Computational and Applied Mathematics (2024) 43:398
https://doi.org/10.1007/s40314-024-02915-z

New constructions of MSRD codes

Umberto Martínez-Peñas[1]

Received: 13 February 2024 / Revised: 15 July 2024 / Accepted: 20 August 2024 /
Published online: 6 September 2024

Abstract

In this work, we provide four methods for constructing new maximum sum-rank distance (MSRD) codes. The first method, a variant of cartesian products, allows faster decoding than known MSRD codes of the same parameters. The other three methods allow us to extend or modify existing MSRD codes in order to obtain new explicit MSRD codes for sets of matrix sizes (numbers of rows and columns in different blocks) that were not attainable by previous constructions. In this way, we show that MSRD codes exist (by giving explicit constructions) for new ranges of parameters, in particular with different numbers of rows and columns at different positions.

Keywords Linearized Reed–Solomon codes · Maximum sum-rank distance codes · Rank metric · Sum-rank metric

Mathematics Subject Classification 15B33 · 94B05 · 94B65

1 Introduction

The sum-rank metric, defined in Nóbrega and Uchôa-Filho (2010) and implicitly considered earlier in Lu and Kumar (2005), has recently attracted considerable attention in Coding Theory due to its applications in reliable and secure multishot network coding (Martínez-Peñas and Kschischang 2019b; Nóbrega and Uchôa-Filho 2010), PMDS codes for repair in distributed storage (Cai et al. 2022; Gopi and Guruswami 2022; Martínez-Peñas and Kschischang 2019a), rate-diversity optimal space-time codes (Lu and Kumar 2005; Shehadeh and Kschischang 2022), and multilayer crisscross error correction (Martínez-Peñas 2022b), among others.

The size or dimension (when linear) of codes also satisfy a Singleton bound with respect to their minimum sum-rank distance (Byrne et al. 2021, Th. III.2). Codes attaining this bound

Supported by a María Zambrano contract by the University of Valladolid, Spain (Contract no. E-47-2022-0001486), by Grant TED2021-130358B-I00 funded by MCIN/AEI/10.13039/501100011033 and by the “European Union NextGenerationEU/PRTR” and by MICIU/AEI/ 10.13039/501100011033 and ERDF/EU (Grant no. PID2022–138906NB-C21).

✉ Umberto Martínez-Peñas
umberto.martinez@uva.es

1 IMUVa-Mathematics Research Institute, University of Valladolid, Valladolid, Spain

are therefore optimal with respect to the size-distance tradeoff and are called maximum sum-rank distance (MSRD) codes. Linearized Reed–Solomon codes (Martínez-Peñas 2018) are the first MSRD codes that can be decoded in polynomial time over a field of subexponential size in the code length (Martínez-Peñas and Kschischang 2019b). Afterwards, a number of alternative MSRD codes have appeared in the literature (Byrne et al. 2021; Chen 2023; Martínez-Peñas 2018, 2022a, 2023; Neri 2022; Neri et al. 2023; Santonastaso and Sheekey 2023; Santonastaso and Zullo 2023), covering other ranges of parameters (different field sizes and/or matrix sizes).

In this work, we provide four methods for constructing new MSRD codes. The first method (Sect. 3) consists of a special arrangement of cartesian products of preexisting MSRD codes and allows faster decoding than known MSRD codes of the same parameters. The other three methods (Sects. 4, 5 and 6) allow us to extend or modify existing MSRD codes in order to obtain new explicit MSRD codes for sets of matrix sizes (numbers of rows and columns in different blocks) that were not attainable by previous constructions. Furthermore, the constructions in Sects. 5 and 6 admit different numbers of rows and columns at different positions. Not many explicit MSRD constructions with this feature were known before (Byrne et al. 2021; Chen 2023). In Sect. 7, we compare the concrete examples of MSRD codes obtained in this work with the known MSRD codes from the literature. In particular, we show that the parameters of MSRD codes from the literature can all be attained by our constructions, whereas our constructions of MSRD codes attain new ranges of parameters (numbers of rows and columns).

2 Preliminaries

In this preliminary section, we revisit the basic properties of codes in the sum-rank metric (Sect. 2.1) and some known constructions of MSRD codes (Sect. 2.2). For tutorials and surveys on the topic, we refer to Gorla et al. (2023); Martínez-Peñas et al. (2022).

Let $\mathbb{F}_q$ denote the finite field of size q, denote by $\mathbb{F}_q^{m \times n}$ the space of matrices of size $m \times n$ over $\mathbb{F}_q$, for positive integers m and n, and set $\mathbb{F}_q^n = \mathbb{F}_q^{1 \times n}$. We also denote $\mathbb{N} = \{0, 1, 2, \ldots\}$, $[n] = \{1, 2, \ldots, n\}$ and $[m, n] = \{m, m+1, \ldots, n\}$ for positive integers m and n with $m \leq n$. In the following, $\langle \cdot \rangle_{\mathbb{F}_q}$ and $\dim_{\mathbb{F}_q}$ denote linear span and dimension over $\mathbb{F}_q$.

2.1 The sum-rank metric

Fix positive integers $\ell, m_1 \geq m_2 \geq \cdots \geq m_\ell$ and $n_i \leq m_i$, for $i \in [\ell]$. We will consider the sum-rank metric in the space $\prod_{i=1}^{\ell} \mathbb{F}_q^{m_i \times n_i}$, where we will call each factor $\mathbb{F}_q^{m_i \times n_i}$ a rank block, thus ℓ is the number of (rank) blocks. For $C = (C_1, \ldots, C_\ell) \in \prod_{i=1}^{\ell} \mathbb{F}_q^{m_i \times n_i}$, we define its sum-rank weight as

$$\mathrm{wt}(C) = \sum_{i=1}^{\ell} \mathrm{Rk}(C_i),$$

where Rk denotes the rank function. The sum-rank metric is defined as $\mathrm{d}(C, D) = \mathrm{wt}(C - D)$, for $C, D \in \prod_{i=1}^{\ell} \mathbb{F}_q^{m_i \times n_i}$. For a code (i.e., a subset) $\mathcal{C} \subseteq \prod_{i=1}^{\ell} \mathbb{F}_q^{m_i \times n_i}$, we define its minimum sum-rank distance as

$$\mathrm{d}(\mathcal{C}) = \min\{\mathrm{d}(C, D) : C, D \in \mathcal{C}, C \neq D\}.$$

For an $\mathbb{F}_q$-linear code $\mathcal{C} \subseteq \prod_{i=1}^{\ell} \mathbb{F}_q^{m_i \times n_i}$, its minimum sum-rank distance coincides with its minimum sum-rank weight, that is, $\mathrm{d}(\mathcal{C}) = \min\{\mathrm{wt}(C) : C \in \mathcal{C}, C \neq 0\}$.

Observe that, when $\ell = 1$, the sum-rank metric recovers the rank metric, and when $m_1 = n_1 = \cdots = m_\ell = n_\ell = 1$, the sum-rank metric recovers the Hamming metric.

As in the case of the Hamming metric, there exists a Singleton bound that relates the minimum sum-rank distance and the size of a code without involving the field size (except for taking logarithms or dimensions). For a code (linear or non-linear) $\mathcal{C} \subseteq \prod_{i=1}^{\ell} \mathbb{F}_q^{m_i \times n_i}$ with $|\mathcal{C}| \geq 2$, let $\mathrm{d}(\mathcal{C}) = \sum_{i=1}^{j-1} n_i + \delta + 1$, where $j \in [\ell]$ and $0 \leq \delta \leq n_j - 1$. The Singleton bound for the sum-rank metric, proven in (Byrne et al. 2021, Th. III.2), reads

$$\log_q |\mathcal{C}| \leq \sum_{i=j}^{\ell} m_i n_i - m_j \delta. \tag{1}$$

Notice that, if $\mathcal{C}$ is $\mathbb{F}_q$-linear, then $\log_q |\mathcal{C}| = \dim_{\mathbb{F}_q}(\mathcal{C})$. A code $\mathcal{C} \subseteq \prod_{i=1}^{\ell} \mathbb{F}_q^{m_i \times n_i}$ is called a Maximum Sum-Rank Distance (MSRD) code if it meets the Singleton bound (1). See Sect. 2.2 for some known explicit constructions.

When $m = m_1 = \cdots = m_\ell$, we may consider the space $\mathbb{F}_{q^m}^n$, where $n = n_1 + \cdots + n_\ell$, instead of $\prod_{i=1}^{\ell} \mathbb{F}_q^{m_i \times n_i}$, due to the following. Given an ordered basis $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_m) \in \mathbb{F}_{q^m}^m$ of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, we define the $\mathbb{F}_q$-linear vector space isomorphism $M_{\boldsymbol{\gamma}}^r : \mathbb{F}_{q^m}^r \longrightarrow \mathbb{F}_q^{m \times r}$ given by

$$M_{\boldsymbol{\gamma}}^r(\mathbf{c}) = \begin{pmatrix} c_{1,1} & c_{1,2} & \ldots & c_{1,r} \\ c_{2,1} & c_{2,2} & \ldots & c_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m,1} & c_{m,2} & \ldots & c_{m,r} \end{pmatrix}, \tag{2}$$

for $\mathbf{c} = (c_1, \ldots, c_r) \in \mathbb{F}_{q^m}^r$, where $c_{i,j} \in \mathbb{F}_q$, for $i \in [m]$ and $j \in [r]$, are the unique scalars such that $c_j = \sum_{i=1}^{m} \gamma_i c_{i,j}$, for $j \in [r]$. Now, if we set $\mathbf{n} = (n_1, \ldots, n_\ell)$, we may extend the previous map to another $\mathbb{F}_q$-linear vector space isomorphism $M_{\boldsymbol{\gamma}}^{\mathbf{n}} : \mathbb{F}_{q^m}^n \longrightarrow \prod_{i=1}^{\ell} \mathbb{F}_q^{m \times n_i}$ by

$$M_{\boldsymbol{\gamma}}^{\mathbf{n}}(\mathbf{c}) = \left(M_{\boldsymbol{\gamma}}^{n_1}(\mathbf{c}_1), \ldots, M_{\boldsymbol{\gamma}}^{n_\ell}(\mathbf{c}_\ell) \right), \tag{3}$$

for a vector $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_\ell) \in \mathbb{F}_{q^m}^n$, where $\mathbf{c}_i \in \mathbb{F}_{q^m}^{n_i}$, for $i \in [\ell]$. We may also define its sum-rank weight as

$$\mathrm{wt}(\mathbf{c}) = \mathrm{wt}\left(M_{\boldsymbol{\gamma}}^{\mathbf{n}}(\mathbf{c}) \right) = \sum_{i=1}^{\ell} \mathrm{Rk}\left(M_{\boldsymbol{\gamma}}^{n_i}(\mathbf{c}_i) \right).$$

Therefore, we may define the sum-rank metric in $\mathbb{F}_{q^m}^n$ simply as $\mathrm{d}(\mathbf{c}, \mathbf{d}) = \mathrm{wt}(\mathbf{c} - \mathbf{d})$, for $\mathbf{c}, \mathbf{d} \in \mathbb{F}_{q^m}^n$. The advantage of considering the sum-rank metric in $\mathbb{F}_{q^m}^n$ is that we may consider $\mathbb{F}_{q^m}$-linear codes in such an ambient space. Notice that most constructions of MSRD codes are $\mathbb{F}_{q^m}$-linear codes in $\mathbb{F}_{q^m}^n$ Martínez-Peñas (2018, 2022a, 2023); Neri (2022); Santonastaso and Zullo (2023), see Sect. 2.2. However, in this manuscript we will construct $\mathbb{F}_q$-linear MSRD codes where not all $m_1, \ldots, m_\ell$ are equal. Only a few constructions in this case are known (Byrne et al. 2021; Chen 2023).

Observe that, when considering the sum-rank metric in $\mathbb{F}_{q^m}^n$ as above, we need to specify the vector $\mathbf{n} = (n_1, \ldots, n_\ell)$, which we call the sum-rank length partition. Otherwise, the map $M_{\boldsymbol{\gamma}}^{\mathbf{n}}$ is not well defined.

2.2 Some known MSRD codes

We now briefly describe the general $\mathbb{F}_{q^m}$-linear MSRD codes in $\mathbb{F}_{q^m}^n$ introduced in Martínez-Peñas (2022a). They generalize linearized Reed–Solomon codes (Martínez-Peñas 2018), which were the first $\mathbb{F}_{q^m}$-linear MSRD codes whose field sizes q^m are subexponential in the code length n. In general, the MSRD codes in Martínez-Peñas (2022a) are the ones with the smallest finite-field sizes q^m for the given parameters known so far. Moreover, they have the longest block length ℓ compared to q and the matrix sizes, among known MSRD codes. Constructions 2, 3 and 4 in this manuscript (Sects. 4, 5 and 6, respectively) will allow us to extend the block length or modify the matrix sizes of such MSRD codes in non-trivial ways.

Since we are looking for long MSRD codes and an MSRD code can easily be shortened (Martínez-Peñas 2019, Sect. 3.3), we will consider the following codes with the longest lengths possible. Let μ and r be positive integers, define $\ell = \mu(q-1)$ and $n = \ell r$, and consider the sum-rank length partition $\mathbf{n} = (r, \ldots, r)$ (ℓ times). For $k \in [n]$, define the matrix in $\mathbb{F}_{q^m}^{k \times n}$ given by

$$M_k(\mathbf{a}, \boldsymbol{\beta}) = \left(\begin{array}{ccc|c|ccc} \beta_1 & \ldots & \beta_{\mu r} & \ldots & \beta_1 & \ldots & \beta_{\mu r} \\ \beta_1^q a_1 & \ldots & \beta_{\mu r}^q a_1 & \ldots & \beta_1^q a_{q-1} & \ldots & \beta_{\mu r}^q a_{q-1} \\ \beta_1^{q^2} a_1^{\frac{q^2-1}{q-1}} & \ldots & \beta_{\mu r}^{q^2} a_1^{\frac{q^2-1}{q-1}} & \ldots & \beta_1^{q^2} a_{q-1}^{\frac{q^2-1}{q-1}} & \ldots & \beta_{\mu r}^{q^2} a_{q-1}^{\frac{q^2-1}{q-1}} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \beta_1^{q^{k-1}} a_1^{\frac{q^{k-1}-1}{q-1}} & \ldots & \beta_{\mu r}^{q^{k-1}} a_1^{\frac{q^{k-1}-1}{q-1}} & \ldots & \beta_1^{q^{k-1}} a_{q-1}^{\frac{q^{k-1}-1}{q-1}} & \ldots & \beta_{\mu r}^{q^{k-1}} a_{q-1}^{\frac{q^{k-1}-1}{q-1}} \end{array} \right), \quad (4)$$

where $a_1, \ldots, a_{q-1} \in \mathbb{F}_{q^m}^*$ are such that $N_{q^m,q}(a_i) \neq N_{q^m,q}(a_j)$ if $i \neq j$ (where $N_{q^m,q}(a) = a \cdot a^q \cdots a^{q^{m-1}} = a^{\frac{q^m-1}{q-1}}$, for $a \in \mathbb{F}_{q^m}$, is the norm of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$), and where $\beta_1, \ldots, \beta_{\mu r} \in \mathbb{F}_{q^m}^*$ are such that, if we set $\mathcal{H}_i = \langle \beta_{(i-1)r+1}, \beta_{(i-1)r+2}, \ldots, \beta_{ir} \rangle_{\mathbb{F}_q} \subseteq \mathbb{F}_{q^m}$, then

1. $\dim_{\mathbb{F}_q}(\mathcal{H}_i) = r$, and
2. $\mathcal{H}_i \cap \left(\sum_{j \in \Gamma} \mathcal{H}_j \right) = \{0\}$, for any set $\Gamma \subseteq [\mu]$, such that $i \notin \Gamma$ and $|\Gamma| \leq \min\{k, \mu\} - 1$,

for all $i \in [\mu]$.

With these assumptions, the $\mathbb{F}_{q^m}$-linear code $\mathcal{C}_k(\mathbf{a}, \boldsymbol{\beta}) = \{\mathbf{x} M_k(\mathbf{a}, \boldsymbol{\beta}) : \mathbf{x} \in \mathbb{F}_{q^m}^k\} \subseteq \mathbb{F}_{q^m}^n$ has dimension k (over $\mathbb{F}_{q^m}$) and is MSRD by (Martínez-Peñas 2022a, Th. 3.12). We refer the reader to (Martínez-Peñas 2022a, Sect. 4) for concrete examples of choices of $a_1, \ldots, a_{q-1}$ and $\beta_1, \ldots, \beta_{\mu r}$ (in particular for the longest values of r and μ, and thus of ℓ, given q and m). Recall that, by (Martínez-Peñas 2019, Th. 5), the dual code $\mathcal{C}_k(\mathbf{a}, \boldsymbol{\beta})^\perp$ is also MSRD. However, generator matrices of such codes are not known in general.

Linearized Reed–Solomon codes (Martínez-Peñas 2018) correspond to the above MSRD codes when $\mu = 1$, that is, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_r)$ and the two conditions on $\mathcal{H}_1$ simply mean that $\beta_1, \ldots, \beta_r$ are $\mathbb{F}_q$-linearly independent.

3 Construction 1: Cartesian products

In general, cartesian products of MSRD codes are not MSRD. However, we now present a particular case where they are indeed MSRD. The main interest in this construction is that, when the component codes are linearized Reed–Solomon codes, we will see that the resulting code admits decoding algorithms that are faster than those of other MSRD codes of the same parameters.

Construction 1 *Consider (linear or non-linear) codes* $\mathcal{C}_1, \ldots, \mathcal{C}_t \subseteq \prod_{i=1}^{\ell} \mathbb{F}_q^{m_i \times m_i}$, *where* $m_1 \geq \cdots \geq m_\ell$. *Consider their cartesian product arranged as follows:*

$$\mathcal{C} = \left\{ \left(\begin{pmatrix} C_{1,1} \\ \vdots \\ C_{t,1} \end{pmatrix}, \ldots, \begin{pmatrix} C_{1,\ell} \\ \vdots \\ C_{t,\ell} \end{pmatrix} \right) : (C_{k,1}, \ldots, C_{k,\ell}) \in \mathcal{C}_k, k \in [t] \right\} \subseteq \prod_{i=1}^{\ell} \mathbb{F}_q^{(tm_i) \times m_i}, \tag{5}$$

and consider the sum-rank metric in $\prod_{i=1}^{\ell} \mathbb{F}_q^{(tm_i)\times m_i}$ *by taking ranks in each block of matrices* $\mathbb{F}_q^{(tm_i)\times m_i}$, *for* $i \in [\ell]$. *Observe that this is different than simply considering* $\left(\prod_{i=1}^{\ell} \mathbb{F}_q^{m_i \times m_i}\right)^t$ *with the rank blocks* $\mathbb{F}_q^{m_i \times m_i}$.

As in the classical case, we have the following basic result. The proof is straightforward.

Lemma 1 *If* $d_k = \mathrm{d}(\mathcal{C}_k)$, *for* $k \in [t]$, *then*

$$\log_q |\mathcal{C}| = \sum_{k=1}^{t} \log_q |\mathcal{C}_k| \quad \text{and} \quad \mathrm{d}(\mathcal{C}) = \min\{d_1, \ldots, d_t\}.$$

In particular, we obtain MSRD codes in the following particular case.

Theorem 1 *If* $\mathcal{C}_i$ *is MSRD for* $i \in [t]$, $|\mathcal{C}_1| = \cdots = |\mathcal{C}_t|$ *and* $d = d_1 = \cdots = d_t$, *then* $\mathcal{C}$ *is MSRD. More precisely,* $\mathrm{d}(\mathcal{C}) = d = \sum_{i=1}^{j-1} m_i + \delta + 1$, *where* $j \in [\ell]$ *and* $0 \leq \delta \leq m_j - 1$, *and*

$$\log_q |\mathcal{C}| = t \left(\sum_{i=j}^{\ell} m_i^2 - m_j \delta \right).$$

Proof Since $\mathcal{C}_k$ is MSRD of distance d, we have $\log_q |\mathcal{C}_k| = \sum_{i=j}^{\ell} m_i^2 - m_j\delta$, for $k \in [t]$, thus $\log_q |\mathcal{C}| = t\left(\sum_{i=j}^{\ell} m_i^2 - m_j\delta\right)$, and we are done, since the Singleton bound in this case is

$$\log_q |\mathcal{C}| \leq \sum_{i=j}^{\ell} (tm_i)m_i - (tm_j)\delta = t \left(\sum_{i=j}^{\ell} m_i^2 - m_j\delta \right).$$

□

Consider now $\ell \in [q-1]$ and let $\mathcal{D} \subseteq \mathbb{F}_{q^m}^{\ell m}$ be an $\mathbb{F}_{q^m}$-linear linearized Reed–Solomon code (Martínez-Peñas 2018) (see also Sect. 2.2) of minimum sum-rank distance $d \in [\ell m]$. Set $\mathcal{C}_1 = \cdots = \mathcal{C}_t = M_{\boldsymbol{\gamma}}^{\mathbf{m}}(\mathcal{D}) \in (\mathbb{F}_q^{m\times m})^\ell$, in the cartesian-product construction from (5), for $\mathbf{m} = (m, \ldots, m)$ and for an ordered basis $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_m)$ of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. Then the code

$$\mathcal{C} \subseteq (\mathbb{F}_q^{(tm)\times m})^\ell$$

from (5) is $\mathbb{F}_q$-linear and MSRD of minimum sum-rank distance d.

The only MSRD codes with such parameters and with a known efficient decoder are linearized Reed–Solomon codes $\mathcal{C}' \subseteq \mathbb{F}_{q^{tm}}^{\ell m} \cong (\mathbb{F}_q^{(tm)\times m})^\ell$ of minimum sum-rank distance d.

However, decoding $\mathcal{C}$ is always more efficient than decoding $\mathcal{C}'$, since $\mathcal{C}$ requires decoding t linearized Reed–Solomon codes over $\mathbb{F}_{q^m}$, $\mathcal{C}'$ requires decoding one linearized Reed–Solomon code over $\mathbb{F}_{q^{tm}}$, in both cases of code length ℓm, and there are no algorithms for multiplication in $\mathbb{F}_{q^{tm}}$ of linear complexity (or lower) in t over $\mathbb{F}_{q^m}$.

For instance, if we use the Welch-Berlekamp decoder from Martínez-Peñas and Kschischang (2019b), then decoding $\mathcal{C}'$ requires $\mathcal{O}((\ell m)^2)$ operations in $\mathbb{F}_{q^{tm}}$, while decoding $\mathcal{C}$ requires $\mathcal{O}(t(\ell m)^2)$ operations in $\mathbb{F}_{q^m}$. Assume that one multiplication in $\mathbb{F}_{q^{tm}}$ costs about $\mathcal{O}(t^2)$ operations in $\mathbb{F}_{q^m}$. Then decoding $\mathcal{C}'$ requires $\mathcal{O}((t\ell m)^2)$ operations in $\mathbb{F}_{q^m}$, while decoding $\mathcal{C}$ requires $\mathcal{O}(t(\ell m)^2)$ operations in $\mathbb{F}_{q^m}$.

4 Construction 2: Combining bases

Now we provide a construction that combines two linear codes by "glueing" their bases.

Construction 2 *Let*

$$\mathcal{C}_1 \subseteq \prod_{i=1}^{\ell} \mathbb{F}_q^{m_i \times n_i} \quad \textit{and} \quad \mathcal{C}_2 \subseteq \prod_{i=1}^{t} \mathbb{F}_q^{m_{\ell+i} \times n_{\ell+i}}$$

be $\mathbb{F}_q$*-linear codes of dimensions* k_1 *and* k_2*, respectively. Set also* $d_1 = \mathrm{d}(\mathcal{C}_1)$ *and* $d_2 = \mathrm{d}(\mathcal{C}_2)$.

Let $\{B_{j,1}, \ldots, B_{j,k_j}\}$ *form a basis of* $\mathcal{C}_j$*, for* $j = 1, 2$*. Consider the* $\mathbb{F}_q$*-linear code* $\mathcal{C} \subseteq \prod_{i=1}^{\ell+t} \mathbb{F}_q^{m_i \times n_i}$ *with basis*

$$\{(B_{1,1}, B_{2,1}), \ldots, (B_{1,k}, B_{2,k})\},$$

where $k = \min\{k_1, k_2\}$ *and where* $(B_{1,i}, B_{2,i})$ *means concatenation of the tuples* $B_{1,i}$ *and* $B_{2,i}$.

The code $\mathcal{C}$ satisfies the following result, whose proof is straightforward.

Lemma 2 *It holds that*

$$\dim(\mathcal{C}) = \min\{k_1, k_2\} \quad \textit{and} \quad \mathrm{d}(\mathcal{C}) \geq d_1 + d_2.$$

Proof The claim on dimensions is clear since the tuples $(B_{1,1}, B_{2,1}), \ldots, (B_{1,k}, B_{2,k})$ are $\mathbb{F}_q$-linearly independent. Next, a nonzero codeword in $\mathcal{C}$ is of the form

$$\left(\sum_{i=1}^{k} \lambda_i B_{1,i}, \sum_{i=1}^{k} \lambda_i B_{2,i} \right), \tag{6}$$

for some $\lambda_1, \ldots, \lambda_k \in \mathbb{F}_q$, not all zero. In particular, $\sum_{i=1}^{k} \lambda_i B_{1,i}$ and $\sum_{i=1}^{k} \lambda_i B_{2,i}$ are nonzero codewords in $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, thus their sum-rank weights are at least d_1 and d_2, respectively. Hence the sum-rank weight of the codeword in (6) is at least $d_1 + d_2$ and we are done. □

Now assume that $m_1 \geq \cdots \geq m_{\ell+t}$ and $n_i \leq m_i$ for $i \in [\ell + t]$. Assume also that $\mathcal{C}_1$ and $\mathcal{C}_2$ are MSRD with

$$d_1 = \sum_{i=1}^{\ell} n_i \quad \text{and} \quad d_2 = \sum_{i=\ell+1}^{j-1} n_i + \delta + 1,$$

for $j \in [\ell + 1, \ell + t]$ and $0 \leq \delta \leq m_j - 1$. In particular, $k_1 = m_\ell$ by the Singleton bound (1). Finally, assume also that $m_\ell \geq k_2$. In this case, we have the following.

Theorem 2 *With assumptions as in the above paragraph, the code $\mathcal{C}$ is MSRD with*

$$\mathrm{d}(\mathcal{C}) = \sum_{i=1}^{j-1} n_i + \delta + 1 \quad \textit{and} \quad \dim(\mathcal{C}) = \sum_{i=j}^{\ell+t} m_i n_i - m_j \delta.$$

Proof Trivial from Lemma 2 and the parameters of $\mathcal{C}_1$ and $\mathcal{C}_2$. □

Observe that the main parameter restrictions are

$$\mathrm{d}(\mathcal{C}) > \sum_{i=1}^{\ell} n_i \quad \text{and} \quad m_\ell \geq \sum_{i=j}^{\ell+t} m_i n_i - m_j \delta.$$

We also note that Construction 2 can be iterated any given number of times.

In Sect. 7, we will show how Construction 2 generalizes constructions from the literature.

5 Construction 3: Using lattices of MSRD codes

In this section, we provide a construction of $\mathbb{F}_q$-linear MSRD codes based on lattices of (shorter) MSRD codes. We describe the general construction in Sect. 5.1 and provide concrete examples in Sect. 5.2.

5.1 The general construction

Consider the parameters $m_1 \geq \cdots \geq m_\ell$ and $n_i \leq m_i$ for $i \in [\ell]$. We further assume that $m = m_s = m_{s+1} = \cdots = m_\ell$, for some $s \in [\ell]$. Set $n = n_1 + \cdots + n_\ell$ and let $d \in [n]$ be such that

$$d - t \geq \sum_{i=1}^{s-1} n_i + 1, \tag{7}$$

for some positive integer t. Consider an $\mathbb{F}_q$-linear MSRD code $\mathcal{C}_\varnothing \subseteq \prod_{i=1}^{\ell} \mathbb{F}_q^{m_i \times n_i}$ of distance $\mathrm{d}(\mathcal{C}_\varnothing) = d$, let $\{B_{u,v}\}_{u=1,v=1}^{t,m} \subseteq \prod_{i=1}^{\ell} \mathbb{F}_q^{m_i \times n_i}$ be a set of $\mathbb{F}_q$-linearly independent tuples such that $\mathcal{C}_\varnothing \cap \langle B_{i,j} : i \in [t], j \in [m] \rangle_{\mathbb{F}_q} = 0$, and define the $\mathbb{F}_q$-linear code

$$\mathcal{C}_I = \mathcal{C}_\varnothing \oplus \langle B_{i,j} : i \in I, j \in [m] \rangle_{\mathbb{F}_q}, \tag{8}$$

for $I \subseteq [t]$. Observe that this imposes the restriction $tm + \dim_{\mathbb{F}_q}(\mathcal{C}_\varnothing) \leq \sum_{i=1}^{\ell} m_i n_i$. Given $I \subseteq [t]$, we have by definition that

$$\dim(\mathcal{C}_I) = \dim(\mathcal{C}_\varnothing) + m|I| = m(n - d + 1 + |I|).$$

We will further assume that $\mathrm{d}(\mathcal{C}_I) = d - |I|$. This implies that $\mathcal{C}_I$ is MSRD due to the Singleton bound (1), since such a bound is $m(n - d + 1 + |I|)$ in this case, since $d - |I| \geq d - t \geq \sum_{i=1}^{s-1} n_i + 1$ by (7), and $m_s = \cdots = m_\ell = m$. Observe that the family $\{\mathcal{C}_I\}_{I \subseteq [t]}$ forms a lattice of MSRD codes isomorphic to the lattice of subsets of $[t]$ by the map $I \mapsto \mathcal{C}_I$.

We now proceed to obtain a new $\mathbb{F}_q$-linear MSRD code of distance d but longer than $\mathcal{C}_\varnothing$. To that end, we consider additional lengths $m_{\ell+1}, \ldots, m_{\ell+\ell_t}, n_{\ell+1}, \ldots, n_{\ell+\ell_t}$, for integers $0 = \ell_0 < \ell_1 < \ell_2 < \cdots < \ell_t$ such that

$$m_{\ell+\ell_{i-1}+1} n_{\ell+\ell_{i-1}+1} + \cdots + m_{\ell+\ell_i} n_{\ell+\ell_i} \leq m, \tag{9}$$

for $i \in [t]$. Consider now $\mathbb{F}_q$-linear subspaces $\mathcal{V}_j \subseteq \mathbb{F}_q^m$ such that $\dim(\mathcal{V}_j) = m_{\ell+j} n_{\ell+j}$, for $j \in [\ell_t]$, and such that

$$\mathcal{V}_{\ell_{i-1}+1}, \mathcal{V}_{\ell_{i-1}+2}, \ldots, \mathcal{V}_{\ell_i}$$

form a direct sum inside $\mathbb{F}_q^m$, for $i \in [t]$. This is possible thanks to condition (9). Finally, consider $\mathbb{F}_q$-linear vector space isomorphisms

$$\varphi_j : \mathcal{V}_j \longrightarrow \mathbb{F}_q^{m_{\ell+j} \times n_{\ell+j}},$$

for $j \in [\ell_t]$.

The main construction of this section is as follows.

Construction 3 *We construct the $\mathbb{F}_q$-linear code $\mathcal{C} \subseteq \prod_{i=1}^{\ell+\ell_t} \mathbb{F}_q^{m_i \times n_i}$ as a direct sum of two subcodes $\mathcal{C}_1$ and $\mathcal{C}_2$. First, let $\mathcal{C}_1 \subseteq \prod_{i=1}^{\ell+\ell_t} \mathbb{F}_q^{m_i \times n_i}$ be equal to $\mathcal{C}_\varnothing$ but adding zeros to each codeword in the ith block for every $i \in [\ell+1, \ell+\ell_t]$. Second, let*

$$\mathcal{C}_2 = \bigoplus_{i=1}^{t} \bigoplus_{j=\ell_{i-1}+1}^{\ell_i} \left\{ \left(\sum_{k=1}^{m} \alpha_k B_{i,k}, 0, \ldots, \underbrace{\varphi_j(\boldsymbol{\alpha})}_{(\ell+j)th\ block}, \ldots, 0 \right) : \boldsymbol{\alpha} \in \mathcal{V}_j \right\} \subseteq \prod_{i=1}^{\ell+\ell_t} \mathbb{F}_q^{m_i \times n_i},$$

where we use the notation $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{F}_q^m$. Finally, define $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$.

We next show that the code $\mathcal{C}$ is an $\mathbb{F}_q$-linear MSRD code of minimum distance d.

Theorem 3 *The code $\mathcal{C}$ from Construction 3 is an $\mathbb{F}_q$-linear MSRD code of minimum sum-rank distance $\mathrm{d}(\mathcal{C}) = d$ and dimension $\dim_{\mathbb{F}_q}(\mathcal{C}) = m(n-d+1) + \sum_{i=\ell+1}^{\ell+\ell_t} m_i n_i$.*

Proof First, let

$$\mathcal{D}_j = \left\{ \left(\sum_{k=1}^{m} \alpha_k B_{i,k}, 0, \ldots, \underbrace{\varphi_j(\boldsymbol{\alpha})}_{(\ell+j)\text{th block}}, \ldots, 0 \right) : \boldsymbol{\alpha} \in \mathcal{V}_j \right\} \subseteq \prod_{i=1}^{\ell+\ell_t} \mathbb{F}_q^{m_i \times n_i},$$

for $j \in [\ell_{i-1}+1, \ell_i]$ and $i \in [t]$. Clearly $\mathcal{D}_j$ is an $\mathbb{F}_q$-linear subspace isomorphic to $\mathcal{V}_j$ and thus of dimension $m_{\ell+j} n_{\ell+j}$. Observe now that all the subspaces

$$\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{\ell_t}$$

form a direct sum inside $\prod_{i=1}^{\ell+\ell_t} \mathbb{F}_q^{m_i \times n_i}$, since a nonzero codeword in $\mathcal{D}_j$ has a nonzero component in the $(\ell+j)$th block for some $j \in [\ell_{i-1}+1, \ell_i]$, for some $i \in [t]$, and is identically zero in all the other rank blocks with indices in $[\ell+1, \ell+\ell_t]$. Therefore we indeed have that

$$\mathcal{C}_2 = \bigoplus_{i=1}^{t} \bigoplus_{j=\ell_{i-1}+1}^{\ell_i} \mathcal{D}_j.$$

In particular, we have that

$$\dim(\mathcal{C}_2) = \sum_{i=1}^{t} \sum_{j=\ell_{i-1}+1}^{\ell_i} \dim(\mathcal{D}_j) = \sum_{i=\ell+1}^{\ell+\ell_t} m_i n_i.$$

Similarly, since every nonzero codeword in $\mathcal{C}_2$ contains a nonzero element in at least one of the blocks in the positions $j \in [\ell+1, \ell+\ell_t]$ and $\mathcal{C}_1$ is identically zero in those positions, we also deduce that $\mathcal{C}_1 \cap \mathcal{C}_2 = 0$. In particular, it holds indeed that $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$, and

$$\dim(\mathcal{C}) = \dim(\mathcal{C}_1) + \dim(\mathcal{C}_2) = m(n-d+1) + \sum_{i=\ell+1}^{\ell+\ell_t} m_i n_i.$$

Now we show that the minimum distance of $\mathcal{C}$ is d. A codeword in $\mathcal{C}$ is of the form

$$C = \left(D + \sum_{i=1}^{t} \sum_{j=\ell_{i-1}+1}^{\ell_i} \sum_{k=1}^{m} \alpha_{j,k} B_{i,k}, \varphi_1(\boldsymbol{\alpha}_1), \ldots, \varphi_{\ell_t}(\boldsymbol{\alpha}_{\ell_t}) \right),$$

where $D \in \mathcal{C}_\varnothing$ and $\boldsymbol{\alpha}_j = (\alpha_{j,1}, \ldots, \alpha_{j,m}) \in \mathcal{V}_j$, for $j \in [\ell_t]$. Set

$$I = \{i \in [t] \mid \exists j \in [\ell_{i-1}+1, \ell_i] \text{ such that } \boldsymbol{\alpha}_j \neq \mathbf{0}\}.$$

Then we have

$$C = \left(D + \sum_{i \in I} \sum_{j=\ell_{i-1}+1}^{\ell_i} \sum_{k=1}^{m} \alpha_{j,k} B_{i,k}, \varphi_1(\boldsymbol{\alpha}_1), \ldots, \varphi_{\ell_t}(\boldsymbol{\alpha}_{\ell_t}) \right).$$

On the first ℓ blocks, we have the codeword

$$D + \sum_{i \in I} \sum_{j=\ell_{i-1}+1}^{\ell_i} \sum_{k=1}^{m} \alpha_{j,k} B_{i,k} \in \mathcal{C}_I. \tag{10}$$

Given $i \in I$, observe that $\sum_{k=1}^{m} \left(\sum_{j=\ell_{i-1}+1}^{\ell_i} \alpha_{j,k} \right) B_{i,k} \neq 0$, since $B_{i,1}, \ldots, B_{i,m}$ are $\mathbb{F}_q$-linearly independent, $\mathcal{V}_{\ell_{i-1}+1}, \ldots, \mathcal{V}_{\ell_i}$ form a direct sum inside $\mathbb{F}_q^m$ and there is at least one $j \in [\ell_{i-1}+1, \ell_i]$ such that $\boldsymbol{\alpha}_j \neq \mathbf{0}$. In particular,

$$\sum_{i \in I} \left(\sum_{j=\ell_{i-1}+1}^{\ell_i} \sum_{k=1}^{m} \alpha_{j,k} B_{i,k} \right) \neq 0$$

since $\{B_{i,j}\}_{i=1,j=1}^{t,m}$ are $\mathbb{F}_q$-linearly independent. Combining this fact with $\mathcal{C}_\varnothing \cap \langle B_{i,j} : i \in [t], j \in [m] \rangle_{\mathbb{F}_q} = 0$, we conclude that the codeword in (10) is zero if, and only if, $D = 0$ and $I = \varnothing$, which is equivalent to C being zero. Hence if C is nonzero, then

$$\mathrm{wt}\left(D + \sum_{i \in I} \sum_{j=\ell_{i-1}+1}^{\ell_i} \sum_{k=1}^{m} \alpha_{j,k} B_{i,k} \right) \geq \mathrm{d}(\mathcal{C}_I) = d - |I|.$$

Finally, since there is at least one $j \in [\ell_{i-1}+1, \ell_i]$ such that $\boldsymbol{\alpha}_j \neq \mathbf{0}$, for every $i \in I$, then

$$\mathrm{wt}(\varphi_1(\boldsymbol{\alpha}_1), \ldots, \varphi_{\ell_t}(\boldsymbol{\alpha}_{\ell_t})) \geq |I|,$$

and we conclude that $\mathrm{wt}(C) \geq d$ if C is nonzero. In other words, $\mathrm{d}(\mathcal{C}) \geq d$, but equality must hold by the Singleton bound (1), thus $\mathrm{d}(\mathcal{C}) = d$ and we are done. □

5.2 Concrete examples

Lattices of MSRD codes were studied in Martínez-Peñas (2023) in order to extend the MSRD codes from Martínez-Peñas (2022a), i.e., those from Sect. 2.2. However, the extensions from Martínez-Peñas (2023) only added blocks of matrices of size $1 \times m$. Using the technique from Sect. 5.1, we now give extensions of the MSRD codes from Sect. 2.2 for new ranges of parameters, providing new constructions of MSRD codes.

Consider $m = m_1 = \cdots = m_\ell$ and $r = n_1 = \cdots = n_\ell \leq m$ for $i \in [\ell]$, and set $n = \ell r$. Let k and t be positive integers such that $t + k \leq n$ and let $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{t+k} \in \mathbb{F}_{q^m}^n$ be $\mathbb{F}_{q^m}$-linearly independent. For $I \subseteq [t]$, define the $\mathbb{F}_{q^m}$-linear code $\mathcal{D}_I = \langle \mathbf{g}_i : i \in I \rangle_{\mathbb{F}_{q^m}} \oplus \langle \mathbf{g}_{t+1}, \ldots, \mathbf{g}_{t+k} \rangle_{\mathbb{F}_{q^m}} \subseteq \mathbb{F}_{q^m}^n$, and assume that it is MSRD, that is,

$$\dim_{\mathbb{F}_{q^m}}(\mathcal{D}_I) = k + |I| \quad \text{and} \quad \mathrm{d}(\mathcal{D}_I) = n - k - |I| + 1.$$

If $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_m)$ forms an ordered basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ and we define $\mathcal{C}_I = M_{\boldsymbol{\gamma}}^{\mathbf{n}}(\mathcal{D}_I) \subseteq \prod_{i=1}^{\ell} \mathbb{F}_q^{m \times n_i}$, then $\{\mathcal{C}_I\}_{I \subseteq [t]}$ forms a lattice of $\mathbb{F}_q$-linear MSRD codes as in Sect. 5.1, where $\mathrm{d}(\mathcal{C}_\varnothing) = d = n - k + 1$ and $\mathrm{d}(\mathcal{C}_I) = d - |I|$, for $I \subseteq [t]$. In Construction 3, we set $B_{i,j} = M_{\boldsymbol{\gamma}}^{\mathbf{n}}(\gamma_j \mathbf{g}_i)$, for $i \in [t]$ and $j \in [m]$, and the condition $tm + \dim_{\mathbb{F}_q}(\mathcal{C}_\varnothing) \leq mn$ is satisfied. Note also that we may take $s = 1$ since $m_1 = \cdots = m_\ell = m$ and $d - t \geq 1$.

When $t = 2$, one way of constructing the vectors $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{t+k} \in \mathbb{F}_{q^m}^n$ is as follows. Consider

$$\begin{pmatrix} \mathbf{g}_1 \\ \hline \mathbf{g}_3 \\ \mathbf{g}_4 \\ \vdots \\ \mathbf{g}_{k+2} \\ \hline \mathbf{g}_2 \end{pmatrix} = \left(\begin{array}{lll|c|lll} \beta_1 & \ldots & \beta_{\mu r} & \ldots & \beta_1 & \ldots & \beta_{\mu r} \\ \beta_1^q a_1 & \ldots & \beta_{\mu r}^q a_1 & \ldots & \beta_1^q a_{q-1} & \ldots & \beta_{\mu r}^q a_{q-1} \\ \beta_1^{q^2} a_1^{\frac{q^2-1}{q-1}} & \ldots & \beta_{\mu r}^{q^2} a_1^{\frac{q^2-1}{q-1}} & \ldots & \beta_1^{q^2} a_{q-1}^{\frac{q^2-1}{q-1}} & \ldots & \beta_{\mu r}^{q^2} a_{q-1}^{\frac{q^2-1}{q-1}} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \beta_1^{q^k} a_1^{\frac{q^k-1}{q-1}} & \ldots & \beta_{\mu r}^{q^k} a_1^{\frac{q^k-1}{q-1}} & \ldots & \beta_1^{q^k} a_{q-1}^{\frac{q^k-1}{q-1}} & \ldots & \beta_{\mu r}^{q^k} a_{q-1}^{\frac{q^k-1}{q-1}} \\ \beta_1^{q^{k+1}} a_1^{\frac{q^{k+1}-1}{q-1}} & \ldots & \beta_{\mu r}^{q^{k+1}} a_1^{\frac{q^{k+1}-1}{q-1}} & \ldots & \beta_1^{q^{k+1}} a_{q-1}^{\frac{q^{k+1}-1}{q-1}} & \ldots & \beta_{\mu r}^{q^{k+1}} a_{q-1}^{\frac{q^{k+1}-1}{q-1}} \end{array} \right),$$

where $\ell = \mu(q-1)$, $n = \ell r$, and $a_1, \ldots, a_{q-1}, \beta_1, \ldots, \beta_{\mu r} \in \mathbb{F}_{q^m}^*$ satisfy the properties stated after equation (4). With these assumptions, $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3 \ldots, \mathbf{g}_{k+2} \in \mathbb{F}_{q^m}^n$ are $\mathbb{F}_{q^m}$-linearly independent and the $\mathbb{F}_{q^m}$-linear codes $\mathcal{D}_I = \langle \mathbf{g}_i : i \in I \rangle_{\mathbb{F}_{q^m}} \oplus \langle \mathbf{g}_3, \ldots, \mathbf{g}_{k+2} \rangle_{\mathbb{F}_{q^m}} \subseteq \mathbb{F}_{q^m}^n$, for $I \subseteq \{1, 2\}$, are MSRD by (Martínez-Peñas 2022a, Th. 3.12) and (Martínez-Peñas 2023, Lemma 5).

In (Martínez-Peñas 2023, Cor. 8), it was shown how to extend these MSRD codes by adding $t = 2$ rank blocks each formed by matrices of sizes $1 \times m$ (i.e., adding a Hamming-metric block $\mathbb{F}_{q^m}^2$). With Construction 3, we may extend them to obtain an $\mathbb{F}_q$-linear MSRD code $\mathcal{C} \subseteq \prod_{i=1}^{\ell+\ell_2} \mathbb{F}_q^{m_i \times n_i}$ with $\mathrm{d}(\mathcal{C}) = d$ by adding $t = 2$ sets of blocks of any sizes $m_{\ell+1} \times n_{\ell+1}, \ldots, m_{\ell+\ell_2} \times n_{\ell+\ell_2}$, with the only restrictions

$$m_{\ell+1} \times n_{\ell+1} + \cdots + m_{\ell+\ell_1} \times n_{\ell+\ell_1} \leq 1 \times m,$$
$$m_{\ell+\ell_1+1} \times n_{\ell+\ell_1+1} + \cdots + m_{\ell+\ell_2} \times n_{\ell+\ell_2} \leq 1 \times m,$$

where $0 < \ell_1 < \ell_2$, hence achieving more flexibility in how we may extend such MSRD codes. In particular, the extension may be obtained by adding a block with a sum-rank metric that is not the Hamming metric, in contrast with Martínez-Peñas (2023). This is the first known extension of the MSRD codes from Martínez-Peñas (2022a) by adding rank blocks of matrices of sizes different than $1 \times m$.

In (Martínez-Peñas 2023, Sect. 7), the MSRD extension as above adding a Hamming-metric block $\mathbb{F}_{q^m}^2$ was shown to be a one-weight code in some cases (that is, a code whose nonzero codewords all have the same sum-rank weight). The same result holds for the general code $\mathcal{C}$ as above. The following proposition is straightforward by (Martínez-Peñas 2023, Prop. 13).

Proposition 1 *Let $\mathcal{C} \subseteq \prod_{i=1}^{\ell+\ell_2} \mathbb{F}_q^{m_i \times n_i}$ be as above and assume that $\dim_{\mathbb{F}_q}(\mathcal{C}) = 2m$. Then $\mathcal{C}$ is a one-weight code if, and only if, $\ell_1 = 1$, $\ell_2 = 2$ and $\bigcup_{i=1}^{\mu} \mathcal{H}_i = \mathbb{F}_{q^m}$, where $\mathcal{H}_1, \ldots, \mathcal{H}_\mu$ are as in Sect. 2.2.*

A family of lattices of MSRD codes for $t = 3$ can be obtained as follows, although only for $k = 0$ (i.e., $\mathcal{D}_\varnothing = 0$), m odd and q even. Consider

$$\begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{pmatrix} = \left(\begin{array}{ccc|c|ccc} \beta_1 & \ldots & \beta_{\mu r} & \ldots & \beta_1 & \ldots & \beta_{\mu r} \\ \beta_1^q a_1 & \ldots & \beta_{\mu r}^q a_1 & \ldots & \beta_1^q a_{q-1} & \ldots & \beta_{\mu r}^q a_{q-1} \\ \beta_1^{q^2} a_1^{q+1} & \ldots & \beta_{\mu r}^{q^2} a_1^{q+1} & \ldots & \beta_1^{q^2} a_{q-1}^{q+1} & \ldots & \beta_{\mu r}^{q^2} a_{q-1}^{q+1} \end{array}\right), \tag{11}$$

where $\ell = \mu(q-1)$, $n = \ell r$, and $a_1, \ldots, a_{q-1}, \beta_1, \ldots, \beta_{\mu r} \in \mathbb{F}_{q^m}^*$ satisfy the properties stated after equation (4). If we further assume that m is odd and q is even, then it was shown in the proof of (Martínez-Peñas 2023, Th. 5) that $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3 \in \mathbb{F}_{q^m}^n$ are $\mathbb{F}_{q^m}$-linearly independent and $\mathcal{D}_I = \langle \mathbf{g}_i : i \in I \rangle_{\mathbb{F}_{q^m}} \subseteq \mathbb{F}_{q^m}^n$, for $I \subseteq \{1, 2, 3\}$, are MSRD. Notice that in this case $\mathcal{D}_\varnothing = 0$, $d = n + 1$ and $\mathrm{d}(\mathcal{D}_I) = d - |I| = n + 1 - |I|$, for $I \subseteq \{1, 2, 3\}$.

In (Martínez-Peñas 2023, Th. 3), it was shown how to extend these MSRD codes by adding $t = 3$ rank blocks each formed by matrices of sizes $1 \times m$ (i.e., adding a Hamming-metric block $\mathbb{F}_{q^m}^3$). With Construction 3, we may extend them to obtain an $\mathbb{F}_q$-linear MSRD code $\mathcal{C} \subseteq \prod_{i=1}^{\ell+\ell_3} \mathbb{F}_q^{m_i \times n_i}$ with $\mathrm{d}(\mathcal{C}) = d$ by adding $t = 3$ sets of blocks of any sizes $m_{\ell+1} \times n_{\ell+1}, \ldots, m_{\ell+\ell_3} \times n_{\ell+\ell_3}$, with the only restrictions

$$\begin{aligned} m_{\ell+1} \times n_{\ell+1} + \cdots + m_{\ell+\ell_1} \times n_{\ell+\ell_1} &\leq 1 \times m, \\ m_{\ell+\ell_1+1} \times n_{\ell+\ell_1+1} + \cdots + m_{\ell+\ell_2} \times n_{\ell+\ell_2} &\leq 1 \times m, \\ m_{\ell+\ell_2+1} \times n_{\ell+\ell_2+1} + \cdots + m_{\ell+\ell_3} \times n_{\ell+\ell_3} &\leq 1 \times m, \end{aligned}$$

where $0 < \ell_1 < \ell_2 < \ell_3$, hence achieving more flexibility in how we may extend such MSRD codes, as in the case $t = 2$ shown earlier.

6 Construction 4: Using systematic MSRD codes

In this section, we provide a construction of $\mathbb{F}_q$-linear MSRD codes based on systematic generator matrices of $\mathbb{F}_{q^m}$-linear MSRD codes in $\mathbb{F}_{q^m}^n$. We describe the general construction in Sect. 6.1 and provide concrete examples in Sects. 6.2 and 6.3.

6.1 The general construction

Consider the parameters $m = m_1 = \cdots = m_\ell$ and $n_i \leq m$, for $i \in [\ell]$. Let also $t \in [m]$, define $n = n_1 + \cdots + n_\ell$ and let $\mathcal{D}_0 \subseteq \mathbb{F}_{q^m}^{n+t}$ be an $\mathbb{F}_{q^m}$-linear MSRD code of distance $\mathrm{d}(\mathcal{D}_0) = d - t \geq 1$, for some $d \in [t+1, t+n]$, for the sum-rank length partition $(n_1, \ldots, n_\ell, t)$. Hence $\dim_{\mathbb{F}_{q^m}}(\mathcal{D}_0) = n - d + 1 + 2t$. We will set $k = n + t - d + 1$. Consider a generator matrix of $\mathcal{D}_0$ of the form

$$G_0 = \left(\begin{array}{c|cccc} \mathbf{g}_1 & 1 & 0 & \dots & 0 \\ \mathbf{g}_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{g}_t & 0 & 0 & \dots & 1 \\ \hline \mathbf{g}_{t+1} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{g}_{t+k} & 0 & 0 & \dots & 0 \end{array} \right) \in \mathbb{F}_{q^m}^{(t+k)\times(n+t)}, \tag{12}$$

where $\mathbf{g}_1, \dots, \mathbf{g}_{t+k} \in \mathbb{F}_{q^m}^n$. Such a generator matrix exists by Gaussian elimination and the fact that the last $\dim_{\mathbb{F}_{q^m}}(\mathcal{D}_0) \geq t$ positions form an information set of $\mathcal{D}_0$ since it is MSRD, thus MDS (see Martínez-Peñas et al. 2022, Ch. 1). Notice that G_0 is only a systematic generator matrix if $k = 0$. However, we will still call it systematic for simplicity.

Assume that there is an $\mathbb{F}_q$-linear subspace $\mathcal{V} \subseteq \mathbb{F}_{q^m}^t$ and a vector space isomorphism

$$\phi : \mathcal{V} \longrightarrow \prod_{i=\ell+1}^{\ell+u} \mathbb{F}_q^{m_i \times n_i}, \tag{13}$$

for positive integers u, $m \geq m_{\ell+1} \geq \cdots \geq m_{\ell+u}$ and $n_i \leq m_i$, for $i \in [\ell+1, \ell+u]$, such that

$$\mathrm{wt}(\phi(\boldsymbol{\lambda})) \geq \mathrm{wt}(\boldsymbol{\lambda}), \tag{14}$$

for all $\boldsymbol{\lambda} \in \mathcal{V}$. We will provide examples of such an isomorphism in Sect. 6.2. Notice that a necessary condition for its existence is

$$tm \geq m_{\ell+1}n_{\ell+1} + \cdots + m_{\ell+u}n_{\ell+u}.$$

The main construction of this section is as follows.

Construction 4 *Fix an ordered basis $\boldsymbol{\gamma} \in \mathbb{F}_{q^m}^m$ of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, set $\mathbf{n} = (n_1, \dots, n_\ell)$ and define the code in $\prod_{i=1}^{\ell+u} \mathbb{F}_q^{m_i \times n_i}$ given by*

$$\mathcal{C} = \left\{ \left(M_{\boldsymbol{\gamma}}^{\mathbf{n}} \left(\sum_{i=1}^{t+k} \lambda_i \mathbf{g}_i \right), \phi(\lambda_1, \dots, \lambda_t) \right) : (\lambda_1, \dots, \lambda_t) \in \mathcal{V}, \lambda_{t+1}, \dots, \lambda_{t+k} \in \mathbb{F}_{q^m} \right\}.$$

We next show that the code $\mathcal{C}$ is an $\mathbb{F}_q$-linear MSRD code of minimum distance d.

Theorem 4 *The code $\mathcal{C}$ from Construction 4 is an $\mathbb{F}_q$-linear MSRD code of minimum sum-rank distance* $\mathrm{d}(\mathcal{C}) = d$ *and dimension* $\dim_{\mathbb{F}_q}(\mathcal{C}) = m(n-d+1) + \sum_{i=\ell+1}^{\ell+u} m_i n_i$.

Proof Similarly to Construction 3 and Theorem 3, we may write the code as the direct sum $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$, where

$$\mathcal{C}_1 = M_{\boldsymbol{\gamma}}^{\mathbf{n}} \left(\langle \mathbf{g}_{t+1}, \dots, \mathbf{g}_{t+k} \rangle_{\mathbb{F}_{q^m}} \right) \times 0,$$

where 0 is the zero subspace in $\prod_{i=\ell+1}^{\ell+u} \mathbb{F}_q^{m_i \times n_i}$, and

$$\mathcal{C}_2 = \left\{ \left(M_{\boldsymbol{\gamma}}^{\mathbf{n}} \left(\sum_{i=1}^{t} \lambda_i \mathbf{g}_i \right), \phi(\lambda_1, \dots, \lambda_t) \right) : (\lambda_1, \dots, \lambda_t) \in \mathcal{V} \right\}.$$

It holds that $\mathcal{C}_1 \cap \mathcal{C}_2 = 0$, since any nonzero codeword in $\mathcal{C}_2$ has a nonzero component in at least one of the last u rank blocks, whereas $\mathcal{C}_1$ is identically zero in such positions. Thus $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$. Next, the claim on the dimension of $\mathcal{C}$ follows from the fact that $\dim_{\mathbb{F}_q}(\mathcal{C}_1) = m(n-d+1)$ and

$$\dim_{\mathbb{F}_q}(\mathcal{C}_2) = \dim_{\mathbb{F}_q}(\mathcal{V}) = \sum_{i=\ell+1}^{\ell+u} m_i n_i,$$

since ϕ is a vector space isomorphism.

Now let

$$C = \left(M_{\boldsymbol{\gamma}}^{\mathbf{n}} \left(\sum_{i=1}^{t+k} \lambda_i \mathbf{g}_i \right), \phi(\boldsymbol{\lambda}) \right) \in \mathcal{C} \setminus 0,$$

for $\lambda_1, \ldots, \lambda_{t+k} \in \mathbb{F}_{q^m}$, where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_t) \in \mathcal{V}$. We have that

$$\mathbf{c} = \left(\sum_{i=1}^{t+k} \lambda_i \mathbf{g}_i, \boldsymbol{\lambda} \right) \in \mathcal{D}_0,$$

which is nonzero since C is nonzero. Finally, we have that

$$\begin{aligned} \mathrm{wt}(C) &= \mathrm{wt}\left(M_{\boldsymbol{\gamma}}^{\mathbf{n}} \left(\sum_{i=1}^{t+k} \lambda_i \mathbf{g}_i \right) \right) + \mathrm{wt}\left(\phi(\boldsymbol{\lambda})\right) \\ &\geq \mathrm{wt}\left(\sum_{i=1}^{t+k} \lambda_i \mathbf{g}_i \right) + \mathrm{wt}(\boldsymbol{\lambda}) = \mathrm{wt}(\mathbf{c}) \geq \mathrm{d}(\mathcal{D}_0) = d, \end{aligned}$$

where the first inequality holds by (14). Therefore, $\mathrm{d}(\mathcal{C}) \geq d$, and by the Singleton bound (1), equality must hold. □

6.2 Concrete examples for the isomorphism ϕ

We start with a construction of the map ϕ from (13), i.e., a construction of an $\mathbb{F}_q$-linear subspace $\mathcal{V} \subseteq \mathbb{F}_{q^m}^t$ and a vector space isomorphism $\phi : \mathcal{V} \longrightarrow \prod_{i=\ell+1}^{\ell+u} \mathbb{F}_q^{m_i \times n_i}$ such that $\mathrm{wt}(\phi(\boldsymbol{\lambda})) \geq \mathrm{wt}(\boldsymbol{\lambda})$, for all $\boldsymbol{\lambda} \in \mathbb{F}_{q^m}^t$. The idea will be to partition matrices into disjoint submatrices.

Definition 1 Given $X \subseteq [m]$ and $Y \subseteq [t]$, define $\pi_{X,Y} : \mathbb{F}_q^{m \times t} \longrightarrow \mathbb{F}_q^{|X| \times |Y|}$ as the map such that $\pi_{X,Y}(C)$ is the submatrix of $C \in \mathbb{F}_q^{m \times t}$ formed by its entries in the positions $(i, j) \in X \times Y$.

Definition 2 Consider $X_1, \ldots, X_u \subseteq [m]$ and $Y_1, \ldots, Y_u \subseteq [t]$ such that $(X_i \times Y_i) \cap (X_j \times Y_j) = \varnothing$ if $i \neq j$. Next, define the surjective $\mathbb{F}_q$-linear map $\pi : \mathbb{F}_q^{m \times t} \longrightarrow \prod_{i=\ell+1}^{\ell+u} \mathbb{F}_q^{m_i \times n_i}$ by

$$\pi(C) = \left(\pi_{X_1,Y_1}(C), \ldots, \pi_{X_u,Y_u}(C) \right),$$

for $C \in \mathbb{F}_q^{m \times t}$.

We illustrate this definition with the following example.

Example 1 Consider the case $m = 4$, $t = 5$ and $u = 5$, and choose the following partition

$$\begin{array}{ll} X_1 = \{1,2,3\}, & Y_1 = \{1,2,3\}, \\ X_2 = \{4\}, & Y_2 = \{1,2\}, \\ X_3 = \{1\}, & Y_3 = \{4,5\}, \\ X_4 = \{2,3\}, & Y_4 = \{4,5\}, \\ X_5 = \{4\}, & Y_5 = \{3,4,5\}. \end{array}$$

Observe that $(X_i \times Y_i) \cap (X_j \times Y_j) = \varnothing$ if $i \neq j$. Now, the map

$$\pi : \mathbb{F}_q^{4\times 5} \longrightarrow \prod_{i=1}^{5} \mathbb{F}_q^{|X_i|\times|Y_i|}$$

from Definition 2 essentially consists in partitioning a matrix from $\mathbb{F}_q^{4\times 5}$ as follows:

$$\left(\begin{array}{|ccc|cc|} \hline c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} & c_{1,5} \\ \cline{4-5} c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} & c_{2,5} \\ \cline{4-5} c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} & c_{3,5} \\ \hline c_{4,1} & c_{4,2} & \multicolumn{1}{|c}{c_{4,3}} & c_{4,4} & c_{4,5} \\ \hline \end{array} \right).$$

In this example, each set X_i consists of consecutive numbers in $[m]$, and similarly for the sets Y_i. Furthermore, in this example $[m] \times [t] = \bigcup_{i=1}^{5} X_i \times Y_i$. However, these two properties do not need to hold according to Definition 2.

Let the notation and assumptions be as in Definition 2. By the well-known properties of ranks of matrices and their submatrices, it holds that

$$\mathrm{Rk}(C) \leq \sum_{i=1}^{u} \mathrm{Rk}(\pi_{X_i,Y_i}(C)), \tag{15}$$

for all $C \in \mathbb{F}_q^{m\times t}$. Therefore, we may define the map ϕ and the subspace $\mathcal{V}$ as follows.

Definition 3 Consider $X_1, \ldots, X_u \subseteq [m]$ and $Y_1, \ldots, Y_u \subseteq [t]$ such that $(X_i \times Y_i) \cap (X_j \times Y_j) = \varnothing$ if $i \neq j$. Let $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_m)$ be an ordered basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, and set

$$\mathcal{U} = \left\{ (c_{i,j})_{i=1,j=1}^{m,t} \in \mathbb{F}_q^{m\times t} : c_{i,j} = 0, \text{ for } (i,j) \in ([m]\times[t]) \setminus \bigcup_{s=1}^{u} (X_s \times Y_s) \right\}.$$

Finally, define $\mathcal{V} = (M_{\boldsymbol{\gamma}}^t)^{-1}(\mathcal{U}) \subseteq \mathbb{F}_{q^m}^t$ and the map $\phi : \mathcal{V} \longrightarrow \prod_{i=\ell+1}^{\ell+u} \mathbb{F}_q^{m_i\times n_i}$ given by

$$\phi(\boldsymbol{\lambda}) = \pi\left(M_{\boldsymbol{\gamma}}^t(\boldsymbol{\lambda})\right),$$

for $\boldsymbol{\lambda} \in \mathcal{V}$, where π is as in Definition 2.

The following result is straightforward using (15).

Proposition 2 *The map* $\phi : \mathcal{V} \longrightarrow \prod_{i=\ell+1}^{\ell+u} \mathbb{F}_q^{m_i\times n_i}$ *from Definition 3 is a vector space isomorphism such that* $\mathrm{wt}(\phi(\boldsymbol{\lambda})) \geq \mathrm{wt}(\boldsymbol{\lambda})$, *for all* $\boldsymbol{\lambda} \in \mathbb{F}_{q^m}^t$.

6.3 Concrete examples of MSRD codes

We now provide examples of systematic matrices as in (12), and therefore examples of MSRD codes coming from Construction 4. We will make use of the $\mathbb{F}_{q^m}$-linear MSRD codes from Sect. 2.2.

Consider positive integers $m = m_1 = \cdots = m_\ell$ and $r = n_1 = \cdots = n_\ell = t \leq m$. Assume also that $\ell + 1 = \mu(q-1)$ and let $n = n_1 + \cdots + n_\ell = \ell r$, for some positive integer μ. Let $a_1, \ldots, a_{q-1}, \beta_1, \ldots, \beta_{\mu r} \in \mathbb{F}_{q^m}^*$ satisfy the properties stated after equation (4). Set $k = n + t - d + 1$ for some $d \in [t+1, t+n]$. We may choose $\mathcal{D}_0 \subseteq \mathbb{F}_{q^m}^{n+t}$ in Construction 4 as the $\mathbb{F}_{q^m}$-linear MSRD code with generator matrix $M_{t+k}(\mathbf{a}, \boldsymbol{\beta}) \in \mathbb{F}_{q^m}^{(t+k)\times(n+t)}$, given in Sect. 2.2, or the $\mathbb{F}_{q^m}$-linear MSRD code with parity-check matrix $M_{n-k}(\mathbf{a}, \boldsymbol{\beta}) \in \mathbb{F}_{q^m}^{(t+k)\times(n+t)}$, for the sum-rank length partition $(n_1, \ldots, n_\ell, t) = (r, \ldots, r)$ ($\ell + 1$ times). Observe that $\mathrm{d}(\mathcal{D}_0) = d - t \geq 1$ and $\dim_{\mathbb{F}_{q^m}}(\mathcal{D}_0) = t + k$. Finally, by Gaussian elimination, we may obtain a generator matrix of $\mathcal{D}_0$ as in (12), for some $\mathbf{g}_1, \ldots, \mathbf{g}_{t+k} \in \mathbb{F}_{q^m}^n$.

The next step is to choose a matrix partition in order to define the vector space isomorphism ϕ as in Sect. 6.2. Let u be a positive integer and choose $X_1, \ldots, X_u \subseteq [m]$ and $Y_1, \ldots, Y_u \subseteq [t]$ such that $(X_i \times Y_i) \cap (X_j \times Y_j) = \varnothing$ if $i \neq j$. Define the $\mathbb{F}_q$-linear subspace $\mathcal{V} \subseteq \mathbb{F}_{q^m}^t$ and the vector space isomorphism $\phi : \mathcal{V} \longrightarrow \prod_{i=\ell+1}^{\ell+u} \mathbb{F}_q^{m_i \times n_i}$ as in Definition 3.

By Construction 4, we obtain an $\mathbb{F}_q$-linear MSRD code $\mathcal{C} \subseteq \prod_{i=1}^{\ell+u} \mathbb{F}_q^{m_i \times n_i}$ of minimum sum-rank distance $\mathrm{d}(\mathcal{C}) = d \in [t+1, t+n]$ and dimension $\dim_{\mathbb{F}_q}(\mathcal{C}) = m(n-d+1) + \sum_{i=\ell+1}^{\ell+u} m_i n_i$, where

$$\begin{aligned} \ell = \mu(q-1) - 1, \quad & r = n_1 = \cdots = n_\ell \leq m = m_1 = \cdots = m_\ell, \\ m_{\ell+j} = |X_j| \quad & \text{and} \quad n_{\ell+j} = |Y_j|, \end{aligned}$$

for $j \in [u]$. The possible values of μ and r in this construction (which come from the code $\mathcal{D}_0$ from Martínez-Peñas (2022a)) are described in (Martínez-Peñas 2022a, Table 1).

As a concrete example, we may choose $\mu = 1$ and $r = m$, corresponding to linearized Reed–Solomon codes (Martínez-Peñas 2018 (first row in Martínez-Peñas 2022a, Table 1). In this case, we obtain an $\mathbb{F}_q$-linear MSRD code in $\prod_{i=1}^{\ell+u} \mathbb{F}_q^{m_i \times n_i}$, as above, of minimum sum-rank distance $d \in [t+1, t+n]$, where

$$\begin{aligned} \ell = q - 2, \quad & r = n_1 = \cdots = n_\ell = m_1 = \cdots = m_\ell, \\ m_{\ell+j} = |X_j| \quad & \text{and} \quad n_{\ell+j} = |Y_j|, \end{aligned}$$

for $j \in [u]$.

Remark 1 By (Martínez-Peñas 2023, Th. 1), the vectors $\mathbf{g}_1, \ldots, \mathbf{g}_{t+k} \in \mathbb{F}_{q^m}^n$ from the systematic generator matrix in (12) are such that the $\mathbb{F}_{q^m}$-linear codes $\mathcal{D}_I = \langle \mathbf{g}_i : i \in I \rangle_{\mathbb{F}_{q^m}} \oplus \langle \mathbf{g}_{t+1}, \ldots, \mathbf{g}_{t+k} \rangle_{\mathbb{F}_{q^m}} \subseteq \mathbb{F}_{q^m}^n$, for $I \subseteq [t]$, are all MSRD with $\dim_{\mathbb{F}_{q^m}}(\mathcal{D}_I) = k + |I|$. Thus we would be in the scenario of Sect. 5.2. However, using Construction 3 in this case, we may extend such codes by adding any matrix sizes $m_{\ell+1} \times n_{\ell+1}, \ldots, m_{\ell+u} \times n_{\ell+u}$, where

$$m_{\ell+\ell_{i-1}+1} n_{\ell+\ell_{i-1}+1} + \cdots + m_{\ell+\ell_i} n_{\ell+\ell_i} \leq m,$$

for $i \in [t]$, for integers $0 = \ell_0 < \ell_1 < \ell_2 < \cdots < \ell_t = u$. In particular, $m_{\ell+1} n_{\ell+1} + \cdots + m_{\ell+u} n_{\ell+u} \leq tm$.

However, the reader may easily verify that, using Construction 4, we have more flexibility in the choice of the matrix sizes $m_{\ell+1} \times n_{\ell+1}, \ldots, m_{\ell+u} \times n_{\ell+u}$ to extend the MSRD codes $\mathcal{D}_I$. For instance, it is still necessary that $m_{\ell+1} n_{\ell+1} + \cdots + m_{\ell+u} n_{\ell+u} \leq tm$, but we can easily partition matrices in order to obtain $m_{\ell+\ell_{i-1}+1}n_{\ell+\ell_{i-1}+1}+\cdots+m_{\ell+\ell_i}n_{\ell+\ell_i} > m$ for some $i \in [t]$, which is not possible with Construction 3.

This is due to the fact that we are using a stronger property than (Martínez-Peñas 2023, Th. 1), namely, we are using that $\mathcal{D}_0$ is MSRD for the sum-rank length partition $(n_1, \ldots, n_\ell, t)$ for $t > 1$.

Remark 2 Conversely, it is natural to ask whether we may use Construction 4 for the doubly and triply extended MSRD codes that we could obtain via (Martínez-Peñas 2023, Th. 1) from the lattices of MSRD codes in Sect. 5.2. However, such doubly and triply MSRD codes using (Martínez-Peñas 2023, Th. 1) are extended by adding a Hamming-metric block (and extensions by adding a rank-metric block are not possible Martínez-Peñas 2023, Prop. 11). Thus Construction 4 would not be applicable in this case.

The previous two remarks show that, due to the concrete examples from Sects. 5.2 and 6.3, one cannot always use Construction 4 instead of Construction 3 and viceversa.

7 Comparisons with previous MSRD codes

In this section, we briefly compare the concrete examples of MSRD codes that can be obtained via Constructions 1, 2, 3 and 4 with the known MSRD codes in the literature (Byrne et al. 2021; Chen 2023; Martínez-Peñas 2018, 2022a, 2023; Neri 2022; Neri et al. 2023; Santonastaso and Sheekey 2023; Santonastaso and Zullo 2023). For simplicity, we will simply show that the parameters of the MSRD codes in those works can be obtained via Constructions 1, 2, 3 and 4, whereas our constructions give rise to MSRD codes for strictly larger sets of parameters.

First, as stated at the end of Sect. 3, Construction 1 does not cover new parameters, but can be decoded faster than linearized Reed–Solomon codes for the same parameters.

Second, (Byrne et al. 2021, Const. VII.3) can be obtained applying Construction 2 recursively by choosing $\ell = t = 1$.

Next, the MSRD codes from Neri (2022); Santonastaso and Sheekey (2023) cover the same parameters as the MSRD codes from Martínez-Peñas (2022a). Now, the codes from Martínez-Peñas (2022a) correspond to those in Sect. 6.3 when choosing the trivial matrix partition $X_1 = [m]$, $Y_1 = [t]$ and $u = 1$ in order to construct the map ϕ from Sect. 6.2. Thus it is clear that the concrete MSRD codes from Sect. 6.3 (built via Construction 4) cover a strictly larger set of parameters.

Doubly extended linearized Reed–Solomon codes (Neri et al. 2023) are a particular case of the doubly and triply extended MSRD codes from Martínez-Peñas (2023). Now, the doubly extended MSRD codes from Martínez-Peñas (2023) correspond to those in Sect. 5.2 when choosing $\ell_1 = 1$, $\ell_2 = 2$, $m_{\ell+1} = m_{\ell+2} = m$ and $n_{\ell+1} = n_{\ell+2} = 1$. Similarly, the triply extended MSRD codes from Martínez-Peñas (2023) correspond to those in Sect. 5.2 when choosing $\ell_1 = 1$, $\ell_2 = 2$, $\ell_3 = 3$, $m_{\ell+1} = m_{\ell+2} = m_{\ell+3} = m$ and $n_{\ell+1} = n_{\ell+2} = n_{\ell+3} = 1$. Hence it is clear that the concrete MSRD codes from Sect. 5.2 (built via Construction 3) cover a strictly larger set of parameters.

The recent MSRD codes from (Chen 2023, Sect. 5.2) can be obtained via Construction 2, where the code $\mathcal{C}_2$ is the concrete MSRD code from Sect. 5.2 choosing $a_1 = 1$ and puncturing the blocks corresponding to $a_2, \ldots, a_{q-1}$ (i.e., choosing the generator matrix of a Gabidulin code Gabidulin 1985), and restricting added blocks to square matrices, i.e., $m_{\ell+1} = n_{\ell+1}, \ldots, m_{\ell+\ell_2} = n_{\ell+\ell_2}$. Notice that the code $\mathcal{C}_1$ in Construction 2 needs to be a trivial code of dimension m_ℓ by Theorem 2.

Finally, notice that Construction 4 cannot be obtained via Construction 3 by Remark 1. Similarly, Construction 3 cannot be obtained via Construction 4 by Remark 2. In particular, the concrete MSRD codes in Sects. 5.2 and 6.3 cover different sets of parameters.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature. The funding has been received from Universidad de Valladolid with Grant no. E-47-2022-0001486; Ministerio de Ciencia e Innovación with Grant nos. TED2021-130358B-I00, PID2022-138906NB-C21.

Declarations

Conflict of interest The author declares no Conflict of interest.

References

Byrne E, Gluesing-Luerssen H, Ravagnani A (2021) Fundamental properties of sum-rank-metric codes. IEEE Trans Info Theory 67(10):6456–6475

Cai H, Miao Y, Schwartz M, Tang X (2022) A construction of maximally recoverable codes with order-optimal field size. IEEE Trans Info Theory 68(1):204–212

Chen H (2023) New explicit good linear sum-rank-metric codes. IEEE Trans Info Theory 69(10):6303–6313

Gabidulin EM (1985) Theory of codes with maximum rank distance. Prob Info Transm 21(1):1–12

Gopi S, Guruswami V (2022) Improved maximally recoverable LRCs using skew polynomials. IEEE Trans Info Theory 68(11):7198–7214

Gorla E, Martínez-Peñas U, Salizzoni F (2023) Sum-rank metric codes. Preprint: arXiv:2304.12095

Lu H-F, Kumar PV (2005) A unified construction of space-time codes with optimal rate-diversity tradeoff. IEEE Trans Info Theory 51(5):1709–1730

Martínez-Peñas U (2018) Skew and linearized Reed–Solomon codes and maximum sum rank distance codes over any division ring. J Algebra 504:587–612

Martínez-Peñas U (2019) Theory of supports for linear codes endowed with the sum-rank metric. Des Codes Crypto 87:2295–2320

Martínez-Peñas U (2022) A general family of MSRD codes and PMDS codes with smaller field sizes from extended Moore matrices. SIAM J Disc Math 36(3):1868–1886

Martínez-Peñas U (2022) Multilayer crisscross error and erasure correction. Preprint: arXiv:2203.07238

Martínez-Peñas U (2023) Doubly and triply extended MSRD codes. Finite Fields App 91:102272

Martínez-Peñas U, Kschischang FR (2019) Universal and dynamic locally repairable codes with maximal recoverability via sum-rank codes. IEEE Trans Info Theory 65(12):7790–7805

Martínez-Peñas U, Kschischang FR (2019) Reliable and secure multishot network coding using linearized Reed–Solomon codes. IEEE Trans Info Theory 65(8):4785–4803

Martínez-Peñas U, Shehadeh M, Kschischang FR (2022) Codes in the sum-rank metric, fundamentals and applications. Found Trends Commun Inform Theory 19(5):814–1031

Neri A (2022) Twisted linearized Reed–Solomon codes: a skew polynomial framework. J Algebra 609:792–839

Neri A, Santonastaso P, Zullo F (2023) The geometry of one-weight codes in the sum-rank metric. J Combin Theory S A 194:105703

Nóbrega RW, Uchôa-Filho BF (2010) Multishot codes for network coding using rank-metric codes. In: Proc. Third IEEE Int. Workshop Wireless Network Coding, pages 1–6

Santonastaso P, Sheekey J (2023) On MSRD codes, h-designs and disjoint maximum scattered linear sets. arXiv:2308.00378

Santonastaso P, Zullo F (2023) On subspace designs. EMS Surv Math Sci 11(1):1–62
Shehadeh M, Kschischang FR (2022) Space-time codes from sum-rank codes. IEEE Trans Info Theory 68(3):1614–1637

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.