Ekeland variational principles for vector equilibrium problems

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ABSTRACT
This work concerns Ekeland variational principles for scalar and vector cyclically antimonotone bifunctions on complete metric spaces. The scalar results work for extended bifunctions and they are obtained by a generalized version of the Dancs-Hegedüs-Medvegyev's fixed point theorem. As a result, weaker lower-semicontinuity assumptions have been considered, that generalize the concept of strictly decreasingly lower-semicontinuous real-valued function. The vector results are derived from the previous ones by a scalarization approach and are based on new notions of cyclical antimonotonicity, lower boundedness and strictly decreasingly lower-semicontinuity for vector bifunctions. Several results in the literature are improved since they are stated by weaker assumptions.

KEYWORDS
Ekeland variational principle; equilibrium problem; strictly decreasingly lower-semicontinuity; cyclically antimonotone bifunction; triangle inequality property

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1. Introduction

Lots of basic results in mathematical programming have been generalized to equilibrium problems since they were introduced in 1994 by Blum and Oettli [1]. These contributions are really valuable, because equilibrium problems encompass several fundamental issues in applied mathematics, like optimization problems, variational inequalities, saddle point theorems, Nash equilibrium problems, fixed-point theorems, complementary problems and so on.

The Ekeland variational principle is one of the aforementioned basic results. The first equilibrium version of the Ekeland variational principle was stated by Oettli and Théra [2] in 1993 and it works for bifunctions satisfying the so-called triangle inequality property.

Although this seminal Ekeland variational principle has been reformulated and generalized in several ways (see [3–5]), only in the recent works [6–8] the triangle inequality

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assumption has been weakened via the cyclical antimonotonicity condition. Notice that an unconstrained variational inequality problem can be reformulated as an equilibrium problem satisfying the triangle inequality property if and only if the linear operator that defines the variational inequality is constant (see [7,9]). This assertion illustrates how strong the triangle inequality property is in the setting of an equilibrium problem.

This paper addresses Ekeland variational principles for vector equilibrium problems. Analogously to the scalar case, one can find in the literature these results for vector bifunctions that fulfill the triangle inequality property (see [10–14]) and also for cyclically antimonotone vector bifunctions (see [14–16]). In [14,15], the obtained vector equilibrium versions of the Ekeland variational principle depend strongly on the existence of the supremum for each upper bounded set. As a result, they can be applied provided that the final space of the bifunction is a real bounded complete linear space, i.e., provided that the ordering cone is strongly minihedral (see [14,15]).

The approach in [16] is different from the previous one and it is based on the scalarization of the nominal equilibrium problem through linear functionals in the positive polar cone of the ordering cone. Therefore, the Ekeland variational principles in [16] can be applied to vector bifunctions whose final space is locally convex.

The main objective of this work is to derive Ekeland variational principles more general than the ones in [14–16]. In this sense, the most powerful results are Theorems 3.9, 4.12 and 4.17. For our aim, a new concept of cyclically antimonotone vector bifunction is introduced by a scalarization approach. We underline that in our main results, both the lower-semicontinuity assumptions and the lower boundedness assumptions involve only the objective bifunction of the problem.

This work is structured as follows. In Section 2, the setting of the paper is introduced and a basic Ekeland variational principle for strictly decreasingly lower-semicontinuous extended-real-valued functions is recalled. This result is obtained from a generalized version of the so-called Dancs-Hegedüs-Medvegyev's fixed point theorem. In Section 3, several equilibrium versions of the Ekeland variational principle are derived for cyclically antimonotone extended-real-valued bifunctions. Therefore, they can be applied to extended-real-valued equilibrium problems whose objective bifunction does not satisfy the triangle inequality property. In Section 4, we address Ekeland variational principles for vector equilibrium problems, which are derived by a new notion of cyclically antimonotone vector bifunction and the results of the previous section.

The main Ekeland variational principles of this paper encompass and extend some others in the literature for scalar and vector bifunctions, like the ones by Miholca [15] and Qiu [16], because weaker assumptions are considered. In particular, it is worth noticing several new strictly decreasingly lower-semicontinuity notions not only for real bifunctions but also for vector bifunctions.

2. Preliminaries

Let Y be a real locally convex Hausdorff topological linear space and consider the preorder $\leq_D$ in Y defined by a closed convex cone $D \subset Y$:

$$y_1, y_2 \in Y, \quad y_1 \leq_D y_2 \iff y_2 - y_1 \in D. \tag{1}$$

The following property is obvious:

$$y_1, y_2, z_1, z_2 \in Y, \quad y_1 \leq_D y_2,\, z_1 \leq_D z_2 \Rightarrow y_1 + z_1 \leq_D y_2 + z_2. \tag{2}$$

Recall that D is said to be pointed if $D \cap (-D) = \{0\}$. In the sequel, $\mathbb{R}_+$ refers to the set of nonnegative real numbers. The positive polar cone of D is denoted by D^+, i.e.,

$$D^+ := \{\lambda \in Y^* : \lambda(d) \geq 0, \forall d \in D\},$$

where Y^* stands for the topological dual space of Y.

This work addresses the so-called vector equilibrium problem (VEP): Find $\bar{x} \in X$ such that

$$x \in X \backslash \{\bar{x}\},\ f(\bar{x}, x) \leq_D 0 \Rightarrow f(\bar{x}, x) = 0,$$

where X is a nonempty set and $f : X \times X \to Y$ is a bifunction. A point $\bar{x} \in X$ is said to be a strict solution of (VEP) if

$$f(\bar{x}, x) \not\leq_D 0 \quad \forall x \in X \backslash \{\bar{x}\}.$$

The formulations of the Ekeland variational principle in the framework of problem (VEP) that are studied in this paper look for strict solutions of the equilibrium problem whose vector objective bifunction is the following perturbation of the nominal vector bifunction: $f(\cdot,\cdot) + d(\cdot,\cdot)q : X \times X \to Y$, where (X, d) is a metric space and $q \in D\backslash\{0\}$.

If $Y = \mathbb{R}$ and $D = \mathbb{R}_+$, then (VEP) reduces to the following scalar equilibrium problem:

$$\text{Find } \bar{x} \in X \text{ such that } f(\bar{x}, x) \geq 0, \quad \forall x \in X \backslash \{\bar{x}\}.$$

As usual, the effective domain of an extended-real-function $g : X \to \mathbb{R} \cup \{\pm\infty\}$ is denoted by $\operatorname{dom} g$, i.e.,

$$\operatorname{dom} g := \{x \in X : g(x) < +\infty\}$$

and g is said to be proper if $\operatorname{dom} g \neq \emptyset$ and $g(x) > -\infty$ for all $x \in \operatorname{dom} g$.

The main mathematical tool of this work is the subsequent Ekeland variational principle. It is a simple consequence of the next generalization of the well-known Dancs-Hegedüs-Medvegyev's fixed point theorem (see [17, Theorem 3.1]).

Recall that $(x_n) \subset X$ is said to be a Picard sequence of a dynamical system $S : X \rightrightarrows X$ if $x_{n+1} \in S(x_n)$ for all $n \in \mathbb{N}$. It is said to be distinct if $x_{n+1} \neq x_n$, for all $n \in \mathbb{N}$.

Theorem 2.1. *Let (X, d) be a metric space and consider $x_0 \in X$ and a dynamical system $S : X \rightrightarrows X$ satisfying the conditions:*

(A1) $x \in S(x)$ for all $x \in S(x_0) \cup \{x_0\}$;
(A2) $x_2 \in S(x_1) \Rightarrow S(x_2) \subset S(x_1)$ for all $x_1 \in S(x_0)$;
(A3) For each Picard sequence $(x_n) \subset X$ of S whose initial point is x_0, we have that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$;
(A4) For each distinct and Cauchy Picard sequence (x_n) of S whose initial point is x_0, there exists $\bar{x} \in S(x_n)$ for all $n \in \mathbb{N}$;

Then there is a point $\bar{x} \in S(x_0)$ such that $S(\bar{x}) = \{\bar{x}\}$.

Notice that in Theorem 2.1 it is not assumed that the metric space is complete. As a result, assumption (A4) is different from the one imposed in [18, Corollary 2.4]: For each Picard sequence (x_n) of S being convergent to $\bar{x}$ and whose initial point is x_0, it follows that $\bar{x} \in S(x_n)$ for all $n \in I\!N$.

The next lower-semicontinuity notion was introduced in [19, Definition 9].

Definition 2.2. Let X be a topological space. A proper extended-real-valued function $g : X \to I\!R \cup \{+\infty\}$ is said to be strictly decreasingly lower-semicontinuous at a point $\bar{x} \in X$ ($<$-lsc at $\bar{x}$ in short form), if for every sequence $(x_n) \subset X$ converging to $\bar{x}$, one has

$$\forall n \in I\!N, g(x_{n+1}) < g(x_n) \Longrightarrow \forall n \in I\!N, g(\bar{x}) \le g(x_n).$$

The function g is called $<$-lsc when it is $<$-lsc at x, for all $x \in X$.

Remark 1. A close notion to the previous one is that of decreasingly lower-semicontinuous functions introduced by Kirk and Saliga [20] (called by them lower-semicontinuity from above) meaning that $g(\bar{x}) \le \lim_{n\to\infty} g(x_n)$ for every sequence (x_n) being convergent to $\bar{x}$ and satisfying $g(x_{n+1}) \le g(x_n)$, for all $n \in I\!N$.

Let us consider a function $g : I\!R \to I\!R$ defined by

$$g(x) := \begin{cases} 2^n & \text{if } n < x < n+1 \text{ and } n \in \mathbb{Z}, \\ \frac{1}{2}(2^{n-1} + 2^n) & \text{if } x = n \text{ and } n \in \mathbb{Z}. \end{cases}$$

Obviously, g is not either lower-semicontinuous or decreasingly lower-semicontinuous at every point $x = n$ for $n \in \mathbb{Z}$. But, g is $<$-lsc since there is no any convergent sequence (x_n) such that the sequence $(g(x_n))$ is strictly decreasing.

It is important to note that the sum of a $<$-lsc function and a continuous function might not be $<$-lsc. Consider the function g above and the continuous function $h : I\!R \to I\!R$ defined by $h(x) = -x + \frac{5}{2}$. The sequence (x_n) with $x_n := 1 - \frac{1}{n}$ converges to $\bar{x} = 1$. For each $n \ge 2$ we have

$$g(x_n) + h(x_n) = \frac{5}{2} + \frac{1}{n} \quad \text{and} \quad g(\bar{x}) + h(\bar{x}) = 3.$$

Since the sequence $(g(x_n)+h(x_n))$ is strictly decreasing, but $g(x_n)+h(x_n) < g(\bar{x})+h(\bar{x})$ for all $n \in I\!N$, $n \ge 3$, the function $g + h$ is not $<$-lsc at $\bar{x} = 1$.

Theorem 2.3. *Let (X, d) be a complete metric space and $g : X \to I\!R \cup \{+\infty\}$ be a proper extended-real-valued function. Assume that the function g is bounded from below and $<$-lsc. Then, for any $x_0 \in \operatorname{dom} g$, there is $\bar{x} \in X$ such that*

(a) $g(\bar{x}) + d(x_0, \bar{x}) \le g(x_0)$;
(b) $g(x) + d(\bar{x}, x) > g(\bar{x})$, *for all* $x \in X\backslash\{\bar{x}\}$.

Note that in [19, Corollary 3] Bao et al. established a version of the Ekeland variational principle in pseudo-quasimetric spaces which says that this variational principle holds for the class of $<$-lsc functions.

3. Ekeland variational principles for extended-real-valued bifunctions

As far as we know, the first Ekeland variational principle for an equilibrium problem was stated by Oettli and Théra [2]. This result can be applied to an extended-real-valued bifunction $h : X \times X \to I\!R \cup \{+\infty\}$ that is diagonal null, i.e., $h(x, x) = 0$ for all $x \in X$, and satisfies the so-called triangle inequality property:

$$h(x_1, x_3) \leq h(x_1, x_2) + h(x_2, x_3), \quad \forall x_1, x_2, x_3 \in X. \tag{3}$$

The notion of cyclically antimonotone real-valued function was introduced by Castellani and Giuli [7, Definition 2.11] in order to state Ekeland variational principles for an equilibrium problem whose objective bifunction is finite and does not satisfy the triangle inequality property.

Next, this concept is recalled in a slightly more general setting because it involves an extended-real-valued bifunction. This fact is needed not only to encompass the seminal Oettli and Théra's Ekeland variational principle, but also to apply it to vector equilibrium problems via a scalarization approach based on the well-known Gerstewitz's functional (see [21]).

Definition 3.1. Let $X \neq \emptyset$ and $h : X \times X \to I\!R \cup \{+\infty\}$. We say that h is cyclically antimonotone if for each finite nonempty set $\{x_1, x_2, \dots, x_n\} \subset X$ it follows that

$$h(x_1, x_2) + h(x_2, x_3) + \cdots + h(x_{n-1}, x_n) + h(x_n, x_1) \geq 0. \tag{4}$$

In the subsequent theorem, we characterize the cyclically antimonotone extended-real-valued bifunctions by a similar result as [7, Theorem 2.13]. It is proved for the convenience of the reader, since some technical adjustments must be carried out in order to avoid the improper value $+\infty - \infty$.

Theorem 3.2. *Let $X \neq \emptyset$ and $h : X \times X \to I\!R \cup \{+\infty\}$. If there exists an extended-real-valued function $g : X \to I\!R \cup \{+\infty\}$ such that*

$$\begin{cases} \operatorname{dom} h(\cdot, x) \subset \operatorname{dom} g, & \forall x \in X, \\ h(x_1, x_2) + g(x_2) \geq g(x_1), & \forall x_1, x_2 \in X, \end{cases} \tag{5}$$

then h is cyclically antimonotone. Conversely, if h is cyclically antimonotone and $\operatorname{dom} h(x_0, \cdot) = X$ *for some $x_0 \in X$, then there exists an extended-real-valued function* $g : X \to I\!R \cup \{+\infty\}$ *such that*

$$\begin{cases} \operatorname{dom} h(\cdot, x_0) \subset \operatorname{dom} g, \\ h(x_1, x_2) + g(x_2) \geq g(x_1), & \forall x_1, x_2 \in X. \end{cases} \tag{6}$$

Proof. Suppose that an extended-real-valued function $g : X \to I\!R \cup \{+\infty\}$ fulfills statement (5) and consider an arbitrary finite nonempty set $\{x_1, x_2, \dots, x_n\} \subset X$. Define $x_{n+1} := x_1$. Then, for each $j \in \{1, 2, \dots, n\}$ we have that

$$h(x_j, x_{j+1}) + g(x_{j+1}) \geq g(x_j). \tag{7}$$

If there exists $j \in \{1, 2, \dots, n\}$ such that $h(x_j, x_{j+1}) = +\infty$, then condition (4) is satisfied. Otherwise, by the first assertion of (5) we deduce that $g(x_j) < +\infty$, for all $j \in \{1, 2, \dots, n\}$ and adding relation (7) for $j = 1, 2, \dots, n$ we see that (4) is true.

Conversely, suppose that h is cyclically antimonotone and there exists a point $x_0 \in$

X such that $\operatorname{dom} h(x_0, \cdot) = X$. Define $g_{x_0} : X \to I\!R \cup \{\pm\infty\}$,

$$g_{x_0}(x) := \inf\{h(x, x_n) + h(x_n, x_{n-1}) + \cdots + h(x_1, x_0) : \forall n \in I\!N, \\ \forall \{x_1, x_2, \ldots, x_n\} \subset X\} \tag{8}$$

(the value $h(x, x_0)$ is considered whenever $n = 0$). It is clear that g_{x_0} satisfies the first assertion of (6). Moreover, as h is cyclically antimonotone, for each finite nonempty set $\{x_1, x_2, \ldots, x_n\} \subset X$ we have that

$$h(x, x_n) + h(x_n, x_{n-1}) + \cdots + h(x_1, x_0) \geq -h(x_0, x) > -\infty.$$

Thus, $g_{x_0}(x) > -\infty$, for all $x \in X$. Finally, in order to check the second statement of (6), let $\{u_1, u_2, \ldots, u_n\} \subset X$ be an arbitrary finite nonempty set and consider two points $x_1, x_2 \in X$. It follows that

$$h(x_1, x_2) + h(x_2, u_n) + h(u_n, u_{n-1}) + \cdots + h(u_1, x_0) \geq g_{x_0}(x_1).$$

As the set $\{u_1, u_2, \ldots, u_n\} \subset X$ is arbitrary, we have that

$$h(x_1, x_2) + g_{x_0}(x_2) \geq g_{x_0}(x_1)$$

and the proof is completed. □

Remark 2. 1. Notice that the first hypothesis of (5) could be replaced with the condition $\operatorname{dom} h(x, \cdot) \subset \operatorname{dom} g$, for all $x \in X$. Moreover, $g_{x_0}(x_0) < +\infty$ and if h fulfills the triangle inequality property, then we have $g_{x_0}(x) = h(x, x_0)$, for all $x \in X$.

2. Notice that a bifunction $h : X \times X \to I\!R \cup \{+\infty\}$ is cyclically antimonotone if and only if the bifunction $h' : X \times X \to I\!R \cup \{+\infty\}$, $h'(x_1, x_2) := h(x_2, x_1)$ is cyclically antimonotone. For this reason, Theorem 3.2 reduces to [7, Theorem 2.13] when the bifunction h is finite.

The first part of Remark 2 motivates the subsequent two corollaries.

Corollary 3.3. *An extended-real-valued bifunction $h : X \times X \to I\!R \cup \{+\infty\}$ is cyclically antimonotone provided that it satisfies the triangle inequality property (3) and there exists $x_0 \in X$ such that $\operatorname{dom} h(\cdot, x_0) = X$.*

Proof. Consider an arbitrary point $x_0 \in X$ satisfying $\operatorname{dom} h(\cdot, x_0) = X$ and define $g : X \to I\!R$, $g(x) = h(x, x_0)$, for all $x \in X$. Let $x_1, x_2 \in X$. By the triangle inequality property (3) we have that

$$h(x_1, x_2) + g(x_2) = h(x_1, x_2) + h(x_2, x_0) \geq h(x_1, x_0) = g(x_1).$$

Therefore, the result follows by applying the first part of Theorem 3.2. □

Corollary 3.4. *An extended-real-valued bifunction $h : X \times X \to I\!R \cup \{+\infty\}$ is cyclically antimonotone if it satisfies the triangle inequality (3) and is bounded from below in the second argument for each value of the first argument.*

Proof. Define $g(x) = \inf_{z \in X} h(x, z)$, for all $x \in X$. We have $g > -\infty$ because of the lower boundedness assumption. Let $x_1, x_2 \in X$. By the triangle inequality property

(3), for each $x \in X$ we have that

$$h(x_1, x_2) + h(x_2, x) \geq h(x_1, x) \geq g(x_1)$$

and since $x \in X$ is arbitrary we deduce that $h(x_1, x_2) + g(x_2) \geq g(x_1)$.

Consider $z, x \in X$ such that $h(z, x) < +\infty$. It follows that $g(z) \leq h(z, x) < +\infty$. Therefore, $\operatorname{dom} h(\cdot, x) \subset \operatorname{dom} g$ and the result follows by applying the first part of Theorem 3.2. □

In the next proposition, the function g_{x_0} introduced in the proof of Theorem 3.2 is related with any other function g that satisfies assertion (5).

Proposition 3.5. *Let $h : X \times X \to \mathbb{R} \cup \{+\infty\}$ be an extended-real-valued bifunction satisfying $\operatorname{dom} h(x_0, \cdot) = X$ for some $x_0 \in X$. Suppose that $g : X \to \mathbb{R} \cup \{+\infty\}$ fulfills condition (5). Then $g(x_0) < +\infty$ and the function g_{x_0} defined in (8) fulfills $g_{x_0}(x) \geq g(x) - g(x_0)$, for all $x \in X$.*

Proof. As $\operatorname{dom} h(x_0, \cdot) = X$ and $\operatorname{dom} h(\cdot, x) \subset \operatorname{dom} g$ for all $x \in X$, we deduce that $g(x_0) < +\infty$.

Let $x \in X$. If $x \notin \operatorname{dom} g_{x_0}$, the result is obvious. Suppose that $x \in \operatorname{dom} g_{x_0}$ and let $\{x_1, x_2, \ldots, x_n\} \subset X$ be any finite set such that

$$h(x, x_n) + h(x_n, x_{n-1}) + \cdots + h(x_2, x_1) + h(x_1, x_0) < +\infty.$$

Then, $h(x, x_n) < +\infty$ and $h(x_j, x_{j-1}) < +\infty$, for all $j \in \{1, 2, \ldots, n\}$. By (5) it follows that $g(x) < +\infty$, $g(x_j) < +\infty$, $h(x, x_n) \geq g(x) - g(x_n)$ and $h(x_j, x_{j-1}) \geq g(x_j) - g(x_{j-1})$, for all $j \in \{1, 2, \ldots, n\}$. Therefore,

$$\begin{aligned} h(x, x_n) &+ h(x_n, x_{n-1}) + \cdots + h(x_2, x_1) + h(x_1, x_0) \\ &\geq g(x) - g(x_n) + \textstyle\sum_{j=1}^{n} (g(x_j) - g(x_{j-1})) \\ &= g(x) - g(x_0) \end{aligned}$$

and the result follows. □

Next, several Ekeland variational principles for extended-real-valued bifunctions are obtained.

Theorem 3.6. *Let (X, d) be a complete metric space and $h : X \times X \to \mathbb{R} \cup \{+\infty\}$ be an extended-real-valued bifunction. Suppose that there exists a proper extended-real-valued function $g : X \to \mathbb{R} \cup \{+\infty\}$ such that*

$$h(x_1, x_2) + g(x_1) \geq g(x_2), \quad \forall\, x_1, x_2 \in X. \tag{9}$$

Assume that g is bounded from below and $<$-lsc. Then, for each $x_0 \in \operatorname{dom} g$ there exists $\bar{x} \in \operatorname{dom} g$ satisfying

(a) $d(x_0, \bar{x}) \leq g(x_0) - g(\bar{x}) \leq h(\bar{x}, x_0)$;
(b) $h(\bar{x}, x) + d(\bar{x}, x) > 0$, $\forall x \in X \backslash \{\bar{x}\}$.

Proof. Consider a point $x_0 \in \operatorname{dom} g$. By Theorem 2.3 we deduce that there is a point

$\bar{x} \in X$ satisfying

$$g(\bar{x}) + d(x_0, \bar{x}) \leq g(x_0), \tag{10}$$
$$g(x) + d(\bar{x}, x) > g(\bar{x}), \quad \forall\, x \in X \backslash \{\bar{x}\}. \tag{11}$$

Condition (10) implies $\bar{x} \in \operatorname{dom} g$ and by assumption (9) it follows that

$$d(x_0, \bar{x}) \leq g(x_0) - g(\bar{x}) \leq h(\bar{x}, x_0).$$

Analogously, condition (11) and assumption (9) imply

$$h(\bar{x}, x) \geq g(x) - g(\bar{x}) > -d(\bar{x}, x), \quad \forall\, x \in X \backslash \{\bar{x}\},$$

and the proof finishes. □

Condition (9) is satisfied provided that the bifunction h fulfills the triangle inequality property. This fact motivates the subsequent result.

Corollary 3.7. *Let (X, d) be a complete metric space and $h : X \times X \to I\!R \cup \{+\infty\}$ be an extended-real-valued bifunction. Suppose that h fulfills the triangle inequality (3) and there exists $\hat{x} \in X$ such that $h(\hat{x}, \cdot)$ is proper, bounded from below and $<$-lsc. Then, for all $x_0 \in X$, $h(\hat{x}, x_0) < +\infty$, there exists $\bar{x} \in X$ satisfying*

(a) $h(\hat{x}, \bar{x}) + d(x_0, \bar{x}) \leq h(\hat{x}, x_0)$;
(b) $d(x_0, \bar{x}) \leq h(\bar{x}, x_0)$;
(c) $h(\bar{x}, x) + d(\bar{x}, x) > 0$, $\forall x \in X \backslash \{\bar{x}\}$.

Proof. Consider a point $\hat{x} \in X$ such that the extended-real-valued function $g_{\hat{x}} := h(\hat{x}, \cdot) : X \to I\!R \cup \{+\infty\}$ is proper, bounded from below and $<$-lsc. As h satisfies the triangle inequality property, for each $x_1, x_2 \in X$ we have that

$$h(x_1, x_2) + g_{\hat{x}}(x_1) = h(x_1, x_2) + h(\hat{x}, x_1) \geq h(\hat{x}, x_2) = g_{\hat{x}}(x_2).$$

Then, the result follows by applying Theorem 3.6. □

Remark 3. By taking $x_0 = \hat{x}$, Corollary 3.7 encompasses [5, Theorem 2.1], where a finite diagonal null bifunction h is considered. In addition, instead of the $<$-lsc assumption, it is supposed that the set

$$F(x) := \{y \in X : h(x, y) + d(x, y) \leq 0\}$$

is closed, for all $x \in X$. This result can be derived by applying Theorem 2.1 to the dynamical system $F : X \rightrightarrows X$. Indeed, conditions (A1), (A2) and (A4) are obviously satisfied. Concerning (A3), notice that for each Picard sequence $(x_n) \subset X$, the triangle inequality implies that

$$\begin{aligned}\sum_{n=0}^{k} d(x_n, x_{n+1}) &\leq \sum_{n=0}^{k}(-h(x_n, x_{n+1})) \\ &\leq \sum_{n=0}^{k}(h(\hat{x}, x_n) - h(\hat{x}, x_{n+1})) \\ &= h(\hat{x}, x_0) - h(\hat{x}, x_{k+1}) \\ &\leq h(\hat{x}, x_0) - \inf_{x\in X} h(\hat{x}, x) \\ &< +\infty\end{aligned}$$

and so $d(x_n, x_{n+1}) \to 0$, i.e. condition (A3) is true too.

Next, we state the main result of this section. It is an equilibrium version of the Ekeland variational principle whose assumptions only involve the bifunction. The next notion is the counterpart for bifunctions of the concept in Definition 2.2.

Definition 3.8. Let X be a topological space. An extended-real-valued bifunction $h : X \times X \to I\!R \cup \{+\infty\}$ is said to be strictly decreasingly lower-semicontinuous at a point $\bar{x} \in X$ ($\times^{>}$-lsc at $\bar{x}$ in short form) if for every sequence $(x_n) \subset X$ converging to $\bar{x}$, one has

$$\forall n \in I\!N, h(x_{n+1}, x_n) > 0 \Longrightarrow \forall n \in I\!N, h(x_n, \bar{x}) \leq 0. \tag{12}$$

The function h is called $\times^{>}$-lsc if it is $\times^{>}$-lsc at x, for all $x \in X$.

In short notation $\times^{>}$, symbol $\times$ underlines that the notion involves a bifunction, and the superscript $^{>}$ denotes the binary relation involved in the left-hand side of condition (12).

Notice that a real-valued function $g : X \to I\!R$ is $<$-lsc if and only if the bifunction $h : X \times X \to I\!R$, $h(x_1, x_2) = g(x_2) - g(x_1)$, is $\times^{>}$-lsc.

Theorem 3.9. *Let (X, d) be a complete metric space and $h : X \times X \to I\!R \cup \{+\infty\}$ be an extended-real-valued bifunction. Suppose that h is cyclically antimonotone and there exists $\hat{x} \in X$ such that $h(\cdot, \hat{x})$ is bounded from above. Assume that one of the following assumptions holds:*

(H1) h is $\times^{>}$-lsc.
(H2) For every $x \in X$, $h(x, x) \leq 0$ and $h(\cdot, x)$ is upper semicontinuous at x.

Then, for all $x_0 \in X$, $h(\hat{x}, x_0) < +\infty$, there exists $\bar{x} \in X$ satisfying

(a) $d(x_0, \bar{x}) \leq h(\bar{x}, x_0)$;
(b) $h(\bar{x}, x) + d(\bar{x}, x) > 0$, $\forall x \in X \backslash \{\bar{x}\}$.

Proof. Consider the bifunction $h' : X\times X \to I\!R\cup\{+\infty\}$, $h'(x_1, x_2) = h(x_2, x_1)$, for all $x_1, x_2 \in X$. As h is cyclically antimonotone, it follows that h' is cyclically antimonotone too. Let $\hat{x} \in X$ be such that $h'(\hat{x}, \cdot)$ is bounded from above. In particular, we have that $\operatorname{dom} h'(\hat{x}, \cdot) = X$. By Theorem 3.2, we deduce that there exists an extended-real-valued function $g : X \to \mathbb{R} \cup \{+\infty\}$ such that

$$\begin{cases} \operatorname{dom} h'(\cdot, \hat{x}) \subset \operatorname{dom} g, \\ h'(x_1, x_2) + g(x_2) \geq g(x_1), \quad \forall x_1, x_2 \in X. \end{cases} \tag{13}$$

Assertions in (13) and the upper boundedness of $h'(\hat{x}, \cdot)$ yield to the lower boundedness of g. Indeed, there exists $M \in \mathbb{R}$ such that $h'(\hat{x}, x) \leq M$, for all $x \in X$. Moreover, $\hat{x} \in \operatorname{dom} g$ as $h'(\hat{x}, \hat{x}) < +\infty$. Therefore, for each $x \in X$,

$$g(x) \geq g(\hat{x}) - h'(\hat{x}, x) \geq g(\hat{x}) - M,$$

and g is bounded from below.

Next, we prove that g is $<$-lsc in two cases according to (H1) and (H2).

Case 1: Assume that (H1) is satisfied. Fix an arbitrary sequence (x_n) converging to $\bar{x}$ such that

$$\forall n \in \mathbb{N}, g(x_{n+1}) < g(x_n).$$

The second assertion of (13) yields

$$\forall n \in \mathbb{N}, h'(x_n, x_{n+1}) \geq g(x_n) - g(x_{n+1}) > 0.$$

By (12), we have

$$\forall n \in \mathbb{N}, h'(\bar{x}, x_n) \leq 0.$$

Again, the second assertion of (13) ensures that

$$\forall n \in \mathbb{N}, g(x_n) \geq h'(\bar{x}, x_n) + g(x_n) \geq g(\bar{x}).$$

Therefore, g is $<$-lsc.

Case 2: Assume that (H2) is satisfied. The second assertion of (13) and the upper semicontinuity of $h(\cdot, x)$ at $x \in X$ imply that g is lower-semicontinuous. Indeed, consider an arbitrary point $x \in X$. Let us check that $g(x) \leq \liminf_{u \to x} g(u)$. As $h'(x, x) \leq 0$ and $h'(x, \cdot)$ is upper semicontinuous at x, for each $\varepsilon > 0$ there exists a neighborhood U of x such that $h'(x, u) \leq \varepsilon$, for all $u \in U$. In particular, we see that $h'(x, \cdot)$ is finite in U and by (13) we deduce that

$$g(u) \geq g(x) - h'(x, u) \geq g(x) - \varepsilon, \quad \forall u \in U$$

and the lower-semicontinuity of g is proved. Thus, g is $<$-lsc.

Let $x_0 \in X$ be such that $h(\hat{x}, x_0) < +\infty$. By the first statement of (13) we see that $x_0 \in \operatorname{dom} g$ and the result follows by applying Theorem 3.6. □

Remark 4. 1. Theorem 3.9 improves [7, Corollary 2.17] as the upper boundedness and upper semicontinuity assumptions are weaker.

2. Let X be a topological space and $h : X \times X \to \mathbb{R} \cup \{+\infty\}$ be an extended-real-valued diagonal null bifunction such that for all $x \in X$, $h(\cdot, x) : X \to \mathbb{R} \cup \{+\infty\}$ is upper semicontinuos at x. The proof of Theorem 3.9 shows that each function

$g : X \to I\!R \cup \{+\infty\}$ fulfilling $h(x_1, x_2) + g(x_1) \geq g(x_2)$, for all $x_1, x_2 \in X$, is lower-semicontinuous. This assertion improves the first implication of [7, Theorem 2.16], where only the opposite of function g_{x_0} defined in (8) is considered.

Remark 5. 1. Notice that Theorem 3.6 is a consequence of Theorem 2.3. Reciprocally, Theorem 2.3 can be stated by applying Theorem 3.6 to the extended-real-valued bifunction $h : X \times X \to I\!R \cup \{+\infty\}$, $h(x_1, x_2) = g(x_2) - g(x_1)$, for all $x_1 \in \operatorname{dom} g$ and $x_2 \in X$, and $h(x_1, x_2) = c$ otherwise, where $c \in I\!R \cup \{+\infty\}$ is arbitrary. Therefore, Theorem 3.6 and Theorem 2.3 are equivalent results.

2. Theorem 3.6 improves [9, Corollary 3.7] and the version of [6, Theorem 6] for metric spaces instead of quasi-metric spaces, because a weaker lower-semicontinuity assumption is required and it can be applied to extended-real-valued bifunctions.

3. Corollary 3.7 generalizes [3, Theorem 2.2], [4, Theorem 2.1] and [2, Theorem 1] because weaker hypotheses are assumed. On the one hand, h is assumed to be $<$-lsc instead of lower-semicontinuous. Moreover, this condition is required for a fixed point in the first argument, instead of for each point in the first argument. On the other hand, the lower boundedness assumption is required for a fixed point in the first argument, instead of for each point in the first argument. Finally, the function h is not assumed to be diagonal null.

4. Ekeland variational principles for vector bifunctions

First at all, a notion of cyclically antimonotone vector bifunction is introduced. It is based on a nonlinear scalarization approach. Recall that Y is assumed to be a real locally convex Hausdorff topological linear space and $D \subset Y$ is a closed convex cone.

Consider an arbitrary nonempty set $E \subset Y$, $q \in Y \backslash \{0\}$ and the so-called Gerstewitz's scalarization function $\varphi_E^q : Y \to I\!R \cup \{\pm\infty\}$, defined as follows (see [21,22] and the references therein):

$$\varphi_E^q(y) = \inf\{t \in I\!R : y \in tq - E\},$$

where $\inf \emptyset = +\infty$.
We denote

$$S(\varphi_E^q, r, \mathcal{R}) := \{y \in Y : \varphi_E^q(y) \mathcal{R} r\}, \quad \forall r \in I\!R, \quad \forall \mathcal{R} \in \{\leq, <, =\}.$$

In addition, $\operatorname{cl} E$ and $\operatorname{cone} E$ refer to the closure and the cone generated by E, respectively, and $\operatorname{core} E$ and $\operatorname{vcl}_q E$ stand for the algebraic interior of E and the vector closure of E in direction q (see [23,24]), respectively, i.e.,

$$\begin{aligned} \operatorname{core} E &:= \{y \in Y : \forall v \in Y\, \exists \delta > 0 \text{ s.t. } y + [0, \delta] v \subset E\}, \\ \operatorname{vcl}_q E &:= \{y \in Y : \forall t > 0\, \exists t' \in [0, t] \text{ s.t. } y + t'q \in E\}, \end{aligned}$$

and for each $\lambda \in Y^* \backslash \{0\}$,

$$H_+^\lambda := \{y \in Y : \lambda(y) \geq 0\}.$$

Recall that E is called algebraically solid if $\operatorname{core} E \neq \emptyset$ and φ_E^q is said to be $\leq_E$-monotone if for each $y_1, y_2 \in Y$, $y_1 \leq_E y_2$, it follows that $\varphi_E^q(y_1) \leq \varphi_E^q(y_2)$. Here,

$\leq_E$ extends the binary relation $\leq_D$ defined in (1) for the closed convex cone D to an arbitrary set E, i.e.,

$$y_1, y_2 \in Y, \quad y_1 \leq_E y_2 \iff y_2 - y_1 \in E.$$

In addition, the function φ_E^q is said to be positively homogeneous (resp., subadditive, convex) if $\varphi_E^q(\alpha y) = \alpha\varphi_E^q(y)$, for all $y \in Y$ and $\alpha > 0$ (resp., $\varphi_E^q(y_1 + y_2) \leq \varphi_E^q(y_1) + \varphi_E^q(y_2)$, $\varphi_E^q(\alpha y_1 + (1-\alpha)y_2) \leq \alpha\varphi_E^q(y_1) + (1-\alpha)\varphi_E^q(y_2)$, for all $y_1, y_2 \in Y$, for all $\alpha \in (0,1)$). In these definitions we assume $+\infty - \infty = -\infty + \infty = +\infty$.

The following properties are well-known. For parts (i)-(ix) see [21, Lemma 3 and Theorems 4, 8 and 14] and [22, Theorem 2.3.1 and Proposition 2.3.7], whereas part (x) can be easily checked. We denote $E^+ := (\operatorname{cone} E)^+$.

Lemma 4.1. *We have that*

(i) φ_E^q *is translative, i.e.,* $\varphi_E^q(y + tq) = \varphi_E^q(y) + t$, *for all* $y \in Y$ *and for all* $t \in \mathbb{R}$.
(ii) $S(\varphi_E^q, 0, \leq) = (-\infty, 0]q - \operatorname{vcl}_q E$.
(iii) $S(\varphi_E^q, 0, <) = (-\infty, 0)q - \operatorname{vcl}_q E$.
(iv) $S(\varphi_E^q, 0, =) = (-\operatorname{vcl}_q E)\backslash((-\infty, 0)q - \operatorname{vcl}_q E)$.
(v) φ_E^q *is* $\leq_E$*-monotone iff* $E + E \subset [0, +\infty)q + \operatorname{vcl}_q E$.
(vi) *Assume that* $\varphi_E^q > -\infty$. *Then,* φ_E^q *is subadditive iff* $\operatorname{vcl}_q E + \operatorname{vcl}_q E \subset [0, +\infty)q + \operatorname{vcl}_q E$.

If E is a closed convex cone, then the next properties are also satisfied:

(vii) φ_E^q *is positively homogeneous and convex.*
(viii) *If* $q \notin -E$, *then* φ_E^q *is proper and* $\varphi_E^q(0) = 0$.
(ix) *If E is not a linear subspace then* $\operatorname{dom}\varphi_E^q = Y$ *if and only if* $\{q, -q\} \cap \operatorname{core} E \neq \emptyset$.
(x) *Consider* $\lambda \in E^+$ *and* $q \in Y$ *such that* $\lambda(q) = 1$. *We have* $E \subset H_+^\lambda$ *and* $\varphi_{H_+^\lambda}^q(y) = \lambda(y)$ *for all* $y \in Y$.

Definition 4.2. A vector bifunction $f : X \times X \to Y$ is said to be D-cyclically antimonotone if there exists $q \in Y\backslash(-D)$ such that the proper extended-real-valued bifunction $\varphi_D^q \circ f : X \times X \to \mathbb{R} \cup \{+\infty\}$ is cyclically antimonotone.

Let us denote

$$\mathcal{Q}(f) \quad := \quad \{q \in Y\backslash(-D) : \varphi_D^q \circ f \text{ is cyclically antimonotone}\}.$$

Then, f is D-cyclically antimonotone if and only if $\mathcal{Q}(f) \neq \emptyset$.

Remark 6. Consider $Y = \mathbb{R}$, $D = \mathbb{R}_+$ and an arbitrary positive real number q. Then $\varphi_{\mathbb{R}_+}^q(y) = y/q$, for all $y \in \mathbb{R}$, and the function $\varphi_{\mathbb{R}_+}^q \circ f$ is cyclically antimonotone if and only if f is cyclically antimonotone. Thus, the notion of D-cyclically antimonotone vector bifunction encompasses the corresponding scalar concept.

In the next result, we provide some sufficient conditions for the cyclical antimonotonicity concept introduced in Definition 4.2.

Theorem 4.3. *Consider a vector bifunction* $f : X \times X \to Y$ *and the next assertions:*

(i) *There exists a vector function* $g : X \to Y$ *satisfying* $f(x, y) \geq_D g(y) - g(x)$, *for all* $x, y \in X$.

(ii) *For each finite nonempty set* $\{x_1, x_2, \dots, x_n\} \subset X$ *we have that*

$$f(x_1, x_2) + f(x_2, x_3) + \cdots + f(x_{n-1}, x_n) + f(x_n, x_1) \geq_D 0.$$

(iii) *For all* $\lambda \in D^+\backslash\{0\}$, $\lambda \circ f$ *is cyclically antimonotone.*

(iv) *There exists* $\lambda \in D^+\backslash\{0\}$ *such that* $\lambda \circ f$ *is cyclically antimonotone.*

(v) *For each finite nonempty set* $\{x_1, x_2, \dots, x_n\} \subset X$ *we have that*

$$f(x_1, x_2) + f(x_2, x_3) + \cdots + f(x_{n-1}, x_n) + f(x_n, x_1) \leq_D 0$$
$$\Rightarrow f(x_1, x_2) + f(x_2, x_3) + \cdots + f(x_{n-1}, x_n) + f(x_n, x_1) = 0.$$

(vi) *For each* $q \in D\backslash(-D)$, $\varphi_D^q \circ f$ *is cyclically antimonotone.*

(vii) f *is* D*-cyclically antimonotone.*

(viii) *There exists* $q \in Y\backslash(-D)$ *and a real-valued function* $g : X \to I\!R$ *such that for each* $x, y \in X$,

$$f(x, y) + tq \not\leq_D g(y)q - g(x)q, \quad \forall t > 0.$$

The following implications are true: (i) $\Rightarrow$ *(ii)* $\Leftrightarrow$ *(iii)* $\Rightarrow$ *(iv), (ii)* $\Rightarrow$ *(vi) and (v)* $\Rightarrow$ *(vi); If* D *is strongly minihedral, i.e., for every subset of* Y *which is bounded from above has a supremum, then (ii)* $\Rightarrow$ *(i) is also true.*

Moreover, (vii) $\Rightarrow$ *(viii) provided that there exists* $q \in \mathcal{Q}(f)$ *such that* $f(X, x_0) \cup f(x_0, X) \subset I\!Rq - D$ *for some* $x_0 \in X$.

In addition, if D *is not a linear space, then (vi) implies (vii), and if* D *is pointed, then (iii)* $\Rightarrow$ *(v).*

In addition, if $Y = I\!R$ *and* $D = I\!R_+$, *then all statements above are equivalent.*

Proof. Suppose that assertion (i) is true. Consider an arbitrary finite nonempty set $\{x_1, x_2, \dots, x_n, x_{n+1}\} \subset X$ such that $x_{n+1} = x_1$. As D is a convex cone, by property (2) it follows that

$$\sum_{j=1}^{n} f(x_j, x_{j+1}) \geq_D \sum_{j=1}^{n} (g(x_{j+1}) - g(x_j)) = 0$$

and statement (ii) holds true. For the converse implication when D is strongly minihedral, see [15, Theorem 4.4] or [14, Theorem 4.1].

Moreover, by the Bipolar Theorem we have that

$$\sum_{j=1}^{n} f(x_j, x_{j+1}) \geq_D 0 \iff \sum_{j=1}^{n} (\lambda \circ f)(x_j, x_{j+1}) \geq 0, \quad \forall \lambda \in D^+\backslash\{0\}$$

and the equivalence (ii) $\Leftrightarrow$ (iii) is proved.

It is obvious that assertion (iii) implies assertion (iv). In addition, $(ii) \Rightarrow (vi)$ follows since by parts (v), (vi) and (viii) of Lemma 4.1 the function φ_D^q is $\leq_D$-monotone, subadditive and $\varphi_D^q(0) = 0$.

Assume that D is pointed. Then, relation (iii) $\Rightarrow$ (v) is obvious, since (ii) and (iii) are equivalent.

Next, assume that statement (v) is fulfilled and let $q \in D\backslash(-D)$. Consider an arbitrary finite nonempty set $\{x_1, x_2, \ldots, x_n\} \subset X$ and the point

$$y := f(x_1, x_2) + f(x_2, x_3) + \cdots + f(x_{n-1}, x_n) + f(x_n, x_1).$$

By parts (vi) and (viii) of Lemma 4.1 we obtain

$$\varphi_D^q(f(x_1, x_2)) + \varphi_D^q(f(x_2, x_3)) + \cdots + \varphi_D^q(f(x_{n-1}, x_n)) + \varphi_D^q(f(x_n, x_1)) \geq \varphi_D^q(y).$$

If $y = 0$, then $\varphi_D^q(y) = 0$ by Lemma 4.1(viii). Suppose that $y \neq 0$. Then, by statement (v) we see that $y \notin -D$. By part (ii) of Lemma 4.1 we have that $\varphi_D^q(y) > 0$. Therefore, assertion (v) implies (vi).

If D is not a linear space, then $D \backslash (-D) \neq \emptyset$, and part (vii) is an obvious consequence of part (vi).

Let us check that part (vii) implies part (viii). Assume that the vector bifunction f is D-cyclically antimonotone and there exist $q \in \mathcal{Q}(f)$ and $x_0 \in X$ such that $f(X, x_0) \cup f(x_0, X) \subset I\!Rq - D$. Define $h : X \times X \to I\!R \cup \{+\infty\}$, $h(x_1, x_2) = (\varphi_D^q \circ f)(x_2, x_1)$. As $q \in \mathcal{Q}(f)$, we have that $\varphi_D^q \circ f$ is cyclically antimonotone, and so it is obvious that h is cyclically antimonotone too. Since $f(X, x_0) \subset I\!Rq - D$ it follows that $\operatorname{dom} h(x_0, \cdot) = X$. Then by Theorem 3.2 we deduce that there exists an extended-real-valued function $g : X \to I\!R \cup \{+\infty\}$ such that

$$\begin{cases} \operatorname{dom} h(\cdot, x_0) \subset \operatorname{dom} g, \\ h(x_1, x_2) + g(x_2) \geq g(x_1), \quad \forall x_1, x_2 \in X. \end{cases} \tag{14}$$

Thus, $\operatorname{dom} (\varphi_D^q \circ f)(x_0, \cdot) \subset \operatorname{dom} g$. As $f(x_0, X) \subset I\!Rq - D$ we see that $\operatorname{dom} (\varphi_D^q \circ f)(x_0, \cdot) = X$ and it follows that g is finite. Then, as a result of the second assertion of (14) we deduce that

$$(\varphi_D^q \circ f)(x, y) + g(x) \geq g(y), \quad \forall x, y \in X.$$

Since φ_D^q is translative (see Lemma 4.1(i)), we deduce that

$$\varphi_D^q(f(x, y) + (g(x) - g(y))q) \geq 0, \quad \forall x, y \in X.$$

Thus,

$$f(x, y) + (g(x) - g(y))q \notin S(\varphi_D^q, 0, <), \quad \forall x, y \in X$$

and by using Lemma 4.1(iii) again we obtain

$$f(x, y) + (g(x) - g(y))q \notin (-\infty, 0)q - D, \quad \forall x, y \in X.$$

Therefore,

$$f(x, y) + tq \not\leq_D g(y)q - g(x)q, \quad \forall x, y \in X, \forall t > 0.$$

Next, we check that conditions (i)-(viii) are equivalent as long as $Y = I\!R$ and $D = I\!R_+$. Since D is pointed, implications (iii) $\Rightarrow$ (v) and (vi) $\Rightarrow$ (vii) are satisfied. Since λ is a positive number, parts (ii) and (iv) are equivalent. Moreover, since $I\!Rq - D = Y$ for all $q \neq 0$, part (vii) implies part (viii). The last assertion can be rewritten as follows: there exists a point $q > 0$ and a real-valued-function $g : X \to I\!R$ such that for each $x, y \in X$ and $t > 0$ it follows that

$$f(x, y) + tq > g(y)q - g(x)q.$$

This assertion is equivalent to say that there exists a point $q > 0$ and a real-valued function $g : X \to I\!R$ such that for each $x, y \in X$,

$$f(x, y) \geq g(y)q - g(x)q$$

and assertion (i) is obtained. Therefore, implication (viii) ⇒ (i) holds true. This finishes the proof. □

Remark 7. 1. Assertion *(ii)* ⇒ *(i)* of Theorem 4.3 was stated in [15, Theorem 4.4] and [14, Theorem 4.1] by a constructive proof based on the function $g_{x_0} : X \to Y$,

$$g_{x_0}(x) := -\inf\{f(x, x_n) + f(x_n, x_{n-1}) + \cdots + f(x_1, x_0) : \forall n \in I\!N, \\ \forall\{x_1, x_2, \ldots, x_n\} \subset X\},$$

where $x_0 \in X$ is arbitrary. To be precise, the function g_{x_0} is well-defined whenever D is strongly minihedral and f fulfills assertion (ii), and it satisfies $f(x, y) \geq_D g_{x_0}(y) - g_{x_0}(x)$, for all $x, y \in X$.

2. Condition $f(X, x_0) \cup f(x_0, X) \subset I\!Rq - D$ of implication (vii) ⇒ (viii) can be dropped whenever φ^q_D is finite. When D is not a linear subspace and $q \notin -D$, this happens if and only if $q \in \text{core}D$ (see Lemma 4.1(ix)).

3. Assertion *(ii)* ⇒ *(vi)* of Theorem 4.3 has been stated in [14, Proposition 4.1] when D is proper (i.e., $D \neq Y$), algebraically solid and $q \in \text{core}\, D$. Notice that the assertion of Theorem 4.3 is more general as $\text{core}\, D \subset D\backslash(-D)$ and $\text{core}\, D$ could be empty.

4. Recently Miholca [15] and Qiu [16] introduced a concept of cyclically antimonotone vector bifunction via assertion (ii) of Theorem 4.3. From that theorem it is clear that the notion of D-cyclically antimonotone vector bifunction is more general.

5. An example of vector variational inequality problem where condition (i) of Theorem 4.3 is satisfied was introduced in [9] in connection with the so-called strong supergradients of a cone concave vector mapping (see [9,25]).

Analogously, Qiu [16, Remark 3.8 and Theorem 3.7] defined a vector bifunction as strongly cyclically antimonotone if it satisfies assertion (i) of Theorem 4.3. This notion is stronger than Miholca's concept. The main results of [9] have been obtained by assuming this type of strong cyclical antimonotonicity. Thus, their counterparts in this paper improve them as they are stated via a more general cyclical antimonotonicity notion (compare, for instance, [9, Theorems 3.6 and 4.1] with Theorems 4.12 and 4.14, respectively).

In the rest of this section, we state some Ekeland variational principles for vector bifunctions that are more general than some others recently published in the literature.

Theorem 4.4. *Let (X, d) be a complete metric space and $f : X \times X \to Y$ be a vector bifunction. Consider $q \in Y\backslash(-D)$ and suppose that there exists a proper extended-real-valued function $g : X \to I\!R \cup \{+\infty\}$ such that $\varphi^q_D \circ f : X \times X \to I\!R \cup \{+\infty\}$*

satisfies

$$(\varphi_D^q \circ f)(x_1, x_2) + g(x_1) \geq g(x_2), \quad \forall x_1, x_2 \in X.$$

Assume that g is bounded from below and $<$-lsc. Then, for each $x_0 \in \operatorname{dom} g$ there exists $\bar{x} \in \operatorname{dom} g$ such that

(a) $g(\bar{x}) + d(x_0, \bar{x}) \leq g(x_0)$*;*
(b) $d(x_0, \bar{x}) \leq (\varphi_D^q \circ f)(\bar{x}, x_0)$*;*
(c) $f(\bar{x}, x) + d(\bar{x}, x)q \not\leq_D 0$, $\forall x \in X \backslash \{\bar{x}\}$.

Proof. Assertions (a) and (b) are deduced by applying Theorem 3.6 to $h := \varphi_D^q \circ f$. Moreover, for each $x \in X \backslash \{\bar{x}\}$ we have that

$$(\varphi_D^q \circ f)(\bar{x}, x) + d(\bar{x}, x) > 0.$$

By Lemma 4.1(i) we have that

$$\varphi_D^q(f(\bar{x}, x) + d(\bar{x}, x)q) > 0.$$

Then, Lemma 4.1(ii) implies that $f(\bar{x}, x) + d(\bar{x}, x)q \notin -D$ and the proof is completed. □

Corollary 4.5. *Let (X, d) be a complete metric space and $f : X \times X \to Y$ be a vector bifunction. Consider $\lambda \in D^+$ and $q \in Y$ such that $\lambda(q) = 1$. Suppose that there exists a proper extended-real-valued function $g : X \to \mathbb{R} \cup \{+\infty\}$ such that the real-valued bifunction $\lambda \circ f : X \times X \to \mathbb{R}$ satisfies*

$$(\lambda \circ f)(x_1, x_2) + g(x_1) \geq g(x_2), \quad \forall x_1, x_2 \in X. \tag{15}$$

Assume that g is bounded from below and $<$-lsc. Then, for each $x_0 \in \operatorname{dom} g$ there exists $\bar{x} \in \operatorname{dom} g$ such that

(a) $g(\bar{x}) + d(x_0, \bar{x}) \leq g(x_0)$*;*
(b) $d(x_0, \bar{x}) \leq (\lambda \circ f)(\bar{x}, x_0)$*;*
(c) $f(\bar{x}, x) + d(\bar{x}, x)q \not\leq_D 0$, $\forall x \in X \backslash \{\bar{x}\}$.

Proof. Let us consider the closed convex cone $K := H_+^\lambda$. As $\lambda(q) = 1$ we have that $q \notin -K$. In addition, by Lemma 4.1(x) we deduce that $\lambda = \varphi_K^q$. Then, parts (a) and (b) are obtained by applying parts (a) and (b) of Theorem 4.4 to the cone K.

Part (c) follows by Theorem 4.4(c) and the next implications, which are true due to $D \subset H_+^\lambda$:

$$\begin{aligned}
& \forall x \in X \backslash \{\bar{x}\}, f(\bar{x}, x) + d(\bar{x}, x)q \not\leq_{H_+^\lambda} 0 \\
\iff & \forall x \in X \backslash \{\bar{x}\}, f(\bar{x}, x) + d(\bar{x}, x)q \notin -H_+^\lambda \\
\Longrightarrow & \forall x \in X \backslash \{\bar{x}\}, f(\bar{x}, x) + d(\bar{x}, x)q \notin -D \\
\iff & \forall x \in X \backslash \{\bar{x}\}, f(\bar{x}, x) + d(\bar{x}, x)q \not\leq_D 0.
\end{aligned}$$

This finishes the proof. □

Corollary 4.5 improves [16, Theorem 3.5] since a weaker lower-semicontinuity assumption is considered (see Remark 1).

Condition (15) is fulfilled whenever the vector bifunction f satisfies the triangle inequality property with respect to the partial order $\leq_D$:

$$f(x_1, x_2) \leq_D f(x_1, x_3) + f(x_3, x_2), \quad \forall x_1, x_2, x_3 \in X. \tag{16}$$

This remark motivates the subsequent corollary.

Corollary 4.6. *Let (X, d) be a complete metric space and $f : X \times X \to Y$ be a vector bifunction that satisfies the triangle inequality property (16). If there exists $\hat{x} \in X$ and $\hat{\lambda} \in D^+ \backslash \{0\}$ such that $(\hat{\lambda} \circ f)(\hat{x}, \cdot) : X \to \mathbb{R}$ is $<$-lsc and bounded from below, then for each $q \in Y$, $\hat{\lambda}(q) = 1$ and $x_0 \in X$ there exists $\bar{x} \in X$ such that*

(a) $(\hat{\lambda} \circ f)(\hat{x}, \bar{x}) + d(x_0, \bar{x}) \leq (\hat{\lambda} \circ f)(\hat{x}, x_0)$;
(b) $d(x_0, \bar{x}) \leq (\hat{\lambda} \circ f)(\bar{x}, x_0)$;
(c) $f(\bar{x}, x) + d(\bar{x}, x)q \not\leq_D 0$, $\forall x \in X \backslash \{\bar{x}\}$.

Proof. Let $g_{\hat{\lambda},\hat{x}} : X \to \mathbb{R}$ be the real-valued function $g_{\hat{\lambda},\hat{x}}(x) := (\hat{\lambda} \circ f)(\hat{x}, x)$, for all $x \in X$. Since $\hat{\lambda} \in D^+$, and f fulfills the triangle inequality property (16), for each $x_1, x_2 \in X$ it follows that

$$(\hat{\lambda} \circ f)(x_1, x_2) + g_{\hat{\lambda},\hat{x}}(x_1) \geq g_{\hat{\lambda},\hat{x}}(x_2).$$

Then the corollary follows by applying Corollary 4.5. □

Remark 8. The lower-semicontinuity and lower boundedness assumptions of Corollary 4.6 are satisfied in several settings.

For instance, recall that a vector function $g : X \to Y$ from a topological space X to a partially ordered real locally convex Hausdorff topological linear space $(Y, \leq_D)$ is said to be lower-semicontinuous if for any $x \in X$ and any 0-neighborhood V in Y there exists a neighborhood U of x in X such that $g(U) \subset g(x) + V + D$ (see [16,22]).

Then, if there exists $\hat{x} \in X$ such that the vector function $f(\hat{x}, \cdot) : X \to Y$ is lower-semicontinuous, then the real-valued function $(\lambda \circ f)(\hat{x}, \cdot) : X \to \mathbb{R}$ is lower-semicontinuous (see [16, Proposition 2.7]) and so it is $<$-lsc.

Analogously, g is called topologically D-bounded if for any 0-neighborhood V in Y there exists $r > 0$ such that $g(X) \subset rV + D$ (see [22,26]). Then, if there exists $\hat{x} \in X$ such that the vector function $f(\hat{x}, \cdot)$ is topologically D-bounded, it follows that the real-valued function $(\lambda \circ f)(\hat{x}, \cdot)$ is bounded from below, for all $\lambda \in D^+$ (see [26, Proposition 4.6]).

For each $q \in Y \backslash \{0\}$, notation $\leq_D^q$ stands for the next preorder:

$$y_1, y_2 \in Y, \quad y_1 \leq_D^q y_2 \iff y_2 - y_1 \in [0, +\infty)q + D.$$

Notice that $\leq_D^q$ and $\leq_D$ coincide whenever $q \in D$ since in this case $[0, +\infty)q + D = D$. Then, $\leq_D^q$ encompasses the preorder $\leq_D$ introduced in (1). Furthermore, by parts (i) and (ii) of Lemma 4.1, for each $y \in Y$ and $s \in \mathbb{R}$ we have that

$$y \leq_D^q sq \iff \varphi_D^q(y) \leq s. \tag{17}$$

In addition, $\leq_D^q$ can be also viewed as a special case of the preorder $\leq_D$ provided that $q \notin -D$, since in this case the set $[0, +\infty)q + D$ is a closed convex cone.

Lemma 4.7. *The set $[0, +\infty)q + D$ is a convex cone. If, in addition, $q \notin -D$, then it is also closed.*

Proof. It is obvious that $[0, +\infty)q+D$ is a convex cone, as it is the sum of two convex cones.

Assume that $q \notin -D$ and consider two nets $(t_i) \subset I\!R_+$ and $(d_i) \subset D$ such that $t_i q + d_i \to y$. We claim that (t_i) is bounded. Otherwise, there exists a subnet (t_{i_j}) such that $t_{i_j} \to +\infty$. Define $y_i := t_i q + d_i$, for all i. Then, since D is closed, we have that

$$q = \lim_{i_j \to \infty} (q - (1/t_{i_j}) y_{i_j}) \in -D,$$

that is a contradiction. Thus, (t_i) is a bounded net.

As a result, we can suppose, taking a subnet if necessary, that $t_i \to t \geq 0$. Therefore, $d_i = y_i - t_i q \to y - tq$ and then $y - tq \in D$. Thus,

$$y = tq + (y - tq) \in [0, +\infty)q + D$$

and the proof finishes. □

Definition 4.8. Consider $g : X \to Y$ and $q \in Y \backslash \{0\}$.

- (i) The function g is said to be q-order bounded from above if there exists $M \in I\!R$ such that $g(x) \leq_D^q Mq$, for all $x \in X$.
- (ii) Let X be a topological space and $\bar{x} \in X$. The function g is said to be q-order upper semicontinuous at $\bar{x}$ if for each $\varepsilon > 0$ there exists a neighborhood U of $\bar{x}$ in X such that

$$s \in I\!R, g(\bar{x}) \leq_D^q sq \Rightarrow g(x) \leq_D^q (s + \varepsilon)q, \forall x \in U. \tag{18}$$

Remark 9. 1. If $q \in D \backslash \{0\}$, then $g : X \to Y$ is q-order bounded from above if and only if there exists $M \in I\!R$ such that $g(x) \leq_D Mq$ for all $x \in X$.

2. Analogously, if $q \in D \backslash \{0\}$, g is q-order upper semicontinuous at $\bar{x}$ if and only if for each $\varepsilon > 0$ there exists a neighborhood U of $\bar{x}$ in X such that $g(\bar{x}) \leq_D sq$ implies $g(x) \leq_D (s + \varepsilon)q$, $\forall x \in U$.

Definition 4.9. Let X be a topological space and $q \in Y \backslash \{0\}$. The vector bifunction $f : X \times X \to Y$ is said to be q-strictly decreasingly lower-semicontinuous at a point $\bar{x}$ ($\times^{\not\leq_D^q}$-lsc at $\bar{x}$ in short form) if for every sequence $(x_n) \subset X$ converging to $\bar{x}$, one has

$$\forall n \in I\!N,\ f(x_{n+1}, x_n) \not\leq_D^q 0 \Longrightarrow \forall n \in I\!N, f(x_n, \bar{x}) \leq_D^q 0. \tag{19}$$

The function f is called $\times^{\not\leq_D^q}$-lsc if it is $\times^{\not\leq_D^q}$-lsc at x, for all $x \in X$.

If $q \in D$, statement (19) is equivalent to the next one:

$$\forall n \in I\!N, f(x_{n+1}, x_n) \not\leq_D 0 \Longrightarrow \forall n \in I\!N, f(x_n, \bar{x}) \leq_D 0 \tag{20}$$

and we write $\times^{\not\leq_D}$-lsc instead of $\times^{\not\leq_D^q}$-lsc.

Remark 10. In [13, Definition 2.5] and [16, Definition 2.4], a bifunction $f : X \times X \to Y$ is called D-sequentially lower monotone at a point $\bar{x} \in X$ (D-slm at $\bar{x}$ in short form), if for each sequence $(x_n) \subset X$, $x_n \to \bar{x}$, it follows that

$$\forall n \in I\!N, f(x_n, x_{n+1}) \leq_D 0 \Longrightarrow \forall n \in I\!N, f(x_n, \bar{x}) \leq_D 0. \tag{21}$$

For real-valued functions, this notion reduces to the decreasing lower-semicontinuity concept (see Remark 1), whereas the $\times^{\not\leq_D^q}$-lsc notion encompasses the strict version of the previous one that was introduced in Definition 2.2. Indeed, if $Y = I\!R$, $D = I\!R_+$, $g : X \to Y$ and $f(x_1, x_2) = g(x_2) - g(x_1)$, then assertions (20) and (21) state that

$$\forall n \in I\!N, g(x_{n+1}) < g(x_n) \Longrightarrow \forall n \in I\!N, g(\bar{x}) \leq g(x_n),$$

$$\forall n \in I\!N, g(x_{n+1}) \leq g(x_n) \Longrightarrow \forall n \in I\!N, g(\bar{x}) \leq g(x_n)$$

and so the definitions of decreasing lower-semicontinuity and strictly decreasing lower-semicontinuity of g at $\bar{x}$ are obtained.

Lemma 4.10. *Consider $g : X \to Y$ and $q \in Y \setminus (-D)$.*

(i) *g is q-order bounded from above if and only if $\varphi_D^q \circ g$ is bounded from above.*
(ii) *Assume that X is a topological space. g is q-order upper semicontinuous at $\bar{x} \in X$ if and only if $\varphi_D^q \circ g$ is upper semicontinuous at $\bar{x}$.*

Proof. (i) It is a direct consequence of (17).
(ii) By statement (17) we see that condition (18) is equivalent to the next one:

$$s \in I\!R, \varphi_D^q(g(\bar{x})) \leq s \Rightarrow \varphi_D^q(g(x)) \leq s + \varepsilon, \forall x \in U.$$

This assertion is true whenever $g(\bar{x}) \notin \operatorname{dom}\varphi_D^q$, since there is no any real number s such that $\varphi_D^q(g(\bar{x})) \leq s$. Otherwise, it is equivalent to the condition $\varphi_D^q(g(x)) \leq \varphi_D^q(g(\bar{x})) + \varepsilon$, $\forall x \in U$ and the proof finishes. □

The next lemma is proved in an analogous way as the previous one.

Lemma 4.11. *Let X be a topological space and $q \in Y \setminus (-D)$. A vector bifunction $f : X \times X \to Y$ is $\times^{\not\leq_D^q}$-lsc if and only if the extended-real-valued bifunction $\varphi_D^q \circ f$ is $\times^{>}$-lsc.*

Now we are in a position to state the first main result of this section.

Theorem 4.12. *Let (X, d) be a complete metric space and $f : X \times X \to Y$ be a D-cyclically antimonotone vector bifunction. Suppose that there exists $q \in \mathcal{Q}(f)$ such that $f(\cdot, \hat{x}) : X \to Y$ is q-order bounded from above for some $\hat{x} \in X$ and one of the next two conditions holds true:*

(H1) f is $\times^{\not\leq_D^q}$-lsc.
(H2) For every $x \in X$, $f(x, x) \leq_D^q 0$ and $f(\cdot, x)$ is q-order upper semicontinuous at x.

Then, for each $x_0 \in X$, $f(\hat{x}, x_0) \in I\!Rq - D$, there exists $\bar{x} \in X$ such that

(a) $d(x_0, \bar{x}) \leq (\varphi_D^q \circ f)(\bar{x}, x_0)$;
(b) $f(\bar{x}, x) + d(\bar{x}, x)q \not\leq_D^q 0$, $\forall x \in X \setminus \{\bar{x}\}$.

Proof. Consider a vector $q \in \mathcal{Q}(f)$ fulfilling the assumptions of the theorem and the cyclically antimonotone extended-real-valued bifunction $h := \varphi_D^q \circ f$. By Lemmas 4.10 and 4.11 it is clear that $h(\cdot, \hat{x})$ is bounded from above and one of the following next assertions holds true:

(H1) h is $\times^{>}$-lsc.
(H2) For every $x \in X$, $h(x, x) \leq 0$ and $h(\cdot, x)$ is upper semicontinuous at x.

Consider a point $x_0 \in X$ such that $f(\hat{x}, x_0) \in I\!R q - D$. By applying Theorem 3.9 we deduce that there exists $\bar{x} \in X$ satisfying

(a) $d(x_0, \bar{x}) \leq h(\bar{x}, x_0)$;
(b') $h(\bar{x}, x) + d(\bar{x}, x) > 0, \ \forall x \in X \backslash \{\bar{x}\}$.

By parts (i) of Lemma 4.1 and (17), statement (b') above is equivalent to the assertion (b) of the theorem and the proof is completed. □

The following Ekeland variational principle encompasses [15, Theorem 5.1] and [14, Theorem 4.3], where the ordering cone is strongly minihedral and condition (ii) of Theorem 4.3 holds true (since both assumptions ensures the existence of a function $g : X \to Y$ satisfying statement (22), see Theorem 4.3). A version of this result for D-sequentially lower monotone vector bifunctions satisfying the triangle inequality property (16) was stated in [13, Corollary 3.7].

Corollary 4.13. *Let (X, d) be a complete metric space and $f : X \times X \to Y$ be a vector bifunction. Suppose that there exists a vector function $g : X \to Y$ satisfying*

$$g(x_2) - g(x_1) \leq_D f(x_1, x_2), \quad \forall x_1, x_2 \in X. \tag{22}$$

Assume that f is $\times^{\not\leq_D}$-lsc and there exist $q \in D \backslash (-D)$ and a point $\hat{x} \in X$ such that $f(\cdot, \hat{x}) : X \to Y$ is q-order bounded from above. Then, for each $x_0 \in X$, $f(\hat{x}, x_0) \in I\!R q - D$, there exists $\bar{x} \in X$ such that

(a) $d(x_0, \bar{x}) q \leq (\varphi_D^q \circ f)(\bar{x}, x_0)$;
(b) $f(\bar{x}, x) + d(\bar{x}, x) q \not\leq_D 0, \ \forall x \in X \backslash \{\bar{x}\}$.

Proof. By Theorem 4.3 we deduce that f is D-cyclically antimonotone and $D \backslash (-D) \subset \mathcal{Q}(f)$. In addition, the orderings $\leq_D^q$ and $\leq_D$ coincide whenever $q \in D$. Then the result follows by applying Theorem 4.12. □

Corollary 4.13 provides a counterpart to [7, Corollary 2.17], [15, Theorem 5.1] and [14, Theorem 4.3], which involve stronger semicontinuity assumptions. For instance, Corollary 4.13 reduces to Theorem 3.9 when $Y = I\!R$ and $D = I\!R_+$, and this result improves [7, Corollary 2.17] since weaker upper boundedness and upper semicontinuous hypotheses are considered (see part 1 of Remark 4).

Next, a Weierstrass theorem for weak solutions of a vector equilibrium problem is stated as an application of the notions of D-cyclically antimonotone vector bifunction and q-strictly decreasingly lower semicontinuity. Suppose that $\operatorname{core} D \neq \emptyset$ and denote $\widetilde{D} := \operatorname{core} D \cup \{0\}$. It is said that a point $\bar{x} \in X$ is a weak solution of problem (VEP), denoted $\bar{x} \in \mathrm{W}(f, D)$, if

$$x \in X \backslash \{\bar{x}\}, \ f(\bar{x}, x) \leq_{\widetilde{D}} 0 \Rightarrow f(\bar{x}, x) = 0.$$

Theorem 4.14. *Let X be a compact topological space and $f : X \times X \to Y$ be a D-cyclically antimonotone vector bifunction. Suppose that there exists $q \in \mathcal{Q}(f)$ and $x_0 \in X$ such that f is $\times^{\not\leq_D^q}$-lsc and $f(X, x_0) \subset I\!R q + D$. Then,* $\mathrm{W}(f, D) \neq \emptyset$.

Proof. Consider $q \in \mathcal{Q}(f)$ and $x_0 \in X$ satisfying the hypotheses of the theorem and define $h, h' : X \times X \to I\!R \cup \{+\infty\}$, $h(x_1, x_2) := (\varphi_D^q \circ f)(x_1, x_2)$, $h'(x_1, x_2) = h(x_2, x_1)$, for all $x_1, x_2 \in X$. Clearly, h' is cyclically antimonotone and by the assumptions we have that $h'(x_0, X) = (\varphi_D^q \circ f)(X, x_0) \subset I\!R$. Then, by the second part of Theorem 3.2 we deduce that there exists $g : X \to I\!R \cup \{+\infty\}$ such that

$$\begin{cases} \operatorname{dom} h'(\cdot, x_0) \subset \operatorname{dom} g, \\ h'(x_1, x_2) + g(x_2) \geq g(x_1), \quad \forall x_1, x_2 \in X. \end{cases} \tag{23}$$

By Lemma 4.11 we see that h is $\times^{>}$-lsc. As a result, in the proof of Theorem 3.9 we have deduced that g is $<$-lsc. Then, by applying the Weierstrass theorem to g we obtain that $\arg\min_X g \neq \emptyset$.

We claim that $\arg\min_X g \subset \mathrm{W}(f, D)$. Indeed, take a point $\bar{x} \in \arg\min_X g$. By (23) it follows that

$$h(\bar{x}, x) = h'(x, \bar{x}) \geq g(x) - g(\bar{x}) \geq 0, \quad \forall x \in X.$$

If $\bar{x} \notin \mathrm{W}(f, D)$, then there exists $x \in X$ such that $f(\bar{x}, x) \in -\operatorname{core} D$. Therefore, $h(\bar{x}, x) = \varphi_D^q(f(\bar{x}, x)) < 0$, that is a contradiction. Thus, $\mathrm{W}(f, D) \neq \emptyset$ and the proof finishes. □

Remark 11. Theorem 4.14 improves the Weierstrass theorem for weak solutions of problem (VEP) in [9, Theorem 4.1]. Indeed, this result assumes the vector bifunction f to satisfy the following strong cyclical antimonotonicity condition (see part 5 of Remark 7) for the set $E = q + D$, $q \in D \backslash (-D)$: there exists $g : X \to Y$ such that

$$f(x, y) \geq_E g(y) - g(x), \quad \forall x, y \in X,$$

and it never holds true when f is diagonal null.

To complete this section, we illustrate how the result in Theorem 4.12 could be further extended from a convex ordering cone $D \subset Y$ to a domination set $E \subset Y$ and a vector $q \in Y \backslash \{0\}$ which satisfy the following conditions:

(E1) $0 \in E$ and $E + E \subset E$.
(E2) $E + [0, +\infty)q \subset E$.
(E3) E is vectorial closed in direction q, i.e., $\operatorname{vcl}_q E = E$.

Notice that (E1) ensures that the binary relation $\leq_E$ is a preorder. In addition, by Lemma 4.1(v)(vi), assumptions (E1) and (E3) imply that φ_E^q is $\leq_E$-monotone and also subadditive on each nonempty set $F \subset Y$ such that $\varphi_E^q(y) > -\infty$, for all $y \in F$. It is worth underlining that the next results are derived without considering any topological structure in Y.

Conditions (E1)-(E3) can be fulfilled by nonconvex sets that are not a cone. For instance, consider $Y = I\!R^2$, $q = (0, 1)$ and

$$E = [0, +\infty)q \cup \{(y_1, y_2) \in I\!R^2 : y_1 \geq 0, y_2 \geq 1\}.$$

It is easy to check that E is neither a convex set nor a cone. However, it satisfies conditions (E1)-(E3).

From now on, for each $y_1, y_2 \in Y$ we say that $y_1 <_E^q y_2$ if $y_2 - y_1 \in (0, +\infty)q + E$. In addition, a set $F \subset Y$ is said to be (E, q)-lower bounded if there exists $M \in \mathbb{R}$ such that $y \not\leq_E Mq$, for all $y \in F$. By parts (i) and (ii) of Lemma 4.1 it is clear that $F \subset Y$ is (E, q)-lower bounded if and only if φ_E^q is bounded from below on F.

Lemma 4.15. *Let $E \subset Y$ and $q \in Y \backslash \{0\}$ be satisfying conditions (E2) and (E3). Consider a point $y \in Y$. If a sequence $(r_n) \subset \mathbb{R}$ converges to $r \in \mathbb{R}$ and $y + r_n q \leq_E 0$ for all $n \in \mathbb{N}$, then $y + rq \leq_E 0$.*

Proof. Suppose that $r \leq r_m$ for some $m \in \mathbb{N}$. Then, by assumption (E2) we deduce

$$y + rq = (r - r_m)q + y + r_m q \in (-\infty, 0]q - E = -E$$

and the result follows.

On the contrary, assume that $r > r_n$, for all $n \in \mathbb{N}$. In this case we claim that $-y - rq \in \operatorname{vcl}_q E$. Indeed, define $t_n := r - r_n$ for all $n \in \mathbb{N}$. It is clear that $t_n \geq 0$, $t_n \to 0$ and

$$-(y + rq) + t_n q = -y - r_n q \in E.$$

Thus, the assertion is true.

As E is vectorial closed in direction q, we deduce that $-y - rq \in E$ and so $y + rq \leq_E 0$. The proof is completed. □

Definition 4.16. Let X be a topological space. A function $g : X \to Y$ is said to be $<_E^q$-strictly decreasingly lower-semicontinuous at a point $\bar{x} \in X$ ($<_E^q$-lsc at $\bar{x}$ in short form) if for every sequence (x_n) in X converging to $\bar{x}$ one has

$$\forall n \in \mathbb{N}, g(x_{n+1}) <_E^q g(x_n) \Longrightarrow \forall n \in \mathbb{N}, g(\bar{x}) \leq_E g(x_n).$$

The function g is called $<_E^q$-lsc if it is $<_E^q$-lsc at x, for all $x \in X$.

Remark 12. The concept above is a vector version of the notion of strictly decreasing lower-semicontinuity of a real-valued function introduced in Definition 2.2 (see [13, Definition 2.4], [16, Definition 2.2] and the references therein for a vector counterpart of the concept of decreasing lower-semicontinuity of a real-valued function recalled in Remark 1). For instance, if $E = D$ and $q \in D \backslash (-D)$, then $g(x_1) \neq g(x_2)$ whenever $g(x_1) <_D^q g(x_2)$.

The next theorem is the second main contribution in this section.

Theorem 4.17. (Revised vectorial version of the exact EVP). *Let (X, d) be a complete metric space, $f : X \times X \to Y$ be a vector function and the pair (E, q) as above. Suppose that there exists a vector function $g : X \to Y$ satisfying*

$$g(x_2) - g(x_1) \leq_E f(x_1, x_2), \quad \forall x_1, x_2 \in X. \tag{24}$$

Consider the set-valued map $S : X \rightrightarrows X$ with values

$$S(x) := \{u \in X \ : \ g(u) + d(x, u)q \leq_E g(x)\}.$$

Assume that g is $<^q_E$-lsc and the set $g(S(x)) - g(x)$ is (E,q)-lower bounded, for all $x \in X$. Then, for each $x_0 \in X$ there exists $\bar{x} \in X$ such that

(a) $g(\bar{x}) + d(x_0, \bar{x})q \leq_E g(x_0)$;
(b) $d(x_0, \bar{x})q \leq_E g(x_0) - g(\bar{x}) \leq_E f(\bar{x}, x_0)$;
(c) $f(\bar{x}, x) + d(\bar{x}, x)q \not\leq_E 0$, $\forall x \in X \backslash \{\bar{x}\}$.

Proof. We have to check the hypotheses of Theorem 2.1 on S and an arbitrary point $x_0 \in X$. It is easy to check that condition (A1) follows of $0 \in E$ and condition (A2) follows as a result of the triangle inequality property of the metric d, the preorder $\leq_E$ and (E2).

Next, we check (A3). For each Picard sequence (x_n) of S whose initial point is x_0, we have

$$\sum_{k=0}^{n} d(x_k, x_{k+1})q \leq_E \sum_{k=0}^{n} (g(x_k) - g(x_{k+1})) = g(x_0) - g(x_{n+1})$$

and thus

$$\sum_{k=0}^{n} d(x_k, x_{k+1})q - (g(x_0) - g(x_{n+1})) \in -E.$$

Taking into account the properties (i) and (ii) of φ^q_E in Lemma 4.1, we have

$$\begin{aligned} & \sum_{k=0}^{n} d(x_k, x_{k+1}) + \varphi^q_E(g(x_{n+1}) - g(x_0)) \\ = \quad & \varphi^q_E \left(\sum_{k=0}^{n} d(x_k, x_{k+1})q + g(x_{n+1}) - g(x_0) \right) \leq 0 \end{aligned}$$

and thus

$$\sum_{k=0}^{n} d(x_k, x_{k+1}) \leq -\varphi^q_E(g(x_{n+1}) - g(x_0)) < +\infty$$

due to the boundedness assumption imposed in the theorem.

Finally, in order to check condition (A4), let (x_n) be a distinct and Cauchy Picard sequence of S whose initial point is x_0. Since X is complete, it converges to some element $\bar{x}$. Taking into account the definition of S, we have

$$g(x_{n+1}) + d(x_n, x_{n+1})q \leq_E g(x_n) \ \text{ for all } \ n \in \mathbb{N}$$

clearly implying

$$g(x_{n+1}) <^q_E g(x_n) \ \text{ for all } \ n \in \mathbb{N}.$$

Since g is $<^q_E$-lsc, one has

$$g(\bar{x}) \leq_E g(x_n) \ \text{ for all } \ n \in \mathbb{N}.$$

As $x_{n+k} \in S(x_n)$ for all $n, k \in I\!N$, we have

$$g(\bar{x}) + d(x_n, x_{n+k})q \leq_E g(x_{n+k}) + d(x_n, x_{n+k})q \leq_E g(x_n).$$

Since k was arbitrary, passing to limit as $k \to +\infty$ and applying Lemma 4.15 we arrive at

$$g(\bar{x}) + d(x_n, \bar{x})q \leq_E g(x_n)$$

clearly verifying that $\bar{x} \in S(x_n)$. Since n was arbitrary, condition (A4) holds true.

Theorem 2.1 ensures the existence of $\bar{x} \in X$ satisfying

(i) $\bar{x} \in S(x_0)$, i.e., $g(\bar{x}) + d(x_0, \bar{x})q \leq_E g(x_0)$;
(ii) $S(\bar{x}) = \{\bar{x}\}$, i.e., $g(x) + d(\bar{x}, x)q \not\leq_E g(\bar{x})$ for all $x \in X \backslash \{\bar{x}\}$.

Obviously, (i) is equivalent to (a). Taking into account (24) and the preorder $\leq_E$, we have

$$d(x_0, \bar{x})q \leq_E g(x_0) - g(\bar{x}) \leq_E f(\bar{x}, x_0)$$

clearly verifying (b).

To complete the proof, we prove (c) by contradiction. Assume that (c) does not hold. Then, we could find $x \neq \bar{x}$ such that

$$f(\bar{x}, x) + d(\bar{x}, x)q \leq_E 0.$$

By (24), we have

$$g(x) - g(\bar{x}) + d(\bar{x}, x)q \leq_E f(\bar{x}, x) + d(\bar{x}, x)q \leq_E 0$$

which contradicts (ii). □

In the particular case $E = D$, if condition (24) holds true, then g is $<_D^q$-lsc provided that f is $\times^{\not\leq_D}$-lsc. Concerning the boundedness assumption of Theorem 4.17, notice that set $S(x)$ is bounded whenever the set $g(S(x)) - g(x)$ is (E, q)-lower bounded, since

$$\varphi_E^q(g(u) - g(x)) \leq \varphi_E^q(-d(u, x)q) \leq -d(u, x), \quad \forall u \in S(x).$$

Actually, we have the next sufficient conditions.

Lemma 4.18. *Suppose that $f : X \times X \to Y$ and $g : X \to Y$ satisfy condition* (24) *for $E = D$. If f is $\times^{\not\leq_D}$-lsc at $\bar{x} \in X$, then g is $<_D^q$-lsc at $\bar{x}$ too, for all $q \in Y \backslash (-D)$.*

Proof. Let $(x_n) \subset X$ be a sequence converging to $\bar{x}$ and satisfying $g(x_{n+1}) <_D^q g(x_n)$, for all $n \in I\!N$. Then, $g(x_n) - g(x_{n+1}) \in (0, +\infty)q + D$, for all $n \in I\!N$.

If there exists $m \in I\!N$ such that $f(x_{m+1}, x_m) \leq_D 0$, by (24) we deduce that $g(x_m) \leq_D g(x_{m+1})$. Therefore, $((0, +\infty)q + D) \cap (-D) \neq \emptyset$ and so $q \in -D$, that is a contradiction.

Then, $f(x_{n+1}, x_n) \not\leq_D 0$, for all $n \in I\!N$ and so $f(x_n, \bar{x}) \leq_D 0$, since f is $\times^{\not\leq_D}$-lsc at $\bar{x}$. Thus, by (24) we see that $g(\bar{x}) - g(x_n) \leq_D 0$ for all $n \in I\!N$ and the proof is complete. □

Lemma 4.19. *Consider a function $g : X \to Y$, the pair (E, q) as above and a nonempty set $A \subset X$.*

(i) *If the set $g(A)$ is (E, q)-lower bounded, then $g(A) - g(x)$ is also (E, q)-lower bounded, for all $x \in g^{-1}(\mathbb{R}q - E)$.*
(ii) *Consider $\lambda \in E^+$ such that $\lambda(q) > 0$. If λ is bounded from below in $g(A)$, then $g(A)$ is (E, q)-lower bounded.*

Proof. (i) Assume that $g(A)$ is (E, q)-lower bounded, i.e., there exists $M \in \mathbb{R}$ such that $g(a) \not\leq_E Mq$, for all $a \in A$. Consider a point $x \in g^{-1}(\mathbb{R}q - E)$. There exists $t \in \mathbb{R}$ such that $g(x) \leq_E tq$. We claim that $g(a) - g(x) \not\leq_E (M - t)q$, for all $a \in A$. Indeed, let us suppose, reasoning by contradiction, that there exists $u \in A$ such that $g(u) - g(x) \leq_E (M - t)q$. Then,

$$g(u) = (g(u) - g(x)) + g(x) \leq_E (M - t)q + tq = Mq,$$

that is a contradiction. Therefore, the set $g(A) - g(x)$ is (E, q)-lower bounded and the proof finishes.

(ii) Suppose that λ is bounded from below in $g(A)$. Then, there exists $m \in \mathbb{R}$ such that $\lambda(g(a)) > m$, for all $a \in A$. Define $M := m/\lambda(q)$. We claim that $g(a) \not\leq_E Mq$, for all $a \in A$. Indeed, if there is a point $u \in A$ such that $g(u) \leq_E Mq$, then $\lambda(g(u)) \leq m$, that is a contradiction.

As $g(a) \not\leq_E Mq$, for all $a \in A$, we deduce that set $g(A)$ is (E, q)-lower bounded and the proof finishes. □

Remark 13. The (E, q)-lower boundedness condition is more effective on the set $g(S(x)) - g(x)$, for all $x \in X$, than on the whole image set $g(X)$. Let us illustrate this claim with an example. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $g : X \to Y$ is defined by $g(x) = (x/2, 0)$ for all $x \in \mathbb{R}$, $q = (1, 0)$ and $E = [0, +\infty)q$. Obviously, $g(X) = \mathbb{R} \times \{0\}$, $\operatorname{dom} \varphi_E^q = \mathbb{R} \times \{0\}$ and $\varphi_E^q(y, 0) = y$ for all $y \in \mathbb{R}$. It is clear that

$$\inf_{x \in X} \varphi_E^q(g(x)) = -\infty$$

and then $g(X)$ is not (E, q)-lower bounded. However, for each $x \in X$, we have $S(x) = \{x\}$ and then

$$\inf_{u \in S(x)} \varphi_E^q(g(u) - g(x)) = \varphi_E^q((0, 0)) = 0.$$

Thus, the set $g(S(x)) - g(x)$ is (E, q)-lower bounded, for all $x \in X$.

Remark 14. 1. Assume that $q \notin -\operatorname{cl}\operatorname{cone} E$. Then, condition (24) implies that the function $\varphi_E^q \circ f : X \times X \to \mathbb{R} \cup \{+\infty\}$ is cyclically antimonotone. Indeed, consider a finite nonempty set $\{x_1, x_2, \ldots, x_n, x_{n+1}\} \subset X$ such that $x_{n+1} = x_1$. By condition (24) we have that

$$\sum_{j=1}^{n} f(x_j, x_{j+1}) \geq_E \sum_{j=1}^{n} (g(x_{j+1}) - g(x_j)) = 0.$$

Since φ_E^q is subadditive, $\leq_E$-monotone and $\varphi_E^q(0) = 0$ (see parts (v), (vi) and (viii) of

Lemma 4.1), it follows that

$$\begin{aligned}\sum_{j=1}^{n}(\varphi_E^q \circ f)(x_j, x_{j+1}) &\geq \varphi_E^q\left(\sum_{j=1}^{n} f(x_j, x_{j+1})\right)\\ &\geq \varphi_E^q(0)\\ &= 0.\end{aligned}$$

Therefore, $\varphi_E^q \circ f$ is cyclically antimonotone.

2. Suppose that $E = D$. In order to compare the boundedness assumptions of Theorems 4.12 and 4.17, notice that $f(\cdot, \hat{x})$ is q-order bounded from above whenever we could find $M_1 \in I\!R$ such that $f(x, \hat{x}) - M_1 q \in (-\infty, 0]q - D$ for all $x \in X$, i.e.,

$$-f(X, \hat{x}) + M_1 q \subset [0, +\infty)q + D. \tag{25}$$

By condition (24) with $x_1 = x$ and $x_2 = \hat{x}$, we have

$$-f(x, \hat{x}) \leq_D g(x) - g(\hat{x})$$

and then, applying the scalarization function φ_D^q we deduce

$$\varphi_D^q(-f(x, \hat{x})) \leq \varphi_D^q(g(x) - g(\hat{x})).$$

Thus, for ensuring the boundedness from below of $\varphi_D^q(g(x) - g(\hat{x}))$ when x belongs to $S(\hat{x})$, we could assume for some $M_2 \in I\!R$ that

$$\varphi_D^q(-f(x, \hat{x})) > M_2, \quad \forall x \in X.$$

It is equivalent to

$$(-f(X, \hat{x}) - M_2 q) \cap ((-\infty, 0]q - D) = \emptyset.$$

This assumption is weaker than the q-order boundedness from above considered in (25).

3. Theorem 4.17 extends the main results of [14, Section 3] when a complete metric space (X, d) and the distance d are considered instead of a left complete quasi-metric space and a W-distance, respectively. Even in this particular case, Theorem 4.17 improves [14, Theorems 3.1 and 3.2] as more general lower boundedness and lower-semicontinuity assumptions are required.

Indeed, let Y be a real linear space. Suppose that the ordering cone D is proper, convex and algebraically solid, and consider an arbitrary $q \in \operatorname{core} D$. Then, the set $E := \operatorname{vcl}_q D$ is a convex cone that fulfills properties (E1)-(E3). Consider a vector bifunction $f : X \times X \to Y$ satisfying the triangle inequality property (16) and such that for each $x \in X$ and $y \in Y$, the real-valued function $(\varphi_D^q \circ f)(x, \cdot) : X \to I\!R$ is bounded from below and the next sublevel set is closed:

$$S(f(x, \cdot), y) := \{u \in X : f(x, u) \leq_E y\}.$$

Fix an arbitrary point x_0. By [14, Lemma 2.2] we see that $X_0 := S(f(x_0, \cdot) + d(x_0, \cdot)q, 0)$ is closed. Suppose that $X_0 \neq \emptyset$ (otherwise, $\bar{x} = x_0$ fulfills the assertions of [14, Theorem

3.1]). Let us apply Theorem 4.17 to the complete metric space (X_0, d). Consider an arbitrary point $a \in X_0$ and the function $g_a : X_0 \to Y$, $g_a(x) = f(a, x)$, for all $x \in X_0$. For each $x_1, x_2 \in X_0$, by the triangle inequality property (16) we have that

$$g_a(x_2) = f(a, x_2) \leq_E f(a, x_1) + f(x_1, x_2) = g_a(x_1) + f(x_1, x_2)$$

and condition (24) holds true. As $\varphi_E^q \circ g_a = (\varphi_D^q \circ f)(a, \cdot)$ (see [21, Lemma 3]), by Lemma 4.19 we deduce that the set $g_a(S(x)) - g_a(x)$ is (E, q)-lower bounded, for all $x \in X_0$. In addition, it is easy to check from the closedness of the sublevel sets $S(f(x, \cdot), y)$ that g_a is $<_E^q$-lsc.

Then, by Theorem 4.17 we deduce that there exists $\bar{x} \in X_0$ such that

$$f(\bar{x}, x) + d(\bar{x}, x)q \not\leq_E 0, \quad \forall x \in X_0 \backslash \{\bar{x}\}. \tag{26}$$

As $\bar{x} \in X_0$, statement (a) of [14, Theorem 3.1] is obtained. Assume, reasoning by contradiction, that there is a point $x' \in X \backslash X_0$ such that

$$f(\bar{x}, x') + d(\bar{x}, x')q \leq_E 0.$$

Then, by the triangle inequality property (16) and statement (a) of [14, Theorem 3.1] it follows that

$$\begin{aligned} f(x_0, x') + d(x_0, x')q &\leq_D f(x_0, \bar{x}) + f(\bar{x}, x') + d(x_0, \bar{x})q + d(\bar{x}, x')q \\ &\leq_E f(\bar{x}, x') + d(\bar{x}, x')q \\ &\leq_E 0 \end{aligned}$$

and so $x' \in X_0$, that is a contradiction. Therefore, statement (26) is true for all $x \in X \backslash \{\bar{x}\}$ and [14, Theorem 3.1] is stated as a result of Theorem 4.17.

Reasoning in the same way can be checked that [10, Theorem 3.1] and [14, Theorem 3.2] are a consequence of Theorem 4.17 when a complete metric space (X, d) and the distance d are considered instead of a left complete quasi-metric space and a W-distance, respectively.

It is worth noticing that Theorem 4.17 could be applied to problems whose ordering cone is not algebraically solid. However, [14, Theorems 3.1 and 3.2] cannot be applied in that setting. In addition, these results can be also compared with Theorem 4.12 by considering Y endowed with the so-called core convex topology τ_c. Recall that (Y, τ_c) is a real locally convex Hausdorff topological linear space satisfying $\operatorname{int}_{\tau_C} D = \operatorname{core} D$ (see [27, Proposition 6.3.1] and [28]) and $\operatorname{cl}_{\tau_c} D = \operatorname{vcl}_q D$ (see [28, Lemma 3.1] and [24, Proposition 2.3]).

Recall that given a function $g : X \to Y$, $x_0 \in X$ and $y \in Y$, we are denoting:

$$\begin{aligned} S(x_0) &:= \{x \in X \ : \ g(x) + d(x_0, x)q \leq_E g(x_0)\}, \\ S(g, y) &:= \{x \in X : g(x) \leq_E y\}. \end{aligned}$$

By conditions (E1) and (E2) it follows that $S(x_0) \subset S(g, g(x_0))$.

Next, a version of [9, Theorem 3.6] for $<_E^q$-lsc functions is obtained, where E is the vector closure of a convex cone $C \subset Y$ in direction $q \in C \backslash \{0\}$. It is a result of Theorem 4.17 and Lemma 4.19. Therefore, Theorem 4.17 encompasses [9, Theorem 3.6].

Corollary 4.20. *Consider a complete metric space* (X,d), $x_0 \in X$, $f : X \times X \to Y$, *a convex cone* $C \subset Y$, $q \in C \backslash \{0\}$, $E = \mathrm{vcl}_q C$ *and* $\lambda \in C^+$ *such that* $\lambda(q) > 0$. *Suppose that there exists a vector function* $g : X \to Y$ *satisfying*

$$g(x_2) - g(x_1) \leq_E f(x_1, x_2), \quad \forall x_1, x_2 \in X. \tag{27}$$

Assume that $S(x_0)$ *is closed,* g *is* $<^q_E$*-lsc at* x*, for all* $x \in S(x_0)$ *and*

$$\inf\{\lambda(g(x)) : x \in S(g, g(x_0))\} > -\infty. \tag{28}$$

Then, there exists $\bar{x} \in X$ *such that*

(a) $g(\bar{x}) + d(x_0, \bar{x})q \leq_E g(x_0)$;
(b) $d(x_0, \bar{x})q \leq_E g(x_0) - g(\bar{x}) \leq_E f(\bar{x}, x_0)$;
(c) $f(\bar{x}, x) + d(\bar{x}, x)q \not\leq_E 0$, $\forall x \in X \backslash \{\bar{x}\}$.

Proof. It is easy to check that E satisfies conditions (E1)-(E3). Let us apply Theorem 4.17 to the metric space (X_0, d) instead of (X, d), where $X_0 = S(x_0)$.

As $S(x_0)$ is closed, the metric space (X_0, d) is complete. By the assumptions we have that $g : X_0 \to Y$ is $<^q_E$-lsc. Moreover, by applying part (ii) of Lemma 4.19 to $A = S(g, g(x_0))$ and $g - g(x_0)$, we see that the boundedness condition of Theorem 4.17 is fulfilled.

Then, Theorem 4.17 can be applied and we deduce that there exists $\bar{x} \in X_0$ satisfying statements (a), (b) and (c) for all $x \in X_0 \backslash \{\bar{x}\}$. Suppose, reasoning by contradiction, that there exists $x' \in X \backslash X_0$ such that

$$f(\bar{x}, x') + d(\bar{x}, x')q \leq_E 0.$$

By (27) and part (a) we have that

$$\begin{aligned} g(x') + d(x_0, x')q &\leq_E g(x') + d(x_0, \bar{x})q + d(\bar{x}, x')q \\ &\leq_E g(\bar{x}) + f(\bar{x}, x') + d(\bar{x}, x')q + d(x_0, \bar{x})q \leq_E 0 \\ &\leq_E g(\bar{x}) + d(x_0, \bar{x})q \\ &\leq_E g(x_0), \end{aligned}$$

that is a contradiction, since $x' \notin X_0$. Then, statement (c) is true, for all $x \in X \backslash \{\bar{x}\}$ and the proof finishes. □

Remark 15. 1. Notice that the closedness assumptions of Corollary 4.20 and [9, Theorem 3.6] are different from the one in Theorem 4.17. For instance, consider $X = Y = I\!R$, the usual distance $d(x_1, x_2) = |x_1 - x_2|$, for all $x_1, x_2 \in I\!R$, $C = I\!R_+$, $q = 1$ and $g : I\!R \to I\!R$,

$$g(t) = \begin{cases} -t & \text{if} \quad t \leq -1, \\ 0 & \text{if} \quad t \in (-1, 0), \\ 1 + t & \text{if} \quad t \geq 0. \end{cases}$$

It is clear that $E = \mathrm{vcl}_q C = I\!R_+$ and so $\leq_E$ and $<^q_E$ coincide with the usual orderings $\leq$ and $<$ in $I\!R$, respectively. It is easy to obtain that $S(0) = (-1, 0]$ and so $S(0)$ is not closed. However, g is $<^q_E$-lsc.

2. Corollary 4.20 encompasses [11, Theorem 1]. Indeed, consider $C = D$, $q \in D\backslash\{0\}$ and $\lambda \in D^+$ such that $\lambda(q) = 1$. Assume that f satisfies the triangle inequality property (16) with respect to the partial order $\leq_D$. Moreover, consider a point $x_0 \in X$ and the function $g_{x_0} : X \to Y$ given by $g_{x_0}(x) = f(x_0, x)$. Suppose that λ is bounded from below in the image set $g_{x_0}(X)$ and the sublevel set $S(g_{x_0}, y)$ is closed, for all $y \in Y$. Then, it is clear that g_{x_0} fulfills conditions (27) and (28). In addition, $S(x_0)$ is closed and g_{x_0} is $<_D^q$-lsc (the first assertion follows by [11, Lemma 2] and the second one is trivial). As a result, Corollary 4.20 can be applied and the assertions of [11, Theorem 1] are obtained.

In a similar way, it is easy to check that Corollary 4.20 generalizes [15, Theorem 3.4].

3. Corollary 4.20 encompasses [16, Theorem 3.7], where a stronger lower-semicontinuity assumption is considered (see Remarks 12 and 1).

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