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Inverses of k-Toeplitz matrices with applications to resonator arrays with multiple receivers[☆]

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ABSTRACT

We find closed-form algebraic formulas for the elements of the inverses of tridiagonal 2- and 3-Toeplitz matrices which are symmetric and have constant upper and lower diagonals. These matrices appear, respectively, as the impedance matrices of resonator arrays in which a receiver is placed over every 2 or 3 resonators. Consequently, our formulas allow to compute the currents of a wireless power transfer system in closed form, allowing for a simple, exact and symbolic analysis thereof. Small numbers are chosen for illustrative purposes, but the elementary linear algebra techniques used can be extended to k-Toeplitz matrices of this special form with k arbitrary, hence resonator arrays with a receiver placed over every k resonators can be analysed in the same way.

1. Introduction

This work concerns the theory and applications of some special tridiagonal matrices, known in the literature as tridiagonal k-Toeplitz matrices. Those are tridiagonal matrices of order n, say, where the entries along the main diagonal and its adjacent diagonals are periodic sequences of period k, so that they have the form

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$$\begin{pmatrix} a_1 & b_1 & & & & & & & \\ c_1 & \ddots & \ddots & & & & & & \\ & \ddots & a_k & b_k & & & & & \\ & & c_k & a_1 & b_1 & & & & \\ & & & c_1 & \ddots & \ddots & & & \\ & & & & \ddots & a_k & b_k & & \\ & & & & & c_k & a_1 & b_1 & \\ & & & & & & c_1 & \ddots & \ddots \\ & & & & & & & \ddots & \end{pmatrix},$$

where the entries are real or complex numbers, with $b_j c_j \neq 0$ for $j = 1, \ldots, k$. These matrices have proved to be a very useful tool in many contexts of pure and applied mathematics, e.g., in partial differential equations (appearing in the discretization of elliptic or parabolic partial differential equations by finite difference methods), in chain models of quantum physics ([7]), and in sound propagation theory ([9,10]). Gover in [13] solved the eigenproblem associated with such matrices for the special case $k = 2$. Gover's results were recasted by Marcellán and Petronilho in [14] using tools from the theory of orthogonal polynomials. Later, in [15] these authors solved the associated eigenproblem for the case $k = 3$, using again tools from orthogonal polynomials theory and polynomial mappings (see also the work [7] by Álvarez-Nodarse et al.). The eigenproblem of a general tridiagonal k-Toeplitz matrix was solved by da Fonseca and Petronilho ([12]), motivated by the need of finding explicit formulas for the entries of the inverses of such matrices (whenever they are nonsingular), the special case $k = 3$ having been considered previously by these same authors in [11]. Recently these explicit formulas for the entries of the inverses have proved to be very useful in real world problems involving circuit models (e.g. [6]). Such formulas for the entries were obtained in [11,12] as expressions involving polynomial mappings and Chebyshev polynomials of the second kind, a fact that (despite the beauty of such formulas) may be regarded as an additional difficulty in their applications, especially for those which are not so familiar with the theory of orthogonal polynomials.

Our aim in this contribution is twofold. On the one hand we will determine, without using the theory of orthogonal polynomials, explicit algebraic expressions for the entries of the inverses of symmetric 2- and 3-Toeplitz matrices which have constant and equal upper and lower diagonals ($b_1 = \cdots = b_k = c_1 = \cdots = c_k$). To do so we will only resort to elementary linear algebra: we will compute the determinants of such matrices by linear recurrence relations, and then apply those determinants to compute the minors appearing in the cofactor matrix, which directly relate to the elements of the inverse. It is clear that the methods found here can be applied to k-Toeplitz matrices with constant and equal upper and lower diagonals, for arbitrary k.

On the other hand, we will apply these results to achieve closed formulas for wireless power transfer (WPT) systems using resonator arrays with multiple receivers. WPT systems have been going through intensive research lately, as they allow one to avoid electrical contact and transfer power in rough environments with water, dust or dirt. Nowadays they are being used in several applications as electrical vehicle charging ([1]), mobile devices charging ([18]) and powering biomedical devices ([22]). However, they have the drawback that, in case of misalignment or distance from the transmitter to the receiver, the efficiency and power transmitted can drop abruptly. So, in order to overcome this inconvenience, arrays of resonators can be used to transfer power over longer distances ([16,17,23]). In these arrays the first resonator is usually connected to a power source and transmits power through magnetic coupling to the other resonators of the array, which are arranged in a plane with parallel axes, and a receiver is placed over the array to absorb the power transmitted ([2–5,17,20]). In the literature, these arrays have been examined mostly using magnetoinductive wave theory ([17,20]) or through the circuit analysis of the array ([3,23]), in which the array is represented by an impedance matrix which contains the impedance of each resonator and the mutual inductances between pairs of resonators ([2,17,19,23,24]). In [5,6] the inversion of the impedance matrix is performed using generic tridiagonal matrices. In this way, it is possible to determine closed-form expressions for equivalent impedance, the power transmitted and the efficiency of these systems. However all these works consider only one receiver placed over the array. Instead, the array could possibly transmit power to several receivers at the same time. In this paper we study and give closed-form algebraic formulas for the currents, power transmission and efficiency in an array powering multiple receivers placed over every two or three resonators. These small numbers have been chosen for the sake of simplicity of the exposition, but the same methods work equally well for arrays with receivers placed every k resonators, with k arbitrary.

2. Description of the circuit

In this paper we consider an array with N identical resonators and some identical receivers placed over them. If the lth resonator has a receiver over it (Fig. 1(A,B)) then an impedance $\hat{Z}_d$ is added, which is the impedance of the receiver as seen from the resonator ([17,20]). The last resonator (Nth) is connected to a termination impedance $\hat{Z}_T$. The first resonator is connected to a voltage source $\hat{V}_s$, which we consider to generate an ideal sinusoidal voltage, as has been done in other WPT

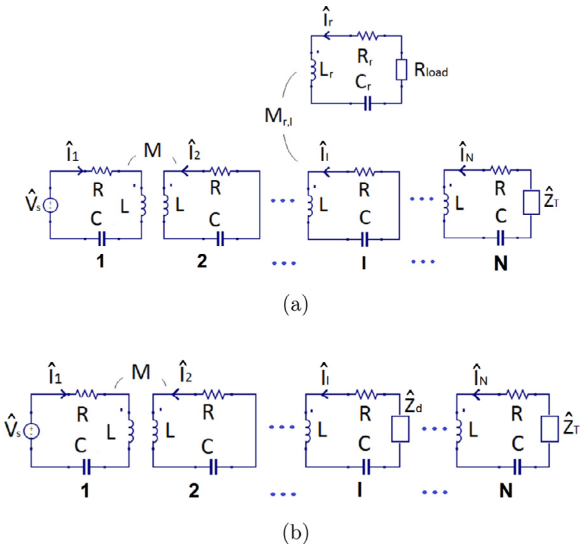


Fig. 1. Circuit representations of a possible configuration of the considered resonator array: (A) Receiver over the lth cell. (B) Receiver represented by an impedance $\hat{Z}_d$.

works ([8,21]). The impedance of each resonator is given by $\hat{Z} = R + j\omega L + 1/j\omega C$, being L the inductance of the resonator, R its intrinsic resistance and C the added capacitance. At the resonant angular frequency $\omega_0 = 2\pi f_0 = 1/\sqrt{LC}$, the impedance of each resonator becomes equal to its resistance ($\hat{Z} = R$). The mutual inductance between adjacent resonators is given by M, whereas the one between non-adjacent resonators is neglected, as its value is much smaller compared to M in arrays arranged in a plane with parallel axes ([17,20]). Then the equivalent circuit can be written in matrix form as $\hat{\mathbf{V}} = \hat{\mathbf{Z}}_\mathbf{m}\hat{\mathbf{I}}$ with $\hat{\mathbf{V}} = [\hat{V}_s, 0, \dots, 0]^T$, $\hat{\mathbf{I}} = [\hat{I}_1, \dots, \hat{I}_N]^T$ and the matrix $\hat{\mathbf{Z}}_\mathbf{m}$ a symmetric tridiagonal matrix:

$$\hat{\mathbf{Z}}_\mathbf{m} = \begin{bmatrix} \hat{Z} & j\omega M & \cdots & 0 & \cdots & 0 \\ j\omega M & \hat{Z} & \ddots & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & j\omega M & \ddots & 0 \\ 0 & 0 & j\omega M & \hat{Z}+\hat{Z}_d & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & j\omega M \\ 0 & 0 & 0 & 0 & j\omega M & \hat{Z}+\hat{Z}_T \end{bmatrix}, \tag{2.1}$$

where a term $\hat{Z} + \hat{Z}_d$ appears as the lth element of the diagonal whenever the lth resonator has a receiver placed over it. In order to determine the current vector $\hat{\mathbf{I}}$ (i.e., the currents flowing in the resonators) as $\hat{\mathbf{I}} = \hat{\mathbf{Z}}_\mathbf{m}^{-1}\hat{\mathbf{V}}$ we need to determine the inverse matrix $\hat{\mathbf{Z}}_\mathbf{m}^{-1}$. Actually, as all components of $\hat{\mathbf{V}}$ are 0 except for the first one, we only need to determine the first column of $\hat{\mathbf{Z}}_\mathbf{m}^{-1}$. Nevertheless, since long lines of resonators may be used, some attenuation can be expected along the array, so in practice it may become necessary to add voltage sources at several points (and then $\hat{\mathbf{V}}$ would have more than one nonzero element). For this reason we determine all entries of $\hat{\mathbf{Z}}_\mathbf{m}^{-1}$ (see Sections 3.2.1 and 4.2.1). After determining the current in each resonator, one can determine the power transmitted to a receiver.

In this paper we are interested in the case in which the receivers are periodically placed over the resonators, that is, with a receiver placed over every k resonators. For simplicity of the analysis we will only consider explicitly the cases $k = 2$ and $k = 3$, and no terminal impedance besides the one which eventually comes from a receiver placed over the Nth resonator. We also note that, since the mathematical analysis (undertaken in Sections 3 and 4 for $k = 2, 3$ respectively) finds the inverse of any symmetric k-Toeplitz matrix with constant upper and lower diagonals, it actually allows to find the currents in any system with arbitrary impedances $a_1, \dots, a_k$ over the first k resonators and periodically repeating afterwards (in particular, the case with several identical receivers per period can be handled in the same manner). The current, power and efficiency formulas with arbitrary periodic impedance matrix for the cases $k = 2, 3$ would then be formally the same as those in Section 5, but with different impedance parameters, depending on the corresponding impedance matrix.

2.1. Receiver over every 2 resonators

Consider a receiver placed over each resonator of even index (see Fig. 2). In this case the impedance matrix of the array is a 2-Toeplitz matrix with constant and equal upper and lower diagonals:

$$\hat{\mathbf{Z}}_{\mathbf{m}} = \begin{bmatrix} \hat{Z} & j\omega M & 0 & \dots & \dots & \dots & \dots & 0 \\ j\omega M & \hat{Z}+\hat{Z}_d & j\omega M & 0 & \dots & \dots & \dots & 0 \\ 0 & j\omega M & \hat{Z} & j\omega M & 0 & \dots & \dots & 0 \\ 0 & 0 & j\omega M & \hat{Z}+\hat{Z}_d & j\omega M & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & j\omega M \\ 0 & 0 & \dots & \dots & \dots & 0 & j\omega M & \hat{Z}+\hat{Z}_T \end{bmatrix},$$

with $\hat{Z}_T = \begin{cases} 0, & \text{if } N \text{ is odd} \\ \hat{Z}_d, & \text{if } N \text{ is even} \end{cases}$.

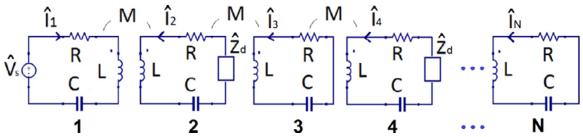


Fig. 2. Circuit of a resonator array with a receiver placed over each resonator of even index (shown here for N odd).

2.2. Receiver over every 3 resonators

When there is a receiver placed over each resonator of index a multiple of 3 (Fig. 3), the impedance matrix of the array is a 3-Toeplitz matrix with constant and equal upper and lower diagonals:

$$\hat{\mathbf{Z}}_{\mathbf{m}} = \begin{bmatrix} \hat{Z} & j\omega M & 0 & \dots & \dots & 0 \\ j\omega M & \hat{Z} & j\omega M & 0 & \dots & 0 \\ 0 & j\omega M & \hat{Z}+\hat{Z}_d & j\omega M & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 0 & j\omega M & \hat{Z}+\hat{Z}_T \end{bmatrix},$$

with $\hat{Z}_T = \begin{cases} 0 \text{ if } N \neq 3p \\ \hat{Z}_d \text{ if } N = 3p \end{cases}$

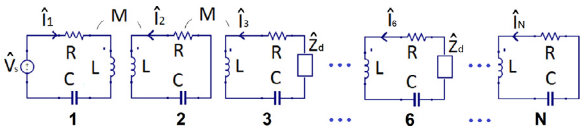


Fig. 3. Circuit of a resonator array with a receiver placed over each resonator of index multiple of 3 (shown here for N not a multiple of 3).

3. 2-Toeplitz matrix

Recall that the (i, j)*th cofactor* of the matrix $A \in \mathrm{M}_n(\mathbb{C})$ is

$$C_{ij} = (-1)^{i+j} \det(A_{ij}),$$

where $A_{ij} \in \mathrm{M}_{n-1}(\mathbb{C})$ is the submatrix of A formed by removing the ith row and the jth column. Then the *cofactor matrix* of A is the matrix $C(A) = (C_{ij}) \in \mathrm{M}_n(\mathbb{C})$. The inverse of a regular matrix A can be computed as

$$A^{-1} = \frac{\mathrm{adj}(A)}{\det A},$$

where the *adjugate matrix* of A is $\mathrm{adj}(A) = C(A)^T$, the transpose of its cofactor matrix.

Denote $M_n(a_1, a_2, b) = \begin{pmatrix} a_1 & b & 0 & \dots & \dots & \dots & \dots & 0 \\ b & a_2 & b & 0 & \dots & \dots & \dots & 0 \\ 0 & b & a_1 & b & 0 & \dots & \dots & 0 \\ 0 & 0 & b & a_2 & b & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & b \\ 0 & 0 & \dots & \dots & \dots & 0 & b & \alpha \end{pmatrix} \in \mathrm{M}_n(\mathbb{C})$, where α is a_1 when n is odd and a_2 when n is even. We compute the inverse of $M_n(a_1, a_2, b)$ with the previous formula. In this case the adjugate matrix is just the cofactor matrix, since $M_n(a_1, a_2, b)$ is symmetric.

3.1. Determinant

Recall that the *Laplace expansion* along the jth column gives the determinant of a matrix $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{C})$ as $\det A = a_{1j}C_{1j} + \cdots a_{nj}C_{nj}$ (Laplace expansion along a row is analogous). Let $D(n) = \det(M_n(a_1, a_2, b))$. From Laplace expansion along the last column we see that

$$D(n) = \alpha D(n-1) - bD' = \alpha D(n-1) - b^2 D(n-2),$$

where D' has been computed by Laplace expansion along its last row. We get two linear recurrence equations for $D(n)$: for $D(2k)$ and for $D(2k-1)$. Written in matrix form:

$$\begin{pmatrix} D(2k) \\ D(2k-1) \end{pmatrix} = \begin{pmatrix} a_2 & -b^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} D(2k-1) \\ D(2k-2) \end{pmatrix},$$

$$\begin{pmatrix} D(2k-1) \\ D(2k-2) \end{pmatrix} = \begin{pmatrix} a_1 & -b^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} D(2k-2) \\ D(2k-3) \end{pmatrix}.$$

Put $A = \begin{pmatrix} a_2 & -b^2 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} a_1 & -b^2 \\ 1 & 0 \end{pmatrix}$. Since A gives $D(2k)$ from $D(2k-1)$ and B gives $D(2k-1)$ from $D(2k-2)$, which has again even argument, by induction we get

$$\begin{pmatrix} D(2k) \\ D(2k-1) \end{pmatrix} = (AB)^{k-1} \begin{pmatrix} D(2) \\ D(1) \end{pmatrix} = (AB)^{k-1} \begin{pmatrix} a_1 a_2 - b^2 \\ a_1 \end{pmatrix}.$$

Denote $a^2 = a_1 a_2$. Let us diagonalize AB (when possible) as PDP^{-1} with D diagonal, so that

$$\begin{pmatrix} D(2k) \\ D(2k-1) \end{pmatrix} = PD^{k-1}P^{-1} \begin{pmatrix} a^2 - b^2 \\ a_1 \end{pmatrix}.$$

The characteristic polynomial of AB is

$$X^2 + (2b^2 - a^2)X + b^4,$$

its eigenvalues

$$r_{1,2} = \frac{a^2}{2} - b^2 \pm \frac{\sqrt{a^2(a^2 - 4b^2)}}{2}.$$

A sufficient condition for diagonalization is $a^2 \neq 4b^2$, as this implies $r_1 \neq r_2$. In that case, a matrix of eigenvectors is

$$P = \begin{pmatrix} \frac{r_1 + b^2}{a_1} & \frac{r_2 + b^2}{a_1} \\ 1 & 1 \end{pmatrix}$$

with determinant

$$\det(P) = \frac{r_1 - r_2}{a_1}$$

and inverse (computed via the adjugate matrix)

$$P^{-1} = \frac{1}{\det(P)} \begin{pmatrix} 1 & -\frac{r_2 + b^2}{a_1} \\ -1 & \frac{r_1 + b^2}{a_1} \end{pmatrix}.$$

Now

$$PD^{k-1} = \begin{pmatrix} \frac{r_1+b^2}{a_1} r_1^{k-1} & \frac{r_2+b^2}{a_1} r_2^{k-1} \\ r_1^{k-1} & r_2^{k-1} \end{pmatrix},$$

$$P^{-1}\begin{pmatrix} a^2-b^2 \\ a_1 \end{pmatrix} = \frac{1}{\det(P)}\begin{pmatrix} a^2-2b^2-r_2 \\ -a^2+2b^2+r_1 \end{pmatrix} = \frac{1}{\det(P)}\begin{pmatrix} r_1 \\ -r_2 \end{pmatrix},$$

since $a^2 - 2b^2 = \mathrm{tr}(AB) = r_1 + r_2$. Putting all the results together we get

$$D(2k) = \frac{1}{r_1 - r_2}((r_1+b^2)r_1^k - (r_2+b^2)r_2^k),$$

$$D(2k-1) = \frac{a_1}{r_1 - r_2}(r_1^k - r_2^k).$$

The matrix $M_n(a_1, a_2, b)$ will be invertible if and only if its determinant is nonzero, which will be the case precisely when $r_1^k \neq r_2^k$ if $n = 2k-1$ and when $(r_1+b^2)r_1^k \neq (r_2+b^2)r_2^k$ if $n = 2k$. Observe that the sufficient condition for diagonalization $a^2 \neq 4b^2$ is not enough to assure invertibility, as the example with $n = 3,\ b = 1,\ a_1 = 1,\ a_2 = 2$ shows:

$$\mathrm{rank}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 2 < 3$$

since the second row is the sum of the first and third ones.

3.2. Elements of the inverse

As stated in Section 2, when there is only a voltage source at the first resonator we only need to compute the first column of the inverse, equivalently the first row (since the matrix is symmetric), so for ease of reasoning we first explain how to compute the first row of the cofactor matrix. We give the computation for an arbitrary element of the inverse, which solves the more general problem of having several voltage sources in arbitrary positions, at the end of this section.

When we compute cofactor C_{1j} of $M(a_1, a_2, b)$ via its submatrix M_{1j}, by construction the first row and the jth column of M are removed, the elements of the $1, \ldots, j-1$th columns get bumped up one position upwards, and the rest of elements from $M(a_1, a_2, b)$ keep their original relative positions in M_{1j}, but with both their indices lowered by one (e.g., the (j, j)th element becomes the $(j-1, j-1)$th one). Thus M_{1j} will have the element b in its diagonal positions $(1, 1), \ldots, (j-1, j-1)$, zero in the columns below those b, and the unaltered lower right block M' of $M_n(a_1, a_2, b)$ from the (j, j)th element onwards:

$$M(a_1,a_2,b) = \left(\begin{array}{cccc|c} a_1 & b & & & \\ b & a_2 & b & & \\ 0 & b & \ddots & b & \\ \vdots & \ddots & b & a_j & \\ \hline 0 & \cdots & 0 & b & M' \end{array}\right) \Rightarrow M_{1j} = \left(\begin{array}{ccc|c} b & a_2 & b & \\ 0 & b & \ddots & \\ \vdots & \ddots & b & \\ \hline 0 & \cdots & 0 & M' \end{array}\right)$$

So M_{1j} is diagonally composed of an upper triangular block with constant diagonal b and a lower right block M' which is another 2-Toeplitz matrix (either $M_{n-j}(a_1, a_2, b)$ or $M_{n-j}(a_2, a_1, b)$, depending on the parity of j), so its determinant is the product of the determinants of these two blocks, which are known. Denote now by $D_n(a_1, a_2, b)$ the determinant of the matrix $M_n(a_1, a_2, b)$. Recall that $\alpha = \alpha(n)$ equals a_1 when n is odd and a_2 when n is even; write $\alpha_2(n)$ for the function with the opposite behaviour. By the exposition above, the elements q_{1j} of the first row of the adjugate matrix of $M_n(a_1, a_2, b)$ are of the form

$$q_{1j} = (-1)^{j-1} b^{j-1} D_{n-j}(\alpha_2(j), \alpha(j), b).$$

To get the elements m_{1j} of the first row of the inverse we just need to divide by the determinant of the whole matrix:

$$m_{1j} = (-b)^{j-1}\frac{D_{n-j}(\alpha_2(j), \alpha(j), b)}{D_n(a_1, a_2, b)}$$

(when $D_n(a_1, a_2, b) \neq 0$).

Example. For the matrix $M_8(a_1, a_2, b)$ we have

$$m_{15} = \frac{(-b)^4 D_3(a_2, a_1, b)}{D_8(a_1, a_2, b)} = \frac{b^4 a_2 \left(r_1^2 - r_2^2\right)}{\left(r_1+b^2\right) r_1^4 - \left(r_2+b^2\right) r_2^4}$$

(when $D_8(a_1, a_2, b) \neq 0$ and $a^2 \neq 4b^2$). Observe that in the numerator we get a_2 instead of a_1 because in $D_3(a_2, a_1, b)$ the elements a_2, a_1 are swapped. Note also that r_1, r_2 are symmetric with respect to a_1, a_2.

3.2.1. General case

The technique applied above allows to find any element of the inverse. Since the matrix is symmetric we may suppose $j \geq i$. In general, the submatrix M_{ij} which gives rise to the cofactor C_{ij} is an upper block-triangular matrix with three diagonal blocks: a first matrix $M_{i-1}(a_1, a_2, b)$, a middle upper triangular matrix of order $j-i$ with constant diagonal b, and an ending matrix $M_{n-j}(\alpha_2(j), \alpha(j), b)$. Recall that the determinant of a block-triangular matrix equals the product of the determinants of its diagonal blocks, so that the (i, j)th element m_{ij} of the inverse is

$$m_{ij} = (-b)^{j-i} \frac{D_{i-1}(a_1, a_2, b) D_{n-j}(\alpha_2(j), \alpha(j), b)}{D_n(a_1, a_2, b)}, \quad i \leq j$$

(when $D_n(a_1, a_2, b) \neq 0$).

4. 3-Toeplitz matrix

Let $M_n(a_1, a_2, a_3, b) = \begin{pmatrix} a_1 & b & 0 & \dots & \dots & 0 \\ b & a_2 & b & 0 & \dots & 0 \\ 0 & b & a_3 & b & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 0 & b & \alpha \end{pmatrix} \in \mathbb{M}_n(\mathbb{C})$,

where $\alpha = \begin{cases} a_1, & n \equiv 1 \pmod 3 \\ a_2, & n \equiv 2 \pmod 3 \\ a_3, & n \equiv 3 \pmod 3 \end{cases}$. Note that the impedance matrix in Section 2.2 is of the special form $M_n(a_1, a_1, a_2, b)$, but the cases $M_n(a_1, a_2, a_1, b)$ and $M_n(a_2, a_1, a_1, b)$ will be needed when computing its inverse. We compute the inverse of $M_n(a_1, a_2, a_3, b)$ via its adjugate matrix, which is again its cofactor matrix.

4.1. Determinant

Let $D(n) = \det(M_n(a_1, a_2, a_3, b))$. By Laplace expansion along the last column we see that

$$D(n) = \alpha D(n-1) - bD' = \alpha D(n-1) - b^2 D(n-2),$$

where D' has been computed by Laplace expansion along the last row. We get three linear recurrence equations for $D(n)$: for $D(3k)$, $D(3k-1)$ and $D(3k-2)$. Written in matrix form:

$$\begin{pmatrix} D(3k) \\ D(3k-1) \end{pmatrix} = \begin{pmatrix} a_3 & -b^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} D(3k-1) \\ D(3k-2) \end{pmatrix},$$

$$\begin{pmatrix} D(3k-1) \\ D(3k-2) \end{pmatrix} = \begin{pmatrix} a_2 & -b^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} D(3k-2) \\ D(3(k-1)) \end{pmatrix},$$

$$\begin{pmatrix} D(3k-2) \\ D(3(k-1)) \end{pmatrix} = \begin{pmatrix} a_1 & -b^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} D(3(k-1)) \\ D(3(k-1)-1) \end{pmatrix}.$$

Put $A_i = \begin{pmatrix} a_i & -b^2 \\ 1 & 0 \end{pmatrix}$, $a^3 = a_1 a_2 a_3$, $s = a_1 + a_2 + a_3$, $d = a_1 a_2 - b^2$. Observe that a and s are symmetric in a_1, a_2, a_3, but $d = d(a_1, a_2)$ is not; hence for $M(a_1, a_1, a_2)$ we will have to consider $d(a_1, a_1)$, while for $M(a_1, a_2, a_1)$ and $M(a_2, a_1, a_1)$ we will consider $d(a_1, a_2)$. By induction we get

$$\begin{pmatrix} D(3k) \\ D(3k-1) \end{pmatrix} = (A_3 A_2 A_1)^{k-1} \begin{pmatrix} D(3) \\ D(2) \end{pmatrix} = (A_3 A_2 A_1)^{k-1} \begin{pmatrix} a^3 + (a_2 - s)b^2 \\ d \end{pmatrix}.$$

Let us diagonalize $A_3 A_2 A_1$ (when possible) as PDP^{-1} with D diagonal. The characteristic polynomial of $A_3 A_2 A_1$ is

$$X^2 + (sb^2 - a^3)X + b^6,$$

its eigenvalues

$$r_{1,2} = \frac{a^3 - sb^2}{2} \pm \frac{\sqrt{(a^3 - sb^2)^2 - 4b^6}}{2}.$$

Observe that r_1, r_2 are also symmetric with respect to a_1, a_2, a_3, being functions of a and s. A sufficient condition for diagonalization is $a^3 - sb^2 \neq \pm 2b^3$, as this implies $r_1 \neq r_2$. In that case, a matrix of eigenvectors is

$$P = \begin{pmatrix} \frac{r_1 + a_2 b^2}{d} & \frac{r_2 + a_2 b^2}{d} \\ 1 & 1 \end{pmatrix}$$

with determinant

$$\det(P) = \frac{r_1 - r_2}{d}$$

and inverse (computed via the adjugate matrix)

$$P^{-1} = \frac{1}{\det(P)} \begin{pmatrix} 1 & -\dfrac{r_2 + a_2 b^2}{d} \\ -1 & \dfrac{r_1 + a_2 b^2}{d} \end{pmatrix}.$$

Now

$$PD^{k-1} = \begin{pmatrix} \dfrac{r_1 + a_2 b^2}{d} r_1^{k-1} & \dfrac{r_2 + a_2 b^2}{d} r_2^{k-1} \\ r_1^{k-1} & r_2^{k-1} \end{pmatrix},$$

$$P^{-1} \begin{pmatrix} a^3 + (a_2 - s) b^2 \\ d \end{pmatrix} = \frac{1}{\det(P)} \begin{pmatrix} a^3 - s b^2 - r_2 \\ -a^3 + s b^2 + r_1 \end{pmatrix} = \frac{1}{\det(P)} \begin{pmatrix} r_1 \\ -r_2 \end{pmatrix},$$

since $a^3 - sb^2 = \operatorname{tr}(A_3 A_2 A_1) = r_1 + r_2$. Putting all the results together we get

$$D(3k) = \frac{1}{r_1 - r_2} ((r_1 + a_2 b^2) r_1^k - (r_2 + a_2 b^2) r_2^k),$$

$$D(3k-1) = \frac{d}{r_1 - r_2} (r_1^k - r_2^k).$$

We still have to compute $D(3k-2)$, which equals $a_1 D(3(k-1)) - b^2 D(3(k-1)-1)$:

$$D(3k-2) = \frac{1}{r_1 - r_2} ((a_1 r_1 + b^4) r_1^{k-1} - (a_1 r_2 + b^4) r_2^{k-1}).$$

4.2. Elements of the inverse

The computation of the elements of the inverse matrix from the determinants of the submatrices giving the (i, j)th minors is analogous to the case of 2-Toeplitz matrices studied in Section 3.2, the main difference being that the ending block of the block-triangular matrix can now be $M_{n-j}(a_1, a_2, a_3, b)$, $M_{n-j}(a_2, a_3, a_1, b)$ or $M_{n-j}(a_3, a_1, a_2, b)$, depending on the residue of j modulo 3. Denote by $D_n(a_1, a_2, a_3, b)$ the determinant of the matrix $M_n(a_1, a_2, a_3, b)$ and by $\sigma_j(a_1, a_2, a_3)$ the jth cyclic permutation of (a_1, a_2, a_3) to the left, i.e., $\sigma_0(a_1, a_2, a_3) = (a_1, a_2, a_3)$, $\sigma_1(a_1, a_2, a_3) = (a_2, a_3, a_1)$, $\sigma_2(a_1, a_2, a_3) = (a_3, a_1, a_2)$, $\sigma_3(a_1, a_2, a_3) = (a_1, a_2, a_3)$, etc. The elements q_{1j} of the first row of the adjugate matrix of $M_n(a_1, a_2, a_3, b)$ are of the form

$$q_{1j} = (-b)^{j-1} D_{n-j}(\sigma_j(a_1, a_2, a_3), b).$$

To get the elements m_{1j} of the first row of the inverse we just need to divide by the determinant of the whole matrix:

$$m_{1j} = (-b)^{j-1} \frac{D_{n-j}(\sigma_j(a_1, a_2, a_3), b)}{D_n(a_1, a_2, a_3, b)}$$

(when $D_n(a_1, a_2, a_3, b) \neq 0$).

Since the parameter d is not symmetric with respect to a_1, a_2, a_3, care with d must be taken when $n - j \equiv -1 \pmod 3$.

Example. For the matrix $M_8(a_1, a_2, a_3, b)$ we have

$$m_{14} = (-b)^3 \frac{D_4(a_2, a_3, a_1, b)}{D_8(a_1, a_2, a_3, b)} = -b^3 \frac{(a_2 r_1 + b^4) r_1 - (a_2 r_2 + b^4) r_2}{d(a_1, a_2) \cdot (r_1^3 - r_2^3)}$$

(when $D_n(a_1, a_2, a_3, b) \neq 0$ and $a^3 - sb^2 \neq \pm 2b^3$).

Recall that we are chiefly interested in matrices of the form $M_n(a_1, a_1, a_2)$. In this particular case, the elements m_{ij} of the first row of the inverse are

$$m_{1j} = \frac{(-b)^{j-1} D_{n-j}(\sigma_j(a_1, a_1, a_2), b)}{D_n(a_1, a_1, a_2, b)}$$

(when $D_n(a_1, a_1, a_2, b) \neq 0$).

If $n \equiv -1 \pmod 3$ then $d(a_1, a_1) = a_1^2 - b^2$ will appear in the denominator. If $n - j \equiv -1 \pmod 3$, then in the numerator it will appear either $d(a_1, a_1)$ if $j \equiv 0 \pmod 3$ or $d(a_1, a_2) = a_1 a_2 - b^2$ if $j \equiv 1, 2 \pmod 3$.

4.2.1. General case

Since $M_n(a_1, a_2, a_3, b)$ is symmetric we may suppose $j \geq i$. The submatrix M_{ij} which gives rise to the cofactor C_{ij} is an upper block-triangular matrix with three diagonal blocks: a first matrix $M_{i-1}(a_1, a_2, a_3, b)$, a middle upper triangular matrix of order $j-i$ with constant diagonal b, and an ending matrix $M_{n-j}(\sigma_j(a_1, a_2, a_3), b)$. Hence the (i, j)th element m_{ij} of the inverse is

$$m_{ij} = (-b)^{j-i}\frac{D_{i-1}(a_1, a_2, a_3, b)D_{n-j}(\sigma_j(a_1, a_2, a_3), b)}{D_n(a_1, a_2, a_3, b)}, \;\; i \leq j$$

(when $D_n(a_1, a_2, a_3, b) \neq 0$).

5. Application of the mathematical results

In this section we will use the generic expressions obtained for the elements of the inverse of the tridiagonal matrix to determine the expressions for the currents, power transmitted and efficiency of the resonator array. Subsequently we will use said expressions to illustrate the mathematical results and understand how the behaviour of the system changes with the variation of its characteristics and parameters. In particular, we will analyse the behaviour of the system for different values of the receiver impedance R_d.

5.1. Expressions for the currents in the resonators

5.1.1. Receiver over every 2 resonators

The formulas of the 2-Toeplitz case with circuit parameters

$$b = j\omega M, \;\; a_1 = \hat{Z}, \;\; a_2 = \hat{Z} + \hat{Z}_d$$

produce the values of the currents in the $(2k)$th resonators.

For even N, $N = 2p$, we have

$$\begin{aligned}\hat{I}_{2k} &= -\hat{V}_s(j\omega M)^{2k-1}\frac{s_1r_1^{p-k} - s_2r_2^{p-k}}{s_1r_1^p - s_2r_2^p},\\ \hat{I}_1 &= -\hat{V}_s(\hat{Z} + \hat{Z}_d)\frac{r_1^p - r_2^p}{s_1r_1^p - s_2r_2^p}\end{aligned}$$

with

$$\begin{aligned}s_{1,2} &= \frac{1}{2}\hat{Z}(\hat{Z} + \hat{Z}_d) \pm \frac{1}{2}\sqrt{\hat{Z}(\hat{Z} + \hat{Z}_d)(\hat{Z}(\hat{Z} + \hat{Z}_d) + 4(\omega M)^2)},\\ r_{1,2} &= s_{1,2} + (\omega M)^2.\end{aligned}$$

For N odd, $N = 2p - 1$, we have

$$\begin{aligned}\hat{I}_{2k} &= -\hat{V}_s(j\omega M)^{2k-1}\frac{r_1^{p-k} - r_2^{p-k}}{r_1^p - r_2^p},\\ \hat{I}_1 &= -\frac{\hat{V}_s}{\hat{Z}}\frac{s_1r_1^{p-1} - s_2r_2^{p-1}}{r_1^p - r_2^p}\end{aligned}$$

with $r_{1,2}$, $s_{1,2}$ as before.

5.1.2. Receiver over every 3 resonators

The formulas of the 3-Toeplitz case with circuit parameters

$$b = j\omega M, \;\; a_1 = a_2 = \hat{Z}, \;\; a_3 = \hat{Z} + \hat{Z}_d$$

produce the values of the currents in the $(3k)$th resonators. For $N = 3p$ we have

$$\begin{aligned}\hat{I}_{3k} &= \hat{V}_s(-j\omega M)^{3k-1}\frac{s_3r_3^{p-k} - s_4r_4^{p-k}}{s_3r_3^p - s_4r_4^p},\\ \hat{I}_1 &= \hat{V}_s(\hat{Z}(\hat{Z} + \hat{Z}_d) + (\omega M)^2)\frac{r_3^p - r_4^p}{s_3r_3^p - s_4r_4^p}\end{aligned}$$

with

$$\alpha = \hat{Z}^2(\hat{Z}+\hat{Z}_d) + (\omega M)^2(3\hat{Z}+\hat{Z}_d),$$
$$r_{3,4} = \frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 + 4(\omega M)^6},$$
$$s_{3,4} = r_{3,4} - \hat{Z}(\omega M)^2.$$

For $N = 3p-1$:

$$\hat{I}_{3k} = \hat{V}_s(-j\omega M)^{3k-1}\frac{r_3^{p-k} - r_4^{p-k}}{r_3^p - r_4^p},$$
$$\hat{I}_1 = \frac{\hat{V}_s}{\hat{Z}^2 + (\omega M)^2}\frac{t_3 r_3^{p-1} - t_4 r_4^{p-1}}{r_3^p - r_4^p}$$

with $t_{3,4} = \hat{Z}r_{3,4} + (\omega M)^4$.

Finally, for $N = 3p-2$:

$$\hat{I}_{3k} = \hat{V}_s(-j\omega M)^{3k-1}\frac{t_3 r_3^{p-k-1} - t_4 r_4^{p-k-1}}{t_3 r_3^{p-1} - t_4 r_4^{p-1}},$$
$$\hat{I}_1 = \hat{V}_s\frac{u_3 r_3^{p-1} - u_4 r_4^{p-1}}{t_3 r_3^{p-1} - t_4 r_4^{p-1}}$$

with $u_{3,4} = s_{3,4} - \hat{Z}_d(\omega M)^2$.

5.2. Analysis of the currents on the resonators

For simplicity we consider now that the resonators and receivers have the same resonant frequency ω_0 and that the array is working at such frequency ($\hat{Z} = R$ and $\hat{Z}_d = R_d$ are real), and the voltage source having a root-mean-square (RMS) value of 1V (V_s =1V). In order to illustrate the mathematical results obtained, we offer some examples of the RMS values of the currents for each case (I_{2k} and I_{3k}), using the values from [5] ($R = 0.11\Omega$, $\omega_0 M = -1.43$).

5.2.1. Receiver over every 2 resonators

Using the expressions from previous sections, we calculate the RMS values of the currents for four different values of $\hat{Z}_d = R_d$ (R, $5R$, $10R$, $100R$), for N even (Fig. 4(A)) and N odd (Fig. 4(B)). As we can see from Fig. 4, the currents have higher values if the value of R_d is lower. Also, when the number of resonators of the array is even, the currents drop less abruptly as we move to the end of the array, compared with the odd case.

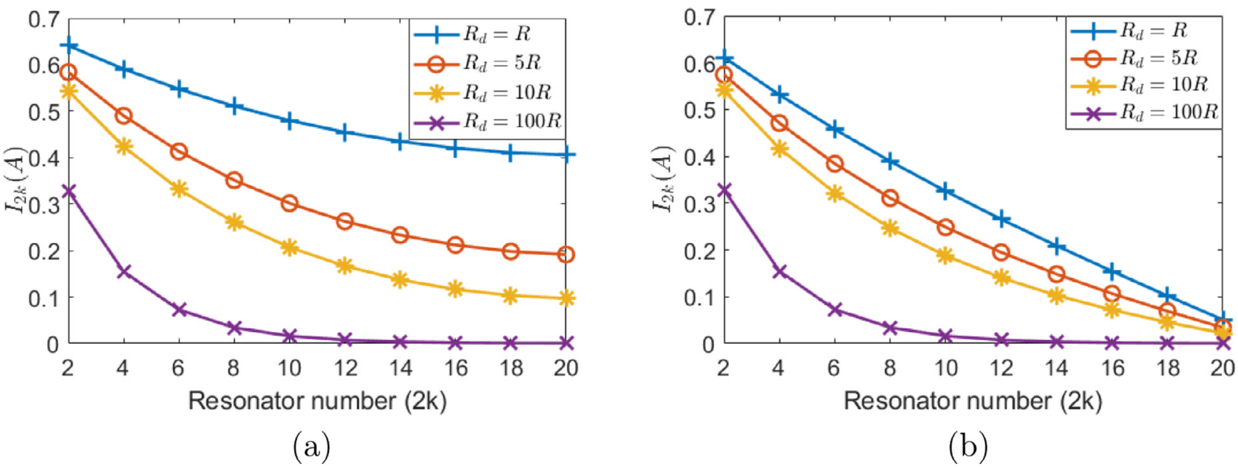


Fig. 4. Comparison of the currents in the 2kth resonators, for (A) $N = 20$ and (B) $N = 21$, for different values of R_d.

5.2.2. Receiver over every 3 resonators

Similarly, by using the expressions from previous sections we obtain the RMS values of the currents for different values of R_d and N (we use $N = 21$, $N = 22$ and $N = 23$). The results are shown in Fig. 5.

As seen with the values of I_{2k}, the values of the currents decrease as the value of R_d increases, in this case with alternating highs and lows for a fixed R_d. The cases $N = 3p-3$ and $N = 3p-1$ present approximately the same behaviour, in contrast with the $N = 3p-2$ case, in which the alternating behaviour is reversed. When k is odd, we have higher values

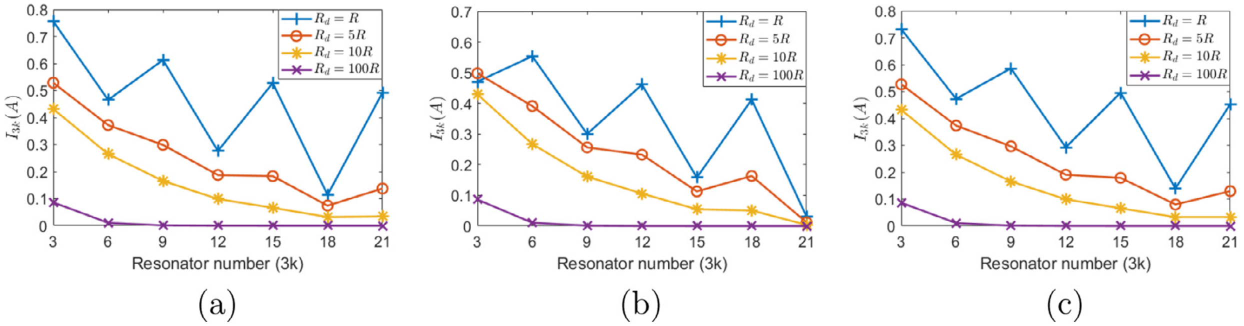


Fig. 5. Comparison of the currents in the 3*k*th resonators, for (A) $N = 21$, (B) $N = 22$ and (C) $N = 23$ for different values of R_d.

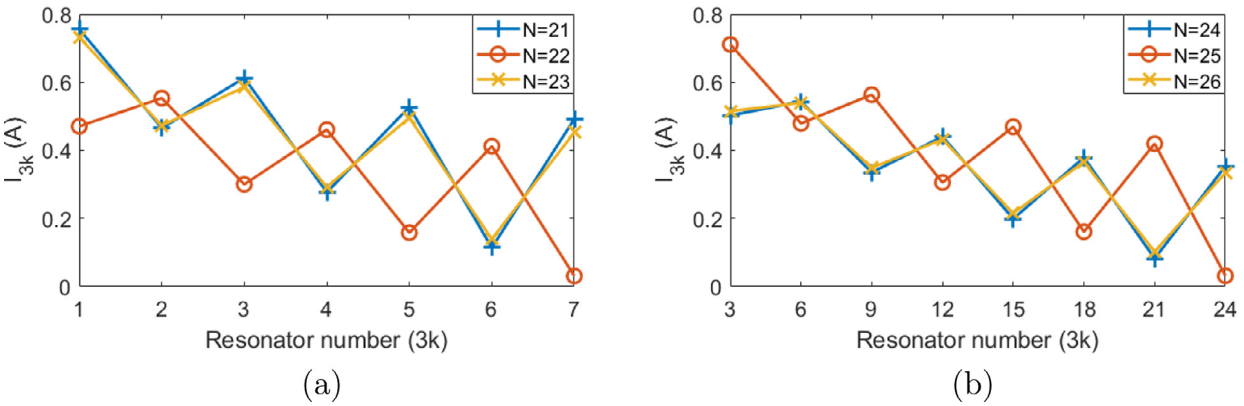


Fig. 6. Comparison of the currents in the 3*k*th resonators, for (A) *N*=21 to 23, (B) *N*=24 to 26.

for $N = 21$ and $N = 23$ than for $N = 22$, and the other way around when k is even; this happens in general for p odd, the behaviour is reversed for p even (See Fig. 6). The differences between the values at the same k for $N = 3p - 3, 3p - 1$ and $N = 3p - 2$ get bigger as k grows.

5.3. Expressions for the power transmitted and efficiency of the system

We use the expressions obtained for the currents in the 2*k*th and 3*k*th resonators to calculate the power transmitted to each receiver over those resonators. Considering I_{2k} and I_{3k} the RMS values for the currents on the 2*k*th and 3*k*th resonators, respectively, and considering the array operating at resonant frequency ω_0 and that the receivers have the same resonant frequency as the resonators, meaning that $\hat{Z}_d$ is real ($\hat{Z}_d = R_d$), we determine the power transmitted for the period 2 and 3 cases as being, respectively,

$$P_{R_d,2k} = I_{2k}^2 R_d \text{ and } P_{R_d,3k} = I_{3k}^2 R_d.$$

Afterwards, using the RMS values of voltage (V_s) and of the current (I_1) in the first resonator, we get the expression for the efficiency of the system by adding the power transmitted to all receivers and dividing it by the input power:

$$\eta_{2k} = \frac{P_{out}}{P_{in}} = \frac{\sum P_{R_d,2k}}{V_s I_1} \text{ and } \eta_{3k} = \frac{P_{out}}{P_{in}} = \frac{\sum P_{R_d,3k}}{V_s I_1}.$$

5.3.1. Receiver over every 2 resonators

For $N = 2p$ we have

$$P_{R_d,2k} = V_s^2 R_d (\omega_0 M)^{4k-2} \frac{(s_1 r_1^{p-k} - s_2 r_2^{p-k})^2}{(s_1 r_1^p - s_2 r_2^p)^2},$$

$$\eta_{2k} = \frac{R_d}{(R + R_d)(r_1^p - r_2^p)(s_1 r_1^p - s_2 r_2^p)} \sum_{k=1}^{p} (\omega_0 M)^{4k-2} (s_1 r_1^{p-k} - s_2 r_2^{p-k})^2.$$

For $N = 2p - 1$:

$$P_{R_d,2k} = V_s^2 R_d (\omega_0 M)^{4k-2} \frac{(r_1^{p-k} - r_2^{p-k})^2}{(r_1^p - r_2^p)^2},$$

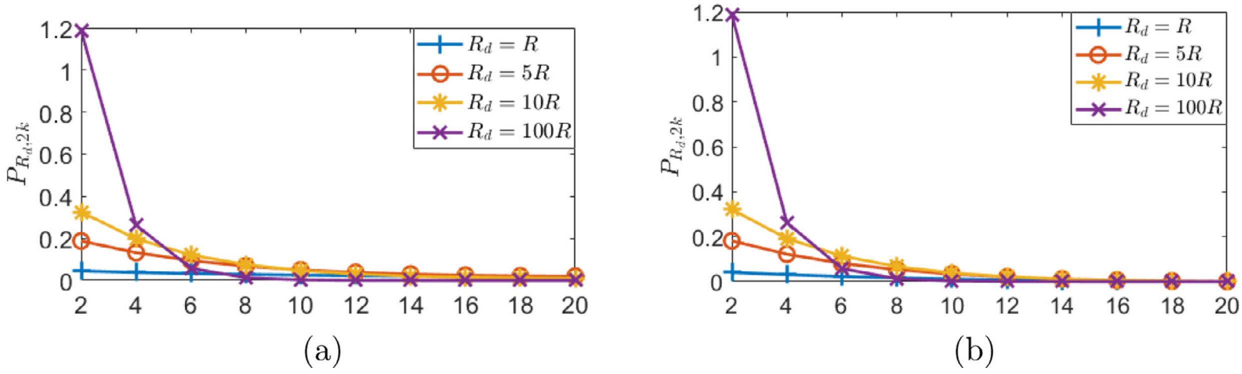


Fig. 7. Comparison of the power transmitted to the receivers over the 2kth resonators, for (A) $N = 20$ and (B) $N = 21$, for different values of R_d.

$$\eta_{2k} = \frac{RR_d}{(r_1^p - r_2^p)(s_1 r_1^{p-1} - s_2 r_2^{p-1})} \sum_{k=1}^{p-1} (\omega_0 M)^{4k-2} (r_1^{p-k} - r_2^{p-k})^2.$$

5.3.2. Receiver over every 3 resonators

For $N = 3p$ we have

$$P_{R_d,3k} = V_s^2 R_d (\omega_0 M)^{6k-2} \frac{(s_3 r_3^{p-k} - s_4 r_4^{p-k})^2}{(s_3 r_3^p - s_4 r_4^p)^2},$$

$$\eta_{3k} = \frac{R_d}{(R(R+R_d) + (\omega_0 M)^2)(r_3^p - r_4^p)(s_3 r_3^p - s_4 r_4^p)} \sum_{k=1}^{p} (\omega_0 M)^{6k-2} (s_3 r_3^{p-k} - s_4 r_4^{p-k})^2.$$

For $N = 3p - 1$:

$$P_{R_d,3k} = V_s^2 R_d (\omega_0 M)^{6k-2} \frac{(r_3^{p-k} - r_4^{p-k})^2}{(r_3^p - r_4^p)^2},$$

$$\eta_{3k} = \frac{R_d (R^2 + (\omega_0 M)^2)}{(r_3^p - r_4^p)(t_3 r_3^{p-1} - t_4 r_4^{p-1})} \sum_{k=1}^{p-1} (\omega_0 M)^{6k-2} (r_3^{p-k} - r_4^{p-k})^2.$$

For $N = 3p - 2$:

$$P_{R_d,3k} = V_s^2 R_d (\omega_0 M)^{6k-2} \frac{(t_3 r_3^{p-k-1} - t_4 r_4^{p-k-1})^2}{(t_3 r_3^{p-1} - t_4 r_4^{p-1})^2},$$

$$\eta_{3k} = \frac{R_d}{(t_3 r_3^{p-1} - t_4 r_4^{p-1})(u_3 r_3^{p-1} - u_4 r_4^{p-1})} \sum_{k=1}^{p-1} (\omega_0 M)^{6k-2} (t_3 r_3^{p-k-1} - t_4 r_4^{p-k-1})^2.$$

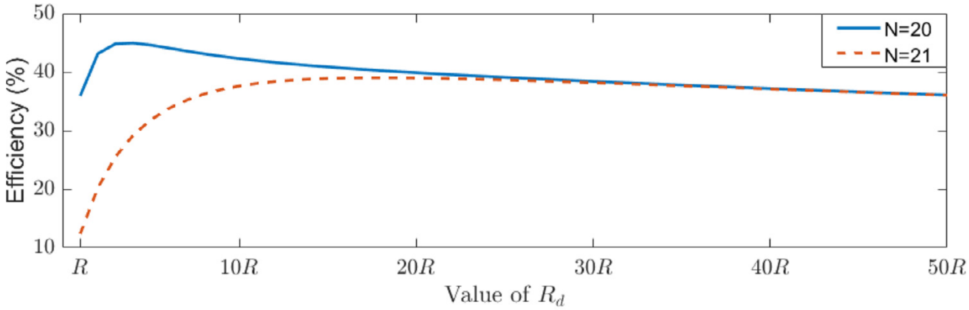
5.4. Analysis of the power transmitted to the receivers and efficiency of the system

As done previously, we use the values from [5], consider that the resonators and receivers have the same resonant frequency ω_0 with the array working at such frequency, and an RMS value of 1V for the voltage source ($V_s = 1$V).

5.4.1. Receiver over every 2 resonators

In Fig. 7 we show the values of the power for four different values of $\hat{Z}_d = R_d$ (R, $5R$, $10R$, $100R$), for N even and N odd. In Fig. 8 we plot the efficiency as a function of R_d. The power transmitted to each of the resonators is approximately the same whether N is odd or even. The maximum power transmitted is obtained when $R_d = 100R$, while the higher average power between all the receivers is obtained when $R_d = 10R$. In contrast, we observe that the efficiency is higher when the number of resonators of the array is even. This phenomenon disappears as N grows, both efficiency curves becoming identical for N big enough: already for $N = 60$ and $N = 61$ the maximum difference in efficiency is less than 1%.

Considering the efficiency as a function of x with $R_d = xR$, we can get the maximum efficiency from the closed-form formula either symbolically or numerically. For $N = 20$, the maximum is 0.450, found with $R_d = 3.62R$, while for $N = 21$ the maximum is 0.390, found with $R_d = 17.88R$. The calculation could be done with arbitrary parameters. In addition, when R_d



(a)

Fig. 8. Comparison of the efficiency of the system as a function of R_d, for $N = 20$ and $N = 21$.

tends to infinity (the rest of parameters being fixed), r_1 becomes the dominant eigenvalue and η_{2k} behaves essentially as $\frac{(\omega_0 M)^2}{R{R_d}^{-1}}$ (for both N even and odd); in particular $\eta_{2k} \to 0$ as $R_d \to \infty$.

5.4.2. Receiver over every 3 resonators

In Fig. 9 we show the values of the power for four different values of $\hat{Z}_d = R_d$ (R, $5R$, $10R$, $100R$) and different values of N. In Fig. 10 we plot the efficiency as a function of R_d.

The power transmitted also has different profile for the case $N = 3p - 1$ compared to the $N = 3p, 3p - 2$ cases, which are similar. Differently from the P_{2k,R_d} case, the maximum power transmitted is obtained when $R_d = 10R$, while the higher average power between all the receivers is obtained when $R_d = 5R$.

The efficiency behaviour is almost the same for the three values of N. Considering the efficiency as a function of x with $R_d = xR$ we find that the maximum is 0.746 for all three, found approximately with $R_d = 24.87R$. The calculation could be done with arbitrary parameters. The values obtained for the period 3 case are higher than the ones for the period 2 case, however this could be due to the fact that less receivers are being used. In addition, when R_d tends to infinity (the rest of parameters being fixed), r_3 becomes the dominant eigenvalue and η_{3k} behaves essentially as $\frac{(\omega_0 M)^4}{R(R^2+(\omega_0 M)^2)}{R_d}^{-1}$ (independently of the character of N modulo 3); in particular $\eta_{3k} \to 0$ as $R_d \to \infty$.

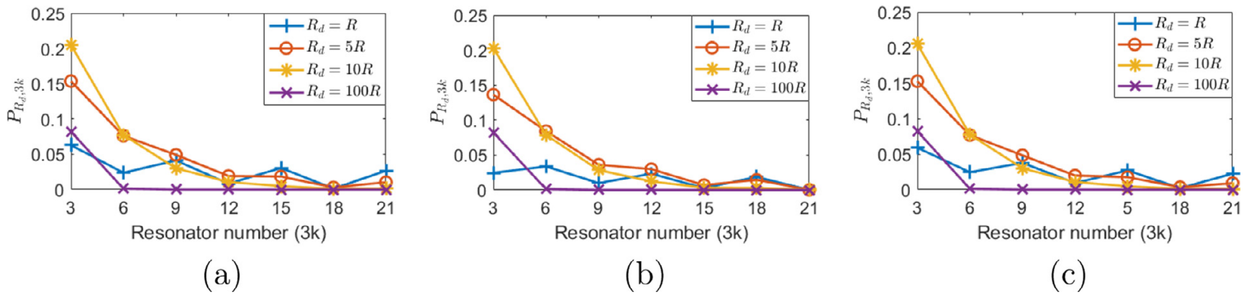
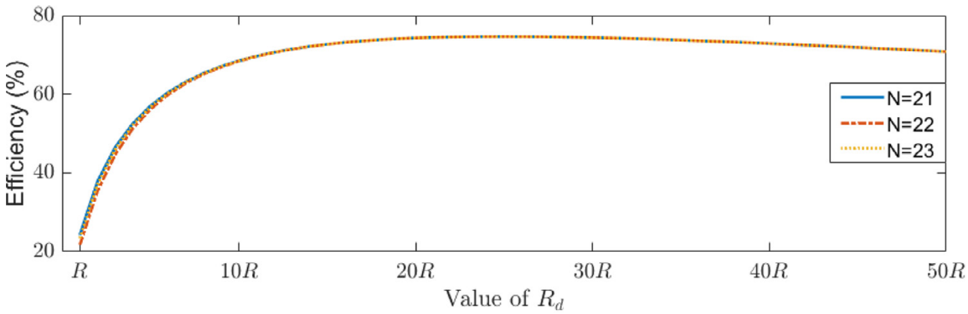


Fig. 9. Comparison of the power transmitted to the receivers over the 3kth resonators, for (A) $N = 21$, (B) $N = 22$ and (C) $N = 23$ for different values of R_d.



(a)

Fig. 10. Comparison of the efficiency of the system as a function of R_d, for $N = 21$, $N = 22$ and $N = 23$.

6. Conclusions

In this paper, the inversion of special 2- and 3-Toeplitz matrices is used to analyse and assess the power transfer capability of a resonator array with multiple receivers. By replacing the generic parameters with the parameters of the circuit, it is possible to obtain closed-form expressions for the currents in the resonators, power transfer and efficiency of the system. Using these expressions, some examples were made in order to illustrate the mathematical results obtained and show their practical applicability. It was found that, for the same lengths of the array, the efficiency is higher when considering a receiver over every three resonators. However, higher values of power transmission are obtained when using an array with a receiver over every two resonators. Also, the efficiency profiles for $N = 2p$ and $N = 2p - 1$ are quite different when N is small. The results obtained in this work allow one to better understand the behaviour of an array with multiple resonators in order to expand the applications for these types of systems. The closed expressions obtained can help electrical engineers to design systems composed of these resonator arrays, since they allow for abstract, general reasoning over all circuits, not depending on the data of a specific case, in contrast with numerical methods. In particular we can easily predict the behaviour of the system when one of the parameters is changed, we can study the limit behaviour, and maximization of the efficiency with the impedance of the receivers as a function of the impedance of the resonators can be done in a symbolic way. In addition, since the formulas are rational functions, the only source of numerical instability are denominators, the number of significant digits is controllable, and computation is fast. Finally, the same general analysis can be done mutatis mutandis for resonator arrays having any number of receivers placed periodically every k resonators, with k arbitrary.

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