Generalized ε-quasi solutions of set optimization problems

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Abstract

We introduce notions of generalized ε-quasi solutions to approximate set type solutions of set optimization problems. We study their properties, consistency and limit behavior as approximations to efficient and strict weak efficient solutions. Moreover, we prove an existence result for such solutions and a bound for their asymptotic cone. Finally, we obtain optimality conditions for them.

Keywords: set optimization, approximate solution, generalized ε-quasi solution, asymptotic map, optimality condition

1 Introduction

Let us consider the scalar optimization problem:

$$(\tilde{\mathcal{P}}) \quad \begin{cases} \text{Minimize } f(x) \\ \text{subject to } x \in C \end{cases}$$

where $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper function and $C \subset \operatorname{dom} f$ is a nonempty set. Loridan introduced in [26] two approximate solution notions for this problem. Let $\varepsilon > 0$ be fixed. A vector $\bar{x} \in C$ is said to be

- an ε-approximate solution of this problem, if $f(\bar{x}) \leq f(x) + \varepsilon$ for all $x \in C$;
- an ε-quasi solution of this problem, if $f(\bar{x}) \leq f(x) + \varepsilon\|x - \bar{x}\|$ for all $x \in C$.

By Ekeland's Variational Principle (see [10]), if f is lsc and bounded from below and C is closed, we obtain that some ε-quasi solutions are near to approximate solutions; i.e., for every scalar $\alpha > 0$ and ε-approximate solution $\bar{x}$ there exists a (ε/α)-quasi solution $\hat{x}$ such that $\|\hat{x} - \bar{x}\| \leq \alpha$. Thus, when solving numerically a scalar optimization problem, one can consider ε-quasi solutions with a small error ε, since some of them are near to feasible points whose objective value is almost optimal. For some problems, the limit of ε-quasi solutions when $\varepsilon \searrow 0$ is a true solution and for the convex case, true solutions can be thought as a limit of ε-quasi solutions (see [8, Remark 3.1]). Thus, ε-quasi solutions are good candidates for the notion of approximate solution (see also [3, Corollary 2.14]).

Loridan [27] extended the notion of ε-quasi solution to vector optimization problems whose final space is finite dimensional ordered by components. Gutiérrez et al. [18] generalized it to the same setting and a closed pointed convex ordering cone. Gao et al. [11] and Gutiérrez et al. [16] extended it to vector optimization problems with a Hausdorff locally convex final space ordered by an arbitrary proper convex cone, which is assumed to be pointed in [11]. It is important to point out that ε-quasi solutions are better than ε-approximate ones since the necessary optimality conditions are described exactly at each point and so they can be used in a more practical and effective way, just like the usual multiplier rules (see [13, Remark 4.13]).

From the above background, we conclude that ε-quasi solutions are useful from both a computational as well as a theoretical viewpoint. They also allow us to deal with scalar and vector optimization problems with no solutions or when it is not possible to apply/check any existence theorem or when it is difficult to calculate the solutions and some inaccuracy can be assumed by the decision maker. One expects the same state of affairs for optimization problems with set-valued objective maps.

In the last few years, optimization problems with set-valued objective maps or set optimization problems (SOPs for short) have attracted the attention of the mathematical community due to their vast applications in economics, finance, game theory, interval and fuzzy optimization, optimal control and differential inclusions, among others (see [21]). For these problems there exist vector-type and set-type solution notions. Most of the approximate solution notions for SOPs from the literature have been developed for vector-type solutions (see for instance [15, 33] among others). The study of approximate solutions of set-type is less developed.

In this paper, we approximate set-type solutions of a finite dimensional SOP by new concepts of generalized ε-quasi solutions. Our definitions encompass not only those for vector optimization problems, but also most of the well-known approximate solutions for SOPs from the literature. Gutiérrez et al. [14] defined three notions of approximate minimizers for sets using the lower-type set relation. Gutiérrez et al. [17] defined strict approximate solutions of SOPs under the set criterion and studied their limit behavior when the error tends to zero. They proved a general existence result and used it to obtain approximate Ekeland variational principles. Khahn and Quy [22] obtained versions of the Ekeland variational principle for set-valued maps not only for usual vector-type minimizers but also for set-type ones. Qiu and He [32] obtained new versions of such Ekeland variational principles. In this paper, we employ a different approach to derive the existence of ε-quasi solutions of SOPs.

The paper is structured as follows. In Section 2, we fix the notation and recall some preliminaries. We recall continuity and convexity notions for set-valued maps. We state the SOP and recall set-type solution notions for it. In Section 3, we introduce notions of generalized ε-quasi ℓ-solutions for set-type solutions of SOPs w.r.t. the lower set less relation. We study their properties and establish conditions that ensure the consistency of the definitions. We also study their convergence properties when varying the error ε and when approximating the ordering cone by means of Henig dilating cones. In Section 4, we obtain existence results. To this end, we use a scalarization procedure. In addition, a bound for the asymptotic cone of the set of generalized strict weak ε-quasi ℓ-solutions is derived. Finally, Section 5 is devoted to obtain optimality conditions for generalized strict weak ε-quasi ℓ-solutions.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{B}$ the closed unit ball in $\mathbb{R}^m$. We denote by $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{co} A$, $\operatorname{cone} A$, δ_A, σ_A, and A^∞, respectively, the interior, closure, convex hull, generated cone, indicator function, support function, and asymptotic cone of a set $A \subset \mathbb{R}^m$, see [5]. For $\varepsilon \geq 0$, we write $\varepsilon_k \searrow \varepsilon$ (resp. $\varepsilon_k \to \varepsilon_+$) if $\varepsilon_k \geq \varepsilon_{k+1} \geq \varepsilon$ (resp. $\varepsilon_k \geq \varepsilon$) for all k and $\varepsilon_k \to \varepsilon$. The nonnegative orthant of $\mathbb{R}^m$ will be denoted by $\mathbb{R}^m_+$.

In the sequel, D is a solid ($\operatorname{int} D \neq \emptyset$) pointed ($D \cap (-D) = \{0\}$) closed convex cone in $\mathbb{R}^m$. We have $D + D = D$, $\lambda D = D$ and $\lambda(\mathbb{R}^m \backslash (-\operatorname{int} D)) \subset \mathbb{R}^m \backslash (-\operatorname{int} D)$ for all $\lambda > 0$; $D + \operatorname{int} D = \operatorname{int} D$; $\mathbb{R}^m \backslash (-\operatorname{int} D) + D = \mathbb{R}^m \backslash (-\operatorname{int} D)$; and $D \subset \mathbb{R}^m \backslash (-\operatorname{int} D)$. The following properties for a set A in $\mathbb{R}^m$ hold: $\operatorname{cl}(A + D) + D = \operatorname{cl}(A + D)$ and $\operatorname{int}(A + D) + D = \operatorname{int}(A + D) = A + \operatorname{int} D$, $\operatorname{int} \operatorname{cl}(A + D) = \operatorname{int}(A + D)$ and $\operatorname{cl}(A + D) = \operatorname{cl}(A + \operatorname{int} D)$, see [6].

The (positive) polar cone of D is $D^+ := \{\xi \in \mathbb{R}^m \colon \langle \xi, p \rangle \geq 0, \forall p \in D\}$, its strict (positive) polar cone is $D^{s+} := \{\xi \in \mathbb{R}^m \colon \langle \xi, p \rangle > 0, \forall p \in D \backslash \{0\}\}$ and $D^- := -D^+$. By the bipolar theorem, we have

$$p \in D \iff \langle \xi, p \rangle \geq 0, \forall \xi \in D^+. \tag{1}$$

As D is pointed, we have $\operatorname{int} D^+ \neq \emptyset$ (see [34, Exercise 6.22]) and $\operatorname{int} D^+ = D^{s+}$. Moreover,

$$p \in \operatorname{int} D \iff \langle \xi, p \rangle > 0, \, \forall \xi \in D^+ \backslash \{0\}. \tag{2}$$

If $y, z \in \mathbb{R}^m$, we denote by $y \leq_D z$ if $z - y \in D$ and $y \ll_D z$ if $z - y \in \operatorname{int} D$.

As usual, we simplify the notation of a singleton $\{y\}$ by y and according we denote $A + y := A + \{y\}$, for all $A \subset \mathbb{R}^m$ and $y \in \mathbb{R}^m$. Recall that A is said to be D-proper if $A + D \neq \mathbb{R}^m$; D-convex if $A + D$ is a convex set; D-closed if $A + D$ is a closed set; and D-bounded from below if $A \subset \rho\mathbb{B} + D$ for some $\rho > 0$.

To compare sets A, B from $\mathbb{R}^m$, we recall the preorder $\leq_D^\ell$ defined by

$$A \leq_D^\ell B \overset{def}{\iff} B \subset A + D \iff B + D \subset A + D. \tag{3}$$

We also recall the set relation $\ll_D^\ell$ defined by

$$A \ll_D^\ell B \overset{def}{\iff} B \subset A + \operatorname{int} D \iff B + D \subset A + \operatorname{int} D. \tag{4}$$

Clearly, $\mathbb{R}^m \leq_D^\ell A \leq_D^\ell \emptyset$ for every set $A \subset \mathbb{R}^m$. Hence, the sets $\mathbb{R}^m$ and $\emptyset$ play the same role in $(2^{\mathbb{R}^m}, \leq_D^\ell)$ as $-\infty$ and $+\infty$ in $(\mathbb{R} \cup \{\pm\infty\}, \leq)$.

We recall the equivalence relation $\sim_D^\ell$ for nonempty sets defined by

$$A \sim_D^\ell B \overset{def}{\iff} A \leq_D^\ell B \text{ and } B \leq_D^\ell A \iff A + D = B + D. \tag{5}$$

To simplify notation, we introduce the set relation $\lnsim_D^\ell$ defined by

$$A \lnsim_D^\ell B \overset{def}{\iff} A \leq_D^\ell B \text{ and } A \not\sim_D^\ell B \iff B + D \subsetneq A + D. \tag{6}$$

We recall a convergence notion for a sequence of sets $\{A_k\}$ from $\mathbb{R}^m$ (see [34]): $\limsup_k A_k := \{x \colon \exists x^{k_j} \in A_{k_j} \to x\}$ is its outer limit and $\liminf_k A_k := \{x \colon \exists x^k \in A_k \to x\}$ is its inner limit, where $\{A_{k_j}\}$ is a subsequence of $\{A_k\}$ and $x^k \in A_k \to x$ denotes $x^k \in A_k$ for all k large enough and $x^k \to x$. We say that $\{A_k\}$ converges to a set $A \subset \mathbb{R}^m$ in the sense of Painlevé-Kuratowski, denoted by $A_k \to A$ or $\lim_k A_k = A$, if $\limsup_k A_k \subset A \subset \liminf_k A_k$.

We recall a result about the convergence of monotone sequences of sets.

Proposition 2.1 [34, Exercise 4.3]

(a) *If $\{A_k\}$ is nondecreasing; i.e., $A_k \subset A_{k+1}$, then $\lim_k A_k = \operatorname{cl} \bigcup_k A_k$.*

(b) *If $\{A_k\}$ is nonincreasing; i.e., $A_{k+1} \subset A_k$, then $\lim_k A_k = \bigcap_k \operatorname{cl} A_k$.*

We denote by $F \colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ a set-valued map that associates to $x \in \mathbb{R}^n$ a set $F(x) \subset \mathbb{R}^m$. Its domain is $\operatorname{dom} F := \{x : F(x) \neq \emptyset\}$ ($F(x) = \emptyset$ for $x \notin \operatorname{dom} F$). We denote by $F(A) := \bigcup_{x \in A} F(x)$ the image of A under F,

$\operatorname{gph} F := \{(x, y) \colon y \in F(x)\}$ its graph, $\operatorname{epi} F := \{(x, y) : F(x) \leq^{\ell}_{D} y\}$ its ℓ-epigraph, $\operatorname{lev}(F, y) := \{x : F(x) \leq^{\ell}_{D} y\}$ its ℓ-level set of height $y \in \mathbb{R}^m$ (to simplify, we write the singleton set $\{y\}$ by y), and $F_D \colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ its profile map defined by $F_D(x) := F(x) + D$.

Whenever "N" denotes some property of sets in $\mathbb{R}^m$, it is said that F is "N"-valued if $F(x)$ has the property "N" for every x. For instance, F is closed-valued if $F(x)$ is closed for every x. We say that F is closed (resp. convex) if its graph is closed (resp. convex).

We recall some boundedness notions for maps. We say that F is bounded if $F(\mathbb{R}^n)$ is bounded; ℓ-bounded from below if there exists b such that $b \leq^{\ell}_{D} F(x)$ for all x; D-bounded from below if $F(\mathbb{R}^n)$ is D-bounded from below; locally bounded at x if for some neighborhood U of x the set $F(U)$ is bounded; and locally bounded if it is so at every x. We list some properties of these notions.

Proposition 2.2 (a) *If F is bounded, then F is D-bounded from below.*

(b) *If F is D-bounded from below, then F is D-bounded-valued from below.*

(c) *The following conditions are equivalent:*

- (i) *There exists b such that $b \leq_D y$ for all $y \in F(\mathbb{R}^n)$;*
- (ii) *F is ℓ-bounded from below;*
- (iii) *F is D-bounded from below.*

Proof Parts (a)–(b) are easy to check. Equivalence $(i) \Leftrightarrow (ii)$ and implication $(ii) \Rightarrow (iii)$ in (c) are trivial. We prove implication $(iii) \Rightarrow (ii)$. By hypothesis $F(\mathbb{R}^n) \subset \rho\mathbb{B} + D$ for some $\rho > 0$. We assert that for a fixed $q \in \operatorname{int} D$, we have $\rho\mathbb{B} \subset -kq + D$ for some $k \in \mathbb{N}$ and this implies (ii) by taking $b := -kq$. On the contrary, if we suppose that $\rho\mathbb{B} \not\subset -kq + D$ for all $k \in \mathbb{N}$, then there exists $u^k \in \rho\mathbb{B}$ such that $u^k \notin -kq + D$. Hence $\frac{1}{k}u^k + q \notin \operatorname{int} D$ for all k and after taking the limit, we obtain $q \notin \operatorname{int} D$, a contradiction. □

Remark 2.3 *In [32], the notions of D-boundedness from below and ℓ-boundedness from below are called quasi D-lower boundedness and D-lower boundedness, respectively. Therein it is shown that, without the solidness of D, they are different concepts. In [24, 32] one can find weaker boundedness notions from below via linear scalarizations.*

A map F is said to be inner semicontinuous (isc) at x if $F(x) \subset \liminf_k F(x^k)$ for every $x^k \to x$; outer semicontinuous (osc) at x if $\limsup_k F(x^k) \subset F(x)$ for every $x^k \to x$; upper semicontinuous (usc) at x if for any open set V containing $F(x)$, there exists an open set U containing x such that $F(U) \subset V$. It is said to be isc (resp. osc, usc) if it is so at every x. Clearly, F is osc iff it is closed. We list some properties of these notions.

Proposition 2.4 [4, 34]

(a) If F is compact-valued, then F is D-closed-valued.

(b) If F is usc and compact-valued, then F is locally bounded and osc.

(c) F is locally bounded iff $F(B)$ is bounded for every bounded set B iff whenever $y^k \in F(x^k)$ for all k and $\{x^k\}$ is bounded, then $\{y^k\}$ is bounded.

(d) If F is usc at x and $F(x)$ is closed, then F is osc at x. The reverse implication holds if, in addition, F is locally bounded at x.

We recall a continuity notion for set-valued maps defined via $\leq_D^\ell$ (see [20] and the references therein).

Definition 2.5 *A map F is said to be ℓ-outer semicontinuous (ℓ-osc) at x if $F(x) \leq_D^\ell \limsup_k (F(x^k) + D)$ for every $x^k \to x$. It is said to be ℓ-osc if it is so at every x.*

There exist various weaker notions of semicontinuity for set-valued maps, as K-lower semicontinuity and K-sequentially lower monotonicity (see [32] and references therein). We employ ℓ-outer semicontinuity since it is characterized by the closedness of the ℓ-epigraph that usually is assumed when approximating SOPs. We recall some properties of this notion.

Proposition 2.6 [20]

(a) F is ℓ-osc iff F is D-closed-valued and it has closed ℓ-level sets iff $\operatorname{epi} F$ *is closed.*

(b) If F is osc and locally bounded, then F is ℓ-osc.

We recall a convexity notion for set-valued maps defined via $\leq_D^\ell$ (see [20] and [35] for some remarks on the origin of these notions).

Definition 2.7 *A map F is said to be ℓ-convex, if for all $x, x' \in \mathbb{R}^n$ and $t \in (0,1)$ one has $F(tx + (1-t)x') \leq_D^\ell tF(x) + (1-t)F(x')$.*

Clearly, the domain of an ℓ-convex map is convex. It is easy to check that F is convex iff $tF(x) + (1-t)F(x') \subset F(tx + (1-t)x')$ for all $x, x' \in \mathbb{R}^n$ and $t \in (0,1)$. Convex maps are ℓ-convex. We list some properties of this notion.

Proposition 2.8 [20]

(a) F is ℓ-convex iff $\operatorname{epi} F$ *is convex iff F_D is convex.*

(b) If F is ℓ-convex, then F is D-convex-valued, $F_D(\mathbb{R}^n)$ is convex and F has convex ℓ-level sets.

A set optimization problem reads as follows:

$$(\mathcal{P}) \quad \begin{cases} \text{Minimize } F(x) \\ \text{subject to } x \in C \end{cases}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a set-valued map and $C \subset \operatorname{dom} F$ is a nonempty set.

This problem arises naturally in a variety of theoretical and practical problems. There exist two types of solutions for this problem (see [20, 21]): vector-type ones, when the preferences are defined on elements of $\mathbb{R}^m$ by the binary relations $\leq_D$ and $\ll_D$, and set-type ones, when the preferences are defined on elements of $2^{\mathbb{R}^m}$ by the binary relations $\leq_D^\ell$ and $\ll_D^\ell$. In this paper, we focus on the latter.

Definition 2.9 *A vector $\bar{x} \in C$ is said to be:*

- *an ℓ-efficient solution of $(\mathcal{P})$, denoted by $\bar{x} \in \ell\text{-}\mathrm{E}(\mathcal{P}, D)$, if there is no $x \in C$ such that $F(x) \lneq_D^\ell F(\bar{x})$.*
- *a strict ℓ-efficient solution of $(\mathcal{P})$, denoted by $\bar{x} \in \ell\text{-}\mathrm{SE}(\mathcal{P}, D)$, if there is no $x \in C$ with $x \neq \bar{x}$ such that $F(x) \leq_D^\ell F(\bar{x})$.*
- *a weakly ℓ-efficient solution of $(\mathcal{P})$, denoted by $\bar{x} \in \ell\text{-}\mathrm{WE}(\mathcal{P}, D)$, if $x \in C$ and $F(x) \ll_D^\ell F(\bar{x})$ imply $F(\bar{x}) \ll_D^\ell F(x)$.*
- *a strict weakly ℓ-efficient solution of $(\mathcal{P})$, denoted by $\bar{x} \in \ell\text{-}\mathrm{SWE}(\mathcal{P}, D)$, if there is no $x \in C$ with $x \neq \bar{x}$ such that $F(x) \ll_D^\ell F(\bar{x})$.*

We have the following relationships between these notions:

$$\ell\text{-}\mathrm{SE}(\mathcal{P}, D) \subset \ell\text{-}\mathrm{E}(\mathcal{P}, D) \subset \ell\text{-}\mathrm{WE}(\mathcal{P}, D) \text{ and } \ell\text{-}\mathrm{SWE}(\mathcal{P}, D) \subset \ell\text{-}\mathrm{WE}(\mathcal{P}, D). \quad (7)$$

Notice that one could assume without loss of generality that the values of F are free disposal sets w.r.t. the cone D, i.e., $F(x) + D = F(x)$, for all $x \in C$, in the sense that all above solution sets do not change if we replace F by F_D.

If F is not D-proper-valued; i.e., $F(\bar{x}) + D = \mathbb{R}^m$ for some $\bar{x} \in C$, then $F(\bar{x}) \leq_D^\ell F(x)$ for all $x \in C$. This means that $\bar{x}$ is an ideal or strong solution of problem $(\mathcal{P})$. In this case, it is easy to check that

$$\ell\text{-}\mathrm{E}(\mathcal{P}, D) = \ell\text{-}\mathrm{WE}(\mathcal{P}, D) = \{x \in C : F(x) + D = \mathbb{R}^m\}$$

and the problem is trivial from a theoretical point of view since by assuming that the values of F are free disposal w.r.t. the cone D, then the ℓ-efficient and weakly ℓ-efficient solutions of problem $(\mathcal{P})$ are just the feasible points whose value is the whole space $\mathbb{R}^m$. Then, in the sequel, we will assume that all the maps are D-proper-valued.

If F is D-closed-valued, then $F(x) \not\ll_D^\ell F(x)$ for all $x \in C$ that implies $\ell\text{-}\mathrm{WE}(\mathcal{P}, D) = \ell\text{-}\mathrm{SWE}(\mathcal{P}, D)$ (see [20, Lemma 1]). Hence $\bar{x} \in \ell\text{-}\mathrm{SWE}(\mathcal{P}, D)$ iff there is no $x \in C$ such that $F(x) \ll_D^\ell F(\bar{x})$.

We recall an existence result of set-type solutions for problem $(\mathcal{P})$.

Theorem 2.10 [23] *If C is compact and F has closed ℓ-level sets, then ℓ-$\mathrm{E}(\mathcal{P}, D)$ is nonempty.*

If F is ℓ-osc, then F is D-closed-valued and it has closed ℓ-level sets by Proposition 2.6(a); thus, ℓ-$\mathrm{SWE}(\mathcal{P}, D) = \ell$-$\mathrm{WE}(\mathcal{P}, D)$. So, under the hypotheses of Theorem 2.10, by inclusion (7), we conclude that ℓ-$\mathrm{SWE}(\mathcal{P}, D)$ is nonempty. On the other hand, by [20, Proposition 5], if F is ℓ-osc and locally bounded, then ℓ-$\mathrm{SWE}(\mathcal{P}, D)$ is closed.

3 Generalized ε-quasi solutions

We define some new notions of generalized ε-quasi solutions for SOPs that extend those from [18] given for vector optimization problems.

Let $\varepsilon \geq 0$, $G \subset \mathbb{R}^m$ be a nonempty set, and $\varphi : \mathbb{R}^n \to [0, +\infty)$ be a continuous function such that $\varphi(x) > 0$ if $x \neq 0$. We denote $\mathcal{A} := (G, \varphi)$.

Definition 3.1 *A vector $\bar{x} \in C$ is said to be a generalized (w.r.t. $\mathcal{A}$)*

- *ε-quasi ℓ-solution of $(\mathcal{P})$, denoted by $\bar{x} \in \ell$-$\mathrm{E}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$, if there is no $x \in C$ and $e \in G$ such that $F(x) + \varepsilon\varphi(x - \bar{x})e \lnsim^{\ell}_{D} F(\bar{x})$.*
- *strict ε-quasi ℓ-solution of $(\mathcal{P})$, denoted by $\bar{x} \in \ell$-$\mathrm{SE}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$, if there is no $x \in C$ and $e \in G$ such that $F(x) + \varepsilon\varphi(x - \bar{x})e \leq^{\ell}_{D} F(\bar{x})$.*
- *weak ε-quasi ℓ-solution of $(\mathcal{P})$, denoted by $\bar{x} \in \ell$-$\mathrm{WE}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$, if $x \in C$, $e \in G$ and $F(x) + \varepsilon\varphi(x - \bar{x})e \ll^{\ell}_{D} F(\bar{x})$ imply $F(\bar{x}) \ll^{\ell}_{D} F(x) + \varepsilon\varphi(x - \bar{x})e$.*
- *strict weak ε-quasi ℓ-solution of $(\mathcal{P})$, denoted by $\bar{x} \in \ell$-$\mathrm{SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$, if there is no $x \in C$ and $e \in G$ such that $F(x) + \varepsilon\varphi(x - \bar{x})e \ll^{\ell}_{D} F(\bar{x})$.*
- *proper ε-quasi ℓ-solution of $(\mathcal{P})$, denoted by $\bar{x} \in \ell$-$\mathrm{PE}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$, if $\bar{x} \in \ell$-$\mathrm{E}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$ and there exists a solid pointed convex cone $K \subset \mathbb{R}^m$ such that $D\backslash\{0\} \subset \operatorname{int} K$ and $\bar{x} \in \ell$-$\mathrm{E}(\mathcal{P}, K, \mathcal{A}, \varepsilon)$.*

Remark 3.2 1. *Two important instances of φ are $\varphi(\cdot) \equiv 1$ and $\varphi(\cdot) = \|\cdot\|$. We provide more examples of function φ in the paragraph before Proposition 4.4.*

2. *When defining that $\bar{x}$ is a proper ε-quasi ℓ-solution of $(\mathcal{P})$ we consider that $\bar{x}$ is from ℓ-$\mathrm{E}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$. We do this, since it is not implied by the remaining part of the definition, in contrast to the case of vector-valued functions (see [18,* Proposition 3.1(b)]). *Indeed, for $F\colon \mathbb{R} \rightrightarrows \mathbb{R}^2$ defined by*

$$F(x) = \begin{cases} [0,1] \times [0,1], & \text{if } x = 0; \\ [-1,1] \times [-1,1], & \text{if } x \neq 0, \end{cases}$$

$C = \mathbb{R}$, $D = \operatorname{cone}\operatorname{co}\{(1,2),(2,1)\}$, $K = \mathbb{R}^2_+$, $G = \{(1,1)\}$, $\varphi \equiv 1$ and $\varepsilon = 1$, we have

$$F(x) + \varepsilon\varphi(x - \bar{x})e = F(x) + (1,1) = \begin{cases} [1,2] \times [1,2], & \text{if } x = 0; \\ [0,2] \times [0,2], & \text{if } x \neq 0. \end{cases}$$

Clearly, $D\backslash\{0\} \subset \operatorname{int} K$, $0 \in \ell$-$\mathrm{E}(\mathcal{P}, K, \mathcal{A}, \varepsilon)$ since $F(0) + (1,1) \not\lnsim^{\ell}_{K} F(0)$ and $F(0) \sim^{\ell}_{K} F(x) + (1,1)$ for every $x \neq 0$. However, we have $0 \notin \ell$-$\mathrm{E}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$ since $F(x) + (1,1) \lnsim^{\ell}_{D} F(0)$ for $x \neq 0$.

We establish properties of these notions that follow from the definitions.

Lemma 3.3 *Consider problem* $(\mathcal{P})$, $\mathcal{A} = (G, \varphi)$ *and* $\varepsilon \geq 0$.

(*a*) $\ell\text{-I}(\mathcal{P}, D, \mathcal{A}, 0) = \ell\text{-I}(\mathcal{P}, D)$ *for* $\mathrm{I} \in \{\mathrm{E}, \mathrm{WE}\}$,

$$\ell\text{-SE}(\mathcal{P}, D, \mathcal{A}, \varepsilon) \cup \ell\text{-PE}(\mathcal{P}, D, \mathcal{A}, \varepsilon) \subset \ell\text{-E}(\mathcal{P}, D, \mathcal{A}, \varepsilon),$$
$$\ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon) \subset \ell\text{-WE}(\mathcal{P}, D, \mathcal{A}, \varepsilon).$$

If $K \subset \mathbb{R}^m$ *is a solid pointed convex cone such that* $D\backslash\{0\} \subset \operatorname{int} K$, *then*

$$\ell\text{-SWE}(\mathcal{P}, K, \mathcal{A}, \varepsilon) \subset \ell\text{-SE}(\mathcal{P}, D\backslash\{0\}, \mathcal{A}, \varepsilon).$$

(*b*) *If* F *is* D*-closed-valued, then*

$$\begin{aligned} \ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, 0) &= \ell\text{-SWE}(\mathcal{P}, D), \\ \ell\text{-WE}(\mathcal{P}, D, \mathcal{A}, \varepsilon) &= \ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon), \\ \ell\text{-E}(\mathcal{P}, D, \mathcal{A}, \varepsilon) &\subset \ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon). \end{aligned} \tag{8}$$

(*c*) *If* $K \subset \mathbb{R}^m$ *is a solid pointed convex cone such that* $D\backslash\{0\} \subset \operatorname{int} K$ *and* F *is* K*-proper and* $\operatorname{cl} K$*-closed-valued, then*

$$\ell\text{-WE}(\mathcal{P}, K, \mathcal{A}, \varepsilon) \subset \ell\text{-SE}(\mathcal{P}, D\backslash\{0\}, \mathcal{A}, \varepsilon). \tag{9}$$

Proof Part (*a*) is trivial. The equalities in (*b*) are trivial and the inclusion follows similarly as in (*c*).

(*c*) If on the contrary, we suppose that there exist $\bar{x} \in \ell\text{-WE}(\mathcal{P}, K, \mathcal{A}, \varepsilon)$, $e \in G$ and $x \in C$ such that $F(x) + \varepsilon\varphi(x - \bar{x})e \leq^{\ell}_{D\backslash\{0\}} F(\bar{x})$. Hence $F(\bar{x}) \subset F(x) + \varepsilon\varphi(x - \bar{x})e + D\backslash\{0\}$ and as $D\backslash\{0\} \subset \operatorname{int} K$ and $\bar{x} \in \ell\text{-WE}(\mathcal{P}, K, \mathcal{A}, \varepsilon)$, we have $F(\bar{x}) + \operatorname{int} K = F(x) + \varepsilon\varphi(x - \bar{x})e + \operatorname{int} K$. Therefore

$$\begin{aligned} F(\bar{x}) + \operatorname{cl} K &\subset F(x) + \varepsilon\varphi(x - \bar{x})e + D\backslash\{0\} + \operatorname{cl} K \\ &\subset F(x) + \varepsilon\varphi(x - \bar{x})e + \operatorname{int} K = F(\bar{x}) + \operatorname{int} K \subset F(\bar{x}) + K, \end{aligned}$$

a contradiction since $F(\bar{x})$ is K-proper and $\operatorname{cl} K$-closed. □

Remark 3.4 1. *From Lemma 3.3, we see that it suffices to deal with generalized proper and strict weak* ε*-quasi* ℓ*-solutions since the first ones are contained in the set of* ε*-quasi* ℓ*-solutions that in turn are contained in the second ones under* D*-closed valuedness.*

2. *It is worth underlining that* $\ell\text{-SE}(\mathcal{P}, D\backslash\{0\}, \mathcal{A}, 0)$ *coincides with the set* $\ell\text{-E}(\mathcal{P}, D, \mathcal{A}, 0)$ *whenever* F *is single-valued.*

3. *Under the assumptions imposed in* (9)*, we have that proper* ε*-quasi* ℓ*-solutions are strict* ε*-quasi* ℓ*-solutions w.r.t.* $D\backslash\{0\}$. *For instance, for the data in Remark 3.2*(2)*, we have* $0 \in \ell\text{-SE}(\mathcal{P}, D\backslash\{0\}, \mathcal{A}, \varepsilon)$. *These assumptions hold true when* $\varepsilon = 0$ *and* F *is single-valued.*

4. *Clearly, if* F *is* K*-closed-valued, then* F *is* $\operatorname{cl} K$*-closed-valued.*

We establish some monotonicity properties of the solution sets.

Proposition 3.5 *Consider problem* $(\mathcal{P})$, $\mathcal{A} = (G, \varphi)$ *and* $\varepsilon \geq 0$.

(a) If $G \subset D$ *and* $\varepsilon < \varepsilon'$*, then* $\ell\text{-I}(\mathcal{P}, D, \mathcal{A}, \varepsilon) \subset \ell\text{-I}(\mathcal{P}, D, \mathcal{A}, \varepsilon')$ *for* $\text{I} \in \{\text{E}, \text{SWE}, \text{PE}\}$*.*

(b) If $\mathcal{A}' = (G', \varphi)$ *with* $G \leq_D^\ell G'$*, then* $\ell\text{-I}(\mathcal{P}, D, \mathcal{A}, \varepsilon) \subset \ell\text{-I}(\mathcal{P}, D, \mathcal{A}', \varepsilon)$ *for* $\text{I} \in \{\text{E}, \text{SWE}, \text{PE}\}$*. Hence,* $\ell\text{-I}(\mathcal{P}, D, \mathcal{A}, \varepsilon) = \ell\text{-I}(\mathcal{P}, D, \mathcal{A}', \varepsilon)$ *when* $G \sim_D^\ell G'$*.*

(c) If $G \subset D$ *and* $\mathcal{A}' = (G, \varphi')$ *with* $\varphi \leq \varphi'$*, then* $\ell\text{-I}(\mathcal{P}, D, \mathcal{A}, \varepsilon) \subset \ell\text{-I}(\mathcal{P}, D, \mathcal{A}', \varepsilon)$ *for* $\text{I} \in \{\text{E}, \text{SWE}, \text{PE}\}$*.*

(d) If $D \subset D'$ *with* $D' \subset \mathbb{R}^m$ *being a solid pointed convex cone, then*

$$\ell\text{-SWE}(\mathcal{P}, D', \mathcal{A}, \varepsilon) \subset \ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon).$$

Proof We prove (a)–(c) for $\text{I} = \text{E}$. For the others the proofs run similarly.

(a) Consider $\bar{x} \in C$. If $\bar{x} \notin \ell\text{-E}(\mathcal{P}, D, \mathcal{A}, \varepsilon')$, then there exist $x \in C$ and $e \in G$ such that $F(\bar{x}) + D \subsetneq F(x) + \varepsilon'\varphi(x - \bar{x})e + D$. As $G \subset D$, we have $(\varepsilon' - \varepsilon)\varphi(x - \bar{x})e \in D$ and thus

$$\begin{aligned} F(x) + \varepsilon'\varphi(x - \bar{x})e + D &= F(x) + \varepsilon\varphi(x - \bar{x})e + (\varepsilon' - \varepsilon)\varphi(x - \bar{x})e + D \\ &\subset F(x) + \varepsilon\varphi(x - \bar{x})e + D. \end{aligned}$$

Hence, $F(\bar{x}) + D \subsetneq F(x) + \varepsilon\varphi(x - \bar{x})e + D$; i.e., $\bar{x} \notin \ell\text{-E}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$.

(b) Consider $\bar{x} \in C$. If $\bar{x} \notin \ell\text{-E}(\mathcal{P}, D, \mathcal{A}', \varepsilon)$, then there exist $x \in C$ and $e' \in G'$ such that $F(\bar{x}) + D \subsetneq F(x) + \varepsilon\varphi(x - \bar{x})e' + D$. As there exists $e \in G$ with $e' \in e + D$, we have $F(\bar{x}) + D \subsetneq F(x) + \varepsilon\varphi(x - \bar{x})e + D$; thus, $\bar{x} \notin \ell\text{-E}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$. The remaining assertion is trivial.

(c) Consider $\bar{x} \in C$. If $\bar{x} \notin \ell\text{-E}(\mathcal{P}, D, \mathcal{A}', \varepsilon)$, then there exist $x \in C$ and $e \in G$ such that $F(\bar{x}) + D \subsetneq F(x) + \varepsilon\varphi'(x - \bar{x})e + D$. As $G \subset D$, we have $\varepsilon(\varphi' - \varphi)(x - \bar{x})e \in D$ and thus

$$\begin{aligned} F(x) + \varepsilon\varphi'(x - \bar{x})e + D &= F(x) + \varepsilon\varphi(x - \bar{x})e + \varepsilon(\varphi' - \varphi)(x - \bar{x})e + D \\ &\subset F(x) + \varepsilon\varphi(x - \bar{x})e + D. \end{aligned}$$

Hence, $F(\bar{x}) + D \subsetneq F(x) + \varepsilon\varphi(x - \bar{x})e + D$; i.e., $\bar{x} \notin \ell\text{-E}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$.

(d) Consider $\bar{x} \in C$. If $\bar{x} \notin \ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$, then there exist $x \in C$ and $e \in G$ such that $F(\bar{x}) \subset F(x) + \varepsilon\varphi(x - \bar{x})e + \operatorname{int} D$. As $\operatorname{int} D \subset \operatorname{int} D'$, we have $F(\bar{x}) \subset F(x) + \varepsilon\varphi(x - \bar{x})e + \operatorname{int} D'$; thus, $\bar{x} \notin \ell\text{-SWE}(\mathcal{P}, D', \mathcal{A}, \varepsilon)$. □

We study the closedness of the set of generalized strict weak ε-quasi ℓ-solutions.

Proposition 3.6 *If* F *is* ℓ*-osc and locally bounded, and* C *is closed, then* $\ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$ *is closed for all* $\varepsilon \geq 0$*.*

Proof Let $x^k \in \ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon) \to \bar{x}$. For $x \in C$ and $e \in G$ being fixed, we have $F(x^k) \not\subset F(x) + \varepsilon\varphi(x - x^k)e + \operatorname{int} D$. There exists $y^k \in F(x^k)$ such that $y^k \notin F(x) + \varepsilon\varphi(x - x^k)e + \operatorname{int} D$. As F is locally bounded, the sequence $\{y^k\}$ is bounded; thus, $y^k \to \bar{y}$ for some $\bar{y}$ up to subsequences. As $y^k \in F(x^k) + D \to \bar{y}$, we have $\bar{y} \in \limsup_k (F(x^k) + D)$ that by ℓ-outer semicontinuity of F implies $\bar{y} \in F(\bar{x}) + D$. On the other hand, as $y^k - \varepsilon\varphi(x - x^k)e \in \mathbb{R}^m \setminus \operatorname{int}(F(x) + D)$, after taking the limit, we obtain $\bar{y} \notin F(x) + \varepsilon\varphi(x - \bar{x})e + \operatorname{int} D$. Hence $F(\bar{x}) + D \not\subset F(x) + \varepsilon\varphi(x - \bar{x})e + \operatorname{int} D$; thus, $\bar{x} \in \ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$ since $\bar{x} \in C$ (recall that C is closed), $x \in C$ and $e \in G$ were arbitrary. □

We establish conditions on $\mathcal{A} = (G, \varphi)$ that allow us to obtain consistent notions of generalized ε-quasi ℓ-solutions. This result extends [25, Proposition 1.3] obtained for vector optimization problems.

Proposition 3.7 *Consider problem* $(\mathcal{P})$ *and* $\mathcal{A} = (G, \varphi)$. *Then*

(a) ℓ-$\mathrm{E}(\mathcal{P}, D, \mathcal{A}, \varepsilon) = \emptyset$, *for all* $\varepsilon > 0$ *under any of the following items:*

- (i) $G \cap (-D\backslash\{0\}) \neq \emptyset$, $\varphi(0) > 0$ *and* F *is* D*-bounded valued.*
- (ii) $G \cap (-\operatorname{int} D) \neq \emptyset$ *and* $\varphi(0) > 0$.
- (iii) $G \cap (-\operatorname{int} D) \neq \emptyset$, $\lim_{\|x\| \to +\infty} \varphi(x) = +\infty$, C *is unbounded and* F *is bounded.*

(b) ℓ-$\mathrm{SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon) = \emptyset$, *for all* $\varepsilon > 0$ *under any of the following items:*

- (i) $G \cap (-\operatorname{int} D) \neq \emptyset$ *and* $\varphi(0) > 0$.
- (ii) $G \cap (-\operatorname{int} D) \neq \emptyset$, $\lim_{\|x\| \to +\infty} \varphi(x) = +\infty$, C *is unbounded and* F *is bounded.*

Proof (a) On the contrary, suppose that ℓ-$\mathrm{E}(\mathcal{P}, D, \mathcal{A}, \varepsilon) \neq \emptyset$ for some $\varepsilon > 0$.
(i) For $\bar{x}$ in this set and $e \in G \cap (-D\backslash\{0\})$ (note that $\varepsilon\varphi(0)e \neq 0$), we have:

$$F(\bar{x}) \not\subset F(\bar{x}) + \varepsilon\varphi(0)e + D \quad \text{or} \quad F(\bar{x}) + D = F(\bar{x}) + \varepsilon\varphi(0)e + D.$$

If the first case holds true, then $0 \notin \varepsilon\varphi(0)e + D$, that is a contradiction, since $e \in -D$ and $\varepsilon\varphi(0) > 0$.
If the second case holds true, then it is not difficult to check that $F(\bar{x}) + D = F(\bar{x}) + k\varepsilon\varphi(0)e + D$ for all $k \in \mathbb{N}$. As F is D-bounded valued, there exists $\rho > 0$ such that $F(\bar{x}) \subset \rho\mathbb{B} + D$. Hence $F(\bar{x}) + k\varepsilon\varphi(0)e \subset \rho\mathbb{B} + D$ for all $k \in \mathbb{N}$. For $\bar{y} \in F(\bar{x})$, we have $\bar{y} + k\varepsilon\varphi(0)e \in \rho\mathbb{B} + D$ for all $k \in \mathbb{N}$. After dividing by k and taking the limit, we obtain $\varepsilon\varphi(0)e \in D$, a contradiction.
(ii) For $\bar{x}$ in this set and $e \in G \cap (-\operatorname{int} D)$ (note that $\varepsilon\varphi(0)e \in -\operatorname{int} D$), we get

$$F(\bar{x}) \not\subset F(\bar{x}) + \varepsilon\varphi(0)e + D \quad \text{or} \quad F(\bar{x}) + D = F(\bar{x}) + \varepsilon\varphi(0)e + D.$$

If the first case holds true, then $0 \notin \varepsilon\varphi(0)e + D$, that is a contradiction since $\varepsilon\varphi(0)e \in -D$.
If the second case holds true, then

$$F(\bar{x}) + D = F(\bar{x}) + \bigcup_{k\in\mathbb{N}} (k\varepsilon\varphi(0)e + D).$$

As $\varepsilon\varphi(0)e \in -\operatorname{int} D$, it is easy to check that $\mathbb{R}^m = \bigcup_{k\in\mathbb{N}} (k\varepsilon\varphi(0)e + D)$. Then, $F(\bar{x}) + D = \mathbb{R}^m$, that is a contradiction since $F(\bar{x})$ is D-proper.
(iii) For $\bar{x}$ in this set, $e \in G \cap (-\operatorname{int} D)$ and $\{x^k\} \subset C$ with $\|x^k\| \to +\infty$, for each k one has

$$F(\bar{x}) \not\subset F(x^k) + \varepsilon\varphi(x^k - \bar{x})e + D \quad \text{or} \quad F(\bar{x}) + D = F(x^k) + \varepsilon\varphi(x^k - \bar{x})e + D.$$

If the first case holds true for a subsequence $\{x^{k_j}\}$, then $F(\bar{x}) \not\subset F(x^{k_j}) + \varepsilon\varphi(x^{k_j} - \bar{x})e + D$ and there exists $y^{k_j} \in F(\bar{x})$ such that $y^{k_j} \notin F(x^{k_j}) + \varepsilon\varphi(x^{k_j} - \bar{x})e + D$ for all j. For $w^{k_j} \in F(x^{k_j})$, we have $y^{k_j} \notin w^{k_j} + \varepsilon\varphi(x^{k_j} - \bar{x})e + D$ for such j. As $\{y^{k_j}\}$ and $\{w^{k_j}\}$ are bounded, after dividing by $\varphi(x^{k_j} - \bar{x})$ and taking the limit, we obtain $-\varepsilon e \in \operatorname{cl}(\mathbb{R}^m\backslash D) = \mathbb{R}^m\backslash \operatorname{int} D$, a contradiction.
Therefore, the second case holds true eventually, i.e., there exists k_0 such that $F(x^k) + \varepsilon\varphi(x^k - \bar{x})e \subset F(\bar{x}) + D$, for all $k \geq k_0$. For $w^k \in F(x^k)$ there exists $y^k \in F(\bar{x})$ such

that $w^k + \varepsilon\varphi(x^k - \bar{x})e \in y^k + D$, for all $k \geq k_0$. As $\{y^k\}$ and $\{w^k\}$ are bounded, after dividing by $\varphi(x^k - \bar{x})$ and taking the limit, we obtain $\varepsilon e \in D$, a contradiction.

(b) On the contrary, suppose that ℓ-SWE$(\mathcal{P}, D, \mathcal{A}, \varepsilon) \neq \emptyset$ for some $\varepsilon > 0$.
(i) For $\bar{x}$ in this set and $e \in G \cap (-\operatorname{int} D)$, we have $F(\bar{x}) \not\subset F(\bar{x}) + \varepsilon\varphi(0)e + \operatorname{int} D$. This implies $0 \notin \varepsilon\varphi(0)e + \operatorname{int} D$, that is a contradiction since $\varepsilon\varphi(0)e \in -\operatorname{int} D$.
(ii) For $\bar{x}$ in this set, $e \in G \cap (-\operatorname{int} D)$ and $\{x^k\} \subset C$ such that $\|x^k\| \to +\infty$, we have $F(\bar{x}) \not\subset F(x^k) + \varepsilon\varphi(x^k - \bar{x})e + \operatorname{int} D$ for all k. There exists $y^k \in F(\bar{x})$ such that $y^k \notin F(x^k) + \varepsilon\varphi(x^k - \bar{x})e + \operatorname{int} D$. For $w^k \in F(x^k)$, we have $y^k \notin w^k + \varepsilon\varphi(x^k - \bar{x})e + \operatorname{int} D$. As $\{y^k\}$ and $\{w^k\}$ are bounded, after dividing by $\varphi(x^k - \bar{x})$ and taking the limit, we obtain $-\varepsilon e \in \operatorname{cl}(\mathbb{R}^m \setminus \operatorname{int} D) = \mathbb{R}^m \setminus \operatorname{int} D$, a contradiction. □

It is worth pointing out that (b)–(ii) implies (a)–(iii) by (8) whenever F is D-closed-valued.

In what follows, in order to have consistent notions of generalized ε-quasi ℓ-solutions, we consider:

Assumption 1: $G \subset \mathbb{R}^m$ is a nonempty set such that $G \cap (-D) = \emptyset$.

As in [18, Lemma 3.1], we capture generalized proper ε-quasi ℓ-solutions by using a family of Hening dilating cones defined as follows: Let B be a compact base of D; i.e., $B \subset \mathbb{R}^m$ is a compact convex set satisfying $0 \notin B$ and $\operatorname{cone} B = D$ (such a set exists, see [12]) and $c = d(0, B) > 0$, we define

$$D_\gamma := \operatorname{cone}(B + \gamma\mathbb{B}), \text{ for } \gamma \in (0, c).$$

Each set D_γ is a solid pointed closed convex cone, $D = \cap_{\gamma \in (0,c)} D_\gamma$ and $D\setminus\{0\} \subset D_\gamma \setminus \{0\} \subset \operatorname{int} D_{\gamma'}$ for $0 < \gamma < \gamma' < c$. From this and Proposition 2.1(b), we have $\lim_k D_{\gamma_k} = D$ for every $\gamma_k \searrow 0$.

Proposition 3.8 *Consider problem* $(\mathcal{P})$, $\mathcal{A} = (G, \varphi)$, $\varepsilon \geq 0$ *and* $\gamma_k \in (0, c) \searrow 0$.

(a) $\lim_k \ell\text{-SWE}(\mathcal{P}, D_{\gamma_k}, \mathcal{A}, \varepsilon) = \operatorname{cl} \bigcup_k \ell\text{-SWE}(\mathcal{P}, D_{\gamma_k}, \mathcal{A}, \varepsilon) \subset \operatorname{cl} \ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$.

(b) *If F is K-closed and K-proper valued for any solid pointed convex cone $K \subset \mathbb{R}^m$ such that $D\setminus\{0\} \subset \operatorname{int} K$, then*

$$\Big(\bigcup\nolimits_k \ell\text{-E}(\mathcal{P}, D_{\gamma_k}, \mathcal{A}, \varepsilon)\Big) \cap \ell\text{-E}(\mathcal{P}, D, \mathcal{A}, \varepsilon) \subset \ell\text{-PE}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$$
$$\subset \Big(\bigcup\nolimits_k \ell\text{-E}(\mathcal{P}, D_{\gamma_k}\setminus\{0\}, \mathcal{A}, \varepsilon)\Big) \cap \ell\text{-E}(\mathcal{P}, D, \mathcal{A}, \varepsilon).$$

(c) $\limsup_k (\ell\text{-E}(\mathcal{P}, D_{\gamma_k}, \mathcal{A}, \varepsilon) \cap \ell\text{-E}(\mathcal{P}, D, \mathcal{A}, \varepsilon)) \subset \operatorname{cl} \ell\text{-PE}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$.

Proof (a) As $\{\ell\text{-SWE}(\mathcal{P}, D_{\gamma_k}, \mathcal{A}, \varepsilon)\}$ is nondecreasing by Proposition 3.5(d), we have $\lim_k \ell\text{-SWE}(\mathcal{P}, D_{\gamma_k}, \mathcal{A}, \varepsilon) = \operatorname{cl} \cup_k \ell\text{-SWE}(\mathcal{P}, D_{\gamma_k}, \mathcal{A}, \varepsilon)$ by Proposition 2.1(a). The inclusion follows also from Proposition 3.5(d).

(b) Inclusion $\big(\bigcup_k \ell\text{-E}(\mathcal{P}, D_{\gamma_k}, \mathcal{A}, \varepsilon)\big) \cap \ell\text{-E}(\mathcal{P}, D, \mathcal{A}, \varepsilon) \subset \ell\text{-PE}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$ is a direct consequence of the definitions, since $D\setminus\{0\} \subset \operatorname{int} D_{\gamma_k}$ for all k. We prove the second inclusion. If $\bar{x} \in \ell\text{-PE}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$, then $\bar{x} \in \ell\text{-E}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$ and there exists a solid pointed convex cone $K \subset \mathbb{R}^m$ such that $D\setminus\{0\} \subset \operatorname{int} K$, and $\bar{x} \in \ell\text{-E}(\mathcal{P}, K, \mathcal{A}, \varepsilon)$. In the proof of [18, Lemma 3.1] it is shown that for such a cone K there exists

$\gamma_k \in (0, c)$ such that $D_{\gamma_k}\backslash\{0\} \subset \operatorname{int} K$. From this, by Lemma 3.3, we obtain $\bar{x} \in \ell\text{-E}(\mathcal{P}, K, \mathcal{A}, \varepsilon) \subset \ell\text{-E}(\mathcal{P}, D_{\gamma_k}\backslash\{0\}, \mathcal{A}, \varepsilon)$ and the second inclusion follows.

(c) The inclusion is an obvious consequence of (b). □

Remark 3.9 *It is worth pointing out that the above properties can be written in terms of limits of solution maps. For instance, part* (a) *can be written as:*

$$\limsup_{\gamma\searrow 0} \ell\text{-SWE}(\mathcal{P}, D_\gamma, \mathcal{A}, \varepsilon) \subset \operatorname{cl} \bigcup_{\gamma\in(0,c)} \ell\text{-SWE}(\mathcal{P}, D_\gamma, \mathcal{A}, \varepsilon) \subset \operatorname{cl} \ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon).$$

Indeed, we have

$$\begin{aligned} \limsup_{\gamma\searrow 0} \ell\text{-SWE}(\mathcal{P}, D_\gamma, \mathcal{A}, \varepsilon) &= \bigcup_{\{\gamma_k\searrow 0\}} \operatorname{cl} \bigcup_k \ell\text{-SWE}(\mathcal{P}, D_{\gamma_k}, \mathcal{A}, \varepsilon) \\ &\subset \operatorname{cl} \bigcup_{\{\gamma_k\searrow 0\}} \bigcup_k \ell\text{-SWE}(\mathcal{P}, D_{\gamma_k}, \mathcal{A}, \varepsilon) \\ &= \operatorname{cl} \bigcup_{\gamma\in(0,c)} \ell\text{-SWE}(\mathcal{P}, D_\gamma, \mathcal{A}, \varepsilon) \\ &\subset \operatorname{cl} \ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon). \end{aligned}$$

We write the above properties in terms of sequences in order to highlight the limit behavior of the approximations.

We study the behavior of generalized ε-quasi ℓ-solutions when $\varepsilon_k \searrow \varepsilon$.

Theorem 3.10 *Consider problem* ($\mathcal{P}$), $\mathcal{A} = (G, \varphi)$ *with* $G \subset D\backslash\{0\}$, $\varepsilon \geq 0$ *and* $\varepsilon_k \searrow \varepsilon$.

(a) $\operatorname{cl} \ell\text{-I}(\mathcal{P}, D, \mathcal{A}, \varepsilon) \subset \lim_k \ell\text{-I}(\mathcal{P}, D, \mathcal{A}, \varepsilon_k) = \bigcap_k \operatorname{cl} \ell\text{-I}(\mathcal{P}, D, \mathcal{A}, \varepsilon_k)$ *for* $\mathrm{I} \in \{\mathrm{E}, \mathrm{SWE}, \mathrm{PE}\}$.

(b) *If* F *is compact-valued, then* $\ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon) = \bigcap_k \ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon_k)$.

(c) *If* F *is osc, locally bounded and* C *is closed, then*

$$\lim_k \ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon_k) = \ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon).$$

Proof (a) As $\{\ell\text{-I}(\mathcal{P}, D, \mathcal{A}, \varepsilon_k)\}$ is nonincreasing by Proposition 3.5(a), we have $\lim_k \ell\text{-I}(\mathcal{P}, D, \mathcal{A}, \varepsilon_k) = \bigcap_k \operatorname{cl} \ell\text{-I}(\mathcal{P}, D, \mathcal{A}, \varepsilon_k)$ by Proposition 2.1(b). The inclusion follows again from Proposition 3.5(a).

(b) The left-side set is contained in the right-side one by Proposition 3.5(a). We prove the reverse inclusion. On the contrary, if there exists $\bar{x}$ in the right-hand side set but not in the left-side one, then $F(\bar{x}) \subset F(x) + \varepsilon\varphi(x - \bar{x})e + \operatorname{int} D$ for some $x \in C$ and $e \in G$; i.e., $F(\bar{x}) - \varepsilon\varphi(x - \bar{x})e \subset \operatorname{int}(F(x) + D)$. As the left-side set is compact, there exists $\delta > 0$ such that $F(\bar{x}) - \varepsilon\varphi(x - \bar{x})e + \delta\mathbb{B} \subset \operatorname{int}(F(x) + D)$. From this, we obtain $F(\bar{x}) - \varepsilon_k\varphi(x - \bar{x})e \subset \operatorname{int}(F(x) + D)$ for some k, i.e., $F(\bar{x}) \subset F(x) + \varepsilon_k\varphi(x - \bar{x})e + \operatorname{int} D$, a contradiction.

(c) This follows from (a)–(b) and Proposition 3.6, since F is compact-valued by Proposition 2.4(d) and F is ℓ-osc by Proposition 2.6(b). □

4 Existence results

To study the existence of generalized ε-quasi ℓ-solutions of problem $(\mathcal{P})$, we extend to set-valued maps the approach used in [25] based on linear scalarizations. To do this, we use the scalarization function $F_\xi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ defined for F and $\xi \in D^+\backslash\{0\}$ by

$$F_\xi(x) := \inf_{y \in F(x)} \langle \xi, y \rangle, \quad \text{for all } x \in \mathbb{R}^n$$

(we set $\inf_\emptyset \langle \xi, y \rangle = +\infty$). In [21] it is reported that this function was introduced by Dien in [7]. It is easy to check that $\operatorname{dom} F_\xi = \operatorname{dom} F$ and $F_\xi(x) = -\sigma_{F(x)}(-\xi)$ for all x. From the latter and by properties of support functions (see [5, Proposition 1.3.2]), we obtain that for a fixed x the function $\xi \mapsto F_\xi(x)$ is usc, positively homogeneous and superadditive and $(F_D)_\xi = F_\xi$. If $\overline{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is the map such that $\overline{F}(x) := \operatorname{cl} F(x)$ for all x, then $\overline{F}_\xi = F_\xi$.

We obtain other properties of the scalarization function to be used later. Part (a) is a direct consequence of [32, Proposition 3.4]. Part (b) extends [28, Lemma 3.18(2)] where the map F is assumed to be osc and locally bounded. It appears also in [29, Proposition 2.3] under upper D-continuity.

Proposition 4.1 *Consider F and $\xi \in D^+\backslash\{0\}$. Then*

(a) *If F is ℓ-bounded from below, then $\inf_{\mathbb{R}^n} F_\xi > -\infty$.*

(b) *If F is ℓ-osc and locally bounded, then F_ξ is lsc.*

(c) *If F is ℓ-convex, then F_ξ is convex.*

Proof (a) By hypothesis $F(x) \subset b + D$ for all $x \in \mathbb{R}^n$ for some $b \in \mathbb{R}^m$. From this, we obtain $F_\xi(x) \geq \inf_{d \in D} \langle \xi, b + d \rangle \geq \langle \xi, b \rangle$ for all $x \in \mathbb{R}^n$.

(b) Let $x^k \to x$ and $\alpha := \liminf_k F_\xi(x^k)$. Clearly, $\alpha \in \mathbb{R}$ as F is locally bounded. As $\alpha_n := \inf_{k \geq n} F_\xi(x^k) \leq \alpha$ for all n, we can choose a subsequence $\{x^{k_n}\}_n$ such that $F_\xi(x^{k_n}) < \alpha + \frac{1}{n}$ for all n. As $F_\xi(x^{k_n}) = \inf_{y \in F(x^{k_n})} \langle \xi, y \rangle$, there exists $y^{k_n} \in F(x^{k_n})$ such that $\langle \xi, y^{k_n} \rangle < \alpha + \frac{1}{n}$. The sequence $\{y^{k_n}\}_n$ is bounded by the local boundedness of F. Hence $y^{k_n} \to y$ for some y, up to subsequences. As $y^{k_n} \in F(x^{k_n})$, we have $y \in \limsup_k (F(x^k) + D)$ that by the ℓ-outer semicontinuity of F implies $y \in F(x) + D$. After taking the limit to the last inequality, we obtain $\langle \xi, y \rangle \leq \alpha$. As $y = \tilde{y} + d$ for $\tilde{y} \in F(x)$ and $d \in D$, we have $F_\xi(x) \leq \langle \xi, \tilde{y} \rangle \leq \langle \xi, y \rangle \leq \alpha$. Hence $F_\xi(x) \leq \alpha$ and we are done.

(c) See [29, Proposition 2.2(a)]. □

We show that some ε-quasi solutions of the scalarization function are generalized ε-quasi ℓ-solutions of problem $(\mathcal{P})$. To this end, for a given fixed scalar $\varepsilon > 0$, we define the set of (ε, φ)-quasi solutions of F_ξ on C by

$$(\varepsilon, \varphi)\text{-}\arg\min_C F_\xi := \{\bar{x} \in C : F_\xi(\bar{x}) \leq F_\xi(x) + \varepsilon\varphi(x - \bar{x}), \, \forall x \in C\}.$$

We denote also $\tau_G(\eta) := \inf_{e\in G}\langle \eta, e\rangle$ for all $\eta \in \mathbb{R}^m$.

We now prove that certain (ε, φ)-quasi solutions of F_ξ on C are generalized ε-quasi ℓ-solutions of problem $(\mathcal{P})$. To this end, as $F_\xi(x) = -\sigma_{F(x)}(-\xi)$, we use properties of support functions (see [5, Proposition 1.3.3]).

Proposition 4.2 *Consider problem $(\mathcal{P})$ with F being D-bounded-valued from below, $\mathcal{A} = (G, \varphi)$ and $\varepsilon > 0$. Then*

$$\bigcup_{\substack{\delta\in[0,\varepsilon)\\ \xi\in D^+\backslash\{0\},\tau_G(\xi)>0}} (\delta\tau_G(\xi), \varphi)\text{-}\arg\min_C F_\xi \subset \ell\text{-}\mathrm{E}(\mathcal{P}, D, \mathcal{A}, \varepsilon).$$

Proof Consider $\xi \in D^+\backslash\{0\}$ such that $\tau_G(\xi) > 0$ and $\delta \in [0, \varepsilon)$. If $\bar{x} \notin \ell\text{-}\mathrm{E}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$, then we have $F(\bar{x}) \subset F(x) + \varepsilon\varphi(x - \bar{x})e + D$ and $F(\bar{x}) + D \neq F(x) + \varepsilon\varphi(x - \bar{x})e + D$ for some $x \in C$ and $e \in G$. The latter implies that $\bar{x} \neq x$ if $\varphi(0) = 0$. By properties of support functions, we have $\sigma_{F(\bar{x})}(\eta) \leq \sigma_{F(x)+\varepsilon\varphi(x-\bar{x})e+D}(\eta) = \sigma_{F(x)}(\eta) + \varepsilon\varphi(x - \bar{x})\langle\eta, e\rangle + \delta_{D^-}(\eta)$ for all $\eta \in \mathbb{R}^m$. For every $\xi \in D^+\backslash\{0\}$, we have $-\xi \in -D^+\backslash\{0\} = D^-\backslash\{0\}$ and from above we obtain $F_\xi(\bar{x}) \geq F_\xi(x)+\varepsilon\varphi(x-\bar{x})\langle\xi, e\rangle$. We have $\varepsilon\varphi(x - \bar{x})\langle\xi, e\rangle \geq \varepsilon\tau_G(\xi)\varphi(x - \bar{x}) > \delta\tau_G(\xi)\varphi(x - \bar{x})$ for $0 \leq \delta < \varepsilon$ since $\tau_G(\xi)\varphi(x - \bar{x}) > 0$. As F is D-bounded-valued from below, we have $F_\xi(x) > -\infty$ and then $F_\xi(\bar{x}) > F_\xi(x) + \delta\tau_G(\xi)\varphi(x - \bar{x})$; i.e., $\bar{x} \notin (\delta\tau_G(\xi), \varphi)\text{-}\arg\min_C F_\xi$ and $\bar{x}$ is not in the left-hand side set. □

Corollary 4.3 *Consider problem $(\mathcal{P})$ with F being D-bounded-valued from below, $\mathcal{A}=(G, \varphi)$ with G compact and $\varepsilon>0$.*

(*a*) $$\bigcup_{\xi\in D^+\backslash\{0\},\, \delta\in[0,\varepsilon)} (\delta\tau_G(\xi), \varphi)\text{-}\arg\min_C F_\xi \subset \ell\text{-}\mathrm{E}(\mathcal{P}, D, \mathcal{A}, \varepsilon), \text{ if } G \subset \operatorname{int} D.$$

(*b*) $$\bigcup_{\xi\in \operatorname{int} D^+,\, \delta\in[0,\varepsilon)} (\delta\tau_G(\xi), \varphi)\text{-}\arg\min_C F_\xi \subset \ell\text{-}\mathrm{E}(\mathcal{P}, D, \mathcal{A}, \varepsilon), \text{ if } G \subset D\backslash\{0\}.$$

Proof As G is compact, we have $\tau_G(\xi) > 0$ when $\xi \in D^+\backslash\{0\}$ and $G \subset \operatorname{int} D$, or when $\xi \in \operatorname{int} D^+$ and $G \subset D\backslash\{0\}$. □

From Proposition 4.2 and Corollary 4.3, we see that to prove the existence of generalized ε-quasi ℓ-solutions of problem $(\mathcal{P})$, one can prove the existence of (ε, φ)-quasi solutions of F_ξ. To do this, we recall an extension of the Ekeland variational principle due to Qiu [31]. We write it in our finite dimensional framework. Let $\psi : [0, +\infty) \to [0, +\infty)$ be a subadditive ($\psi(s+t) \leq \psi(s)+\psi(t)$ for all $s, t \in [0, +\infty)$), nondecreasing, continuous function such that $\psi(0) = 0$ and $\psi(t) > 0$ for all $t > 0$. Let Ψ denote the class of all such functions. Some functions from this class are: $\psi(t) = \alpha t$, $\psi(t) = \alpha t/(1 + \beta t)$, $\psi(t) = \alpha \sqrt[n]{t}$, $\psi(t) = \alpha \ln(1 + \beta t)$, and $\psi(t) = \alpha \arctan \beta t$ where $n \in \mathbb{N}$, $\alpha > 0$ and $\beta > 0$ are constants (see [30]).

Proposition 4.4 [31] *If $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ is a proper, lsc, bounded from below function and $\psi\in\Psi$, then for any $x^0\in\operatorname{dom} f$ there exists $\bar{x}\in\mathbb{R}^n$ such that*

$$f(\bar{x})+\psi(\|x^0-\bar{x}\|)\le f(x^0) \text{ and } f(\bar{x})<f(x)+\psi(\|x-\bar{x}\|),\ \forall x\in\mathbb{R}^n,\ x\ne\bar{x}.$$

We now obtain an existence result of generalized ε-quasi ℓ-solutions via the above scalarization and the Ekeland variational principle.

Theorem 4.5 *Consider problem $(\mathcal{P})$, $\mathcal{A}=(G,\varphi)$ with $\varphi(\cdot):=\psi(\|\cdot\|)$ where $\psi\in\Psi$ and G is closed with $G\subset D\backslash\{0\}$ or G is convex with $0\notin\operatorname{cl}G$ and $D\subset\operatorname{cone}G$. If F is locally bounded, ℓ-osc, ℓ-bounded from below, then $\ell\text{-E}(\mathcal{P},D,\mathcal{A},\varepsilon)\ne\emptyset$, for all $\varepsilon>0$.*

Proof If $\xi\in D^+\backslash\{0\}$, then F_ξ is proper, lsc, bounded from below by Proposition 4.1. When G is closed with $G\subset D\backslash\{0\}$, we claim that $\tau_G(\xi)>0$ whenever $\xi\in\operatorname{int}D^+$. Indeed, otherwise there exists $\{e^k\}\subset G$ such that $\langle\xi,e^k\rangle\to 0_+$. If $\{e^k\}$ is bounded, then $e^k\to\bar{e}$ for some $\bar{e}$, up to subsequences; thus, $\langle\xi,\bar{e}\rangle=0$. As G is closed, we have $\bar{e}\in G$ and thus $\langle\xi,\bar{e}\rangle>0$, a contradiction. On the other hand, if $\{e^k\}$ is unbounded, then we may consider that $\|e^k\|\to+\infty$. Hence $e^k/\|e^k\|\to\bar{e}$ for some $\bar{e}\ne0$, up to subsequences. Clearly, $\langle\xi,\bar{e}\rangle=0$. As $e^k/\|e^k\|\in D$ and D is closed, we have $\bar{e}\in D\backslash\{0\}$ and thus $\langle\xi,\bar{e}\rangle>0$, a contradiction. By applying Proposition 4.4 to $f=F_\xi$ with $\xi\in\operatorname{int}D^+$ and $\delta\tau_G(\xi)\psi$ instead of ψ where $\delta\in(0,\varepsilon)$ (notice that $\delta\tau_G(\xi)\psi\in\Psi$ since $\delta\tau_G(\xi)>0$), we have $(\delta\tau_G(\xi),\varphi)\text{-}\operatorname{arg\,min}_C F_\xi\ne\emptyset$. Hence $\ell\text{-E}(\mathcal{P},D,\mathcal{A},\varepsilon)\ne\emptyset$ by Proposition 4.2.

When G is convex with $0\notin\operatorname{cl}G$ and $D\subset\operatorname{cone}G$, by the separation theorem there exists $\bar{\xi}\in\operatorname{int}D^+$ such that $\tau_G(\bar{\xi})>0$. Then, as in the previous part, the result follows by applying Proposition 4.4 to $f=F_{\bar{\xi}}$ and $\delta\tau_G(\bar{\xi})\psi$ instead of ψ where $\delta\in(0,\varepsilon)$. ☐

Remark 4.6 *It is worth pointing out that under the hypothesis of Theorem 4.5, we have $\ell\text{-SWE}(\mathcal{P},D,\mathcal{A},\varepsilon)\ne\emptyset$, for all $\varepsilon>0$ by (8) and Proposition 2.6(a).*

We obtain properties of ℓ-boundedness from below that complement those in Proposition 2.2. To do this, we recall the notion of asymptotic map of F that is a map, denoted by F^∞, such that $\operatorname{epi}F^\infty:=(\operatorname{epi}F)^\infty$ (see [20, Definition 6]). Part (a) extends a necessary condition in scalar minimization from [2, Proposition 15.1.2].

Lemma 4.7 (a) *If F is ℓ-bounded from below, then $0\le_D^\ell F^\infty(v)$ for all v; hence F^∞ is also ℓ-bounded from below.*

(b) *If $F(\mathbb{R}^n)^\infty\cap\{\xi\}^-=\{0\}$ for all $\xi\in D^+\backslash\{0\}$, then F is ℓ-bounded from below.*

Proof (a) By hypothesis $F(\mathbb{R}^n)\subset b+D$ for some b. If $w\in F^\infty(v)$, then $(v,w)\in\operatorname{epi}F^\infty=(\operatorname{epi}F)^\infty$ and there exist $t_k\to+\infty$ and $(x^k,y^k)\in\operatorname{epi}F$ such that

$(1/t_k)(x^k, y^k) \to (v, w)$. As $y^k \in F(x^k) + D$, we have $\frac{y^k - b}{t_k} \in D$ and after taking the limit, we obtain $w \in D$. As w was arbitrary, we have $F^\infty(v) \subset D$.

(b) On the contrary, suppose that F is not ℓ-bounded from below. Hence for $q \in \operatorname{int} D$ being fixed, for every $k \in \mathbb{N}$ there exists $x^k \in \operatorname{dom} F$ such that $F(x^k) \not\subset -kq + D$; thus, there exists $y^k \in F(x^k)$ such that $y^k \notin -kq + D$. By a separation theorem (see, for instance, [34, Theorem 2.39]), for each k, there exists $\xi^k \in \mathbb{R}^m \backslash \{0\}$ such that $\langle \xi^k, y^k \rangle < \langle \xi^k, -kq + d \rangle$ for all $d \in D$. From this, we deduce that $\xi^k \in D^+ \backslash \{0\}$ and since we may consider that $\|\xi^k\| = 1$, we have $\xi^k \to \xi$ for some $\xi \in D^+ \backslash \{0\}$, up to subsequences. If $\{y^k\}$ is bounded, then dividing the last inequality by k and taking the limit, we obtain $\langle \xi, q \rangle \leq 0$, a contradiction to (2). Hence we may consider that $\|y^k\| \to +\infty$; thus, $y^k / \|y^k\| \to w$ for some $w \neq 0$, up to subsequences and we have $w \in F(\mathbb{R}^n)^\infty$. As $\langle \xi^k, y^k \rangle < 0$ for all k, dividing by $\|y^k\|$ and taking the limit, we obtain $\langle \xi, w \rangle \leq 0$; i.e., $w \in \{\xi\}^-$, a contradiction. □

We study the boundedness of the set of generalized strict weak ε-quasi ℓ-solutions of $(\mathcal{P})$. To do this, we obtain a bound for the asymptotic cone of this set. We recall an "asymptotic compactness condition" (see [20, Definition 7] and the references therein). A map F satisfies condition (CR1) if for every sequences $t_k \to +\infty$, $v^k \to v$ with $v \neq 0$, $t_k v^k \in C$ and $z^k \in F(t_k v^k)$ for all k, the sequence $\{z^k / t_k\}$ has a convergent subsequence. We assume the convention $\{+\infty \cdot e\} = \emptyset$.

Proposition 4.8 *Consider problem* $(\mathcal{P})$, $\mathcal{A} = (G, \varphi)$ *with* $G \subset D \backslash \{0\}$ *and* $\varepsilon > 0$. *If* F *satisfies condition* (CR1)*, then*

$$\ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon)^\infty \subset \bigcap_{e \in G} \{v \in \mathbb{R}^n : -\varepsilon(-\varphi)^\infty(-v)e \not\ll^\ell_D F^\infty(v)\}. \tag{10}$$

Proof First, we claim that $v = 0$ belongs to the right-hand side set. Indeed, as $(-\varphi)^\infty(0)$ is equal to 0 or $-\infty$ by [5, Proposition 2.5.1], it follows that $\{-\varepsilon(-\varphi)^\infty(0)e\}$ is equal to $\{0\}$ or $\{+\infty \cdot e\} = \emptyset$. Since $0 \in F^\infty(0) + D$ we have that $\emptyset \neq F^\infty(0) + D \not\subset \operatorname{int} D$, and so $-\varepsilon(-\varphi)^\infty(0)e \not\ll^\ell_D F^\infty(0)$.

If $v \neq 0$ belongs to the left-hand side set, then there exist $t_k \to +\infty$ and $x^k \in \ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$ such that $x^k / t_k \to v$ (we have $\|x^k\| \to +\infty$, up to subsequences). Hence $F(x^k) \not\subset F(x) + \varepsilon\varphi(x - x^k)e + \operatorname{int} D$ for fixed $x \in C$ and $e \in G$; thus, for a fixed $y \in F(x)$, we have $F(x^k) \not\subset y + \varepsilon\varphi(x - x^k)e + \operatorname{int} D$; i.e.,

$$\exists y^k \in F(x^k) \text{ such that } y^k \notin y + \varepsilon\varphi(x - x^k)e + \operatorname{int} D \text{ for all } k. \tag{11}$$

By condition (CR1), we have $y^k / t_k \to w$ for some w, up to subsequences. From this and since $(x^k, y^k) \in \operatorname{epi} F$, we obtain $(v, w) \in (\operatorname{epi} F)^\infty = \operatorname{epi} F^\infty$; thus, $w \in F^\infty(v) + D$. Dividing the right-hand side part in (11) by t_k and after taking the limit of the subsequence that yields the limsup for the expression with φ, we obtain

$$-w + \varepsilon(\limsup_k \frac{1}{t_k}\varphi(x - x^k))e \in \mathbb{R}^m \backslash (-\operatorname{int} D) \tag{12}$$

whenever $\limsup_k \frac{1}{t_k}\varphi(x - x^k) \in \mathbb{R}$.

By the formula for the asymptotic function in [5, Theorem 2.5.1], we obtain

$$\limsup_k \frac{\varphi(x - x^k)}{t_k} = -\liminf_k \frac{-\varphi(t_k \frac{x - x^k}{t_k})}{t_k} \leq -(-\varphi)^\infty(-v).$$

If $(-\varphi)^\infty(-v) = -\infty$, then $\{-\varepsilon(-\varphi)^\infty(-v)e\}$ is equal to $\{+\infty \cdot e\} = \emptyset$ and the result follows since $w \in F^\infty(v) + D$ (and so $F^\infty(v) \neq \emptyset$). Otherwise, $\limsup_k \frac{\varphi(x-x^k)}{t_k}$ and $(-\varphi)^\infty(-v)$ are real numbers and we have

$$\varepsilon\Big[-(-\varphi)^\infty(-v) - \limsup_k \frac{1}{t_k}\varphi(x-x^k)\Big]e \in D, \tag{13}$$

since we have $G \subset D\backslash\{0\}$. After adding expressions (12) and (13), we obtain that $-w - \varepsilon(-\varphi)^\infty(-v)e \in \mathbb{R}^m\backslash(-\operatorname{int} D)$; i.e., $w \notin -\varepsilon(-\varphi)^\infty(-v)e + \operatorname{int} D$. Therefore, $F^\infty(v) + D \not\subset -\varepsilon(-\varphi)^\infty(-v)e + \operatorname{int} D$ and v is in the right-hand side set. □

Corollary 4.9 *Under the hypotheses of Proposition 4.8, if*

$$\bigcap_{e\in G}\{v \in \mathbb{R}^n : -\varepsilon(-\varphi)^\infty(-v)e \not\leqslant^\ell_D F^\infty(v)\} = \{0\}, \tag{14}$$

then ℓ-SWE$(\mathcal{P}, D, \mathcal{A}, \varepsilon)$ *is bounded.*

Proof By Proposition 4.8, we have ℓ-SWE$(\mathcal{P}, D, \mathcal{A}, \varepsilon)^\infty = \{0\}$ that implies the boundeness of the solution set. □

Remark 4.10 1. *If* $\varphi(x) = 1$ (*resp.* $\varphi(x) = \|x\|$) *for all* x, *then* $(-\varphi)^\infty(v) = 0$ (*resp.* $(-\varphi)^\infty(v) = -\|v\|$) *for all* v.

2. *The bound* (10) *with* $\varepsilon = 0$ *appears in* [20, Corollary 3] *under condition* (CR1)*; i.e.,* ℓ-SWE$(\mathcal{P}, D)^\infty \subset \{v \in \mathbb{R}^n : 0 \not\leqslant^\ell_D F^\infty(v)\}$.

3. *Clearly, condition* (14) *holds if there exists some vector* $e \in G$ *such that* $\{v \in \mathbb{R}^n : -\varepsilon(-\varphi)^\infty(-v)e \not\leqslant^\ell_D F^\infty(v)\} = \{0\}$.

5 Optimality conditions

We establish optimality conditions for generalized strict weak ε-quasi ℓ-solutions of $(\mathcal{P})$. To do this, we recall some notions from [1, 21]. For a nonempty set $A \subset \mathbb{R}^m$, we denote by $\operatorname{IMin} A := \{a \in A : a \leq_D a' \text{ for all } a' \in A\}$ its ideal minimal point and by $T(A, \bar{x}) := \{u\colon \exists t_k \to 0_+, \exists u^k \to u : \bar{x} + t_k u^k \in A, \forall k\}$ its contingent cone at $\bar{x} \in \operatorname{cl} A$. If A is convex, then $T(A, \bar{x}) = \operatorname{cl}\operatorname{cone}(A - \bar{x})$. The contingent derivative of F at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is a set-valued map $DF(\bar{x}, \bar{y}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ such that $\operatorname{gph}(DF(\bar{x}, \bar{y})) = T(\operatorname{gph} F, (\bar{x}, \bar{y}))$. By [21, Theorem 11.3.7], we have $DF(\bar{x}, \bar{y})(u) + D \subset DF_D(\bar{x}, \bar{y})(u)$ for all u.

The Dini-Hadamard upper (resp. right-sided) directional derivative of φ at $\bar{x}$ in the direction v is defined by

$$\varphi^{DH+}(\bar{x};\ v) := \limsup_{(t,u)\to(0_+,v)} \frac{\varphi(\bar{x}+tu) - \varphi(\bar{x})}{t}$$

$$\Big(\text{resp. } \varphi'_+(\bar{x};\ v) := \lim_{t\to 0_+} \frac{\varphi(\bar{x}+tv) - \varphi(\bar{x})}{t}\Big).$$

Next, we assume $C = \operatorname{dom} F$ (otherwise, we redefine the objective map by $F(x) = \emptyset$, for all $x \notin C$, resulting in an equivalent formulation of the problem).

Theorem 5.1 *Consider problem* $(\mathcal{P})$, $\mathcal{A} = (G, \varphi)$ *with* $G \subset D\backslash\{0\}$ *and* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ *such that* $\operatorname{dom} DF_D(\bar{x}, \bar{y}) \neq \emptyset$.

(*a*) *Assume* $\varphi(0) = 0$. *If* $\bar{x} \in \ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$ *and* $\operatorname{IMin} F(\bar{x}) = \{\bar{y}\}$, *then*

$$DF_D(\bar{x}, \bar{y})(u) + \varepsilon\varphi^{DH+}(0;\ u)G \subset \mathbb{R}^m\backslash(-\operatorname{int} D),\ \forall u \in T(C, \bar{x}). \tag{15}$$

(*b*) *If* F *is* ℓ*-convex,* φ *is convex and*

$$DF_D(\bar{x}, \bar{y})(u) + \varepsilon\varphi'_+(0;\ u)G \subset \mathbb{R}^m\backslash(-\operatorname{int} D),\ \forall u \in T(C, \bar{x}), \tag{16}$$

then $\bar{x} \in \ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$.

Proof (*a*) Clearly, condition (15) holds when $\varphi^{DH+}(0;\ u) = +\infty$ (note that $\varphi^{DH+}(0;\ u) \in [0, +\infty]$ since $\varphi(0) = 0$) or $u \in T(C, \bar{x})\backslash\operatorname{dom} DF_D(\bar{x}, \bar{y})$ (recall that $\operatorname{dom} DF_D(\bar{x}, \bar{y}) \subset T(C, \bar{x})$).

Let $u \in \operatorname{dom} DF_D(\bar{x}, \bar{y})$ and $\varphi^{DH+}(0;\ u)$ be finite. If $v \in DF_D(\bar{x}, \bar{y})(u)$, then there exist $t_k \to 0_+$ and $(u^k, v^k) \to (u, v)$ such that $\bar{y} + t_k v^k \in F(\bar{x} + t_k u^k) + D$ for all k (see [21, Proposition 11.1.8]). As $\bar{x} \in \ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$ and $\bar{x} + t_k u^k \in C$ for all k, we have $F(\bar{x}) \not\subset F(\bar{x} + t_k u^k) + \varepsilon\varphi(t_k u^k)e + \operatorname{int} D$ for $e \in G$. Hence $F(\bar{x}) \not\subset \bar{y} + t_k v^k + \varepsilon\varphi(t_k u^k)e + \operatorname{int} D$ for such k. From this and $F(\bar{x}) \subset \bar{y} + D$, we obtain $\bar{y} + D \not\subset \bar{y} + t_k v^k + \varepsilon\varphi(t_k u^k)e + \operatorname{int} D$; thus, $t_k v^k + \varepsilon\varphi(t_k u^k)e \notin -\operatorname{int} D$ and so

$$v^k + \varepsilon\Big(\frac{\varphi(0 + t_k u^k) - \varphi(0)}{t_k}\Big)e \in \mathbb{R}^m\backslash(-\operatorname{int} D),\ \text{for such } k.$$

Moreover, as

$$\varepsilon\Big(\sup_{j \geq k} \frac{\varphi(0 + t_j u^j) - \varphi(0)}{t_j} - \frac{\varphi(0 + t_k u^k) - \varphi(0)}{t_k}\Big)e \in D,$$

after adding these expressions, we have

$$v^k + \varepsilon\Big(\sup_{j \geq k} \frac{\varphi(0 + t_j u^j) - \varphi(0)}{t_j}\Big)e \in \mathbb{R}^m\backslash(-\operatorname{int} D),\ \text{for such } k.$$

After taking the limit, we obtain

$$v + \varepsilon\Big(\limsup_k \frac{\varphi(0 + t_k u^k) - \varphi(0)}{t_k}\Big)e \in \mathbb{R}^m\backslash(-\operatorname{int} D).$$

As $\varepsilon(\varphi^{DH+}(0;\ u) - \limsup_k \frac{\varphi(0+t_k u^k)-\varphi(0)}{t_k})e \in D$, after adding the last two expressions, we get $v + \varepsilon\varphi^{DH+}(0;\ u)e \in \mathbb{R}^m\backslash(-\operatorname{int} D)$ and condition (15) holds.

(*b*) On the contrary, suppose that condition (16) holds but it holds that $\bar{x} \notin \ell\text{-SWE}(\mathcal{P}, D, \mathcal{A}, \varepsilon)$. Hence there exist $x \in C$ and $e \in G$ such that $F(\bar{x}) \subset F(x) + \varepsilon\varphi(x - \bar{x})e + \operatorname{int} D$. As F is ℓ-convex, we have $F(x) - \bar{y} \subset DF_D(\bar{x}, \bar{y})(x - \bar{x})$ by [21, Corollary 11.1.23]. Moreover, as φ is convex, we have $\varphi(x - \bar{x}) \geq \varphi(0) + \varphi'_+(0;\ x - \bar{x})$; thus, $\varepsilon(\varphi(x - \bar{x}) - \varphi'_+(0;\ x - \bar{x}))e \in D$ since $e \in D\backslash\{0\}$. From these relationships and condition (16) for $u = x - \bar{x}$, we obtain $F(\bar{x}) - \bar{y} \subset \mathbb{R}^m\backslash(-\operatorname{int} D) + \operatorname{int} D$, a contradiction since $0 \in F(\bar{x}) - \bar{y}$. □

Remark 5.2 1. *In part* (*a*) *we can put* F *instead of* F_D. *In such a case the reverse implication holds under the convexity of* F (*see* [21, Theorem 11.1.22]).

2. *Interval-type, box-type and cone-valued-type maps* [9, 19] *are important instances of maps whose images have an ideal element.*

3. [21, Proposition 11.1.9] *Let* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. *If* $\bar{x} \in \operatorname{int} C$ *and* F *is Lipschitz around* $\bar{x}$ *(i.e., there exist* $L > 0$ *and a neighborhood* U *of* $\bar{x}$ *such that* $F(x) \subset F(z) + L\|x - z\|\mathbb{B}$ *for all* $x, z \in U$*), then* $\operatorname{dom} DF(\bar{x}, \bar{y}) = \mathbb{R}^n$.

4. *Theorem 5.1 extends* [26, Propositions 3.1–3.2] *and* [8, Propositions 3.1–3.2] *for scalar functions and is related to* [11, Theorem 4.2] *for vector functions. Point out that if* $\varphi(x) = \|x\|$, *then* $\varphi^{DH+}(0;\ u) = \|u\|$.

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