Exact and Approximate Vector Ekeland Variational Principles.

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ABSTRACT
The paper concerns with exact and approximate Ekeland variational principles for vector-valued functions and bifunctions, that are derived via linear and nonlinear scalarization processes by an approximate scalar formulation of the Ekeland variational principle and a revised version of Dancs-Hegedüs-Medvegyev's fixed point theorem. Both results are also interesting in themselves and involve really mild assumptions. As a result, the obtained Ekeland variational principles improve some recent results in the literature since weaker assumptions are required.

KEYWORDS
Ekeland variational principle; Dancs-Hegedüs-Medvegyev's fixed point theorem; vector optimization; scalarization; strictly decreasing cone lower semicontinuity

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1. Introduction

The celebrated Ekeland variational principle (EVP) is an important mathematical tool with many applications in different fields in nonlinear analysis and optimization theory. The best references for those are given by Ekeland himself: his survey article [11] and his book with J.-P. Aubin [4]. It is well established that Dancs-Hegedüs-Medvegyev's fixed point theorem [9, Theorem 3.1] has served as a significant tool in proving versions of the EVP and their extensions to vector and set optimization; see, for example, [17] and the references therein. Recently, Bao and Théra [6] proposed a revised version of Dancs-Hegedüs-Medvegyev's result which not only unifies the existing generalized versions of this theorem including the one-sequence version in [17, Lemma 3.4] and the preorder principle in [21, Theorem 2.1], but also improves them as weaker assumptions are required.

In [14, Theorem 4.1] Gutiérrez et al. established exact and approximate EVPs for vector equilibrium problems without assuming any topology on the final space of the bifunction of the problem. For this aim, the domination set is assumed to be free-

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disposal. This kind of sets were introduced in mathematical economics by Debreu (see [10]). The main tool is Theorem 3.5, a result for the existence of a kind of strict fixed point of a set-valued mapping.

This paper has two folds: (1) we propose a general approximate scalar EVP and a refined version of Dancs-Hegedüs-Medvegyev's fixed point theorem which would lead to better exact EVPs and (2) we deduce approximate and exact EVPs for vector-valued functions. The obtained results improve other recent ones of the literature since weaker assumptions are required. For instance, the final space of the vector functions is not equipped with any topology and a new lower semicontinuity notion named strictly $\leq_D$-decreasing lower semicontinuity is considered, that generalizes the so-called sequentially submonotonicity (see [7,12,19,20]).

Moreover, the role of some mathematical tools and assumptions usually required on the vector EVPs are clarified. For instance, by Corollary 2.9 we can emphasize that [14, Theorem 3.5] can be viewed as another form of Dancs-Hegedüs-Medvegyev's fixed point theorem.

The paper is organized as follows. In Section 2, the framework and some general notations are presented. In addition, we develop two basic mathematical tools from which the approximate and exact EVPs are obtained: an approximate version of the scalar EVP (see [8, Proposition 2.5]) and a revised version of Dancs-Hegedüs-Medvegyev's fixed point theorem (see, [9, Theorem 3.1]). In Section 3, an approximate vector EVP is stated. Its assumptions are weaker than the ones considered in the previous results of the literature. Section 4 is devoted to a new exact vector EVP based on free-disposal domination sets and its corollaries. It is worth underlining the one based on approximate solutions of the problem, where no lower boundedness assumptions are required. We also prove that the vector EVP is satisfied for strictly decreasing lower-semicontinuous vector-valued functions.

2. Preliminaries

Let $\mathcal{T}$ be a topological space and consider a set $H \subseteq \mathcal{T}$. Throughout this paper, $\operatorname{cl} H$ denotes the closure of H. $\mathbb{R}^n_+$ stands for the nonnegative orthant of $\mathbb{R}^n$ and $\mathbb{R}_+ := \mathbb{R}^1_+$. We assume $0 \in \mathbb{N}$ and (x_n) denotes the sequence $\{x_n : n \in \mathbb{N}\}$.

Let A be an arbitrary nonempty set. Given an extended real function $\varphi : A \to \mathbb{R} \cup \{\pm\infty\}$, $\operatorname{dom} \varphi$ stands for the effective domain of φ; i.e.,

$$\operatorname{dom} \varphi := \{a \in A : \varphi(a) < +\infty\}.$$

The function φ is said to be proper if $\operatorname{dom} \varphi \neq \emptyset$ and $\varphi(a) > -\infty$, for all $a \in A$. As it is usual we assume $\inf_{a \in A} \varphi(a) = +\infty$ whenever $A = \emptyset$. A bifunction $\eta : A \times A \to \mathbb{R}$ is said to satisfy the triangle inequality property if

$$\eta(a_1, a_3) \leq \eta(a_1, a_2) + \eta(a_2, a_3), \quad \forall a_1, a_2, a_3 \in A.$$

Consider a dynamic system $\Phi : A \rightrightarrows A$. Then, $(a_n) \subseteq A$ is called a generalized Picard sequence of Φ if $a_{n+1} \in \Phi(a_n)$, for all $n \in \mathbb{N}$.

The algebraic dual space of a real linear space Y is denoted by Y'. Moreover, $G^+ \subseteq Y'$ (resp. $G^{s+} \subseteq Y'$) stands for the positive (resp. strict positive) polar cone of a nonempty set $G \subseteq Y$; i.e.,

$$G^+ := \{\lambda \in Y' \ : \ \forall y \in G, \lambda(y) \geq 0\}$$
$$\Big(\text{resp. } G^{s+} := \bigcup_{\delta > 0} \{\lambda \in Y' \ : \ \forall y \in G \backslash \{0\}, \lambda(y) \geq \delta\} \cup \{0\}\Big).$$

For each $\lambda \in Y'$, we denote $\tau_G(\lambda) = \inf_{y \in G} \lambda(y)$.

The preference relation defined by a domination set $\emptyset \neq E \subseteq Y$ is denoted by $\leq_E$; i.e.,

$$y_1, y_2 \in Y, \quad y_1 \leq_E y_2 \iff y_2 - y_1 \in E.$$

Then, a function $\xi : Y \to I\!R \cup \{\pm\infty\}$ is said to be nondecreasing with respect to $\leq_E$ ($\leq_E$-nondecreasing for short) if

$$y_1, y_2 \in Y, \quad y_1 \leq_E y_2 \Rightarrow \xi(y_1) \leq \xi(y_2).$$

A bifunction $h : A \times A \to Y$ is said to fulfill the triangle inequality property with respect to $\leq_E$ if

$$h(a_1, a_3) \leq_E h(a_1, a_2) + h(a_2, a_3), \quad \forall a_1, a_2, a_3 \in A.$$

Moreover, h is said to be diagonal null if $h(a, a) = 0$, for all $a \in A$.

The sublevel set of a vector-valued function $g : A \to Y$ at $y \in Y$ with respect to the relation $\leq_E$ is denoted as follows:

$$[g \leq_E y] := \{a \in A : g(a) \leq_E y\}.$$

A set $E \subseteq Y$ is said to be free disposal with respect to a convex cone $D \subseteq Y$ if $E + D = E$. If additionally $0 \notin E$, then E is said to be an improvement set. The generated cone by E is denoted $\operatorname{cone} E$. Moreover, the so-called algebraic interior and recession cone of E are denoted by $\operatorname{core} E$ and 0^+E, respectively, i.e.,

$$\operatorname{core} E := \{y \in Y : \forall v \in Y, \exists \gamma > 0 \text{ s.t. } y + [0, \gamma]v \subseteq E\},$$
$$0^+E := \{v \in Y : y + tv \in E, \forall y \in E, \forall t \in I\!R_+\}.$$

Recall that the vector closure of E in the direction of a vector $q \in Y$ is the set (see [22])

$$\operatorname{vcl}_q E := \{y \in Y : \forall t > 0 \, \exists t' \in [0, t] \text{ s.t. } y + t'q \in E\}.$$

We say that E is algebraic solid (resp. q-vectorial closed) if $\operatorname{core} E \neq \emptyset$ (resp. $\operatorname{vcl}_q E = E$). It is easy to check that $\operatorname{vcl}_q E$ is q-vectorial closed (i.e., $\operatorname{vcl}_q \operatorname{vcl}_q E = \operatorname{vcl}_q E$). The vector closure of E is the set $\operatorname{vcl} E := \bigcup_{q \in Y} \operatorname{vcl}_q E$ (see [1]). We have the next basic properties.

Lemma 2.1. *Suppose that* $\operatorname{cone} E$ *is proper (i.e.,* $\operatorname{cone} E \neq Y$*) and convex and* $q \in \operatorname{core} \operatorname{cone} E$*. Then,* $\operatorname{core} \operatorname{cone} E + \operatorname{vcl}_q \operatorname{cone} E = \operatorname{core} \operatorname{cone} E$ *and* $q \notin -\operatorname{vcl} \operatorname{cone} E$*.*

Proof. The first assertion is a direct consequence of [15, Proposition 18]. For the second one, notice that $\operatorname{vcl}\operatorname{cone} E = \operatorname{vcl}_q \operatorname{cone} E$ (see [22, Proposition 2.3]). Then, if $q \in -\operatorname{vcl}\operatorname{cone} E$ it follows that

$$\begin{aligned} 0 &\in \operatorname{core}\operatorname{cone} E + \operatorname{vcl}\operatorname{cone} E \\ &= \operatorname{core}\operatorname{cone} E + \operatorname{vcl}_q \operatorname{cone} E \\ &= \operatorname{core}\operatorname{cone} E \end{aligned}$$

that is a contradiction, since $\operatorname{cone} E$ is proper. □

In this work, the EVPs concerning with vector-valued functions are stated by linear and nonlinear scalarization. For the second approach, the so-called Gertewitz's scalarization functional $\varphi_E^q : Y \to I\!R \cup \{\pm\infty\}$ is considered. For each $q \in Y\backslash\{0\}$, φ_E^q is defined as follows:

$$\varphi_E^q(y) := \inf\{t \in I\!R : y \in tq - E\}. \tag{1}$$

In the subsequent lemma we provide some basic properties of φ_E^q (see [15, Proposition 2, Theorem 4 and Theorem 8]).

Lemma 2.2. *It follows that*

(i) *If* $q \notin -\operatorname{vcl}\operatorname{cone} E$, *then* $\varphi > -\infty$.
(ii) φ_E^q *is* q*-translative, i.e.,* $\varphi_E^q(y + tq) = \varphi_E^q(y) + t$, *for all* $y \in Y$ *and* $t \in I\!R$.
(iii) $\varphi_E^q(0) > 0$ *if and only if* $0 \notin \operatorname{cone}\{q\} + \operatorname{vcl}_q E$.
(iv) *Consider* $\emptyset \neq C \subseteq Y$. *Then,* φ_E^q *is* $\leq_C$*-nondecreasing if and only if* $E + C \subseteq [0, +\infty)q + \operatorname{vcl}_q E$.

The next two theorems state the basic mathematical tools from which the approximate and exact EVPs of Sections 3 and 4 are obtained. We begin with an approximate version of the scalar EVP (see [8, Proposition 2.5]). Assertion (b) of this result shows the existence of approximate strict solutions of the perturbed problem.

Theorem 2.3 (Approximate scalar EVP)**.** *Let* A *be a nonempty set,* $\varphi : A \to I\!R \cup \{\pm\infty\}$ *be a proper function,* $\eta : A \times A \to I\!R$ *be satisfying the triangle inequality property and* $\tau > 0$. *Assume that* $a_0 \in \operatorname{dom}\varphi$ *and*

$$c := \inf_{a \in S(a_0)} \{\varphi(a) + \eta(a_0, a)\} > -\infty, \tag{2}$$

where

$$S(a_0) := \{a \in A \ : \ \varphi(a) + \eta(a_0, a) \prec \varphi(a_0) - \tau\}$$

and $\prec \in \{\leq, <\}$. *Then, there exists* $\bar{a} \in A$ *such that*

(a) $\varphi(\bar{a}) + \eta(a_0, \bar{a}) \prec \varphi(a_0) - \tau$ *or* $\bar{a} = a_0$,
(b) $\varphi(a) + \eta(\bar{a}, a) \not\prec \varphi(\bar{a}) - \tau$, $\forall a \in A$.

Proof. Consider the set-valued mapping $S : A \rightrightarrows A$ given by

$$S(a) := \{u \in A \ : \ \varphi(u) + \eta(a, u) \prec \varphi(a) - \tau\}.$$

If $S(a_0) = \emptyset$, then $\bar{a} = a_0$ satisfies (a) and (b). Otherwise, we could choose $a_1 \in S(a_0)$. Then, we have $\varphi(a_1) + \eta(a_0, a_1) \prec \varphi(a_0) - \tau$. If $S(a_1) = \emptyset$, we set $\bar{a} = a_1$. We claim that there is a finite number k such that $S(a_k) = \emptyset$. Arguing by contradiction, assume that there is a sequence (a_n) such that $a_{n+1} \in S(a_n)$ for all $n \in I\!N$, i.e.,

$$\varphi(a_{n+1}) + \eta(a_n, a_{n+1}) \prec \varphi(a_n) - \tau. \tag{3}$$

Clearly, $(\varphi(a_n)) \subseteq I\!R$. Moreover, $(S(a_n))$ is a sequence of nested sets because of the triangle inequality property of η. Summing up inequality (3) for $n = 0, \dots, k$ while taking into account the triangle inequality property of η, we have

$$\varphi(a_{k+1}) + \eta(a_0, a_{k+1}) \leq \varphi(a_{k+1}) + \sum_{n=0}^{k} \eta(a_n, a_{n+1}) \prec \varphi(a_0) - (k+1)\tau.$$

The boundedness assumption ensures that $-\infty < c \leq \varphi(a_0) - (k+1)\tau$ and thus k is a finite number. Finally, $a_k \in S(a_{k-1}) \subseteq S(x_0)$ and the proof is complete. □

Next we state a formulation of Theorem 2.3 involving approximate solutions of optimization problems.

Definition 2.4. Let (P) be the optimization problem defined by a proper objective function $\varphi : A \to I\!R \cup \{+\infty\}$ on a nonempty set A:

$$\text{Min}\{\varphi(a) : a \in A\}. \tag{P}$$

Consider $\varepsilon \geq 0$. A point $a_0 \in A$ is said to be an approximate (resp. strict approximate) solution of problem (P) with error ε, and it is denoted $x_0 \in \mathrm{A}(\varphi, \varepsilon)$ (resp. $x_0 \in \mathrm{SA}(\varphi, \varepsilon)$) if $\varphi(a_0) - \varepsilon \leq \varphi(a)$ (resp. $\varphi(a_0) - \varepsilon < \varphi(a)$) for all $a \in A \backslash \{a_0\}$.

Corollary 2.5 (Approximate scalar EVP)**.** *Let A be a nonempty set, $\varphi : A \to I\!R \cup \{\pm\infty\}$ be a proper function, $\eta : A \times A \to I\!R_+$ be satisfying the triangle inequality property, $\gamma > 0$, $\varepsilon > 0$ and $\delta > 0$. Then, for each $a_0 \in \mathrm{A}(\varphi, \varepsilon + \delta)$ there exists $\bar{a} \in A$ such that*

- *(a)* $\varphi(\bar{a}) + \frac{\varepsilon}{\gamma}\eta(a_0, \bar{a}) \leq \varphi(a_0) - \delta$ *or* $\bar{a} = a_0$,
- *(b) If* $\bar{a} \neq a_0$, *then* $\eta(a_0, \bar{a}) \leq \gamma$,
- *(c)* $\varphi(a) + \frac{\varepsilon}{\gamma}\eta(\bar{a}, a) > \varphi(\bar{a}) - \delta$, $\forall a \in A$.

If in addition η is diagonal null, then assertion (c) states that $\bar{a} \in \mathrm{SA}(\varphi + \eta(\bar{a}, \cdot), \delta)$.

Proof. Consider a point $a_0 \in \mathrm{A}(\varphi, \varepsilon)$ and the bifunction $\eta' : A \times A \to I\!R_+$, $\eta' := \frac{\varepsilon}{\gamma}\eta$. Let us apply Theorem 2.5 by considering the bifunction η' instead η and $\tau = \delta$.

Clearly, η' also satisfies the triangle inequality property. Moreover, assumption (2) of Theorem 2.5 is fulfilled. Indeed, since the values of η' are nonnegative and a_0 is an approximate solution with error ε, we have that:

$$\inf_{a \in S(a_0)} \{\varphi(a) + \eta'(a_0, a)\} \geq \inf_{a \in A} \{\varphi(a) + \eta'(a_0, a)\} \geq \inf_{a \in A} \{\varphi(a)\} \geq \varphi(a_0) - \varepsilon - \delta.$$

Then, by applying Theorem 2.5 with $\prec=\leq$ we deduce the existence of a point $\bar{a} \in A$ satisfying:

(a) $\varphi(\bar{a}) + \dfrac{\varepsilon}{\gamma}\eta(a_0, \bar{a}) \leq \varphi(a_0) - \delta$ or $\bar{a} = a_0$,

(c) $\varphi(a) + \dfrac{\varepsilon}{\gamma}\eta(\bar{a}, a) > \varphi(\bar{a}) - \delta$, $\forall a \in A$.

If $\bar{a} \neq a_0$, as $a_0 \in \mathrm{A}(\varphi, \varepsilon)$, by statement (a) we see that

$$\frac{\varepsilon}{\gamma}\eta(a_0, \bar{a}) \leq \varphi(a_0) - \varphi(\bar{a}) - \delta \leq \varepsilon$$

and so statement (b) is obtained. Finally, it is obvious by assertion (c) that $\bar{a} \in \mathrm{SA}(\varphi + \eta(\bar{a}, \cdot), \delta)$ as long as η is diagonal null and the proof finishes. □

It is worth noticing that in contrast to the usual (exact) EVP, the point $\bar{a}$ is a strict approximate solution of the perturbation $\varphi + (\varepsilon/\gamma)\eta(\bar{a}, \cdot)$ instead of a strict exact solution whenever η is diagonal null, since neither completeness conditions nor lower-semicontinuity assumptions are assumed.

Theorem 2.3 encompasses [8, Proposition 2.5] (see Corollary 2.5). In comparison with this result, notice that a function η satisfying the triangle inequality property is considered instead of a metric. In addition, the lower boundedness assumption is weaker whenever bifunction η is nonnegative. For example, the function $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi(x) = x$ is unbounded from below, but for each $x_0 \in \mathbb{R}$, the set $S(x_0)$ is empty provided that the usual distance is considered. Furthermore, the initial set A of function φ is an arbitrary set.

Subsequently we provide a revised version of the so-called Dancs-Hegedüs-Medvegyev's fixed point theorem (see [9, Theorem 3.1]) for a dynamic system $\Phi : X \rightrightarrows X$ not enjoying the properties $\Phi(x)$ is closed and $x \in \Phi(x)$, for all $x \in X$.

Theorem 2.6 (Dancs-Hegedüs-Medvegyev's fixed point theorem)**.** *Let (X, d) be a complete metric space, $x_0 \in X$ and $\Phi : X \rightrightarrows X$ be a dynamic system. Suppose that each generalized Picard sequence (x_n) of Φ whose starting point is x_0 satisfies the following conditions:*

(B1) $\Phi(x_{n+1}) \subseteq \mathrm{cl}\,\Phi(x_n)$ *for all* $n \in \mathbb{N}$.
(B2) $d(x_n, x_{n+1}) \to 0$ *as* $n \to +\infty$.
(B3) *If* $x_n \to x$ *and* $x_{n+1} \neq x_n$ *for all* $n \in \mathbb{N}$, *then* $\Phi(x) \subseteq \mathrm{cl}\,\Phi(x_n)$ *for all* $n \in \mathbb{N}$.

Then, there is a fixed point $\bar{x} \in \mathrm{cl}\,\Phi(x_0) \cup \{x_0\}$ of the dynamic system Φ; i.e., $\Phi(\bar{x}) \subseteq \{\bar{x}\}$. Assume furthermore that $\Phi(\bar{x}) \neq \emptyset$, then $\Phi(\bar{x}) = \{\bar{x}\}$.

Proof. If the distance d satisfies condition (B2), then the equivalent distance $d' = d/(1 + d)$ also does, so we can suppose d is bounded on X. Denote the diameter of a subset A in X by $\delta(A)$; i.e.,

$$\delta(A) := \sup\{d(a, u) : a, u \in A\}.$$

If $\Phi(x_0) \subseteq \{x_0\}$, then $\bar{x} = x_0$. Otherwise, choose $x_1 \in \Phi(x_0)$ and $x_1 \neq x_0$ such that

$$d(x_0, x_1) \geq \delta(\Phi(x_0))/2 - 1/2.$$

Assume that there are $x_1, \ldots, x_n$ with $x_{k+1} \in \Phi(x_k)$ and $x_{k+1} \neq x_k$ for $k = 0, \ldots, n-1$.

If $\Phi(x_n) \subseteq \{x_n\}$, then $\bar{x} = x_n$. Otherwise, choose $x_{n+1} \in \Phi(x_n)$ and $x_{n+1} \neq x_n$ such that

$$d(x_n, x_{n+1}) \geq \delta(\Phi(x_n))/2 - 1/2^{n+1}. \tag{4}$$

If there is a generalized Picard sequence (x_n) of Φ satisfying condition (4), conditions (B1) and (B2) yield

$$\Phi(x_{n+1}) \subseteq \operatorname{cl}\Phi(x_n) \subseteq \operatorname{cl}\Phi(x_0) \text{ and } \delta(\Phi(x_n)) \to 0 \text{ as } n \to +\infty$$

clearly justifying that (x_n) is a Cauchy sequence. Since the space X is complete, there is a limit $\bar{x} \in \operatorname{cl}\Phi(x_0)$. By condition (B3), we have

$$\Phi(\bar{x}) \subseteq \operatorname{cl}\Phi(x_n) \text{ for all } n \in I\!N. \tag{5}$$

We show that $\Phi(\bar{x}) \subseteq \{\bar{x}\}$ by contradiction. Assume that there is some $y \neq \bar{x}$ such that $y \in \Phi(\bar{x})$. By (5) we have $y \in \operatorname{cl}\Phi(x_n)$ for all $n \in I\!N$ and thus $d(x_{n+1}, y) \leq \delta(\Phi(x_n)) \to 0$ as $n \to +\infty$. This means that $y = \bar{x}$. This contradiction completes the proof of the theorem. □

Example 2.7. Let (X_1, d) be the complete metric space given by the set $X_1 = \{1/n : n \in I\!N\backslash\{0\}\} \cup \{0\}$ and the metric $d(x_1, x_2) = |x_1 - x_2|$, for all $x_1, x_2 \in X_1$. Consider the point $x_0 = 1$ and the following dynamic system $\Phi_1 : X_1 \rightrightarrows X_1$:

$$\Phi_1(x) = \begin{cases} \{0\} & \text{if} \quad x = 0, \\ \{1/m : m \in I\!N\backslash\{0\}, m > n\} & \text{if} \quad x = 1/n \text{ and } n \text{ is even}, \\ \{1/m : m \in I\!N\backslash\{0\}, m > n\} \cup \{0\} & \text{if} \quad x = 1/n \text{ and } n \text{ is odd}. \end{cases}$$

As all assumptions of Theorem 2.6 are satisfied, it follows that there exists a point $\bar{x} \in X_1$ such that $\Phi_1(\bar{x}) \subseteq \{\bar{x}\}$. Obviously, $\bar{x} = 0$ and $\Phi_1(0) = \{0\}$. Notice that $1/n \notin \Phi_1(1/n)$, for all $n \in I\!N\backslash\{0\}$, $\Phi_1(1/n)$ is not closed provided that n is even, and $1/m \in \Phi_1(1/n)$, $\Phi_1(1/m) \not\subseteq \Phi_1(1/n)$ as long as n is even and $m > n$ is odd. Thus, assumptions (3.1), (3.2) and (3.3) in the original Dancs-Hegedüs-Medvegyev's fixed point theorem (see [9, Theorem 3.1]) are not fulfilled and so it cannot be applied. For instance, for all $n \in I\!N\backslash\{0\}$, $\frac{1}{2n+1} \in \Phi_1(\frac{1}{2n})$ but $\Phi_1(\frac{1}{2n+1}) \not\subseteq \Phi_1(\frac{1}{2n})$.

Assumption (B2) cannot be dropped. Indeed, the dynamic system $\Phi_2 : X_1 \rightrightarrows X_1$, $\Phi_2(x) = X_1$ for all $x \in X_1$, satisfies assumptions (B1) and (B3), but it has not any fixed point. It is obvious that hypothesis (B2) is not fulfilled.

Analogously, hypothesis (B3) cannot be removed. For instance, the dynamic system $\Phi_3 : X_1 \rightrightarrows X_1$, $\Phi_3(0) = X_1$ and $\Phi_3(1/n) = \{1/m : m \in I\!N\backslash\{0\}, m > n\}$ fulfills (B1) and (B2), but is doesn't satisfy (B3). Clearly, Φ_3 has not any fixed point.

Finally, assumption (B1) is also needed. For instance, let

$$X_2 = \left\{ s_n := \sum_{m=1}^{n} 1/m : n \in I\!N\backslash\{0\} \right\}.$$

It is obvious that (X_2, d) is a complete metric space. The dynamic system $\Phi_3 : X_2 \rightrightarrows X_2$, $\Phi_3(s_n) = \{s_{n+1}\}$, for all $n \in I\!N\backslash\{0\}$ fulfills hypotheses (B2) and (B3), but it has not any fixed point. It is easy to check that $\Phi_3(s_{n+1}) \not\subseteq \Phi_3(s_n)$, for all $n \in I\!N\backslash\{0\}$.

In the subsequent result we state the usual “located” version of Dancs-Hegedüs-Medvegyev's fixed point theorem (the fixed point $\bar{x}$ is the starting point x_0 or it belongs to the image $\Phi(x_0)$ of the starting point x_0). The proof coincides with the proof of Theorem 2.6 and it is omitted.

Corollary 2.8. *Let (X, d) be a complete metric space, $x_0 \in X$ and $\Phi : X \rightrightarrows X$ be a dynamic system. Suppose that each generalized Picard sequence (x_n) of Φ whose starting point is x_0 satisfies the following conditions:*

(B1) $\Phi(x_{n+1}) \subseteq \Phi(x_n)$ *for all* $n \in I\!N$.
(B2) $d(x_n, x_{n+1}) \to 0$ *as* $n \to +\infty$.
(B3) *If* $x_n \to x$ *and* $x_{n+1} \neq x_n$ *for all* $n \in I\!N$, *then* $\Phi(x) \subseteq \Phi(x_n)$ *for all* $n \in I\!N$ *and* $x \in \Phi(x_0)$.

Then, there is a fixed point $\bar{x} \in \Phi(x_0) \cup \{x_0\}$ of the dynamic system Φ; i.e., $\Phi(\bar{x}) \subseteq \{\bar{x}\}$. Assume furthermore that $\Phi(\bar{x}) \neq \emptyset$, then $\Phi(\bar{x}) = \{\bar{x}\}$.

Remark 1. The counterpart in [6, Theorem 4.1] of conditions (B2) and (B3) of Corollary 2.8 when one reduces that theorem to the metric setting are:

(E2) $\lim_{n\to\infty} \sup_{u\in\Phi(x_n)} d(x_n, u) = 0$.
(E3) There is some element $\bar{x} \in X$ such that $\bar{x} \in \Phi(x_n)$ for all $n \in I\!N$.
(E5) $\Phi(\bar{x}) \subseteq \Phi(x_n)$, for all $n \in I\!N$.

Clearly, $x_n \to \bar{x}$ and (E2) is equivalent to (B2) since (B1) holds true. Thus, (E2) and (E3) can be formulated as follows:

(E2) $d(x_n, x_{n+1}) \to 0$ as $n \to +\infty$.
(E3) If $x_n \to \bar{x}$, then $\bar{x} \in \Phi(x_n)$ for all $n \in I\!N$.
(E5) $\Phi(\bar{x}) \subseteq \Phi(x_n)$, for all $n \in I\!N$.

These three conditions are stronger than (B1) and (B2) provided that $\Phi(\bar{x}) = \emptyset$. For instance, consider the complete metric space (X_1, d) introduced in Example 2.7 and the dynamic system $\Phi : X_1 \rightrightarrows X_1$,

$$\Phi(x) = \begin{cases} \emptyset & \text{if} \quad x = 0, \\ \{1/m : m \in I\!N\backslash\{0\}, m \geq n\} & \text{if} \quad x = 1/n, n \in I\!N\backslash\{0\}. \end{cases}$$

We have that assumptions (B1), (B2) and (B3) of Corollary 2.8 are satisfied, but condition (E3) above is not fulfilled.

The fixed point theorem stated in [14, Theorem 3.5] is a particular case of Theorem 2.6. This assertion is proved in the next corollary, as an illustration of the power of Theorem 2.6.

Corollary 2.9. *Let (X, d) be a complete metric space and consider a set-valued mapping $S : X \rightrightarrows X$, a function $m : X \to I\!R \cup \{\pm\infty\}$ and $x_0 \in X$ such that $S : S(x_0) \rightrightarrows S(x_0)$, $S(x_0)\backslash\{x_0\} \neq \emptyset$ and $m : S(x_0)\backslash\{x_0\} \to [0, +\infty]$. Assume the following conditions:*

(i) *If $x \in S(x_0)$ and $y \in S(x)$, then $S(y) \subseteq S(x)$.*
(ii) *There exists $x_1 \in S(x_0)\backslash\{x_0\}$ such that $m(x_1) < +\infty$.*
(iii) *For each $x \in S(x_0)$, $m(x) < +\infty$, and $y \in S(x)\backslash\{x\}$, $m(y) < m(x)$.*
(iv) *The set $S(x)$ is closed for all $x \in S(x_0)$, and there exists $\alpha > 0$ such that for*

each $x \in S(x_0)$ and $y \in S(x)$ such that $m(x), m(y) \in I\!R$, it follows that

$$\alpha d(x, y) \leq m(x) - m(y). \tag{6}$$

Then, there exists a point $\bar{x} \in S(x_0)$ such that $S(\bar{x}) \subseteq \{\bar{x}\}$.

Proof. Take a point $x_1 \in X$ satisfying condition (ii). If $S(x_1) \subseteq \{x_1\}$, then the result follows by defining $\bar{x} := x_1$. Otherwise, by (i) we see that $S : S(x_1) \rightrightarrows S(x_1)$ and by (iv) we have that $S(x_1)$ is closed. Then, Theorem 2.6 can be applied to the data $X' := S(x_1)$, an arbitrary point $x'_0 \in X'$ and $\Phi := S$ as long as assumptions (B1), (B2) and (B3) are fulfilled. Let us check these hypotheses.

Given a generalized Picard sequence $(x_n) \subseteq X'$ of Φ, for each n we have $x_{n+1} \in S(x_n)$. Then, (B1) is a direct consequence of (i), and (B3) follows by (i) and the closedness of the sets $S(x)$, for all $x \in S(x_0)$.

Fix $n \geq 1$, we have $x_{n+1} \in S(x_1)$ with $m(x_1) < +\infty$. By condition (iii), we have $m(x_{n+1}) \leq m(x_1)$. Then, for each n, condition (6) yields

$$\alpha d(x_n, x_{n+1}) \leq m(x_n) - m(x_{n+1}).$$

Summing these inequalities from $n = 1$ to $n = k$, we have

$$\alpha \sum_{n=1}^{k} d(x_n, x_{n+1}) \leq m(x_1) - m(x_{k+1}) \leq m(x_1).$$

Since k was arbitrary, we could pass the relation as $k \to +\infty$ to get

$$\alpha \sum_{n=1}^{+\infty} d(x_n, x_{n+1}) \leq m(x_1) < +\infty$$

clearly verifying that $d(x_n, x_{n+1}) \to 0$ as $n \to +\infty$ and thus condition (B2) is satisfied. □

3. Approximate Ekeland variational principles

The main results of this section are the next two approximate EVP for vector-valued functions, which are derived from Theorem 2.3 by applying nonlinear and linear scalarization techniques, respectively.

Theorem 3.1. *Let A be a nonempty set, Y be a real linear space and $g : A \to Y$ be a vector-valued function. Consider two nonempty sets $E, G \subseteq Y$ and $q \in Y \backslash \{0\}$ such that $E + (E \cup G) \subseteq E$, E is free-disposal with respect to* $\operatorname{cone}\{q\}$, *$q \notin -\operatorname{vcl} \operatorname{cone} E$ and $0 \notin \operatorname{vcl}_q E$. Let $\eta : A \times A \to I\!R$ be satisfying the triangle inequality property.*

Consider $a_0 \in A$ such that

$$\inf\{\varphi_E^q(g(a) - g(a_0)) + \eta(a_0, a) \ : \ a \in S_{E,\eta}(a_0)\} > -\infty, \tag{7}$$

where

$$S_{E,\eta}(a_0) := \{a \in A \ : \ g(a) + \eta(a_0, a)q \leq_E g(a_0)\}.$$

Then there exists $\bar{a} \in A$ such that

(a) $g(\bar{a}) + \eta(a_0, \bar{a})q \leq_E g(a_0)$ *or* $\bar{a} = a_0$,
(b) $\varphi_E^q(g(a) - g(a_0)) + \eta(\bar{a}, a) \not\leq \varphi_E^q(g(\bar{a}) - g(a_0)) - \varphi_E^q(0)$, $\forall a \in S_{E,\eta}(a_0)$,
(c) $g(a) + \eta(\bar{a}, a)q \not\leq_{E \cup G} g(\bar{a}) - \varphi_E^q(0)q$, $\forall a \in A$.

Proof. In order to apply Theorem 2.3, consider the data $A' = S_{E,\eta}(a_0)$, $\varphi : A' \to \mathbb{R} \cup \{\pm\infty\}$, $\varphi := \varphi_E^q \circ (g - g(a_0))$, $\tau = \varphi(a_0) = \varphi_E^q(0)$ and $\prec = \leq$.

If $A' = \emptyset$, then the result follows by taking $\bar{a} := a_0$. Suppose that $A' \neq \emptyset$. Let us check the assumptions of Theorem 2.3.

We claim that φ is proper. Indeed, by the definition of function φ_E^q we see that

$$\varphi(a) = \varphi_E^q(g(a) - g(a_0)) \leq -\eta(a_0, a), \quad \forall a \in A'$$

and so $\operatorname{dom} \varphi \neq \emptyset$. In addition, assumption $q \notin -\operatorname{vcl} \operatorname{cone} E$ implies that $\varphi > -\infty$ (see Lemma 2.2(i)). Thus, φ is a proper function.

As E is nonempty and free-disposal with respect to cone $\{q\}$ we deduce that $\mathbb{R}q \cap E \neq \emptyset$ and so $\varphi_E^q(0) < +\infty$. Then, $a_0 \in \operatorname{dom} \varphi$. Moreover, by Lemma 2.2(iii) we know that $\varphi_E^q(0) > 0$ provided that $0 \notin \operatorname{cone}\{q\} + \operatorname{vcl}_q E$, which holds true by the assumptions. Indeed, as $\operatorname{vcl} \operatorname{cone} E$ is a cone and $0 \notin \operatorname{vcl}_q E$ we have that

$$\begin{aligned} q \notin -\operatorname{vcl} \operatorname{cone} E &\Rightarrow \operatorname{cone}\{q\} \backslash \{0\} \cap (-\operatorname{vcl} \operatorname{cone} E) = \emptyset \\ &\Rightarrow \operatorname{cone}\{q\} \backslash \{0\} \cap (-\operatorname{vcl} E) = \emptyset \\ &\Rightarrow \operatorname{cone}\{q\} \backslash \{0\} \cap (-\operatorname{vcl}_q E) = \emptyset \\ &\Rightarrow \operatorname{cone}\{q\} \cap (-\operatorname{vcl}_q E) = \emptyset. \end{aligned}$$

Thus, $0 \notin \operatorname{cone}\{q\} + \operatorname{vcl}_q E$ and so $\varphi_E^q(0) > 0$.

Next we compute the set $S(a_0)$. As φ_E^q is q-translative (see Lemma 2.2(ii)), for each point $a \in S_{E,\eta}(a_0)$ we have that

$$\begin{aligned} \varphi(a) + \eta(a_0, a) &= \varphi_E^q(g(a) - g(a_0)) + \eta(a_0, a) \\ &= \varphi_E^q(g(a) - g(a_0) + \eta(a_0, a)q) \\ &\leq 0. \end{aligned}$$

Therefore,

$$S_{E,\eta}(a_0) \subseteq \{a \in A : \varphi(a) + \eta(a_0, a) \leq 0\}$$

and so

$$\begin{aligned} S(a_0) &= \{a \in A' : \varphi(a) + \eta(a_0, a) \leq \varphi(a_0) - \tau\} \\ &= \{a \in A : \varphi(a) + \eta(a_0, a) \leq 0\} \cap S_{E,\eta}(a_0) \\ &= S_{E,\eta}(a_0). \end{aligned}$$

Then,

$$\inf_{a \in S(a_0)} \{\varphi(a) + \eta(a_0, a)\} = \inf_{a \in S_{E,\eta}(a_0)} \{\varphi_E^q(g(a) - g(a_0)) + \eta(a_0, a)\} > -\infty$$

and all hypotheses of Theorem 2.3 are fulfilled. Thus, there exists $\bar{a} \in A'$ such that

(b) $\varphi(a) + \eta(\bar{a}, a) \not\leq \varphi(\bar{a}) - \tau, \ \forall a \in A'$

and the second assertion of the theorem holds true.

As $\bar{a} \in A'$, we have $g(\bar{a}) + \eta(a_0, \bar{a})q \leq_E g(a_0)$ and the first statement of the theorem holds true too. In order to state the third one, suppose reasoning by contradiction that there exists $a \in A$ such that

$$g(a) + \eta(\bar{a}, a)q \leq_{E \cup G} g(\bar{a}) - \varphi_E^q(0)q. \tag{8}$$

We claim that $a \in A'$. Indeed, since $\bar{a} \in A'$ and η satisfies the triangle inequality property we deduce that

$$\begin{aligned} g(a) + \eta(a_0, a)q &\leq_{\text{cone}\,\{q\}} g(a) + \eta(\bar{a}, a)q + \eta(a_0, \bar{a})q \\ &\leq_{E \cup G} g(\bar{a}) + \eta(a_0, \bar{a})q - \varphi_E^q(0)q \\ &\leq_E g(a_0) - \varphi_E^q(0)q \end{aligned}$$

Then, since E is free-disposal with respect to cone $\{q\}$ and $E + (E \cup G) \subseteq E$, it follows that

$$\begin{aligned} g(a_0) - (g(a) + \eta(a_0, a)q) &\in \text{cone}\,\{q\} + (E \cup G) + E + \varphi_E^q(0)q \\ &\subseteq (E \cup G) + E \\ &\subseteq E. \end{aligned}$$

Thus, $g(a) + \eta(a_0, a)q \leq_E g(a_0)$, i.e., $a \in A'$.

Moreover, we have that

$$E + (E \cup G) \subseteq E \subseteq \text{vcl}_q E \subseteq [0, +\infty)q + \text{vcl}_q E.$$

Therefore, φ_E^q is $\leq_{E \cup G}$-nondecreasing (see Lemma 2.2(iv)). As φ_E^q is q-translative, by statement (8) we have that

$$\begin{aligned} \varphi(a) + \eta(\bar{a}, a) &= \varphi_E^q(g(a) - g(a_0)) + \eta(\bar{a}, a) \\ &= \varphi_E^q(g(a) - g(a_0) + \eta(\bar{a}, a)q) \\ &\leq \varphi_E^q(g(\bar{a}) - g(a_0) - \varphi_E^q(0)q) \\ &= \varphi_E^q(g(\bar{a}) - g(a_0)) - \varphi_E^q(0) \\ &= \varphi(\bar{a}) - \varphi_E^q(0), \end{aligned}$$

which is contrary to assertion (b). Therefore, part (c) holds true and the proof finishes. □

Remark 2. 1. By Lemma 2.1 it follows that assumption $q \notin -\text{vcl}\,\text{cone}\,E$ is fulfilled whenever $\text{cone}\,E$ is proper and convex and $q \in \text{core}\,\text{cone}\,E$. Moreover, it is easy to check that this hypothesis can be replaced by the conditions $\varphi_E^q \circ (g - g(a_0)) > -\infty$ and $\text{cone}\,\{q\} \cap (-\text{vcl}_q\,E) = \emptyset$.

2. It is obvious that $E + 0^+E \subseteq E$. Thus, in Theorem 3.1, one can consider $G = 0^+E$ provided that $E + E \subseteq E$. In this case it is worth noticing in order to apply assertion (c) that $E \not\subseteq 0^+E$ in general. For instance, consider $Y = \mathbb{R}^2$, $q = (1/4, 2)$ and let

$E \subseteq Y$ be the epigraph of the function $\varphi : I\!R \to I\!R$,

$$\varphi(x) = \begin{cases} +\infty & \text{if} \quad x \in (-\infty, 0) \\ 1 & \text{if} \quad x \in [0, 1/2] \\ 2x & \text{if} \quad x \in (1/2, 1] \\ 2 & \text{if} \quad x \in (1, 2] \\ x & \text{if} \quad x \in (2, +\infty). \end{cases}$$

We have that E is free-disposal with respect to $\operatorname{cone}\{q\}$, $E + E \subseteq E$ and $0 \notin E = \operatorname{vcl}_q E$. Moreover,

$$\operatorname{cone} E = \{(y_1, y_2) \in I\!R^2_+ : y_2 \geq y_1\},$$
$$0^+ E = \{(y_1, y_2) \in I\!R^2_+ : y_2 \geq 2y_1\}.$$

Therefore, $\operatorname{cone} E$ is a proper convex cone, $q \in \operatorname{core} \operatorname{cone} E$ and $E \not\subseteq 0^+ E$.

As an illustration of the second part of Remark 2, we state a formulation of Theorem 3.1 based on Kutateladze's approximate solutions of vector optimization problems (see [13,18]).

Definition 3.2. Let Y be a real linear space and $D \subseteq Y$ be a convex cone. Consider $\varepsilon \geq 0$, $q \in D \backslash (-D)$ and a function $g : A \to Y$ from an arbitrary nonempty set A. A point $a_0 \in A$ is said to be a Kutateladze's approximate solution of g in the direction q with precision ε, denoted by $a_0 \in \operatorname{EK}(g, \varepsilon q)$, if

$$a \in A, \quad g(a) \leq_D g(a_0) - \varepsilon q \Rightarrow g(a) = g(a_0) - \varepsilon q.$$

Corollary 3.3. *Let A be a nonempty set, Y be a real linear space and $g : A \to Y$ be a vector-valued function. Consider a convex cone $D \subseteq Y$, $q \in D \backslash (-D)$ and $\varepsilon, \delta, \gamma > 0$. Let $\eta : A \times A \to I\!R_+$ be satisfying the triangle inequality property.*

Then, for each $a_0 \in \operatorname{EK}(g, (\varepsilon + \delta)q)$ there exists $\bar{a} \in A$ such that

(a) $g(\bar{a}) + \dfrac{\varepsilon}{\gamma}\eta(a_0, \bar{a})q \leq_D g(a_0) - \delta q$ *or* $\bar{a} = a_0$,
(b) *if* $\bar{a} \neq a_0$, *then* $\eta(a_0, \bar{a}) \leq \gamma$,
(c) $g(a) + \dfrac{\varepsilon}{\gamma}\eta(\bar{a}, a)q \not\leq_D g(\bar{a}) - \delta q, \ \forall a \in A$.

Proof. Consider a point $a_0 \in \operatorname{EK}(g, (\varepsilon + \delta)q)$, $E = \delta q + D$, $G = 0^+ E$ and $\dfrac{\varepsilon}{\gamma}\eta$ instead of η. Let us check that the assumptions of Theorem 3.1 are fulfilled (we consider the hypotheses $\varphi^q_E \circ (g - g(a_0)) > -\infty$ and $\operatorname{cone}\{q\} \cap (-\operatorname{vcl}_q E) = \emptyset$ instead of $q \notin -\operatorname{vcl} \operatorname{cone} E$, see Remark 2).

Clearly, $\dfrac{\varepsilon}{\gamma}\eta$ satisfies the triangle inequality property. In addition, as $q \in D$ and D is a convex cone we deduce

$$E + E = \delta q + D + \delta q + D \subseteq \delta q + D = E$$

and

$$\operatorname{cone}\{q\} + E = \operatorname{cone}\{q\} + \delta q + D = \delta q + D = E.$$

We claim that

$$\operatorname{cone}\{q\} \cap (-\operatorname{vcl}_q E) = \emptyset. \tag{9}$$

Indeed, suppose by contradiction that there exists $t \geq 0$ such that $tq \in -\operatorname{vcl}_q E$. Then, there exists a sequence $(t_n) \subseteq I\!R_+$, $t_n \to 0$, such that $-tq+t_nq \in \delta q+D$, for all $n \in I\!N$. Therefore, $(-(\delta+t)+t_n)q \in D$. Since $\delta+t>0$ and $t_n \to 0$, there exist $n_0 \in I\!N$ big enough such that $-(\delta+t)+t_{n_0}<0$ and so we deduce that $q \in -D$, a contradiction.

Thus, assertion (9) holds true. In particular, we obtain that $0 \notin \operatorname{vcl}_q E$.

As φ_D^q is q-translative (see Lemma 2.2(ii)), we have that

$$\begin{aligned}
\varphi_E^q(y) &= \inf\{t \in I\!R : y \in tq - \delta q - D\} \\
&= \inf\{t \in I\!R : y + \delta q \in tq - D\} \\
&= \varphi_D^q(y+\delta q) \\
&= \varphi_D^q(y) + \delta, \quad \forall y \in Y.
\end{aligned}$$

It follows that $\varphi_D^q(0)=0$. Indeed, as $0 \in -D$ we have that $\varphi_D^q(0) \leq 0$. If $\varphi_D^q(0)<0$, then there exists $t>0$ such that $0 \in -tq-D$. Thus, $q \in -D$, that is a contradiction. Therefore, $\varphi_E^q(0)=\delta$.

Since $a_0 \in \operatorname{EK}(g,(\varepsilon+\delta)q)$, we have that

$$y_a := g(a)-g(a_0)+(\varepsilon+\delta)q \notin -D\backslash\{0\}, \quad \forall a \in A. \tag{10}$$

If $\varphi_D^q(y_a)<0$, then there exists $t>0$ such that $y_a \in -tq-D \subsetneq -D\backslash\{0\}$, a contradiction. Thus, $\varphi_D^q(y_a) \geq 0$ and since is φ_D^q is q-translative we obtain

$$\varphi_E^q(g(a)-g(a_0)) = \varphi_D^q(g(a)-g(a_0))+\delta = \varphi_D^q(y_a)-\varepsilon \geq -\varepsilon, \quad \forall a \in A.$$

In particular,

$$\begin{aligned}
&\inf\{\varphi_E^q(g(a)-g(a_0)) + \frac{\varepsilon}{\gamma}\eta(a_0,a) \;:\; a \in S_{E,\frac{\varepsilon}{\gamma}\eta}(a_0)\} \\
\geq &\inf\{\varphi_E^q(g(a)-g(a_0)) \;:\; a \in A\} \\
\geq &-\infty.
\end{aligned}$$

Therefore, all assumptions of Theorem 3.1 hold true and it follows that there exists $\bar{a} \in A$ such that

(a) $g(\bar{a}) + \frac{\varepsilon}{\gamma}\eta(a_0,\bar{a})q \leq_E g(a_0)$ or $\bar{a}=a_0$,

(c) $g(a) + \frac{\varepsilon}{\gamma}\eta(\bar{a},a)q \not\leq_{E\cup 0^+E} g(\bar{a})-\delta q$, $\forall a \in A$.

Clearly, assertion (a) is equivalent to the next one:

(a) $g(\bar{a}) + \frac{\varepsilon}{\gamma}\eta(a_0,\bar{a})q \leq_D g(a_0)-\delta q$ or $\bar{a}=a_0$.

Moreover, it is obvious that $0^+(\delta q+D)=D$. Thus, $E \cup 0^+E = D$ and assertion (c) results:

(c) $g(a) + \frac{\varepsilon}{\gamma}\eta(\bar{a},a)q \not\leq_D g(\bar{a})-\delta q$, $\forall a \in A$.

Finally, if $\bar{a} \neq a_0$, by applying assertion (a) and statement (10) with $a = \bar{a}$ we see that

$$g(\bar{a}) - g(a_0) + \delta q \in \left(-\frac{\varepsilon}{\gamma}\eta(a_0, \bar{a})q - D\right) \backslash (-\varepsilon q - D\backslash\{0\}).$$

Therefore, $\frac{\varepsilon}{\gamma}\eta(a_0, \bar{a}) \leq \varepsilon$ and assertion (b) is proved, which finishes the proof. □

Notice that Corollary 3.3 reduces to Corollary 2.5 when $Y = I\!R$, $D = I\!R_+$ and $q = 1$.

Theorem 3.4 (Approximate vectorial EVP)**.** *Let A be a nonempty set, Y be a real linear space and $g : A \to Y$ be a vector-valued function. Consider two nonempty sets $E \subseteq Y\backslash\{0\}$ and $G \subseteq Y$ such that $(G^+ \cap E^{s+})\backslash\{0\} \neq \emptyset$. Let $h : A \times A \to Y$ be satisfying the triangle inequality property with respect to $\leq_G$.*

Let $a_0 \in A$ and $\lambda \in (G^+ \cap E^{s+})\backslash\{0\}$ be such that

$$\inf\{\lambda(g(a)) + \lambda(h(a_0, a)) \ : \ a \in S_\lambda(a_0)\} > -\infty, \tag{11}$$

where

$$S_\lambda(a_0) := \{a \in A : \lambda(g(a)) + \lambda(h(a_0, a)) \leq \lambda(g(a_0)) - \tau_E(\lambda)\}. \tag{12}$$

Then there exists $\bar{a} \in A$ such that

(a) $\lambda(g(\bar{a})) + \lambda(h(a_0, \bar{a})) \leq \lambda(g(a_0)) - \tau_E(\lambda)$ *or* $\bar{a} = a_0$,
(b) $\lambda(g(a)) + \lambda(h(\bar{a}, a)) > \lambda(g(\bar{a})) - \tau_E(\lambda)$, $\forall a \in A$,
(c) $g(a) + h(\bar{a}, a) \not\leq_E g(\bar{a})$, $\forall a \in A$.

Proof. Under the assumptions made, there is $\lambda \in (G^+ \cap E^{s+})\backslash\{0\}$ satisfying (11). By the definition of E^{s+} and $0 \notin E$ we have that $\tau_E(\lambda) > 0$.

Consider the functions $\varphi := \lambda \circ g$, $\eta := \lambda \circ h$ and $\tau := \tau_E(\lambda)$. As $\lambda \in G^+$ and h fulfills the triangle inequality property with respect to $\leq_G$, it follows that η satisfies the triangle inequality property. Then by applying Theorem 2.3 we see that there exists a point $\bar{a} \in A$ satisfying parts (a) and (b).

Next, we show that $\bar{a}$ fulfills (c) as well. Arguing by contradiction, assume that there is some $a \in A$ such that

$$g(a) + h(\bar{a}, a) \leq_E g(\bar{a}).$$

Since λ is a linear function we have

$$\lambda(g(a)) + \lambda(h(\bar{a}, a)) \leq \lambda(g(\bar{a})) - \tau_E(\lambda)$$

that is a contradiction to (b). Therefore, the proof is completed. □

Remark 3. Roughly speaking, assumptions $0 \notin E$ and $E^{s+}\backslash\{0\} \neq \emptyset$ imply that $0 \notin \operatorname{cl} E$ whenever Y is a real topological linear space. If the real linear space Y is not equipped with any topology, then one can claim that $0 \notin \operatorname{vcl}_q E$, for all $q \in Y$.

Corollary 3.5. *Let A be a nonempty set, Y be a real linear space and $g : A \to Y$ be a vector-valued function. Consider two nonempty sets $E \subseteq Y\backslash\{0\}$ and $G \subseteq Y$ such* *that $E+(G\cup E)\subseteq E$ and $(G^+\cap E^{s+})\backslash\{0\}\neq\emptyset$. Let $h:A\times A\to Y$ be satisfying the triangle inequality property with respect to $\leq_G$.*

Let $a_0\in A$ and $\lambda\in(G^+\cap E^{s+})\backslash\{0\}$ such that

$$\inf\{\lambda(g(a))+\lambda(h(a_0,a))\ :\ a\in S_{E,h}(a_0)\}>-\infty, \tag{13}$$

where

$$S_{E,h}(a_0):=\{a\in A\ :\ g(a)+h(a_0,a)\leq_E g(a_0)\}.$$

Then there exists $\bar{a}\in A$ such that

(a) $g(\bar{a})+h(a_0,\bar{a})\leq_E g(a_0)$ *or* $\bar{a}=a_0$,
(b) $g(a)+h(\bar{a},a)\not\leq_E g(\bar{a})$, $\forall a\in A$.

Proof. Consider $a_0\in A$ and $\lambda\in(G^+\cap E^{s+})\backslash\{0\}$ satisfying (13). The result is trivial whenever $S_{E,h}(a_0)=\emptyset$. Thus, let us assume that $S_{E,h}(a_0)\neq\emptyset$.

In order to apply Theorem 3.4 to the set $S_{E,h}(a_0)$ instead of A and the functions $g:S_{E,h}(a_0)\to Y$ and $h:S_{E,h}(a_0)\times S_{E,h}(a_0)\to Y$, notice that the set $S_\lambda(a_0)$ of (12) reduces to $S_{E,h}(a_0)$. Indeed, for every $a\in S_{E,h}(a_0)$ we have $g(a_0)-g(a)-h(a_0,a)\in E$ and thus

$$\lambda(g(a_0))-\lambda(g(a))-\lambda(h(a_0,a))\geq\tau_E(\lambda)$$

or equivalently

$$\lambda(g(a))+\lambda(h(a_0,a))\leq\lambda(g(a_0))-\tau_E(\lambda).$$

Therefore,

$$\begin{aligned}S_\lambda(a_0)&=\{a\in S_{E,h}(a_0):\lambda(g(a))+\lambda(h(a_0,a))\leq\lambda(g(a_0))-\tau_E(\lambda)\}\\&=\{a\in A:\lambda(g(a))+\lambda(h(a_0,a))\leq\lambda(g(a_0))-\tau_E(\lambda)\}\cap S_{E,h}(a_0)\\&=S_{E,h}(a_0).\end{aligned}$$

Thus, Theorem 3.4 ensures the existence of $\bar{a}\in S_{E,h}(a_0)$ such that

(c') $g(a)+h(\bar{a},a)\not\leq_E g(\bar{a})\ \forall a\in S_{E,h}(a_0)$.

Assertion $\bar{a}\in S_{E,h}(a_0)$ implies part (a). Next, we show that $\bar{a}$ satisfies part (b) as well. Arguing by contradiction, assume that there is some $a\in A$ such that

$$g(a)+h(\bar{a},a)\leq_E g(\bar{a}). \tag{14}$$

Let us check that $a\in S_{E,h}(a_0)$. By the triangle inequality property of h with respect to $\leq_G$ we have

$$g(a)+h(a_0,a)\leq_G g(a)+h(a_0,\bar{a})+h(\bar{a},a)\leq_E g(\bar{a})+h(a_0,\bar{a})\leq_E g(a_0).$$

Since $E+(G\cup E)\subseteq E$ we deduce $g(a)+h(a_0,a)\leq_E g(a_0)$; i.e., $a\in S_{E,h}(a_0)$. Thus, (14) is a contradiction to (c'), and the proof is completed. $\square$

Remark 4. Notice that assumptions $E + G \subseteq E$ and $(G^+ \cap E^{s+})\backslash\{0\} \neq \emptyset$ could be connected. For example, if G is coradiant (i.e., $tG \subseteq G$, for all $t > 1$) and $E + G \subseteq E$ then $E^+ \subseteq G^+$. Indeed, let $\lambda \in E^+$ and consider two points $y \in G$ and $e_0 \in E$. We have that

$$y = \lim_{n\to\infty} (1/n)(e_0 + ny)$$

and so

$$\lambda(y) = \lim_{n\to\infty} (1/n)\lambda(e_0 + ny).$$

As $\lambda \in E^+$ and $e_0 + ny \in E + G$ for all $n \in I\!N$, $n \geq 2$, we see that $\lambda(y) \geq 0$. Since y is arbitrary, it follows that $\lambda \in G^+$.

Thus, if G is coradiant, assumptions $E + (G \cup E) \subseteq E$ and $(G^+ \cap E^{s+})\backslash\{0\} \neq \emptyset$ reduce to $E + (G \cup E) \subseteq E$ and $E^{s+}\backslash\{0\} \neq \emptyset$.

In order to illustrate Corollary 3.5, we derive an approximate Ekeland variational principle for vector bifunctions by Al-Homidan et. al (see [2, Theorem 3.4]).

Corollary 3.6. *Let (X, d) be a metric space, $f : X \times X \to Y$ and $x_0 \in X$. Consider $E \subseteq Y$ and $q \in Y\backslash\{0\}$ such that $0 \notin E$, $E + E \subseteq E$ and $E + [0, \infty)q = E$, and let $\lambda \in E^{s+}$. Suppose that there exists $g : X \to Y$ satisfying:*

(i) $f(x_1, x_2) \geq_E g(x_2) - g(x_1)$, *for all* $x_1, x_2 \in X$.
(ii) $\sup\{\lambda(f(x, x_0)) - \lambda(q)d(x_0, x) : g(x) \leq_E g(x_0)\} < +\infty$.

Then, there exists $\bar{x} \in X$ such that

(a) $g(\bar{x}) - g(x_0) + d(x_0, \bar{x})q \leq_E 0$, *or* $\bar{x} = x_0$,
(b) $f(\bar{x}, x) + d(\bar{x}, x)q \not\leq_E 0$, *for all* $x \in X$.

Proof. The result follows by applying Corollary 3.5 to the next data: $A = X$, $G = \text{cone}\{q\}$, $h : X \times X \to Y$, $h(x_1, x_2) = d(x_1, x_2)q$ and $a_0 = x_0$. Indeed, it is obvious that $E + (G \cup E) \subseteq E$ and h satisfies the triangle inequality property with respect to $\leq_G$.

By Remark 4 we deduce that $\lambda \in G^+$. Moreover, as E is free-disposal with respect to cone$\{q\}$ we have $S_{E,h}(x_0) \subseteq \{x \in X : g(x) \leq_E g(x_0)\}$ and then by assumptions (i) and (ii) we deduce that

$$\begin{aligned}
&\inf\{\lambda(g(x)) + \lambda(h(x_0, x)) : x \in S_{E,h}(x_0)\} \\
\geq &\inf\{\lambda(g(x) - g(x_0)) + \lambda(q)d(x_0, x) : g(x) \leq_E g(x_0)\} + \lambda(g(x_0)) \\
\geq &\inf\{\lambda(-f(x, x_0)) + \lambda(q)d(x_0, x)) : g(x) \leq_E g(x_0)\} + \lambda(g(x_0)) \\
\geq &- \sup\{\lambda(f(x, x_0)) - \lambda(q)d(x_0, x)) : g(x) \leq_E g(x_0)\} + \lambda(g(x_0)) \\
> &- \infty.
\end{aligned}$$

Therefore, since all hypotheses of Corollary 3.5 are satisfied, there exists a point $\bar{x} \in X$ such that

(a) $g(\bar{x}) + d(x_0, \bar{x})q \leq_E g(x_0)$, or $\bar{x} = x_0$,
(b') $g(x) + d(\bar{x}, x)q \not\leq_E g(\bar{x})$, $\forall x \in X$.

We claim that assertion (b) is a consequence of statement (b'). Indeed, suppose that there exists $x \in X$ such that $f(\bar{x}, x) + d(\bar{x}, x)q \leq_E 0$. Then, by assumption (i) we have

$$g(x) - g(\bar{x}) + d(\bar{x}, x)q \leq_E f(\bar{x}, x) + d(\bar{x}, x)q \leq_E 0,$$

that is contrary to (b'). Therefore, assertion (b) holds true and the proof is complete. □

Next we obtain a formulation of Corollary 3.5 for Helbig's approximate solutions of vector optimization problems (see [13,16]).

Definition 3.7. Let Y be a real linear space and $D \subseteq Y$ be a convex cone. Consider $\varepsilon \geq 0$, $\lambda \in D^+\backslash\{0\}$ and a function $g : A \to Y$ from an arbitrary nonempty set A. A point $a_0 \in A$ is said to be a Helbig's approximate solution of g by λ with precision ε, denoted by $a_0 \in \mathrm{EH}(g, \lambda, \varepsilon)$, if

$$a \in A, \quad g(a) \leq_D g(a_0) \Rightarrow (\lambda \circ g)(a_0) - \varepsilon \leq (\lambda \circ g)(a).$$

Corollary 3.8. *Let A be a nonempty set, Y be a real linear space and $g : A \to Y$ be a vector-valued function. Consider a convex cone $D \subseteq Y$, $q \in D$, $\lambda \in D^+$ such that $\lambda(q) = 1$ and $\varepsilon, \delta, \gamma > 0$. Let $\eta : A \times A \to I\!R_+$ be satisfying the triangle inequality property.*

Then, for each $a_0 \in \mathrm{EH}(g, \lambda, \varepsilon + \delta)$ there exists $\bar{a} \in A$ such that

(a) *both $g(\bar{a}) + \frac{\varepsilon}{\gamma}\eta(a_0, \bar{a})q \leq_D g(a_0)$ and $\lambda(g(\bar{a})) + \frac{\varepsilon}{\gamma}\eta(a_0, \bar{a}) \leq \lambda(g(a_0)) - \delta$, or $\bar{a} = a_0$,*

(b) *if $\bar{a} \neq a_0$, then $\eta(a_0, \bar{a}) \leq \gamma$,*

(c) *for each $a \in A$ such that $g(a) + \frac{\varepsilon}{\gamma}\eta(\bar{a}, a)q \leq_D g(\bar{a})$ it follows that $\lambda(g(\bar{a})) - \delta < \lambda(g(a)) + \frac{\varepsilon}{\gamma}\eta(\bar{a}, a)$.*

Proof. Consider a point $a_0 \in \mathrm{EH}(g, \lambda, \varepsilon + \delta)$. The result follows by applying Corollary 3.5 to the next data:

$$E = \{y \in D : \lambda(y) \geq \delta\},$$

$G = \mathrm{cone}\,\{q\}$ and $h : A \times A \to Y$, $h(a_1, a_2) = \frac{\varepsilon}{\gamma}\eta(a_1, a_2)q$. All assumptions of Corollary 3.5 hold true. Indeed, it is obvious that $E + (E \cup G) \subseteq E$ and $\lambda \in (G^+ \cap E^{s+})\backslash\{0\}$. Moreover, h satisfies the triangle inequality property with respect to $\leq_G$ since η fulfills the triangle inequality property.

In addition, as $E \subseteq D$, $q \in D$, D is a convex cone and the values of h are nonnegative we have that

$$S_{E,h}(a_0) \subseteq \{a \in A : g(a) \leq_D g(a_0)\}.$$

Therefore,

$$(\lambda \circ g)(a) \geq (\lambda \circ g)(a_0) - (\varepsilon + \delta), \quad \forall a \in S_{E,h}(a_0),$$

since $a_0 \in \mathrm{EH}(g, \lambda, \varepsilon + \delta)$, and then,

$$\begin{aligned}\inf\{\lambda(g(a)) + \lambda(h(a_0, a)) : a \in S_{E,h}(a_0)\} &= \inf\{\lambda(g(a)) + \frac{\varepsilon}{\gamma}\eta(a_0, a) : a \in S_{E,h}(a_0)\} \\ &\geq \inf\{\lambda(g(a)) : a \in S_{E,h}(a_0)\} \\ &\geq (\lambda \circ g)(a_0) - (\varepsilon + \delta) \\ &> -\infty.\end{aligned}$$

Thus, there exists $\bar{a} \in A$ such that

(a) $g(\bar{a}) + \frac{\varepsilon}{\gamma}\eta(a_0, \bar{a})q \leq_E g(a_0)$ or $\bar{a} = a_0$,
(c) $g(a) + \frac{\varepsilon}{\gamma}\eta(\bar{a}, a)q \not\leq_E g(\bar{a})$, $\forall a \in A$.

Clearly, assertions (a) and (b) are equivalent to the next ones:

(a) Both $g(\bar{a}) + \frac{\varepsilon}{\gamma}\eta(a_0, \bar{a})q \leq_D g(a_0)$ and $\lambda(g(\bar{a})) + \frac{\varepsilon}{\gamma}\eta(a_0, \bar{a}) \leq \lambda(g(a_0)) - \delta$, or $\bar{a} = a_0$.
(c) For each $a \in A$ such that $g(a) + \frac{\varepsilon}{\gamma}\eta(\bar{a}, a)q \leq_D g(\bar{a})$, we have that $\lambda(g(\bar{a})) - \delta < \lambda(g(a)) + \frac{\varepsilon}{\gamma}\eta(\bar{a}, a)$.

Finally, if $\bar{a} \neq a_0$, then by the first part of statement (a) we see that $g(\bar{a}) \leq_D g(a_0)$. Then, as $a_0 \in \mathrm{EH}(g, \lambda, \varepsilon + \delta)$, by the second part of statement (a) it follows that

$$\frac{\varepsilon}{\gamma}\eta(a_0, \bar{a}) + \delta \leq \lambda(g(a_0)) - \lambda(g(\bar{a})) \leq \varepsilon + \delta$$

and then $\eta(a_0, \bar{a}) \leq \gamma$, which finishes the proof. □

Notice that Corollary 3.8 encompasses Corollary 2.5 by considering $Y = \mathbb{R}$, $D = \mathbb{R}_+$ and $q = \lambda = 1$.

In [14, Theorem 4.1] Gutiérrez et al. established an approximate EVP for a vector-valued bifunction via an improvement set E. The subsequent result shows that Corollary 3.5 encompasses this EVP.

Corollary 3.9 (Approximate vectorial EVP for bifunctions)**.** *Let (X, d) be a metric space, Y be a real linear space, $E \subseteq Y$ be an improvement set with respect to a convex cone $D \subseteq Y$ such that $E+E \subseteq E$. Consider a vector-valued bifunction $f : X \times X \to Y$, $x_0 \in X$ and $q \in D\backslash\{0\}$.*

Assume that there exists $\lambda \in E^{s+}\backslash\{0\}$ such that

$$\inf\{\lambda(f(x_0, x)) + \lambda(q)d(x_0, x) \ : \ x \in S_{E,q}(x_0)\} > -\infty, \tag{15}$$

where $S_{E,q}(x_0) := \{x \in X : f(x_0, x) + d(x_0, x)q \leq_E f(x_0, x_0)]$. Then, there exists $\bar{x} \in X$ such that

(a) *$f(x_0, \bar{x}) + d(x_0, \bar{x})q \leq_E f(x_0, x_0)$ or $\bar{x} = x_0$,*
(b) *$f(x_0, x) + d(\bar{x}, x)q \not\leq_E f(x_0, \bar{x})$, $\forall x \in X$.*

Proof. Define $A := X$, $g : A \to Y$, $g(a) := f(x_0, a)$, for all $a \in A$, $h : A \times A \to Y$, $h(a, u) := d(a, u)q$, for all $a, u \in A$, and $G := \mathrm{cone}\,\{q\}$.

It is easy to check that h satisfies the triangle inequality property with respect to $\leq_G$. Moreover, since $E+E \subseteq E$, $E+D=E$ and $q \in D$ we have that $E+(G\cup E) \subseteq E$. On the other hand, as G is coradiant and $E+G \subseteq E$ it follows that $G^+ \cap E^{s+} = E^{s+}$ (see Remark 4).

Thus, all assumptions of Corollary 3.5 are fulfilled and its assertions prove this result. □

Remark 5. Notice that in Corollary 3.9, the metric space (X, d) could be replaced with a nonempty set A and a function $\eta : A \times A \to \mathbb{R}$ satisfying the triangle inequality property. In addition, the requirements E being an improvement set with respect to a convex cone $D \subseteq Y$ and $q \in D\backslash\{0\}$ can be also clarified by considering an arbitrary point $q \in Y\backslash\{0\}$ and an improvement set E with respect to the cone generated by $\{q\}$.

On the other hand, the assumptions $E + D = E$ and $E + E \subseteq E$ could be dropped by applying Theorem 3.4 instead of Corollary 3.5. In this case, the hypothesis $\lambda(q) \geq 0$ has to be assumed. Notice that the corresponding bounded from below hypothesis (11) will be in general stronger than (15) and also the conclusion (a) will be weaker.

Corollary 3.9 also reduces to Corollary 3.6 by considering the bifunction $\bar{f} : X\times X \to Y$, $\bar{f}(x_1, x_2) = g(x_2) - g(x_1)$.

4. Exact Ekeland variational principles

In this section, as a simple consequence of Theorem 2.6, we provide a revised vectorial version of the exact EVP and its corollaries. The subsequent concept is required.

Definition 4.1. Let Y be a real linear space and consider a nonempty set $G \subseteq Y$ and a function $\xi : Y \to \mathbb{R} \cup \{\pm\infty\}$. We say that a function $\rho : \operatorname{cone} G \to \mathbb{R}$ is an additive minorant of ξ in $\operatorname{cone} G$ (additive minorant of ξ for short) if

$$y \in Y, z \in \operatorname{cone} G, \quad \xi(y + z) \geq \xi(y) + \rho(z).$$

In the subsequent lemma we provide some classes of functions that fulfill this definition.

Lemma 4.2. *Let Y be a real linear space and consider a nonempty set $G \subseteq Y$.*

- (i) *If $\rho_i : \operatorname{cone} G \to \mathbb{R}$ is an additive minorant of $\xi_i : Y \to \mathbb{R} \cup \{+\infty\}$, i=1,2, then $\rho_1 + \rho_2$ is an additive minorant of $\xi_1 + \xi_2$.*
- (ii) *Consider an arbitrary superadditive function $\xi : Y \to \mathbb{R} \cup \{+\infty\}$. Then, each function $\rho : \operatorname{cone} G \to \mathbb{R}$ such that $\rho \leq \xi$ is an additive minorant of ξ. In particular, this assertion can be applied to each function ξ in the algebraic dual space Y' of Y.*
- (iii) *Let $G = \operatorname{cone}\{q\}$ be the cone generated by $q \in Y\backslash\{0\}$. The function $\rho : G \to \mathbb{R}$, $\rho(tq) = t$, for all $t \in \mathbb{R}_+$ is an additive minorant for each q-translative function; in particular, the Gertewitz's scalarization function φ_E^q defined by (1).*

Subsequently we formulate a refined vectorial version of the exact EVP for a general free disposal set E in which the usual closedness assumption of the dynamic system $S_{E,q}$ is weakened.

Theorem 4.3 (Revised vectorial version of the exact EVP)**.** *Let (X,d) be a complete metric space, Y be a real linear space, $g : X \to Y$ be a vector-valued function, E be a nonempty set in Y and $q \in Y\backslash\{0\}$ with $E+E \subseteq E$ and*

$$E + (0,+\infty)q \subseteq E. \tag{16}$$

Assume that there is a $\leq_E$-nondecreasing function $\xi : Y \to I\!R \cup \{\pm\infty\}$ and a nonzero positively homogeneous additive minorant function ρ of ξ in $\operatorname{cone}\{q\}$. Consider the set-valued mapping $S_{E,q} : X \rightrightarrows X$,

$$S_{E,q}(x) := \{y \in X \ : \ g(y) + d(x,y)q \leq_E g(x)\}$$

and a point $x_0 \in \operatorname{dom}(\xi \circ g)$ such that

$$c_2 := \inf\{\xi(g(x)) \ : \ x \in S_{E,q}(x_0)\} > -\infty \tag{17}$$

and the extended monotonicity closedness assumption is fulfilled at x_0 in the sense that for every generalized Picard sequence (x_n) of $S_{E,q}$ starting from x_0, converging to x, and $x_{n+1} \neq x_n$ for all $n \in I\!N$, one has $S_{E,q}(x) \subseteq \operatorname{cl} S_{E,q}(x_n)$ for all $n \in I\!N$. Then there exists $\bar{x} \in X$ such that

(a) $\bar{x} \in \operatorname{cl} S_{E,q}(x_0)$ *or* $\bar{x} = x_0$,
(b) $g(x) + d(\bar{x},x)q \not\leq_E g(\bar{x})$, $\forall x \in X\backslash\{\bar{x}\}$.

We will prove this theorem by applying Theorem 2.6. That's the reason for the next two propositions.

Proposition 4.4. *For each $x \in X$ and $y \in S_{E,q}(x)$ it follows that $S_{E,q}(y) \subseteq S_{E,q}(x)$. In particular we have that $S_{E,q}$ satisfies condition (B1) of Theorem 2.6.*

Proof. Take arbitrary elements $x \in X$, $y \in S_{E,q}(x)$ and $z \in S_{E,q}(y)$. By the definition of $S_{E,q}$ and $\leq_E$, we have

$$g(z) + d(y,z)q \in g(y) - E \quad \text{and} \quad g(y) + d(x,y)q \in g(x) - E.$$

Combining these two inclusions together while taking into account $E + E \subseteq E$, we have

$$g(z) + d(x,z)q \in g(x) + (d(x,z) - d(x,y) - d(y,z))q - E.$$

The triangle inequality property of the metric d ensures that $d(x,z)-d(x,y)-d(y,z) \leq 0$ and thus condition (16) forces $(d(x,z) - d(x,y) - d(y,z))q - E \subseteq -E$ and

$$g(z) + d(x,z)q \in g(x) - E$$

clearly implying $z \in S_{E,q}(x)$. Since z was arbitrary in $S_{E,q}(y)$, $S_{E,q}(y) \subseteq S_{E,q}(x)$. The validity of condition (B1) in Theorem 2.6 is verified. □

Proposition 4.5. *$S_{E,q}$ satisfies condition (B2) of Theorem 2.6 for all starting point $x_0 \in \operatorname{dom}(\xi \circ g)$.*

Proof. For each generalized Picard sequence $(x_n) \subseteq X$ (if exists), we have $x_{n+1} \in S_{E,q}(x_n)$ for all $n \in \mathbb{N}$; i.e.,

$$g(x_{n+1}) + d(x_n, x_{n+1})q \in g(x_n) - E \quad (n \in I\!N).$$

Adding this relation for $n = 0, 1, \ldots, k$ while taking into account $E + E \subseteq E$, we have

$$g(x_{k+1}) + \sum_{n=0}^{k} d(x_n, x_{n+1})q \in g(x_0) - E.$$

Let $\xi : Y \to I\!R \cup \{\pm\infty\}$ be a $\leq_E$-nondecreasing function satisfying property (17) and consider a nonzero positively homogeneous additive minorant function ρ of ξ in cone $\{q\}$. It follows that

$$\xi(g(x_{k+1})) + \rho(q)\sum_{n=0}^{k} d(x_n, x_{n+1}) \leq \xi(g(x_{k+1}) + \sum_{n=0}^{k} d(x_n, x_{n+1})q) \leq \xi(g(x_0)). \quad (18)$$

By Proposition 4.4 we see that $(x_n) \subseteq S_{E,q}(x_0)$. Then, by property (17) we deduce that

$$\rho(q)\sum_{n=0}^{k} d(x_n, x_{n+1}) \leq \xi(g(x_0)) - \xi(g(x_{k+1})) \leq \xi(g(x_0)) - c_2.$$

Since $\xi(g(x_0)) - c_2$ is a number, $\rho(q) > 0$ and k was arbitrary, we could pass the last inequality to limit as $k \to +\infty$ to conclude that

$$\sum_{n=0}^{+\infty} d(x_n, x_{n+1}) < +\infty$$

clearly ensuring that $d(x_n, x_{n+1}) \to 0$ as $n \to +\infty$; i.e., (B2) is fulfilled. □

Proof of Theorem 4.3. The set-valued mapping $S_{E,q}$ satisfies conditions (B1), (B2) and (B3) of Theorem 2.6 due to Propositions 4.4, 4.5 and the imposed extended monotonicity closedness property of $S_{E,q}$ at x_0. Employing the revised Dancs-Hegedüs-Medvegyev's fixed point theorem, there is $\bar{x} \in \operatorname{cl} S_{E,q}(x_0) \cup \{x_0\}$ such that $S_{E,q}(\bar{x}) \subseteq \{\bar{x}\}$. The proof is complete. □

Remark 6. 1. Theorem 4.3 encompasses [14, Theorem 4.1(ii)] by taking $E = \operatorname{vcl}_q D$ and $g(x) = f(x_0, x)$, where D is a convex cone in Y and $f : X \times X \to Y$. Notice that the closedness of the sets $S_{E,q}(x)$ for $x \in S_{E,q}(x_0)$ is not required in Theorem 4.3.

2. If the boundedness condition (17) holds for a function $\xi \in E^{+s}$ (and so the domination set E leads to approximate solutions of the problem), it would be better to use the approximate EVP established in Corollary 3.5 since it doesn't require the extended monotonicity closedness assumption imposed on $S_{E,q}$. Moreover, the boundedness assumption is even weaker than the one in Theorem 4.3, as

$$\inf\{\xi(g(x)) \ : \ x \in S_{E,q}(x_0)\} > -\infty$$

implies

$$\inf\{\xi(g(x)) + \xi(q)d(x_0, x) \ : \ x \in S_{E,q}(x_0)\} > -\infty.$$

3. Assumption (16) automatically follows when E is a free disposal set with respect to a convex cone D and $q \in D\backslash\{0\}$. Actually, condition (16) means that E is a free disposal set with respect to the ray generated by q, which is a convex cone. Therefore, this new EVP holds for a large class of free disposal sets. For instance, let $E \subseteq I\!R^2$ be the epigraph of the function $\varphi : I\!R_+ \to I\!R$ with $\varphi(0) = 0$ and $\varphi(x) = 1$ for all $x \neq 0$. Clearly, E is neither a cone nor a convex set. However, $E + E \subseteq E$ and condition (16) with $q = (0, 1)$ is satisfied.

The next result is a located version of Theorem 4.3. Its proof coincides with the proof of this theorem, but applying Corollary 2.8 instead of Theorem 2.6.

Corollary 4.6. *Consider the same setting and hypotheses as in Theorem 4.3 and assume the strong extended monotonicity closedness assumption at x_0: for every generalized Picard sequence (x_n) of $S_{E,q}$ whose starting point is x_0, converging to x and $x_{n+1} \neq x_n$ for all $n \in I\!N$, one has $S_{E,q}(x) \subseteq S_{E,q}(x_n)$ for all $n \in I\!N$ and $x \in S_{E,q}(x_0)$. Then there exists $\bar{x} \in X$ such that*

(a) $g(\bar{x}) + d(x_0, \bar{x},)q \leq_E g(x_0)$ *or* $\bar{x} = x_0$,
(b) $g(x) + d(\bar{x}, x)q \not\leq_E g(\bar{x})$, $\forall x \in X\backslash\{\bar{x}\}$.

By Corollary 4.6, one can state Ekeland variational for vector bifunctions. Next, as an illustration, we derive a generalization of [2, Theorem 3.6], where the ordering cone D could not satisfy the q-vectorial closedness property.

Corollary 4.7. *Let (X, d) be a complete metric space, $f : X \times X \to Y$ and $x_0 \in X$. Consider a convex cone $D \subseteq Y$, $q \in D\backslash\{0\}$ and $\lambda \in D^+$ such that $\lambda(q) > 0$. Suppose that there exists $g : X \to Y$ satisfying:*

(i) $f(x_1, x_2) \geq_D g(x_2) - g(x_1)$, *for all* $x_1, x_2 \in X$.
(ii) $\inf\{\lambda(g(x)) : g(x) \leq_D g(x_0)\} > -\infty$.
(iii) *The sets $S_{D,q}(x)$ are closed, for each $x \in X$ such that $g(x) \leq_D g(x_0)$.*

Then, there exists $\bar{x} \in X$ such that

(a) $g(\bar{x}) - g(x_0) + d(x_0, \bar{x})q \leq_D 0$, *or* $\bar{x} = x_0$,
(b) $f(\bar{x}, x) + d(\bar{x}, x)q \not\leq_D 0$, *for all* $x \in X\backslash\{\bar{x}\}$.

Proof. Let us apply Corollary 4.6 to the following data: $E = D$, $\xi := \lambda$ and $\rho := \lambda$. We only check assumption (17) and the strong extended monotonicity closedness hypothesis of the dynamic system $S_{D,q}$ at x_0 as the other ones are obvious.

Clearly,

$$S_{D,q}(x_0) \subseteq \{x \in X : g(x) \leq_D g(x_0)\} \tag{19}$$

since $q \in D$ and D is a convex cone. Thus, by assumption (ii) we have that

$$\inf\{\lambda(g(x)) : x \in S_{D,q}(x_0)\} \geq \inf\{\lambda(g(x)) : g(x) \leq_D g(x_0)\} > -\infty$$

and assumption (17) is fulfilled.

Let (x_n) be a generalized Picard sequence of $S_{E,q}$ whose starting point is x_0 and suppose that $x_n \to x$. For each $n \in I\!N$, by Proposition 4.4 we see that $x_{n+k} \in S_{D,q}(x_n) \subseteq S_{D,q}(x_0)$, for all $k \in I\!N$, $k \geq 1$. Then, by (19) and assumption (iii) it follows that $x \in S_{D,q}(x_n)$, and by applying Proposition 4.4 again we deduce that $S_{D,q}(x) \subseteq S_{D,q}(x_n)$. Therefore, the dynamic system $S_{D,q}$ satisfies the strong extended monotonicity closedness hypothesis at x_0.

By Corollary 4.6, there exists $\bar{x} \in X$ such that

(a) $g(\bar{x}) + d(x_0, \bar{x},)q \leq_D g(x_0)$ or $\bar{x} = x_0$,
(b') $g(x) + d(\bar{x}, x)q \not\leq_D g(\bar{x})$, $\forall x \in X \backslash \{\bar{x}\}$.

Condition (b') implies assertion (b). Indeed, suppose that there exists $x \in X \backslash \{\bar{x}\}$ such that $f(\bar{x}, x) + d(\bar{x}, x)q \leq_D 0$. Then, by assumption (i) we see that

$$g(x) - g(\bar{x}) + d(\bar{x}, x)q \leq_D f(\bar{x}, x) + d(\bar{x}, x)q \leq_D 0,$$

that is contrary to (b'). Therefore (b) holds true and the proof finishes. □

The main role of function ξ in Theorem 4.3 is assumption (17) that could not be satisfied for each $\leq_E$-nondecreasing function $\xi : Y \to I\!R \cup \{\pm\infty\}$. This issue can be easily addressed whenever the starting point x_0 is a Kutateladze's approximate solution, as it is showed in the subsequent result.

Corollary 4.8. *Let (X, d) be a complete metric space, Y be a real linear space, $D \subseteq Y$ be a convex cone and $g : X \to Y$ be a vector-valued function. Consider $q \in D \backslash (-D)$, $\varepsilon \geq 0$ and a point $x_0 \in \mathrm{EK}(g, \varepsilon q)$. Suppose that the set-valued mapping $S_{D,q} : X \rightrightarrows X$ satisfies the extended monotonicity closedness assumption at x_0. Then there exists $\bar{x} \in X$ such that*

(a) $\bar{x} \in \mathrm{cl}\, S_{D,q}(x_0)$ *or* $\bar{x} = x_0$,
(b) $g(x) + d(\bar{x}, x)q \not\leq_D g(\bar{x})$, $\forall x \in X \backslash \{\bar{x}\}$.

If $S_{D,q}$ fulfills the strong extended monotonicity closedness assumption at x_0, then assertion (a) can be replaced by the next stronger one:

(a') $g(\bar{x}) + d(x_0, \bar{x})q \leq_D g(x_0)$.

Proof. The result follows by applying Theorem 4.3 and Corollary 4.6 to the following data: $\tilde{g} := g - g(x_0)$, $E := D$ and $\xi := \varphi_D^q$.

Indeed, as it was already mentioned after Definition 4.1, function φ_D^q is q-translative. Then, by Lemma 4.2(iii) it has a nonzero positively homogeneous additive minorant function in cone$\{q\}$. Moreover, φ_D^q is $\leq_D$-nondecreasing (see Lemma 2.2(iv)) and it is clear from the definition and assumption $q \notin -D$ that $\varphi_D^q(0) = 0$. Then, $x_0 \in \mathrm{dom}\,(\varphi_D^q \circ \tilde{g})$.

Finally, since $x_0 \in \mathrm{EK}(g, \varepsilon q)$, we have that $g(x) - g(x_0) \notin -\varepsilon q - D \backslash \{0\}$, for all $x \in X$, and so $\varphi_D^q(\tilde{g}(x)) \geq \varepsilon$, for all $x \in X$. Therefore, assumption (17) is fulfilled with $c_2 := \varepsilon$. □

Remark 7. Corollary 4.8 improves the main results of [23, Section 3], where the convex cone D is assumed to be algebraic solid and $q \in \mathrm{core}\, D$. Indeed, on the one hand, it follows by the proof of Corollary 4.7 that the closedness of the set $S_{D,q}(x)$, for all $x \in X$, implies that the dynamic system $S_{D,q}$ satisfies the strong extended monotonicity closedness property at every point $x \in X$.

On the other hand, if $\varphi_D^q \circ g$ is bounded from below and $\varphi_D^q \circ g > c$, $c \in I\!R$, then for each $x \in X$ we have that $x \in \mathrm{EK}(g, (\varphi_D^q(g(x)) - c)q)$. Indeed, if there exists $x' \in X$ such that $g(x') \leq_D g(x) - (\varphi_D^q(g(x)) - c)q$, as φ_D^q is $\leq_D$-nondecreasing and q-translative (see Lemma 2.2) we have

$$\varphi_D^q(g(x')) \leq \varphi_D^q(g(x) - (\varphi_D^q(g(x)) - c)q) = \varphi_D^q(g(x)) - \varphi_D^q(g(x) + c = c,$$

that is a contradiction.

Thus, Corollary 4.8 encompasses the formulation of [23, Corollary 3.1] for a complete metric space (X, d) and $w = (1/\varepsilon)d$ (see [23, Lemma 2.2] with $f(x_1, x_2) = \phi(x_2) - \phi(x_1)$ to check the closedness of the sets $S_{D,q}(x)$, for all $x \in X$). It is worth noticing that in the less general setting of a complete metric space instead of a left complete quasi metric space and a distance instead of a W-distance, Corollary 4.8 improves [23, Corollary 3.1] since neither the algebraic solidness nor the q-vectorial closedness of D are required.

Both [23, Theorem 3.1] and [23, Theorem 3.2] are also improved by Corollary 4.8, since [23, Theorem 3.2] is a consequence of [23, Corollary 3.1] (compare, for instance, with Corollaries 3.6 and 4.7 in this work) and [23, Theorem 3.1] results of applying [23, Theorem 3.2] to the function $\phi : X \to Y$, $\phi(x) = f(x_0, x)$ whenever f is diagonal null, i.e., $f(x, x) = 0$ for all $x \in X$.

In [3, Section 3], similar results to the ones in [23, Section 3] were stated in the stronger framework of a locally convex Hausdorff topological linear space Y ordered by a closed convex cone D with nonempty interior. By the same reasons as above, Corollary 4.8 can be viewed as an improvement of them.

As a consequence, we will obtain the following EVP for vector-valued functions being strictly decreasing lower-semicontinuous. This type of lower-semicontinuity generalizes the decreasing lower-semicontinuity property used in [7]; the latter is known also as sequentially submonotonicity (see [12,19,20]). Recall that a sequence $(y_n) \subseteq Y$ is said to be $\leq_D$-decreasing (resp. strictly $\leq_D$-decreasing) if $y_{n+1} \leq y_n$ (resp. $y_{n+1} \leq y_n$ and $y_{n+1} \neq y_n$) for all $n \in I\!N$.

Definition 4.9 (Strictly $\leq_D$-decreasing lower-semicontinuity)**.** Let X be a topological space, Y be a real linear space and $D \subseteq Y$ be a convex cone. A vector-valued function $g : X \to Y$ is said to be $\leq_D$-decreasing (resp. strictly $\leq_D$-decreasing) lower semicontinuous (lsc for short) if for every convergent sequence $(x_n) \subseteq X$ with the limit $\bar{x}$ such that $(g(x_n))$ is $\leq_D$-decreasing (resp. strictly $\leq_D$-decreasing), one has

$$g(\bar{x}) \leq_D g(x_n), \ \forall n \in I\!N.$$

Obviously, each $\leq_D$-decreasing lsc function is also strictly $\leq_D$-decreasing lsc, but the converse implication is not true in general. For instance, consider $X = Y = I\!R$, $D = I\!R_+$ and the function $g : I\!R \to I\!R$,

$$g(x) := \begin{cases} 0 & \text{if} \quad x < 0, \\ 2 & \text{if} \quad x = 0, \\ 1 & \text{if} \quad x > 0. \end{cases}$$

It is easy to check that g is strictly $\leq_D$-decreasing lsc since there is no sequence $(x_n) \subseteq X$ such that $(g(x_n))$ is strictly $\leq_D$-decreasing. However, g is not $\leq_D$-decreasing lsc. Indeed, the sequence $x_n := 1/n$, for all $n \in \mathbb{N}\backslash\{0\}$, converges to $\bar{x} = 0$, $(g(x_n))$ is $\leq_D$-decreasing, but $g(\bar{x}) \not\leq_D g(x_n)$, for all $n \in \mathbb{N}\backslash\{0\}$.

Corollary 4.10 (EVP for strictly $\leq_D$-decreasing lsc functions)**.** *Let (X, d) be a complete metric space, Y be a real linear space, $D \subseteq Y$ be a convex cone and $g : X \to Y$ be a vector-valued function. Consider $q \in D\backslash(-D)$, $\varepsilon \geq 0$ and a point $x_0 \in \mathrm{EK}(g, \varepsilon q)$. Suppose that g is strictly $\leq_D$-decreasing lsc and D is q-vectorial closed. Then there exists $\bar{x} \in X$ such that*

(a) $g(\bar{x}) + d(x_0, \bar{x})q \leq_D g(x_0)$,
(b) $g(x) + d(\bar{x}, x)q \not\leq_D g(\bar{x})$, $\forall x \in X\backslash\{\bar{x}\}$.

Proof. By Corollary 4.8, it is sufficient to check the validity of the stronger extended monotonicity closedness assumption at x_0 of the set-valued mapping $S_{D,q}$. For this aim, fix a generalized Picard sequence (x_n) of $S_{D,q}$ with starting point x_0 such that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$ and $x_n \to x$. Taking into account the definition of $S_{D,q}$, we have

$$g(x_{n+1}) + d(x_n, x_{n+1})q \leq_D g(x_n), \quad \forall n \in \mathbb{N}.$$

Then, as $q \in D\backslash(-D)$, for each $n \in \mathbb{N}$ we see that $g(x_{n+1}) \leq_D g(x_{n+1})$ and $g(x_{n+1}) \neq g(x_n)$. Thus, the sequence $(g(x_n))$ is strictly $\leq_D$-decreasing. Since g is strictly $\leq_D$-decreasing lsc and (x_n) converges to x, we have $g(x) \leq_D g(x_n)$ for all $n \in \mathbb{N}$. As $x_{n+k} \in S_{D,q}(x_n)$ for all $n, k \in \mathbb{N}$, we have

$$g(x) + d(x_n, x_{n+k})q \leq_D g(x_{n+k}) + d(x_n, x_{n+k})q \leq_D g(x_n). \tag{20}$$

We claim that, for each $n \in \mathbb{N}$, $x \in S_{D,q}(x_n)$, i.e.,

$$g(x) + d(x_n, x)q \leq_D g(x_n). \tag{21}$$

Indeed, as $q \in D$, if there exists $k \in \mathbb{N}$ such that $d(x_n, x) < d(x_n, x_{n+k})$, by (20) it follows that

$$g(x) + d(x_n, x)q \leq_D g(x) + d(x_n, x_{n+k})q \leq_D g(x_n)$$

and statement (21) holds true. Otherwise, for each $k \in \mathbb{N}$, $t_k := d(x_n, x) - d(x_n, x_{n+k}) \geq 0$, $t_k \to 0$ and

$$g(x_n) - g(x) - d(x_n, x)q + t_k q \in D.$$

Therefore, $g(x_n) - g(x) - d(x_n, x)q \in \mathrm{vcl}_q\, D = D$ and (21) is also satisfied.

By Proposition 4.4 with $E = D$ we see that $S_{D,q}(x) \subseteq S_{D,q}(x_n)$. Since n was arbitrary, the stronger extended monotonicity closedness assumption at x_0 of the set-valued mapping $S_{D,q}$ holds true. □

Note that in [5, Corollary 3] Bao et al. established a version of EVP for strict decreasing lsc extended-real-valued functions in pseudo-quasimetric spaces.

5. Conclusions

In this paper, we state a really general approximate scalar EVP and we weaken several assumptions of Dancs-Hegedüs-Medvegyev's fixed point theorem in [9, Theorem 3.1]. Both results allow us to establish better approximate and exact EVPs for vector-valued functions in vector optimization with free-disposal domination sets and deduce from them a number of particular versions of vector EVPs which include known and new results. In particular, an EVP for strictly $\leq_D$-decreasing lsc vector-valued functions is deduced. The results in this paper can be easily extended to the setting of quasi-metric spaces.

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