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FURTHER RESULTS ON QUASI EFFICIENT SOLUTIONS IN MULTIOBJECTIVE OPTIMIZATION

CÉSAR GUTIÉRREZ

Mathematics Research Institute (IMUVa), University of Valladolid, Edificio LUCIA, Paseo de Belen S/N, 47011 Valladolid, Spain

Abstract. This work focuses on quasi efficient solutions of a vector optimization problem. Namely, it is assumed the objective function to map values between finite dimensional spaces and the ordering in the final space is defined by a pointed convex cone. In this setting, several new concepts of quasi efficient solutions are introduced and their basic properties are obtained. After that, it is studied their limit behavior in order to deduce when they approximate well other kind of efficient solutions of the problem. Finally, in convex problems, linear scalarization results are stated that relate the introduced quasi efficient solutions with solutions of scalar optimization problems. The concepts and results of the paper extend and clarify several ones of the recent literature.

Keywords. Generalized convexity; Linear scalarization; Multiobjective optimization; Quasi efficient solution.

1. INTRODUCTION

To the best of our knowledge, the concept of quasi solution of an optimization problem was introduced by Loridan in [32]. These solutions, whose existence can be guarantied by the Ekeland variational principle [8], are important as they can be identified as approximate stationary points (see [1, 6, 9, 24, 32]). In other words, they are in the core of many numerical algorithms to solve optimization problems.

Loridan [31] extended this type of solutions to multiobjective optimization problems and the Pareto order. After that, Gupta and Mehra [14], Gutiérrez et al. [23] and Gao et al. [13] generalized Loridan's concept to vector optimization problems and orderings defined by pointed convex cones, and Gutiérrez et al. [20] to vector optimization problems whose objective function is set-valued and the ordering is provided by a free-disposal set. Since then, other authors have introduced some refinements of these seminal generalizations (see [15, 26, 27]). Recently, the notion of quasi solution has been defined for set optimization problems [22].

In the setting of vector optimization, the study of quasi efficient solutions is meaningful since not only concern feasible points that are close to be stationary, but also they encompass many different types of nondominated solutions of the problem, as efficient solutions, weak/proper/approximate efficient solutions and others (see [23]). By this unifying reason,

E-mail address: cesargv@uva.es
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authors are interested to characterize quasi efficient solutions by linear and nonlinear scalarizations (see [13, 20, 26, 31]) and multiplier rules (see [3, 10, 13, 15, 20, 23, 27, 29, 30]), to study their limit behavior when the precision tends to zero (see [23, 26, 27]) and to state existence results (see [13, 14, 23]) and duality assertions (see [3, 10, 14, 29, 30]).

The aim of this work is to introduce new concepts of quasi efficiency and by them to derive further results on quasi solutions of a multiobjective optimization problem whose final space is ordered by a pointed convex cone. Namely, in Section 2, the basic notations and mathematical tools of the paper are introduced. In addition, the main quasi efficient solution concepts of the literature are recalled. After that, in Section 3, new concepts of quasi efficient solution of a vector optimization problem are defined and related with the ones given up to now in the literature. Moreover, their basic properties are stated. In Section 4, the limit behavior of the quasi efficient solution sets obtained by the new definitions is studied. It is proved that they allow approximating well the sets of efficient and weak efficient solutions of the problem. As a result, a notion of proper quasi efficiency is introduced and its basic properties are obtained. Finally, in Section 5, linear scalarization results are derived in convex problems, which characterize weak and proper quasi efficient solutions as solutions of scalar optimization problems. The obtained results generalize and clarify many others of the recent literature.

2. Preliminaries

Throughout, $\operatorname{int} M$, $\operatorname{bd} M$, M^c and $\operatorname{cone} M$ denote the topological interior, the boundary, the complement and the generated cone by a set $M \subset \mathbb{R}^p$, respectively. We say that M is solid whenever $\operatorname{int} M \neq \emptyset$. Moreover, $\mathbb{B}_p$ denotes the open Euclidean unit ball of $\mathbb{R}^p$, $\mathbb{R}^p_+$ refers to the nonnegative orthant of $\mathbb{R}^p$ and $\mathbb{R}_+ := \mathbb{R}^1_+$. Recall that M is said to be a coradiant set if $\alpha M \subset M$, for all $\alpha > 1$ (see [17]). The shadow generated by M is the set $\operatorname{shw} M := \bigcup_{\alpha > 1} \alpha M$ (see [36]). Clearly, M is coradiant if and only if $\operatorname{shw} M \subset M$. In addition, $M \subset \operatorname{shw} M$ whenever M is open and so, in this case, M is coradiant if and only if $\operatorname{shw} M = M$. Recall that M is free-disposal with respect to a convex cone $K \subset \mathbb{R}^p$ if $M + K = M$ (see [5]). If additionally $0 \notin M$, then M is called an improvement set (see [2]). Moreover, M is said to be K-convex if $M + K$ is convex.

The inner and outer limits of a sequence (M_m) of nonempty subsets of $\mathbb{R}^p$ are the sets

$$\liminf_{m\to\infty} M_m := \{y \in \mathbb{R}^p : \exists M_m \ni y_m \to y\},$$
$$\limsup_{m\to\infty} M_m := \{y \in \mathbb{R}^p : \exists M_{m_k} \ni y_{m_k} \to y\}.$$

Consider a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ and $\emptyset \neq Q \subset \mathbb{R}^n$. The domain and the graph of F are the sets

$$\operatorname{dom} F := \{x \in \mathbb{R}^n : F(x) \neq \emptyset\},$$
$$\operatorname{gph} F := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^p : y \in F(x)\}.$$

Moreover, $F|_Q : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ stands for the mapping $F|_Q(x) = F(x)$ if $x \in Q$ and $F|_Q(x) = \emptyset$ otherwise. Clearly, $\operatorname{dom} F|_Q = (\operatorname{dom} F) \cap Q$. Given $F_1, F_2 : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$, we denote $F_1 \subset F_2$ if $\operatorname{gph} F_1 \subset \operatorname{gph} F_2$, i.e., $F_1(x) \subset F_2(x)$, for all $x \in \mathbb{R}^n$. Moreover, we say that $F_1 \subset F_2$ in Q if $F_1|_Q \subset F_2|_Q$, and $F_1 = F_2$ in Q whenever $F_1 \subset F_2$ and $F_2 \subset F_1$ in Q. In addition, αF_1, $F_1^c, F_1 \cap F_2, \operatorname{int} F_1, \operatorname{shw} F_1 : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ stand for the set-valued mappings $(\alpha F_1)(x) := \alpha F_1(x)$, $F_1^c(x) := F_1(x)^c$, $(F_1 \cap F_2)(x) := F_1(x) \cap F_2(x)$, $\operatorname{int} F_1(x) := \operatorname{int}(F_1(x))$ and $\operatorname{shw} F_1(x) := \operatorname{shw}(F_1(x))$, for all $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Notice that $\operatorname{gph}(F_1 \cap F_2) = \operatorname{gph} F_1 \cap \operatorname{gph} F_2$.

The inner and outer limits of $F:\mathbb{R}^n \rightrightarrows \mathbb{R}^p$ at a point $x_0\in\mathbb{R}^n$ are, respectively, the sets

$$\operatorname*{Lim\,inf}_{x\to x_0} F(x) := \{y\in\mathbb{R}^p : \forall x_m\to x_0\,\exists y_m\to y,\,(x_m,y_m)\in \operatorname{gph} F\ \forall m\},$$
$$\operatorname*{Lim\,sup}_{x\to x_0} F(x) := \{y\in\mathbb{R}^p : \exists((x_m,y_m))\subset \operatorname{gph} F, (x_m,y_m)\to(x_0,y)\}.$$

Recall that F is said to be open (resp. closed, graph-convex) if $\operatorname{gph} F$ is an open (resp. closed, convex) set, and F is said to be inner (resp. outer) semicontinuous at x_0 if

$$F(x_0)\subset \operatorname*{Lim\,inf}_{x\to x_0} F(x) \quad (\text{resp. } \operatorname*{Lim\,sup}_{x\to x_0} F(x)\subset F(x_0)).$$

One has that F is outer semicontinuous everywhere if and only if F is closed (see [33, Theorem 5.7]). In particular, if F is closed, then F is closed-valued. Moreover, if F is solid-valued and convex-valued, then the set-valued mapping $\operatorname{int} F$ is open if and only F is inner semicontinuous everywhere (see [33, Theorem 5.9]). Notice that F is graph-convex if and only if

$$\alpha F(x)+(1-\alpha)F(u)\subset F(\alpha x+(1-\alpha)u),\quad \forall x,u\in\mathbb{R}^n,\forall\alpha\in(0,1).$$

In particular, F is convex-valued whenever it is graph-convex.

Given a sequence of set-valued mappings $F_m:\mathbb{R}^n \rightrightarrows \mathbb{R}^p$, where $\operatorname{dom} F_m=\mathbb{R}^n$ for all $m\in\mathbb{N}$, its graphical inner and outer limits $\operatorname*{g\text{-}lim\,inf}_{m\to\infty} F_m, \operatorname*{g\text{-}lim\,sup}_{m\to\infty} F_m:\mathbb{R}^n \rightrightarrows \mathbb{R}^p$ are, respectively, the set-valued mappings

$$\operatorname*{g\text{-}lim\,inf}_{m\to\infty} F_m(x) := \{y\in\mathbb{R}^p : \exists\, \operatorname{gph} F_m \ni (x_m,y_m)\to(x,y)\},$$
$$\operatorname*{g\text{-}lim\,sup}_{m\to\infty} F_m(x) := \{y\in\mathbb{R}^p : \exists\, \operatorname{gph} F_{m_k} \ni (x_{m_k},y_{m_k})\to(x,y)\}.$$

This paper addresses the constrained vector optimization problem

$$\operatorname{Min}\{f(x): x\in S\}, \tag{P$_D$}$$

where $f:\mathbb{R}^n\to\mathbb{R}^p$, $\emptyset\neq S\subset\mathbb{R}^n$ and $D\subset\mathbb{R}^p$ stands for a pointed ($D\cap(-D)=\{0\}$) closed convex cone that models the practitioner's preferences in the objective space $\mathbb{R}^p$ through the following partial order:

$$y_1,y_2\in\mathbb{R}^p,\quad y_1\leq_D y_2 :\iff y_2-y_1\in D.$$

Problem (P_D) in the trivial case where $D=\{0\}$ is denoted by (P). It means that there is not any fixed preference structure in the final space. As a result, every feasible point $x\in S$ is a solution of the problem as its value $f(x)$ cannot be improved.

Roughly speaking, solutions of problem (P_D) are defined by considering notions of minimal point with respect to the partial order $\leq_D$ or more generally, with respect to another binary relation in $\mathbb{R}^p$, which is related with $\leq_D$ in some sense. For instance, the following concept is a reformulation of [23, Definition 3.1]. Let $\emptyset\neq G\subset\mathbb{R}^p$, $\varphi:\mathbb{R}^n\to\mathbb{R}$ and denote $\mathscr{D}:=(D,G,\varphi)$ if $D\neq\{0\}$ and $\mathscr{D}:=(G,\varphi)$ otherwise.

Definition 2.1. A feasible point $x_0\in S$ is said to be a quasi minimizer of problem (P_D) with respect to $\mathscr{D}=(D,G,\varphi)$, denoted by $x_0\in \mathrm{E}(f,S,\mathscr{D})$, if

$$f(x)+\varphi(x-x_0)e-f(x_0)\notin -D,\quad \forall x\in S\backslash\{x_0\},\forall e\in G. \tag{2.1}$$

Remark 2.1. (i) Condition (2.1) can be extended to $x = x_0$ whenever $\varphi(0)G \cap (-D) = \emptyset$; otherwise, it never could be satisfied for all $x \in S$. In other words, if that extension is not considered, then additional assumptions that relate the elements of $\mathscr{D}$ are not needed a priory.

(ii) Notice that (2.1) can be replaced with the condition

$$f(x) + h(x,x_0)e - f(x_0) \notin -D, \quad \forall x \in S\backslash\{x_0\}, \forall e \in G, \tag{2.2}$$

where $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. However, both conditions provide the same notion by considering the functions φ and h are "parameters". Indeed, if (2.1) is true, then (2.2) is satisfied by taking the function $h(x,u) = \varphi(x-u)$, for all $x,u \in \mathbb{R}^n$. Conversely, if (2.2) holds true, then (2.1) is fulfilled by defining $\varphi(x) = h(x+x_0,x_0)$, for all $x \in \mathbb{R}^n$. This reformulation was introduced in [13, 20] and it has recently been studied in [26, 27].

(iii) Definition 2.1 encompasses many concepts of solution of problem (P_D). Let us show some examples, where the function $\varphi(x) = 1$, for all $x \in \mathbb{R}^n$, is denoted by $\mathbb{1}$:

(a) If $\varphi = \mathbb{1}$, then quasi minimizers of problem (P_D) correspond to efficient solutions with respect to the domination set $G + D$. Then, one can recover the sets of efficient, weak efficient and Henig proper efficient solutions (see [34, Definition 3.1.1 and Remark 3.1.1] and [25, Definition 2.1]) by considering $G = D\backslash\{0\}$, $G = \operatorname{int} D$ and $G = \operatorname{int} K$ for a dilating cone K of D (i.e., $K \subsetneq \mathbb{R}^p$ is a convex cone such that $D\backslash\{0\} \subset \operatorname{int} K$), respectively. These relations are direct consequences of the equalities $D\backslash\{0\} + D = D\backslash\{0\}$, $\operatorname{int} D + D = \operatorname{int} D$ and $\operatorname{int} K + D = \operatorname{int} K$. The corresponding sets of efficient, weak efficient and Henig proper efficient solutions of problem (P_D) will be denoted in the sequel by $\mathrm{E}(f,S,D)$, $\mathrm{WE}(f,S,D)$ and $\mathrm{HPE}(f,S,D)$, respectively.

This case also covers the efficiency concepts defined by a free-disposal ordering set G with respect to D (in particular, an improvement set, see [16, Definition 3.1] or [19, Definition 4.1]).

(b) More generally, the trivial case $D = \{0\}$ along with the function $\varphi = \mathbb{1}$ lead to the usual condition of nondomination with respect to the ordering set G:

$$(f(S\backslash\{x_0\}) - f(x_0)) \cap (-G) = \emptyset. \tag{2.3}$$

Examples of this setting are the concept of approximate efficiency with respect to a coradiant set (see [17, Definition 3.2]) or a set-valued mapping $G : \mathbb{R}_+ \rightrightarrows \mathbb{R}^p$ (see [18, Definition 7]) and the notion of global S-minimizer due to Flores-Bazán and Hernández (see [12]).

Let us underline that in this case, the partial order $\leq_D$ obviously does not play any role and the preference structure that is being considered in the solution concept depends only on the set G. Moreover, if one expects obtaining any relationships of the set of quasi minimizers of problem (P) with respect to $\mathscr{D} = (G, \mathbb{1})$ and the set of efficient solutions with respect to a partial order $\leq_K$ defined by an ordering convex cone $K \subset \mathbb{R}^p$, then some properties linking the sets G and K must be required.

(c) In [26, Definition 2(i)], the notion of (G,h)-quasi efficient solution is introduced. Namely, for a set $G \subset \mathbb{R}^p$ (denoted by C in [26, Definition 2]) and a function $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ such that $h(x,z) > 0$ whenever $x \neq z$, a point $x_0 \in S$ is said to be a (G,h)-quasi efficient solution of problem (P_D) if there is not $x \in S\backslash\{x_0\}$ such that $f(x_0) \in f(x) + h(x,x_0)G$, i.e.,

$$f(x) + h(x,x_0)e - f(x_0) \neq 0, \quad \forall x \in S\backslash\{x_0\}, \forall e \in G. \tag{2.4}$$

Clearly, Definition 2.1 reduces to this concept in the trivial case $D = \{0\}$ (see part (ii) above). Therefore, as in the previous part (b), the preference structure that is being considered in condition (2.4) depends only on the set G and the function h, and problem (P) is analyzed. Namely, since function h ranges in $\mathbb{R}_+$, the values of (G,h)-quasi efficient solutions are nondominated with respect to certain directions in cone G.

3. A NEW QUASI EFFICIENCY NOTION

Next, we define a new concept of quasi efficiency for problem (P), which is motivated by the general quasi order introduced in [11, Definition 3.1]. For this aim, a set-valued mapping $C : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is considered and assumed to fulfill $\operatorname{dom} C = \mathbb{R}^n \times \mathbb{R}^n$. Moreover, we denote $\Delta_n := \bigcup_{x\in\mathbb{R}^n} \{(x,x)\} \subset \mathbb{R}^n \times \mathbb{R}^n$ and $S^2\backslash\Delta_n := (S\times S)\backslash\Delta_n$.

Definition 3.1. A point $x_0 \in S$ is said to be a quasi efficient solution of problem (P), and it is denoted by $x_0 \in \mathrm{QE}(f,S,C)$, if

$$f(x_0) \notin f(x) + C(x,x_0), \quad \forall x \in S\backslash\{x_0\}. \tag{3.1}$$

Remark 3.1. (i) Obviously, $\mathrm{QE}(f,S,C_1) = \mathrm{QE}(f,S,C_2)$ whenever $C_1 = C_2$ in $S^2\backslash\Delta_n$.

(ii) The notion of quasi efficient solution of problem (P) reduces to the concept of quasi minimizer of problem (P_D) with respect to $\mathscr{D} = (D,G,\varphi)$ by considering the set-valued mapping $C(x,u) = \varphi(x-u)G + D$, for all $x,u \in \mathbb{R}^n$ (notice that the set-valued map C is related to D as its values are free-disposal sets with respect to D). Therefore, it generalizes all solution concepts involved in Remark 2.1(iii).

In particular, notice by part (c) of Remark 2.1(iii) that Definition 3.1 encompasses [26, Definition 2] through the set-valued mapping $C : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$, $C(x,u) = h(x,u)G$, for all $x,u \in \mathbb{R}^n$ (set G is referred to C in [26, Definition 2]). In addition, Definition 3.1 is more general. For instance, consider $p = 2$, $\mu(x,y) := \|x-y\|/(1+\|x-y\|)$ and the set-valued mapping

$$C(x,y) = (\{1\} \times [-\mu(x,y),\mu(x,y)]) + \operatorname{cone}\{(1,0)\}, \quad \forall x,y \in \mathbb{R}^n. \tag{3.2}$$

It is easy to check that there is not any nonempty set $G \subset \mathbb{R}^p$ and any function $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$, $h(x,y) > 0$ whenever $x \neq y$, such that $C = hG$.

(iii) Moreover, Definition 3.1 covers some notions of efficient solution with respect to a variable ordering structure (see [7]). Indeed, consider a cone convex-valued mapping $D : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$. Then, Definition 3.1 applied to the set-valued mappings $C(x,u) = D(f(x))\backslash\{0\}$, $C(x,u) = \operatorname{int} D(f(x))$ and $C(x,u) = \operatorname{int} K(f(x))$ for a dilating cone $K(f(x))$ of $D(f(x))$ (resp. $C(x,u) = D(f(u))\backslash\{0\}$, $C(x,u) = \operatorname{int} D(f(u))$ and $C(x,u) = \operatorname{int} K(f(u))$) correspond to the feasible point x_0 being a nondominated, weak nondominated and Henig proper nondominated (resp. minimal, weak minimal and Henig proper minimal) solution of problem (P) with respect to the variable ordering structure D (see [7, Definition 2.33]).

(iv) Finally, let us notice the particular case where $0 \in C(x,x_0)$, for all $x \in S\backslash\{x_0\}$, which implies that $f(x_0) \neq f(x)$, for all $x \in S\backslash\{x_0\}$ whenever $x_0 \in \mathrm{QE}(f,S,C)$. This is a sort of strict efficient solution. In other words, condition $0 \notin C(x,x_0)$ as long as $x \neq x_0$ seems to be a natural assumption to deal with a more general kind of efficiency concept.

The following theorem shows basic properties of the quasi solutions of problem (P). For each $u \in S$, $C^c|_S(u,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ refers to the mapping $C^c|_S(u,x) = C^c(u,x)$ if $x \in S$ and $C^c|_S(u,x) = \emptyset$ otherwise.

Theorem 3.1. *The next statements are true.*

(i) *Consider* $C_1, C_2 : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$. *If* $C_1 \subset C_2$ *in* $S^2 \backslash \Delta_n$, *then* $\mathrm{QE}(f,S,C_2) \subset \mathrm{QE}(f,S,C_1)$.

(ii) *Let* $(C_i)_{i\in I}, C$ *be a collection of set-valued mappings, where* $C_i, C : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *for all* $i \in I$, *and* $C = \bigcup_{i\in I} C_i$ *in* $S^2 \backslash \Delta_n$. *Then,* $\mathrm{QE}(f,S,C) = \bigcap_{i\in I} \mathrm{QE}(f,S,C_i)$.

(iii) *Let* $(C_i)_{i\in I}, C$ *be a collection of set-valued mappings, where* $C_i, C : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *for all* $i \in I$, *and* $C = \bigcap_{i\in I} C_i$ *in* $S^2 \backslash \Delta_n$. *Then,* $\mathrm{QE}(f,S,C) \supset \bigcup_{i\in I} \mathrm{QE}(f,S,C_i)$.

(iv) *Consider* $C : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$. *Suppose that the feasible set* S *is closed, the set-valued mapping* $C^c|_S(u,\cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *is outer semicontinuous at* $x \in S\backslash\{u\}$, *for all* $(x,u) \in S^2 \backslash \Delta_n$, *and* f *is continuous. Then the set* $\mathrm{QE}(f,S,C)$ *is closed.*

(v) *Suppose that the feasible set* S *is closed,* f *is continuous and* $C(u,\cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *is open, for all* $u \in S$. *Then the set* $\mathrm{QE}(f,S,C)$ *is closed.*

Proof. (i) Let $x \in S$ be such that $x \notin \mathrm{QE}(f,S,C_1)$. Then, there exists $u \in S\backslash\{x\}$ such that $f(x) \in f(u) + C_1(u,x)$. Therefore, $(u,x,f(x)-f(u)) \in \mathrm{gph}\, C_1|_{S^2\backslash\Delta_n}$. As $C_1 \subset C_2$ in $S^2\backslash\Delta_n$, we have that $(u,x,f(x)-f(u)) \in \mathrm{gph}\, C_2|_{S^2\backslash\Delta_n}$ and so $f(x) \in f(u) + C_2(u,x)$. Thus, $x \notin \mathrm{QE}(f,S,C_2)$ and part (i) is proved.

(ii) Clearly, $\mathrm{gph}\, C|_{S^2\backslash\Delta_n} = \bigcup_{i\in I} \mathrm{gph}\, C_i|_{S^2\backslash\Delta_n}$. Therefore, $C_i \subset C$ in $S^2\backslash\Delta_n$ for all $i \in I$ and part (i) implies that $\mathrm{QE}(f,S,C) \subset \bigcap_{i\in I} \mathrm{QE}(f,S,C_i)$. Conversely, we claim that $\mathrm{QE}(f,S,C)^c \subset (\bigcap_{i\in I} \mathrm{QE}(f,S,C_i))^c$. Indeed, if $x \in S$ and $x \notin \mathrm{QE}(f,S,C)$, then there exists $u \in S\backslash\{x\}$ such that $(u,x,f(x)-f(u)) \in \mathrm{gph}\, C|_{S^2\backslash\Delta_n}$. Thus, there exists $i_0 \in I$ such $(u,x,f(x)-f(u)) \in \mathrm{gph}\, C_{i_0}|_{S^2\backslash\Delta_n}$ and so $x \notin \mathrm{QE}(f,S,C_{i_0})$, i.e., $x \in (\bigcap_{i\in I} \mathrm{QE}(f,S,C_i))^c$ and the proof of part (ii) is completed.

(iii) Since $C \subset C_i$ in $S^2\backslash\Delta_n$ for all $i \in I$, part (i) implies that $\bigcup_{i\in I} \mathrm{QE}(f,S,C_i) \subset \mathrm{QE}(f,S,C)$ and the result is stated.

(iv) Consider a sequence $(x_m) \subset \mathrm{QE}(f,S,C)$ such that $x_m \to x_0$. We have that $x_0 \in S$ since the feasible set S is assumed to be closed. Fix a point $u \in S\backslash\{x_0\}$. Clearly, $x_m \neq u$ eventually. As x_m is a quasi efficient solution of problem (P) with respect to C we have that $(x_m, f(x_m) - f(u)) \in \mathrm{gph}\, C|_S^c(u,\cdot)$ eventually. By the continuity of f we see that $f(x_m) \to f(x_0)$. In addition, the set-valued mapping $C^c|_S(u,\cdot)$ is outer semicontinuous at x_0. Therefore,

$$f(x_0) - f(u) \in \operatorname*{Lim\,sup}_{x\to x_0} C^c|_S(u,x) \subset C^c|_S(u,x_0),$$

i.e., $f(x_0) \notin f(u) + C(u,x_0)$. Since u was arbitrarily chosen in $S\backslash\{x_0\}$ we deduce that $x_0 \in \mathrm{QE}(f,S,C)$ and the proof of part (iv) finishes.

(v) Clearly, $\mathrm{gph}\, C^c|_S(u,\cdot) = (\mathrm{gph}\, C(u,\cdot))^c \cap (S \times \mathbb{R}^p)$, for all $u \in \mathbb{R}^n$. As the graph of the set-valued mapping $C(u,\cdot)$ is assumed to be an open set, for all $u \in S$, and S is closed we deduce that $\mathrm{gph}\, C^c|_S(u,\cdot)$ is closed, and the result follows by applying part (iv). □

Remark 3.2. In order to apply part (v) of Theorem 3.1, let us recall that the set-valued mapping $\mathrm{int}\, C(u,\cdot)$ is open whenever $C(u,\cdot)$ is inner semicontinuous everywhere, solid-valued and convex-valued. Consider, for instance, a continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}_+$ and two nonempty sets $G, D \subset \mathbb{R}^p$ such that D is a solid convex cone and G is D-convex. Define the set-valued mapping $A^{\varphi}_{G,D} : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ given by $A^{\varphi}_{G,D}(u,x) := \varphi(u-x)G + D$. It is easy to check that $A^{\varphi}_{G,D}(u,\cdot)$ is inner semicontinuous everywhere, solid-valued and convex-valued. Therefore, the set-valued mapping $\mathrm{int}\, A^{\varphi}_{G,D}(u,\cdot)$ is open and so $\mathrm{QE}(f,S,\mathrm{int}\, A^{\varphi}_{G,D})$ is closed provided that f is continuous and S is closed. Notice that

$$\operatorname{int} A^{\varphi}_{G,D}(u,x) = \operatorname{int}(A^{\varphi}_{G,D}(u,x)) = \operatorname{int}(\varphi(u-x)G+D) = \varphi(u-x)G+\operatorname{int} D, \quad \forall x,u \in \mathbb{R}^n.$$

Parts (i) and (iii) of Theorem 3.1 suggest a way to derive inner and outer approximations to the set $\mathrm{QE}(f,S,C)$. Specifically, the following sandwich type rule holds true whenever $C_1 \subset C \subset C_2$ in $S^2 \backslash \Delta_n$:

$$\mathrm{QE}(f,S,C_2) \subset \mathrm{QE}(f,S,C) \subset \mathrm{QE}(f,S,C_1). \tag{3.3}$$

Therefore, if C is solid-valued in $S^2\backslash\Delta_n$ (i.e., the set $C(x,u)$ is solid, for all $x,u \in S^2\backslash\Delta_n$), then a good outer approximation to $\mathrm{QE}(f,S,C)$ could be obtained by considering the set-valued mapping $C_1(x,u) = \operatorname{int} C(x,u)$, for all $x,u \in \mathbb{R}^n$ (notice that $C_1(x,u)$ and $C(x,u)$ are very close to each other as $C(x,u)\backslash C_1(x,u) \subset \operatorname{bd} C(x,u)$). Concerning the inner approximation to $\mathrm{QE}(f,S,C)$, one could consider any collection $C_i : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}$ such that $C \subset C_i$ in $S^2\backslash\Delta_n$, for all i. In this case, the set-valued mapping given by $C_2(x,u) = \bigcap_i C_i(x,u)$, for all $x,u \in \mathbb{R}^n$, satisfies

$$\bigcup_i \mathrm{QE}(f,S,C_i) \subset \mathrm{QE}(f,S,C_2) \subset \mathrm{QE}(f,S,C).$$

Clearly, the closer $\bigcap_i C_i$ is to C, the better $\bigcup_i \mathrm{QE}(f,S,C_i)$ is as inner approximation to $\mathrm{QE}(f,S,C)$. In addition, one could deal with set-valued mappings C_i such that the corresponding quasi efficient set $\mathrm{QE}(f,S,C_i)$ satisfies good properties; for instance, it is a closed set. All the above remarks motivate the additional concepts of quasi efficient solution of problem (P) introduced in Definitions 3.2 and 4.2.

Definition 3.2. Consider $x_0 \in S$ and let $C : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a solid-valued mapping in $(S\backslash\{x_0\}) \times \{x_0\}$. The point x_0 is said to be a weak quasi efficient solution of problem (P), and it is denoted by $x_0 \in \mathrm{WQE}(f,S,C)$, if $x_0 \in \mathrm{QE}(f,S,\operatorname{int} C)$, i.e., if

$$f(x_0) \notin f(x) + \operatorname{int} C(x,x_0), \quad \forall x \in S\backslash\{x_0\}. \tag{3.4}$$

Remark 3.3. (i) Obviously, $\mathrm{WQE}(f,S,C_1) = \mathrm{WQE}(f,S,C_2)$ whenever $\operatorname{int} C_1 = \operatorname{int} C_2$ in $S^2\backslash\Delta_n$. Namely, if we have $\operatorname{int} C_1(\cdot,x_0) = \operatorname{int} C_2(\cdot,x_0)$ in $S\backslash\{x_0\}$, then $x_0 \in \mathrm{WQE}(f,S,C_1)$ if and only if $x_0 \in \mathrm{WQE}(f,S,C_2)$.

(ii) In [26, Definition 2(ii)], the notion of (G,h)-quasi weak efficient solution is defined. Specifically, for a solid set $G \subset \mathbb{R}^p$ (denoted by C in [26, Definition 2]) and a function $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ such that $h(x,z) > 0$ whenever $x \neq z$, a point $x_0 \in S$ is said to be a (G,h)-quasi weak efficient solution of problem (P) if there is not $x \in S\backslash\{x_0\}$ such that $f(x_0) \in f(x) + h(x,x_0)\operatorname{int} G$. Clearly, (G,h)-quasi weak efficient solutions are weak quasi efficient solutions with respect to the set-valued mapping $C(u,x) = h(u,x)G$, for all $u,x \in \mathbb{R}^n$, since $\operatorname{int} C = h \operatorname{int} G$ in $S^2\backslash\Delta_n$.

(iii) Definition 3.2 also encompasses the notion of weak generalized ε-quasi-minimizer of problem (P_D) due to Gutiérrez et al. [23, Definition 3.1(b)], which corresponds to the set-valued mapping $C = A^{\varepsilon\varphi}_{G,D}$ (see Remark 3.2).

The following result shows basic properties concerning quasi efficient solutions with respect to mapping $\operatorname{shw} C$, which will be useful to derive relationships between efficient solutions and quasi efficient solutions of problem (P_D).

Proposition 3.1. *We have that*

(i) $\mathrm{QE}(f,S,\mathrm{shw}\,C)=\bigcap_{\alpha>1}\mathrm{QE}(f,S,\alpha C)$ *and if* C *is coradiant-valued in the set* $S^2\backslash\Delta_n$*, then* $\mathrm{QE}(f,S,C)\subset\mathrm{QE}(f,S,\mathrm{shw}\,C)$*; if* C *is open-valued in* $S^2\backslash\Delta_n$*, then* $\mathrm{QE}(f,S,\mathrm{shw}\,C)=\bigcap_{\alpha\geq 1}\mathrm{QE}(f,S,\alpha C)$*; if, in addition,* C *is coradiant-valued in* $S^2\backslash\Delta_n$*, then* $\mathrm{QE}(f,S,C)=\bigcap_{\alpha>1}\mathrm{QE}(f,S,\alpha C)$*.*

(ii) $\bigcap_{\alpha>1}\mathrm{WQE}(f,S,\alpha C)\subset\mathrm{WQE}(f,S,C)$ *and the equality is true provided that* $\mathrm{int}\,C$ *is coradiant-valued in* $S^2\backslash\Delta_n$*.*

Proof. (i) The first equality and the inclusion are obvious consequences of the definition of the set-valued mapping $\mathrm{shw}\,C$ and parts (i) and (ii) of Theorem 3.1. For the second equality, notice from the openness assumption that $C(u,x)\subset\mathrm{shw}\,C(u,x)$, for all $(u,x)\in S^2\backslash\Delta_n$. Thus, $\mathrm{shw}\,C=\bigcup_{\alpha\geq 1}\alpha C$ in $S^2\backslash\Delta_n$ and the equality follows again by Theorem 3.1(ii). Finally, the last equality also is a direct consequence of Theorem 3.1(ii) as $C=\bigcup_{\alpha>1}\alpha C$ in $S^2\backslash\Delta_n$.

(ii) Clearly, the open-valued mapping $\mathrm{int}\,C$ satisfies that $\mathrm{int}\,C\subset\mathrm{shw}\,\mathrm{int}\,C$ in $S^2\backslash\Delta_n$. Thus, the inclusion is a result of parts (i) and (ii) of Theorem 3.1, and the equality follows by the last claim of part (i). □

Let us observe that parts (a) and (b) of [26, Proposition 3] are particular cases of Proposition 3.1. Specifically, they can be deduced by taking $C=hG$ in Proposition 3.1 and noticing that $\mathrm{int}\,C=h\,\mathrm{int}\,G$ in $S^2\backslash\Delta_n$ (see Remarks 3.1 and 3.3).

4. Relationships with efficient solutions

Theorem 3.1 allows us to derive basic assertions concerning the approximation to efficient and weak efficient solutions of problem (P) through quasi efficient and weak quasi efficient solutions. Next two results illustrate this claim. Recall that a point $x_0\in S$ is said to be an efficient solution (resp. weak efficient solution) of problem (P_D), denoted by $x_0\in\mathrm{E}(f,S,D)$ (resp. $x_0\in\mathrm{WE}(f,S,D)$), if there is not a feasible point $u\in S$ such that $f(x_0)\in f(u)+D\backslash\{0\}$ (resp. $f(x_0)\in f(u)+\mathrm{int}\,D$, see part (iii)(a) of Remark 2.1). In dealing with weak efficient solutions we assume that D is solid.

Proposition 4.1. *Consider a nonempty set* $G\subset\mathbb{R}^p\backslash\{0\}$ *and a sequence* $\varphi_m:\mathbb{R}^n\to\mathbb{R}_+$ *such that* $\varphi_m(x)>0$ *for all* $x\neq 0$ *and pointwisely converges to zero in* $\mathbb{R}^n\backslash\{0\}$*. Define the set-valued mapping* $A_G^{\varphi_m}:\mathbb{R}^n\times\mathbb{R}^n\rightrightarrows\mathbb{R}^p$ *given by* $A_G^{\varphi_m}(u,x):=\varphi_m(u-x)G$*. If* G *is coradiant (resp.* G *is solid and* $\mathrm{int}\,G$ *is coradiant), then*

$$\bigcap_m \mathrm{QE}(f,S,A_G^{\varphi_m})=\mathrm{E}(f,S,\mathrm{cone}\,G) \tag{4.1}$$

$$(\textit{resp.}\ \bigcap_m \mathrm{WQE}(f,S,A_G^{\varphi_m})=\mathrm{E}(f,S,\mathrm{cone}\,\mathrm{int}\,G)). \tag{4.2}$$

Proof. Assume that G is coradiant. In this case we claim that

$$\bigcup_m \varphi_m(x)G=(\mathrm{cone}\,G)\backslash\{0\},\quad \forall x\in\mathbb{R}^n\backslash\{0\}. \tag{4.3}$$

As a result we have that $\bigcup_m A_G^{\varphi_m}(u,x)=(\mathrm{cone}\,G)\backslash\{0\}$ for all $(u,x)\in S^2\backslash\Delta_n$ and assertion (4.1) follows by Theorem 3.1(ii). In order to check claim (4.3), notice that $\varphi_m(x)G\subset(\mathrm{cone}\,G)\backslash\{0\}$ for all $x\in\mathbb{R}^n\backslash\{0\}$ and $m\in\mathbb{N}$ since $\varphi(x)>0$ and $0\notin G$. Conversely, for each $\alpha>0$ and

$x \in \mathbb{R}^m \backslash \{0\}$ there exists $m \in \mathbb{N}$ such that $\varphi_m(x) < \alpha$ as the sequence φ_m pointwisely converges to zero in $\mathbb{R}^n \backslash \{0\}$. Since G is coradiant we see that $\alpha G \subset \varphi_m(x)G$. Therefore,

$$(\operatorname{cone} G)\backslash\{0\} = \bigcup_{\alpha>0} \alpha G \subset \bigcup_m \varphi_m(x)G$$

and assertion (4.3) is proved.

Equality (4.2) is an obvious consequence of (4.1) by replacing G with $\operatorname{int} G$. □

Remark 4.1. (i) By Remark 2.1(ii) we see that Proposition 4.1 reduces to part (a) of [26, Theorem 2]. In addition, statement (4.2) corrects part (b) of [26, Theorem 2], which does not work. For instance, consider problem (P_D) with the next data: $n = p = 2$, $f(x_1,x_2) = (x_1,0)$, $S = \mathbb{R}^2$, $D = \{(y_1,y_2) \in \mathbb{R}^2 : y_1 \geq |y_2|\}$, $h_n(x_1,x_2) = 1/n$ and

$$G = \{(y_1,y_2) \in \mathbb{R}^2 : y_1 \geq 1, y_1 \geq |y_2| \geq 1/y_1\} \cup \operatorname{shw}\{(1,0)\}.$$

Clearly, set G is solid and coradiant. In addition, $G \subset D\backslash\{0\}$, $\operatorname{cone} G = D$ and the sequence (h_n) pointwisely converges to zero. Thus, all assumptions of [26, Theorem 2(b)] are satisfied. However, it is easy to check $\bigcap_n \mathrm{WQE}(f,S,h_nG) = \mathbb{R}^2$ and $\mathrm{WE}(f,S,D) = \emptyset$.

(ii) As it has already been noticed in parts (b) and (c) of Remark 2.1(iii), some additional conditions linking set G with the ordering cone D have to be considered in order to sets $\mathrm{QE}(f,S,A_G^{\varphi_m})$ and $\mathrm{WQE}(f,S,A_G^{\varphi_m})$ approximate the sets $\mathrm{E}(f,S,D)$ and $\mathrm{WE}(f,S,D)$ of efficient and weak efficient solutions of problem (P_D). This assertion is clear from statements (4.1) and (4.2), where condition $G \subset D$ should be required to approximate efficient and weak efficient solutions of problem (P_D) via quasi efficient solutions in $\mathrm{QE}(f,S,A_G^{\varphi_m})$ and $\mathrm{WQE}(f,S,A_G^{\varphi_m})$.

Corollary 4.1. *Let $G \subset \mathbb{R}^p\backslash\{0\}$ be a nonempty open and coradiant set. Suppose $h_m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ satisfies $h_m > 0$ in $S^2\backslash\Delta_n$, for all $m \in \mathbb{N}$. Consider $x_0 \in S$ and a sequence $(x_m) \subset S$ such that $h_m(\cdot,x_m)$ pointwisely converges to zero in S, $f(x_m) \to f(x_0)$ and $x_m \in \mathrm{QE}(f,S,h_mG)$. Then $x_0 \in \mathrm{E}(f,S,\operatorname{cone} G)$.*

Proof. In order to apply Proposition 4.1, let us define $\varphi_m : \mathbb{R}^n \to \mathbb{R}_+$, $\varphi_m(x) := 1/m$, for all $x \in \mathbb{R}^n$. It is obvious that φ_m pointwisely converges to zero in $\mathbb{R}^n$.

We claim that $x_0 \in \mathrm{QE}(f,S,A_G^{\varphi_m})$, for all $m \in \mathbb{N}$. Otherwise, one can find $m_0 \in \mathbb{N}$ and $u \in S\backslash\{x_0\}$ such that $f(x_0) \in f(u) + (1/m_0)G$. As G is open and $f(x_m) \to f(x_0)$ there exists $m_1 \in \mathbb{N}$ such that $f(x_m) - f(u) \in (1/m_0)G$, for all $m \geq m_1$. As $0 \notin G$ it follows that $x_m \neq u$ for all $m \geq m_1$ and since $h_m(u,x_m) \to 0$ there exists $\mathbb{N} \ni m_2 \geq m_1$ such that $0 < h_{m_2}(u,x_{m_2}) < 1/m_0$. Clearly, $(1/m_0)G \subset h_{m_2}(u,x_{m_2})G$ since G is coradiant, and so $f(x_{m_2}) - f(u) \in h_{m_2}(u,x_{m_2})G$, which is a contradiction as $x_{m_2} \in \mathrm{QE}(f,S,h_{m_2}G)$.

Finally, by Proposition 4.1 we see that

$$x_0 \in \bigcap_m \mathrm{QE}(f,S,A_G^{\varphi_m}) = \mathrm{E}(f,S,\operatorname{cone} G)$$

and the result is proved. □

Remark 4.2. Corollary 4.1 reduces to [26, Theorem 3(b)] by considering the set $G = q + \operatorname{int} D$, where $q \in D\backslash\{0\}$ and D is assumed to be solid. Notice that G is nonempty, open, coradiant, $0 \notin G$ and $\mathrm{E}(f,S,\operatorname{cone} G) = \mathrm{WE}(f,S,D)$ as $\operatorname{cone} G = \operatorname{int} D \cup \{0\}$.

Next we provide more general results concerning the relationships of quasi efficient solutions with efficient solutions of problem (P_D). The notion of cone lower semicontinuity due to Combari et al. [4] will be required. Some relationships with other cone lower semicontinuity concepts can be seen in [23, Remark 3.2 and Proposition 3.3] and [21, Theorem 3.1].

Definition 4.1. [4, Definition 3.1 and Proposition 3.3] The function $f : S \subset \mathbb{R}^n \to \mathbb{R}^p$ is said to be cone lower semicontinuos at $x_0 \in S$ with respect to a convex cone $K \subset \mathbb{R}^p$ (K-lsc at x_0 in short form) if for each sequence $(x_n) \subset S$, $x_n \to x_0$, there exists a sequence $(b_n) \subset \mathbb{R}^p$ such that $b_n \to f(x_0)$ and $b_n \leq_K f(x_n)$. If f is K-lsc at x, for all $x \in S$, then f is said to be K-lsc in S.

Theorem 4.1. *Consider problem (P_D) and a sequence $C_m : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ of set-valued mappings. Assume that f is D-lsc in S. We have that:*

(i) If $C_m(u,x)$ is free-disposal with respect to D, for all $m \in \mathbb{N}$ and for all $(u,x) \in S^2 \backslash \Delta_n$, S is closed and $C : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ satisfies $\operatorname{g-}\limsup_{m\to\infty} C^c_m(u,\cdot) \subset C^c(u,\cdot)$ in $S\backslash\{u\}$, for all $u \in S$, then

$$\limsup_{m\to\infty} \mathrm{QE}(f,S,C_m) \subset \mathrm{QE}(f,S,C).$$

(ii) If $C_m(u,x)$ is convex and free-disposal with respect to D, for all $m \in \mathbb{N}$ and for all $(u,x) \in S^2\backslash\Delta_n$, S is closed and $C : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ satisfies $C(u,\cdot)$ is open-valued in $S\backslash\{u\}$ and $C(u,x) \subset \liminf_{m\to\infty} C_m(u,x_m)$ for all sequence $(x_m) \subset S$, $x_m \to x$ and $x \in S\backslash\{u\}$, for all $u \in S$, then

$$\limsup_{m\to\infty} \mathrm{QE}(f,S,C_m) \subset \mathrm{QE}(f,S,C).$$

(iii) If S is compact, $x_0 \in S$, $C_m(u,x_0)$ is free-disposal with respect to D for all $m \in \mathbb{N}$ whenever $u \in S\backslash\{x_0\}$ and $C : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ satisfies $\operatorname{g-}\limsup_{m\to\infty} C_m(\cdot,x_0) \subset C(\cdot,x_0)$ in $S\backslash\{x_0\}$ and $0 \notin \operatorname{g-}\limsup_{m\to\infty} (C_m(\cdot,x_0)+D)(x_0)$, then

$$x_0 \in \mathrm{QE}(f,S,C) \Rightarrow x_0 \in \mathrm{QE}(f,S,C_m) \text{ eventually.}$$

(iv) Consider a set-valued mapping $C_0 : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ such that for each $(u,\bar{x}) \in S^2\backslash\Delta_n$, $C_0(u,\bar{x})$ is open and there exists $r > 0$ and $k \in \mathbb{N}$ satisfying

$$C_0(u,\bar{x}) \subset \bigcap_{x\in\bar{x}+r\mathbb{B}_n} C_m(u,x), \quad \forall m \geq k. \tag{4.4}$$

Assume that $C_m(u,x)$ is free-disposal with respect to D, for all $m \in \mathbb{N}$ and for all $(u,x) \in S^2\backslash\Delta_n$ and S is closed. Suppose $C : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ satisfies $C = \bigcup_m C_m$ in $S^2\backslash\Delta_n$. Then,

$$\begin{aligned}\mathrm{QE}(f,S,C) = \bigcap_{m\in\mathbb{N}} \mathrm{QE}(f,S,C_m) &\subset \liminf_{m\to\infty} \mathrm{QE}(f,S,C_m)\\ &\subset \limsup_{m\to\infty} \mathrm{QE}(f,S,C_m) \subset \mathrm{QE}(f,S,C_0).\end{aligned} \tag{4.5}$$

Proof. (i) Consider a point $x \in \limsup_{m\to\infty} \mathrm{QE}(f,S,C_m)$. Then there exists a sequence $(x_{m_k}) \subset S$ such that $x_{m_k} \in \mathrm{QE}(f,S,C_{m_k})$ for all m_k and $x_{m_k} \to x$. We have that $x \in S$ since S is closed. In addition, for each $u \in S\backslash\{x\}$, we may assume that $x_{m_k} \neq u$, for all m_k. As $x_{m_k} \in \mathrm{QE}(f,S,C_{m_k})$

we have that $f(x_{m_k}) \notin f(u) + C_{m_k}(u, x_{m_k})$, for each $u \in S \backslash \{x\}$. As f is D-lsc at x, there exists a sequence $(b_{m_k}) \subset \mathbb{R}^p$ such that $b_{m_k} \to f(x)$ and $b_{m_k} \leq_D f(x_{m_k})$, for all m_k. We claim that

$$b_{m_k} \notin f(u) + C_{m_k}(u, x_{m_k}), \quad \forall m_k. \tag{4.6}$$

Indeed, if there exists m_k such that (4.6) is false, then

$$f(x_{m_k}) = b_{m_k} + (f(x_{m_k}) - b_{m_k}) \in b_{m_k} + D \subset f(u) + C_{m_k}(u, x_{m_k}) + D = f(u) + C_{m_k}(u, x_{m_k}),$$

that is a contradiction. Therefore, $\operatorname{gph} C^c_{m_k}(u, \cdot) \ni (x_{m_k}, b_{m_k} - f(u)) \to (x, f(x) - f(u))$, and we deduce taking into account the hypothesis that

$$f(x) - f(u) \in \operatorname*{g\text{-}lim\,sup}_{m\to\infty} [C^c_m(u, \cdot)](x) \subset C^c(u, x).$$

As u is an arbitrary feasible point different from x, we conclude that $x \in \mathrm{QE}(f, S, C)$ and the proof is completed.

(ii) The assumptions on the set-valued mappings C_m and C imply that $\operatorname*{g\text{-}lim\,sup}_{m\to\infty} C^c_m(u, \cdot) \subset C^c(u, \cdot)$ in $S \backslash \{u\}$, for all $u \in S$, and the result follows by part (i). Indeed, consider $(u, x) \in S^2 \backslash \Delta_n$ and $\operatorname{gph} C^c_{m_k}(u, \cdot) \ni (x_{m_k}, y_{m_k}) \to (x, y)$ and suppose by contradiction that $y \in C(u, x)$. As $y_{m_k} \notin C_{m_k}(u, x_{m_k})$ and $C_{m_k}(u, x_{m_k})$ is convex, by the separation theorem there exists $\lambda_{m_k} \in \mathbb{R}^p$ such that $\|\lambda_{m_k}\| = 1$ and $\langle \lambda_{m_k}, z_{m_k} \rangle \leq \langle \lambda_{m_k}, y_{m_k} \rangle$, for all $z_{m_k} \in C_{m_k}(u, x_{m_k})$. We can suppose $\lambda_{m_k} \to \lambda \in \mathbb{R}^p \backslash \{0\}$. Therefore,

$$\langle \lambda, z \rangle \leq \langle \lambda, y \rangle, \quad \forall z \in \liminf_{k\to\infty} C_{m_k}(u, x_{m_k}). \tag{4.7}$$

As $C(u, x)$ is open and $y \in C(u, x)$, there exists $\alpha > 0$ such that $y + \alpha\lambda \in C(u, x)$. By the assumptions we see that $C(u, x) \subset \liminf_{k\to\infty} C_{m_k}(u, x_{m_k})$ and statement (4.7) implies

$$\langle \lambda, y \rangle + \alpha \|\lambda\|^2 = \langle \lambda, y + \alpha\lambda \rangle \leq \langle \lambda, y \rangle,$$

that is a contradiction.

(iii) Consider $x_0 \in \mathrm{QE}(f, S, C)$ and suppose on the contrary that there exists a subsequence (C_{m_k}) such that $x_0 \notin \mathrm{QE}(f, S, C_{m_k})$, for all m_k. Therefore, there exists $u_{m_k} \in S \backslash \{x_0\}$ such that $f(x_0) \in f(u_{m_k}) + C_{m_k}(u_{m_k}, x_0)$. Since S is assumed to be compact we can suppose without loss of generality that $u_{m_k} \to u \in S$. In addition, as f is D-lsc at u there exists a sequence $(b_{m_k}) \subset \mathbb{R}^p$ satisfying $b_{m_k} \leq_D f(u_{m_k})$ and $b_{m_k} \to f(u)$.

If $u \neq x_0$, then we can suppose $u_{m_k} \neq x_0$ and so

$$f(x_0) \in f(u_{m_k}) + C_{m_k}(u_{m_k}, x_0) \subset b_{m_k} + C_{m_k}(u_{m_k}, x_0) + D = b_{m_k} + C_{m_k}(u_{m_k}, x_0).$$

Therefore, $(u_{m_k}, f(x_0) - b_{m_k}) \in \operatorname{gph} C_{m_k}(\cdot, x_0)$ and clearly,

$$f(x_0) - f(u) \in \operatorname*{g\text{-}lim\,sup}_{m\to\infty} C_m(\cdot, x_0)(u).$$

It follows that $f(x_0) \in f(u) + C(u, x_0)$, which is a contradiction since $x_0 \in \mathrm{QE}(f, S, C)$.

Therefore $u = x_0$, and by the same reasonings as above we deduce that $0 \in \operatorname*{g\text{-}lim\,sup}_{m\to\infty} (C_m(\cdot, x_0) + D)(x_0)$, that is also a contradiction.

(iv) The equality in (4.5) follows by Theorem 3.1(ii) and the first and second inclusions are obvious. Finally, we claim that $\operatorname*{g\text{-}lim\,sup}_{m\to\infty} C^c_m(u, \cdot) \subset C^c_0(u, \cdot)$ in $S \backslash \{u\}$, for all $u \in S$ and the last inclusion in (4.5) is true by part (i). Indeed, consider $(u, \bar{x}) \in S^2 \backslash \Delta_n$ and $y \in \operatorname*{g\text{-}lim\,sup}_{m\to\infty} C^c_m(u, \cdot)(\bar{x})$.

Then, there exists a sequence $((x_{m_k},y_{m_k}))_{m_k} \subset \mathbb{R}^n \times \mathbb{R}^p$ such that $\operatorname{gph} C^c_{m_k}(u,\cdot) \ni (x_{m_k},y_{m_k}) \to (\bar{x},y)$. Suppose by contradiction that $y \in C_0(u,\bar{x})$. As the set $C_0(u,\bar{x})$ is open, we have that $(x_{m_k},y_{m_k}) \in (\bar{x}+r\mathbb{B}_n)\times C_0(u,\bar{x})$ eventually, and by condition (4.4) we see that there exists $y_{m_k} \in C_{m_k}(u,x_{m_k})$, that is a contradiction. □

Remark 4.3. (i) Concerning assumption (4.4), which supports the last inclusion in (4.5), notice it implies that $C_0(u,x) \subset \liminf_{m\to\infty} C_m(u,x_m)$ for all $(u,x) \in S^2\backslash\Delta_n$ and for all sequence $(x_m) \subset S$, $x_m \to x$. This condition is supposed in part (ii) of Theorem 4.1, along with the convexity of the sets $C_m(u,x)$, for all $m \in \mathbb{N}$ and for all $(u,x) \in S^2\backslash\Delta_n$. However, this convexity assumption is not required in part (iv) of Theorem 4.1.

(ii) Assumption $C(u,x) \subset \liminf_{m\to\infty} C_m(u,x_m)$ for all $(u,x) \in S^2\backslash\Delta_n$ and for all sequence $(x_m) \subset S$, $x_m \to x$, concerns the "inner part" of the so-called continuous limit of the set-valued mappings $C_m(u,\cdot):\mathbb{R}^n \to \mathbb{R}^p$ (see [33, Definition 5.41]).

Theorem 4.2. *Consider problem* (P_D) *and a set* $G \subset \mathbb{R}^p$ *such that* $G+D$ *is coradiant. Assume that* f *is* D*-lsc in* S*. We have that:*

(i) Assume a nonempty set $H \subset G+D$, $r \in \mathbb{R}_+$ *and* $h_m:\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}_+$ *satisfying* $h_m(u,x) > 0$ *for all* $m \in \mathbb{N}$ *and* $\liminf_{k\to\infty} h_{m_k}(u,x_{m_k}) \le r$, *for all* $(u,x) \in S^2\backslash\Delta_n$ *and for all sequence* (x_{m_k}) *in* S, $x_{m_k} \to x$. *Let* $C:\mathbb{R}^n\times\mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *be* $C(x,u) = \bigcup_{t>r} t(\operatorname{cl}(H^c))^c$, *for all* $(u,x) \in S^2\backslash\Delta_n$. *If* S *is closed we have that*

$$\limsup_{m\to\infty} \operatorname{QE}(f,S,h_mG+D) \subset \operatorname{QE}(f,S,C).$$

(ii) Let $x_0 \in S$, $r \in \mathbb{R}_+$ *and* $h_m:\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}_+$ *satisfy* $\liminf_{m\to\infty} h_m(u_m,x_0) > r$ *for all convergent sequence* (u_m) *in* S*. Suppose* $0 \notin \operatorname{cl}(G+D)$ *and define* $C:\mathbb{R}^n\times\mathbb{R}^n \rightrightarrows \mathbb{R}^p$, $C(x,u) = \bigcup_{\alpha>r} \alpha\operatorname{cl}(G+D)$. *If* S *is compact, we have that*

$$x_0 \in \operatorname{QE}(f,S,C) \Rightarrow x_0 \in \operatorname{QE}(f,S,h_mG+D) \text{ eventually}. \tag{4.8}$$

(iii) Consider $x_0 \in S$, $h_m:\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}_+$ *and a sequence* $(r_m) \subset \mathbb{R}_+\backslash\{0\}$, $r_m \to r$ *and* $r_m > r$ *for all* $m \in \mathbb{N}$*. Suppose* $0 \notin \operatorname{cl}(G+D)$ *and define* $C:\mathbb{R}^n\times\mathbb{R}^n \rightrightarrows \mathbb{R}^p$, $C(x,u) = \bigcup_{\alpha>r} \alpha\operatorname{cl}(G+D)$. *If* S *is compact, then there exists a subsequence* (h_{m_k}) *such that*

$$x_0 \in \operatorname{QE}(f,S,C) \Rightarrow x_0 \in \bigcap_k \operatorname{QE}(f,S,(h_{m_k}+r_k)G+D). \tag{4.9}$$

Proof. (i) We state this part as a consequence of Theorem 4.1(i). Then, let us check that its assumptions are fulfilled.

Clearly, the values of the mapping $h_mG+D:\mathbb{R}^n\times\mathbb{R}^n \rightrightarrows \mathbb{R}^p$ are free-disposal with respect to D, for all $m \in \mathbb{N}$. Consider $(u,x) \in S^2\backslash\Delta_n$ and $y \in \operatorname{g-}\limsup_{m\to\infty}(h_m(u,\cdot)G+D)^c(x)$. There exists a sequence $(x_{m_k},y_{m_k}) \in \mathbb{R}^n\times\mathbb{R}^p$ such that $x_{m_k} \to x$ and $(h_{m_k}(u,x_{m_k})G+D)^c \ni y_{m_k} \to y$. As we have $\liminf_{k\to\infty} h_{m_k}(u,x_{m_k}) \le r$, for each $\alpha > 0$ we can suppose by passing a subsequence if necessary that $0 < h_{m_k}(u,x_{m_k}) < r+\alpha$ for all m_k, and so

$$(r+\alpha)H \subset (r+\alpha)(G+D) \subset h_{m_k}(u,x_{m_k})(G+D) = h_{m_k}(u,x_{m_k})G+D,$$

since $G+D$ is coradiant. Therefore, $y_{m_k} \in ((r+\alpha)H)^c$ for all m_k and it follows that $y \in (r+\alpha)\,\mathrm{cl}\,(H^c)$. As $\alpha > 0$ is arbitrary we deduce

$$y \in \bigcap_{t>r} t\,\mathrm{cl}\,(H^c) = \left(\bigcup_{t>r} t(\mathrm{cl}\,(H^c))^c\right)^c = C(u,x)^c.$$

We have that $\operatorname{g-}\limsup_{m\to\infty}(h_m(u,\cdot)G+D)^c(x) \subset C(u,x)^c$ and the result follows by applying Theorem 4.1(i).

(ii) This result is a particular case of Theorem 4.1(iii). Indeed, let $(u_m) \subset S$ be a convergent sequence to u. By the assumptions we see that there exists $\alpha > r$ such that $h_m(u_m,x_0) > \alpha$ eventually. In addition, as the set $G+D$ is coradiant it follows that $\alpha_2 G+D \subset \alpha_1 G+D$ whenever $0<\alpha_1<\alpha_2$. Thus,

$$\limsup_{m\to\infty}(h_m(u_m,x_0)G+D) \subset \alpha\,\mathrm{cl}\,(G+D)$$

and so, for each $u \in S$, we have that

$$[\operatorname{g-}\limsup_{m\to\infty}(h_m(\cdot,x_0)G+D)](u) \subset \bigcup_{\alpha>r} \alpha\,\mathrm{cl}\,(G+D).$$

Therefore, the hypotheses of Theorem 4.1(iii) hold true and the proof finishes.

(iii) Consider an arbitrary element r_k of the sequence (r_m) and define the sequence $h'_m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$, $h'_m = h_m + r_k$, for all $m \in \mathbb{N}$. Clearly, $\liminf_{m\to\infty} h'_m(u_m,x_0) \geq r_k > r$ for all convergent sequence (u_m) in S, and by part (ii) we see that $x_0 \in \mathrm{QE}(f,S,h'_m G+D)$ eventually provided that $x_0 \in \mathrm{QE}(f,S,C)$. Therefore it is possible to choose a subsequence (h_{m_k}) such that $x_0 \in \mathrm{QE}(f,S,(h_{m_k}+r_k)G+D)$, for all $k \in \mathbb{N}$ and the proof finishes. □

Next we illustrate the previous theorem in a setting where the calculations are easy to obtain.

Corollary 4.2. *Consider problem (P_D) and a coradiant and free-disposal set $G \subset \mathbb{R}^p \backslash \{0\}$ with respect to the cone D. Assume that f is D-lsc in S. We have that:*

(i) Let $h_m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be satisfying $h_m(u,x) > 0$ for all $m \in \mathbb{N}$ and $\liminf_{k\to\infty} h_{m_k}(u,x_{m_k}) = 0$, for all $(u,x) \in S^2\backslash\Delta_n$ and for all sequence (x_{m_k}) in S, $x_{m_k} \to x$. If G is solid and S is closed, then

$$\mathrm{E}(f,S,\mathrm{cone}\,G) \subset \bigcap_{m\in\mathbb{N}} \mathrm{QE}(f,S,h_m G) \subset \liminf_{m\to\infty} \mathrm{QE}(f,S,h_m G) \subset \limsup_{m\to\infty} \mathrm{QE}(f,S,h_m G) \subset \mathrm{E}(f,S,\mathrm{cone}\,(\mathrm{int}\,G)), \tag{4.10}$$

$$\mathrm{E}(f,S,\mathrm{cone}\,(\mathrm{int}\,G)) = \bigcap_{m\in\mathbb{N}} \mathrm{QE}(f,S,h_m\,\mathrm{int}\,G) = \lim_{m\to\infty} \mathrm{QE}(f,S,h_m\,\mathrm{int}\,G). \tag{4.11}$$

In particular, the set $\mathrm{E}(f,S,\mathrm{cone}\,(\mathrm{int}\,G))$ is closed.

(ii) Let $x_0 \in S$, $h_m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be satisfying $\liminf_{m\to\infty} h_m(u_m,x_0) > 0$ for all convergent sequence (u_m) in S. If $0 \notin \mathrm{cl}\,G$ and S is compact, we have that

$$x_0 \in \mathrm{E}(f,S,\mathrm{cone}\,(\mathrm{cl}\,G)) \Rightarrow x_0 \in \mathrm{QE}(f,S,h_m G+D) \text{ eventually.} \tag{4.12}$$

(iii) Consider $x_0 \in S$, $h_m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ and a sequence $(\varepsilon_k) \subset \mathbb{R}_+ \backslash \{0\}$ such that $\varepsilon_k \to 0$. If $0 \notin \operatorname{cl} G$ and S is compact, then there exists a subsequence (h_{m_k}) fulfilling

$$x_0 \in \mathrm{E}(f,S,\operatorname{cone}(\operatorname{cl} G)) \Rightarrow x_0 \in \bigcap_k \mathrm{QE}(f,S,(h_{m_k}+\varepsilon_k)G). \tag{4.13}$$

Proof. (i) In order to apply Theorem 4.2(i) with $H := \operatorname{int} G$ and $r = 0$, notice that

$$C(x,u) = \bigcup_{t>0} t(\operatorname{cl}(H^c))^c = \operatorname{cone}([\operatorname{cl}(H^c)]^c)\backslash\{0\} = \operatorname{cone}(\operatorname{int} G)\backslash\{0\}, \quad \forall (u,x) \in S^2\backslash\Delta_n.$$

Thus, $\mathrm{QE}(f,S,C) = \mathrm{E}(f,S,\operatorname{cone}(\operatorname{int} G))$ and we have

$$\limsup_{m\to\infty} \mathrm{QE}(f,S,h_m G) = \limsup_{m\to\infty} \mathrm{QE}(f,S,h_m G + D) \subset \mathrm{E}(f,S,\operatorname{cone}(\operatorname{int} G)),$$

since the set-valued mappings $h_m G + D$ and $h_m G$ coincide in $S^2\backslash\Delta_n$. In order to state (4.10), notice that $\mathrm{E}(f,S,\operatorname{cone} G) \subset \mathrm{QE}(f,S,h_m G)$, for all $m \in \mathbb{N}$. Indeed, suppose by contradiction that $x \in \mathrm{E}(f,S,\operatorname{cone} G)$ and there exist $m \in \mathbb{N}$ and $u \in S\backslash\{x\}$ such that $f(x) \in f(u) + h_m(u,x)G$. As $h_m(u,x) > 0$ and $0 \notin G$ we have that

$$f(u) - f(x) \in -h_m(u,x)G \subset -(\operatorname{cone} G)\backslash\{0\}$$

that is a contradiction since $x \in \mathrm{E}(f,S,\operatorname{cone} G)$.

Clearly, (4.11) results by applying (4.10) to $\operatorname{int} G$ instead of G, which is also a solid coradiant and free-disposal set with respect to the cone D. As a simple result of assertion (4.11) we say that set $\mathrm{E}(f,S,\operatorname{cone}(\operatorname{int} G))$ is closed as it is the limit of a sequence of sets.

(ii)-(iii) By applying parts (ii) and (iii) of Theorem 4.2 we see that $\mathrm{QE}(f,S,C) = \mathrm{E}(f,S,\operatorname{cone}(\operatorname{cl} G))$ since

$$C(x,u) = \bigcup_{\alpha>0} \alpha \operatorname{cl} G = \operatorname{cone}(\operatorname{cl} G)\backslash\{0\}, \quad \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Then statement (4.8) (resp. (4.9))) reduces to (4.12) (resp. (4.13)) and parts (ii) and (iii) are proved. □

Remark 4.4. Part (i) of Corollary 4.2 reduces to [23, Theorem 3.1(b)] in the case $\varepsilon = 0$ by considering a nonincreasing sequence $(\varepsilon_n) \subset \mathbb{R}_+\backslash\{0\}$ such that $\varepsilon_n \to 0$ and by defining $h_n(x,u) = \varepsilon_n \varphi(x-u)$, for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^n$ (see also [23, Corollary 3.1]).

Let us illustrate Corollary 4.2 with a simple example.

Example 4.1. Consider problem (P_D) with the next data: $f : \mathbb{R}^n \to \mathbb{R}^{n+1}$, $f(x) = (x, \|x\|^2)$, for all $x \in \mathbb{R}^n$, where $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$, $S = \mathbb{R}^n_+$, $D = \mathbb{R}^{n+1}_+$, $q = (0,0,\ldots,0,1) \in \mathbb{R}^{n+1}$, $G = q + \mathbb{R}^{n+1}_+$ and $h_m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$, $h_m(u,x) = (1/m)(\|u-x\|^2+1)$, for all $u,x \in \mathbb{R}^n$, for all $m \in \mathbb{N}$, $m \geq 2$. Clearly, all assumptions of Corollary 4.2(i) are fulfilled. In addition, $\operatorname{cone}(\operatorname{int} G) = \operatorname{int} \mathbb{R}^{n+1}_+ \cup \{0\}$ and it is easy to check that

$$\mathrm{E}(f,S,\operatorname{cone}(\operatorname{int} G)) = \mathrm{WE}(f,S,\mathbb{R}^{n+1}_+) = \operatorname{bd} \mathbb{R}^n_+.$$

Let us obtain the set $\mathrm{QE}(f,S,h_m \operatorname{int} G)$. Consider $x \in \mathbb{R}^n_+$. We have that

$$\begin{aligned} f(x) \notin f(u) + h_m(u,x) \operatorname{int} G &\iff (x, \|x\|^2) \notin (u, \|u\|^2) + (1/m)(\|u-x\|^2+1)q + \operatorname{int} \mathbb{R}^{n+1}_+ \\ &\iff \begin{cases} x \notin u + \operatorname{int} \mathbb{R}^n_+ \text{ or} \\ \|x\|^2 \leq \|u\|^2 + (1/m)\|u-x\|^2 + 1/m, \end{cases} \quad \forall u \in \mathbb{R}^n_+\backslash\{x\}. \end{aligned} \tag{4.14}$$

Obviously, condition (4.14) holds true whenever $x \in \operatorname{bd}\mathbb{R}^n_+$. Therefore, if $x \in \operatorname{int}\mathbb{R}^n_+$, condition (4.14) is fulfilled if and only if

$$\|x\|^2 \le \|u\|^2 + (1/m)\|u-x\|^2 + 1/m, \quad \forall u \in x - \operatorname{int}\mathbb{R}^n_+, \tag{4.15}$$

that is equivalent to the next statement

$$\|x\|^2 \le \inf_{d\in\mathbb{R}^n_+} (\|x-d\|^2 + (1/m)\|d\|^2 + 1/m).$$

It is not hard to obtain that

$$\inf_{d\in\mathbb{R}^n_+} (\|x-d\|^2 + (1/m)\|d\|^2 + 1/m) = \frac{1}{m+1}\|x\|^2 + 1/m \tag{4.16}$$

and so, for each $x \in \operatorname{int}\mathbb{R}^n_+$, inequality (4.15) holds true if and only $\|x\| \le \frac{\sqrt{m+1}}{m}$. Thus, it follows that

$$\operatorname{QE}(f,S,h_m \operatorname{int} G) = \operatorname{bd}\mathbb{R}^n_+ \cup \frac{\sqrt{m+1}}{m}(\operatorname{cl}\mathbb{B}_n \cap \mathbb{R}^n_+).$$

Notice that

$$\begin{aligned}\bigcap_{m\in\mathbb{N}} \operatorname{QE}(f,S,h_m \operatorname{int} G) &= \lim_{m\to\infty} \operatorname{QE}(f,S,h_m \operatorname{int} G)\\ &= \lim_{m\to\infty}\left(\operatorname{bd}\mathbb{R}^n_+ \cup \frac{\sqrt{m+1}}{m}(\operatorname{cl}\mathbb{B}_n \cap \mathbb{R}^n_+)\right)\\ &= \operatorname{bd}\mathbb{R}^n_+ = \operatorname{WE}(f,S,\mathbb{R}^{n+1}_+).\end{aligned}$$

The previous results show that quasi efficient solutions exhibit a good behavior as inner and outer approximation of efficient solutions of problem (P_D) whenever the set-valued mapping C is free-disposal-valued with respect to the ordering cone D. This fact and the claims previous to Definition 3.2 motivate the next notion of proper quasi efficiency. For this aim, the family of dilating cones of D is required:

$$\mathscr{F}(D) := \{K \subsetneq \mathbb{R}^p : K \text{ is a convex cone s.t. } D\backslash\{0\} \subset \operatorname{int} K\}.$$

Definition 4.2. Consider problem (P_D) and let $C : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be free-disposal-valued with respect to D in $S^2\backslash\Delta_n$. A point $x_0 \in S$ is said to be a Henig proper quasi efficient solution of problem (P_D), and it is denoted by $x_0 \in \operatorname{HQE}(f,S,C,D)$, if there exists $K \in \mathscr{F}(D)$ such that $x_0 \in \operatorname{WQE}(f,S,C+K)$, i.e.,

$$\operatorname{HQE}(f,S,C,D) := \bigcup_{K\in\mathscr{F}(D)} \operatorname{WQE}(f,S,C+K). \tag{4.17}$$

Remark 4.5. (i) The convenience to define Henig proper quasi efficiency whenever the set-valued mapping C is free-disposal-valued with respect to D has already been noticed in the literature (see, for instance, the last paragraph in [26, page 384]).

(ii) Clearly, for each $K \in \mathscr{F}(D)$, it follows that $\operatorname{int}(C(x,u)+K) = C(x,u) + \operatorname{int} K$, for all $x,u \in \mathbb{R}^n$, since K is a solid convex cone. Thus, we have that

$$\operatorname{HQE}(f,S,C,D) = \bigcup_{K\in\mathscr{F}(D)} \operatorname{QE}(f,S,C+\operatorname{int} K) \subset \operatorname{QE}(f,S,C+(D\backslash\{0\})). \tag{4.18}$$

In particular, the set-valued mapping $C(x,u) = D$, for all $x,u \in \mathbb{R}^n$, which leads to the so-called Henig proper solutions of problem (P_D), satisfies $\operatorname{QE}(f,S,C+(D\backslash\{0\})) = \operatorname{E}(f,S,D)$ and so

Henig proper quasi efficient solutions are efficient solutions too. In general, if the set-valued mapping C fulfills $C(x,u)+(D\backslash\{0\})=C(x,u)$, for all $x,u\in S^2\backslash\Delta_n$, which is free-disposal-valued with respect to D in $S^2\backslash\Delta_n$, then the set of Henig proper quasi efficient solutions of problem (P) is useful as an inner approximation of the set of quasi efficient solutions of problem (P).

(iii) In the case where C is free-disposal-valued with respect to D in $S^2\backslash\Delta_n$ and D is solid, we have that $\operatorname{int}C=C+\operatorname{int}D\subset C+(D\backslash\{0\})$ in $S^2\backslash\Delta_n$. Thus, by Theorem 3.1(i) and statement (4.18) we see that $\mathrm{HQE}(f,S,C,D)\subset\mathrm{WQE}(f,S,C)$.

(iv) Definition 4.2 reduces to [26, Definition 3] by the set-valued mapping $C=hG+D$ (recall that set G is referred to C in [26, Definition 3]). In other words, [26, Definition 3] can be equivalently formulated by assuming G to be free-disposal with respect to D. Analogously, claim (4.18) generalizes [26, assertion (4)] and along with Proposition 3.1(ii) extends [26, Proposition 3(c)].

The next concept is useful not only to better understanding the concept of Henig proper quasi efficiency, but also to obtain this kind of solutions in practical problems.

Definition 4.3. We say that $\mathscr{B}\subset\mathscr{F}(D)$ generates $\mathscr{F}(D)$ if for each $K\in\mathscr{F}(D)$ there exists $K'\in\mathscr{B}$ such that $K'\backslash\{0\}\subset\operatorname{int}K$.

Notice $\mathscr{F}(D)$ generates $\mathscr{F}(D)$ as $\operatorname{int}K\cup\{0\}\in\mathscr{F}(D)$ as long as $K\in\mathscr{F}(D)$. Next it is stated that all Henig proper quasi efficient solutions can be obtained by considering the cones in a generating family $\mathscr{B}$.

Proposition 4.2. *Suppose that $\mathscr{B}$ generates $\mathscr{F}(D)$. Then,*

$$\mathrm{HQE}(f,S,C,D)=\bigcup_{K\in\mathscr{B}}\mathrm{QE}(f,S,C+(K\backslash\{0\}))=\bigcup_{K\in\mathscr{B}}\mathrm{QE}(f,S,C+\operatorname{int}K) \tag{4.19}$$

$$=\bigcup_{K\in\mathscr{F}(D)}\mathrm{QE}(f,S,C+(K\backslash\{0\})). \tag{4.20}$$

Proof. As $\mathscr{B}\subset\mathscr{F}(D)$ and $C+\operatorname{int}K\subset C+(K\backslash\{0\})$ whenever $K\subsetneq\mathbb{R}^p$ is a cone, by Theorem 3.1(i) we have that

$$\bigcup_{K\in\mathscr{B}}\mathrm{QE}(f,S,C+(K\backslash\{0\}))\subset\bigcup_{K\in\mathscr{B}}\mathrm{QE}(f,S,C+\operatorname{int}K)\subset\mathrm{HQE}(f,S,C,D).$$

Since $\mathscr{B}$ generates $\mathscr{F}(D)$, for each $K\in\mathscr{F}(D)$ there exists $K'\in\mathscr{B}$ satisfying $K'\backslash\{0\}\subset\operatorname{int}K$. Thus, by Theorem 3.1(i) we deduce that $\mathrm{QE}(f,S,C+\operatorname{int}K)\subset\mathrm{QE}(f,S,C+(K'\backslash\{0\}))$. Thus,

$$\mathrm{HQE}(f,S,C,D)=\bigcup_{K\in\mathscr{F}(D)}\mathrm{QE}(f,S,C+\operatorname{int}K)\subset\bigcup_{K\in\mathscr{B}}\mathrm{QE}(f,S,C+(K\backslash\{0\}))$$

and (4.19) is stated. Finally, equality (4.20) results from (4.19) since $\mathscr{F}(D)$ generates $\mathscr{F}(D)$. □

The notion of Henig proper quasi efficiency is useless as long as $D\backslash\{0\}$ is open since it coincides with the concepts of quasi efficiency and weak quasi efficiency. The same assertion holds true when one considers Henig proper quasi efficient solutions with respect to the dilating set-valued mapping $C+\operatorname{int}K$, $K\in\mathscr{F}(D)$, instead of C. Both claims are showed in the next two results.

Proposition 4.3. *Suppose that $D\backslash\{0\}$ is open and let $C:\mathbb{R}^n\times\mathbb{R}^n\rightrightarrows\mathbb{R}^p$ be free-disposal-valued with respect to D in $S^2\backslash\Delta_n$. Then,*

$$\mathrm{HQE}(f,S,C,D)=\mathrm{QE}(f,S,C+(D\backslash\{0\}))=\mathrm{WQE}(f,S,C).$$

Proof. Clearly, $D\backslash\{0\}$ is open if and only if $D=\operatorname{int}D\cup\{0\}$. Thus, $D\backslash\{0\}=\operatorname{int}D$ and $\mathscr{B}=\{D\}$ generates $\mathscr{F}(D)$. Then by (4.19) we have

$$\mathrm{HQE}(f,S,C,D)=\mathrm{QE}(f,S,C+(D\backslash\{0\}))=\mathrm{QE}(f,S,C+\operatorname{int}D).$$

As the set $C(x,u)$ is free-disposal, for all $(x,u)\in S^2\backslash\Delta_n$, it follows that $C+\operatorname{int}D=\operatorname{int}(C+D)=\operatorname{int}C$ in $S^2\backslash\Delta_n$. Therefore, by Remark 3.3(i) we see that $\mathrm{QE}(f,S,C+\operatorname{int}D)=\mathrm{WQE}(f,S,C)$, which finishes the proof. □

Corollary 4.3. *Let $C:\mathbb{R}^n\times\mathbb{R}^n\rightrightarrows\mathbb{R}^p$ be free-disposal-valued with respect to D in $S^2\backslash\Delta_n$. Then,*

$$\mathrm{HQE}(f,S,C+\operatorname{int}K,D)=\mathrm{WQE}(f,S,C+K),\quad \forall K\in\mathscr{F}(D).$$

Proof. Consider $\bar{K}\in\mathscr{F}(D)$. It is easy to check that $\mathscr{F}(\operatorname{int}\bar{K}\cup\{0\})\subset\mathscr{F}(D)$ and $C+\operatorname{int}\bar{K}$ is free-disposal-valued with respect to $\operatorname{int}\bar{K}\cup\{0\}$ in $S^2\backslash\Delta_n$. Thus, Proposition 4.3 implies that

$$\mathrm{WQE}(f,S,C+\bar{K})=\mathrm{HQE}(f,S,C+\operatorname{int}\bar{K},\operatorname{int}\bar{K}\cup\{0\})\subset\mathrm{HQE}(f,S,C+\operatorname{int}\bar{K},D). \tag{4.21}$$

In addition, as $\operatorname{int}\bar{K}+(D\backslash\{0\})=\operatorname{int}\bar{K}$ and $C+\operatorname{int}\bar{K}=\operatorname{int}(C+\bar{K})$, by statement (4.18) we have that

$$\mathrm{HQE}(f,S,C+\operatorname{int}\bar{K},D)\subset\mathrm{QE}(f,S,C+\operatorname{int}\bar{K}+(D\backslash\{0\}))=\mathrm{WQE}(f,S,C+\bar{K}) \tag{4.22}$$

and the proof is completed. □

Corollary 4.3 reduces to [26, Proposition 1(iii)] by considering the set-valued mapping $C=hG+\operatorname{int}K$, where G is a nonempty set in $\mathbb{R}^p$ –denoted by C in [26, Proposition 1(iii)]– and $K\in\mathscr{F}(D)$.

A key mathematical tool to deal with Henig proper quasi efficient solutions via a generating family $\mathscr{B}$ is the concept of approximating family of cones (see [35, Definition 3.1]).

Definition 4.4. It is said that a sequence $(D_m)\subset\mathbb{R}^p$ of nonincreasing (with respect to the inclusion) solid closed pointed convex cones approximates D if $D\backslash\{0\}\subset\operatorname{int}D_m$ and $D=\bigcap_m D_m$. For a closed cone $K\subset\mathbb{R}^p$, if in addition,

$$D\cap K=\{0\}\Rightarrow D_m\cap K=\{0\}\text{ eventually}, \tag{4.23}$$

then it is said that (D_m) separates D from K. The sequence (D_m) is called approximating and separating for D if it approximates D and separates D from K, for all closed cones $K\subset\mathbb{R}^p$.

Remark 4.6. It is well-known the existence of approximating and separating sequences for D, which have been explicitly built in the literature (see, for instance, [25, 28, 35]). The next result shows these sequences of cones generate $\mathscr{F}(D)$.

Proposition 4.4. *We have that each approximating and separating sequence of cones for D generates $\mathscr{F}(D)$.*

Proof. Let (D_m) be an approximating and separating sequence of cones for D. By the definition we see that $(D_m) \subset \mathscr{F}(D)$. In addition, for each $K \in \mathscr{F}(D)$, it follows that $D\backslash\{0\} \subset \operatorname{int} K$, i.e., $D \cap (\operatorname{int} K)^c = \{0\}$. As $(\operatorname{int} K)^c$ is a closed cone, by the separating property (4.23) we deduce that $D_m \cap (\operatorname{int} K)^c = \{0\}$ eventually. Thus, there exists $m \in \mathbb{N}$ such that $D_m\backslash\{0\} \subset \operatorname{int} K$ and the result is proved. □

In the sequel we state that the sets in the sequence $(\mathrm{QE}(f,S,C+\operatorname{int} D_m))_m$ are good inner approximations of the quasi efficient solutions of problem (P_D) whenever C is free-disposal-valued with respect to D in $S^2\backslash\Delta_n$ and (D_m) is an approximating and separating sequence of cones for D. Recall that the recession cone of a nonempty set $F \subset \mathbb{R}^p$ is the cone

$$0^+F := \{y \in \mathbb{R}^p : \exists\,(y_n) \subset F, \exists\,(t_n) \subset \mathbb{R}_+, t_n \to 0, \text{ s.t. } t_n y_n \to y\}.$$

Theorem 4.3. *Let (D_m) be an approximating and separating sequence of cones for D, and $C : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be free-disposal-valued with respect to D in $S^2\backslash\Delta_n$. We have that*

$$\mathrm{HQE}(f,S,C,D) = \bigcup_m \mathrm{QE}(f,S,C+\operatorname{int} D_m) \tag{4.24}$$

$$\subset \mathrm{QE}(f,S,\bigcap_m (C+\operatorname{int} D_m)) \tag{4.25}$$

$$\subset \mathrm{QE}(f,S,C+D\backslash\{0\}). \tag{4.26}$$

If, in addition, $C(x,u)$ is closed and $0^+C(x,u) \cap (-D) = \{0\}$, for all $(x,u) \in S^2\backslash\Delta_n$, then $\mathrm{QE}(f,S,C) \subset \mathrm{QE}(f,S,\bigcap_m (C+\operatorname{int} D_m))$.

Proof. Equality (4.24) is a direct consequence of Proposition 4.4 and (4.19) and inclusions (4.25) and (4.26) follow by Theorem 3.1(i), since

$$C + D\backslash\{0\} \subset \bigcap_m (C+\operatorname{int} D_m) \subset C+\operatorname{int} D_m, \quad \forall m \in \mathbb{N}.$$

Moreover, by considering the assumptions of the second part of the theorem to be fulfilled, we claim that

$$\bigcap_m (C(x,u)+\operatorname{int} D_m) \subset C(x,u), \quad \forall (x,u) \in S^2\backslash\Delta_n, \tag{4.27}$$

and the last statement of this theorem follows by part (i) of Theorem 3.1. Indeed, consider a point $\bar{y} \in \bigcap_m (C(x,u)+\operatorname{int} D_m)$. Therefore, there exist sequences $(y_m) \subset C(x,u)$ and $(d_m) \subset \mathbb{R}^p$, $d_m \in \operatorname{int} D_m$ such that $\bar{y} = y_m + d_m$, for all $m \in \mathbb{N}$.

It follows that the sequence (d_m) is bounded. On the contrary, suppose that $t_m := \|d_m\| \to +\infty$. Then we can assume that $d_m/t_m \to d \in \mathbb{R}^p\backslash\{0\}$. As the sequence (D_m) approximates D we deduce that $d \in D\backslash\{0\}$. Thus, $y_m/t_m \to -d \in 0^+C(x,u) \cap (-D\backslash\{0\})$, which is a contradiction.

By the boundedness of the sequence (d_m), we can assume without loss of generality that d_m converges to a point d that belongs to D since the sequence (D_m) approximates D. Clearly, $y_m \to y = \bar{y} - d$ and as $C(x,u)$ is assumed to be closed we have that $y \in C(x,u)$. Therefore, $\bar{y} \in C(x,u) + D = C(x,u)$ and the proof finishes. □

Remark 4.7. Theorem 4.3 extends the claims in [26, Remark 2], in the sense that the computation of Henig proper quasi efficient solutions taking into account only a sequence of known dilating cones (see Remark 4.6) is possible not only for polyhedral ordering cones, but also for each closed pointed convex cone D.

Theorem 4.4. *Let (D_m) be an approximating and separating sequence of cones for D and consider $h:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_+$ and a nonempty set $G\subset\mathbb{R}^p$. The next assertions hold true:*

(i) We have that

$$\mathrm{HQE}(f,S,hG+D,D)=\bigcup_m \mathrm{QE}(f,S,hG+\operatorname{int}D_m)\subset \mathrm{QE}(f,S,hG+D\backslash\{0\}). \tag{4.28}$$

(ii) If $G+D$ is closed, $h>0$ in $S^2\backslash\Delta_n$ and $0^+(G+D)\cap(-D)=\{0\}$, then

$$\mathrm{QE}(f,S,hG+D)\subset \mathrm{QE}(f,S,\bigcap_m(hG+\operatorname{int}D_m)).$$

(iii) If $G\subset D\backslash\{0\}$, then

$$\bigcup_{\varepsilon\in\mathscr{E}(h)} \mathrm{QE}(f,S,(h-\varepsilon)G+D\backslash\{0\})\subset \mathrm{QE}(f,S,hG+D),$$

where $\mathscr{E}(h):=\{\varepsilon:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_+ : 0<\varepsilon(x,u)<h(x,u),\forall(x,u)\in S^2\backslash\Delta_n\}$.

Proof. (i) Clearly, the values of the mapping $C:=hG+D:\mathbb{R}^n\times\mathbb{R}^n\rightrightarrows\mathbb{R}^p$ are free-disposal sets with respect to D, since $D+D=D$. In addition, $C+\operatorname{int}D_m=hG+\operatorname{int}D_m$ as $D+\operatorname{int}D_m=\operatorname{int}D_m$, for all $m\in\mathbb{N}$, and $D+D\backslash\{0\}=D\backslash\{0\}$ since D is assumed to be pointed. Then, statement (4.28) is obtained by applying statements (4.24) and (4.26) .

(ii) By assumptions, it is obvious that $C(x,u)$ is closed and

$$0^+C(x,u)=0^+(h(x,u)G+D)=0^+(h(x,u)(G+D))=h(x,u)0^+(G+D)=0^+(G+D)$$

in $S^2\backslash\Delta_n$. Therefore,

$$0^+C(x,u)\cap(-D)=0^+(G+D)\cap(-D)=\{0\},\quad (x,u)\in S^2\backslash\Delta_n$$

and by the second part of Theorem 4.3 we deduce that $\mathrm{QE}(f,S,hG+D)\subset \mathrm{QE}(f,S,\bigcap_m(hG+\operatorname{int}D_m))$.

(iii) Since $G\subset D\backslash\{0\}$, in $S^2\backslash\Delta_n$ we have that

$$hG+D\subset(h-\varepsilon)G+\varepsilon G+D\subset(h-\varepsilon)G+D\backslash\{0\}+D=(h-\varepsilon)G+D\backslash\{0\},\quad \forall\varepsilon\in\mathscr{E}(h),$$

and by Theorem 3.1(i) we see that $\mathrm{QE}(f,S,(h-\varepsilon)G+D\backslash\{0\})\subset \mathrm{QE}(f,S,hG+D)$, which finishes the proof. □

5. Linear scalarization

In this last section, we are going to deduce linear scalarization results for weak quasi efficient solutions of problem (P) and Henig proper quasi efficient solutions of problem (P_D), i.e., necessary and sufficient conditions that characterize this kind of efficient solutions by quasi solutions of associated scalar optimization problems. As usual, the necessary conditions are deduced by considering that the objective function f of the problem fulfils certain generalized cone convexity assumptions. Recall that the positive (resp. strict positive) polar cone of a convex cone

$K \subset \mathbb{R}^p$ is the set

$$K^+ := \{\lambda \in \mathbb{R}^p : \langle \lambda, y \rangle \geq 0, \forall y \in K\}$$
$$(\text{resp. } K^{+s} := \{\lambda \in \mathbb{R}^p : \langle \lambda, y \rangle > 0, \forall y \in K \backslash \{0\}\}).$$

In addition, the epigraph of a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ with respect to a convex cone $K \subset \mathbb{R}^p$ is the set:

$$\operatorname{epi}_K F := \{(x,z) \in \mathbb{R}^n \times \mathbb{R}^p : z \in F(x) + K\}.$$

For a nonempty set $Q \subset \mathbb{R}^n$, notice that

$$\operatorname{epi}_K F|_Q := \{(x,z) \in Q \times \mathbb{R}^p : z \in F(x) + K\}$$

and so a point $(x,z) \in \mathbb{R}^n \times \mathbb{R}^p$ belongs to $\operatorname{epi}_K F|_Q$ if and only if $x \in Q$ and there exists $y \in F(x)$ such that $y \leq_K z$.

Definition 5.1. Let Q be a nonempty subset of $\mathbb{R}^n$. The set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is said to be:

(i) Closely cone convex with respect to a convex cone $K \subset \mathbb{R}^p$ (closely K-convex in short form) on Q, if the set $\operatorname{cl} \operatorname{epi}_K F|_Q$ is convex.

(ii) Nearly cone subconvexlike with respect to a convex cone $K \subset \mathbb{R}^p$ (nearly K-subconvexlike in short form) on Q, if the set $\operatorname{cl} \operatorname{cone}(F(Q) + K)$ is convex.

(iii) Generalized cone subconvexlike with respect to a solid convex cone $K \subset \mathbb{R}^p$ (generalized K-subconvexlike in short form) on Q, if the set $\operatorname{cone} F(Q) + \operatorname{int} K$ is convex.

Remark 5.1. In [16] and the references therein, one can find several relationships and properties of the above generalized convexity concepts. In particular, it is well-known that the notions of nearly K-subconvexlikeness and generalized K-subconvexlikeness are equivalent as long as the convex cone K is solid.

Theorem 5.1. *Consider problem* (P)*, a point* $x_0 \in S$ *and* $C : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *such that* $S \backslash \{x_0\} \subset \operatorname{dom} \operatorname{int} C(\cdot, x_0)$. *Assume the set-valued mapping* $\operatorname{int} C(\cdot, x_0) : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *is open, graph-convex and free-disposal-valued with respect to a convex cone* $K \subset \mathbb{R}^p$ *in S,* $0 \notin \operatorname{int} C(x_0, x_0)$ *and* $f - f(x_0)$ *is closely K-convex on S. In addition, suppose that* $\operatorname{int} S \neq \emptyset$ *or* $\mathbb{R}^n \backslash \{x_0\} \subset \operatorname{dom} \operatorname{int} C(\cdot, x_0)$.

If $x_0 \in \operatorname{WQE}(f, S, C)$*, then there exists* $\lambda \in K^+ \backslash \{0\}$ *such that*

$$\langle \lambda, f(x) \rangle + \inf_{z \in C(x,x_0)} \langle \lambda, z \rangle \geq \langle \lambda, f(x_0) \rangle, \quad \forall x \in S \backslash \{x_0\}. \tag{5.1}$$

In addition, if $C(x_0, x_0)$ *is solid, then* (5.1) *can be extended to* $x = x_0$ *and it follows that* $\lambda \in (\operatorname{cone} C(x_0, x_0)^+ \cap K^+) \backslash \{0\}$.

Proof. As x_0 is a weak quasi efficient solution of problem (P) and $0 \notin \operatorname{int} C(x_0, x_0)$, we have that

$$(x, f(x) - f(x_0)) \notin \operatorname{gph}(-\operatorname{int} C(\cdot, x_0)), \quad \forall x \in S.$$

Therefore,

$$\bigcup_{x \in S} \{(x, f(x) - f(x_0))\} \bigcap \operatorname{gph}(-\operatorname{int} C(\cdot, x_0)) = \emptyset. \tag{5.2}$$

As the set-valued mapping $\operatorname{int} C(\cdot, x_0)$ is free-disposal-valued with respect to K in S, statement (5.2) can be rewritten as follows:

$$\left[\left(\bigcup_{x \in S} \{(x, f(x) - f(x_0))\} \right) + (\{0\} \times K) \right] \bigcap \operatorname{gph}(-\operatorname{int} C(\cdot, x_0)) = \emptyset. \tag{5.3}$$

Notice that

$$\left(\bigcup_{x\in S}\{(x,f(x)-f(x_0))\}\right)+(\{0\}\times K)=\bigcup_{x\in S}\{x\}\times(f(x)-f(x_0)+K)=\operatorname{epi}_K(f|_S-f(x_0)),$$

where $f|_S-f(x_0)$ denotes the set-valued mapping from $\mathbb{R}^n$ to $\mathbb{R}^p$ defined by $(f|_S-f(x_0))(x)=\{f(x)-f(x_0)\}$ for all $x\in S$ and $(f|_S-f(x_0))(x)=\emptyset$ otherwise.

Then, by (5.3) we have that

$$\operatorname{epi}_K(f|_S-f(x_0))\subset(\operatorname{gph}(-\operatorname{int}C(\cdot,x_0)))^c.$$

Clearly, the right-hand side of the above inclusion is a closed set as the set-valued mapping $\operatorname{int}C(\cdot,x_0)$ is open. Therefore, one can replace the left-hand side of the above inclusion by $\operatorname{cl}\operatorname{epi}_K(f|_S-f(x_0))$ and we see that

$$\operatorname{cl}\operatorname{epi}_K(f|_S-f(x_0))\bigcap\operatorname{gph}(-\operatorname{int}C(\cdot,x_0))=\emptyset.$$

The set $\operatorname{cl}\operatorname{epi}_K(f|_S-f(x_0))$ is convex since $f-f(x_0)$ is closely K-convex on S. In addition, $\operatorname{gph}(-\operatorname{int}C(\cdot,x_0))$ is solid and convex as the set-valued mapping $\operatorname{int}C(\cdot,x_0)$ is open and graph-convex. Then, $\operatorname{cl}\operatorname{epi}_K(f|_S-f(x_0))$ and $\operatorname{gph}(-\operatorname{int}C(\cdot,x_0))$ can be properly separated (see [33, Theorem 2.39]), i.e., there exists $(\mu,\lambda)\in\mathbb{R}^n\times\mathbb{R}^p\backslash\{(0,0)\}$ such that

$$\langle\mu,x\rangle+\langle\lambda,f(x)-f(x_0)+y\rangle\geq\langle\mu,u\rangle+\langle\lambda,z\rangle,\quad\forall x\in S,y\in K,u\in S\backslash\{x_0\},z\in-C(u,x_0).\tag{5.4}$$

In addition, if $\operatorname{int}C(x_0,x_0)\neq\emptyset$, then (5.4) can be extended to $u=x_0$.

We claim that $\lambda\neq0$. Indeed, if $\lambda=0$ and $\operatorname{int}S\neq\emptyset$, we have $\langle\mu,x-u\rangle\geq0$, for all $x\in S$ and $u\in S\backslash\{x_0\}$. Thus, by considering $\bar{x}\in\operatorname{int}S$ and a point $y\in\mathbb{R}^p\backslash\{0\}$ there exists $\alpha>0$ such that $\bar{x}\pm\alpha y\in S\backslash\{x_0\}$ and so by taking $x:=\bar{x}$ and $u:=\bar{x}\pm\alpha y$ we deduce $\langle\mu,y\rangle=0$. As point y is arbitrary we obtain $\mu=0$, which is a contradiction.

Suppose that $\lambda=0$ and $\mathbb{R}^n\backslash\{x_0\}\subset\operatorname{dom}\operatorname{int}C(\cdot,x_0)$. Then condition $u\in S\backslash\{x_0\}$ in (5.4) can be extended to $u\in\mathbb{R}^n\backslash\{x_0\}$ and by taking $x=x_0$ we see that $\langle\mu,x_0\rangle\geq\langle\mu,u\rangle$, for all $u\in\mathbb{R}^n$, and so $\mu=0$, which is a contradiction.

In addition, from inequality (5.4) with $u=x\in S\backslash\{x_0\}$ and $y=0$ it follows that

$$\langle\lambda,f(x)\rangle+\inf_{z\in C(x,x_0)}\langle\lambda,z\rangle\geq\langle\lambda,f(x_0)\rangle,\quad\forall x\in S\backslash\{x_0\},\tag{5.5}$$

and statement (5.1) is proved. By statement (5.4), we see that $\inf_{y\in K}\langle\lambda,y\rangle>-\infty$. Therefore, $\lambda\in K^+$, since $\lambda\neq0$ and K is a cone. In addition, if $C(x_0,x_0)$ is solid, then (5.4) also holds true for $u=x_0$. Thus, by considering $u=x=x_0$ and $y=0$ in (5.4) we deduce $\lambda\in\operatorname{cone}C(x_0,x_0)^+\backslash\{0\}$ and the proof finishes. □

Theorem 5.2. *Consider problem* (P)*, a point* $x_0\in S$*,* $C:\mathbb{R}^n\times\mathbb{R}^n\rightrightarrows\mathbb{R}^p$ *and let* $K\subset\mathbb{R}^p$ *be a solid convex cone. Suppose* $C(x_0,x_0)\cap(-\operatorname{int}K)=\emptyset$ *and* $f-f(x_0)+C(\cdot,x_0)$ *is nearly P-subconvexlike on the feasible set S, for a convex cone* $P\subset K$*.*

If $x_0\in\operatorname{WQE}(f,S,C+K)$*, then there exists* $\lambda\in(\operatorname{cone}C(x_0,x_0)^+\cap K^+)\backslash\{0\}$ *such that*

$$\langle\lambda,f(x)\rangle+\inf_{z\in C(x,x_0)}\langle\lambda,z\rangle\geq\langle\lambda,f(x_0)\rangle,\quad\forall x\in S.\tag{5.6}$$

Proof. Since $x_0\in\operatorname{WQE}(f,S,C+K)$ and $C(x_0,x_0)\cap(-\operatorname{int}K)=\emptyset$, we have that

$$f(x)-f(x_0)\notin-(C(x,x_0)+\operatorname{int}K),\quad\forall x\in S.$$

As $P \subset K$ is a convex cone, it is obtained that $P + \operatorname{int} K = \operatorname{int} K$. Thus,

$$(f - f(x_0) + C(\cdot, x_0))(S) + P \subset (-\operatorname{int} K)^c$$

and as $(-\operatorname{int} K)^c$ is a closed cone we deduce that

$$\operatorname{cl}\operatorname{cone}((f - f(x_0) + C(\cdot, x_0))(S) + P) \cap (-\operatorname{int} K) = \emptyset. \tag{5.7}$$

By the nearly P-subconvexlikeness of $f - f(x_0) + C(\cdot, x_0)$ on S we have that the left-hand side set in the last intersection is convex. Therefore, as in the proof of the previous theorem, the involved sets in (5.7) can be properly separated, i.e., there exists $\lambda \in \mathbb{R}^p \backslash \{0\}$ such that

$$\langle \lambda, f(x) - f(x_0) + z \rangle \geq \langle \lambda, y \rangle, \quad \forall x \in S, z \in C(x, x_0), y \in -K,$$

and the result follows by considering the same reasoning as in the proof of Theorem 5.1. □

Corollary 5.1. *Consider problem* (P)*, a point $x_0 \in S$, $C : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ and let $K \subset \mathbb{R}^p$ be a solid convex cone. Suppose that $C(\cdot, x_0) : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is free-disposal-valued with respect to the cone $\operatorname{int} K \cup \{0\}$ in S, $0 \notin \operatorname{int} C(x_0, x_0)$ and $f - f(x_0) + C(\cdot, x_0)$ is nearly P-subconvexlike on the feasible set S, for a convex cone $P \subset K$.*

If $x_0 \in \mathrm{WQE}(f, S, C)$, then there exists $\lambda \in (\operatorname{cone} C(x_0, x_0)^+ \cap K^+) \backslash \{0\}$ such that

$$\langle \lambda, f(x) \rangle + \inf_{z \in C(x, x_0)} \langle \lambda, z \rangle \geq \langle \lambda, f(x_0) \rangle, \quad \forall x \in S. \tag{5.8}$$

Proof. Since $C(x, x_0)$ is free-disposal with respect to the cone $\operatorname{int} K \cup \{0\}$, we have that

$$\operatorname{int} C(x, x_0) = \operatorname{int}(C(x, x_0) + (\operatorname{int} K \cup \{0\})) = C(x, x_0) + \operatorname{int}(\operatorname{int} K \cup \{0\}) = C(x, x_0) + \operatorname{int} K, \quad \forall x \in S.$$

Therefore, as $x_0 \in \mathrm{WQE}(f, S, C)$ and $0 \notin \operatorname{int} C(x_0, x_0)$ it follows that $x_0 \in \mathrm{WQE}(f, S, C + K)$ and $C(x_0, x_0) \cap (-\operatorname{int} K) = \emptyset$. Then, all assumptions of Theorem 5.2 are fulfilled and, as a consequence of this result, assertion (5.8) is obtained and the proof is completed. □

Given $\lambda \in \mathbb{R}^p \backslash \{0\}$, we denote

$$H_\lambda^{\geq} := \{y \in \mathbb{R}^p : \langle \lambda, y \rangle \geq 0\},$$
$$H_\lambda^{>} := \{y \in \mathbb{R}^p : \langle \lambda, y \rangle > 0\}.$$

Proposition 5.1. *Consider problem* (P) *and a point $x_0 \in S$. If there exists $\lambda \in \mathbb{R}^p \backslash \{0\}$ such that inequality* (5.1) *holds true, then $x_0 \in \mathrm{WQE}(f, S, C + H_\lambda^{\geq})$. In addition, if inequality* (5.1) *is strict, then $x_0 \in \mathrm{QE}(f, S, C + H_\lambda^{\geq})$.*

Proof. On the contrary, suppose that $x_0 \notin \mathrm{WQE}(f, S, C + H_\lambda^{\geq})$. Then there exist $x \in S \backslash \{x_0\}$ and $y \in \operatorname{int}(C + H_\lambda^{\geq})(x, x_0)$ such that $f(x_0) = f(x) + y$. As $\operatorname{int}(C + H_\lambda^{\geq})(x, x_0) = C(x, x_0) + H_\lambda^{>}$ there exist points $v \in C(x, x_0)$ and $h \in H_\lambda^{>}$ such that $y = v + h$. Thus,

$$\langle \lambda, f(x_0) \rangle = \langle \lambda, f(x) \rangle + \langle \lambda, v \rangle + \langle \lambda, h \rangle > \langle \lambda, f(x) \rangle + \inf_{z \in C(x, x_0)} \langle \lambda, z \rangle,$$

which is contrary to inequality (5.1). Thus, $x_0 \in \mathrm{WQE}(f, S, C + H_\lambda^{\geq})$.

The case where $x_0 \in \mathrm{QE}(f, S, C + H_\lambda^{\geq})$ along with inequality (5.1) holds true strictly can be stated by the same way, and the proof finishes. □

Proposition 5.2. *Consider problem* (P_D) *and a point* $x_0 \in S$. *The next assertions hold true:*

(i) If D *is solid,* $S \backslash \{x_0\} \subset \operatorname{dom} \operatorname{int} C(\cdot, x_0)$, $\operatorname{int} C(\cdot, x_0) : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *is free-disposal-valued with respect to* $\operatorname{int} D \cup \{0\}$ *in* $S \backslash \{x_0\}$, *then*

$$x_0 \in \bigcup_{\lambda \in D^+ \backslash \{0\}} \mathrm{QE}(f, S, C + H_\lambda^>) \Rightarrow x_0 \in \mathrm{WQE}(f, S, C).$$

(ii) If $C(\cdot, x_0) : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *is free-disposal-valued with respect to* D *in* $S \backslash \{x_0\}$, *then*

$$x_0 \in \bigcup_{\lambda \in D^+ \backslash \{0\}} \mathrm{QE}(f, S, C + H_\lambda^{\geq}) \Rightarrow x_0 \in \mathrm{QE}(f, S, C), \tag{5.9}$$

$$x_0 \in \bigcup_{\lambda \in D^{+s}} \mathrm{QE}(f, S, C + H_\lambda^{\geq}) \Rightarrow x_0 \in \mathrm{HQE}(f, S, C). \tag{5.10}$$

Proof. (i) Let us recall that (see [22])

$$\operatorname{int} D = \bigcap_{\lambda \in D^+ \backslash \{0\}} \{y \in \mathbb{R}^p : \langle \lambda, y \rangle > 0\}. \tag{5.11}$$

Since $\operatorname{int} C(\cdot, x_0) : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is free-disposal-valued with respect to $\operatorname{int} D \cup \{0\}$ in $S \backslash \{x_0\}$, we have that

$$\begin{aligned} \operatorname{int} C(x, x_0) &= \operatorname{int} C(x, x_0) + \bigcap_{\lambda \in D^+ \backslash \{0\}} \{y \in \mathbb{R}^p : \langle \lambda, y \rangle > 0\} \\ &\subset \bigcap_{\lambda \in D^+ \backslash \{0\}} (C(x, x_0) + H_\lambda^>), \quad \forall x \in S \backslash \{x_0\}, \end{aligned}$$

and by parts (i) and (iii) of Theorem 3.1 and Remark 3.4 we see that

$$x_0 \in \bigcup_{\lambda \in D^+ \backslash \{0\}} \mathrm{QE}(f, S, C + H_\lambda^>) \Rightarrow x_0 \in \mathrm{WQE}(f, S, C). \tag{5.12}$$

(ii) The proof of implication (5.9) follows the same reasonings as the previous part. Finally, statement (5.10) holds true by the definition, since $H_\lambda^{\geq} \in \mathscr{F}(D)$ whenever $\lambda \in D^{+s}$, which finishes the proof. □

The next sufficient conditions for quasi solutions of problem (P_D) are a direct consequence of Propositions 5.1 and 5.2.

Corollary 5.2. *Consider problem* (P_D) *and a point* $x_0 \in S$. *The next assertions hold true:*

(i) Assume D *is solid,* $S \backslash \{x_0\} \subset \operatorname{dom} \operatorname{int} C(\cdot, x_0)$ *and* $\operatorname{int} C(\cdot, x_0) : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *is free-disposal-valued with respect to* $\operatorname{int} D \cup \{0\}$ *in* $S \backslash \{x_0\}$. *If there exists* $\lambda \in D^+ \backslash \{0\}$ *such that*

$$\langle \lambda, f(x) \rangle + \inf_{z \in C(x, x_0)} \langle \lambda, z \rangle \geq \langle \lambda, f(x_0) \rangle, \quad \forall x \in S \backslash \{x_0\}, \tag{5.13}$$

then $x_0 \in \mathrm{WQE}(f, S, C)$.

(ii) Suppose $C(\cdot, x_0) : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *is free-disposal-valued with respect to* D *in* $S \backslash \{x_0\}$. *We have that*

(a) If there exists $\lambda \in D^+ \backslash \{0\}$ *such that*

$$\langle \lambda, f(x) \rangle + \inf_{z \in C(x, x_0)} \langle \lambda, z \rangle > \langle \lambda, f(x_0) \rangle, \quad \forall x \in S \backslash \{x_0\},$$

then $x_0 \in \mathrm{QE}(f, S, C)$.

(b) If there exists $\lambda \in D^{+s}$ such that assertion (5.13) is fulfilled, then $x_0 \in \mathrm{HQE}(f,S,C)$.

Theorem 5.1 and Corollaries 5.1 and 5.2 allow us to characterize weak and Henig proper quasi efficient solutions of problem (P_D) in the convex case. Namely, we have the following results.

Theorem 5.3. *Consider problem (P_D), a set-valued mapping $C:\mathbb{R}^n\times\mathbb{R}^n \rightrightarrows \mathbb{R}^p$ and a point $x_0 \in S$. Suppose D is solid, $S\backslash\{x_0\} \subset \operatorname{dom}\operatorname{int} C(\cdot,x_0)$ and one of the following conditions is satisfied:*
(A1) The set-valued mapping $\operatorname{int} C(\cdot,x_0):\mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is open, graph-convex and free-disposal-valued with respect to D in S, $0 \notin \operatorname{int} C(x_0,x_0)$ and $f-f(x_0)$ is closely D-convex on S. In addition, $\operatorname{int} S \neq \emptyset$ or $\mathbb{R}^n\backslash\{x_0\} \subset \operatorname{dom}\operatorname{int} C(\cdot,x_0)$.
(A2) The mapping $C(\cdot,x_0):\mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is free-disposal-valued with respect to $\operatorname{int} D \cup \{0\}$ in S, $0 \notin \operatorname{int} C(x_0,x_0)$ and $f-f(x_0)+C(\cdot,x_0)$ is generalized D-subconvexlike on the feasible set S.
Then $x_0 \in \mathrm{WQE}(f,S,C)$ if and only if there exists $\lambda \in D^+\backslash\{0\}$ such that

$$\langle \lambda, f(x)\rangle + \inf_{z\in C(x,x_0)} \langle \lambda, z\rangle \geq \langle \lambda, f(x_0)\rangle, \quad \forall x \in S\backslash\{x_0\}. \tag{5.14}$$

If, in addition, $C(x_0,x_0)$ is solid –this happens whenever condition (A2) holds true, then (5.14) can be extended to $x = x_0$ and $\lambda \in (\operatorname{cone} C(x_0,x_0)^+ \cap D^+)\backslash\{0\}$.

Proof. The necessary condition is a direct application of Remark 5.1, Theorem 5.1 and Corollary 5.1 by considering the cone D instead of K.

In order to deduce the sufficient condition, notice that both conditions (A1) and (A2) imply that $\operatorname{int} C(\cdot,x_0):\mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is free-disposal-valued with respect to $\operatorname{int} D \cup \{0\}$ in $S\backslash\{x_0\}$. Thus, it is a direct consequence of Corollary 5.2(i) and the proof is finished. □

Theorem 5.4. *Consider problem (P_D) and a point $x_0 \in S$. Assume the set-valued mapping $C(\cdot,x_0):\mathbb{R}^n\times\mathbb{R}^n \rightrightarrows \mathbb{R}^p$ to be free-disposal-valued with respect to D in $S\backslash\{x_0\}$, $C(x_0,x_0)\cap(-\operatorname{int} K) = \emptyset$, for all K in a family $\mathscr{B}$ that generates $\mathscr{F}(D)$, and $f-f(x_0)+C(\cdot,x_0)$ to be nearly D-subconvexlike on the feasible set S.*
Then $x_0 \in \mathrm{HQE}(f,S,C,D)$ if and only if there exists $\lambda \in D^{+s}$ such that

$$\langle \lambda, f(x)\rangle + \inf_{z\in C(x,x_0)} \langle \lambda, z\rangle \geq \langle \lambda, f(x_0)\rangle, \quad \forall x \in S. \tag{5.15}$$

Proof. Let $x_0 \in \mathrm{HQE}(f,S,C,D)$. Then, there exists a cone $K \in \mathscr{B}$ such that $x_0 \in \mathrm{WQE}(f,S,C+K)$. Clearly, assumptions of Theorem 5.2 are fulfilled and so there exists $\lambda \in (\operatorname{cone} C(x_0,x_0)^+ \cap K^+)\backslash\{0\}$ such that

$$\langle \lambda, f(x)\rangle + \inf_{z\in C(x,x_0)} \langle \lambda, z\rangle \geq \langle \lambda, f(x_0)\rangle, \quad \forall x \in S.$$

In addition, as $D\backslash\{0\} \subset \operatorname{int} K$, by (5.11) we have that $\lambda \in D^{+s}$ and the necessary condition is obtained. The sufficient one follows by part (ii)(b) of Corollary 5.2, and the proof is completed. □

Remark 5.2. Notice that the assumption $C(x_0,x_0)\cap(-\operatorname{int} K) = \emptyset$, for all $K \in \mathscr{B}$, is satisfied as long as $C(x_0,x_0) \subset D$.

The next example illustrates Theorems 5.3 and 5.4.

Example 5.1. Let us consider problem (P_D) with the following data: $n=1$, $p=2$, $f(t)=(t,t^2)$, $S=\mathbb{R}$, $D=\mathbb{R}^2_+$, $t_0=0$ and $C(t,s)=|t-s|(-1,1)+\mathbb{R}^2_+$. Clearly, all assumptions of Theorem 5.4 are fulfilled. Particularly, notice that

$$\operatorname{cl}\operatorname{cone}((f-f(0)+C(\cdot,0))(\mathbb{R})+\mathbb{R}^2_+)=\{(y_1,y_2)\in\mathbb{R}^2 : y_2\geq 0, y_1+2y_2\geq 0\},$$

which is a convex set. By applying Theorem 5.4 we deduce that $0\in \mathrm{HQE}(f,S,C)$ if and only if there exists $\lambda=(\lambda_1,\lambda_2)\in \operatorname{int}\mathbb{R}^2_+$ such that

$$\lambda_1 t+\lambda_2 t^2+\inf_{d\in\mathbb{R}^2_+}\langle\lambda,|t|(-1,1)+d\rangle\geq 0,\quad \forall t\in\mathbb{R}. \tag{5.16}$$

Clearly,

$$\inf_{d\in\mathbb{R}^2_+}\langle\lambda,|t|(-1,1)+d\rangle=-\lambda_1|t|+\lambda_2|t|$$

and so (5.16) is equivalent to the inequality

$$\lambda_1 t+\lambda_2 t^2+|t|(-\lambda_1+\lambda_2)\geq 0,\quad \forall t\in\mathbb{R}.$$

that holds true, for instance, for $\lambda=(1,2)$. Therefore, $0\in\mathrm{HQE}(f,S,C)$. Notice that [26, Theorem 6 and Corollary 2] cannot be applied to derive this claim, since the set $G=(-1,1)+\mathbb{R}^2_+$ is not coradiant.

Since D is solid, Theorem 5.3 with assumption (A2) can be applied to deduce that $0\in \mathrm{WQE}(f,S,C)$ (observe that the convexity assumptions in Theorem 5.4 and (A2) of Theorem 5.3 coincide, see Remark 5.1). This conclusion is also a result of Remark 4.5(iii). As in the previous assertion, [20, Theorem 4.1] and [26, Theorem 5(a)] cannot be applied to obtain $0\in\mathrm{WQE}(f,S,C)$ since the set $G=(-1,1)+\mathbb{R}^2_+$ is not coradiant.

Theorems 5.3 and 5.4 reduce to the next results for the set-valued mapping $C=hG+D$. For each nonempty set $G\subset\mathbb{R}^p$ and $\lambda\in\mathbb{R}^p$, we denote $\tau_G(\lambda)=\inf_{z\in G}\langle\lambda,z\rangle$.

Corollary 5.3. *Consider problem (P_D), a nonempty set $G\subset\mathbb{R}^p$, a function $h:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_+$ such that $h>0$ in $S^2\backslash\Delta_n$ and a point $x_0\in S$.*

(i) Suppose D is solid, $f-f(x_0)+h(\cdot,x_0)G$ is generalized D-subconvexlike on the feasible set S and $h(x_0,x_0)=0$ or $G\cap(-\operatorname{int}D)=\emptyset$. Then $x_0\in\mathrm{WQE}(f,S,hG+D)$ if and only if there exists $\lambda\in\operatorname{cone}G^+\cap D^+$ if $h(x_0,x_0)>0$ and $\lambda\in D^+$ otherwise, such that

$$\langle\lambda,f(x)\rangle+k_{h,G}(\lambda,x)\geq\langle\lambda,f(x_0)\rangle,\quad\forall x\in S, \tag{5.17}$$

where $k_{h,G}(\lambda,x)=\tau_G(\lambda)h(x,x_0)$ if $x\neq x_0$, or $x=x_0$ and $h(x_0,x_0)>0$, and $k_{h,G}(\lambda,x_0)=0$ if $h(x_0,x_0)=0$.

(ii) Assume $f-f(x_0)+h(\cdot,x_0)G$ is nearly D-subconvexlike on the feasible set S, $h(x_0,x_0)G\cap(-\operatorname{int}K)=\emptyset$, for all K in a family $\mathscr{B}$ that generates $\mathscr{F}(D)$. Then $x_0\in\mathrm{HQE}(f,S,hG+D)$ if and only if there exists $\lambda\in D^{+s}$ such that inequality (5.17) is fulfilled.

Proof. (i) Clearly, the set-valued mapping $C:\mathbb{R}^n\times\mathbb{R}^n\rightrightarrows\mathbb{R}^p$, $C(x,u):=h(x,u)G+D$, for all $(x,u)\in\mathbb{R}^n\times\mathbb{R}^n$, is free-disposal-valued with respect to $\operatorname{int}D\cup\{0\}$ in S. In addition, $\operatorname{int}C(x_0,x_0)=h(x_0,x_0)G+\operatorname{int}D$ and $0\notin\operatorname{int}C(x_0,x_0)$ as $h(x_0,x_0)=0$ or $G\cap(-\operatorname{int}D)=\emptyset$. Moreover, $f-f(x_0)+C(\cdot,x_0)$ is generalized D-subconvexlike on the feasible set S, since $f-f(x_0)+h(\cdot,x_0)G$ is generalized D-subconvexlike on S.

Therefore, Theorem 5.3 can be applied and so we see that $x_0 \in \mathrm{WQE}(f,S,hG+D)$ if and only if there exists $\lambda \in (\operatorname{cone} C(x_0,x_0)^+ \cap D^+)\backslash\{0\}$ such that

$$\langle \lambda, f(x)\rangle + \inf_{z\in h(x,x_0)G+D} \langle \lambda, z\rangle \geq \langle \lambda, f(x_0)\rangle, \quad \forall x \in S.$$

This statement coincides with (5.17) since $\inf_{z\in h(x,x_0)G+D} \langle \lambda, z\rangle = k_{h,G}(\lambda,x)$ and $C(x_0,x_0)^+ \cap D^+ = \operatorname{cone} G^+ \cap D^+$ if $h(x_0,x_0) > 0$, and $C(x_0,x_0)^+ \cap D^+ = D^+$ otherwise, which finishes the proof of part (i).

Part (ii) is obtained by applying Theorem 5.4 to $C = hG+D$, and the proof is finished. □

Remark 5.3. (i) Clearly, inequality (5.17) implies that $\lambda \in \operatorname{cone} G^+$ whenever $S\backslash\{x_0\} \neq \emptyset$ and G is coradiant. Therefore, in this case, $k_{h,G}(\lambda,x) = h(x,x_0)\tau_G(\lambda)$, for all $x \in S$. If, in addition, G is free-disposal with respect to D, then $\operatorname{cone} G^+ \subset D^+$ and so $\operatorname{cone} G^+ \cap D^+ = \operatorname{cone} G^+$.

(ii) The necessary condition of Corollary 5.3(i) improves [20, Theorem 4.1] and [26, Theorem 5(a)] since set G is not required to be coradiant (see part (i) above to make the comparison). Moreover, the convexity assumption involves an equivalent formulation of the nearly cone convexity, which is easier to check in some cases.

In addition, notice that [26, Theorem 5(b)] is superfluous as it coincides with [26, Theorem 5(a)] by replacing D with $\operatorname{cone} C$. Indeed, as C is assumed to be coradiant it follows that $C + \operatorname{cone} C = C$ and so [26, Theorem 5(b)] is the same as [26, Theorem 5(a)] by considering $D = \operatorname{cone} C$.

The same remarks as in the above paragraph can be observed regarding Corollary 5.3(i), [20, Corollary 4.1] and [26, Corollary 1], and part (b) in [26, Corollary 1], which also coincides with [26, Corollary 1(a)].

(iii) Corollary 5.3(ii) can be compared with [26, Theorem 6 and Corollary 2]. Indeed, on the one hand, Definition 4.2 reduces to [26, Definition 3] by the set-valued mapping $C = hG+D$ (recall that set G is referred to C in [26, Definition 3], see Remark 4.5(iv)). On the other hand, by a careful reading of the proof of [26, Theorem 6], one can see that it works only if condition $h(x_0,x_0)G \cap (-\operatorname{int} K) = \emptyset$ is required, for all $K \in \mathscr{F}(D)$. Therefore, Corollary 5.3(ii) improves [26, Theorem 6 and Corollary 2] as set G is not required to be coradiant.

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