision maker are defined by a convex cone with a nonempty topological interior. They can be applied to non quasiconvex (in particular nonconvex) problems and also to problems where the ordering cone is not polyhedral. This setting essentially encompasses other notions of solution in vector optimization that can be reformulated as weak efficiency concepts, as for instance, Henig global proper efficiency and super efficiency (see [1]).

In the literature on vector optimization, one can find a great amount of papers containing Weierstrass-type theorems and coercive existence results for weak efficient solutions of finite dimensional convex vector optimization problems, where the ordering cone is the nonnegative orthant (i.e., in Pareto multiobjective problems, see [2–7] and the references therein). These results have been extended to problems with arbitrary convex ordering cones in [8–12] and in particular with polyhedral ordering cones in [13, 14]. In addition, the convexity requirements have been weakened in [8, 9, 11] by considering certain quasiconvexity assumptions.

Therefore, these results cannot be applied to both noncoercive and non quasiconvex problems. So, the main objective in this work is to derive existence results in these settings for arbitrary convex ordering cones. Our approach combines the well-known direct method of the calculus of variations with a regularization of the objective function that generalizes a previous one introduced in [15] in scalar optimization problems. This research line was partially suggested therein.

Both methods are performed by using suitable mathematical tools. Let us underline two realizations of the Gerstewitz scalarization function (see [16–18]), asymptotic cones and functions, and a new coercivity notion for vector valued functions. To define the last notion, it is needed to extend the image space by adding a maximal element. This approach has been considered in other papers with other goals (see, for instance, [5, 19–21]). It is important to point out that the extended image space is endowed with a topology, in order to deal with the coercivity notion.

The main results of the paper rely on the closedness and boundedness properties of colevel and level sets. These sets allow us to combine the new coercivity notion with the classical Combari-Laghdir-Thibault lower semicontinuity notion of vector functions (see [22]) and with a cone boundedness notion in order to carry out the above mentioned methods. In particular, a simple formulation for the Combari-Laghdir-Thibault lower semicontinuity notion is stated through the sequentially closedness of colevel sets.

The paper is structured as follows. In Section 2, we fix the problem and the notation. Moreover, we recall basic concepts and tools of vector optimization and asymptotic analysis. Sequential asymptotic cones and functions allow us to deal with problems with unbounded data. The announced two realizations of the Gerstewitz scalarization function are introduced and their main properties are stated. In Section 3, we introduce lower semicontinuity notions for vector valued functions. We characterize them and compare with the existing ones in the literature. Section 4 concerns with a deep study of other basic mathematical tools for obtaining the existence results. They revolve around a notion of boundedness for sets and a new coercivity notion for vector valued functions. We also obtain bounds for the sequential asymptotic cones of level and colevel sets that allow us to study their boundedness. In Section 5, we obtain coercive and noncoercive existence results and compare them with similar ones from the literature. Finally, in Section 5, we provide a summary of conclusions and suggest some research lines to develop.

2 Preliminaries

Throughout this paper $(X, \|\cdot\|)$ is a real normed space and Y is a Hausdorff real locally convex topological vector space, which is assumed to be quasi ordered by a convex cone $P \subset Y$ as usual:

$$y_1 \leq y_2 \overset{def}{\iff} y_2 - y_1 \in P.$$

We denote the closure, the topological interior and the convex hull of a set $M \subset Y$ by $\mathrm{cl}M$, $\mathrm{int}M$ and $\mathrm{conv}M$, respectively. The convex cone P is said to be solid if $\mathrm{int}P \neq \emptyset$.

The topological dual space of Y is denoted by Y^* and the (positive) polar cone of P by P^+, i.e., $P^+ := \{\xi \in Y^* : \xi(y) \geq 0, \forall\, y \in P\}$. Moreover, $\mathbb{R}^p_+$ stands for the nonnegative orthant of $\mathbb{R}^p$ and $\mathbb{R}_+ := \mathbb{R}^1_+$.

Whenever P is solid, a preference relation $\preceq$ weaker than $\leq$ can be defined by replacing the ordering cone P with $\mathrm{int}P \cup \{0\}$; i.e.

$$y_1 \preceq y_2 \overset{def}{\iff} y_2 - y_1 \in \mathrm{int}P \cup \{0\}.$$

We also define

$$y_1 \prec y_2 \overset{def}{\iff} y_1 \preceq y_2 \text{ and } y_1 \neq y_2.$$

If P is proper (i.e., $P \neq Y$), then it follows that

$$y_1 \prec y_2 \iff y_2 - y_1 \in \mathrm{int}P.$$

In this paper, we study weak efficient (non-dominated) solutions of the following vector optimization problem:

$$\min\nolimits_P \{f(x) : x \in C\}, \qquad (\tilde{\mathcal{P}})$$

where $f : X \to Y$ is a vector valued function and $C \subset X$ is a nonempty set.

We recall that $\bar{x} \in C$ is said to be a weak efficient solution of problem $(\tilde{\mathcal{P}})$, denoted by $\bar{x} \in \mathrm{WE}(f, C)$, if

$$x \in C \text{ and } f(x) \preceq f(\bar{x}) \implies f(x) = f(\bar{x})$$

or equivalently

$$f(x) \not\prec f(\bar{x}), \quad \forall x \in C.$$

Weak efficient solutions may not be convenient from a practical point of view, since they would be improved; i.e., feasible points with better objective values for the preference relation $\leq$ could exist. However, some vector optimization problems are so difficult to tackle that practitioners look for weak efficient solutions instead of non-dominated solutions. In addition, weak efficient solutions are really important from a theoretical point of view, since they allow us to obtain necessary conditions for non-dominated solutions (see [17, 23, 24]).

In order to deal with weak efficient solutions of constrained and unconstrained vector optimization problems in a unified way, we consider the extended image space $\overline{Y} := Y \cup \{+\infty_P\}$, where $+\infty_P$ refers to a greatest element of Y with respect to the relation $\preceq$. We assume the following operations and relationships, for all $y \in \overline{Y}$ and $\alpha > 0$ (see [19]):

$$\begin{aligned} y + (+\infty_P) &= +\infty_P + y = +\infty_P, \\ \alpha(+\infty_P) &= +\infty_P, \\ y &\preceq +\infty_P. \end{aligned}$$

The relations $\leq$, $\preceq$ and $\prec$ are generalized to $\overline{Y}$ as follows: for all $y, z \in \overline{Y}$,

$$\begin{aligned} y \leq z &\overset{def}{\iff} z \in (y + P) \cup \{+\infty_P\}, \\ y \preceq z &\overset{def}{\iff} z \in (y + \operatorname{int} P) \cup \{y\} \cup \{+\infty_P\}, \\ y \prec z &\overset{def}{\iff} y \neq +\infty_P \text{ and } z \in (y + \operatorname{int} P) \cup \{+\infty_P\}, \end{aligned}$$

where for the last one, we assume that P is proper.

The following family of sets is a base of neighborhoods of $+\infty_P$ (see [19]):

$$\beta_q(+\infty_P) := \{(rq + \operatorname{int} P) \cup \{+\infty_P\} : r > 0\}, \text{ where } q \in \operatorname{int} P.$$

It is important to point out that it does not depend on the choice of q. Indeed, by applying equality (4) below to $q_1, q_2 \in \operatorname{int} P$ and an arbitrary $\alpha < 0$, we see that there exists $t > 0$ such that $-q_1 \in -tq_2 + P$, i.e., $tq_2 \in q_1 + P$. Thus, for each $r > 0$ there exists $s > 0$ such that $(sq_2 + \operatorname{int} P) \cup \{+\infty_P\} \subset (rq_1 + \operatorname{int} P) \cup \{+\infty_P\}$.

Problem $(\tilde{\mathcal{P}})$ equipped with the concept of weak efficient solution can be reformulated as an unconstrained problem by considering the extended image space $\overline{Y}$ and the objective function $f + \delta_C^Y : X \to \overline{Y}$, where $\delta_C^Y : X \to \overline{Y}$ denotes the indicator vector function of C, i.e., $\delta_C^Y(x) = 0$ if $x \in C$ and $\delta_C^Y(x) = +\infty_P$ if $x \notin C$. Indeed, $\mathrm{WE}(f + \delta_C^Y, X) = \mathrm{WE}(f, C)$.

Therefore, in the sequel, we consider the following extended unconstrained vector optimization problem:

$$\min\nolimits_P \{F(x) : x \in X\}, \tag{$\mathcal{P}$}$$

where $F : X \to \overline{Y}$ is an extended vector valued function assumed to be proper, i.e., $\operatorname{dom} F := \{x \in X : F(x) \neq +\infty_P\}$ is nonempty and the convex cone P is solid. In addition, as $\operatorname{int} P = \operatorname{int}(\operatorname{cl} P)$, we can assume without loss of generality that P is closed. Moreover, we can also assume that P is proper, because of $\mathrm{WE}(F, X) = \emptyset$ otherwise. As a result, we have $(-P) \cap \operatorname{int} P = \emptyset$.

We define two sets that are very important for our approach. For $y \in Y$

$$\begin{aligned} \mathrm{lev}_y F &:= \{x \in X \colon F(x) \leq y\}, \\ \mathrm{colev}_y F &:= \{x \in X \colon y \not\prec F(x)\}, \end{aligned}$$

denote the level and colevel sets of F at height $y \in Y$, respectively. Clearly, $x \in \mathrm{lev}_{F(x)} F \cap \mathrm{colev}_{F(x)} F$, for all $x \in \mathrm{dom} F$, and

$$\mathrm{lev}_y F \subset \mathrm{colev}_y F \subset \mathrm{dom} F, \quad \forall y \in Y. \tag{1}$$

Several results of this paper are derived by scalarization through two realizations of the well-known Gerstewitz scalarization function. Let $q \in \operatorname{int} P$ (from now on we consider such a q), the function $\varphi_P^q : Y \to \mathbb{R} \cup \{\pm\infty\}$ is defined in [17] for $y \in Y$ by (we set $\inf \emptyset = +\infty$):

$$\varphi_P^q(y) := \inf \Big\{ t \in \mathbb{R} : y \in tq - P \Big\}.$$

This function is finite-valued, $\leq$-increasing (resp. $\prec$-increasing); i.e., $y_1, y_2 \in Y$ and $y_1 \leq y_2$ (resp. $y_1 \prec y_2$) imply $\varphi_P^q(y_1) \leq \varphi_P^q(y_2)$ (resp. $\varphi_P^q(y_1) < \varphi_P^q(y_2)$); subadditive, positively homogeneous; i.e., $\varphi_P^q(\alpha y) = \alpha \varphi_P^q(y)$, for all $y \in Y$, $\alpha > 0$; $\varphi_P^q(y + \alpha q) = \varphi_P^q(y) + \alpha$ for all $y \in Y$, $\alpha \in \mathbb{R}$; and $\{y \in Y \colon \varphi_P^q(y) < 0\} = -\operatorname{int} P$ (see [17, Corollary 2.3.5]).

By using this scalarization function we define two extended scalarization functions $\varphi^q, \psi^q : \overline{Y} \to \mathbb{R} \cup \{\pm\infty\}$ for $y \in \overline{Y}$ by (we set $\sup \emptyset = -\infty$):

$$\begin{aligned} \varphi^q(y) &:= \begin{cases} \varphi_P^q(y), & \text{if } y \in Y, \\ +\infty, & \text{if } y = +\infty_P, \end{cases} \\ \psi^q(y) &:= \sup \Big\{ t \in \mathbb{R} : y \in (tq + P) \cup \{+\infty_P\} \Big\}. \end{aligned}$$

The first one is useful to study level sets and the second one to study colevel and solution sets. Clearly, $\varphi^q = \varphi_P^q$ and $\psi^q = -\varphi_{-P}^{-q}$ on Y and $\psi^q(+\infty_P) = \varphi_P^q(+\infty_P) = +\infty$.

The following dual reformulations of these functions hold for $y \in Y$ (see [16, Proposition 1.53]):

$$\varphi^q(y) = \sup\{\xi(y) : \xi \in P^+(q)\}, \tag{2}$$
$$\psi^q(y) = \inf\{\xi(y) : \xi \in P^+(q)\}, \tag{3}$$

where $P^+(q) := \{\xi \in P^+ : \xi(q) = 1\}$. This set is a weak*-compact base of P^+ (see [17, Lemma 2.2.17]). Clearly, $\psi^q \leq \varphi^q$. We point out that φ^q has been employed in [14] to study vector optimization problems under convexity assumptions. We will show that ψ^q is suitable for studying these problems without such assumptions.

Next, some basic properties of φ^q and ψ^q are derived.

Proposition 2.1 *(a)* φ^q, $\psi^q > -\infty$ *and* $\operatorname{dom}\varphi^q = \operatorname{dom}\psi^q = Y$.

(b) φ^q, ψ^q *are both positively homogeneous,* $\leq$*-increasing and* $\prec$*-increasing on* $\overline{Y}$*. In addition,* φ^q *is convex.*

(c) $\varphi^q(y + \alpha q) = \varphi^q(y) + \alpha$, $\psi^q(y + \alpha q) = \psi^q(y) + \alpha$, *for all* $y \in \overline{Y}$, $\alpha \in \mathbb{R}$.

(d) $\{y \in \overline{Y} : \varphi^q(y) < \alpha\} = \alpha q - \operatorname{int}P$ *and* $\{y \in \overline{Y} : \psi^q(y) > \alpha\} = (\alpha q + \operatorname{int}P) \cup \{+\infty_P\}$, *for all* $\alpha \in \mathbb{R}$.

Proof *(a)* We prove the assertion only for ψ^q since for φ^q is obvious. For each $\alpha \in \mathbb{R}$ we have

$$Y = \bigcup_{t \leq \alpha} (tq + P). \tag{4}$$

Indeed, for each $y \in Y$ there exists $s > 0$ such that $q + sy \in P$. Thus, $y \in -(1/s)q + P$. If $-1/s \leq \alpha$, then take $t := -1/s$. Otherwise consider $t := \alpha$, since we have

$$y \in -(1/s)q + P = \alpha q + (-1/s - \alpha)q + P \subset \alpha q + P.$$

Therefore, assertion (4) is satisfied and as a result it follows that for each $y \in Y$ there exists $t \in \mathbb{R}$ such that $y \in tq + P$. Thus, $\psi^q > -\infty$ on Y.

Clearly $\psi^q(+\infty_P) = +\infty$. Suppose that $y \in Y$ and $\psi^q(y) = +\infty$. Then there exists a sequence $(t_k) \subset \mathbb{R}_+ \backslash \{0\}$ such that $t_k \to +\infty$ and $y \in t_k q + P$, for all k. Then, $q \in (1/t_k)y - P$ for all k and so $q \in -\operatorname{cl}P = -P$. Therefore, $q \in (-P) \cap \operatorname{int}P$, which is a contradiction.

(b)–*(d)* These properties follow from properties of φ^q_P, the equality $\psi^q = -\varphi^{-q}_{-P}$ on Y and part *(a)*. □

From part *(d)* above, we obtain the following formulas for level and colevel sets at height rq for $r \in \mathbb{R}$:

$$\operatorname{lev}_{rq}F = \{x \in X : (\varphi^q \circ F)(x) \leq r\}, \tag{5}$$

$$\operatorname{colev}_{rq}F = \{x \in X : (\psi^q \circ F)(x) \leq r\}. \tag{6}$$

Lemma 2.1 *For every* $y \in Y$ *there exists* $r > 0$ *such that*

$$\operatorname{lev}_y F \subset \operatorname{lev}_{rq}F \quad \textit{and} \quad \operatorname{colev}_y F \subset \operatorname{colev}_{rq}F.$$

Proof Let $y \in Y$. We apply (4) to $-y$ and $\alpha < 0$ to deduce that there exists $r > 0$ such that $y \in rq - P$. We prove the first inclusion. If $x \in \operatorname{lev}_y F$, then $y \in F(x) + P$; thus, $F(x) \in y - P \subset rq - P$; i.e., $x \in \operatorname{lev}_{rq}F$. We prove the second inclusion. If $x \in \operatorname{colev}_y F$, then $F(x) \notin (y + \operatorname{int}P) \cup \{+\infty_P\}$. As $rq \in y + P$, we have $rq + \operatorname{int}P \subset y + \operatorname{int}P$; thus, $F(x) \notin (rq + \operatorname{int}P) \cup \{+\infty_P\}$; i.e., $x \in \operatorname{colev}_{rq}F$. □

Remark 2.1 As $\operatorname{dom}F \neq \emptyset$ and $x \in \operatorname{lev}_{F(x)}F \cap \operatorname{colev}_{F(x)}F$, for all $x \in \operatorname{dom}F$, by Lemma 2.1 we see that there exists $r > 0$ such that $\operatorname{lev}_{rq}F \neq \emptyset$ and $\operatorname{colev}_{rq}F \neq \emptyset$.

Given a sequence $(x^k) \subset X$, we denote by $x^k \overset{w}{\to} x$ its convergence to x in the weak topology $w = \sigma(X, X^*)$, by $x^k \overset{s}{\to} x$ its convergence to x in the norm topology $s = \sigma(X, \|\cdot\|)$, and by $x^k \overset{\sigma}{\to} x$ its convergence to x in an arbitrary Hausdorff topology σ on X coarser than s and compatible with the linear structure of X. We say that a nonempty set $M \subset X$ is sequentially σ-closed (resp. sequentially σ-compact) iff M is sequentially closed (resp. sequentially compact) by the topology σ.

We shall consider the following condition on the decision space (X, σ).

Assumption ($\mathbf{H}_\sigma$): *The closed unit ball $\mathbb{B}$ of the space $(X, ||\cdot||)$ is sequentially σ-compact.*

This assumption holds for instance when X is a reflexive Banach space and $\sigma = w$ or when X is the dual Z^* of a normed space Z and σ is the weak* topology on X. The latter encompasses cases when X is non reflexive. See [25] for real-world problems for which these instances are satisfied.

We now recall some asymptotic notions that allow us to deal with problems with "unbounded" data. For a nonempty set $C \subset X$, we denote by C^∞_σ its sequentially asymptotic cone (with respect to the topology σ, see [26, 27]), defined as

$$C^\infty_\sigma := \left\{ v \in X : \exists (x^k) \subset C,\, \exists t_k \to +\infty \text{ s.t. } x^k/t_k \overset{\sigma}{\to} v \right\},$$

where $(x^k) \subset X$ and $(t_k) \subset \mathbb{R}_+$ are sequences. We consider that $\emptyset^\infty_\sigma := \{0\}$. We list some properties of this notion, see [26, 27].

Proposition 2.2 (a) *If C is a sequentially σ-closed cone, then $C^\infty_\sigma = C$.*
(b) *If C is convex and sequentially σ-closed, then C^∞_σ is a convex sequentially σ-closed cone and $C^\infty_\sigma = \{v \in X : x^0 + tv \in C, \forall\, t > 0\}$ for any $x^0 \in C$.*
(c) *If C is bounded, then $C^\infty_\sigma = \{0\}$. The reverse implication holds if X is finite dimensional.*
(d) *If $C \subset D$, then $C^\infty_\sigma \subset D^\infty_\sigma$.*
(e) *If I is an arbitrary index set, then*
  (i) $[\bigcap_{i\in I} C_i]^\infty_\sigma \subset \bigcap_{i\in I} [C_i]^\infty_\sigma$*, where the equality holds if each C_i is convex and sequentially σ-closed and their intersection is nonempty.*
  (ii) $\bigcup_{i\in I} [C_i]^\infty_\sigma \subset [\bigcup_{i\in I} C_i]^\infty_\sigma$*, where the equality holds if I is finite.*

For a proper function $g : X \to \overline{\mathbb{R}}$, its sequentially asymptotic function (with respect to the topology σ, see [26, 27]), denoted by $g^\infty_\sigma : X \to \overline{\mathbb{R}}$ is defined for $v \in X$ as

$$g^\infty_\sigma(v) := \inf \left\{ \liminf_k \frac{g(t_k v^k)}{t_k} : t_k \to +\infty, v^k \overset{\sigma}{\to} v \right\}, \tag{7}$$

where $(v^k) \subset X$ and $(t_k) \subset \mathbb{R}_+$ are sequences. We list some properties of this notion from [26, 27] (various follows straightforwardly from Proposition 2.2). As usual, $\operatorname{epi} g$ stands for the epigraph of g and $\tau_{\mathbb{R}}$ denotes the topology of $\mathbb{R}$.

Proposition 2.3 (a) *g^∞_σ is positively homogeneous.*

(*b*) *If g is convex and sequentially σ-lower semicontinuous (σ-lsc for short), then* $[\mathrm{epi}\, g]^{\infty}_{\sigma\times\tau_{\mathbb{R}}} = \mathrm{epi}(g^{\infty}_{\sigma})$ *and for any* $x^0 \in \mathrm{dom} g$ *and* $v \in X$

$$g^{\infty}_{\sigma}(v) = \lim_{t\to+\infty} \frac{g(x^0+tv)-g(x^0)}{t} = \sup_{t>0} \frac{g(x^0+tv)-g(x^0)}{t}.$$

(*c*) *Let g_i be proper convex sequentially σ-lsc and $\alpha_i \geq 0$ for $i \in \{1,\dots,r\}$ such that* $\cap_{i=1}^{r} \mathrm{dom} g_i \neq \emptyset$. *Then* $(\sum_{i=1}^{r} \alpha_i g_i)^{\infty}_{\sigma} = \sum_{i=1}^{r} \alpha_i (g_i)^{\infty}_{\sigma}$.

(*d*) $\{x \in X : g(x) \leq \lambda\}^{\infty}_{\sigma} \subset \{v \in X :\ g^{\infty}_{\sigma}(v) \leq 0\}$ *for all* $\lambda \in \mathbb{R}$, *where the equality holds whenever g is convex and sequentially σ-lsc, and the level set in the left-hand side is nonempty.*

(*e*) *Let g_i be proper for $i \in I$ with I being an arbitrary index set. Then*
  (*i*) $(\sup_{i\in I} g_i)^{\infty}_{\sigma} \geq \sup_{i\in I} (g_i)^{\infty}_{\sigma}$, *where the equality holds whenever g_i are convex and sequentially σ-lsc and* $\sup_{i\in I} g_i \not\equiv +\infty$.
  (*ii*) $(\inf_{i\in I} g_i)^{\infty}_{\sigma} \leq \inf_{i\in I} (g_i)^{\infty}_{\sigma}$, *where the equality holds whenever I is a finite index set.*

(*f*) *If $g \leq h$, then* $g^{\infty}_{\sigma} \leq h^{\infty}_{\sigma}$.

We say that $g : X \to \overline{\mathbb{R}}$ is σ-coercive if $g^{\infty}_{\sigma}(v) > 0$ for all $v \neq 0$. By part (*d*) we have that, if g is σ-coercive, then $\{x \in X : g(x) \leq \lambda\}^{\infty}_{\sigma} = \{0\}$ for all $\lambda \in \mathbb{R}$, which in turn implies that the level sets are bounded if X is finite dimensional. Moreover, if g is convex and sequentially σ-lsc and has at least one bounded nonempty level set, then g is σ-coercive. Clearly, g has bounded level sets iff g is zero-coercive; i.e., $\lim_{||x||\to+\infty} g(x) = +\infty$.

We recall two cones that play an important role for studying problem $(\mathcal{P})$. For $\xi \in P^+(q)$ and $F : X \to \overline{Y}$, we define the function $\xi \circ F : X \to \mathbb{R} \cup \{+\infty\}$ as usual; i.e., $(\xi \circ F)(x) = \xi(F(x))$ for $x \in \mathrm{dom} F$ and $(\xi \circ F)(x) = +\infty$ elsewhere.

$$Q(\mathcal{P}) := \bigcap_{\xi\in P^+(q)} \{v \in X : (\xi \circ F)^{\infty}_{\sigma}(v) \leq 0\}, \tag{8}$$

$$R(\mathcal{P}) := \bigcup_{\xi\in P^+(q)} \{v \in X : (\xi \circ F)^{\infty}_{\sigma}(v) \leq 0\}. \tag{9}$$

Clearly, $Q(\mathcal{P}) \subset R(\mathcal{P})$. In addition, from (2), (3) and Proposition 2.3(*e*) it follows that for all $v \in X$

$$(\psi^q \circ F)^{\infty}_{\sigma}(v) \leq \inf_{\xi\in P^+(q)} (\xi \circ F)^{\infty}_{\sigma}(v) \leq \sup_{\xi\in P^+(q)} (\xi \circ F)^{\infty}_{\sigma}(v) \leq (\varphi^q \circ F)^{\infty}_{\sigma}(v).$$

Therefore,

$$\begin{aligned}\{v \in X : (\varphi^q \circ F)^{\infty}_{\sigma}(v) \leq 0\} &\subset Q(\mathcal{P}) \\ &\subset R(\mathcal{P}) \subset \{v \in X : (\psi^q \circ F)^{\infty}_{\sigma}(v) \leq 0\}.\end{aligned} \tag{10}$$

When $P^+ = \mathrm{cone}(\mathrm{conv}\{\xi_1, \xi_2, \dots, \xi_m\})$ with $\xi_i \in P^+(q)$ for all i (i.e., P is polyhedral and it is generated by the elements of $B = \{\xi_1, \xi_2, \dots, \xi_m\}$), we will consider the next simpler cone:

$$G_B(\mathcal{P}) := \bigcup_{i=1}^{m} \{v \in X : (\xi_i \circ F)^{\infty}_{\sigma}(v) \leq 0\}.$$

Notice that all above properties also hold true by replacing $R(\mathcal{P})$ with $G_B(\mathcal{P})$.

These cones have been used to obtain coercive existence of efficient and weak efficient solutions of problem $(\mathcal{P})$. The first one has been also used to study its stability under convexity assumptions (see [14] and the references therein).

3 Lower semicontinuity notions for vector valued functions

We define lower semicontinuity notions for vector valued functions that are suitable for performing our approach. We characterize and compare them with the existing ones in the literature.

Definition 3.1 A function $F : X \to \overline{Y}$ is said to be

(a) sequentially (q, σ)-lower semicontinuous via colevel sets (sequentially (q, σ)-lsc in short form) for $q \in \operatorname{int} P$ if $\operatorname{colev}_{rq} F$ is sequentially σ-closed for each $r \in \mathbb{R}$.

(b) weak sequentially σ-lower semicontinuous via colevel sets (weak sequentially σ-lsc in short form) if there exists $q \in \operatorname{int} P$ such that F is sequentially (q, σ)-lsc.

(c) sequentially σ-lower semicontinuous via colevel sets (sequentially σ-lsc in short form) if $\operatorname{colev}_y F$ is sequentially σ-closed for each $y \in Y$.

Remark 3.1 (a) Clearly, if F is sequentially σ-lsc, then it is sequentially (q, σ)-lsc, for all $q \in \operatorname{int} P$. Moreover, if F is sequentially (q, σ)-lsc and $g : X \to \mathbb{R}$ is sequentially σ-lsc, then $F + gq$ is sequentially (q, σ)-lsc (see Proposition 2.1(c) and Lemma 3.1(a) below).

(b) If $Y = \mathbb{R}$ and $P = \mathbb{R}_+$, then level and colevel sets coincide and all lower semicontinuity concepts introduced in Definition 3.1 coincide and encompass the sequential notion of lower semicontinuity of a scalar function.

(c) Let $C \subset X$ be a nonempty set and $f : X \to Y$. For each $y \in Y$ one has $\operatorname{colev}_y(f + \delta_C^Y) = (\operatorname{colev}_y f) \cap C$. Therefore, if C is sequentially σ-closed and f is sequentially (q, σ)-lsc (resp. weak sequentially σ-lsc, sequentially σ-lsc), then so is $f + \delta_C^Y$.

(d) The notion of weak sequentially σ-lower semicontinuity depends on the vector q. Indeed, let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $P = \mathbb{R}^2_+$, σ is the usual topology of $\mathbb{R}$, $q_1 = (1, 1)$, $q_2 = (2, 1)$ and $F : X \to Y$ defined as follows:

$$F(t) = \begin{cases} (-1, -1) - t(1, 0), & \text{if } t < 0, \\ (-1, 0) + t(0, 1), & \text{if } t \geq 0. \end{cases}$$

It is clear that

$$\operatorname{colev}_{rq_1} F = \begin{cases} \emptyset, \text{ if } r < -1, \\ \mathbb{R}, \text{ if } r \geq -1, \end{cases} \operatorname{colev}_{rq_2} F = \begin{cases} \emptyset, & \text{if } \quad r < -1, \\ (-\infty, 0)\,, & \text{if } r \in [-1, -\frac{1}{2}), \\ \mathbb{R}, & \text{if } \quad r \geq -1/2. \end{cases}$$

Therefore, F is sequentially (q_1, σ)-lsc, but it is not sequentially (q_2, σ)-lsc.

(e) Sequentially (q, σ)-lower semicontinuity is weaker than sequentially σ-lower semicontinuity as shown in (d).

There are several lower semicontinuity concepts for vector valued functions in the literature (see [22,24,28]). Between them, the most general is the following one due to Penot and Théra (see [28]) when Y is a real topological vector space.

Definition 3.2 A function $F : X \to \overline{Y}$ is said to be σ-lsc at $x \in X$ in the sense of Penot-Théra if for any $b \in Y$ with $F(x) \in (b + P) \cup \{+\infty_P\}$ and any neighborhood V of b, there exists a neighborhood U of x in σ such that $F(U) \subset (V + P) \cup \{+\infty_P\}$.

A sequential version of this notion was introduced later by Combari, Laghdir and Thibault (see [22]).

Definition 3.3 A function $F : X \to \overline{Y}$ is said to be sequentially σ-lsc at $x \in X$ in the sense of Combari-Laghdir-Thibault if for any $b \in Y$ with $F(x) \in (b + P) \cup \{+\infty_P\}$ and any $(x^k) \subset X$ with $x^k \xrightarrow{\sigma} x$, there exists $(b^k) \subset Y$ with $b^k \to b$ such that $F(x^k) \in (b^k + P) \cup \{+\infty_P\}$ for all k.

If $x \in \mathrm{dom}F$, then in Definition 3.3 one can consider $b = F(x)$ (see [22, Proposition 3.3]). On the other hand, if $\mathrm{int}P \neq \emptyset$, then F is sequentially σ-lsc at $x \notin \mathrm{dom}F$ in the sense of Combari-Laghdir-Thibault iff for any $(x^k) \subset X$ with $x^k \xrightarrow{\sigma} x$ one has $F(x^k) \to +\infty_P$.

In addition, it was proved in [22, Proposition 3.6] that Definitions 3.2 and 3.3 are equivalent whenever (X, σ) and Y are metrizable spaces. Moreover, F is σ-lsc at $x \notin \mathrm{dom}F$ in the Penot-Théra sense iff $\lim_{z \to x} F(z) = +\infty_P$.

The next result shows that sequential σ-lower semicontinuity is a really simple reformulation of σ-lower semicontinuity in the sense of Combari-Laghdir-Thibault. It extends [29, Lemma 2.3] (see also [9, Proposition 2.1(b)]), where it was stated that a function $F : K \to Y$ from a nonempty set $K \subset X$ is σ-lsc in the sense of Penot-Théra at every $x \in K$ iff $\mathrm{colev}_y F \cap K$ is closed for every $y \in Y$. In addition, it encompasses [14, Lemma 2.1(d)], where the necessary condition was derived for F mapping X into Y instead of $\overline{Y}$.

Theorem 3.1 *Let Y be a real topological linear space such that* $\mathrm{int}P \neq \emptyset$*. Then, F is sequentially σ-lsc in the sense of Combari-Laghdir-Thibault iff F is sequentially σ-lsc.*

Proof ($\Rightarrow$) On the contrary, suppose that there exist $y \in Y$ and $(x^k) \subset \mathrm{colev}_y F$ such that $x^k \xrightarrow{\sigma} x$ and $x \notin \mathrm{colev}_y F$. If $x \notin \mathrm{dom}F$, then $F(x^k) \to +\infty_P$ and since the set $(y + \mathrm{int}P) \cup \{+\infty_P\}$ is a neighborhood of $+\infty_P$ there exists k_0 such that $F(x^k) \in (y + \mathrm{int}P) \cup \{+\infty_P\}$, for all $k \geq k_0$; thus, $x^k \notin \mathrm{colev}_y F$, for such k, a contradiction. On the other hand, if $x \in \mathrm{dom}F$, then there exists $(b^k) \subset Y$ such that $b^k \to F(x)$ and $F(x^k) \in (b^k + P) \cup \{+\infty_P\}$, for all k. As $F(x) \in y + \mathrm{int}P$ and $b^k \to F(x)$, there exists k_0 such that $b^k \in y + \mathrm{int}P$ for all $k \geq k_0$; thus, $F(x^k) \in (y + \mathrm{int}P) \cup \{+\infty_P\}$, for such k, a contradiction. The result follows.

($\Leftarrow$) Let $(x^k) \subset X$ be such that $x^k \xrightarrow{\sigma} x$. Suppose that $x \in \mathrm{dom}F$. If there exists a subsequence $(x^{k_m}) \subset \mathrm{colev}_{F(x)-\varepsilon q} F$ where $\varepsilon > 0$ and $q \in \mathrm{int}P$, as F is sequentially σ-lsc we have $x \in \mathrm{colev}_{F(x)-\varepsilon q} F$, a contradiction. Then, for each

n there exists a $k_n > \max\{n, k_{n-1}\}$, $k_0 = 1$ such that $F(x^k) \succ F(x) - (1/n)q$, for all $k \geq k_n$. By defining $b^k = F(x^k)$, for all $k = 1, 2, \ldots, k_1 - 1$, and $b^k := F(x) - (1/n)q$, for all $n \geq 1$ and $k = k_n, k_n + 1, \ldots, k_{n+1} - 1$, we have $F(x^k) \in (b^k + P) \cup \{+\infty_P\}$ for all k and $b^k \to F(x)$.

On the other hand, suppose that $F(x) = +\infty_P$. If for every $r > 0$ there exists n such that $F(x^k) \in (rq + \mathrm{int}P) \cup \{+\infty_P\}$, for all $k \geq n$, then as $\{(rq + \mathrm{int}P) \cup \{+\infty_P\} : r > 0\}$ is a base of neighborhoods of $+\infty_P$, we deduce that $F(x^k) \to +\infty_P$, and the proof finishes. Otherwise, there exists $r > 0$ and a subsequence (x^{k_n}) in $\mathrm{colev}_{rq} F$. From this and since F is sequentially σ-lsc, we have $x \in \mathrm{colev}_{rq} F$, which is a contradiction. □

The lower semicontinuity notions introduced in Definition 3.1 can be characterized by the scalarization function ψ^q. To do this, for $y \in Y$ we define the function $F - y : X \to \overline{Y}$ by $(F - y)(x) := F(x) - y$, for all $x \in X$.

Lemma 3.1 (a) *F is sequentially (q, σ)-lsc iff $\psi^q \circ F$ is sequentially σ-lsc.*
(b) *F is weak sequentially σ-lsc iff there exists $q \in \mathrm{int}P$ such that $\psi^q \circ F$ is sequentially σ-lsc.*
(c) *F is sequentially σ-lsc iff $\psi^q \circ (F - y)$ is sequentially σ-lsc, for all $y \in Y$.*

Proof (a) Formula (6) implies that F is sequentially (q, σ)-lsc iff the set $\{x \in X : (\psi^q \circ F)(x) \leq r\}$ is sequentially σ-closed for all $r \in \mathbb{R}$, i.e., $\psi^q \circ F$ is sequentially σ-lsc.

(b) This part is an obvious consequence of part (a).

(c) By definition, F is sequentially σ-lsc iff $\mathrm{colev}_y F$ is sequentially σ-closed for all $y \in Y$ iff $\mathrm{colev}_{rq}(F - y)$ is sequentially σ-closed for all $y \in Y$ and $r \in \mathbb{R}$. The result follows by part (a) applied to function $F - y$. □

4 Boundedness and coercivity notions

We introduce a boundedness notion for sets w.r.t. the preference relation.

Definition 4.1 A nonempty set $A \subset Y$ is said to be bounded from above if there exists $y \in Y$ such that $a \not\succ y$, for all $a \in A$.

The next result provides topological and geometrical interpretations of this notion. In particular, it is shown that it defines the class of $-P$-proper sets. Therefore, it is a really weak upper boundedness notion (see [30]).

Proposition 4.1 (a) *A is bounded from above iff $+\infty_P \notin \mathrm{cl}A$.*
(b) *A is bounded from above iff $A - P \neq Y$, i.e., A is $-P$-proper.*
(c) *If there exists $\xi \in P^+ \backslash \{0\}$ such that $\sup_{y \in A} \xi(y) < +\infty$, then A is bounded from above. The reverse implication holds if $A - P$ is convex.*
(d) *If there exists $y \in Y$ such that $a \preceq y$ for all $a \in A$, then A is bounded from above.*

Proof (a) The sufficient condition is easy to check. Reciprocally, suppose that there exists $y \in Y$ such that $a \not> y$, for all $a \in A$. By (4), there exists $r > 0$ such that $a \not> rq$, for all $a \in A$ and then $+\infty_P \notin \mathrm{cl}A$.

(b) Notice that for each set $A \subset Y$ we have $A - P = Y$ iff $A - \mathrm{int}P = Y$. Clearly, $A - \mathrm{int}P = Y$ implies $A - P = Y$. Reciprocally, as $P + \mathrm{int}P = \mathrm{int}P$, if $A - P = Y$ then $Y = Y - \mathrm{int}P = A - P - \mathrm{int}P = A - \mathrm{int}P$. Then, A is $-P$-proper iff there exists $y \in Y$ such that $y \notin A - \mathrm{int}P$ or equivalently $a \not> y$, for all $a \in A$; i.e., A is bounded from above.

(c) Suppose that there exist $\xi \in P^+\backslash\{0\}$ and $m \in \mathbb{R}$ such that $\xi(a) \leq m$, for all $a \in A$. Let $q \in \mathrm{int}P$ be fixed. As $\xi(q) > 0$, there exists $t > 0$ such that $\xi(tq) \geq m$. Then, $\xi(a - tq) \leq 0$, for all $a \in A$, and as $\xi \in P^+\backslash\{0\}$ it follows that $a - tq \notin \mathrm{int}P \cup \{+\infty_Y\}$, for all $a \in A$. Thus, $a \not> tq$, for all $a \in A$; i.e., A is bounded from above.

Reciprocally, assume that $A - P$ is convex and consider there exists $y \in Y$ such that $a \not> y$, for all $a \in A$. Then, $(A - y) \cap \mathrm{int}P = \emptyset$, that is equivalent to $(A - P - y) \cap \mathrm{int}P = \emptyset$ since $P + \mathrm{int}P = \mathrm{int}P$. By applying a separation result (see [23, Theorem 3.16]), we deduce that there exists $\xi \in Y^*\backslash\{0\}$ such that $\xi(a - d_1 - y) \leq \xi(d_2)$, for all $a \in A$ and $d_1, d_2 \in P$. From this, we obtain that $\xi \in P^+\backslash\{0\}$ and $\sup_{a\in A} \xi(a) \leq \xi(y)$.

(d) This assertion follows as an obvious consequence of part (c). □

The next condition extends a condition given in [15, Theorem 2.1]. This "compactness" condition allows us to obtain coercivity properties and a noncoercive existence result for problem $(\mathcal{P})$.

Assumption (A): *For each sequence $(x^k) \subset X$ such that $\|x^k\| \to +\infty$, $x^k/\|x^k\| \stackrel{\sigma}{\to} v$, and $(F(x^k))$ is bounded from above, we have*
(i) $x^k/\|x^k\| \stackrel{s}{\to} v$.
(ii) There exists a sequence $(r_k) \subset \mathbb{R}$ such that $r_k \in (0, \|x^k\|)$ for all k and $F(x^k) \in (F(x^k - r_k v) + P) \cup \{+\infty_P\}$ eventually.

This condition is related to the sequential asymptotic function of the scalarization function $\psi^q \circ F$.

Proposition 4.2 *If $(x^k) \subset X$ is such that $\|x^k\| \to +\infty$, $x^k/\|x^k\| \stackrel{\sigma}{\to} v$ and $(F(x^k))$ is bounded from above, then $(\psi^q \circ F)^\infty_\sigma(v) \leq 0$, for all $q \in \mathrm{int}P$.*

Proof Consider $q \in \mathrm{int}P$ and $(x^k) \subset X$ such that $\|x^k\| \to +\infty$, $x^k/\|x^k\| \stackrel{\sigma}{\to} v$ and $(F(x^k))$ is bounded from above. By Proposition 4.1(a) there exists $t \in \mathbb{R}$ such that $F(x^k) \notin (tq + \mathrm{int}P) \cup \{+\infty_P\}$, for all k. By Proposition 2.1(d) we have $(\psi^q \circ F)(x^k) \leq t$, for all k, and by (7) with $t_k := \|x^k\|$ and $v^k := x^k/\|x^k\|$ we obtain $(\psi^q \circ F)^\infty_\sigma(v) \leq \liminf_k (\psi^q \circ F)(t_k v^k)/t_k \leq \lim_k t/t_k = 0$. □

The previous result asserts that assumption (A) holds vacuously if $\psi^q \circ F$ is σ-coercive.

We now proceed to study the boundedness of colevel and level sets. To do this, we obtain the following result as a consequence of (1), inequality

$$(\psi^q \circ F)(x) \leq (\xi \circ F)(x) \leq (\varphi^q \circ F)(x), \;\; \forall x \in X, \; \forall \xi \in P^+(q),$$

and Proposition 2.3(f).

Lemma 4.1 *Let $\xi \in P^+(q)$ be arbitrary. The following assertions hold true.*

(a) *F has bounded colevel sets $\Rightarrow$ F has bounded level sets.*
(b) *$\psi^q \circ F$ is zero-coercive $\Rightarrow$ $\xi \circ F$ is zero-coercive $\Rightarrow$ $\varphi^q \circ F$ is zero-coercive.*
(c) *$\psi^q \circ F$ is σ-coercive $\Rightarrow$ $\xi \circ F$ is σ-coercive $\Rightarrow$ $\varphi^q \circ F$ is σ-coercive.*

The next result relates the sequentially asymptotic functions of linear scalarizations with the corresponding ones of φ^q and ψ^q. We recall that a function $F : X \to \overline{Y}$ is said to be P-convex if $F(\lambda x + (1-\lambda)y) \leq \lambda F(x) + (1-\lambda)F(y)$ for all $x, y \in X$ and $\lambda \in (0,1)$. For each $\xi \in P^+\backslash\{0\}$, we have that $\xi \circ F$ is convex (resp. sequentially σ-lsc) provided that F is P-convex (resp. sequentially σ-lsc, see [14, Lemma 2.1(a),(e)]). In addition, $\varphi^q \circ F$ is convex whenever F is P-convex, since φ^q is both convex and increasing (see Proposition 2.1(a)). Moreover, if $Y = \mathbb{R}^p$ and $P = \mathbb{R}^p_+$, then $F = (F_1, F_2, \ldots, F_p)$ is sequentially σ-lsc iff each component F_i is sequentially σ-lsc.

Proposition 4.3 (a) *If F is P-convex and sequentially σ-lsc, then*

$$(\varphi^q \circ F)^\infty_\sigma(v) = \sup_{\xi \in P^+(q)} (\xi \circ F)^\infty_\sigma(v), \ \ \forall v \in X,$$
$$Q(\mathcal{P}) = \{v \in X \colon (\varphi^q \circ F)^\infty_\sigma(v) \leq 0\}.$$

(b) *If P is a polyhedral cone, then*

$$(\psi^q \circ F)^\infty_\sigma(v) = \min_{1 \leq i \leq m} (\xi_i \circ F)^\infty_\sigma(v), \ \ \forall v \in X,$$
$$G_B(\mathcal{P}) = \{v \in X \colon (\psi^q \circ F)^\infty_\sigma(v) \leq 0\}.$$

Proof If P is a polyhedral cone generated by $B = \{\xi_1, \xi_2, \ldots, \xi_m\} \subset P^+(q)$, then we have

$$\inf_{\xi \in P^+(q)} \xi \circ F = \min_{1 \leq i \leq m} \xi_i \circ F.$$

Then, parts (a)–(b) follow from formulas (2)–(3) and Proposition 2.3(e). □

Next, we obtain bounds for the sequential asymptotic cones of colevel and level sets. By Lemma 2.1, without loss of generality, we can only consider heights $y = rq$ with $r \in \mathbb{R}$.

Proposition 4.4 *Let $F : X \to \overline{Y}$, $r \in \mathbb{R}$ and $y \in Y$.*

(a) $(\mathrm{lev}_{rq} F)^\infty_\sigma \subset \{v \in X : (\varphi^q \circ F)^\infty_\sigma(v) \leq 0\} \subset Q(\mathcal{P})$, *where both inclusions become equalities provided F is P-convex and sequentially σ-lsc and in addition* $\mathrm{lev}_{rq} F \neq \emptyset$ *for the first one.*
(b) $(\mathrm{colev}_{rq} F)^\infty_\sigma \subset \{v \in X : (\psi^q \circ F)^\infty_\sigma(v) \leq 0\}$. *If P is a polyhedral cone, then* $(\mathrm{colev}_{rq} F)^\infty_\sigma \subset G_B(\mathcal{P}) = \{v \in X : (\psi^q \circ F)^\infty_\sigma(v) \leq 0\}$.
(c) *If* $\mathrm{lev}_{rq} F$ *(resp.* $\mathrm{colev}_y F$*) is bounded, then it holds that* $(\mathrm{lev}_{rq} F)^\infty_\sigma = \{0\}$ *(resp.* $(\mathrm{colev}_y F)^\infty_\sigma = \{0\}$*). The reverse implications hold if assumptions* (H_σ) *and* (A)-(i) *hold.*

(d) *If $\varphi^q \circ F$ (resp. $\psi^q \circ F$) is σ-coercive and assumptions* (H_σ) *and* (A)-(i) *hold, then F has bounded level (resp. colevel) sets.*

Proof (a) By (5), Proposition 2.3(d) and (10), we have

$$(\mathrm{lev}_{rq} F)^\infty_\sigma = \{x \in X : (\varphi^q \circ F)(x) \le r\}^\infty_\sigma \subset \{v \in X : (\varphi^q \circ F)^\infty_\sigma(v) \le 0\} \subset Q(\mathcal{P}),$$

where the equalities hold under the additional hypothesis by Proposition 4.3(a) and Proposition 2.3(d) again.

(b) By (6) and Proposition 2.3(d), we have

$$(\mathrm{colev}_{rq} F)^\infty_\sigma = \{x \in X : (\psi^q \circ F)(x) \le r\}^\infty_\sigma \subset \{v \in X : (\psi^q \circ F)^\infty_\sigma(v) \le 0\}$$

and the equality follows by Proposition 4.3(b).

(c) The direct implication follows from Proposition 2.2(c). We prove the reverse implication for colevel sets. Suppose on the contrary that $(\mathrm{colev}_y F)^\infty_\sigma = \{0\}$ and $\mathrm{colev}_y F$ is unbounded. There exists a sequence $(x^k) \subset \mathrm{colev}_y F$ such that $||x^k|| \to +\infty$. By assumption (H_σ), we have $x^k/||x^k|| \xrightarrow{\sigma} v$ for some v, up to subsequences. Clearly, $(F(x^k))$ is bounded from above. This and assumption (A)-(i) imply $v \neq 0$, a contradiction since $v \in (\mathrm{colev}_y F)^\infty_\sigma$. The proof of the reverse implication for level sets runs as before but by using Theorem 4.1(d).

(d) This follows from above and Lemma 2.1. □

Proposition 4.4 completes and clarifies the assertions of [14, Lemmas 2.2 and 3.2] involving the asymptotic cone of level sets.

We now introduce a coercivity notion for vector valued functions.

Definition 4.2 A function $F : X \to \overline{Y}$ is said to be zero-coercive, if for each $y \in Y$ there exists $M > 0$ such that $F(x) \succ y$, for all $x \in X$ such that $\|x\| > M$.

We study the boundedness of colevel sets by using the coercivity notion and function $\psi^q \circ F$.

Theorem 4.1 *Consider the following assertions:*

(a) *F is zero-coercive;*
(b) $\displaystyle\lim_{\|x\|\to+\infty} F(x) = +\infty_P$;
(c) *F has bounded colevel sets;*
(d) *$\psi^q \circ F$ is zero-coercive;*
(e) *$\xi \circ F$ is zero-coercive for all $\xi \in P^+(q)$;*
(f) $R(\mathcal{P}) = \{0\}$.

The following implications hold:

(i) (a) $\Longleftrightarrow$ (b) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (d) $\Longrightarrow$ (e).
(ii) *If F is P-convex and sequentially σ-lsc, then* (e) $\Longrightarrow$ (f).
(iii) *If P is a polyhedral cone, then parts (i) and (ii) above also holds true by replacing assertions (e) and (f) with the following ones:*
(e') *$\xi_i \circ F$ is zero-coercive for all $i = 1, 2, \ldots, m$;*
(f') $G_B(\mathcal{P}) = \{0\}$.
In addition, if assumptions (H_σ) *and* (A)-(i) *hold, then* (f') $\Longrightarrow$ (c).

Proof (i): Implication $(a) \Rightarrow (b)$ is obvious.

$(b) \Rightarrow (c)$ Suppose on the contrary that $\mathrm{colev}_{rq}F$ is unbounded for some $r \in \mathbb{R}$ (see Lemma 2.1). Then there exists $(x^k) \subset X$ such that $\|x^k\| \to +\infty$ and $F(x^k) \notin (rq + \mathrm{int}P) \cup \{+\infty_P\}$, for all k, that is contrary to (b).

$(c) \Rightarrow (d)$ This follows from formula (6).

$(d) \Rightarrow (a)$ Let $y \in Y$ be fixed. We apply (4) to $-y$ and $\alpha < 0$ to deduce that there exists $r > 0$ such that $rq \in y + P$. By hypothesis there exists $M > 0$ such that $(\psi^q \circ F)(x) > r$, for all $x \in X$ such that $\|x\| > M$. By Proposition 2.1(d), we deduce that $F(x) \in (rq + \mathrm{int}P) \cup \{+\infty_Y\}$; thus, $F(x) \in (y + \mathrm{int}P) \cup \{+\infty_P\}$, for such x and so F is zero-coercive.

$(d) \Rightarrow (e)$ See Lemma 4.1(b).

(ii): Let $\xi \in P^+(q)$ be fixed. By hypothesis $\xi \circ F$ is convex and sequentially σ-lsc. From this, the level boundedness and [31, Proposition 1(iv)], we conclude that $\xi \circ F$ is σ-coercive. As ξ was arbitrary, we have $R(\mathcal{P}) = \{0\}$.

(iii): The first part is clear. For the second one, by Proposition 4.4(b) and Lemma 2.1, we have $(\mathrm{colev}_y F)^\infty_\sigma = \{0\}$, which in turn by Proposition 4.4(c) implies that $\mathrm{colev}_y F$ is bounded. □

Remark 4.1 (a) Let $Y = \mathbb{R}^p$, $P = \mathbb{R}^p_+$ and $F = (F_1, F_2, \ldots, F_p) : X \to \overline{\mathbb{R}^p}$ be such that each component F_i is convex and sequentially σ-lsc. By Theorem 4.1, we see that F is zero-coercive iff each component F_i is zero-coercive provided that assumptions (H_σ) and (A)-(i) are fulfilled.

(b) The usefulness of the coercivity notion introduced in Definition 4.2 strongly relies on the solidness of the cone P. For instance, it cannot be reformulated by considering the relative interior instead of the interior because it would be superfluous. Indeed, for $Y = \mathbb{R}^2$, $P = \mathbb{R}_+\{(1,0)\}$, F finite-valued and $y_1, y_2 \in Y$ such that $y_1 - y_2 \notin P \cup (-P)$ there are no vectors $x \in X$ such that $F(x) \in (y_i + \mathrm{ri}\,P) \cup \{+\infty_P\}$ for $i = 1, 2$, where $\mathrm{ri}\,P$ stands for the relative interior of P.

We study the boundedness of level sets by using function $\varphi^q \circ F$. The next result completes [14, Proposition 3.2].

Theorem 4.2 *Consider the following assertions:*

(a) *F has bounded level sets;*
(b) *$\varphi^q \circ F$ is zero-coercive;*
(c) *$\varphi^q \circ F$ is σ-coercive;*
(d) *$Q(\mathcal{P}) = \{0\}$.*

The following implications hold:

(i) $(a) \Longleftrightarrow (b)$, $(c) \Longleftarrow (d)$.
(ii) *If F is P-convex and sequentially σ-lsc, then* $(b) \Longrightarrow (c) \Longrightarrow (d)$.
(iii) *If assumptions* (H_σ) *and* (A)-(i) *hold, then* $(d) \Longrightarrow (a)$.

Proof (i): Implication $(a) \Rightarrow (b)$ follows from (5). Implication $(d) \Rightarrow (c)$ follows from Proposition 4.4(a).

$(b) \Rightarrow (a)$ By (5) we conclude that $\text{lev}_{rq} F$ is bounded for every $r \in \mathbb{R}$. The result follows from this and Lemma 2.1.

(ii): Clearly, $\varphi^q \circ F$ is convex and sequentially σ-lsc. Implication $(b) \Rightarrow (c)$ follows from this, level boundedness and [31, Proposition 1(iv)]. Implication $(c) \Rightarrow (d)$ follows from Proposition 4.4(a).

(iii): See Proposition 4.4(d) and part (i). □

We recall a well-known sufficient condition for weak efficient solutions. We give the proof for reader's convenience.

Lemma 4.2 $\bigcup_{q \in \text{int} P} \arg\min_X (\psi^q \circ F) \subset \text{WE}(F, X)$.

Proof On the contrary, suppose that there exist $q \in \text{int} P$ and $\bar{x} \in X$ such that $\bar{x} \in \arg\min_X(\psi^q \circ F)$ and $\bar{x} \notin \text{WE}(F, X)$. Hence there exists $x \in X$ such that $F(x) \prec F(\bar{x})$. Since ψ^q is $\prec$-increasing by Proposition 2.1(b), we have $\psi^q(F(x)) < \psi^q(F(\bar{x}))$, a contradiction. Therefore, $\bar{x} \in \text{WE}(F, X)$. □

Next, we recall another sufficient condition via linear scalarizations and a formula characterizing the weak efficient solutions by means of colevel sets.

Proposition 4.5 $\bigcup_{\xi \in P^+(q)} \arg\min_X (\xi \circ F) \subset \text{WE}(F, X) = \bigcap_{x \in X} \text{colev}_{F(x)} F$.
The set $\text{WE}(F, X)$ *is sequentially* σ*-closed whenever* F *is sequentially* σ*-lsc.*

Proof See [14, Proposition 4.1(b), Remark 4.1]. □

We obtain a bound for the sequential asymptotic cone of the set of weak efficient solutions. It sharpens [14, Proposition 4.4, Corollary 4.1].

Corollary 4.1 (a) $\text{WE}(F, X)^\infty_\sigma \subset \{v \in X : (\psi^q \circ F)^\infty_\sigma(v) \le 0\}$.
(b) *If* F *is* P*-convex and sequentially* σ*-lsc, then for each* $\xi \in P^+ \backslash \{0\}$ *such that* $\arg\min_X(\xi \circ F) \neq \emptyset$ *we have* $\{v \in X : (\xi \circ F)^\infty_\sigma(v) \le 0\} \subset \text{WE}(F, X)^\infty_\sigma$.
(c) *If* $\text{WE}(F, X)$ *is bounded, then* $\text{WE}(F, X)^\infty_\sigma = \{0\}$. *The reverse implication holds if assumptions* (H_σ) *and* (A)-(i) *hold.*
(d) *If* P *is a polyhedral cone, then* $\text{WE}(F, X)^\infty_\sigma \subset G_B(\mathcal{P}) = \{v \in X : (\psi^q \circ F)^\infty_\sigma(v) \le 0\}$.

Proof (a) The bound follows by taking the asymptotic cone to both sides of equality in Proposition 4.5 and applying Propositions 2.2(e) and 4.4(b).

(b) The inclusion follows by taking the asymptotic cone to both sides of the inclusion in Proposition 4.5 and applying Propositions 2.2(e) and 2.3(d) since by hypothesis $\xi \circ F$ is convex and sequentially σ-lsc.

(c) The proof runs similarly as in Proposition 4.4(c).

(d) By Proposition 4.5, we have $\text{WE}(F, X) \subset \text{colev}_{F(x)} F$ for $x \in X$. From this, Lemma 2.1, and Propositions 2.2(d) and 4.4(b), we obtain the bound. □

5 Coercive and noncoercive existence results

We state a Weierstrass theorem for weak efficient solutions of problem ($\mathcal{P}$).

Theorem 5.1 *If F is weak sequentially σ-lsc and* $\text{colev}_{F(x^0)}F$ *is sequentially σ-compact for some $x^0 \in X$, then* $\text{WE}(F,X)$ *is nonempty. If in addition F is sequentially σ-lsc, then* $\text{WE}(F,X)$ *is sequentially σ-compact.*

Proof Let $A := \text{colev}_{F(x^0)}F$ and $q \in \text{int}P$ be such that the scalar function $g := (\psi^q \circ F) : X \to \mathbb{R} \cup \{+\infty\}$ is sequentially σ-lsc (see Lemma 3.1(b)). If $(x^k) \subset A$ is a minimizing sequence of g, i.e., $\lim_k g(x^k) = \inf_{x\in A} g(x) \in \mathbb{R} \cup \{-\infty\}$, then by sequential σ-compactness, we have $x^k \overset{\sigma}{\to} \bar{x}$ up to subsequences for some $\bar{x} \in A$. From this and hypothesis, we have $\bar{x} \in \arg\min_{x\in A} g(x)$. This and Lemma 4.2 imply $\bar{x} \in \text{WE}(F,A)$. We assert that $\bar{x} \in \text{WE}(F,X)$. On the contrary, there exists $x \in X\backslash A$ such that $F(x) \prec F(\bar{x})$. As $F(x^0) \prec F(x)$, we obtain $F(x^0) \prec F(\bar{x})$, a contradiction since $x^0 \in A$.
It is obvious that $\text{WE}(F,X) \subset \text{colev}_{F(x^0)}F$. Thus, by Proposition 4.5 we deduce that $\text{WE}(F,X)$ is sequentially σ-compact provided that F is sequentially σ-lsc. □

Remark 5.1 Theorem 5.1 complements several Weierstrass theorems from the literature. For instance, it generalizes [14, Proposition 4.2] as long as condition (a) of [14, Lemma 4.1] is assumed. It also generalizes [14, Proposition 4.2(c)], where the sequentially σ-compactness is replaced with a boundedness condition and hypothesis (H_σ). Notice that both results require F to be sequentially σ-lower semicontinuous.
In [9, Theorem 3.2, Corollary 3.1], the following lower semicontinuity hypothesis is considered: the set $G(z) := \{x \in X : F(x) - F(z) \notin \text{int}P\}$ is closed for all $z \in X$. Notice that the function F in Remark 3.1(d) satisfies that $G(z)$ is closed for all $z \in X$. Moreover, F is sequentially (q_1,σ)-lsc, but it is not sequentially (q_2,σ)-lsc.

We now establish an existence result of weak efficient solutions of problem ($\mathcal{P}$) without the compactness assumption. To do this, we assume a coercivity condition. To prove it, we employ the direct method of calculus of variations.

Theorem 5.2 *If assumption* (H_σ) *holds, F is zero-coercive and weak sequentially σ-lsc, then* $\text{WE}(F,X)$ *is nonempty and bounded.*

Proof Let $q \in \text{int}P$ be such that the function $g := (\psi^q \circ F) : X \to \mathbb{R} \cup \{+\infty\}$ is sequentially σ-lsc (see Lemma 3.1). By Theorem 4.1, we see that g is also zero-coercive. Let $(x^k) \subset X$ be a minimizing sequence of g, i.e., $g(x^{k+1}) \le g(x^k)$, for all k and $\lim_k g(x^k) = \inf_{x\in X} g(x) \in \mathbb{R} \cup \{-\infty\}$. As g is zero-coercive and sequentially σ-lsc, the level set $\{x \in X : g(x) \le g(x^1)\}$ is bounded and sequentially σ-closed. In particular, it is contained in $r\mathbb{B}$ for some $r > 0$ and thus it is sequentially σ-compact by assumption (H_σ). As (x^k) is contained in the level set, we have $x^k \overset{\sigma}{\to} \bar{x}$, up to subsequences, for some $\bar{x} \in X$. As $(g(x^k))$ is nonincreasing, we have $g(\bar{x}) \le g(x^k)$, for all k; thus, $\inf_{x\in X} g(x) = g(\bar{x})$

and $\bar{x} \in \operatorname{argmin}_X g$. This and Lemma 4.2 imply that $\bar{x} \in \mathrm{WE}(F, X)$. The boundedness of the solution set follows from Proposition 4.5 since the colevel sets are bounded by Theorem 4.1(i). □

Finally, we obtain a noncoercive existence result that extends [15, Theorem 2.1] given for scalar optimization problems. To this end, let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function such that $\rho \circ || \cdot || : X \to \mathbb{R}_+$ is sequentially σ-lsc. In the sequel, we consider the (q, ε, ρ)-regularization of $F : X \to \overline{Y}$ for $q \in \operatorname{int} P$ and $\varepsilon > 0$, denoted by $F^\rho_{q,\varepsilon} : X \to \overline{Y}$ and defined by

$$F^\rho_{q,\varepsilon}(x) := F(x) + \varepsilon\rho(\|x\|)q, \ \ \forall x \in X.$$

Theorem 5.3 *If assumption* (H_σ) *holds,* F *is sequentially* σ*-lsc and satisfies the following properties:*

(a) *Assumption* (A) *is fulfilled.*
(b) $F^\rho_{q,\varepsilon}$ *is zero-coercive, for all* $\varepsilon > 0$.

Then, $\mathrm{WE}(F, X)$ *is nonempty and sequentially* σ*-closed (possibly unbounded).*

Proof By Remark 3.1(a), $F^\rho_{q,\varepsilon}$ is sequentially (q, σ)-lsc, for all $\varepsilon > 0$. Let (ε_k) be a decreasing sequence of positive real numbers such that $\varepsilon_k \to 0$. By Theorem 5.2, there exists $x^k \in \mathrm{WE}(F^\rho_{q,\varepsilon_k}, X)$, for all k. In particular, $x^k \in \operatorname{dom} F^\rho_{q,\varepsilon_k} = \operatorname{dom} F$, for all k.

We now prove that $(F(x^k))$ is bounded from above. For $x \in \operatorname{dom} F$, we have $F^\rho_{q,\varepsilon_k}(x^k) \notin (F^\rho_{q,\varepsilon_k}(x) + \operatorname{int} P) \cup \{+\infty_P\}$; i.e., $F(x^k) \notin (F^\rho_{q,\varepsilon_k}(x) + \operatorname{int} P) \cup \{+\infty_P\}$ for all k. As (ε_k) is decreasing, we have

$$F(x^k) \notin (F^\rho_{q,\varepsilon_1}(x) + \operatorname{int} P) \cup \{+\infty_P\}, \ \ \forall k. \tag{11}$$

Indeed, otherwise it follows that there exists k such that

$$\begin{aligned} F(x^k) &\in (F^\rho_{q,\varepsilon_1}(x) + \operatorname{int} P) \cup \{+\infty_P\} \\ &= (F(x) + \varepsilon_k\rho(\|x\|)q + (\varepsilon_1 - \varepsilon_k)\rho(\|x\|)q + \operatorname{int} P) \cup \{+\infty_P\} \\ &\subset (F(x) + \varepsilon_k\rho(\|x\|)q + \operatorname{int} P) \cup \{+\infty_P\} \\ &= (F^\rho_{q,\varepsilon_k}(x) + \operatorname{int} P) \cup \{+\infty_P\}, \end{aligned}$$

a contradiction. By (11), we have that $(F(x^k))$ is bounded from above.

Next, we prove that (x^k) is bounded. On the contrary, we may suppose that $\|x^k\| \to +\infty$. Then, by assumptions (H_σ) and (A), there exist $v \in X$ and $r_k \in (0, \|x^k\|)$ for all k, such that $x^k/\|x^k\| \xrightarrow{s} v$ and $F(x^k) \in F(x^k - r_k v) + P$ eventually. For $z^k := x^k - r_k v$, we have

$$F^\rho_{q,\varepsilon_k}(x^k) \notin (F^\rho_{q,\varepsilon_k}(z^k) + \operatorname{int} P) \cup \{+\infty_P\}, \ \ \forall\, k.$$

Thus,

$$F(x^k) + \varepsilon_k[\rho(\|x^k\|) - \rho(\|z^k\|)]q \notin (F(z^k) + \operatorname{int} P) \cup \{+\infty_P\}, \ \ \forall\, k$$

and since $F(x^k) \in F(z^k) + P$ and $q \in \text{int}P$ it follows that $\rho(\|x^k\|) < \rho(\|z^k\|)$; thus, $\|x^k\| < \|z^k\|$ eventually since ρ is increasing. From this and

$$\|x^k\| < ||z^k|| = \left\|x^k - \frac{r_k x^k}{||x^k||} + r_k\Big(\frac{x^k}{||x^k||} - v\Big)\right\| \leq ||x^k||\Big(1 - \frac{r_k}{||x^k||}\Big) + r_k\left\|\frac{x^k}{||x^k||} - v\right\|,$$

we have $\|\frac{x^k}{\|x^k\|} - v\| > 1$ eventually, a contradiction since $x^k/\|x^k\| \xrightarrow{s} v$. Therefore, (x^k) is bounded.

By assumption (H_σ), we have $x^k \xrightarrow{\sigma} \bar{x}$, up to subsequences, for some $\bar{x} \in X$. For $x \in \text{dom}F$ being arbitrary, we have

$$F(x^k) - F(x) \notin (\varepsilon_k[\rho(\|x\|) - \rho(\|x^k\|)]q + \text{int}P) \cup \{+\infty_P\}, \ \forall\, k.$$

From this and Proposition 2.1(d), we have

$$\psi^q(F(x^k) - F(x)) \leq \varepsilon_k[\rho(\|x\|) - \rho(\|x^k\|)] \leq \varepsilon_k \rho(\|x\|), \ \forall\, k.$$

As $\psi^q \circ (F - F(x))$ is sequentially σ-closed by Lemma 3.1(c), after taking the limit we obtain $\psi^q(F(\bar{x}) - F(x)) \leq 0$. By Proposition 2.1($d$) it follows that $F(\bar{x}) \notin (F(x) + \text{int}P) \cup \{+\infty_P\}$. Since $x \in \text{dom}F$ was arbitrary, we have $\bar{x} \in \text{WE}(F, X)$ and the solution set is nonempty. It is also sequentially σ-closed by Proposition 4.5. □

Remark 5.2 Theorem 5.3 remains correct if hypothesis (b) is replaced by the following one:

(b') $(\psi^q \circ F)^\infty_\sigma(v) \geq 0$, for all $v \in X$.

Indeed, let us prove that assumption (H_σ) and condition (b') imply condition (b) for $\rho(t) \equiv t$. Reasoning by contradiction, suppose that there exists $\varepsilon > 0$ such that $F^\rho_{q,\varepsilon}$ is not zero-coercive. Then there exists $(x^k) \subset X$ and $m > 0$ such that $\|x^k\| \to +\infty$ and $F^\rho_{q,\varepsilon}(x^k) \notin (mq + \text{int}P) \cup \{+\infty_P\}$, for all k. By assumption (H_σ) there exists $v \in X$ such that $x^k/\|x^k\| \xrightarrow{\sigma} v$. Then, by Proposition 2.1(d) we have $(\psi^q \circ F)(x^k) + \varepsilon\|x^k\| \leq m$, for all k and by applying (7) with $t_k := \|x^k\|$ and $v^k := x^k/\|x^k\|$ it follows that

$$(\psi^q \circ F)^\infty_\sigma(v) + \varepsilon \leq \liminf_k (\psi^q \circ F)(t_k v^k)/t_k + \varepsilon \leq \lim_k m/t_k = 0.$$

Therefore, $(\psi^q \circ F)^\infty_\sigma(v) < 0$, a contradiction.

Next, we apply Theorem 5.3 to establish the existence of weak efficient solutions for a simple non quasiconvex vector optimization problem. This example illustrates that Theorem 5.3 allows to check the existence of weak efficient solutions in problems where most of the results from the literature cannot be applied since they hold under convexity or quasiconvexity assumptions.

Example 5.1 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $P = \mathbb{R}^2_+$, σ is the usual topology of $\mathbb{R}$ and $F(t) = (|t|, \sin|t|)$. Clearly, F is σ-continuous and assumptions (H_σ) and (A)-(i) hold. In addition, the image set $F(\mathbb{R})$ is bounded from above and for each $(t^k) \subset \mathbb{R}$ such that $|t^k| \to +\infty$, we have $F(t^k) \in F(t^k - 2\pi v) + P$ for all k such that $|t^k| > 2\pi$, where $v = 1$ if $t^k \to +\infty$ whereas $v = -1$ if $t^k \to -\infty$. For $\varepsilon > 0$, $q = (1, 1)$ and $\rho(t) = t$, we have $F^\rho_{q,\varepsilon}(t) = (|t| + \varepsilon|t|, \sin|t| + \varepsilon|t|)$. It is clear that $\xi \circ F^\rho_{q,\varepsilon}$ is coercive for every $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2_+\backslash\{0\}$ since

$$(\xi \circ F^\rho_{q,\varepsilon})(t) = \xi_1|t| + \varepsilon(\xi_1 + \xi_2)|t| + \xi_2 \sin|t| \geq \varepsilon(\xi_1 + \xi_2)|t| - \xi_2.$$

This and Theorem 4.1 imply that $F^\rho_{q,\varepsilon}$ is zero-coercive, for all $\varepsilon > 0$. Then, we can apply Theorem 5.3 to see that $\mathrm{WE}(F, X)$ is nonempty and sequentially σ-closed. It is easy to check that

$$\mathrm{WE}(F, X) = [-3\pi/2, -\pi] \cup \{0\} \cup [\pi, 3\pi/2] \cup \{\pm(3\pi/2 + 2k\pi) : k \geq 1\}.$$

Notice that $\xi \circ F$ is not quasiconvex for every $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2_+\backslash\{0\}$ with $\xi_2 \neq 0$. Thus, the nonemptiness of $\mathrm{WE}(F, X)$ cannot be established by existence results for weak efficient solutions that hold under quasiconvexity assumptions (see, for instance, [2–14] and the references therein).

6 Conclusions

We have obtained existence results for weak efficient solutions of vector optimization problems in infinite dimensional spaces with arbitrary ordering cones. To do this, we have employed a regularization approach and new lower semicontinuity, boundedness and coercivity notions for vector valued functions, that are weaker than similar ones from the literature. Thus, the stated results improve most of the existence results for weak efficient solutions scattered in the literature. In particular, they allow us to deal with problems that are neither convex nor quasiconvex. Moreover, they allow us to deal also with noncoercive objective functions and non polyhedral ordering cones.

We have employed tools from asymptotic analysis to deal with unbounded data. To do this, we have obtained bounds for the sequential asymptotic cones of level, colevel and solutions sets and we have calculated the sequential asymptotic functions of linear and nonlinear scalarization functions.

We hope this paper will help to find existence results for other kind of solutions notions for vector optimization problems and will motivate the development of new regularization approaches.

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