Duplication in OWA-generated Positional Aggregation Rules

José Luis García-Lapresta and Miguel Martínez-Panero

Abstract In this paper we deal with positional aggregation rules where the alternatives are socially ordered according to their aggregated positions. These positional values are generated by means of a predetermined aggregation function from the positions in the corresponding individual orderings. Specifically, our interest is focused on OWA-generated positional aggregation rules and, as a first step in our research, we characterize those ones satisfying duplication and propose an overall social order induced by them.

1 Introduction

According to Gärdenfors [13], "positionalist voting functions are those social choice functions where the positions of the alternatives in the agents' preference orders crucially influence the social ordering of the alternatives". This is a vague notion that can be understood in different ways[1]. The most popular case of positional aggregation rules are scoring rules[2], where a score is associated with each position and alternatives are socially ordered by the sum of scores obtained from the individual orderings.

However, scoring rules are not exclusive to capture positionalist features of voting. In fact, our proposal based on aggregation functions (mainly through OWAs) sheds light to some aspects not taken into account in the scoring approach. One of these interesting properties, not satisfied by the scoring rules, is the duplication principle. This property, appearing naturally in several contexts, entails a sort of irrelevance of clone voters in the final result and might not seem suitable at all in voting

PRESAD Research Group, IMUVA, Dep. of Applied Economics, University of Valladolid, Spain, e-mail: {lapresta, panero}@eco.uva.es

[1] See Pattanaik [21], specially Section 3.

[2] See Chebotarev and Shamis [7] for a referenced survey on scoring rules and their characterizations.

scenarios. But it will be shown that it is related to some concrete OWA operators inducing positional voting rules and intended to be used under complete ignorance.

The paper is organized as follows. In Section 2 we introduce the basic notation for the preferences of the agents over the alternatives and their related positions. Section 3 is devoted to aggregation rules and aggregation functions; specifically, we focus our attention on OWAs and show their connections with some well-known voting systems appearing in the literature. The need of taking into account a variable electorate leads us to use extended OWAs (EOWAs) and, with this background, in Section 4 we define duplication and then we characterize those OWA-generated positional aggregation rules satisfying this property. An illustrative example is also presented and, finally, a proposal of an overall social order based on the characterized rules is obtained in a unifying way.

2 Preliminaries

Consider a set of agents $V = \{1,\ldots,m\}$, with $m \in \mathbb{N}$, who show their preferences on a set of alternatives $X = \{x_1,\ldots,x_n\}$, with $n \geq 2$. With $L(X)$ we denote the set of *linear orders* on X, and with $W(X)$ the set of *weak orders* (or *complete preorders*) on X. Given $R \in W(X)$, with $\succ$ and $\sim$ we denote the asymmetric and the symmetric parts of R, respectively. A *profile* is a vector $\boldsymbol{R} = (R_1,\ldots,R_m)$ of weak orders, where R_v contains the preferences of the agent v, with $v = 1,\ldots,m$. Vectors in $\mathbb{R}^n$ are denoted as $\boldsymbol{a} = (a_1,\ldots,a_n)$. Given $\boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^n$, with $\boldsymbol{a} \leq \boldsymbol{b}$ we mean $a_i \leq b_i$ for every $i \in \{1,\ldots,n\}$.

Definition 1. Given $R \in W(X)$, the *position* of alternative $x_i \in X$ is defined as

$$p(x_i) = n - \#\left\{x_j \in X \mid x_i \succ x_j\right\} - \frac{1}{2}\#\left\{x_j \in X \setminus \{x_i\} \mid x_j \sim x_i\right\}. \tag{1}$$

It is equivalent to linearize the weak order and to assign each alternative the average of the positions of the alternatives within the same equivalence class (see, for instance, Smith [23] for a similar procedure in the context of scoring rules).

Example 1. Consider $R \in W(\{x_1,\ldots,x_7\})$ given by

R
$x_2\ x_3\ x_5$
x_1
$x_4\ x_7$
x_6

Then,

$$p(x_2) = p(x_3) = p(x_5) = \frac{1+2+3}{3} = 2 = 7 - 4 - \frac{1}{2}2,$$

$$p(x_1) = 4 = 7 - 3 - \frac{1}{2}0,$$

$$p(x_4) = p(x_7) = \frac{5+6}{2} = 5.5 = 7 - 1 - \frac{1}{2}1,$$

$$p(x_6) = 7 = 7 - 0 - \frac{1}{2}0.$$

Consequently, R is codified by the *positions vector*

$$(p(x_1), p(x_2), p(x_3), p(x_4), p(x_5), p(x_6), p(x_7)) = (4, 2, 2, 5.5, 2, 7, 5.5).$$

Taking into account the positions of the alternatives, every profile $\boldsymbol{R} \in W(X)^m$ has associated a *position matrix* containing the positions of the alternatives for all the agents

$$\begin{pmatrix} p_1(x_1) & p_1(x_2) & \cdots & p_1(x_n) \\ p_2(x_1) & p_2(x_2) & \cdots & p_2(x_n) \\ \cdots & \cdots & \cdots & \cdots \\ p_m(x_1) & p_m(x_2) & \cdots & p_m(x_n) \end{pmatrix},$$

where $p_v(x_i)$ is the position of x_i for agent v. Thus, row v contains the positions of the alternatives according to agent v, and column i contains the positions of the alternative x_i.

3 The aggregation process

Given a domain $D \subseteq W(X)^m$ with $m \in \mathbb{N}$, an *aggregation rule on D* is a mapping $F : D \longrightarrow W(X)$ that satisfies the following conditions:

1. *Anonymity*: For every permutation π on $\{1, \ldots, m\}$ and every profile $\boldsymbol{R} \in D$, it holds
$$F\left(R_{\pi(1)}, \ldots, R_{\pi(m)}\right) = F\left(R_1, \ldots, R_m\right).$$
2. *Neutrality*: For every permutation σ on $\{1, \ldots, n\}$ and every profile $\boldsymbol{R} \in D$, it holds
$$F\left(R_1^\sigma, \ldots, R_m^\sigma\right) = \left(F\left(R_1, \ldots, R_m\right)\right)^\sigma,$$
where R_v^σ and $\left(F\left(R_1, \ldots, R_m\right)\right)^\sigma$ are the orders obtained from R_v and $F(R_1, \ldots, R_m)$, respectively, by relabeling the alternatives according to σ, i.e., $x_{\sigma(i)} \, R_v^\sigma \, x_{\sigma(j)} \Leftrightarrow x_i \, R_v \, x_j$ and $x_{\sigma(i)} \left(F\left(R_1, \ldots, R_m\right)\right)^\sigma x_{\sigma(j)} \Leftrightarrow x_i \, F\left(R_1, \ldots, R_m\right) x_j$.
3. *Unanimity*: For every profile $\boldsymbol{R} \in D$ and all $x_i, x_j \in X$, it holds
$$\left(\forall v \in V \;\; x_i \, R_v \, x_j\right) \Rightarrow x_i \, F(\boldsymbol{R}) \, x_j.$$

Anonymity means a symmetric consideration for the agents; neutrality means a symmetric consideration for the alternatives; and unanimity means that if all the individuals consider an alternative as good as another one, then the social preference coincides with the individual preferences on this issue.

It is worth mentioning that the setting of aggregation rules, where the outcome is a social order (as in Smith [23]), is not the unique framework in Social Choice Theory. Other possibilities can be taken into account, such as social choice correspondences, where the result is the (nonempty) subset of the best alternatives (as in Young [29, 30]; see also Laslier [16] for further rank-based and pairwise-based approaches), or even social choice functions, where a single alternative is assigned to each profile[3].

3.1 Aggregation functions

In our proposal, we have extended the notion of aggregation function to the unbounded interval $[1,\infty)$. On aggregation functions in the standard unit interval, see Calvo *et al.* [5], Beliakov *et al.* [4] and Grabisch *et al.* [14].

Definition 2. $A : [1,\infty)^m \longrightarrow [1,\infty)$ is an *aggregation function* if it satisfies the following conditions:

1. *Boundary condition*: $A(1,\ldots,1) = 1$.
2. *Monotonicity*: $\boldsymbol{a} \leq \boldsymbol{b} \;\Rightarrow\; A(\boldsymbol{a}) \leq A(\boldsymbol{b})$, for all $\boldsymbol{a},\boldsymbol{b} \in [1,\infty)^m$.

If, additionally, A satisfies *idempotency*, i.e., $A(a,\ldots,a) = a$ for every $a \in [1,\infty)$, then A is called *averaging aggregation function*.

It is easy to see that averaging aggregation functions satisfy *compensativeness*:

$$\min\{a_1,\ldots,a_m\} \leq A(a_1,\ldots,a_m) \leq \max\{a_1,\ldots,a_m\},$$

for every $(a_1,\ldots,a_m) \in [1,\infty)^m$. Typical averaging aggregation functions are the arithmetic mean, trimmed means, the median, the maximum, the minimum, etc. In fact, we can gather all these aggregation functions as particular cases of OWA operators[4].

A *weighting vector* of dimension m is a vector $\boldsymbol{w} = (w_1,\ldots,w_m) \in [0,1]^m$ such that $\displaystyle\sum_{i=1}^{m} w_i = 1$.

Definition 3. Given a weighting vector $\boldsymbol{w}$ of dimension m, the *OWA operator associated with* $\boldsymbol{w}$ is the mapping $A_{\boldsymbol{w}} : [1,\infty)^m \longrightarrow [1,\infty)$ defined by

[3] As pointed out by Courtin *et al.* [9], differences in the axiomatic treatment arise depending on the type of social mechanism considered.

[4] The initials in OWA stand for *ordered weighted averaging*, see Yager [25], Yager and Kacprzyk [27] and Yager *et al.* [28]. A characterization of the OWA operators has been given by Fodor *et al.* [10].

$$A_{\boldsymbol{w}}(a_1,\ldots,a_m)=\sum_{i=1}^{m} w_i \cdot a_{[i]},$$

where $a_{[i]}$ is the i-th greatest number of $\{a_1,\ldots,a_m\}$.

As noted before, some well-known aggregation functions are specific cases of OWA operators.

With appropriate weighting vectors $\boldsymbol{w}=(w_1,\ldots,w_m)$ we obtain

1. The *maximum*, for $\boldsymbol{w}=(1,0,\ldots,0)$.
2. The *minimum*, for $\boldsymbol{w}=(0,\ldots,0,1)$.
3. The *arithmetic mean*, for $\boldsymbol{w}=\left(\frac{1}{m},\ldots,\frac{1}{m}\right)$.
4. The *k-trimmed means*:
   - If $k=1$, $\boldsymbol{w}=\left(0,\frac{1}{m-2},\ldots,\frac{1}{m-2},0\right)$.
   - If $k=2$, $\boldsymbol{w}=\left(0,0,\frac{1}{m-4},\ldots,\frac{1}{m-4},0,0\right)$.
   - $\ldots$.
5. The *median*:

   a. If m is odd, $w_i=\begin{cases}1, \text{ if } i=\frac{m+1}{2},\\ 0, \text{ otherwise.}\end{cases}$

   b. If m is even, $w_i=\begin{cases}\frac{1}{2}, \text{ if } i\in\{\frac{m}{2},\frac{m}{2}+1\},\\ 0, \text{ otherwise.}\end{cases}$
6. The *mid-range*, for $\boldsymbol{w}=(0.5,0,\ldots,0,0.5)$.

3.2 Positional aggregation rules

Definition 4. Given an aggregation function $A:[1,\infty)^m \longrightarrow [1,\infty)$ and a profile $\boldsymbol{R}\in W(X)^m$, the *aggregated position* of the alternative $x_i\in X$ is defined as

$$p_A(x_i)=A(p_1(x_i),\ldots,p_m(x_i)),$$

where $p_v(x_i)$ is the position of x_i for agent $v\in V$.

Definition 5. Given an aggregation function $A:[1,\infty)^m \longrightarrow [1,\infty)$, the *positional aggregation rule associated with* A is the mapping $F_A: W(X)^m \longrightarrow W(X)$ defined by $F_A(\boldsymbol{R}) = \succcurlyeq_A$, where

$$x_i \succcurlyeq_A x_j \;\Leftrightarrow\; p_A(x_i)\le p_A(x_j).$$

For example, taking into account some of the OWA operators introduced above, we obtain positional aggregation rules which are connected to (or even replicate) well-known procedures appearing in the literature:

- The arithmetic mean as aggregation operator induces the Borda rule. And it is worth mentioning that the arithmetic mean is also the basis for the *Range Voting* method (Smith [24]), in a decisional context where the alternatives receive numerical assessments one by one.
- The median instead of the arithmetic mean, and linguistic terms instead of numerical values, are used in the *Majority Judgment* voting system supported by Balinski and Laraki [2]. An extension of this procedure using centered OWA operators (Yager [26]) appears in García-Lapresta and Martínez-Panero [12]. Again, in a different scenario, Basset and Persky [3] already proposed to select the alternative with best median evaluation (see also Laslier [18]).
- The maximum leads to a voting system in which each alternative is evaluated according to the worst reached position. Those with the best assigned value are then elected. Such a *maximin* voting system, which advocates the maximin principle of normative economics[5], is characterized by Congar and Merlin [8] and the same idea is also the key for the *leximin* voting system appearing in Laslier [17], although in a different decisional framework (see also Laslier [18]). This is also the case for the Simpson-Kramer method (see Levin and Nalebuff [19]) in a pairwise comparison context. Furthermore, the procedure obtained through the maximum as aggregation operator is also related to the *Coombs* method (where the alternative with the largest number of last positions is sequentially withdrawn), as well as to the antiplurality rule (see Baharad and Nitzan [1]).
- The minimum entails a voting system called *maximax*[6] by Congar and Merlin [8], also characterized by them. Its conception is similar to that of the *Hare* system, also known as *Alternative Vote* (where the alternative with the fewest first positions is sequentially withdrawn). It is also related to the most used (and criticized) system: plurality rule (see Laslier [17]).
- The mid-range OWA operator is related to the basic *best-worst* voting system (see García-Lapresta *et al.* [11]).

It is easy to check the following result.

Proposition 1. *F_A is an aggregation rule for every aggregation function A.*

In order to take into account a variable electorate (for example, to deal with the clonation or appearance of new agents), we introduce some extended notions of those already defined throughout the paper.

Definition 6. An *extended aggregation rule* is a mapping

$$\widetilde{F} : \bigcup_{m \in \mathbb{N}} W(X)^m \longrightarrow W(X)$$

such that $F_m = \widetilde{F}|_{W(X)^m}$ is an aggregation rule for each $m \in \mathbb{N}$ and $F_1(R) = R$.

[5] Rawls [22, p. 328]: "the basic structure is perfectly just when the prospects of the least fortunate are as great as they can be".

[6] The apparent discordance leading the maximum to the maximin voting system, as well as the minimum to the maximax, relies on our positional approach where, contrary to the scoring context, the smallest value is associated with the best position.

Definition 7. An *extended OWA operator (EOWA)* is a family of OWA operators with associated weighting vectors $\boldsymbol{w}^m = (w_1^m, \ldots, w_m^m)$, one for each dimension $m = 1, 2, 3, \ldots$

Following Calvo and Mayor [6] and Mayor and Calvo [20] (see also Beliakov *et al.* [4, pp. 54-56]), we can show graphically an EOWA operator as a weighting triangle

$$\begin{array}{ccccccccc}
 & & & & w_1^1 & & & & \\
 & & & w_1^2 & & w_2^2 & & & \\
 & & w_1^3 & & w_2^3 & & w_3^3 & & \\
 & w_1^4 & & w_2^4 & & w_3^4 & & w_4^4 & \\
w_1^5 & & w_2^5 & & w_3^5 & & w_4^5 & & w_5^5 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

For simplicity, from now on superindexes will be avoided when confusion is not possible.

4 Duplication

Here we introduce a property which, broadly speaking, states that new voters replicating the same preferences of already existing voters will not affect the outcome. This (at first sight) non-compelling property appears as *duplication* in Congar and Merlin [8], where they characterize the maximin procedure.

Definition 8. An extended aggregation rule $\widetilde{F}$ satisfies *duplication* if

$$F_{m+1}(\boldsymbol{R}, R_i) = F_m(\boldsymbol{R})$$

for every profile $\boldsymbol{R} = (R_1, \ldots, R_i, \ldots, R_m) \in W(X)^m$ and every $i \in \{1, \ldots, m\}$.

4.1 A characterization result

It is interesting to find those procedures satisfying duplication, and the following result shows the answer for aggregation rules associated with EOWAs.

Proposition 2. *Given an EOWA operator A, the extended aggregation rule $\widetilde{F}_A$ satisfies duplication if and only if A is a rational convex combination of the maximum and the minimum EOWA operators, i.e., $\mathbf{w} = \alpha(1, \ldots, 0) + (1 - \alpha)(0, \ldots, 1)$ for some $\alpha \in [0, 1] \cap \mathbb{Q}$.*

Proof. $\Leftarrow$) It is straightforward that aggregation rules associated with $\boldsymbol{w} = (1,0,\ldots,0)$ (i.e., maximin), $\boldsymbol{w} = (0,0,\ldots,1)$ (i.e., maximax), and convex combinations of them, $w = (\alpha,0,\ldots,0,1-\alpha)$ with $0 \leq \alpha \leq 1$, satisfy duplication.

$\Rightarrow$) We first prove that if duplication holds, all intermediate weights $w_2,\ldots,w_{m-1}$ should be zero. Our reasoning will deal with a profile consisting in all circular permutations of three ordered alternatives, but it is extensible to any order[7]. Thus, consider the profile

R_1	R_2	R_3
x_1	x_2	x_3
x_2	x_3	x_1
x_3	x_1	x_2

where the associated position matrix is

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}.$$

As each alternative occupies each position exactly once, a global tie arises and the aggregated position for each is $p_A(x_i) = 3w_1^3 + 2w_2^3 + w_3^3,\ i = 1,2,3,$ so that $x_1 \sim x_2 \sim x_3$, being A the aggregation rule corresponding to any EOWA with $\boldsymbol{w}^3 = (w_1^3, w_2^3, w_3^3)$.

Now suppose that agent 1 is replicated, becoming the new situation

R_1	R_2	R_3	$R_4 = R_1$
x_1	x_2	x_3	x_1
x_2	x_3	x_1	x_2
x_3	x_1	x_2	x_3

where the new associated position matrix is

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

Then, the aggregated positions for each alternative are

$$\begin{aligned} p_A(x_1) &= 3w_1^4 + 2w_2^4 + w_3^4 + w_4^4, \\ p_A(x_2) &= 3w_1^4 + 2w_2^4 + 2w_3^4 + w_4^4, \\ p_A(x_3) &= 3w_1^4 + 3w_2^4 + 2w_3^4 + w_4^4. \end{aligned}$$

Taking into account duplication, the tie among all three alternatives holds; hence

[7] These circular permutations yield a *Condorcet cycle*.

$$x_1 \sim_A x_2 \Leftrightarrow w_3^4 = 0,$$
$$x_1 \sim_A x_3 \Leftrightarrow w_2^4 + w_3^4 = 0,$$
$$x_1 \sim_A x_3 \Leftrightarrow w_2^4 = 0.$$

Then, $w_2^4 = w_3^4 = 0$. Once proven that central weights are null (this fact will be taken into account in what follows), what remains is to show that lateral weights should the same at any level, i.e., $w_1^m = \alpha$ and $w_m^m = 1-\alpha$, for all $m \geq 2$. To do this, consider $\alpha = \frac{p}{q}$ with $p, q \in \mathbb{N}$ and $p < q$, expressed as an irreductible fraction, and any profile with m agents and $q+1$ alternatives where the alternative x_1 is at least the best for one agent and the worst for another one, while x_2 occupies the position $p+1$ for all of them. A sketch of such *ad hoc* profile would be

position	R_1	$\dots$	R_i	$\dots$	R_j	$\dots$	R_m
1	$\dots$	$\dots$	x_1	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$p+1$	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$q+1$	$\dots$	$\dots$	$\dots$	$\dots$	x_1	$\dots$	$\dots$

The aggregated positions for the selected alternatives would be

$$p_A(x_1) = \frac{p}{q}(q+1) + \left(1 - \frac{p}{q}\right) = p+1,$$

$$p_A(x_2) = \frac{p}{q}(p+1) + \left(1 - \frac{p}{q}\right)(p+1) = p+1,$$

so that $x_1 \sim_A x_2$, being A the aggregation rule corresponding to any EOWA with such weights.

But now, if we replicate any subset of agents becoming the new weights $\beta \neq \alpha$ and hence $1-\beta \neq 1-\alpha$, then the new aggregated positions would be

$$p_A(x_1) = \beta(q+1) + (1-\beta) \neq p+1,$$

$$p_A(x_2) = \beta(p+1) + (1-\beta)(p+1) = p+1,$$

so that $x_1 \sim_A x_2$ does not hold. Hence, if lateral weights change from one dimension to another, duplication fails. $\square$

In conclusion, under duplication we obtain the class of weighting triangles

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & \alpha & & 1-\alpha & & \\ & \alpha & & 0 & & 1-\alpha & \\ \alpha & & 0 & & 0 & & 1-\alpha \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{array}$$

As particular cases we have:
$\alpha = 1$: maximum (maximin procedure),
$\alpha = 0$: minimum (maximax procedure),
$\alpha = 0.5$: mid-range.

It is worth mentioning that duplication is related to the *Hurwicz criterion* [15] used in decision making under complete uncertainty, where the value of a decision is a convex combination of its lowest possible expected value (pessimistic assessment) and of its highest one (optimistic assessment).

4.2 An illustrative example

Consider the profile

R_1	R_2	R_3
x_2	$x_2\ x_3$	$x_1\ x_3$
x_3	x_1	x_2
x_1		

where the associated position matrix is

$$\begin{pmatrix} 3 & 1 & 2 \\ 3 & 1.5 & 1.5 \\ 1.5 & 3 & 1.5 \end{pmatrix}.$$

If we choose an OWA $A_{\boldsymbol{w}(\alpha)}$ associated with weights $\boldsymbol{w}(\alpha) = (\alpha, 0, 1-\alpha)$, the corresponding social positions for the alternatives would be:

$$p_{A_{\boldsymbol{w}(\alpha)}}(x_1) = 3\alpha + 1.5(1-\alpha) = 1.5\alpha + 1.5,$$

$$p_{A_{\boldsymbol{w}(\alpha)}}(x_2) = 3\alpha + 1(1-\alpha) = 2\alpha + 1,$$

$$p_{A_{\boldsymbol{w}(\alpha)}}(x_3) = 2\alpha + 1.5(1-\alpha) = 0.5\alpha + 1.5.$$

According to the possible values of α, the corresponding social orders are shown in the following table:

$\alpha = 0$	$0 < \alpha < \frac{1}{3}$	$\alpha = \frac{1}{3}$	$\frac{1}{3} < \alpha < 1$	$\alpha = 1$
x_2	x_2	$x_2\ \ x_3$	x_3	x_3
$x_1\ \ x_3$	x_3	x_1	x_2	$x_1\ \ x_2$
	x_1		x_1	

As one could expect, different social orders appear depending on α. In the following section we propose a integrating method to obtain a unified result for each alternative taking into account the different outcomes when α ranges from 0 to 1.

4.3 Overall positions and social order

For the general case with n alternatives and using in a first stage the positional voting rule associated with the OWA $\mathbf{w}(\alpha) = (\alpha, 0, \ldots, 0, 1-\alpha)$, it is possible to assign the corresponding social position $p_{A_{\mathbf{w}(\alpha)}}(x_i)$ to the alternative x_i. Thus, we can introduce the function $\mu_i : [0,1] \longrightarrow \mathbb{R}$ given by $\mu_i(\alpha) = p_{A_{\mathbf{w}(\alpha)}}(x_i)$. Such function is always piecewise constant, and hence Riemann integrable. This fact allows us to define the *overall position of* x_i as

$$p(x_i) = \int_0^1 \mu_i(\alpha)d\alpha.$$

Easy computations lead to the following results in the previous example:

$$p(x_1) = \int_0^1 \mu_1(\alpha)d\alpha = 3,$$

$$p(x_2) = \int_0^1 \mu_2(\alpha)d\alpha = 5/3,$$

$$p(x_3) = \int_0^1 \mu_3(\alpha)d\alpha = 4/3.$$

Thus, the overall social order is $x_3 \succ x_2 \succ x_1$.

In conclusion, for each $\alpha \in [0,1]$ the corresponding positional aggregation rule associated with $p_{A_{\mathbf{w}(\alpha)}}$ only takes into account the best and worst positions for each alternative, yielding different social orders in each case. However, the possible criticism on the influence of the choice of α in the result can be mitigated under this overall approach, where a social order is obtained not corresponding with any specific α, but amalgamating all allowable values for this parameter.

Acknowledgments

The authors would like to thank Jorge Alcalde-Unzu, Jean-François Laslier, Vincent Merlin and Marina Núñez for their comments and suggestions, as well as the Spanish Ministerio de Ciencia e Innovación (Project ECO2012-32178) and ERDF for the funding support.

References

1. Baharad, E., Nitzan, S. (2005): "The inverse plurality rule – an axiomatization". *Social Choice and Welfare* 25, pp. 173-178.
2. Balinski, M., Laraki, R. (2011): *Majority Judgment: Measuring Ranking and Electing*. MIT Press, Cambridge MA.

3. Bassett, G.W., Persky, J. (1999): "Robust voting". *Public Choice* 99, pp. 299-310.
4. Beliakov, G., Pradera, A., Calvo, T. (2007): *Aggregation Functions: A Guide for Practitioners*, Springer, Heidelberg.
5. Calvo, T., Kolesárova, A., Komorníková, M., Mesiar, R. (2002): "Aggregation operators: Properties, classes and construction methods", in: T. Calvo, G. Mayor, R. Mesiar (Eds.), *Aggregation Operators: New Trends and Applications*, Physica-Verlag, Heidelberg, pp. 3–104.
6. Calvo, T., Mayor, G. (1999): Remarks on two types of extended aggregation functions. *Tatra Mountains Mathematical Publications* 16, pp. 235-253.
7. Chebotarev, P.Y., Shamis, E. (1998): "Characterizations of scoring methods for preference aggregation". *Annals of Operations Research* 80, pp. 299-332.
8. Congar, R., Merlin, V. (2012): "A characterization of the maximin rule in the context of voting". *Theory and Decision* 72, pp. 131-147.
9. Courtin, S., Mbih, B., Moyouwou, I. (2012): "Are Condorcet procedures so bad according to the reinforcement axiom?". *Thema Working Paper* 2012-37, Université de Cergy Pontoise.
10. Fodor, J., Marichal, J.L., Roubens, M. (1995): "Characterization of the Ordered Weighted Averaging Operators". *IEEE Transtactions on Fuzzy Systems* 3, pp. 236-240.
11. García-Lapresta, J.L., Marley, A.A.J., Martínez-Panero, M. (2010): "Characterizing best-worst voting systems in the scoring context". *Social Choice and Welfare* 34, pp. 487-496.
12. García-Lapresta, J.L., Martínez-Panero, M. (2009): "Linguistic-based voting through centered OWA operators". *Fuzzy Optimization and Decision Making* 8, pp. 381-393.
13. Gärdenfors, P. (1973): "Positionalist voting functions". Theory and Decision 4, pp. 1-24.
14. Grabisch, M., Marichal, J.L., Mesiar, R., Pap, E. (2009): *Aggregation Functions*. Cambridge University Press, Cambridge.
15. Hurwicz, L. (1951): "A class of criteria for decision-making under ignorance", *Cowles Commission Discussion Paper: Statistics* 356, 1951.
16. Laslier, J.F. (1996): "Rank-based choice correspondences". *Economics Letters* 52, pp. 279-286.
17. Laslier, J.F. (2012): "And the loser is … plurality voting", in: D.S. Felsenthal, M. Machover (Eds.), *Electoral Systems: Paradoxes, Assumptions, and Procedures*, Springer-Verlag, Berlin, pp. 327-351.
18. Laslier, J.F. (2012): "On choosing the alternative with the best median evaluation". *Public Choice* 153, pp. 269-277.
19. Levin, J., Nalebuff, B. (1995): "An introduction to vote-counting schemes". *Journal of Economic Perspectives* 9, pp. 3-26.
20. Mayor, G., Calvo, T. (1997): "On extended aggregation functions". *Proceedings of IFSA 97*, vol. I, Prague, pp. 281-285.
21. Pattanaik, P.K. (2002): "Positional rules of collective decision-making", in K.J. Arrow, A.K. Sen, K. Suzumura (Eds.), *Handbook of Social Choice and Welfare, Volume 1*, Elsevier, Amsterdam, pp. 361-394.
22. Rawls, J. (1973): "Distributive justice", in: E. Phelps (Ed.), *Economic Justice: Selected Readings*, Penguin Education, Harmondsworth, pp. 319-362.
23. Smith, J.H. (1973): "Aggregation of preferences with variable electorate". *Econometrica* 41, pp. 1027-1041.
24. Smith, W.D.: "Range Voting website", in `http://rangevoting.org/`
25. Yager, R.R. (1988): "On ordered weighted averaging operators in multicriteria decision making". *IEEE Transactions on Systems Man and Cybernetics* 8, pp. 183-190.
26. Yager, R.R. (2007): Centered OWA operators. *Soft Computing* 11, pp. 631-639.
27. Yager, R.R., Kacprzyk, J. (eds.) (1997): *The Ordered Weighted Averaging Operators: Theory and Applications*. Kluwer Academic Publishers, Boston.
28. Yager, R.R., Kacprzyk, J., Beliakov, G. (eds.) (2011): *Recent Developments in the Ordered Weighted Averaging Operators: Theory and Practice*. Springer-Verlag, Berlin.
29. Young, H.P. (1974): "An axiomatization of Borda's rule". *Journal of Economic Theory* 9, pp. 43-52.
30. Young, H.P. (1975): "Social choice scoring functions". *SIAM Journal on Applied Mathematics* 28, pp. 824-838.