Universidad de Valladolid

Escuela de Doctorado Universidad de Valladolid

**PROGRAMA DE DOCTORADO EN MATEMÁTICAS**

TESIS DOCTORAL:

# COMBINATORIAL ASPECTS OF SEQUENCES OF BLOW-UPS

Presentada por Daniel Camazón Portela para optar al grado de
Doctor/a por la Universidad de Valladolid

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**Financiación recibida**

- Contrato predoctoral de la Universidad de Valladolid 2018.
- Participación en el proyecto financiado por AEI PGC2018-096446-B-C21.
- Participación en el proyecto financiado por MICIU/AEI/ 10.13039/501100011033 y ERDF/EU PID2022-138906NB-C21.

# Acknowledgments

May the reader forgive me the discourtesy of writing the acknowledgments in spanish.

A mis directores de tesis, Antonio Campillo y Santiago Encinas, por su inestimable ayuda para que esta tesis llegase a buen puerto, por toda su paciencia y dedicación, por todas las conversaciones y largas reuniones mantenidas y por todos los momentos compartidos dentro y fuera de la universidad durante estos años. Me siento tremendamente afortunado de haber tenido la oportunidad de trabajar con dos grandes matemáticos como ellos.

A los miembros del Departamento de Álgebra, Geometría, Topología y Análisis matemático de la Universidad de Valladolid, y a los miembros del grupo de investigación SINGACOM por todo el apoyo que he recibido durante mi periodo de formación doctotal. Especialmente, a Félix Delgado, con el que he tenido la gran suerte de compartir tareas docentes, y que además de ser un excelente matemático es también una excelente persona. Al coordinador del programa de doctorado, Alfonso Gordaliza, por su dedicación y su entrega, por estar siempre disponible y dispuesto a buscar soluciones en momentos difíciles. También a mis compañeros de doctorado Şeyma Bodur, Ignacio de Miguel y María Martín, que poco a poco se convirtieron en una pequeña gran familia, y en especial a Álvaro Samperio, que ha sido como un hermano con el que he tenido la gran suerte de poder compartir todos los buenos y malos momentos que han acontecido en todos estos años.

Me gustaría mencionar también a los miembros del Departamento de Matemáticas de la Universidad Jaume I, Carlos Galindo, Francisco Montserrat, Julio Moyano y Fernando Hernando por su estupenda acogida cada vez que visitado Castellón. Especialmente, a Carlos Jesús Moreno, con el que he compartido muchos momentos y vivencias en congresos y también fuera de la universidad, y que ha sido un hermano mayor siempre dispuesto a brindarme buenos consejos.

Me gustaría también agradecer a los miembros del Departamento de Matemáticas de la Universidad de Almería y a su equipo directivo por su maravillosa acogida y por hacerme sentir en casa. En especial, a Juan Antonio López, que sin apenas conocerme me abrío las puertas y me brindó una oportunidad de trabajar con él. También me gustariá mencionar a todos los profesores con los que he tenido la suerte de compartir tareas docentes durante este año, por todo el apoyo y la confianza que he recibido de su parte.

No quiero olvidarme tampoco de todos los compañeros y compañeras que he conocido durante estos años, y que han sido casa, lejos de casa, en muchos congresos.

Me siento tremendamente privilegiado por toda la gente maravillosa que me ha acompañado en el camino de la vida. A mi madre, por ser siempre un ejemplo de fortaleza, coraje y valentía, por su incansable defensa de la alegría y por la huella que dejó en nosotros. A mi hermana, por hacer de hermana y de madre, por todo su apoyo y su amor incondicional. A mi sobrina, porque que fue y sigue siendo uno de los mejores regalos de la vida.

A Alba, mi compañera de camino, por decidir compartir la vida conmigo y contruir un hogar juntos con Frida, por hacerme tan feliz.

# Abstract

We study sequences of blow-ups at smooth centers $Z_s \xrightarrow{\pi_s} Z_{s-1} \xrightarrow{\pi_{s-1}} \cdots \xrightarrow{\pi_2} Z_1 \xrightarrow{\pi_1} Z_0$ and the associated **sequential morphism** $\pi : Z_s \to Z_0$. To this end, we introduce the key concept of a **final divisor**, that is, an irreducible exceptional component $E_i$ of the exceptional divisor of $\pi$, strict transform of the exceptional divisor $E_i^i$ of $\pi_i$, such that there exists an open set $U_i$ on $Z_i$, with $E_i^i \subset U_i$, such that the restriction of the composition $\pi_{i+1} \circ \pi_{i+2} \circ ... \circ \pi_{s-1} \circ \pi_s$ above $U_i$ defines an isomorphism. Furthermore, we study the admissible proximity relations between two final divisors with non empty intersection.

In the case of sequences of point blow-ups in arbitrary dimension and the corresponding sequential morphisms, we define two equivalence relations on the set of sequential morphisms: the algebraic equivalence and the combinatorial equivalence, which allow us to classify them. By proving a result that characterizes final divisors in terms of some relations defined over the Chow group of zero-cycles of its sky, we are able to recover the sequence of blow-ups, modulo algebraic equivalence, from the associated sequential morphism. As a result, we establish a connection between the corresponding algebraic and combinatorial equivalence classes of these two objects. Moreover, when the ground $Z_0$ is a projective space, we give an explicit presentation of the Chow ring $A^\bullet(Z_s)$ of the sky $Z_s$ of a sequential morphism obtained from point blow-ups and we obtain a surprising result: this Chow ring depends only on the number of blow-ups.

In the case of sequences of point and rational curve blow-ups with ground $\mathbb{P}^3$, we also characterize final divisors by explicitly giving their defining relations over $A_0(Z_s)$, and we introduce an explicit presentation of the Chow ring of its sky $A^\bullet(Z_s)$. By contrast to the case of sequences of point blow-ups, we prove that two sequences of point and

rational curve blow-ups may have non-isomorphic Chow rings even if they have the same length and proximity relations.

# Contents

# Introduction

## Sequences of blow-ups in the literature

Sequences of blow-ups play an essential role in Algebraic Geometry research. Let us mention some works where sequences of blow-ups appear as a key tool for resolution of singularities. In the foundational work [25] Hironaka proved the resolution of singularities in the category of algebraic schemes over a field of characteristic zero. In [40] Villamayor takes those results as starting point and exhibits a constructive resolution of singularities, and in [16] Encinas and Villamayor study a constructive proof of desingularization, as the outcome of a process obtained by successively blowing up the maximum stratum of a function. See [23],[24],[6] for more details and references.

Two questions, concerning factorization of birational morphisms and maps by blow-ups along smooth varieties, are of fundamental importance in birational algebraic geometry. Let $X^{'}$ and $X^{''}$ be complete smooth algebraic varieties which are birationally equivalent. Does there exist a third variety $X$ and birational morphisms $X \to X^{'}$, $X \to X^{''}$, which are compositions of blow-ups along closed smooth irreducible subvarieties? Does there exist a sequence of varieties $X_i$, for $i = 0, ..., n$, such that $X_0 = X^{'}$, $X_n = X^{''}$, and $X_{i+1}$, is obtained from $X_i$ by a blow-up or blow-down along a closed smooth irreducible subvariety?

In [11] Danilov managed to generalize the Zariski theorem [42] proving that every projective and birational morphism between smooth algebraic varieties whose fibres are of dimension $\leq 1$ is a composition of blowing ups at smooth centers of codimension 2. In [36] Sancho proved that a proper and birational morphism $\pi : X^{'} \to X$ between regular schemes whose fibres are of dimension $\leq 1$ factors, locally, through a blowing-up at a

regular center of codimension 2. Furthermore if $\pi$ is projective then $\pi$ is a composition of blowing-ups at regular centers. In [41] Wlodarczyk proved that if $\phi : X_1 \to X_2$ is a toric birational map between two complete smooth toric varieties of the same dimension, then $\phi$ can be decomposed in a sequence of equivariant blow-ups and blow-downs along smooth centers. More recently, in [1] Abramovich, Karu, Matsuki and Wlodarczyk proved that if $\phi : X_1 \to X_2$ is a birational map between complete nonsingular algebraic varieties $X_1$ and $X_2$ over an algebraically closed field $K$ of characteristic zero, and $U \subset X_1$ is an open set where $\phi$ is an isomorphism, then $\phi$ can be factored into a sequence of blow-ups and blow-downs with nonsingular irreducible centers disjoint from $U$.
On the blow-up counterpart, that is blow-downs, in [30] Lascu formulates the equivalent of Castelnuovo-Enriques conditions for the existence of regular contractions in higher dimensional varieties, and Ishii in [27] gives a necessary and sufficient condition for a subvariety of a projective non-singular variety to be contracted in an algebraic variety which is again nonsingular projective, and study some geometric properties of the contraction.

Given a sequence of blow-ups, there is a simple combinatorial object associated to it, the dual complex, that generalizes to higher dimension the well known dual graph associated to a sequence of point blow-ups in two dimensional varieties (see e.g. [8, Sect 4.4]). The dual complex of an simple normal crossing divisor $E$, denoted $D(E)$, is a CW-complex whose vertices correspond to the irreducible components $E_i$ of $E$, and whose cells of dimension $d$ are in correspondence with strata of codimension $d$ in $E$, i.e. the connected components of the intersection $E_{i_0} \cap \cdots \cap E_{i_d}$. Over the complex numbers, one can think of $D(E)$ as the combinatorial part of the topology of $E$. In [9] Castellanos proved that the dual complex with appropriate weights determines the complete geometry of all infinitely near points associated with a given curve singularity. In [7] Campillo and Reguera studied morphisms given by composition of a sequence of point blow-ups of smooth $d-$dimensional varieties in terms of combinatorial information coming from the $d-$ary intersection form on divisors with exceptional support. In [39] Tsuchihashi proves that an arbitrary weighted graph on a compact topological surface is a weighted dual graph of a toric divisor arising as the exceptional set of a resolution of a $3-$dimensional cusp singularity, if and only if it satisfies the monodromy condition and the convexity condition.

Given an embedding of a variety $X$ in a complete variety $V$ as an open and dense part, and the polyhedron $\Pi(D)$ of the "infinite part" $D = V \setminus X$, Danilov proved in [10] that when $D$ is a divisor with transversally crossing components the homotopy type of the polyhedron $\Pi(D)$ does not change under a monoidal transformation of $V$ with center in $D$. With some standard assumptions on resolution of singularities, this assertion shows that the homotopy type of $\Pi(D)$ depends only on $X$ (and is called the polyhedron of $X$ at infinity). In [37] Stepanov proved that if $\pi^{'} : (Y^{'}, E^{'}) \to (X, o)$ and $\pi^{''} : (Y^{''}, E^{''}) \to (X, o)$ are two log-resolutions of an isolated singularity $(X, o)$, then the topological spaces $D(E^{'})$ and $D(E^{''})$ have the same homotopy type. Moreover, in [38] he showed that the highest cohomologies of the dual complex associated to a resolution of an isolated rational singularity vanish, and he proved that the dual complex associated to a resolution of an isolated hypersurface singularity is simply connected. In [12], de Fernex, Kollár and Xu prove that the dual complex of a singularity is well defined, up to homotopy, and in many cases, for instance for isolated singularities, they identify and study a "minimal" representative of the homotopy class that is well defined up to piecewise linear homeomorphism.

In [29] Kóllar proves that every simplicial complex is the dual complex of some simple normal crossing divisor in a smooth variety and he extends earlier results on the existence of singularities with prescribed dual complex. In [4] Arapura, Bakhtary and Wlodarczyk prove that the homotopy type of the dual complex of $E$ depends only the the complement $X \setminus E$, and in fact only on its proper birational class.

Given $X$ a connected, smooth, and proper $K$-variety of dimension $n$, where $K$ is the quotient field of a complete discrete valuation ring $R$ with residue field $k$,in the survey [35] Nicaise studies the connections between $X^{an}$, the Berkovich analytification of $K$, and a *sncd*-model of $X$ over $R$, that is, a regular scheme $\mathcal{X}$ of finite type over $R$, endowed with an isomorphism of $K-$schemes $\mathcal{X}_K \to X$, such that the special fiber $\mathcal{X}_k$ is a divisor with strict normal crossings. In particular, one can attach a subspace $Sk(\mathcal{X})$ of $X^{an}$, to any proper *sncd*$-$model $\mathcal{X}$ of $X$ over $R$, called the Berkovich skeleton of $X$, which is canonically homeomorphic to the dual intersection complex of the strict normal crossings divisor $\mathcal{X}_k$. This skeleton, that can be viewed as the space of real valuations on the function field of $X$ that extend the discrete valuation on $K$ and that are monomial with respect to $\mathcal{X}_k$, controls the homotopy type of $X^{an}$ (it is a strong deformation retract of $X^{an}$), providing an interesting link between the geometry of $X^{an}$ and the birational geometry of models of $X$.

Intersection theory has played a key roll in the development of some important results in birational geometry. In the foundational work [33] Mori realized that given an smooth projective variety $X$, then the part of the cone of curves lying in the open half-space where the intersection number with the canonical class is negative is locally finitely generated, its generators being called the extremal rays of $X$. Moreover, he showed that every extremal ray is spanned by the homology class of a rational curve $[C]$, and that in dimension 3, for every extremal ray $R$ of $X$ there is a unique morphism $g_R : X \to Z$, called the contraction of $R$, such that an irreducible curve $C \subset X$ is mapped to a point by $g_R$ iff $[C] \in R$. As a consequence, new completely different way of thinking about morphisms of varieties arose, and since then everyone imagines an extremal ray or face of a cone.
Regarding the computation of birational invariants, in [2] Aluffi proves that given a birational map between smooth algebraic varieties $\varphi : V \dashrightarrow W$ which does not change the canonical class, then the total homology Chern classes of $V$ and $W$ are push-forwards of the same class from a resolution of indeterminacies of $\varphi$. As an example, Aluffi proves that the push-forward of the total Chern class of a crepant resolution of a singular variety is independent of the resolution. Furthermore, Aluffi in [3] introduces a notion of integration on the category of proper birational maps to a given variety $X$, with value in an associated Chow group, whose applications include new birational invariants and comparison results for Chern classes and numbers of nonsingular birational varieties.

## Overview of the thesis

This thesis is devoted to the study of sequences of blow-ups with regular centers (see Definition 2.1.1):

$$Z_s \xrightarrow{\pi_s} Z_{s-1} \xrightarrow{\pi_{s-1}} \cdots \xrightarrow{\pi_2} Z_1 \xrightarrow{\pi_1} Z_0.$$

In particular, we consider the final exceptional divisor $E$ in $Z_s$ and study how the intersection of the irreducible components $E_i$ can give invariants on the total morphism $\pi : Z_s \to Z_0$. As a consequence, different sequences of blow-ups realizing the same morphism will have the same invariants.

We focus in conditions characterizing when an irreducible component can correspond to the last blow-up of the sequence (final components).
First we restrict to the special case of point blow-ups (Chapters 3 and 4) and we have obtained surprising results on the Chow ring $A^{\bullet}(Z_s)$.
Then, we restrict to the case $Z_0 = \mathbb{P}^3$ and consider sequences of blow-ups with centers that are either points or rational curves.

The thesis is divided into six chapters whose main results are summed up in the rest of the introduction.

## The content of Chapter 2

In Chapter 2 we introduce the basic objects of this research, that is, sequences of blow-ups, sequential morphisms and final divisors. In Section 2.1 we define the key concepts of sequences of blow-ups at smooth centers (Definition 2.1.1) and sequential morphism (Definition 2.1.2). Given a sequence of blow-ups $Z_s \xrightarrow{\pi_s} Z_{s-1} \xrightarrow{\pi_{s-1}} \cdots \xrightarrow{\pi_2} Z_1 \xrightarrow{\pi_1} Z_0$, we will say that $Z_s$ and $Z_0$ are the **sky** and the **ground** of the sequence, respectively. Moreover, we generalize the usual proximity relations for higher dimensional centers (Definitions 2.1.5 and 2.1.7) which lead us to introduce a new proximity relation, the $t-$proximity.
In Section 2.2 we give a short result (Lemma 2.2.1) about the normal bundle of the complete intersection of two irreducible components of the exceptional divisor. Section 2.3 is devoted to the definition of the key concept of final divisor for both sequences of blow-ups and sequential morphisms.

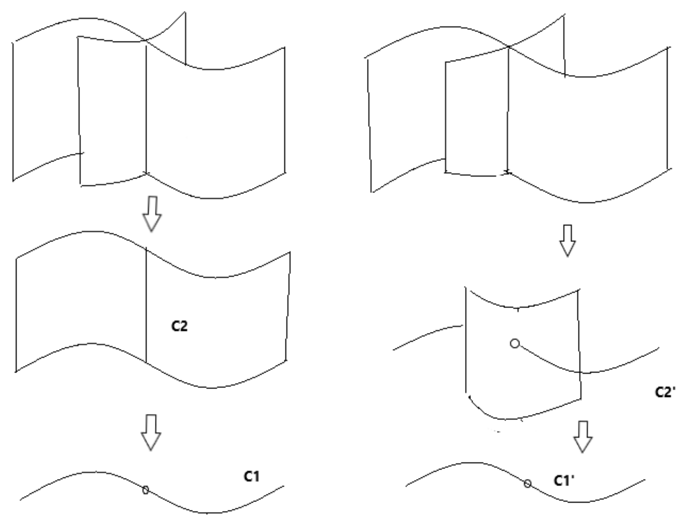
**Definition.** *2.3.1 Let $(Z_s, ..., Z_0, \pi)$ be a sequence of blow-ups as in Definition 2.1.1. The components of the exceptional divisor $E$ in $Z_s$ are $\{E_1, ..., E_s\}$. Assume that $E_i$ is an irreducible component. Set $E_i^i$ to be the image of $E_i$ in $Z_i$. We say that $E_i$ is **final** with respect to $(Z_s, ..., Z_0, \pi)$ if there exists an open set $U_i$ on $Z_i$ such that $E_i^i \subset U_i$, $V_i = \pi_{s,i}^{-1}(U_i) \subset Z_s$, and $\pi_{s,i}|_{V_i} : V_i \to U_i$ is an isomorphism (see Remark 2.1.3 for $\pi_{s,i}$).*

**Definition.** *2.3.3 Let $\pi : Z_s \to Z_0$ be a sequential morphism. We say that an irreducible component $E_i$ of $E$ is **final** if there exists a sequence of blow-ups $(Z_s, ..., Z_0, \pi)$ associated to $\pi : Z_s \to Z_0$ such that $E_i$ is final with respect to this sequence.*

Moreover, we recall the concept of regular projective contraction (Definition 2.3.4), and characterize the only admissible proximity relations between two final divisors with non empty intersection within the next result.

**Theorem.** *2.3.10 Let $E_i, E_j \subset Z_s$ be both final divisors for the sequential morphism $\pi : Z_s \to Z_0$. Then $E_i \cap E_j \neq \emptyset$ if and only if $E_i$ is proximate to $E_j$ and $E_j$ is $t-$proximate to $E_i$, or vice versa.*

Figure 1: Example of two blow up processes which lead to a same sequential morphism with two intersecting final divisors




Finally, in Section 2.4, in order to obtain a combinatorial object associated to the intersections of the exceptional divisor, we have the $n-$ary multilinear intersection form on the abelian group of divisors with exceptional support (Definition 2.4.1), that will be intensively used in Chapters 3, 4 and 6 in order to give a numerical characterization of final divisors.

## The content of Chapter 3

In Chapter 3 we focus on the study of sequences of blow-ups as in Definition 2.1.1 over algebraically closed fields, where all the centers $C_{i+1}$ are points. In section 3.1 we define the notions of algebraic and combinatorial equivalence for both sequences of points blow-ups and sequential morphisms (Definitions 3.1.1, 3.1.2, 3.1.5 and 3.1.7). Roughly speaking, algebraic equivalence is determined by the existence of an isomorphism between the skies, that is between the varieties obtained after the last blow-up of the sequences, whereas combinatorial equivalence is determined by the existence of a permutation relating the $n-$ary intersection forms.

Section 3.2 is devoted to give a numerical characterization of final divisors in terms of values of the $n-$ary intersection form on the abelian groups of divisors with exceptional support.

**Proposition.** *3.2.4 $E_i$ is final if and only if*

$$(e_i)^n = (-1)^{n-1}$$

In Section 3.3, we use the previous result in order to recover the sequences of point blow-ups from the associated sequential morphism modulo algebraic equivalence.

**Theorem.** *3.3.3 Let $\pi : Z_s \to Z_0$ be a sequential morphism. Given the n$-$ary multilinear intersection form we can recover all the sequences of point blow-ups that are associated to sequential morphisms in the same algebraic equivalence class of $\pi : Z_s \to Z_0$.*

Section 3.4 is devoted to prove some relations between algebraic and combinatorial equivalence classes of sequences of point blow-ups and sequential morphisms.

**Proposition.** *3.4.1 Any of the sequences obtained as in 3.3.4, that is, by decomposing a regular projective contraction from a fixed sky $Z_s$ and a fixed simple normal crossing divisor $E$, are associated to sequential morphisms in the same algebraic equivalence class (see Definition 3.1.1).*

**Theorem.** *3.4.5 Two sequences of point blow-ups $(Z_s, ..., Z_0, \pi)$ and $(Z_s^{'}, ..., Z_0^{'}, \pi^{'})$, with $s = s^{'}$, are combinatorially equivalent as in Definition 3.1.7 if and only if their associated sequential morphisms $\pi : Z_s \to Z_0$ and $\pi^{'} : Z_s^{'} \to Z_0^{'}$ are combinatorially equivalent as in Definition 3.1.2, and both statements are true if and only if the associated multilinear maps $\Phi_{Z,E}$ and $\Phi_{Z^{'},E^{'}}$ are equivalent too as in Definition 3.1.2*

**Theorem.** *3.4.7 Given two sequential morphisms $\pi : Z_s \to Z_0$ and $\pi^{'} : Z_{s^{'}}^{'} \to Z_0^{'}$, then they are algebraically equivalent as in Definition 3.1.1 if and only if there are sequences* *of point blow-ups* $(Z_s, ..., Z_0, \pi)$ *and* $(Z'_{s'}, ..., Z'_0, \pi')$ *associated to* $\pi : Z_s \to Z_0$ *and* $\pi' : Z'_{s'} \to Z'_0$ *respectively such that they are algebraically equivalent as in Definition 3.1.5.*

Finally, in Section 3.5, we give two explicit presentations of the Chow ring of the sky of a sequence of point blow-ups $A^\bullet(Z_s)$ when $Z_0 \cong \mathbb{P}^n$. The first one using the classes of the total transforms of the exceptional components as generators and the second one using the classes of the strict transforms ones.

**Theorem.** *3.5.3 The Chow ring of the sky* $A^\bullet(Z_s)$*, when* $Z_0 \cong \mathbb{P}^n$*, is isomorphic to*

$$A^\bullet(Z_s) \cong \mathbb{Z}\left[x_0, x_1, ..., x_s\right] / (\{x_i \cdot x_j\}_{\substack{i,j=0 \\ i \neq j}}^{s}, \{(-1)^n (x_i)^n + (x_0)^n\}_{i=1}^{s}),$$

*by sending* $x_0$ *to the class* $h^{s*}$ *and* $x_i$ *to the class* $e_i^{s*}$ *for* $i = 1, ..., s$.

**Theorem.** *3.5.6 A presentation of* $A^\bullet(Z_s)$*, when* $Z_0 \cong \mathbb{P}^n$*, using* $\left\{\widetilde{h}^s, \{e_i^s\}_{i=1}^{s}\right\}$ *as generators is the following one:*

$$A^\bullet(Z_s) \cong \frac{\mathbb{Z}\left[y_0, y_1, ..., y_s\right]}{\mathcal{A}}, \tag{1}$$

*where*

$$\mathcal{A} = ((\{y_0 \cdot y_i\}_{i=1}^{s}, \left\{(y_i + \sum_{k=i+1}^{s} b_{k,i} y_k) \cdot (y_j + \sum_{l=j+1}^{s} b_{l,j} y_l)\right\}_{\substack{i,j=1 \\ i \neq j}}^{s}, \left\{(y_i)^n + ((-1)^n + \#\{j\}_{j \to i})(y_0)^n\right\}_{i=1}^{s})),$$

*by sending* $y_0$ *to* $h^{s*}$ *and* $y_i$ *to* $e_i^s$ *for* $i = 1, ..., s$.

Furthermore, we prove the surprising fact that when $Z_0 \cong \mathbb{P}^n$ the skies of two sequences of point blow-ups of the same length have isomorphic Chow rings (Corollary 3.5.5).

## The content of Chapter 4

In Chapter 4 we extend the results of the previous one in order to consider sequences of point blow-ups over perfect fields. This more general setting lead us to define in Section 4.1 algebraically and combinatorially compatible partitions of the exceptional divisor.

**Definition.** *4.1.4 Given a sequential morphism $\pi : Z_s \to Z_0$ as in Definition 2.1.2 and a partition $E = \sqcup_{i=1}^{l} F_i$, we will say that the partition is combinatorially compatible with $\pi$ if for each $i = 1, .., l$, and $H_{j_1}, H_{j_2} \in F_i$ there exists $\sigma \in S_m$ such that*

*a* $\sigma(j_1) = j_2$,

*b* $\mathcal{I}_{Z_s,E}(H_{i_1}, H_{i_2}, ..., H_{i_n}) = \mathcal{I}_{Z_s,E}(H_{\sigma(i_1)}, H_{\sigma(i_2)}, ..., H_{\sigma(i_d)})$ $\forall i_1, .., i_n$

**Definition.** *4.1.5 Given a sequence of point blow-ups $(Z_0, ..., Z_s, \pi)$ and a partition of the exceptional divisor $E = \sqcup_{i=1}^{l} F_i$, we will say that the partition is combinatorially compatible with the sequence $(Z_0, ..., Z_s, \pi)$ if for each $i = 1, .., l$ and $H_{j_1}, H_{j_2} \in F_i$ there exists $\sigma \in S_m$ such that*

*a* $\sigma(j_1) = j_2$,

*b* $deg(H_{j_1}) = [K(P_{j_1}) : K] = \left[K(P_{\sigma(j_1)}) : K\right] = deg(H_{\sigma(j_1)})$,

*c if* $H_{j_1} \in F_{i_1}$, $H_{j_k} \in F_{i_k}$ *and* $H_{j_k} \to H_{j_1}$, *then* $H_{\sigma(j_k)} \to H_{\sigma(j_1)}$

**Definition.** *4.1.7 Given a sequential morphism $\pi : Z_s \to Z_0$ as in Definition 2.1.2 and a partition of the exceptional divisor $E = \sqcup_{i=1}^{l} F_i$, we will say that the partition is algebraically compatible with the morphism $\pi$ if there exist a smaller field $\widetilde{K} \subset K$ with $k \subset \widetilde{K}$, there are $\widetilde{K}-$varieties $\widetilde{Z_0}$ and $\widetilde{Z}$ and a $\widetilde{K}-$morphism $\widetilde{Z} \xrightarrow{\widetilde{\pi}} \widetilde{Z_0}$*

$$\begin{array}{ccc} Z \cong \widetilde{Z} \times_{Spec(\widetilde{K})} Spec(K) & \xrightarrow{\pi} & Z_0 \cong \widetilde{Z_0} \times_{Spec(\widetilde{K})} Spec(K) \\ \downarrow{\scriptstyle \beta} & & \downarrow \\ \widetilde{Z} & \xrightarrow{\widetilde{\pi}} & \widetilde{Z_0} \end{array}$$

*such that the exceptional divisor of $\widetilde{\pi}$, $\widetilde{E}$, has irreducible components $\widetilde{H_1}, ..., \widetilde{H_l}$ and for each $i = 1, ..., l$ then $\forall H \in F_i$ $\beta(H) = \widetilde{H_i}$*

**Definition.** *4.1.8 Given a sequence of point blow-ups $(Z_0, ..., Z_s, \pi)$ and a partition of the exceptional divisor $E = \sqcup_{i=1}^{l} F_i$, we sill say that the partition is algebraically compatible with the sequence $(Z_0, ..., Z_s, \pi)$ if there exist a smaller field $\widetilde{K} \subset K$ with $k \subset \widetilde{K}$ and there are $\widetilde{K}-$varieties $\widetilde{Z_i}$ and $\widetilde{K}-$morphisms $\widetilde{Z}_{i+1} \xrightarrow{\widetilde{\pi}_{i+1}} \widetilde{Z}_i$*

$$\begin{array}{ccccccccc} Z_s & \xrightarrow{\pi_s} & Z_{s-1} & \xrightarrow{\pi_{s-1}} & \cdot & \longrightarrow & Z_1 & \xrightarrow{\pi_1} & Z_0 \\ \downarrow{\scriptstyle \beta} & & \downarrow & & & & \downarrow & & \downarrow \\ \widetilde{Z}_s & \xrightarrow{\widetilde{\pi}_s} & \widetilde{Z}_{s-1} & \xrightarrow{\widetilde{\pi}_{s-1}} & \cdot & \longrightarrow & \widetilde{Z}_1 & \xrightarrow{\widetilde{\pi}_1} & \widetilde{Z}_0 \end{array}$$

*where* $Z_i \cong \widetilde{Z}_i \times_{Spec(\widetilde{K})} Spec(K)$ $\forall i = 1, ..., s$ *,such that the exceptional divisor of* $(\widetilde{Z_0}, ..., \widetilde{Z}_l, \widetilde{\pi})$ *has irreducible components* $\widetilde{H_1}, ..., \widetilde{H_l}$ *and for each* $i = 1, ..., l$ *then* $\forall H \in F_i$ $\beta(H) = \widetilde{H_i}$.

The following section runs in parallel with the ones of Chapter 3, that is, Section 4.2 deals with the natural extension of the definitions of algebraic and combinatorial equivalences of sequences of point blow-ups and sequential morphisms, when considering algebraically and combinatorially compatible partitions of the exceptional divisor.

**Definition.** *4.2.1 We say that two algebraically marked sequential morphisms* $(\pi : Z \to Z_0, \sqcup_{i=1}^{l} F_i)_{alg}$ *and* $(\pi' : Z' \to Z_0', \sqcup_{i=1}^{l'} F_i')_{alg}$ *over* $K$ *are algebraically equivalent, and we denote it by* $(\pi : Z \to Z_0, \sqcup_{i=1}^{l} F_i)_{alg} \overset{alg}{\sim}_K (\pi' : Z' \to Z_0', \sqcup_{i=1}^{l'} F_i')_{alg}$ *, if and only if there exist smaller fields* $\widetilde{K}, \widetilde{K'} \subset K$ *with* $\widetilde{K} \cong_k \widetilde{K'}$ *satisfying the conditions of Definition 4.1.7*

$$\begin{array}{ccc} Z_s \cong \widetilde{Z}_s \times_{Spec(\widetilde{K})} Spec(K) & \xrightarrow{\pi} & Z_0 \cong \widetilde{Z_0} \times_{Spec(\widetilde{K})} Spec(K) \\ \downarrow \beta & & \downarrow \\ \widetilde{Z}_s & \xrightarrow{\widetilde{\pi}} & \widetilde{Z_0} \end{array}$$

$$\begin{array}{ccc} Z_s' \cong \widetilde{Z'}_s \times_{Spec(\widetilde{K'})} Spec(K) & \xrightarrow{\pi'} & Z_0' \cong \widetilde{Z_0'} \times_{Spec(\widetilde{K'})} Spec(K) \\ \downarrow \beta' & & \downarrow \\ \widetilde{Z_s'} & \xrightarrow{\widetilde{\pi'}} & \widetilde{Z_0'} \end{array}$$

*and there exist isomorphisms* $a$ *and* $b$ *such that the following diagram is commutative*

$$\begin{array}{ccc} \widetilde{Z} & \overset{b}{\longleftrightarrow} & \widetilde{Z'} \\ \downarrow \widetilde{\pi} & & \downarrow \widetilde{\pi'} \\ \widetilde{Z_0} & \overset{a}{\longleftrightarrow} & \widetilde{Z_0'} \end{array}$$

**Definition.** *4.2.3 Given two combinatorially marked sequential morphisms* $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{comb}$ *and* $(\pi' : Z_s' \to Z_0', \sqcup_{i=1}^{l} F_i')_{comb}$ *we say that the associated multilinear forms* $\Phi_{Z, \sqcup_{i=1}^{l} F_i}$ *and* $\Phi_{Z', \sqcup_{i=1}^{l} F_i'}$ *are equivalent, and we denote it by* $\Phi_{Z, \sqcup_{i=1}^{l} F_i} \sim \Phi_{Z', \sqcup_{i=1}^{l} F_i'}$*, if there exists* $\tau \in \mathcal{S}_l$ *such that*

$$\tau(\Phi_{Z, \sqcup_{i=1}^{l} F_i}) = \Phi_{Z', \sqcup_{i=1}^{l} F_i'}.$$

*Moreover, the combinatorially marked sequential morphisms* $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{comb}$ *and* $(\pi' : Z_s' \to Z_0', \sqcup_{i=1}^{l} F_i')_{comb}$ *are said to be combinatorially equivalent, and we denote* *it by* $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{comb} \overset{comb}{\sim} (\pi^{'} : Z_s^{'} \to Z_0^{'}, \sqcup_{i=1}^{l} F_i^{'})_{comb}$*, when their associated multilinear maps* $\Phi_{Z,\sqcup_{i=1}^{l} F_i}$ *and* $\Phi_{Z^{'},\sqcup_{i=1}^{l} F_i^{'}}$ *are equivalent.*

**Definition.** *4.2.5 We say that two algebraically marked sequences of point blow ups,* $(Z_s, ..., Z_0, \pi, \sqcup_{i=1}^{l} F_i)_{alg}$ *, and* $(Z_{s^{'}}, ..., Z_0^{'}, \pi^{'}, \sqcup_{i=1}^{l^{'}} F_i^{'})_{alg}$*, are algebraically equivalent over* $K$*, and we denote it by* $(Z_s, ..., Z_0, \pi, \sqcup_{i=1}^{l} F_i)_{alg} \overset{alg}{\sim}_K (Z_{s^{'}}, ..., Z_0^{'}, \pi^{'}, \sqcup_{i=1}^{l^{'}} F_i^{'})_{alg}$*, if and only if* $l = l^{'}$ *and there exist smaller fields* $\widetilde{K}, \widetilde{K^{'}} \subset K$ *with* $\widetilde{K} \cong_k \widetilde{K^{'}}$

$$\begin{array}{ccccccccc}
Z_s & \xrightarrow{\pi_s} & Z_{s-1} & \xrightarrow{\pi_{s-1}} & \cdot & \xrightarrow{\pi_2} & Z_1 & \xrightarrow{\pi_1} & Z_0 \\
\downarrow{\scriptstyle \beta} & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\widetilde{Z}_s & \xrightarrow{\widetilde{\pi}_s} & \widetilde{Z}_{s-1} & \xrightarrow{\widetilde{\pi}_{s-1}} & \cdot & \xrightarrow{\widetilde{\pi}_2} & \widetilde{Z}_1 & \xrightarrow{\widetilde{\pi}_1} & \widetilde{Z}_0
\end{array}$$

$$\begin{array}{ccccccccc}
Z_s^{'} & \xrightarrow{\pi_s^{'}} & Z_{s-1}^{'} & \xrightarrow{\pi_{s-1}^{'}} & \cdot & \xrightarrow{\pi_2^{'}} & Z_1^{'} & \xrightarrow{\pi_1^{'}} & Z_0^{'} \\
\downarrow{\scriptstyle \beta^{'}} & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\widetilde{Z}_s^{'} & \xrightarrow{\widetilde{\pi}_s^{'}} & \widetilde{Z}_{s-1}^{'} & \xrightarrow{\widetilde{\pi}_{s-1}^{'}} & \cdot & \xrightarrow{\widetilde{\pi}_2^{'}} & \widetilde{Z}_1^{'} & \xrightarrow{\widetilde{\pi}_1^{'}} & \widetilde{Z}_0^{'}
\end{array}$$

*with* $Z_i \cong \widetilde{Z}_i \times_{Spec(\widetilde{K})} Spec(K)$ *(resp.* $Z_i^{'} \cong \widetilde{Z}_i^{'} \times_{Spec(\widetilde{K}^{'})} Spec(K)$*) and algebraic isomorphisms* $a, b = b_t, b_{t-1}, ..., b_1$*, with* $t \leq s$*, such that there are indexes* $r_1, ..., r_t = s \in \{1, ..., l\}$ *and* $r_1^{'}, ..., r_t^{'} = s^{'} \in \left\{1, ..., s^{'}\right\}$*, where* $Z_{r_i} \to Z_{r_i - 1} \to ... \to Z_{r_{i-1}}$ *(resp.* $Z_{r_i}^{'} \to Z_{r_i-1}^{'} \to ... \to Z_{r_{i-1}}^{'}$*), with* $r_i > r_{i-1}$ *(resp* $r_i^{'} > r_{i-1}^{'}$*), is a brick blow-up* $\forall i = 1...t$ *as in Definition 4.2.4 and the diagram*

$$\begin{array}{ccccccccccc}
\widetilde{Z}_s & \longrightarrow & \widetilde{Z}_{r_{t-1}} & \longrightarrow & \widetilde{Z}_{r_{t-2}} & \longrightarrow & \cdot & \longrightarrow & \widetilde{Z}_{r_1} & \longrightarrow & \widetilde{Z}_0 \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b_{t-1}} & & \downarrow{\scriptstyle b_{t-2}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle a} \\
\widetilde{Z}_{s^{'}}^{'} & \longrightarrow & \widetilde{Z}_{r_{t-1}^{'}}^{'} & \longrightarrow & \widetilde{Z}_{r_{t-2}^{'}}^{'} & \longrightarrow & \cdot & \longrightarrow & \widetilde{Z}_{r_1^{'}}^{'} & \longrightarrow & \widetilde{Z}_0^{'}
\end{array}$$

*is commutative.*

**Definition.** *4.2.6 We say that two combinatorially marked sequences of point blow ups,* $(Z_s, ..., Z_0, \sqcup_{i=1}^{l} F_i, \pi)_{comb}$ *and* $(Z_s^{'}, ..., Z_0^{'}, \sqcup_{i=1}^{l} F_i^{'}, \pi^{'})_{comb}$ *as before with respective partitions* $E = \sqcup_{i=1}^{l} F_i$ *and* $E^{'} = \sqcup_{i=1}^{l} F_i^{'}$ *and irreducible components of the exceptional divisor* $H_1, ..., H_m$*;* $H_1^{'}, ..., H_m^{'}$*, with* $l = l^{'}$*, are combinatorially equivalent, and we denote it by* $(Z_s, ..., Z_0, \sqcup_{i=1}^{l} F_i, \pi)_{comb} \overset{comb}{\sim}_K (Z_s^{'}, ..., Z_0^{'}, \sqcup_{i=1}^{l} F_i^{'}, \pi^{'})_{comb}$*, if and only there is a permutation* $\tau$ *in* $S_l$ *such that for every two different indexes* $i, j$ *one has*

*a* $F_i$ *is proximate to* $F_j$ *if and only if* $F_{\tau(i)}^{'}$ *is proximate to* $F_{\tau(j)}^{'}$*,* b $deg(F_i) = \sum_{H \in F_i} deg(H) = \sum_{H' \in F_i'} deg(H') = deg(F'_{\tau(i)})$

Section 4.3 is devoted to the numerical characterization of final divisors and its natural extension to final elements of a partition.

**Proposition.** *4.3.1 $H_i$ is final if and only if*

$$(h_i)^n = (-1)^r (h_i)^s \cdot (h_j)^r \;\; and \;\; (h_i) \cdot (h_j)^{n-1} > 0$$

*for every $j$ such that $H_i \cap H_j \neq \emptyset$ (see Lemma 3.2.3 for a numerical characterization) and for all natural numbers $r$ and $s$ with $r + s = n$.*

**Proposition.** *4.3.3 Given an algebraically marked sequence $(Z_s, ..., Z_0, \sqcup_{i=1}^{l} F_i, \pi)_{alg}$ then $F_i$ is final if and only if*

$$(F_i)^n = (-1)^r (F_i)^s \cdot (F_j)^r \;\; and \;\; (F_i) \cdot (F_j)^{n-1} > 0$$

*for every $j$ such that $F_i \cap F_j \neq \emptyset$ and for all natural numbers $r$ and $s$ with $r + s = d$.*

Within the next two sections, that is Section 4.4 and Section 4.5, we recover the sequences of point blow-ups from the associated sequential morphism modulo algebraic equivalence, and prove some relations between algebraic and combinatorial equivalence classes of sequences of point blow-ups and sequential morphisms.

**Theorem.** *4.4.2 Let $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{alg}$ be an algebraically marked sequential morphism. Given the $n-ary$ multilinear intersection form associated to the partition $\mathcal{I}_{Z, \sqcup_{i=1}^{l} F_i}$ (see Definition 4.2.2) we can recover all the algebraically marked sequences of point blow-ups that are associated to algebraically marked sequential morphisms in the same algebraic equivalence class of $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{alg}$.*

**Proposition.** *4.5.1 Any of the sequences obtained as in 4.4.3, that is, by decomposing a regular projective contraction from a fixed sky $Z_s$ and a fixed simple normal crossing divisor $E$, are associated to sequential morphisms in the same algebraic equivalence class (see Definition 4.2.1).*

**Theorem.** *4.5.5 Two combinatorially marked sequences of point blow-ups $(Z_s, ..., Z_0, \sqcup_{i=1}^{l} F_i, \pi)_{comb}$ and $(Z_s', ..., Z_0', \sqcup_{i=1}^{l'} F_i', \pi')_{comb}$, with $l = l'$, are combinatorially equivalent over $K$ as in Definition 4.2.6 if and only if their associated* *combinatorially marked sequential morphisms* $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{comb}$ *and* $(\pi^{'} : Z_s^{'} \to Z_0^{'}, \sqcup_{i=1}^{l^{'}} F_i^{'})_{comb}$ *are combinatorially equivalent over* $K$ *as in Definition 4.2.3, and both statements are true if and only if the associated multilinear maps* $\Phi_{Z, \sqcup_{i=1}^{l} F_i}$ *and* $\Phi_{Z^{'}, \sqcup_{i=1}^{l^{'}} F_i^{'}}$ *are equivalent too as in Definition 4.2.3*

**Theorem.** *4.5.8 Given two algebraically marked sequential morphisms* $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{alg}$ *and* $(\pi^{'} : Z_s^{'} \to Z_0^{'}, \sqcup_{i=1}^{l^{'}} F_i^{'})_{alg}$ *, then they are algebraically equivalent over* $K$ *as in Definition 4.2.1 if and only if there exist algebraically marked sequences of point blow-ups* $(Z_s, ..., Z_0, \sqcup_{i=1}^{l} F_i, \pi)_{alg}$ *and* $(Z_s^{'}, ..., Z_0^{'}, \sqcup_{i=1}^{l^{'}} F_i^{'}, \pi^{'})_{alg}$ *associated to* $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{alg}$ *and* $(\pi^{'} : Z_s^{'} \to Z_0^{'}, \sqcup_{i=1}^{l^{'}} F_i^{'})_{alg}$ *respectively such that they are algebraically equivalent over* $K$ *as in Definition 4.2.5.*

## The content of Chapter 5

In Chapter 5 we recall some technical results about rational ruled surfaces, which we will use in Chapter 6. Section 5.1 is devoted to the study of some general properties of vector bundles of rank 2 over curves. In Section 5.2 we review some definitions and results about ruled surfaces, and in section 5.3 we focus on the study of rational ruled surfaces over a smooth rational curve, that is Hirzebruch surfaces $\mathbb{F}_\delta$, and specify the classes of the irreducible non-singular rational curves in its Chow ring.
Let us recall that $S_0$ and $F$ generate the Chow ring of Hirzebruch surface $\mathbb{F}_\delta$. By an abuse of notation we will denote in the same way these curves and their images in $A^\bullet(\mathbb{F}_\delta$.

**Proposition.** *5.3.5 Given a Hirzebruch surface* $\mathbb{F}_\delta$*, then any irreducible non-singular rational curve* $C \subset \mathbb{F}_\delta$ *is of one of the following types*

*A. either a section of class* $S_0 + bF$ *with* $b = 0$ *or* $b \geq \delta$*,*

*B. or a fiber* $F$*,*

*C. or a curve of class* $2S_0 + 2F$ *if* $\delta = 1$*,*

*D. or a curve of class* $aS_0 + F$ *with* $a > 0$ *if* $\delta = 0$*.*

Finally, in Section 5.4 we give a basic example of a Hirzebruch surface arising as the exceptional divisor of the blow-up of $\mathbb{P}^3$ with center a rational curve, and sum up the previous known results about admissible splitting of the normal bundle of a rational curve in $\mathbb{P}^3$.

**Proposition.** *5.4.3 Let $\mathcal{C} \subset \mathbb{P}^3$ be an irreducible rational smooth curve of degree $\gamma$. Then its normal bundle $N_{\mathcal{C}/\mathbb{P}^3}$ satisfies*

$$N_{\mathcal{C}/\mathbb{P}^3} \cong \begin{cases} \mathcal{O}(1) \oplus \mathcal{O}(1) & if \;\; \gamma = 1, \\ \mathcal{O}(4) \oplus \mathcal{O}(2) & if \;\; \gamma = 2, \\ \mathcal{O}(5) \oplus \mathcal{O}(5) & if \;\; \gamma = 3, \\ \mathcal{O}(2\gamma - 1 - a) \oplus \mathcal{O}(2\gamma - 1 + a) & if \;\; \gamma \geq 4, \end{cases}$$

*where $|a| \leq \gamma - 4$.*

# The content of Chapter 6

In Chapter 6 we focus on the study of sequences of blow-ups at either points or rational curves, with $Z_0 \cong \mathbb{P}^3$. Section 6.1 is devoted to establish some numerical properties of rational curves when considered as centers of blow-ups. First of all we define the concepts of "old" and "new" curves. Before giving these definitions, let us recall that whereas we denote by $C_\gamma$ to the center of the blow-up $\pi_\alpha : Z_\alpha \to Z_{\alpha-1}$, we use the notation $\mathcal{C}^\gamma$ to refer to the strict transform of a curve $\mathcal{C}$.

**Definition.** *6.1.1 We will say that a curve $\mathcal{C}^\alpha \subset Z_\alpha$ is an "old" curve if there exists a curve $\mathcal{C} \subset Z_0$ such that $\mathcal{C}^\alpha$ is the strict transform of $\mathcal{C}$ by the sequential morphism $\pi_{\alpha,0} : Z_\alpha \to Z_0$.*
*We will say that an "old" curve $\mathcal{C}^\alpha \subset Z_\alpha$ is unmodified with respect to the sequential morphism $\pi_{\alpha,0} : Z_\alpha \to Z_0$ if the following condition holds:*

$$C_\beta \cap \mathcal{C}^\beta = \emptyset, \tag{2}$$

*for $\beta = 1, ..., \alpha$.*
*On the other hand, we will say that an "old" curve $\mathcal{C}^\alpha \subset Z_\alpha$ is modified by the blow-up $\pi_{\alpha+1} : Z_{\alpha+1} \to Z_\alpha$ if*

a. *either $dim(C_{\alpha+1}) = 0$ and $C_{\alpha+1} \in \mathcal{C}^\alpha$, and in this case we will say that $\pi_{\alpha+1}$ is a modification of type $I$,* *b. or $dim(C_{\alpha+1}) = 1$ and $C_{\alpha+1} \cap \mathcal{C}^{\alpha} \neq \emptyset$, and in this case we will refer to $\pi_{\alpha+1}$ as a modification of type $II$.*

Moreover, we state the particular case of Theorem 2.3.10 where $dim(Z_0) = 3$, and we give necessary conditions for all possible configurations of final divisors in terms of the intersection numbers given by the $n-$ary multilinear intersection form

**Theorem.** *6.1.5 Let $E_i, E_j \subset Z_s$ be both final divisors for the sequential morphism $\pi : Z_s \to Z_0$. Then $E_i \cap E_j \neq \emptyset$ if and only if $E_i$ is proximate to $E_j$ and $E_j$ is $t-$proximate to $E_i$, or vice versa.*

**Proposition.** *6.1.7 Let $E_i \subset Z_s$ be a final divisor for the sequential morphism $\pi : Z_s \to Z_0$, and let $j, k$ be two indices such that $E_i \cap E_j \neq \emptyset$ and $E_i \cap E_k \neq \emptyset$. Then one of the following conditions must be verified, where $\eta_j, \eta_k \in \mathbb{Z}_+$:*

*I. either $dim(C_i) = 1$ and $C_i$ is proximate to $C_j$ and $t-$proximate to $C_k$, or vice versa, and in this case we have that*

$$(e_i + e_j)^2 \cdot e_i = 0$$
$$(e_i)^2 \cdot e_k = -\eta_k$$
$$e_i \cdot (e_k)^2 = 0$$
$$e_i \cdot e_j \cdot e_k = \eta_k$$

*II. or $dim(C_i) = 1$ and $C_i$ is $t-$proximate to both $C_j$ and $C_k$, and in this case we have*

$$e_i \cdot (e_j)^2 = 0$$
$$(e_i)^2 \cdot e_j = -\eta_j$$
$$e_i \cdot (e_k)^2 = 0$$
$$(e_i)^2 \cdot e_j = -\eta_k$$
$$e_i \cdot e_j \cdot e_k = 0$$

*III. or $dim(C_i) = 1$ and $C_i$ is proximate to both $C_j$ and $C_k$, and the following relations* *are satisfied*

$$(e_i + e_j)^2 \cdot e_i = 0$$
$$(e_i + e_k)^2 \cdot e_i = 0$$
$$e_i \cdot (e_k)^2 = -e_i \cdot (e_j)^2,$$
$$(e_i)^2 \cdot e_k = (e_i)^2 \cdot e_j + e_i \cdot (e_j)^2,$$

*IV. or $dim(C_i) = 0$ and $C_i$ is proximate to both $C_j$ and $C_k$, and the following relations are satisfied*

$$(e_i + e_j) \cdot e_i = (e_i + e_k) \cdot e_i = 0$$
$$(e_i)^2 \cdot e_j = (e_i)^2 \cdot e_k = -1$$
$$e_i \cdot (e_j)^2 = e_i \cdot (e_k)^2 = 1$$
$$e_i \cdot e_j \cdot e_k = 1$$

In section 6.2 we establish a numerical criterion that characterizes final divisors in terms of some relations defined over the Chow group $A_0(Z_s)$ of zero-cycles of its sky $Z_s$. Firstly, we prove that the relations defining an admissible configuration of type III hold if and only if $E_i$ is final.

**Proposition.** *6.2.6 Given a sequence of point and rational curve blow-ups $(Z_\alpha, ..., Z_0, \pi)$, let $E_i^\alpha \subset Z_\alpha$ be an irreducible exceptional component . Furthermore, let us suppose that the following conditions are satisfied:*

*a. there exists just two indexes $j, k$, with $E_i^\alpha \cap E_j^\alpha \neq \emptyset$ and $E_i^\alpha \cap E_k^\alpha \neq \emptyset$, that verify the following conditions:*

*a.i.* $(e_k^\alpha)^2 \cdot e_i^\alpha = -(e_j^\alpha)^2 \cdot e_i^\alpha$,

*a.ii.* $e_k^\alpha \cdot (e_i^\alpha)^2 = e_j^\alpha \cdot (e_i^\alpha)^2 + (e_j^\alpha)^2 \cdot e_i^\alpha$,

*a.iii. and* $(e_j^\alpha + e_i^\alpha)^2 \cdot e_i^\alpha = (e_k^\alpha + e_i^\alpha)^2 \cdot e_i^\alpha = 0$,

*a.iv.* $e_i^\alpha \cdot e_j^\alpha \cdot e_k^\alpha = 0$.

*b. there exists at most one index $\beta$, with $E_i^\alpha \cap E_\beta^\alpha \neq \emptyset$, such that $e_i^\alpha \cdot (e_\beta^\alpha)^2 < 0$, if $(e_j^\alpha)^2 \cdot e_i^\alpha \neq 0$, $(e_k^\alpha)^2 \cdot e_i^\alpha \neq 0$, otherwise such an index does not exist,* *c. if there exists any other index $\gamma$, with $\gamma \neq j, k$, such that $E_i^\alpha \cap E_\gamma^\alpha \neq \emptyset$, then the following relations are satisfied, where $\eta \in \mathbb{Z}_+$:*

$$(e_i^\alpha)^2 \cdot e_\gamma^\alpha = -\eta$$
$$e_i^\alpha \cdot (e_\gamma^\alpha)^2 = 0$$
$$e_i^\alpha \cdot e_j^\alpha \cdot e_\gamma^\alpha = e_i^\alpha \cdot e_k^\alpha \cdot e_\gamma^\alpha = \eta.$$

*d. and in the particular case $(e_k^\alpha)^2 \cdot e_i^\alpha = (e_j^\alpha)^2 \cdot e_i^\alpha = 0$, with $e_k^\alpha \cdot (e_i^\alpha)^2 = e_j^\alpha \cdot (e_i^\alpha)^2 = -\lambda < 0$, if the following relations hold:*

$$(e_\gamma^\alpha)^2 \cdot e_j^\alpha = -1,$$
$$e_\gamma^\alpha \cdot (e_j^\alpha)^2 = 0,$$
$$(e_\gamma^\alpha)^2 \cdot e_k^\alpha = -1,$$
$$e_\gamma^\alpha \cdot (e_k^\alpha)^2 = 0,$$

*thus $\# \{\gamma\} \geq \lambda + 1$.*

*Then $E_i^\alpha \cong \mathbb{F}_\delta$, with $\delta = \left|(e_j^\alpha)^2 \cdot e_i^\alpha\right| = \left|(e_k^\alpha)^2 \cdot e_i^\alpha\right|$, and $N_{E_i^\alpha / Z_\alpha} \cong \mathcal{O}_{E_i^\alpha}(-1)$.*

Secondly, we give a proof of the fact that relations defining an admissible configuration of type IV hold if and only if $E_i$ is final.

**Proposition.** *6.2.7 Let $E_i^\alpha \subset Z_\alpha$ be the strict transform of the exceptional irreducible component $E_i^i$. Let us suppose that the following conditions hold:*

*a. there exists two indexes $j, k$, with $E_i^\alpha \cap E_j^\alpha \neq \emptyset$ and $E_i^\alpha \cap E_k^\alpha \neq \emptyset$, verifying*

*a.i.* $(e_j^\alpha)^2 \cdot e_i^\alpha = (e_k^\alpha)^2 \cdot e_i^\alpha = 1$,

*a.ii.* $e_j^\alpha \cdot (e_i^\alpha)^2 = e_k^\alpha \cdot (e_i^\alpha)^2 = -1$,

*a.iii.* $e_i \cdot e_j \cdot e_k = 1$,

*a.iv. and* $(e_i^\alpha + e_j^\alpha)^2 \cdot e_i^\alpha = (e_i^\alpha + e_k^\alpha)^2 \cdot e_i^\alpha = 0$.

*b. if there exists any other index $\gamma$, with $\gamma \neq j, k$, such that $E_i^\alpha \cap E_\gamma^\alpha \neq \emptyset$, the following relations are satisfied:*

$$(e_i^\alpha)^2 \cdot e_\gamma^\alpha = -1$$
$$e_i^\alpha \cdot (e_\gamma^\alpha)^2 = 1$$
$$e_i^\alpha \cdot e_j^\alpha \cdot e_\gamma^\alpha = e_i^\alpha \cdot e_k^\alpha \cdot e_\gamma^\alpha = 1.$$

*Then $E_i^\alpha \cong \mathbb{P}^2$ and $N_{E_i^\alpha/Z_\alpha} \cong \mathcal{O}_{E_i^\alpha}(-1)$.*

Before proving the main result of this section, that is Theorem 6.2.11, we introduce the key concept of an admissible final configuration for an irreducible component $E_i$ of the exceptional divisor, whose definition is fully explained in terms of the intersection numbers of the irreducible components $\{E_1, E_2, \ldots, E_s\}$.

**Definition.** *6.2.10 We will say that an irreducible exceptional component $E_i$ has an admissible final configuration whenever it satisfies:*

*a. If there exists just one index $j$ such that $E_i \cap E_j \neq \emptyset$, then*

*a.i. either $(e_j + e_i)^2 \cdot e_i = 0$ with the following exceptions:*

*a.i.i. $(e_i)^3 = 3e_i \cdot (e_j)^2$, $(e_i)^2 \cdot e_j = -2e_i \cdot (e_j)^2$, and $(e_j)^3 = 0$,*

*a.i.ii. $(e_i)^3 = 2e_i \cdot (e_j)^2$, $(e_i)^2 \cdot e_j = -\frac{3}{2}e_i \cdot (e_j)^2$, and $(e_j)^3 = -\frac{1}{2}e_i \cdot (e_j)^2$,*

*a.ii. or $(e_j)^2 \cdot e_i = 0$ and $e_j \cdot (e_i)^2 = -\eta$.*

*b. If the cardinal set of indexes $\{\gamma\}$ such that $E_i \cap E_\gamma \neq \emptyset$ is greater or equal to 2, $\#\{\gamma\} \geq 2$, then it verifies one of the conditions stated in Proposition 6.1.7 with respect to any pair $\{j, k\} \subset \{\gamma\}$, that is, $E_i$ has an admissible proximity configuration with respect to $E_j$ and $E_k$. Moreover, in case the irreducible exceptional component $E_i$ has an admissible proximity configuration of type III, then it is with respect to at most two irreducible exceptional components, and if it has an admissible proximity configuration of type IV then it is with respect to at most three irreducible exceptional components.*

*c. There exists at most one index $\gamma$ such that $(e_\gamma)^2 \cdot e_i < 0$.*

*d.* *If there exists some index $\beta$ such that $(e_i+e_\beta)^2\cdot e_i = 0$, with $(e_i)^3 > 0$ and $e_i\cdot(e_\beta)^2 = 0$, then $E_i$ verifies the conditions of Proposition 6.1.11 about the cardinality of the set of index $\{\gamma\}$ verifying $E_i \cap E_\gamma \neq \emptyset$. Moreover, if it has an admissible proximity configuration of type III with respect to $E_j$ and $E_\beta$ then $E_i$ verifies Proposition 6.2.1 and Corollary 6.2.2 (if the other hypothesis also hold), or if it has an admissible proximity configuration of type I with respect to $E_\beta$ and $E_j$ then it verifies Lemma 6.2.3 (if the other hypothesis are verified too).*

*e.* *If there exists some index $\lambda$ such that $(e_i)^2\cdot e_\lambda = -1$, $e_i\cdot(e_\lambda)^2 = 0$ that also verifies the above conditions then*

  *e.i.* *if there exists some index $\mu$ such that $E_\lambda$ has an admissible proximity configuration of type III with respect to $E_i$ and $E_\mu$, then $E_i$ already verifies the above conditions and the same relations with respect to all the same indexes but $E_\lambda$ just by replacing $e_i$ by $\bar{e}_i = (e_i + e_\lambda)$ and $e_\mu$ by $\bar{e}_\mu = (e_\mu + e_\lambda)$ in the computations, and it also satisfies $(\bar{e}_i)^2 \cdot \bar{e}_\mu = -1$ and $\bar{e}_i \cdot (\bar{e}_\mu)^2 = 0$,*

  *e.ii.* *otherwise, $E_i$ already verifies the above conditions and the same relations with respect to all the same indexes but $E_\lambda$ just by replacing $e_i$ by $\bar{e}_i = (e_i + e_\lambda)$ in the computations.*

Finally, we prove the main theorem of this section, where we characterize final divisors of sequences of point and rational curves blow-ups in terms of some relations defined over the Chow group of zero-cycles of its sky $A_0(Z_s)$.

**Theorem.** *6.2.11 An irreducible exceptional component $E_i \subset Z_s$ is a final final divisor for the sequential morphism $\pi : Z_s \to Z_0$ if and only if $E_i$ has an admissible final configuration.*

Finally, in section 6.3 we give a presentation of the Chow ring $A^\bullet(Z_s)$ of the sky of a sequence of point and rational curve blow-ups considering the total transforms of the exceptional components as generators, and as a corollary we prove that whereas the Chow ring of a sequence of point blow-ups depends only on the length of the sequence (see Corollary 3.5.5), this is not the case for sequences of point and rational curve blow-ups.

**Theorem.** *6.3.2 The Chow ring of $Z_{\alpha+1}$, $A^\bullet(Z_{\alpha+1})$, is isomorphic to*

$$A^\bullet(Z_{\alpha+1}) \cong \frac{A^\bullet(Z_\alpha)\left[e^{\alpha+1}_{\alpha+1}, w^{\alpha+1}_{\alpha+1}\right]}{\mathcal{J}_{\alpha+1}},$$

*where*

$$\mathcal{J}_{\alpha+1} = (keri^*_{\alpha+1} \cdot e^{\alpha+1}_{\alpha+1}, h^{\alpha+1*} \cdot e^{\alpha+1}_{\alpha+1} - \mu_0 w^{\alpha+1}_{\alpha+1}, \left\{e^{\alpha*}_{\beta} \cdot e^{\alpha+1}_{\alpha+1} - \mu_\beta w^{\alpha+1}_{\alpha+1}\right\}, (w^{\alpha+1}_{\alpha+1})^2,$$
$$h^{\alpha+1*} \cdot w^{\alpha+1}_{\alpha+1}, \left\{e^{\alpha*}_{\beta} \cdot w^{\alpha+1}_{\alpha+1}\right\}^{\alpha}_{\beta=1}, (e^{\alpha+1}_{\alpha+1})^2 - c_1(N_{C_{\alpha+1}/Z_\alpha}) w^{\alpha+1}_{\alpha+1} + [C_{\alpha+1}], e^{\alpha+1}_{\alpha+1} \cdot w^{\alpha+1}_{\alpha+1} + (h^{\alpha+1*})^3),$$

*with $\mu_\beta = e^{\alpha*}_\beta \cdot [C_{\alpha+1}]$.*

**Corollary.** *6.3.3 The Chow ring of the sky $A^\bullet(Z_s)$ is isomorphic to*

$$A^\bullet(Z_s) \cong \frac{\mathbb{Z}\left[h^{s*}, \{e^{s*}_\alpha\}_{\alpha\in\mathcal{I}_1}, \left\{e^{s*}_\beta, w^{s*}_\beta\right\}_{\beta\in\mathcal{I}_2}\right]}{\mathcal{A}},$$

*where*

$$\mathcal{A} = ((h^{s*})^4, \left\{\{h^{s*} \cdot e^{s*}_\alpha\}, \left\{e^{s*}_\alpha \cdot e^{s*}_\beta\right\}_{\alpha\neq\beta}, \left\{-(e^{s*}_\alpha)^3 + (h^{s*})^n\right\}\right\}_{\alpha,\beta\in\mathcal{I}_1},$$
$$\left\{keri^{s*}_\alpha \cdot e^{s*}_\alpha, h^{s*} \cdot e^{s*}_\alpha - \mu_0 w^{s*}_\alpha, \left\{e^{s*}_\beta \cdot e^{s*}_\alpha - \mu_\beta w^{s*}_\alpha\right\}_{\beta<\alpha}, (w^{s*}_\alpha)^2, h^{s*} \cdot w^{s*}_\alpha, \left\{e^{s*}_\beta \cdot w^{s*}_\alpha\right\}_{\beta<\alpha},\right.$$
$$\left.(e^{s*}_\alpha)^2 - c_1(N_{C_\alpha/Z_{\alpha-1}}) w^{s*}_\alpha + [C_\alpha]^{s*}, e^{s*}_\alpha \cdot w^{s*}_\alpha + (h^{s*})^3\right\}_{\alpha,\beta\in\mathcal{I}_2}).$$

**Corollary.** *6.3.4 Given two sequences of blow-ups $(Z_s, ..., Z_0, \pi)$, $(Z'_s, ..., Z'_0, \pi')$, where $Z_0 \cong Z'_0$, of the same length and with indentical proximity relations, then $A^\bullet(Z_s)$ and $A^\bullet(Z'_s)$ may be non-isomorphic.*

This result follows from the fact that, when considering the blow-up $\pi_{\alpha+1} : Z_{\alpha+1} \to Z_\alpha$ of a rational curve $C_\alpha$, there are some relations in the Chow ring $A^\bullet(Z_{\alpha+1})$ that highly depend on the geometry of $C_\alpha$ through the first Chern class of its normal bundle $c_1(N_{C_\alpha/Z_\alpha})$, as it is showed in Theorem 6.3.2.

# Chapter 1

# Preliminaries

In this chapter, we recall some basic definitions and results which will be used to prove the main results.
In the first section of this chapter, we review some basics of Algebraic Geometry such as divisors, vector bundles and projective bundles and blow-ups. Most of the results in this section can be found in [22] as well as in Appendix B of [17]. In the second section we recall definitions and basic result of Intersection Theory, in particular, rational equivalence, Chern and Segre classes, the Chow ring of a projective bundle, excess intersections and intersection theory of blow-ups. The main references for this are [17] and [15].

## 1.1 Divisors and Blow-ups

### 1.1.1 Vector bundles and projective bundles

**Definition 1.1.1.** *A* ***vector bundle*** *$V$ of rank $r$ on a scheme $X$ is a scheme $V$ equipped with a morphism $\pi : V \to X$, satisfying the following condition. There must be an open covering $\{U_i\}$ of $X$ and isomorphisms $\varphi_i$ of $\pi^{-1}(U_i)$ with $U_i \times \mathbb{A}^r$ over $U_i$, such that over $U_i \cap U_j$ the composites $\varphi_i \circ \varphi_j^{-1}$ are linear, i.e., given by a morphism*

$$g_{ij} : U_i \cap U_j \to GL(r, K).$$

*These transitions functions satisfy:* $g_{ik} = g_{ij}g_{jk}$, $g_{ij}^{-1} = g_{ji}$ *and* $g_{ii} = 1$. *Conversely, any such transition functions determine a vector bundle. Data* $(U_i', \varphi_i')$ *determine an isomorphic bundle if all composites* $\varphi_i' \circ \varphi_j$ *are linear on* $U_i' \cap U_j$.

**Definition 1.1.2.** *A section of* $V$ *is a morphism* $s : X \to V$ *such that* $\pi \circ s = id_X$. *If* $V$ *is determined by transition functions* $g_{ij}$, *a section of* $V$ *is determined by a collection of morphisms* $s_i : U_i \to \mathbb{A}^r$, *such that*

$$s_i = g_{ij}s_j$$

*on* $U_i \cap U_j$. *The sheaf of sections of* $V$ *is a locally free sheaf* $\mathcal{V}$ *of* $\mathcal{O}_X-$*modules of rank* $r$. *Conversely, a locally free sheaf* $\mathcal{V}$ *(of constant rank) comes from a vector bundle* $V$, *unique up to isomorphism. This may be seen by using transition functions. For an affine open set* $U \subset X$ *with coordinate ring* $A$, $\pi^{-1}(U)$ *is an affine open set in* $V$, *with coordinate ring the symmetric algebra*

$$Sym_A \Gamma(U, \mathcal{V}^\vee),$$

*where* $\mathcal{V}^\vee = Hom_{\mathcal{O}_X}(\mathcal{V}, \mathcal{O}_X)$, *and* $\Gamma(U, \mathcal{V}^\vee) = H^0(U, \mathcal{V}^\vee)$ *is the space of sections.*

Before continue with the definition of the $Proj$ of a sheaf of graded algebras, we now recall the construction of the $Proj$ of a graded ring.

Let $S = \oplus_{d \geq 0} S_d$ be a graded ring. If we denote by $S_+$ the ideal $\oplus_{d>0} S_d$, then we define the set $ProjS$ to be the set of all homogeneous prime ideals $\mathfrak{p}$, which do not contain all of $S_+$.

If $\mathfrak{a}$ is a homogeneous ideal of $S$, we define the subset $V(\mathfrak{a}) = \{\mathfrak{p} \in ProjS | \ \mathfrak{p} \supseteq \mathfrak{a}\}$. It can be proved that if $\mathfrak{a}$ and $\mathfrak{b}$ are homogeneous ideals in $S$, then $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$. Moreover, if $\{\mathfrak{a}_i\}$ is any family of homogeneous ideals of $S$, then $V(\sum \mathfrak{a}_i) = \cap V(\mathfrak{a}_i)$ (see [22, Lemma 2.4.]), so we can define a topology on $ProjS$ by taking the closed subsets to be the subsets of the form $V(\mathfrak{a})$.

Next we will define a sheaf of rings $\mathcal{O}$ on $ProjS$. For each $\mathfrak{p} \in ProjS$, we consider the ring $S_{(\mathfrak{p})}$ of elements of degree zero in the localized ring $T^{-1}S$, where $T$ is the multiplicative system consisting of all homogeneous elements of $S$ which are not in $\mathfrak{p}$. For any open subset $U \subseteq ProjS$, we define $\mathcal{O}(U)$ to be the set of functions $s : U \to \coprod S_{(\mathfrak{p})}$ such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$, and such that $s$ is locally a quotient of elements of $S$: for each $\mathfrak{p} \in U$, there exists a neighborhood $V$ of $\mathfrak{p}$ in $U$, and homogeneous elements $a, f$ in $S$, of the same degree, such that for all $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, and $\mathrm{s}(\mathfrak{q}) = a/f$ in $S_{(\mathfrak{q})}$. Now it is clear that $\mathcal{O}$ is a presheaf of rings, with the natural restrictions, and it is also clear from the local nature of the definition that $\mathcal{O}$ is a sheaf.

**Definition 1.1.3.** *If $S$ is any graded ring, we define $(ProjS, \mathcal{O})$ to be the topological space together with the sheaf of rings constructed above.*

In fact $ProjS$ is a scheme as it is proved in the following result.

**Proposition 1.1.4.** *[22, Proposition 2.5.] Let $S$ be a graded ring.*

a *For any $\mathfrak{p} \in ProjS$, the stalk $\mathcal{O}_{\mathfrak{p}}$ is isomorphic to the local ring $S_{(\mathfrak{p})}$*

b *For any homogeneous $f \in S_+$, let $D_+(f) = \{\mathfrak{p} \in ProjS| \ f \notin \mathfrak{p}\}$.Then $D_+(f)$ is open in $ProjS$. Furthermore, these open sets cover $ProjS$, and for each such open set, we have an isomorphism of locally ringed spaces*

$$(D_+(f), \mathcal{O}|_{D_+(f)}) \cong SpecS_{(f)}$$

*where $S_{(f)}$ is the subring of elements of degree 0 in the localized ring $S_f$*

c *$ProjS$ is a scheme.*

**Claim 1.1.5.** *Let $X$ be a noetherian scheme and let $\mathcal{S}$ be a quasi-coherent sheaf of $\mathcal{O}_X-$modules, which has a structure of a sheaf of graded $\mathcal{O}_X-$algebras. Thus $\mathcal{S} \cong \oplus_{d \geq 0} \mathcal{S}_d$, where $\mathcal{S}_d$ is the homogeneous part of degree $d$. We assume furthermore that $\mathcal{S}_0 \cong \mathcal{O}_X$, that $\mathcal{S}_1$ is a coherent $\mathcal{O}_X-$module, and that $\mathcal{S}$ is locally generated by $\mathcal{S}_1$ as an $\mathcal{O}_X-$algebra.*

Let $X$ be a scheme and $\mathcal{S}$ a sheaf of graded $\mathcal{O}_X-$algebras satisfying the conditions above. For each open affine subset $U = SpecA$ of $X$, let $\mathcal{S}(U)$ be the graded $A-$algebra $\Gamma(U, \mathcal{S}|_U)$. Then we consider $Proj\mathcal{S}(U)$ and its natural morphism $p_U : Proj\mathcal{S}(U) \to U$. If $f \in A$, and $U_f = SpecA_f$ , then since $\mathcal{S}$ is quasi-coherent, we see that $Proj\mathcal{S}(U_f) \cong p_U^{-1}(U_f)$. It follows that if $U, V$ are two open affine subsets of $X$, then $p_U^{-1}(U \cap V)$ is naturally isomorphic to $p_V^{-1}(U \cap V)$. These isomorphisms allow us to glue the schemes $Proj\mathcal{S}(U)$ together. Thus we obtain a scheme $Proj\mathcal{S}$ together with a morphism $p$ :

$Proj\mathcal{S} \to X$ such that for each open affine $U \subseteq X$, $p^{-1}(U) \cong Proj\mathcal{S}(U)$. Furthermore the invertible sheaves $\mathcal{O}(1)$ on each $Proj\mathcal{S}(U)$ are compatible under this construction, so they glue together to give an invertible sheaf $\mathcal{O}(1)$ on $Proj\mathcal{S}$, canonically determined by this construction.

The cone of $\mathcal{S}$ over $X$ is defined by $C = Spec(\mathcal{S})$ together with the natural morphism $\pi : C \to X$. If $X$ is affine, with coordinate ring $A$, then $\mathcal{S}$ is determined by a graded $A-$algebra, which we denote also by $\mathcal{S}$. If $x_0, ..., x_n$ are generators for $\mathcal{S}_1$, then $\mathcal{S} = A\left[x_0, ..., x_n\right]/\mathcal{I}$ for a homogeneous ideal $\mathcal{I}$. In this case $C$ is the affine sub scheme of $X \times \mathbb{A}^{n+1}$ defined by the ideal $\mathcal{I}$.

The zero section embedding of $X$ in $C$ is determined by the augmentation homomorphism from $\mathcal{S}$ to $\mathcal{O}_X$, which vanishes on $\mathcal{S}_i$ for $i > 0$, and is the canonical isomorphism of $\mathcal{S}_0$ with $\mathcal{O}_X$. If $C = Spec(\mathcal{S})$ is a cone on $X$, and $f : Z \to X$ is a morphism, the pull-back $f^*C = C \times_X Z$ is the cone on $Z$ defined by the sheaf of $\mathcal{O}_Z-$algebras $f^*\mathcal{S}$. If $Z \subset X$ we write $C|_Z$.

Let $z$ be a variable, $S^\bullet[z]$ the graded algebra whose $n^{th}$ graded piece is

$$S^n \oplus S^{n-1}z \oplus ... \oplus S^1 z^{n-1} \oplus S^0 z^n.$$

The corresponding cone is denoted $C \oplus 1$. The projective cone $P(C \oplus 1)$ is called the projective completion of $C$. The element $z$ in $(S^\bullet[z])^1$ determines a regular section of $\mathcal{O}_{C\oplus 1}(1)$ on $P(C \oplus 1)$ whose zero-scheme is canonically isomorphic to $P(C)$. The complement to $P(C)$ in $P(C \oplus 1)$ is canonically isomorphic to $C$. With this embedding in $P(C \oplus 1)$, $P(C)$ is called the hyperplane at infinity.

**Definition 1.1.6.** *Let $X$ be a noetherian scheme, and let $\mathcal{E}$ be a locally free coherent sheaf on $X$. We define the associated projective space bundle $P(\mathcal{E})$ as follows. Let $\mathcal{S} = Sym(\mathcal{E})$ be the symmetric algebra of $\mathcal{E}$, $\mathcal{S} = \oplus_{d\geq 0} Sym_d(\mathcal{E})$. Then $\mathcal{S}$ is a sheaf of graded $\mathcal{O}_X-$algebras satisfying the conditions above 1.1.5, and we define $P(\mathcal{E}) = Proj\mathcal{S}$. Consequently, it comes with a projection morphism $p : P(\mathcal{E}) \to X$, and an invertible sheaf $\mathcal{O}_{P(\mathcal{E})}(1)$. Note that if $\mathcal{E}$ is free of rank $n+1$ over an open set $U$, then $p^{-1}(U) \cong \mathbb{P}^n_U$, so $P(\mathcal{E})$ is a "relative projective space" over $X$.*

**Proposition 1.1.7.** *[22, Proposition 7.11.] Let $X$, $\mathcal{E}$ and $P(\mathcal{E})$ be as in the above definition. Then:*

*a if rank $\mathcal{E} \geq 2$, there is a canonical isomorphism of graded $\mathcal{O}_X-$algebras $\mathcal{S} \cong$*

$\oplus_{l\in\mathbb{Z}}\pi_*(\mathcal{O}(l))$*, with the grading on the right hand side given by* $l$*. In particular,for* $l<0$, $p_*(\mathcal{O}(l))=0$*; for* $l=0$, $p_*(\mathcal{O}_{P(\mathcal{E})})=\mathcal{O}_X$*, and for* $l=1$, $p_*(\mathcal{O}_{P(\mathcal{E})}(1))=\mathcal{E}$*;*

*b there is a natural surjective morphism* $p^*\mathcal{E}\to\mathcal{O}_{P(\mathcal{E})}(1)$*.*

Now, since a vector bundle $V$ on $X$ is the cone associated to the graded sheaf $Sym(\mathcal{V}^\vee)$, where $\mathcal{V}$ is the sheaf of sections of $V$, then we have the following definition

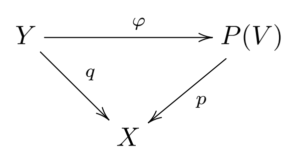
**Definition 1.1.8.** *The* ***projective bundle*** *associated to* $\mathcal{V}$ *is defined by* $P(V)=Proj(Sym(\mathcal{V}^\vee))$*.*

There is a canonical surjection $p^*V^\vee\to\mathcal{O}_{P(V)}(1)$ on $P(V)$, which gives an embedding

$$\mathcal{O}_{P(V)}(-1)\to p^*V.$$

Thus $P(V)$ is the projective bundle of lines in $V$, and $\mathcal{O}_{P(V)}(-1)$ is the universal, or tautological line sub-bundle.

**Proposition 1.1.9.** *[15, Proposition 9.2.] Given a vector bundle* $V$ *on a scheme* $X$*, commutative diagrams of maps of schemes*




*are in natural one-to-one correspondence with line subbundles* $L\subset q^*V$*.*

**Corollary 1.1.10.** *[15, Corollary 9.5.] Let* $X$ *be a scheme. Two projective bundles* $p:P(V)\to X$ *and* $p^{'}:P(V^{'})\to X$ *are isomorphic as* $X-$*schemes if and only if there is a line bundle* $L$ *on* $X$ *such that* $L\otimes V^{'}=V$*. In this case the line bundle* $\mathcal{O}_{P(V)}(-1)$ *corresponds under the isomorphism to* $p^{'*}(L)\otimes\mathcal{O}_{P(V')}(-1)$*.*

**Proposition 1.1.11.** *[17, Appendisx B, B.5.6] If* $W$ *is a sub-bundle of a vector bundle* $V$*, with quotient bundle* $G=V/W$*, there is a canonical embedding of* $P(W)$ *in* $P(V)$*. If* $p:P(V)\to X$ *is the projection, the composite of the canonical maps* $\mathcal{O}_{P(V)}(-1)\to p^*V$ *and* $p^*V\to p^*G$ *corresponds to a section of* $p^*G\otimes\mathcal{O}_{P(V)}(1)$*. This section is regular, and its zero-scheme is* $P(W)$*.*

### 1.1.2 Divisors

Let $(X, \mathcal{O}_X)$ be a scheme (see [22, II.2]). For each affine open set $U$ of $X$, let $K(U)$ be the total quotient ring of the coordinate ring $A(U)$, i.e. the localization of $A(U)$ at the multiplicative system of elements which are not zero divisors. The map $U \to K(U)$ determines a presheaf on $X$, whose associated sheaf of rings is denoted $\mathcal{K}$. Let $\mathcal{K}^*$ denote the (multiplicative) subsheaf of invertible elements in $\mathcal{K}$, and $\mathcal{O}_X^*$ the sheaf of invertible elements $\mathcal{O}_X$. Note that if $X$ is a variety, then $\mathcal{K}$ is the constant sheaf equal to $K(X)$.

**Definition 1.1.12.** *A* ***Cartier divisor*** *$D$ on $X$ is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}_X^*$.*

A Cartier divisor is determined by a collection of affine open sets $U_i$ which cover $X$, and elements $f_i$ in $K(U_i)$, such that $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$ for each $i, j$. Such $f_i$ are called local equations for $D$. The Cartier divisors on $X$ form a group $Div(X)$. The group operation of $Div(X)$ is denoted additively.

If $D \in Div(X)$, the support of $D$, denoted $Supp(D)$, or sometimes $|D|$, is the subset of $X$ consisting of points $x$ such that if $f_i \in K(U_i)$ a local equation with $x \in U_i$ then $f_i \notin \mathcal{O}_{X,x}^*$. The support of a Cartier divisors $D$, like the support of the section of any sheaf, is a closed subset of $X$.

**Definition 1.1.13.** *A Cartier divisor is principal if the corresponding section of $\mathcal{K}^*/\mathcal{O}_X^*$ is the image of a global section of $\mathcal{K}^*$. If $X$ is a variety, the principal divisor of $r \in K(X)^*$ is denoted $div(r)$.*

Since the support of $div(r)$ is a proper closed subset of $X$, there are only a finite number of subvarieties $V$ of codimension one in $X$ such that $r \notin \mathcal{O}_{X,V}^*$.

A Cartier divisor $D$ on a scheme $X$ determines a line bundle on $X$, denoted $\mathcal{O}_X(D)$. The sheaf of sections of $\mathcal{O}_X(D)$ can be defined to be the $\mathcal{O}_X-$subsheaf of $\mathcal{K}$ generated on every $U_i$ as above by $f_i^{-1}$. Equivalently, transition functions for $\mathcal{O}_X(D)$, with respect to the covering $U_i$, are $g_{ij} = f_i/f_j$. The canonical divisor $K_X$ on a non-singular $n-$dimensional variety $X$ is the divisor whose line bundle $\mathcal{O}_X(K_X)$ is $\Omega_X^n = \bigwedge^n(T_X^\vee)$.

**Definition 1.1.14.** *A Cartier divisor $D$ is effective if the local equations $f_i$ are sections of $\mathcal{O}_X$ on $U_i$. In this case there is a canonical section of $\mathcal{O}_X(D)$, which we denote by*

*$s_D$. Regarding $\mathcal{O}_X(D)$ as a subsheaf of $\mathcal{K}$, $s_D$ corresponds to the section 1; with respect to the covering $U_i$, $s_D$ is given by the collection of functions $f_i$, which clearly satisfies $f_i = g_{ij}f_j$ on $U_i \cap U_j$. The section $s_D$ vanishes only on the support of $D$.*

### 1.1.3 Blow-ups

Let $X$ be a closed subscheme of a scheme $Y$, defined by an ideal sheaf $\mathcal{I}$. The normal cone $C_{X/Y}$ to $X$ in $Y$ is the cone over $X$ defined by the graded sheaf of $\mathcal{O}_X$ algebras $\oplus\mathcal{I}^n/\mathcal{I}^{n+1}$:

$$C_{X/Y} = Spec(\oplus_{n\geq 0}\mathcal{I}^n/\mathcal{I}^{n+1}).$$

If the embedding of $X$ in $Y$ is a regular embedding of codimension $d$, then $C_{X/Y}$ is a vector bundle of rank $d$ on $X$, and is denoted also $N_{X/Y}$; the sheaf of sections of $N_{X/Y}$ is $(\mathcal{I}/\mathcal{I}^2)^\vee$.

**Definition 1.1.15.** *The **blow-up** of $Y$ along $X$, denoted $Bl_XY$, is the projective cone over $Y$ of the sheaf of $\mathcal{O}_Y-$algebras $\oplus\mathcal{I}^n$:*

$$Bl_XY = Proj(\oplus_{n\geq 0}\mathcal{I}^n).$$

Let $\widetilde{Y} = Bl_XY$, and let $\pi$ denote the projection from $\widetilde{Y}$ to $Y$. The canonical invertible sheaf (line bundle) $\mathcal{O}(1)$ on the projective cone $\widetilde{Y}$ is the ideal sheaf of $\pi^{-1}(X)$, which is therefore a Cartier divisor on $\widetilde{Y}$, called the exceptional divisor. Let $E = \pi^{-1}(X)$. By construction $E$ is the projective cone of $(\oplus\mathcal{I}^n)\otimes_{\mathcal{O}_Y}\mathcal{O}_X = \oplus\mathcal{I}^n/\mathcal{I}^{n+1}$, so

$$E = P(C_{X/Y})$$

is the projective normal cone to $X$ in $Y$. Moreover, the following result describes how $E$ is embedded in $\widetilde{Y}$.

**Proposition 1.1.16.** *[15, Proposition 13.11.] The normal bundle of $E$ in $\widetilde{Y}$ is*

$$N_{E/\widetilde{Y}} = \mathcal{O}_{\widetilde{Y}}(E)|_E = \mathcal{O}_E(-1). \tag{1.1}$$

Let $\eta$ be the projection from $E = P(C)$ to $X$. If the embedding of $X$ in $Y$ is regular, then the canonical embedding of normal cones $N_{E/Y} \subset \eta^*N_{X/Y}$ is the embedding of the

universal line bundle $\mathcal{O}_E(-1)$ in $\eta^* N_{X/Y}$.
An interesting example of a blow-up preserving the projective bundle structure of $Y$ is described in the following proposition.

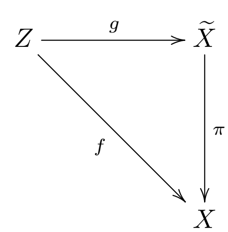
**Proposition 1.1.17.** *[15, Proposition 9.11.] Let $V^{'} \subset V$ be an $r-$dimensional subspace of an $n+1-$dimensional vector space $V$ , and let*

$$W = \mathcal{O}_{\mathbb{P}^{n-r}}(-1) \oplus (V^{'} \otimes \mathcal{O}_{\mathbb{P}^{n-r}}), \tag{1.2}$$

*so that $W$ is a vector bundle of rank $r+1$ on $\mathbb{P}^{n-r} = P(V/V^{'})$. The blow-up $Z$ of $P(V)$ along the $r-1-$dimensional subspace $P(V^{'})$, together with its projection to $\mathbb{P}^{n-r}$, is isomorphic to the projective bundle $p : P(W) \to \mathbb{P}^{n-r}$. Under this isomorphism, the blow-up map $Z \to \mathbb{P}^n$ corresponds to the complete linear series $\left|\mathcal{O}_{P(W)}(1)\right|$.*

In general, $\pi$ induces an isomorphism from $\widetilde{Y} - E$ onto $Y - X$, and is fully characterized by the following universal property.

**Proposition 1.1.18.** *[22, Proposition 7.14 (Universal Property of Blowing Up)] Let $X$ be a noetherian scheme, $\mathcal{I}$ a coherent sheaf of ideals, and $\pi : \widetilde{X} \to X$ the blow-up with respect to $\mathcal{I}$. If $f : Z \to X$ is any morphism such that $f^{-1}\mathcal{I} \cdot \mathcal{O}_Z$is an invertible sheaf of ideals on $Z$, then there exists a unique morphism $g : Z \to \widetilde{X}$ factoring $f$*




If $X \subset Y$ is a closed imbedding, and $f : Y^{'} \to Y$ is a morphism, set $X^{'} = f^{-1}(X)$, $g : X^{'} \to X$ the induced morphism. Then there is a closed imbedding

$$Bl_{X'}Y^{'} \subset Bl_X Y \times_Y Y^{'}$$

If $\widetilde{f}$ is the induced morphism from $Bl_{X'}Y$ to $Bl_X Y$, then $\widetilde{f}^{-1}(E) = E^{'}$, where $E$ and $E^{'}$ are the exceptional divisors.
In particular, if $X \subset Y \subset Z$ are closed imbeddings, there is a canonical imbedding of $Bl_X Y$ in $Bl_X Z$, with the exceptional divisor of $Bl_X Z$ restricting to the exceptional divisor of $Bl_X Y$.

Next result, that will be a key tool for the rest of the work, allow us to explicitly compute the normal bundle of the strict transform of a given subvariety $Y \subset Z$ under some regularity assumptions.

**Proposition 1.1.19.** *[17, Appendix B.6.10.] If $X \subset Y$ and $Y \subset Z$ are regular embeddings, let $\widetilde{Z} = Bl_X Z$, $E$ the exceptional divisor in $\widetilde{Z}$, $\rho$ the projection from $\widetilde{Z}$ to $Z$. Let $\widetilde{Y} = Bl_X Y$. Then $\widetilde{Y} \subset \rho^{-1}(Y)$, $E \subset \rho^{-1}(Y)$, and $\widetilde{Y}$ is the residual scheme to $E$ in $\rho^{-1}(Y)$, i.e., the ideal sheaves of $\widetilde{Y}$, $E$ and $\rho^{-1}(Y)$ in $\widetilde{Z}$ are related by*

$$\mathcal{I}(\widetilde{Y}) \cdot \mathcal{I}(E) = \mathcal{I}(\rho^{-1}(Y)).$$

*In addition, the canonical embedding of $\widetilde{Y}$ in $\widetilde{Z}$ is a regular imbedding, with normal bundle*

$$N_{\widetilde{Y}/\widetilde{Z}} \cong \pi^* N_{Y/Z} \otimes \mathcal{O}(-F)$$

*where $\pi$ is the projection from $\widetilde{Y}$ to $Y$, and $F$ is the exceptional divisor on $\widetilde{Y}$ of such projection.*

## 1.2 Intersection theory

### 1.2.1 Rational equivalence

Let $X$ be an algebraic scheme over a field $k$.

**Definition 1.2.1.** *A $k-$cycle on $X$ is a finite formal sum $\sum n_i [V_i]$ where the $V_i$ are $k-$dimensional subvarieties of $X$, and the $n_i$ are integers. The group of $k-$cycles on $X$, denoted $Z_k X$, is the free abelian group on the $k-$dimensional subvarieties of $X$; to a subvariety $V$ of $X$ corresponds $[V]$ in $Z_k X$.*
*For any $(k+1)-$dimensional subvariety $W$ of $X$, and any $r \in R(W)^*$, define a $k-$cycle $[div(r)]$ on $X$ by*

$$[div(r)] = \sum ord_V(r) [V], \tag{1.3}$$

*the sum over all codimension one subvarieties $V$ of $W$; here $ord_V$ is the order function on $R(W)^*$ defined by the local ring $\mathcal{O}_{V,W}$.*
*A $k-$cycle $\alpha$ is rationally equivalent to zero, written $\alpha \sim 0$, if there are a finite number*

*of* $(k+1)-$*dimensional subvarieties* $W_i$ *of* $X$, *and* $r_i \in R(W)^*$, *such that*

$$\alpha = \sum [div(r_i)] .$$

Since $\left[div(r^{-1})\right] = -[div(r)]$, the cycles rationally equivalent to zero form a subgroup $Rat_k X$ of $Z_k X$. The group of $k-$cycles modulo rational equivalence on $X$ is the factor group

$$A_k X = Z_k X / Rat_k X.$$

**Definition 1.2.2.** *Define* $Z_* X$ *(resp.* $A_* X$*) to be the direct sum of the* $Z_k X$ *(resp.* $A_k X$*) for* $k = 0, 1, ..., dim(X)$*. A cycle (resp. cycle class) on* $X$ *is an element of* $Z_* X$ *(resp.* $A_* X$*).*

If $\alpha$ is a class in $A_* X$, and $k$ is an integer, we denote by $\{\alpha\}_k$ the component $\alpha$ in $A_k X$. Thus $\alpha = \sum_{k \geq 0} \{\alpha\}_k$.

A cycle is positive if it is not zero, and each of its coefficients is a positive integer. A cycle class is positive if it can be represented by a positive cycle.

#### 1.2.1.1 Push-forward and pull-back of cycles

Let $f : X \to Y$ be a proper morphism. For any subvariety $V$ of $X$, the image $W = f(V)$ is then a (closed) subvariety of $Y$. There is an induced imbedding of $R(W)$ in $R(V)$, which is a finite field extension if $W$ has the same dimension as $V$. Set

$$deg(V/W) = \begin{cases} [R(V) : R(W)] & if \ \text{dim(W)=dim(V)} \\ 0 & \text{if } \ \text{dim(W)<dim(V)} \end{cases}$$

where $[R(V) : R(W)]$ denotes the degree of the field extension. Define $f_*[V] = deg(V/W)\,[W]$. This extends linearly to a homomorphism

$$f_* : Z_k X \to Z_k Y.$$

These homomorphisms are functorial: if $g$ is a proper morphism from $Y$ to $Z$, then $(g \circ f)_* = g_* \circ f_*$, as follows from the multiplicativity of degrees of field extensions.

**Proposition 1.2.3.** *[17, Proposition 1.4.] Let $f : X \to Y$ be a proper, surjective morphism of varieties, and let $r \in R(X)^*$. Then*

$$\begin{cases} f_* \left[div(r)\right] = 0 & \text{if } dim(Y) < dim(X) \quad (1.4) \\ f_* \left[div(r)\right] = \left[div(N(r))\right] & \text{if } dim(Y) = dim(X) \quad (1.5) \end{cases}$$

*In* 1.5*, $R(X)$ is a finite extension of $R(Y)$, and $N(r)$ is the norm of $r$, i.e., the determinant of the $R(Y)-$linear endomorphism of $R(X)$ given by multiplication by $r$.*

**Theorem 1.2.4.** *[17, Theorem 1.4.] If $f : X \to Y$ is a proper morphism, and $\alpha$ is a $k-$cycle on $X$ which is rationally equivalent to zero, then $f_*(\alpha)$ is rationally equivalent to zero on $Y$.*

**Definition 1.2.5.** *[17, Definition 1.4.] If $X$ is a complete scheme, i.e., $X$ is proper over $S = Spec(K)$, $K$ the ground field, and $\alpha = \sum_P n_P \left[P\right]$ is a zero-cycle on $X$, the* ***degree*** *of $\alpha$, denoted $deg(\alpha)$, or $\int_X \alpha$, is defined by*

$$deg(\alpha) = \int_X \alpha = \sum_P n_P \left[R(P) : K\right].$$

Equivalently, $deg(\alpha) = p_*(\alpha)$, where $p$ is the structure morphism from $X$ to $S$, and $A_0 S = \mathbb{Z}\left[S\right]$ is identified with $\mathbb{Z}$. By the theorem, rationally equivalent cycles have the same degree. We extend the degree homomorphism to all of $A_* X$,

$$\int_X : A_* X \to \mathbb{Z}$$

by defining $\int_X \alpha = 0$ if $\alpha \in A_k X$, $k > 0$. For any morphism $f : X \to Y$ of complete schemes, and any $\alpha \in A_* X$,

$$\int_X \alpha = \int_Y f_*(\alpha),$$

a special case of functoriality. We often write $\int$ instead of $\int_X$.

Let $f : X \to Y$ be a flat morphism of relative dimension $n$. The examples of primary importance for us will be:

A an open imbedding ($n = 0$),

B the projection of a vector bundle or $\mathbb{A}^n-$bundle, or a projective bundle, to its base,

C the projection from a Cartesian product $X = Y \times Z$ to the first factor, where $Z$ is a purely $n-$dimensional scheme, D any dominant morphism from an $(n+1)-$dimensional variety to a nonsingular curve.

**Remark 1.2.6.** *In this thesis, a flat morphism is always assumed to have relative dimension $n$ for some integer $n \in \mathbb{Z}$.*

For such $f : X \to Y$, and any subvariety $V$ of $Y$, set

$$f^* [V] = \left[f^{-1}(V)\right].$$

Here $f^{-1}(V)$ is the inverse image scheme, a subscheme of $X$ of pure dimension $dim(V)+n$, and $\left[f^{-1}(V)\right]$ is its cycle. This extends by linearity to pull-back homomorphisms

$$f^* : Z_k Y \to Z_{k+n} X.$$

**Lemma 1.2.7.** *[17, Lemma 1.7.1.] If $f : X \to Y$ is flat, then for any subscheme $Z$ of $Y$,*

$$f^* [Z] = \left[f^{-1}(Z)\right].$$

It follows from this lemma that flat pull-backs are functorial: if $f : X \to Y$ and $g : Y \to Z$ are flat, then $g \circ f$ is flat, and $(g \circ f)^* = f^* \circ g^*$. For if $V$ is a subvariety of $Z$, then

$$(g \circ f)^* [V] = \left[(g \circ f)^{-1}(V)\right] = \left[f^{-1} \circ g^{-1}(V)\right] = f^* \left[g^{-1}(V)\right] = f^* \circ g^* [V].$$

**Theorem 1.2.8.** *[17, Theorem 1.7.] Let $f : X \to Y$ be a flat morphism of relative dimension $n$, and $\alpha$ a $k-$cycle on $Y$ which is rationally equivalent to zero. Then $f^*\alpha$ is rationally equivalent to zero in $Z_{k+n}X$. There are therefore induced homomorphisms, the flat pull-backs,*

$$f^* : A_k Y \to A_{k+n} X,$$

*so that $A_*$ becomes a contravariant functor for flat morphisms.*

**Proposition 1.2.9.** *[17, Proposition 1.8.] Let $Y$ be a closed subscheme of a scheme $X$, and let $U = X - Y$. Let i: $Y \to X$, $j : U \to X$ be the inclusions. Then the sequence*

$$A_k Y \xrightarrow{i_*} A_k X \xrightarrow{j^*} A_k U \to 0$$

*is exact for all $k$.*

**Corollary 1.2.10.** *[17, Example 1.8.1.] Let*

$$\begin{array}{ccc} Y' & \xrightarrow{j} & X' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{i} & X \end{array}$$

*be a fibre square, with $i$ a closed imbedding, $p$ proper, such that $p$ induces an isomorphism of $X' - Y'$ onto $X - Y$. Then the sequence*

$$A_kY' \xrightarrow{a} A_kY \oplus A_kX' \xrightarrow{b} A_kX \to 0$$

*is exact, where $a(\alpha) = (q_*(\alpha), -j_*(\alpha))$, $b(\alpha, \beta) = i_*(\alpha) + p_*(\beta)$.*

#### 1.2.1.2 An alternative definition of rational equivalence

Now that the push-forward of cycles i well defined, a more classical definition of $A_*X$ will be given. Let $X$ be a scheme, and let $X \times \mathbb{P}^1$ be the Cartesian product of $X$ with $\mathbb{P}^1$. Let $p$ be the projection from $X \times \mathbb{P}^1$ to $X$. Let $V$ be a $(k+1)-$dimensional subvariety of $X \times \mathbb{P}^1$ such that the projection to the second factor induces a dominant morphism $f$ from $V$ to $\mathbb{P}^1$. For any point $P \in \mathbb{P}^1$ which is rational over the ground field, the scheme-theoretic fiber $f^{-1}(P)$ is a subscheme of $X \times \{P\}$, which $p$ maps isomorphically onto a subscheme of $X$; we denote this subscheme by $V(P)$. Note in particular that $p_* \left[f^{-1}(P)\right] = [V(P)]$ in $Z_kX$. The morphism$f : V \to \mathbb{P}^1$ determines a rational function $f \in R(V)^*$. It follows that

$$\left[f^{-1}(0)\right] - \left[f^{-1}(\infty)\right] = [div(f)],$$

where $0 = (1 : 0)$ and $\infty = (0 : 1)$ are the usual zero and infinity points of $\mathbb{P}^1$. Therefore

$$[V(0)] - [V(\infty)] = p_* [div(f)],$$

which is rationally equivalent to zero on $X$.

**Proposition 1.2.11.** *[17, Proposition 1.6.] A cycle $\alpha$ in $Z_kX$ is rationally equivalent to zero if and only if there are $(k+1)-$dimensional subvarieties $V_1, ..., V_t$ of $X \times \mathbb{P}^1$, such that the projections from $V_i$ to $\mathbb{P}^1$ are dominant, with*

$$\alpha = \sum_{i=1}^{t} [V_i(0)] - [V_i(\infty)]$$

*in $Z_kX$.*

### 1.2.2 Divisors

Let $X$ be an $n-$dimensional variety. A **Weil divisor** on $X$ is an $(n-1)-$cycle on $X$. The Weil divisors form the group $Z_{n-1}X$. Given a Cartier divisor $D$, define the associated Weil divisor $[D]$ of $D$ by setting

$$[D] = \sum ord_V D\,[V]$$

the sum over all codimension one subvarieties $V$ of $X$; note that there are only finitely many $V$ with $ord_V D \neq 0$. The Cartier divisors form an abelian group $Div(X)$: if $D$ and $E$ are given by data $(U_\alpha, f_\alpha)$ and $(U_\alpha, g_\alpha)$, the sum $D+E$ is given by $(U_\alpha, f_\alpha \cdot g_\alpha)$. By the additivity of the order functions, the mapping $D \to [D]$ is a homomorphism

$$Div(X) \to Z_{n-1}(X)$$

Any $f$ in $R(X)^*$ determines a principal Cartier divisor $div(f)$, by taking all local equations equal to $f$. Note that the Weil divisor associated to $div(f)$ is the cycle $[div(f)]$ defined in equation 1.3. Two divisors $D, D^{'}$ are linearly equivalent if they differ by a principal divisor: $D^{'}D + div(f)$. From the definition of rational equivalence, it follows that $[D]$ and $\left[D^{'}\right]$ are rationally equivalent cycles. If $Pic(X)$ denotes the group of linear equivalence classes of Cartier divisors, there is an induced homomorphism

$$Pic(X) \to A_{n-1}(X)$$

This homomorphism is in general neither injective nor surjective.

If $D$ is a Cartier divisor on $X$, and $\alpha$ a $k-$cycle on $X$ we define an intersection class

$$D \cdot \alpha \in A_{k-1}(|D| \cap |\alpha|).$$

By linearity it suffices to define $D \cdot [V]$ if $V$ is a subvariety of $X$. Let $i$ be the inclusion of $V$ in $X$. There are two cases:

A If $V$ is not contained in the support of $D$, then by restricting local equations, $D$ determines a Cartier divisor, denoted $i^*(D)$, on $V$. In this case, set

$$D \cdot [V] = [i^*(D)]\,,$$

the associated Weil divisor of $i^*(D)$ on $V$. In this case $D \cdot [V]$ is a well-defined cycle.

B If $V \subset |D|$, then the line bundle $\mathcal{O}_X(D)$ restricts to a line bundle $i^*\mathcal{O}_X(D)$ on $V$. Choose a Carrier divisor $C$ on $V$ whose line bundle is isomorphic to this line bundle: $\mathcal{O}_V(C) \cong i^*\mathcal{O}_X(D)$, and set

$$D \cdot [V] = [C]$$

the associated Weil divisor of $C$. Since $C$ is well defined up to a principal divisor on $V$, $[C]$ is well defined in $A_{k-1}V$.

This intersection product satisfies the formal properties one would expect for a "cap product". For example:

a If $\alpha \sim \alpha^{'}$, then $D \cdot \alpha = D \cdot \alpha^{'}$ in $A_*(|D|)$.

b If $D - D^{'}$ is principal, then $D \cdot \alpha = D^{'} \cdot \alpha$ in $A_*(|\alpha|)$.

c (Projection formula) If $f : Y \to X$ is a proper surjective morphism of varieties, $D$ a Carrier divisor on $X$, and $\alpha$ a $k-$cycle on $Y$, then

$$f_*^{'}(f^*(D) \cdot \alpha) = D \cdot f_*(\alpha) \tag{1.6}$$

in $A_{k-1}(Z)$, with $Z = |D| \cap f(|\alpha|)$, and $f^{'} : f^{-1}(Z) \to Z$ the morphism induced by $f$. There is a similar compatibility with flat pull-backs.

From $i)$ and $ii)$ it follows that the operation product $D \cdot \alpha$ determines products

$$Pic(X) \otimes A_k X \to A_{k-1}(X).$$

The following proposition is obtained as a particular case of the one above.

**Proposition 1.2.12.** *[13, Proposition 1.10] Let $f : Z \to X$ be a proper surjective morphism. Let $D_1, D_2, ..., D_r$ be Cartier divisors on $X$ with $r = d = dim(X)$. Then, one has*

$$f^*D_1 \cdot f^*D_2 \cdots f^*D_r = deg(f) D_1 \cdot D_2 \cdots D_r$$

*where $deg(f) = [K(Z) : K(X)]$, if $deg(f)$ is finite.*

#### 1.2.2.1 Chern class of a line bundle

Let $L$ be a line bundle on a scheme $X$. For any $k-$dimensional subvariety $V$ of $X$, the restriction $L|_V$ of $L$ to $V$ is isomorphic to $\mathcal{O}_V(C)$ for some Cartier divisor $C$ on $V$, determined up to linear equivalence. The Weil divisor $[C]$ determines a well-defined element in $A_{k-1}(X)$, which we denote by $c_1(L) \cap [V]$:

$$c_1(L) \cap [V] = [C]\,. \tag{1.7}$$

This is extended by linearity to define a homomorphism $\alpha \to c_1(L) \cap \alpha$ from $Z_k(X)$ to $A_{k-1}(X)$. If $L = \mathcal{O}_X(D)$ for a pseudo-divisor $D$ on $X$ (see [17, Definition 2.2.1.]), it follows from the definition of the intersection class that

$$c_1(\mathcal{O}_X(D)) \cap \alpha = D \cdot \alpha$$

in $A_{k-1}(X)$.

**Proposition 1.2.13.** *[17, Proposition 2.5.]*

a *If $\alpha$ is rationally equivalent to zero on $X$, then $c_1(L) \cap \alpha = 0$. There is therefore an induced homomorphism*

$$c_1(L) \cap \_ : A_k X \to A_{k-1} X.$$

b *(Commutativity). If $L$, $L^{'}$ are line bundles on $X$, $\alpha$ a $k-$cycle on $X$, then*

$$c_1(L) \cap (c_1(L^{'}) \cap \alpha) = c_1(L^{'}) \cap (c_1(L)\alpha)$$

*in $A_{k-2}(X)$.*

c *(Projection formula). If $f : X^{'} \to X$ is a proper morphism, $L$ a line bundle on $X$, $\alpha$ a $k-$cycle on $X^{'}$, then*

$$f_*(c_1(f^*L) \cap \alpha) = c_1(L) \cap f_*(\alpha) \tag{1.8}$$

*in $A_{k-1}(X)$.*

d *(Flat pull-back). If $f : X^{'} \to X$ is flat of relative dimension $n$, $L$ a line bundle on $X$, $\alpha$ a $k-$cycle on $X$, then*

$$c_1(f^*L) \cap f^*\alpha = f^*(c_1(L) \cap \alpha)$$

*in $A_{k+n-1}(X^{'})$.*

*e (Additivity). If $L$, $L^{'}$ are line bundles on $X$, $\alpha$ a $k-$cycle on $X$, then*

$$c_1(L \otimes L^{'}) \cap \alpha = c_1(L) \cap \alpha + c_1(L^{'}) \cap \alpha$$

*and*

$$c_1(L^{\vee}) \cap \alpha = -c_1(L) \cap \alpha$$

*in $A_{k-1}X$.*

#### 1.2.2.2 Gysin map for divisors

If $D$ is an effective Cartier divisor on a variety $X$, the restriction of $\mathcal{O}_X(D)$ to $D$ is the normal bundle $N_{D/X}$, and

$$[D] = c_1(\mathcal{O}_x(D)) \cap [X] \,.$$

**Definition 1.2.14.** *[17, Section 2.6] Let $D$ be an effective Cartier divisor on a variety $X$, and let $i : D \to X$ be the inclusion. There are Gysin homorphisms*

$$i^* : Z_k X \to A_{k-1} D$$

*for $k = 1, ..., dim(X)$ defined by the formula*

$$i^*(\alpha) = D \cdot \alpha$$

**Proposition 1.2.15.** *[17, Proposition 2.6.] There are therefore induced homomorphisms:*

$$i^* : A_k X \to A_{k-1} D$$

*for $k = 1, ..., dim(X)$ such that one has*

*a If $\alpha$ is a k-cycle on $X$, then*

$$i_* i^*(\alpha) = c_1(\mathcal{O}_X(D)) \cap \alpha$$

*b If $\alpha$ is a k-cycle on $D$, then*

$$i^* i_*(\alpha) = c_1(N_{D/X}) \cap \alpha \qquad (1.9)$$

*c If $X$ is purely n-dimensional, then*

$$i^*[X] = [D]$$

*in $A_{n-1}D$.*

*d If $L$ is a line bundle on $X$, then*

$$i^*(c_1(L) \cap \alpha) = c_1(i^*L) \cap i^*(\alpha)$$

*in $A_{k-2}(D)$ for any $k-$cycle $\alpha$ on $X$.*

### 1.2.3 Segre classes and Chern classes of vector bundles

Let $V$ be a vector bundle of rank $e+1$ on an algebraic scheme $X$. Let $P(V)$ be the projective bundle of lines in $V$, $p$ the projection from $P(V)$ to $X$, and let $\mathcal{O}(1) = \mathcal{O}_{P(V)}(1)$ denote the canonical line bundle on $P(V)$, i.e., its dual $\mathcal{O}_{P(V)}(-1)$ is the tautological subbundle of $p^*V$. Define homomorphisms $\alpha \to s_i(V) \cap \alpha$ from $A_kX$ to $A_{k-i}X$ by the formula

$$s_i(V) \cap \alpha = p_*(c_1(\mathcal{O}(1))^{e+i} \cap p^*(\alpha)).$$

Here $p^*$ is the flat pull-back from $A_kX$ to $A_{k+e}P(V)$, $c_1(\mathcal{O}(1))^{e+i} \cap \_$ is the iterated first Chern class homomorphism from $A_{k+e}P(V)$ to $A_{k-i}P(V)$, and $p_*$ is the push-forward from $A_{k-i}P(V)$ to $A_{k-i}X$.

Now we define Chern class operator

$$c_l(E) \cap \_ : A_kX \to A_{k-l}X$$

by formally inverting the Segre classes

$$1 + c_1(E) + c_2(E) + ... = (1 + s_1(E) + s_2(E) + ...)^{-1}$$

Explicitly

$$c_p(E) = (-1)^p det \begin{vmatrix} s_1(E) & 1 & 0 & \cdots & 0 \\ s_2(E) & s_1(E) & 1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & 0 \\ & & & s_1(E) & 1 \\ s_p(E) & s_{p-1}(E) & \cdots & \cdots & s_1(E) \end{vmatrix}$$

**Theorem 1.2.16.** *[17, Theorem 3.2.] The Chern classes satisfy the following properties:*

a *(Vanishing) For all vector bundles $E$ on $X$, all $i > rank(E)$,*

$$c_i(E) = 0.$$

b *(Commutativity) For all vector bundles $E$, $F$ on $X$, integers $i$,$j$, and cycles $\alpha$ on $X$,*

$$c_i(E) \cap (c_j(F) \cap \alpha) = c_j(F) \cap (c_i(E) \cap \alpha)$$

c *(Projection formula) Let $E$ be a vector bundle on $X$, $f : X^{'} \to X$ a proper morphism. Then*

$$f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*(\alpha)$$

*for all cycles $\alpha$ on $X^{'}$, all $i$.*

d *(Pull-back) Let $E$ be a vector bundle on $X$, $f : X^{'} \to X$ a flat morphism. Then*

$$c_i(f^*E) \cap f^*\alpha = f^*(c_i(E) \cap \alpha)$$

*for all cycles $\alpha$ on $X$, all $i$.*

e *(Whitney sum) For any exact sequence*

$$0 \to E^{'} \to E \to E^{''} \to 0$$

*of vector bundles on $X$, then*

$$c_t(E) = c_t(E^{'}) \cdot c_t(E^{''}),$$

*i.e.,*

$$c_k(E) = \sum_{i+j=k} c_i(E^{'}) c_j(E^{''});$$

f *(Normalization) If $E$ is a line bundle on a variety $X$, $D$ a Cartier divisor on $X$ with $\mathcal{O}(D) \cong E$, then*

$$c_1(E) \cap [X] = [D].$$

*Note that from c and f it follows that the first Chern class for a line bundle defined here agrees with the definition given in 1.7.*

### 1.2.4 Segre class of a subscheme and excess intersections

The Segre class of a cone $C$, denoted $s(C)$, is the class in $A^{\bullet}(X)$ defined by the formula

$$s(C) = q_*(\sum_{i \geq 0}((c_1(\mathcal{O}(1)))^i \cap [P(C \oplus 1)])$$

**Proposition 1.2.17.** *[17, Proposition 4.1.] If $E$ is a vector bundle on $X$, then*

$$s(E) = c(E)^{-1} \cap [X] \in A^{\bullet}(X)$$

Let $X$ be a closed subscheme of a scheme $Y$.The Segre class of $X$ in $Y$, denoted $s(X, Y)$, is defined to be the Segre class of the normal cone $C_{X/Y}$:

$$s(X, Y) = s(C_{X/Y}) \in A^{\bullet}(X).$$

In case $X$ is regularly imbedded in $Y$, so the normal cone is a vector bundle, it follows from Proposition 1.2.17 that $s(X, Y)$ is the cap product of the total inverse Chern class of the normal bundle with $[X]$.

**Proposition 1.2.18.** *[17, Proposition 4.2.] Let $f : Y^{'} \rightarrow Y$ be a morphism of pure-dimensional schemes, $X \subset Y$ a closed subscheme, $X^{'} = f^{-1}(X)$ the inverse image scheme, $g : X^{'} \rightarrow X$ the induced morphism.*

*a If $f$ is proper, $Y$ irreducible, and $f$ maps each irreducible component of $Y^{'}$ onto $Y$, then*

$$g_*(s(X^{'}, Y^{'})) = deg(Y^{'}/Y)s(X, Y).$$

*b If $f$ is flat, then*

$$g^*(s(X, Y)) = s(X^{'}, Y^{'}).$$

**Corollary 1.2.19.** *[17, Corollary 4.2.2.] Let $X$ be a proper closed subscheme of a variety $Y$. Let $\widetilde{Y}$ be the blow-up of $Y$ along $X$, $\widetilde{X} = P(C_{X/Y})$ the exceptional divisor, $\eta : \widetilde{X} \rightarrow X$ the projection. Then*

$$\begin{aligned} s(X, Y) &= \sum_{k \geq 1}(-1)^{k-1}\eta_*(\widetilde{X}^k) \\ &= \sum_{i \geq 0}\eta_*((c_1(\mathcal{O}(1)))^i \cap \left[P(C_{X/Y})\right]). \end{aligned}$$

For an irreducible subvariety $X$ of a variety $Y$, the coefficient of $[X]$ in the class $s(X,Y)$ is called the multiplicity of $Y$ along $X$, or the algebraic multiplicity of $X$ on $Y$, and is denoted $e_X Y$. If $codim(X,Y) = n > 0$, then

$$\begin{aligned} e_X Y\,[X] &= q_*((c_1(\mathcal{O}(1)))^n \cap [P(C \oplus 1)]) \\ &= p_*((c_1(\mathcal{O}(1)))^{n-1} \cap [P(C)]) \\ &= (-1)^{n-1} p_*(\widetilde{X}^n). \end{aligned}$$

Here $C = C_{X/Y}$, $p$ and $q$ are the projections from $P(C)$ and $P(C \oplus 1)$ to $X$, $\widetilde{Y}$ is the blow-up of $Y$ along $X$, with exceptional divisor $\widetilde{X} = P(C)$. This definition is equivalent to the definition of the multiplicity of the local ring $\mathcal{O}_{X,Y}$ given by Samuel (I).
If $X = P$ is a point, $C = C_{P/Y}$ is the tangent cone to $P$ in $Y$, and

$$e_P Y = \int_{P(C)} (c_1(\mathcal{O}(1)))^{n-1} \cap [P(C)] = deg\,[P(C)]\,.$$

In this case $e_P Y$ is called the multiplicity of $Y$ at $P$.

The excess intersection problems arise in situations in which we wish to describe something about improper intersections, where the intersection has components of dimension greater than expected.

Suppose that $X$ is a smooth projective variety, $D \subset X$ is a Cartier divisor and $i_C : C \to X$ the inclusion morphism of a subvariety $C$ in $X$. If $C$ intersects $D$ generically transversely, then the intersection class $[C] \cdot [D]$ of $D$ and $C$ is $[C \cap D]$; but what if $C$ is contained in $D$?

**Proposition 1.2.20.** *[15, Proposition 13.1.] Suppose that $X$ is a smooth projective variety. Let $i_C : C \to X$ be the inclusion morphism of a subvariety of codimension $k$ in $X$, and let $D \subset X$ be an effective Cartier divisor containing $C$. We have*

$$[C]\,[D] = i_{C*}(\gamma_C) \in A^{k-1}(X),$$

*where $\gamma_C = c_1(N_{D/X}|_C)$*

The previous proposition is in fact a special case of a much more general result.

**Theorem 1.2.21.** *[15, Theorem 13.3. (Excess intersection formula)] If $S \subset X$ is a subvariety of a smooth variety $X$ and $T$ is a locally complete intersection subvariety of*

*X, then*

$$[S] \cdot [T] = \sum_C i_{C*}(\gamma_C),$$

*where:*

a *The sum is taken over the connected components $C$ of $S \cap T$.*

b *$i_C : C \to X$ denotes the inclusion morphism.*

c *$\gamma_C = \{s(C,S)c(N_{T/X}|_C)\}_d \in A_d(C)$, where $d = dimX - codimS - codimT$ is the"expected dimension" of the intersection.*

*If the subvariety $S$ is locally a complete intersection as well, then we have a symmetric form*

$$\gamma_C = \{s(C,X)c(N_{S/X}|_C)c(N_{T/X}|_C)\}_d.$$

One frequently occurring situation in which excess intersection arises is the case of cycles $[A]$ and $[\beta]$ on a smooth variety $X$ that happen to both lie on a proper subvariety $Z \subsetneq X$. Although as cycles on $X$ their intersection cannot even be dimensionally transverse, we can relate their intersection class $[A] \cdot [B] \in A^\bullet(Z)$ in $Z$ to the intersection of their classes on $X$.

**Proposition 1.2.22.** *[15, Proposition 13.6. (Key formula)] Let $i : Z \to X$ be an inclusion of smooth projective varieties of codimension $m$, and let $N_{Z/X}$ be the normal bundle of $Z$ in $X$. If $\alpha \in A^a(Z)$ and $\beta \in A^b(Z)$, then*

$$i_*(\alpha) \cdot i_*(\beta) = i_*(\alpha \cdot \beta \cdot c_m(N_{Z/X})) \in A^{a+b+2m}(X).$$

This proposition follows from the following theorem , that generalizes 1.9 for arbitrary smooth projective subvarieties

**Theorem 1.2.23.** *[15, Theorem 13.7.] Let $i_Z : Z \to X$ be an inclusion of smooth projective varieties of codimension $m$, and let $N_{Z/X}$ be the normal bundle of $Z$ in $X$. For any class $\alpha \in A^\bullet(Z)$ we have*

$$i_Z^*(i_{Z*}(\alpha)) = \alpha \cdot c_m(N_{Z/X}) \in A^{a+m}(Z)$$

### 1.2.5 Chow ring of projective bundles

We start this section by a well known result about the Chow ring of the most basic projective bundle, that is, the one defined over a point.

**Theorem 1.2.24.** *[15, Theorem 2.1.] The Chow ring of $\mathbb{P}^n$ is*

$$A^\bullet(\mathbb{P}^n) = \mathbb{Z}\left[\varsigma\right]/(\varsigma^{n+1}), \tag{1.10}$$

*where $\varsigma \in A^1(\mathbb{P}^n)$ is the rational equivalence class of a hyperplane; more generally, the class of a variety of codimension $k$ and degree $d$ is $d\varsigma^k$.*

Next theorem extends the previous one to projective bundles defined over higher dimensional projective varieties.

**Theorem 1.2.25.** *[15, Theorem 9.6.] Let $V$ be a vector bundle of rank $r+1$ on a smooth projective variety $X$, and let $\varsigma = c_1(\mathcal{O}_{P(V)}(1)) \in A^1(P(V))$, and $p : P(V) \to X$ the projection of the induced projective bundle. The map $p^* : A(X) \to A(P(V))$ is an injective ring homomorphism, and via this map one has the isomorphism of $A(X)$-algebras given by*

$$A(P(V)) \cong A(X)\left[\varsigma\right]/(\varsigma^{r+1} + c_1(V)\varsigma^r + \cdots + c_{r+1}(V))$$

*In particular, the group homomorphism $A(X)^{\oplus r+1} \to A(P(V))$ / given by $(\alpha_0, ..., \alpha_r) \mapsto \sum \varsigma^i p^*(\alpha_i)$ is an isomorphism, so that*

$$A(P(V)) \cong \bigoplus_{i=0}^{r} \varsigma^i A(X)$$

*as groups.*

Continuing with the example appearing in Proposition 1.1.17, next result gives an explicit description of the Chow ring of the blow-up of $\mathbb{P}^n$ at a linear subspace.

**Corollary 1.2.26.** *[15, Corollary 9.12.] Let $Z \subset \mathbb{P}^n \times \mathbb{P}^{n-r}$ be the blow-up of an $r-1-$plane $\Delta$ in $\mathbb{P}^n$. Writing $\alpha, \varsigma \in A^1(Z)$ for the pullbacks of the hyperplane classes on $\mathbb{P}^{n-r}$ and $\mathbb{P}^n$ respectively, we have*

$$A(Z) = \mathbb{Z}\left[\alpha, \varsigma\right]/(\alpha^{n-r+1}, \varsigma^{r+1} - \alpha\varsigma^r). \tag{1.11}$$

*With this notation the class of the exceptional divisor $E \subset Z$, the preimage of $\Delta$ in $Z$, is*

$$[E] = \varsigma - \alpha. \tag{1.12}$$

Finally, given a projective bundle, the following proposition gives a presentation of the class of projective subbundle in terms of the generators its Chow ring.

**Proposition 1.2.27.** *[15, Proposition 9.13] If $X$ is a smooth projective variety and $\mathcal{W} \subset \mathcal{V}$ are vector bundles on $X$ of ranks $s$ and $r$ respectively, then*

$$[P(W)] = \varsigma^{r-s} + \gamma_1 \varsigma^{r-s-1} + ... + \gamma_{r-s} \in A^{r-s}(P(V)),$$

*where $\varsigma = c_1(\mathcal{O}_{P(V)}(1))$ and $\gamma_k = c_k(V/W)$. Moreover, the normal bundle of $P(W)$ in $P(V)$ is $\mathcal{O}_{P(W)}(1) \otimes p^*(V/W)$.*

**Proposition 1.2.28.** *[15, Proposition 9.14.] If $L \subset V$ is a line subbundle of a vector bundle $V$ on a variety $X$, then $P(L) \subset P(V)$ is the image of a section $X \to P(V)$ of the projection $P(V) \to X$, and every section has this form.*

### 1.2.6 Intersection theory of blow-ups

Let $X$ be a regularly imbedded subscheme of a scheme $Y$, of codimension $d$, with normal bundle $N_{X/Y}$. Let $\widetilde{Y}$ be the blow-up of $Y$ along $X$, and let $\widetilde{X} = P(N_{X/Y})$ be the exceptional divisor. We have a fiber square

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{j} & \widetilde{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y \end{array}$$

Since $N_{\widetilde{X}/\widetilde{Y}} = \mathcal{O}_{\widetilde{X}}(-1)$, the excess normal bundle $Q$ is the universal quotient bundle on $P(N_{X/Y})$:

$$Q = g^* N_{X/Y} / N_{\widetilde{X}/\widetilde{Y}} = g^* N_{X/Y} / \mathcal{O}_{\widetilde{X}}(-1).$$

**Proposition 1.2.29.** *[17, Proposition 6.7.]*

*a (Key Formula). For all $x \in A_k X$,*

$$f^* i_*(x) = j_*(c_{d-1}(\mathcal{Q}) \cap g^* x)$$

*in $A_k Y$.*

*b For all $y \in A_k Y$, $f_* f^* y = y$ in $A_k Y$.*

*c If $\widetilde{x} \in A_k \widetilde{X}$, and $g_* \widetilde{x} = j^* j_* \widetilde{x} = 0$, then $x = 0$.*

*d If $\widetilde{y} \in A_k \widetilde{Y}$, and $f_* \widetilde{y} = j^* \widetilde{y} = 0$, then $y = 0$.*

*e There are split exact sequences*

$$0 \to A_k X \xrightarrow{\alpha} A_k \widetilde{X} \oplus A_k Y \xrightarrow{\beta} A_k \widetilde{Y} \to 0 \tag{1.13}$$

*with $\alpha(x) = (c_{d-1}(\mathcal{Q}) \cap g^* x, -i_* x)$, and $\beta(\widetilde{x}, y) = j_* \widetilde{x} + f^* y$. A left inverse for $\alpha$ is given by $(\widetilde{x}, y) \xrightarrow{g}_* (x)$.*

Moreover, we can generalize the pull-back formula for flat morphisms (see Lemma 1.2.7) to the case of blow-ups.

**Theorem 1.2.30.** *[17, Theorem 6.7] (Blow-up Formula). Let $V$ be a $k-$dimensional subvariety of $Y$, and let $\widetilde{V} \subset \widetilde{Y}$ be the proper transform of $V$, i.e. the blow-up of $V$ along $V \cap X$. Then*

$$f^* [V] = \left[\widetilde{V}\right] + j_* \left\{c(\mathcal{Q}) \cap g^* s(V \cap X, V)\right\}_k \tag{1.14}$$

*in $A_k \widetilde{Y}$.*

**Corollary 1.2.31.** *[17, Corollary 6.7.1.] If $X = P$ is a point in $Y$, then*

$$f^* [V] = [V] + e_P V j_* [L],$$

*where $L$ is a $k-$dimensional linear subspace of $E = \mathbb{P}^{d-1}_K$, $K$ the residue field of $\mathcal{O}_{Y,P}$, and $e_P V$ is the multiplicity of $P$ on $V$.*

**Corollary 1.2.32.** *[17, Corollary 6.7.2.] If $dim V \cap X \leq k - d$, then*

$$f^* [V] = \left[\widetilde{V}\right].$$

The following result gives the multiplication rules in the Chow ring $A^\bullet(\widetilde{Y})$, but it does not provide an explicit presentation of the ring.

**Proposition 1.2.33.** *[17, Example 8.3.9.] Let*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{j} & \widetilde{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y \end{array}$$

*be a blow-up diagram, with $Y$, $X$, and therefore $\widetilde{Y}$, $\widetilde{X}$ non singular. The ring structure on $A^{\bullet}(\widetilde{Y})$ is determined by the following rules:*

*a* $f^{*}y \cdot f^{*}y^{'} = f^{*}(y \cdot y^{'})$.

*b* $j_{*}(\widetilde{x}) \cdot j_{*}(\widetilde{x}^{'}) = j_{*}(c_1(j^{*}\mathcal{O}_{\widetilde{Y}}(\widetilde{X})) \cdot \widetilde{x} \cdot \widetilde{x}^{'})$.

*c* $f^{*}(y) \cdot j_{*}(\widetilde{x}) = j_{*}((g^{*}i^{*}y) \cdot \widetilde{x})$.

Finally, the following theorem provides an explicit presentation of $A^{\bullet}(\widetilde{Y})$ under some restrictive assumptions.

**Theorem 1.2.34.** *[28, Appendix Theorem 1.] Suppose the map of bivariant rings*

$$i^{*} : A^{\bullet}(Y) \to A^{\bullet}(X)$$

*is surjective, then $A^{\bullet}(\widetilde{Y})$ is isomorphic to*

$$A^{\bullet}(Y)\,[T]\,/(P(T), (T \cdot Ker(i^{*}))),$$

*where $P(T) \in A^{\bullet}(Y)\,[T]$ is any polynomial whose constant term is $[X]$ and whose restriction to $A^{\bullet}(X)$is the Chern polynomial of the normal bundle $N_{X/Y}$ i.e.*

$$i^{*}(P(T)) = t^{d} + c_1(N_{X/Y})T^{d-1} + \cdots + c_{d-1}(N_{X/Y})T + c_d(N_{X/Y}),$$

*(where $d = codim(X, Y)$). This isomorphism is induced by*

$$f^{*} : A^{\bullet}(Y) \to A^{\bullet}(\widetilde{Y})$$

*and by sending $-T$ to the class of the exceptional divisor.*

# Chapter 2

# Sequences of blow-ups at smooth centers.

In this chapter, we define the basic objects of this research, that is, sequences of blow-ups, sequential morphisms and final divisors.

In the first section of this chapter, apart from defining the key concepts of sequences of blow-ups at smooth centers and sequential morphisms, we also generalize the usual proximity relations for higher dimensional centers. In the second section we give a short result about the normal bundle of the irreducible components of the exceptional divisor.The third section is devoted to the definition of final divisors for both sequences of blow-ups and sequential morphisms, as well as regular projective contractions, and study some properties of the former. Finally, in the last section of this chapter we define the $n-$ary multilinear intersection form on the abelian group of divisors with exceptional support and its associated multilinear form.

The main references for this chapter are [27] and [30].

**Remark 2.0.1.** *Given a variety $Z$ and a subvariety $V \subset Z$ of codimension $d$, from now on we will denote by $v$ to the class $[V] \in A^d(Z)$.*

## 2.1 Sequences of blow-ups and sequential morphisms.

Fix an algebraic closed field $k$. From now on, unless otherwise stated, a variety will mean a reduced projective scheme over $k$.

**Definition 2.1.1.** *A* ***sequence of blow-ups*** *over $K$ is defined as a sequence of blow-ups at smooth closed subvarieties $C_i$ of smooth $d-$dimensional varieties $Z_i$*

$$Z_s \xrightarrow{\pi_s} Z_{s-1} \xrightarrow{\pi_{s-1}} \cdots \xrightarrow{\pi_2} Z_1 \xrightarrow{\pi_1} Z_0,$$

*such that for $i \in \{0, 1, ..., s-1\}$:*

- *a if we denote by $C_{i+1}$ to the center of $\pi_{i+1}$, then $C_{i+1}$ is a smooth subvariety of $Z_i$,*
- *b $codim(C_{i+1}) \geq 2$,*
- *c if we denote by $E_j^j$ the exceptional divisor of $\pi_j$, and for $k > j$ we denote by $E_j^k$ the strict transform of $E_j^j$ in $Z_k$, then $C_{i+1}$ has normal crossings with $\{E_1^i, E_2^i, ..., E_i^i\}$.*

We denote by $\pi$ the composition $\pi_1 \circ \pi_2 \circ ... \circ \pi_{s-1} \circ \pi_s$.

**Definition 2.1.2.** *A morphism $\pi : Z_s \to Z_0$ which can be expressed, in at least one way, as a composition of blow-ups with the conditions in Definition 2.1.1 will be called a* ***sequential morphism*** *.*

**Remark 2.1.3.** *Given a sequence of blow-ups $(Z_s, ..., Z_0, \pi)$, we denote by $\pi_{s,i} : Z_s \to Z_i$ where $\pi_{s,i} = \pi_{i+1} \circ \pi_{i+2} \circ ... \circ \pi_{s-1} \circ \pi_s$.*

**Remark 2.1.4.** *We will refer to $Z_0$ and $Z_s$ as the* ***ground*** *and the* ***sky*** *of the sequential morphism $\pi : Z_s \to Z_0$ respectively. Moreover we will denote by $E_\beta$ the irreducible components over $k$ of the exceptional divisor $E$ of $\pi$, that is we have $E = \bigcup_\beta E_\beta$.*

The centers $C_i$, in general, can have any dimension. We extend the well-known notion of proximity for point blow-ups.

**Definition 2.1.5.** *Given a sequence of blow-ups $(Z_s, ..., Z_0, \pi)$ as in Definition 2.1.1, we say that $C_j$ is* ***proximate*** *(resp.* ***t−proximate****) to $C_i$, and write $C_j \to C_i$ (resp. $C_j \xrightarrow{t} C_i$) if $C_j \subset E_i^{j-1}$ (resp. $C_j \cap E_i^{j-1} \neq \emptyset$ but $C_j \not\subset E_i^{j-1}$).*

Note that, if $C_j$ is either proximate or $t-$proximate to $C_i$ then $j > i$.

**Remark 2.1.6.** *For $j > i$ we denote by $E_i^{j*}$ the total transform of $E_i^i$ by the morphism $\pi_{j,i} : Z_j \to Z_i$. By an abuse of notation $E_i^{i*} = E_i^i$. Note that by definition of the total transform and Theorem 1.2.30, we have*

$$E_i^{k*} = E_i^k + \sum_{j>i} p_{ij} E_j^{k*}$$

*where $p_{ij} = 1$ if $i < j \leq k$ and $C_j$ is proximate to $C_i$ and $p_{ij} = 0$ in any other case.*

**Definition 2.1.7.** *Given a sequential morphism $\pi : Z \to Z_0$ and two irreducible exceptional components $E_i, E_j \subset E$, then we will say that $E_j$ is proximate (resp. $t-$proximate) to $E_i$ if there exists a sequence of blow-ups $(Z_s, ..., Z_0, \pi)$ realizing the sequential morphism $\pi : Z \to Z_0$, such that $C_j$ is proximate (resp. $t-$proximate) to $C_i$.*

## 2.2 A brief note on the normal bundle of the intersection of two exceptional components

Within this short section, we introduce a technical lemma about the splitting of the normal bundle of the complete intersection of two irreducible components of the exceptional divisor, that will be widely used in the rest of this work.

**Lemma 2.2.1.** *Let $D$ and $F$ be two irreducible components of a simple normal crossing divisor $E$ that is regularly embedded in $X$. If we denote by $G = D \cap F$ then*

$$N_{F/X}|_G \cong N_{G/D}$$

*Proof.* Let $i_{G,D} : G \to D$ and $i_{D,X} : D \to X$ be regular embeddings. Then the composite $i_{D,X} \circ i_{G,D}$ is a regular embedding, and there is an exact sequence of vector bundles on $G$ (see [20, Proposition 19.1.5])

$$0 \to N_{G/D} \to N_{G/X} \to N_{D/X}|_G \to 0$$

Since $D$ and $F$ meet regularly in $X$, then (see [18, IV Proposition 3.6.]):

$$N_{G/X} \cong N_{D/X}|_G \oplus N_{F/X}|_G$$

So, we have the following exact sequences of vector bundles on $G$

$$0 \to N_{G/D} \to N_{D/X}|_G \oplus N_{F/X}|_G \to N_{D/X}|_G \to 0$$

Then it follows that $N_{G/D} \cong N_{F/X}|_G$. $\square$

## 2.3 Final divisors. Regular projective contractions.

We start this section by defining one of the key objects of this thesis, that is, final divisors for both sequences of blow-ups and sequential morphisms.

**Definition 2.3.1.** *Let $(Z_s, ..., Z_0, \pi)$ be a sequence of blow-ups as in Definition 2.1.1. The components of the exceptional divisor $E$ in $Z_s$ are $\{E_1, ..., E_s\}$. Assume that $E_i$ is an irreducible component. Set $E_i^i$ to be the image of $E_i$ in $Z_i$. We say that $E_i$ is **finalfinal** with respect to $(Z_s, ..., Z_0, \pi)$ if there exists an open set $U_i$ on $Z_i$ such that $E_i^i \subset U_i$, $V_i = \pi_{s,i}^{-1}(U_i) \subset Z_s$, and $\pi_{s,i}|_{V_i} : V_i \to U_i$ is an isomorphism (see Remark 2.1.3 for $\pi_{s,i}$).*

**Remark 2.3.2.** *Note that $E_i = E_i^*$ is a necessary condition for $E_i$ to be final but it is not a sufficient one, since even if $C_j \xrightarrow{t} C_i$, $E_i^{j*} = E_i^j$ although $E_i$ may not be final.*

**Definition 2.3.3.** *Let $\pi : Z_s \to Z_0$ be a sequential morphism. We say that an irreducible component $E_i$ of $E$ is **finalfinal** if there exists a sequence of blow-ups $(Z_s, ..., Z_0, \pi)$ associated to $\pi : Z_s \to Z_0$ such that $E_i$ is final with respect to this sequence.*

Now we will define a key tool for our study of final divisors, that of a regular projective contraction.

**Definition 2.3.4.** *Let $Z$ and $C$ be two varieties, and let $D$ be a proper closed subvariety of $Z$. Then we say that $D$ is contractable to $C$ within $Z$, if there exist a variety $W$, and a proper birational morphism $f : Z \to W$ such that*

*a $f(D) = C$, and*

*b $D$ is the closed subset of $Z$ which consists of the points where $f$ is not an isomorphism.*

*We call this triple $(Z, f, W)$ a **contraction** of $D$ to $C$, or simply a contraction. We shall say that $D$ is normally (resp. regularly, projectively) contractable to $C$ within $Z$ when moreover*

c *$W$ is a normal (resp. non-singular, projective) variety.*

*In this case we call this triple $(Z, f, W)$ a normal (resp. **regular**, **projective**) contraction of $D$ to $C$.*

The following results give some necessary and sufficient conditions for such a regular projective contraction to exists as well as prove its uniqueness.

**Theorem 2.3.5.** *[27, Theorem 3., Corollary 2.] Let $Z$ be an $n-$dimensional non-singular projective variety, $D$ a divisor on $Z$, and $C$ be an $r-$dimensional non-singular projective variety with $r < n-1$. Then there exists a regular projective contraction of $D$ to $C$ within $Z$ if and only if they satisfy the conditions*

a *$D$ is isomorphic to a projective bundle $P(N)$ for a vector bundle $N$ on $C$. We denote by $p$ the canonical projection of $E$ to $C$ and by $\mathcal{I}_D$ the ideal of $D$ in $\mathcal{O}_Z$,*

b *the normal bundle $N_{D/Z} = \frac{\mathcal{I}_D}{\mathcal{I}_D^2}^{\vee} \cong \mathcal{O}_D(-1)$ and*

c *there is a line bundle $\mathcal{L}^{'}$ on $Z$ generated by its global sections, whose restriction to $D$ is isomorphic to the inverse image by $p$ of an ample line bundle on $C$.*

*Moreover, $f : Z \to W$ is the blow-up of $W$ with center $C$.*

**Proposition 2.3.6.** *[30, Corollaire 2.] Let $f : Z \to W$ a surjective birational morphism, $S(f) \subset Z$ the closed subset of point where $f$ is not biregular, and $D$ an irreducible component of $S(f)$, such that $X$ is factorial over each point of $C = f(D)$. Let $f^{'} : Z \to W^{'}$ a birational morphism, such that $S(f^{'}) = D$ and each fiber of $f^{'}|_D$ is contained in a fiber of $f|_D$. We suppose furthermore that $W^{'}$ is normal over each point of $C^{'} = f^{'}(D)$. Then, it exists a canonical isomorphism $h : W^{'} \to W$ satisfying $h \circ f^{'} = f$.*

Now, we give some necessary algebraic conditions that an irreducible component $E_i$ of the exceptional divisor $E$ have to satisfy in order to be final.

**Proposition 2.3.7.** *Let $E_i$ be an irreducible component of $E$. If $E_i$ is final then it satisfies the three conditions of Theorem 2.3.5, that is*

- *a $E_i$ is isomorphic to a projective bundle $P(N)$ for a vector bundle $N$ on $C_i$. We denote by $p$ the canonical projection of $E_i$ to $C_i$ and by $\mathcal{I}_{E_i}$ the ideal of $E_i$ in $\mathcal{O}_Z$,*
- *b the normal bundle $N_{E_i/Z_s} = \frac{\mathcal{I}_{E_i}}{\mathcal{I}_{E_i}^2}^{\vee} \cong \mathcal{O}_{E_i}(-1)$ and*
- *c there is a line bundle $\mathcal{L}'$ on $Z$ generated by its global sections, whose restriction to $E_i$ is isomorphic to the inverse image by $p$ of an ample line bundle on $C_i$.*

*Proof.* If $E_i$ is final then $E_i$ is isomorphic to a projective bundle $P(N_{C_i/Z_{i-1}})$ over $C_i$. Moreover $N_{E_i/Z} \cong \mathcal{O}_{E_i}(-1)$ by Proposition 1.1.16, and condition c is satisfied by considering $\mathcal{L}' = \pi^*_{s,i-1}\mathcal{L}$, where $\mathcal{L}$ is an ample line bundle over $Z_{i-1}$. □

A natural question arises when dealing with final divisors: Given a sequential morphism $\pi : Z \to Z_0$ is it possible for two irreducible exceptional components $E_i$ and $E_j$ to be final with $E_i \cap E_j \neq \emptyset$? And in this case, which type of proximity relation could exist between them? Moreover, what is the geometric structure of $E_i \cap E_j$ when $E_i$ is final?

**Lemma 2.3.8.** *Let $(Z_s, ..., Z_0, \pi)$ be a sequence of blow-ups such and let $E_i$ be a final component with respect to this sequence. If $E_i \cap E_j \neq \emptyset$ and $E_i$ is proximate to $E_j$, then $E_i \cap E_j \subset E_i$ is isomorphic to a projective subbundle of $E_i$.*

*Proof.* We have that $E_i^i \cap E_j^i = P(N_{C_i/E_j^{i-1}})$, where $N_{C_i/E_j^{i-1}} \subset N_{C_i/Z_{i-1}}$ is a vector subbundle as $C_i$ has normal crossing with $E_j^{i-1}$. Now, the results follows since $E_i$ is final. □

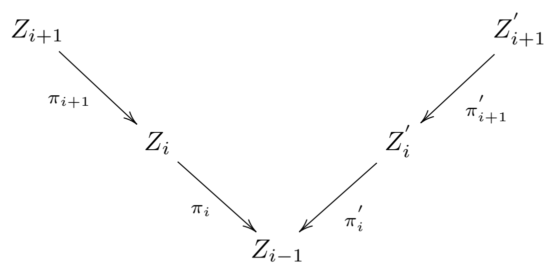
**Theorem 2.3.9.** *Let $(Z_s, ..., Z_0, \pi)$ be a sequence of blow-ups satisfying the following conditions:*

- *a $\pi_i : Z_i \to Z_{i-1}$ is the blow-up at the smooth subvariety $C_i$,*
- *b $D \subset C_i$ is a Cartier divisor and $\pi_{i+1} : Z_{i+1} \to Z_i$ is the blow-up at $\pi_i^{-1}(D)$,*

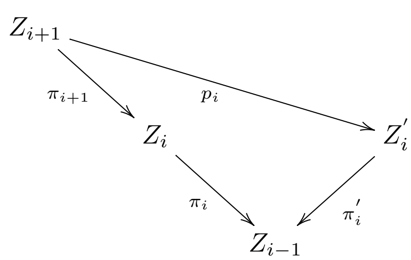
$c$ *and there exists an open subset* $U_{i-1} \subset Z_{i-1}$ *such that* $C_i \subset U_{i-1}$. *If* $U_{i+1} = (\pi_{i+1} \circ \pi_i)^{-1} U_{i-1}$, *then* $E_i^{i+1} \cup E_{i+1}^{i+1} \subset U_{i+1}$, $V_{i+1} = \pi_{s,i+1}^{-1}(U_{i+1}) \subset Z$, *and* $\pi_{s,i+1}|_{V_{i+1}} : V_{i+1} \to U_{i+1}$ *is an isomorphism.*

*Then, the irreducible components* $E_i, E_{i+1}$ *are both finals with respect to the sequential morphism* $\pi : Z_s \to Z_0$ *satisfying* $E_i \xrightarrow{t} E_{i+1}$ *and* $E_{i+1} \to E_i$.

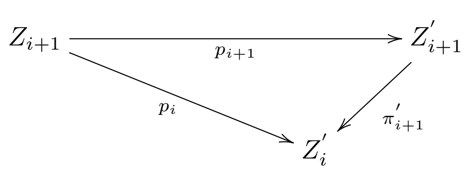
*Proof.* Let us consider the two following sequences of blow-ups:




where $\pi'_i$ denotes the blow-up of $Z_{i-1}$ with center $D$ and $\pi'_{i+1}$ denotes the blow-up of $Z'_i$ with center the strict transform of $C_i$, that is $\widetilde{C}_i$. By the universal property of blow-ups there exists a unique morphism $p_i : Z_{i+1} \to Z'_i$ such that the following diagram commutes




Note that as a consequence, $p_i|_{W_{i+1} \setminus E_i^{i+1} \cup E_{i+1}^{i+1}}$ is an isomorphism. Moreover, if we denote by $\mathcal{I}_{\widetilde{C}_i}$ to the ideal defining $\widetilde{C}_i$, since $C_i \cong \widetilde{C}_i$, $p_i^{-1}(\mathcal{I}_{\widetilde{C}_i})$ is an invertible sheaf. So by the universal property of blow-up there must exist a unique morphism $p_{i+1} : Z_{i+1} \to Z'_{i+1}$ such that

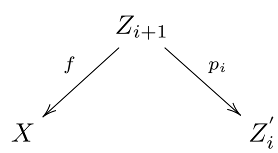



, where $p_{i+1}|_{W_{i+1} \setminus E_i^{i+1} \cup E_{i+1}^{i+1}}$ is an isomorphism.

Now, since $E_i^{i+1} \cong E_i^i$, then $E_i^{i+1}$ is isomorphic to a projective bundle over $C_i$, and consequently over $\widetilde{C}_i$, $p : E_i^{i+1} \to \widetilde{C}_i$. Moreover, by Proposition 1.1.19

$$\begin{aligned} N_{E_i^{i+1}/Z_{i+1}} &\cong \pi_{i+1}^* \mathcal{O}_{E_i^i}(-1) \otimes \mathcal{O}(-E_i^{i+1} \cap E_{i+1}^{i+1}), \\ &\cong \mathcal{O}_{P(N_{C_i/Z_{i-1}})}(-1) \otimes \pi_{i+1}^* \circ \pi_i^* \mathcal{O}(-D), \end{aligned}$$

and considering the vector bundle $N^{'} = N_{C_i/Z_{i-1}}) \otimes \mathcal{O}(-D)$, then $E_i^{i+1} \cong P(N^{'})$ and $N_{E_i^{i+1}/Z_{i+1}} \cong \mathcal{O}_{P(N')}(-1)$. Finally, let $\mathcal{L}$ an ample line bundle on $Z_{i-1}$, then $\pi_{i+1}^* \circ \pi_i^* \mathcal{L}$ will be generated by its global sections and its restriction to $E_i^{i+1}$ will be isomorphic to the inverse image by $\pi_i \circ \pi_{i+1}$ of an ample line bundle on $\widetilde{C}_i$. Therefore, by Theorem 2.3.5 there exists a regular projective contraction of $E_i^{i+1}$ to $\widetilde{C}_i$ within $Z_{i+1}$, $(Z_{i+1}, f, W)$, such that $f : Z_{i+1} \to W$ is the blow-up of $W$ with center $\widetilde{C}_i$. The restriction $f|_{Z_{i+1} \setminus E_i^{i+1}}$ will be an isomorphism and we have the following diagram




Both, $f$ and $p_i$ are birational morphisms, with $S(f) = S(p_i) = E_i^{i+1}$. Moreover, and due to the commutativity of diagram 2.3, a fiber of $p_i|_{E_i^{i+1}}$ is contained in a fiber of $f|_{E_i^{i+1}}$ so by Proposition 2.3.6 there exists a canonical isomorphism $h : W \to Z_i^{'}$ such that $h \circ f = p_i$. As a result, $Z_{i+1}$ must be isomorphic to $Z_{i+1}^{'}$.
Note that $E_{i+1}^{i+1}$ is proximate to $E_i^{i+1}$ with respect to the sequence $(Z_{i+1}, Z_i, Z_{i-1}, \pi)$ and $E_i^{i+1}$ is $t-$proximate to $E_{i+1}^{i+1}$ with respect to the sequence $(Z_{i+1}, Z_i^{'}, z_{i-1}, \pi^{'})$.

□

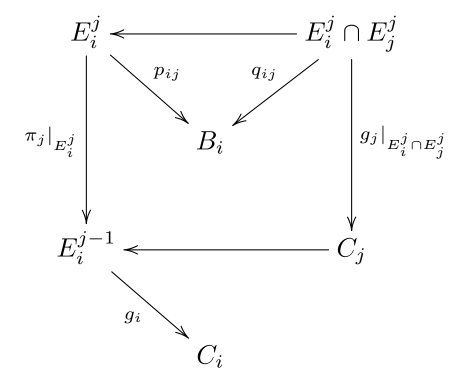
**Theorem 2.3.10.** *Let $E_i, E_j \subset Z_s$ be both final divisors for the sequential morphism $\pi : Z_s \to Z_0$. Then $E_i \cap E_j \neq \emptyset$ if and only if $E_i$ is proximate to $E_j$ and $E_j$ is $t-$proximate to $E_i$, or vice versa.*

*Proof.* Let us suppose that $E_i \cap E_j \neq \emptyset$. Then one of the following conditions must be satisfied:

A either $E_i \to E_j$ and $E_j \to E_i$,

B or $E_i \xrightarrow{t} E_j$ and $E_j \xrightarrow{t} E_i$, C or $E_i \to E_j$ and $E_j \xrightarrow{t} E_i$ (or vice versa).

In the case A, let us consider a sequence of blow-ups, associated to the sequential morphism $\pi : Z_s \to Z_0$, realizing $E_j$ as a final divisor. If we focus on the blow-up corresponding at the $j-$level, that is $\pi_j : Z_j \to Z_{j-1}$, and we restrict it to $E_i^{j-1}$, then we have the following diagram:




, (2.1)

where $E_i^j \cap E_j^j \subset E_i^j$ must be a projective subbundle of $E_i^j$, since $E_i$ is final too. Before going on we should distinguish between the two following cases:

A.i either $codim(C_j, E_i^{j-1}) = 1$, that is, $C_j$ is a divisor of $E_i^{j-1}$,

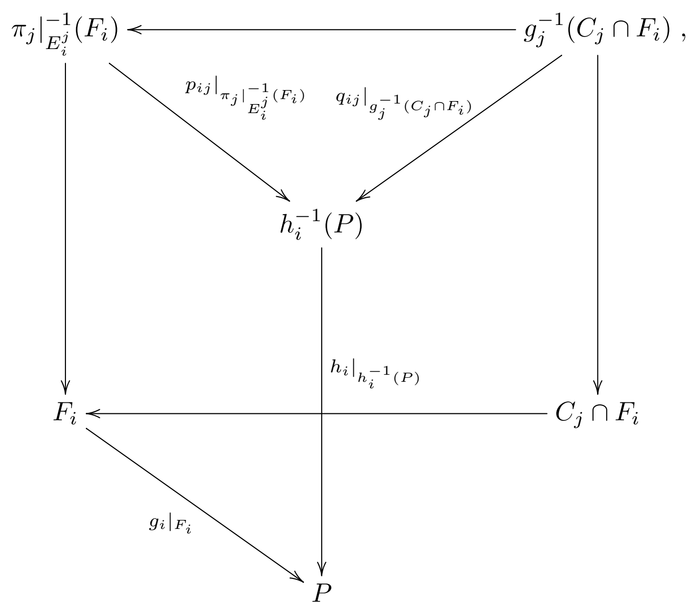
A.ii or $codim(C_j, E_i^{j-1}) > 1$.

In the case A.i, by Proposition 1.1.19 we have that

$$N_{E_i^j/Z_j} = \pi_j^*(N_{E_i^{j-1}/Z_{j-1}}) \otimes \mathcal{O}(-E_i^j \cap E_j^j), \tag{2.2}$$

so the necessary condition to be final $N_{E_i^j/Z_j} \cong \mathcal{O}_{E_i^j}(-1)$ (see Proposition 2.3.7) fails to be true.

In the case A.ii the morphism $\pi_j|_{E_i^j} : E_i^j \to E_i^{j-1}$ defines a divisorial contraction, and as a consequence of [21, Theorem 1.1.], $\pi_j|_{E_i^j} : E_i^j \to E_i^{j-1}$ is a Sarkisov link of type $I$, so there must exists a morphism $h_i : B_i \to C_i$, giving $B_i$ a projective bundle structure over $C_i$. Now, if we denote by $F_i$ to $g_i^{-1}(P)$, where $P \in C_i$ is a point, then we have the following diagram, obtained just by restriction of the previous one:




(2.3)

Since $F_i \cong \mathbb{P}^m$, where $r = codim(C_i, Z_{i-1})$, then by [31, Lemma 2.] the dimension $dim(h_i^{-1}(P))$ is at least $m - dim(C_j \cap F_i) - 1$. Now we will prove that in fact this must be an equality, that is,

$$dim(h_i^{-1}(P)) = m - dim(C_j \cap F_i) - 1. \tag{2.4}$$

Since $g_j^{-1}(C_j \cap F_i)$ has a projective bundle structure over $h_i^{-1}(P)$ then from Theorem 1.2.25 we know that

$$A^\bullet(g_j^{-1}(C_j \cap F_i)) \cong A^\bullet(h_i^{-1}(P))\,[\zeta]\,/(\zeta^{r+1} + c_1(V)\zeta^r + \cdots + c_{r+1}(V)), \tag{2.5}$$

where $V$ is a vector bundle such that $r + 1 = rank(V) \leq dim(C_j \cap F_i) + 1$. Moreover, if $Q \in C_j \cap F_i$ is a point, then we have that $codim(g_j^{-1}(Q), g_j^{-1}(C_j \cap F_i)) = codim(Q, C_j \cap F_i) = dim(C_j \cap F_i)$, so the class $\left[g_j^{-1}(Q)\right] \in A^1(g_j^{-1}(C_j \cap F_i))$ is expressed as

$$\left[g_j^{-1}(Q)\right] = a_0\zeta^{dim(C_j \cap F_i)} + a_1\zeta^{dim(C_j \cap F_i)-1}\cdot\alpha_1 + \ldots + a_{dim(C_j \cap F_i)-1}\zeta\cdot\alpha_{dim(C_j \cap F_i)-1} + \\ + a_{dim(C_j \cap F_i)}\alpha_{dim(C_j \cap F_i)} \text{ in } A^\bullet(g_j^{-1}(C_j \cap F_i)), \tag{2.6}$$

where $\alpha_i \in A^i(h_i^{-1}(P))$ for $i = 1, ..., dim(C_j \cap F_i)$. Now, let us consider a point $O \in q_{ij}|_{g_j^{-1}(C_j \cap F_i)}(g_j^{-1}(Q))$, where $q_{ij}|_{g_j^{-1}(C_j \cap F_i)} : g_j^{-1}(C_j \cap F_i) \to h_i^{-1}(P)$ defines the projective bundle structure. Then the following relation must hold:

$$q_{ij*}|_{g_j^{-1}(C_j \cap F_i)}(\left[q_{ij}|^*_{g_j^{-1}(C_j \cap F_i)}(O)\right] \cdot \left[g_j^{-1}(Q)\right]) = [O] \in A^\bullet(h_i^{-1}(P)), \tag{2.7}$$

but in order to satisfy $q_{ij*}|_{g_j^{-1}(C_j\cap F_i)}(\left[q_{ij}|^*_{g_j^{-1}(C_j\cap F_i)}(O)\right]\cdot [g_j^{-1}(Q)]) \neq 0$ then $a_0 \neq 0$ in equation 2.6, and now it follows from [15, Lemma 9.7.] that $r+1 = rank(V) = dim(C_j \cap F_i)+1$. We can conclude then that $dim(h_i^{-1}(P)) = m - dim(C_j\cap F_i) - 1$. As a consequence, by [31, Theorem 4.], we have that $C_j \cap F_i$ is a linear subspace in $F_i$. Moreover, by Proposition 1.1.17 the pull back by $p_{ij}|_{\pi_j|^{-1}_{E_i^j}(F_i)}$ of the hyperplane class $\xi_i \in A^1(F_i)$ satisfies (see Corollary 1.2.26):

$$p_{ij}|^*_{\pi_j|^{-1}_{E_i^j}(F_i)}(\xi_i) = \left[g_j^{-1}(C_j\cap F_i)\right] + f, \tag{2.8}$$

where $f \in A^1(\pi_j|^{-1}_{E_i^j}(F_i))$ denotes the class of a fiber $F \subset \pi_j|^{-1}_{E_i^j}(F_i)$. Now, by Proposition 1.1.19 we have that:

$$N_{E_i^j/Z_j} \cong \pi_j^*(N_{E_i^{j-1}/Z_{j-1}}) \otimes \mathcal{O}(-E_i^j \cap E - j^j),$$

so in particular, if we restrict to $F_i$:

$$\begin{aligned} N_{E_i^j/Z_j}|_{\pi_j|^{-1}_{E_i^j}(F_i)} &\cong \pi_j|^*_{\pi_j|^{-1}_{E_i^j}(F_i)}(N_{E_i^{j-1}/Z_{j-1}}) \otimes \mathcal{O}_{\pi_j|^{-1}_{E_i^j}(F_i)}(-g_j^{-1}(C_j\cap F_i)),\\ &\cong p_{ij}|^*_{\pi_j|^{-1}_{E_i^j}(F_i)}(\mathcal{O}_{F_i}(-1)) \otimes \mathcal{O}_{\pi_j|^{-1}_{E_i^j}(F_i)}(-g_j^{-1}(C_j\cap F_i)),\\ &\cong \mathcal{O}_{\pi_j|^{-1}_{E_i^j}(F_i)}(-g_j^{-1}(C_j\cap F_i)) \otimes \mathcal{L} \otimes \mathcal{O}_{\pi_j|^{-1}_{E_i^j}(F_i)}(-g_j^{-1}(C_j\cap F_i)), \end{aligned}$$

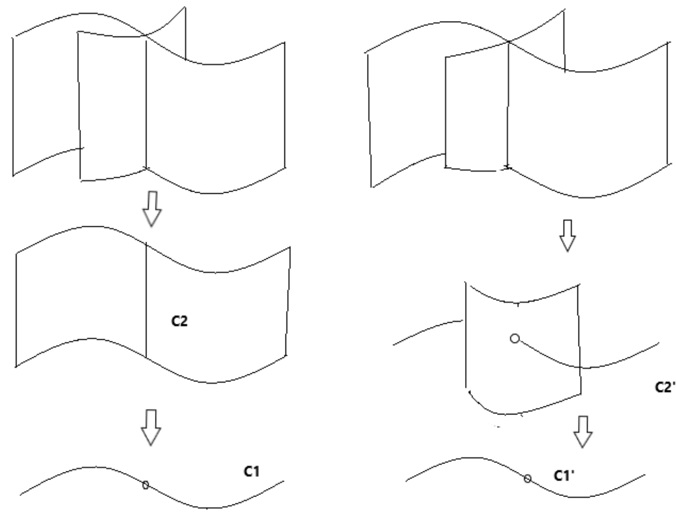
where $\mathcal{L}$ denotes a line bundle in $h_i^{-1}(P)$, so the necessary condition to be final $N_{E_i^j/Z_j} \cong \mathcal{O}(-1)$ (see Proposition 2.3.7) fails to be true.

Now, let us consider the case B. Since $E_i \xrightarrow{t} E_j$ and $E_j \xrightarrow{t} E_i$, then $E_i \cap E_j$ must be isomorphic to a fiber of both $E_i$ and $E_j$. Let us suppose that both are finals and let $(Z_0, ..., Z_s, \pi)$ be a sequence realizing $E_i$ as a final divisor. Then there must exists a regular projective contraction $f : Z_s \to X_{s-1}$ such that $f(E_i) = C_i$. However, if we consider the restriction $f|_{E_j}$ then it can not be a regular projective contraction any more since it contracts $E_i \cap E_j$ whereas $N_{E_i\cap E_j/E_j} \ncong \mathcal{O}_{E_i\cap E_j}(-1)$.

Finally, in the case C, that is $E_i \to E_j$ and $E_j \xrightarrow{t} E_i$ (or vice versa), if $C_j \cap E_i^{j-1}$ is a projective subbundle of rank one or equivalently $C_i' = \pi_j|^{-1}_{E_j^j}(D)$, where $D \in A^1(C_j')$ is a Cartier divisor, then it follows from Theorem 2.3.9 that both are finals.

□

Figure 2.1: Example of two blow up processes which lead to a same sequential morphism with two intersecting final divisors




**Corollary 2.3.11.** *Let $\pi_\alpha : Z_\alpha \to Z_{\alpha-1}$ be the blow-up of $Z_{\alpha-1}$ with center $C_\alpha$ and $\pi_{\alpha+1} : Z_{\alpha+1} \to Z_\alpha$ the blow-up of $Z_\alpha$ with center the image of a section $C_\alpha \to E_\alpha^\alpha$ of the projection $E_\alpha^\alpha \to C_\alpha$. Then $E_\alpha^{\alpha+1}$ is isomorphic to a projective bundle $P(N)$ over $P(N_{C_\alpha/Z_{\alpha-1}}/\mathcal{L})$, where $\mathcal{L}$ is the line bundle corresponding to $C_{\alpha+1}$, and if we denote by $p : P(N_{C_\alpha/Z_{\alpha-1}}/\mathcal{L}) \to C_i$, then $N \cong p^*\mathcal{L} \oplus \mathcal{O}_{P(N_{C_\alpha/Z_{\alpha-1}}/\mathcal{L})}(-1)$. Moreover, $N_{E_\alpha^{\alpha+1}/Z_{\alpha+1}} \cong \mathcal{O}_{P(N)}(-2) \otimes \mathcal{M}$, where $\mathcal{M}$ is the pull-back a line bundle defined over $P(N_{C_\alpha/Z_{\alpha-1}}/\mathcal{L})$.*

**Proposition 2.3.12.** *Let $(Z_s, ..., Z_0, \pi)$ be a sequence of blow-ups such and let $E_i$ be a final component with respect to this sequence. We denote by $e_i, e_j \in A^1(Z_s$ to the classes of $E_i$ and $E_j$ respectively. If $E_i \cap E_j \neq \emptyset$ then one of the following relations holds:*

*A* $e_j \cdot e_i = \varsigma_i$ *if* $dim(C_i) = 0$,

*B* $(e_j + e_i) \cdot e_i = \sum d_j$ *if* $C_i \to C_j$,*with* $dim(C_i) \geq 1$,

*C* $e_j \cdot e_i = \sum d_j$ *if* $C_i \xrightarrow{t} C_j$,

*with* $d_j = j_{i*}(g_i^*(f_j))$, *where* $f_j \in A^1(C_i)$ *denotes a Weil divisor of* $C_i$.

*Proof.* The case A follow directly from Lemma 2.3.8.

For the other cases, let us consider the blow-up $\pi_i : Z_i \to Z_{i-1}$. Then $e_j^{i-1} \in A^1(Z_{i-1})$, and by Proposition 1.2.33 we have

$$\pi_i^*(e_j^{i-1}) \cdot e_i^i = j_{i*}(g_i^*(i_{C_i}^*(e_j^{i-1}))).$$

Moreover, $\pi_i^*(e_j^{i-1}) = e_j^i + e_i^i$ if $C_i \to C_j$ and $\pi_i^*(e_j^{i-1}) = e_j^i$ otherwise. Finally, since $E_i$ is final, there must exist an open $U_i$ on $Z_i$ such that $E_i^i \subset U_i$, $V_i = \pi_{s,i}^{-1}(U_i) \subset Z_s$, and $\pi_{s,i}|_{V_i} : V_i \to U_i$ is an isomorphism, and the result follows. □

## 2.4 The $n-$ary intersection form

This section is devoted to the definition of the $n-$ary multilinear intersection form on the abelian groups of divisors with exceptional support and its associated multilinear form, as we will make an intensive use of them in order to establish the numerical characterization of final divisors and the combinatorial equivalence of sequences of blow-ups and sequential morphisms.

**Definition 2.4.1.** *Given a sequential morphism $\pi : Z_s \to Z_0$, we consider the* ***n−ary multilinear intersection*** *form*

$$\mathcal{I}_{Z_s,E} : \overbrace{\mathbb{E} \times \mathbb{E} \times \cdots \times \mathbb{E}}^{n} \to \mathbb{Z},$$

*defined by intersecting cycles in the sky $Z_s$ and taking degrees, that is*

$$\mathcal{I}_{Z_s,E}(E_{i_1}, E_{i_2}, ..., E_{i_n}) = deg(e_{i_1} \cdot e_{i_2} \cdot e_{i_3} \cdots e_{i_n}),$$

*where $e_{i_1} \cdot e_{i_2} \cdot e_{i_3} \cdots e_{i_n}$ is an intersection class of $0-$cycles in the abelian group $A_0(Z_s)$, and deg stands for the degree.*

**Remark 2.4.2.** *For the sake of simplicity we will denote $e_{i_1} \cdot e_{i_2} \cdots e_{i_n} = \mathcal{I}_{Z_s,E_{Z_s}}(E_{i_1}, E_{i_2}, ..., E_{i_n})$.*

**Definition 2.4.3.** *Given a sequential morphism $\pi : Z_s \to Z_0$ as in Definition 2.1.2, it induces a natural isomorphism $\mathbb{E} \cong \mathbb{Z}^s$, where the standard basis of $\mathbb{Z}^s$ is the image of the $\mathbb{Z}$-basis $\{E_i\}_{i=1}^s$. In this way, the multilinear form of intersection give rise to a* ***multilinear form***

$$\Phi_{Z_s,E} : \overbrace{\mathbb{Z}^s \times \cdots \times \mathbb{Z}^s}^{n} \to \mathbb{Z}$$

*We say that* $\Phi_{Z_s,E}$ *is the multilinear form associated to* $\pi$*. The permutation group* $\mathcal{S}_s$ *acts on the set of multilinear forms* $\mathbb{Z}^s \times \cdots \times \mathbb{Z}^s \to \mathbb{Z}$ *by interchanging the elements of the standard basis of* $\mathbb{Z}^s$*. It is clear that if we denote by* $\Psi_{Z_s,E}$ *to the orbit of* $\Phi_{Z_s,E}$*, then* $\Psi_{Z_s,E}$ *does not depend on the labeling of the elements of the basis* $\{E_i\}$*.*

# Chapter 3

# Sequences of point blow-ups over an algebraically closed field.

In this chapter we focus on the study of sequences of blow-ups as in Definition 2.1.1, that is

$$Z_s \xrightarrow{\pi_s} Z_{s-1} \xrightarrow{\pi_{s-1}} \cdots \xrightarrow{\pi_2} Z_1 \xrightarrow{\pi_1} Z_0,$$

where all the centers $C_{i+1}$ are points. In the first section we define the notion of algebraic and combinatorial equivalence for both sequences of points blow-ups and sequential morphisms. The second section is devoted to give a numerical characterization of final divisors in terms of the values of the $n-$ary intersection form of the abelian groups of divisors with exceptional support. In the next sections, we make use of this previous result in order to recover the sequences of point blow-ups from the associated sequential morphism modulo algebraic equivalence, and prove some relations between algebraic and combinatorial equivalence classes of sequences of point blow-ups and sequential morphisms. Finally, in the last section of this chapter, we give two explicit presentation of the Chow ring of the sky of a sequence of point blow-ups. The first one using the classes of the total transforms of the exceptional components as generators and the second one using the classes of the strict transforms ones. Furthermore, we prove that the skies of two sequences of point blow-ups of the same length have isomorphic Chow rings.

## 3.1 Algebraic and combinatorial equivalence of sequences of point blow-ups and the associated sequential morphisms

First of all we define our notions of equivalence (algebraic and combinatorial) for sequential morphisms (Definitions 3.1.1 and 3.1.2).

**Definition 3.1.1.** *We say that two* ***sequential morphisms*** $\pi : Z_s \to Z_0$ *and* $\pi^{'} : Z^{'}_{s^{'}} \to Z^{'}_0$*, with* $s = s^{'}$*, are* ***algebraically equivalent****, and we denote it by* $\pi \overset{alg}{\sim} \pi^{'}$ *, if and only if there exist isomorphisms* $a$ *and* $b$ *such that the following diagram is commutative*

$$\begin{array}{ccc} Z_s & \overset{b}{\longleftrightarrow} & Z^{'}_{s^{'}} \\ \downarrow \pi & & \downarrow \pi^{'} \\ Z_0 & \overset{a}{\longleftrightarrow} & Z^{'}_0 \end{array}$$

**Definition 3.1.2.** *Given two sequential morphisms* $\pi : Z_s \to Z_0$ *and* $\pi^{'} : Z^{'}_{s^{'}} \to Z^{'}_0$*, with* $s = s^{'}$*, we say that the associated multilinear forms* $\Phi_{Z_s,E}$ *and* $\Phi_{Z^{'}_{s^{'}},E^{'}}$ *are equivalent, and we denote it by* $\Phi_{Z_s,E} \sim \Phi_{Z^{'}_{s^{'}},E^{'}}$*, if there exists* $\tau \in \mathcal{S}_s$ *such that*

$$\tau(\Phi_{Z_s,E}) = \Phi_{Z^{'}_{s^{'}},E^{'}}.$$

*Moreover, the* ***sequential morphisms*** $\pi : Z_s \to Z_0$ *and* $\pi^{'} : Z^{'}_s \to Z^{'}_0$ *are said to be* ***combinatorially equivalent****, and we denote it by* $\pi \overset{comb}{\sim} \pi^{'}$*, when their associated multilinear maps* $\Phi_{Z_s,E}$ *and* $\Phi_{Z^{'}_{s^{'}},E^{'}}$ *are equivalent.*

**Remark 3.1.3.** *If* $\pi : Z_s \to Z_0$ *and* $\pi^{'} : Z^{'}_{s^{'}} \to Z^{'}_0$ *are algebraically equivalent, then* $b(E_i) = E^{'}_{\sigma(i)}$ *for some permutation* $\sigma \in S_s$*, so the multilinear intersection forms are equivalent as in Definition 3.1.2 . However, the converse is not true. For instance, for* $n = 2$ *we consider sequences of five point blow-ups, the first on a rational point of a smooth surface and the other at four different rational points of the exceptional divisor created by the blow-up of the original point. Then the* $5-$*multilinear form, up to a permutation of* $S_5$*, is independent on the choice of the four exceptional points; however, two choices with a different cross-ratio provide sequential morphism which are not algebraically equivalent.*

**Definition 3.1.4.** *Given a variety* $X$ *we will call a* ***brick blow-up*** *with ground* $X$ *to a sequential morphism obtained as a composition of point blow-ups with disjoint centers* $\sqcup_{j=1}^{l} C_j \subset X$*,* $X^{'} = X_l \to X_{l-1} \to ... \to X_1 \to X$*.*

**Definition 3.1.5.** *We say that two* ***sequences of point blow-ups****,* $(Z_s, ..., Z_0, \pi)$ *, and* $(Z^{'}_{s'}, ..., Z^{'}_0, \pi^{'})$*, are* ***algebraically equivalent****, and we denote it by* $(Z_s, ..., Z_0, \pi) \overset{alg}{\sim} (Z^{'}_{s'}, ..., Z^{'}_0, \pi^{'})$*, if and only if* $s = s^{'}$ *and there exist algebraic isomorphisms* $a, b = b_t, b_{t-1}, ..., b_1$*, with* $t \leq s$*, such that there are indexes* $r_1, ..., r_t = s \in \{1, ..., l\}$ *and* $r^{'}_1, ..., r^{'}_t = s^{'} \in \left\{1, ..., s^{'}\right\}$*, where* $Z_{r_i} \to Z_{r_i - 1} \to ... \to Z_{r_{i-1}}$ *(resp.* $Z^{'}_{r_i} \to Z^{'}_{r_i - 1} \to ... \to Z^{'}_{r_{i-1}}$*), with* $r_i > r_{i-1}$ *(resp* $r^{'}_i > r^{'}_{i-1}$*), is a brick blow-up* $\forall i = 1...t$ *as in Definition 3.1.4 and the diagram*

$$\begin{array}{ccccccccccc}
Z_s & \longrightarrow & Z_{r_{t-1}} & \longrightarrow & Z_{r_{t-2}} & \longrightarrow & \cdot & \longrightarrow & Z_{r_1} & \longrightarrow & Z_0 \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b_{t-1}} & & \downarrow{\scriptstyle b_{t-2}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle a} \\
Z_{s'} & \longrightarrow & Z^{'}_{r^{'}_{t-1}} & \longrightarrow & Z^{'}_{r^{'}_{t-2}} & \longrightarrow & \cdot & \longrightarrow & Z^{'}_{r^{'}_1} & \longrightarrow & Z^{'}_0
\end{array}$$

*is commutative.*

**Remark 3.1.6.** *If two sequences of point blow-ups* $(Z_s, ..., Z_0, \pi)$ *and* $(Z^{'}_s, ..., Z^{'}_0, \pi^{'})$ *are algebraically equivalent, then their associated sequential morphisms are also algebraically equivalent. Therefore, in particular, one has* $b(E_i) = E^{'}_{\sigma(i)}$ *where* $\sigma \in S_s$ *is a permutation. Moreover, for two different indexes* $i, j$*, one has that* $E_i$ *is proximate to* $E_j$ *if and only if* $E^{'}_{\sigma(i)}$ *is proximate to* $E^{'}_{\sigma(j)}$*.*

**Definition 3.1.7.** *We say that two* ***sequences of point blow ups****,* $(Z_s, ..., Z_0, \pi)$ *and* $(Z^{'}_s, ..., Z^{'}_0, \pi^{'})$*, with* $s = s^{'}$*, are* ***combinatorially equivalent****, and we denote it by* $(Z_s, ..., Z_0, \pi) \overset{comb}{\sim} (Z^{'}_{s'}, ..., Z^{'}_0, \pi^{'})$*, if and only there is a permutation* $\tau$ *in* $S_s$ *such that for every two different indexes* $i, j$ *one has*

*a* $E_i$ *is proximate to* $E_j$ *if and only if* $E^{'}_{\tau(i)}$ *is proximate to* $E^{'}_{\tau(j)}$*,*

## 3.2 Final divisors: Numerical characterization

**Lemma 3.2.1.** *In the case of sequences of point blow-ups, if two irreducible components* $E_i$ *and* $E_j$ *are both final, then* $E_i \cap E_j = \emptyset$*.*

*Proof.* Set $Pi \in Z_{i-1}$, $P_j \in Z_{j-1}$, to be the points such that $E_i$ maps to $P_i$ and $E_j$ maps to $P_j$. If $E_\beta$ is final, with $\beta \in \{i, j\}$, then $E_\beta \cong \mathbb{P}^{n-1}$ and $N_{E_\beta / Z_s} \cong \mathcal{O}_{E_\beta}(-1)$ by

Proposition 2.3.7.

Let us suppose that $E_i \cap E_j \neq \emptyset$. Then either $P_i$ is proximate to $P_j$ or $P_j$ is proximate to $P_i$. In the first case $E_j \ncong \mathbb{P}^{n-1}$ and $N_{E_j/Z_s} \ncong \mathcal{O}_{E_j}(-1)$ so $E_j$ is not final, whereas in the second case $E_i \ncong \mathbb{P}^{n-1}$ and $N_{E_i/Z_s} \ncong \mathcal{O}_{E_i}(-1)$ so $E_i$ is not final. Through any of them we get to a contradiction. □

The result above makes a huge difference with respect to the more general case of blow-ups at higher dimensional centers, where two final divisors may not have an empty intersection (see Theorem 2.3.9).

**Remark 3.2.2.** *Assume that we have a sequential morphism associated to a sequence of point blow-ups. If an irreducible component $E_\alpha$ of $E$ is final with respect to one representative of the sequences associated to this sequential morphism then it is final with respect to all. This fact drastically changes when more general centers are allowed (see Theorem 2.3.9).*

Before characterizing numerically final divisors, we need a numerical characterization of empty intersections $E_i \cap E_j = \emptyset$.

**Lemma 3.2.3.** *In case of sequences of point blow-ups $E_i \cap E_j = \emptyset$ if and only if $(e_i)^s \cdot (e_j)^r = 0$ for all $r \neq 0$ and $s \neq 0$ with $r + s = n$.*

*Proof.* If $E_i \cap E_j = \emptyset$ then $(e_i)^s \cdot (e_j)^r = 0$ follows directly.

In order to prove the necessary condition, we will prove that $E_i \cap E_j \neq \emptyset$ implies that $\exists r \neq 0, s \neq 0$ such that $(e_i)^s (e_j)^r \neq 0$; it is enough to prove it in the case of a sequence of point blow-ups of length $s = 3$ since the general result follows by induction.

First let $\pi_1 : Z_1 \to Z_0$ be the blow-up with center $P_1$. Now we blow-up $Z_1$ with center $P_2$ such that $P_2 \in E_1^1$, that is such that $P_2$ is proximate to $P_1$. If we denote by $D_{1,2}$ to $E_1^2 \cap E_2^2$, thus we have the following diagram where all the morphisms are regular embeddings

$$\begin{array}{ccccc} & & E_1^2 & & \\ & {\scriptstyle i_{D_{1,2},E_1^2}} \nearrow & & \searrow {\scriptstyle j_{E_1^2,Z_2}} & \\ D_{1,2} & & \xrightarrow{\quad i_{D_{1,2},Z_2} \quad} & & Z_2 \\ & {\scriptstyle i_{D_{1,2},E_2^2}} \searrow & & \nearrow {\scriptstyle j_{E_2^2,Z_2}} & \\ & & E_2^2 & & \end{array}$$

Then it follows by Proposition 1.2.20 that

$$e_1^2 \cdot e_2^2 \cdot e_2^2 = i_{D_{1,2},Z_2*}(c_1(N_{E_1^2/Z_2}|_{D_{1,2}}))$$
$$e_1^2 \cdot e_2^2 \cdot e_1^2 = i_{D_{1,2},Z_2*}(c_1(N_{E_2^2/Z_2}|_{D_{1,2}}))$$

Moreover, we have the following diagram

$$\begin{array}{ccc} D_{1,2} & \xrightarrow{i_{D_{1,2},E_1^2}} & E_1^2 \\ \Big\downarrow & & \Big\downarrow {\scriptstyle \pi_2|_{E_1^1}} \\ P_2 & \xrightarrow{i_{P_2,E_1^1}} & E_1^1 \end{array}$$

so by proposition 1.1.19

$$N_{E_1^2/Z_2} \cong \pi_2^*|_{E_1^1}(N_{E_1^1/Z_1}) \otimes \mathcal{O}(-D_{1,2})$$

Since $\mathcal{O}(D_{1,2})|_{D_{1,2}} \cong N_{D_{1,2}/E_1^2}$, then we have

$$N_{E_1^2/Z_2}|_{D_{1,2}} \cong L \otimes N^\vee_{D_{1,2}/E_1^2},$$

where $L$ denotes a trivial line bundle. Moreover, as a consequence of Lemma 2.2.1 we have $N_{D_{1,2}/E_1^2} \cong N_{E_2^2/Z_2}|_{D_{1,2}}$ so

$$c_1(N_{E_1^2/Z_2}|_{D_{1,2}}) \cong -c_1(N_{E_2^2/Z_2}|_{D_{1,2}})$$

By induction on $r$ and $s$ respectively it follows

$$e_1^2 \cdot (e_2^2)^r = i_{D_{1,2},Z_2*}((c_1(N_{E_2^2/Z_2}|_{D_{1,2}}))^{r-1})$$
$$(e_1^2)^s \cdot e_2^2 = (-1)^{s-1} i_{D_{1,2},Z_2*}((c_1(N_{E_2^2/Z_2}|_{D_{1,2}}))^{s-1})$$

Finally, as $N_{E_2^2/Z_2} \cong \mathcal{O}_{E_2^2}(-1)$ and $D_{1,2} \subset E_2^2$ is a projective sub-bundle, it follows that

$$(e_1^2)^s \cdot (e_2^2)^r = (-1)^{s-1} i_{D_{1,2},Z_2*}(c_1(\mathcal{O}_{D_{1,2}}(-1))^{r+s-2})$$

Note that $(e_1^2)^s \cdot (e_2^2)^r \neq 0$ and furthermore if we denote by $\Delta_{1,2}$ to $i_{D_{1,2},Z_2*}(c_1(\mathcal{O}_{D_{1,2}}(-1))^{r+s-2}))$ then $(e_1^2)^s \cdot (e_2^2)^r = (-1)^{s-1}\Delta_{1,2}$ if $r+s=n$.
Let $\pi_3 : Z_3 \to Z_2$ be the blow-up of $Z_2$ with center $P_3$, such that $P_3 \in E_1^2 \cap E_2^2$, that is $P_3$ is proximate to $P_1$ and to $P_2$. Then it follows that by theorem 1.2.30

$$(e_1^3)^s \cdot (e_2^3)^r = (\pi_3^*(e_1^2) - e_3^3)^s \cdot (\pi_3^*(e_2^2) - e_3^3)^r$$

and due to the Projection formula 1.6, then

$$\begin{aligned}(e_1^3)^s(e_2^3)^r &= (\pi_3^*(e_1^2))^s \cdot (\pi_3^*(e_2^2))^r + (-1)^{r+s}(j_{E_3^3,Z_3*}(c_1(\mathcal{O}_{E_3^3}(-1))^{r+s-1}) \quad (3.1)\\ &= (e_1^2)^s \cdot (e_2^2)^r + (-1)^{r+s}(j_{E_3^3,Z_3*}(c_1(\mathcal{O}_{E_3^3}(-1))^{r+s-1}) \quad (3.2)\end{aligned}$$

Since $(e_1^2)^s \cdot (e_2^2)^r \neq 0$ and furthermore that it is of the form $(e_1^2)^s \cdot (e_2^2)^r = (-1)^{s-1}\Delta_{1,2}$, then it must exist $r, s$ with $r \neq 0$ and $s \neq 0$ such that $(e_1^3)^s(e_2^3)^r \neq 0$.
For the more general case, let us suppose that $\{P_{\alpha_1}, P_{\alpha_2}, ..., P_{\alpha_k}\}$ are proximate to both $P_1$ and $P_2$. Then by iterating equation 3.1

$$(e_1^{\alpha_k})^s \cdot (e_2^{\alpha_k})^r = (e_1^2)^s \cdot (e_2^2)^r + (-1)^n \sum_{j=1}^{k} (j_{E_{\alpha_j}^{\alpha_j},Z_{\alpha_j}*}(c_1(\mathcal{O}_{E_{\alpha_j}^{\alpha_j}}(-1))^{n-1}),$$

so it must exist $r, s$ with $r \neq 0$ and $s \neq 0$ such that $(e_1^{\alpha_k})^s \cdot (e_2^{\alpha_k})^r \neq 0$. □

Now we are ready to characterize numerically when an irreducible component $E_i$ of the exceptional divisor $E$ is final.

**Proposition 3.2.4.** *$E_i$ is final if and only if*

$$(e_i)^n = (-1)^{n-1}$$

*Proof.* Firstly, let us suppose that $E_i$ is final. Then by Proposition 1.2.22 we have

$$(e_i)^n = j_{E_i,Z_s*}((c_1(N_{E_i/Z_s}))^{n-1})$$

but $N_{E_i/Z_s} \cong \mathcal{O}_{E_i}(-1)$, so if we denote by $\varsigma$ to $c_1(\mathcal{O}_{E_i}(-1))$, it follows that

$$(e_i)^n = j_{E_i,Z_s*}((-\varsigma)^{n-1})$$

so we can conclude the result $(e_i)^n = (-1)^{n-1}$.

Now, let us suppose that $E_i$ is not final. Then by Proposition 1.1.19 its normal bundle satisfies

$$N_{E_i/Z_s} = \pi^*_{n,i}|_{E^i_i}(N_{E^i_i/Z_i}) \otimes \bigotimes_{\alpha\to i} \pi^*_{n,\alpha}|_{E^\alpha_i}(\mathcal{O}(-E^\alpha_i \cap E^\alpha_\alpha)),$$

and by the Projection formula 1.6 we have

$$(e_i)^n = j_{E_i,Z_s*}((-\varsigma)^{n-1}) + \sum_{\alpha\to i} j_{E_i,Z_s*}((-E^\alpha_i \cap E^\alpha_\alpha)^{n-1})$$

Since $\sum_{\alpha\to i} j_{E_i,Z_s*}((-E^\alpha_i \cap E^\alpha_\alpha)^{n-1}) \neq 0$, we can conclude that $(e_i)^n \neq (-1)^{n-1}$. □

## 3.3 Recovering of the sequence of point blow-ups

Before continuing, we need to prove the following technical lemma that is crucial for the uniqueness of the regular projective contractions.

**Lemma 3.3.1.** *Let $X$ and $Y$ be two affine normal varieties such that $X = Spec(A)$ and $Y = Spec(B)$. Let $\pi : Z \to X$ and $\pi^{'} : Z^{'} \to Y$ be proper morphisms. If $Z$ is isomorphic to $Z^{'}$ then $A \cong B$.*

*Proof.* By [19, Theorem 3.2.1], since $\pi$ and $\pi^{'}$ are proper morphisms then $\pi_*(\mathcal{O}_Z)$ and $\pi^{'}_*(\mathcal{O}_{Z'})$ are a coherent sheaves on $X$ and $Y$ respectively. Since $X$ and $Y$ are both normal, then $\mathcal{O}_X \cong \pi_*(\mathcal{O}_Z)$ and $\mathcal{O}_Y \cong \pi^{'}_*(\mathcal{O}_{Z'})$, so

$$A \cong H^0(X, \mathcal{O}_X) \cong H^0(Z, \mathcal{O}_Z),$$
$$B \cong H^0(Y, \mathcal{O}_Y) \cong H^0(Z^{'}, \mathcal{O}_{Z'}).$$

Since $H^0(Z, \mathcal{O}_Z) \cong H^0(Z^{'}, \mathcal{O}_{Z'})$, then it follows that $A \cong B$. □

**Proposition 3.3.2.** *Let $(Z_s, ..., Z_0, \pi)$ be a sequence of point blow-ups (as in Definition 2.1.1) of length $s$ and let $E_i$ be an irreducible component of the exceptional divisor $E$ of $\pi$. If $E_i$ is final, then there exists a regular projective contraction $(Z_s, f_s, X_{s-1})$ of $E_i$ to a point such that $f_s(E)$ is a simple normal crossing divisor and $X_{s-1}$ is the sky of a sequence of point blow-ups with ground $Z_0$.*

*Proof.* Since $E_i$ is final there must exist an isomorphism between the two opens sets $U_i \subset Z_i$ and $V_i \subset Z_s$ via $\pi_{s,i}$. After shrinking $U_i$ if necessary, we may assume that

$U_i \setminus E_i^i$ is isomorphic via $\pi_i$ to an open set of $Z_{i-1} \setminus \{P_i\}$ where $P_i = \pi_i(E_i^i)$.

Note that $W_i = \pi_i(U_i)$ is an open set in $Z_{i-1}$. In fact $\pi_i|_{U_i}$ is the blow-up of $W_i$ at $P_i$.

$$\begin{array}{ccc} V_i & \xleftrightarrow{\pi_{s,i}|_{V_i}} & U_i \\ & & \Big\downarrow {\scriptstyle \pi_i|_{U_i}} \\ & & W_i \end{array}$$

Set $\phi = (\pi_i \circ \pi_{s,i})|_{V_i}$ the composition morphism from $V_i$ to $W_i$

$$\begin{array}{ccc} V_i & \xleftrightarrow{\pi_{s,i}|_{V_i}} & U_i \\ & \searrow^{\phi} & \Big\downarrow {\scriptstyle \pi_i|_{U_i}} \\ & & W_i \end{array}$$

where $\phi := \pi_i \circ \pi_{s,i}$.

Set $\overline{W}_i = Z \setminus E_i$. We construct $X_{s-1}$ by gluing $W_i$ and $\overline{W}_i$ along the open isomorphic sets $W_i \setminus \{P_i\} \subset W_i$ and $V_i \setminus E_i \subset \overline{W}_i$. Note that $W_i \setminus \{P_i\} \overset{\pi_i}{\cong} U_i \setminus E_i^i \overset{\pi_{s,i}}{\cong} V_i \setminus H_i$.

Now we define $f_s : Z \to X_{s-1}$, $f_s|_{\overline{W}_i} = Id_{\overline{W}_i}$, $f_s|_{V_i} = \phi$, which is well defined by the isomorphisms.

Finally, it is clear from the construction that if we denote by $D_{X_{s-1}}$ to the image $f_s(E)$, then $D_{X_{s-1}}$ is a simple normal crossing divisor.

*An alternative construction of the contraction.*

Since $E_i$ is final, then $E_i \cong \mathbb{P}^{n-1}$, where $P_i = \pi_{s,i}(E_i)$, and moreover by Proposition 1.1.16 its normal bundle $N_{E_i/Z_s} \cong \mathcal{O}_{E_i}(-1)$. Let $F$ be a very ample line bundle on $Z_s$. Then $F \otimes \mathcal{O}_{E_i} = L_2 \otimes \mathcal{O}_{E_i}(u)$. If we consider the line bundle $L := F \otimes \mathcal{O}(E_i)^{\otimes u}$, then by [27, Corollary 2.] there exists a regular projective contraction $(Z_s, \varphi, X_{s-1}^{'})$ of $E_i$ to a closed point, that we will denote by $P_i^{'}$, with $\varphi|_{E_i} = P_i^{'}$, such that $\varphi$ is defined by the complete linear system $|L|$. To see that $D_{X_{s-1}^{'}} := \varphi(E)$ is still a simple normal crossing divisor we prove that the contraction is unique up to isomorphism. By [27, Theorem 3] $\varphi : Z_s \to X_{s-1}^{'}$ is the blowing up of $X_{s-1}^{'}$ at a point $P_i^{'}$. Let $Y_i$ be an affine open neighborhood of $P_i$ in $X_{s-1}$. If we denote by $Y_i^{'} := \varphi(f_s^{-1}(Y_i))$, then is $Y_i^{'}$ is an affine neighborhood of $P_i^{'}$ in $X_{s-1}^{'}$ and there exist two proper morphisms

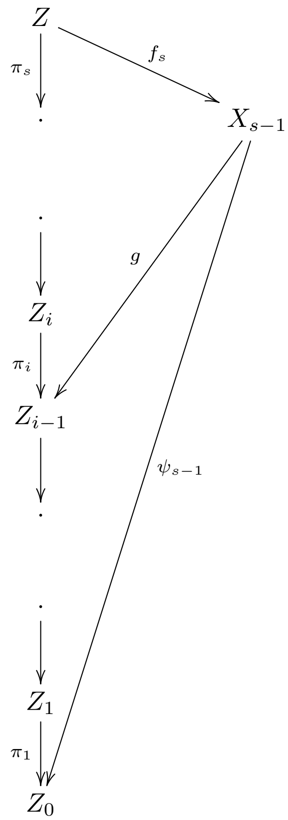
$$\begin{array}{ccc} & f_s^{-1}(Y_i) & \\ {\scriptstyle f_s|_{f_s^{-1}(Y_i)}} \swarrow & & \searrow {\scriptstyle \varphi|_{f_s^{-1}(Y_i)}} \\ Y_i & & Y_i^{'} \end{array}$$

By Lemma 3.3.1 this implies that $Y_i \cong Y_i^{'}$, so its then clear that $D_{X_{s-1}^{'}}$ is a simple normal crossing divisor.

So we have proved that there exists a regular projective contraction $(Z_s, f_s, X_{s-1})$ of $E_i$ to a point $P_i \in X_{s-1}$.

$$\begin{array}{ccc} Z_s & \xrightarrow{f_s} & X_{s-1} \\ \downarrow {\scriptstyle \pi_s} & & \\ \cdot & & \\ \cdot & & \\ \downarrow & & \\ Z_i & & \\ \downarrow {\scriptstyle \pi_i} & & \\ Z_{i-1} & & \\ \downarrow & & \\ \cdot & & \\ \cdot & & \\ \downarrow & & \\ Z_1 & & \\ \downarrow {\scriptstyle \pi_1} & & \\ Z_0 & & \end{array}$$

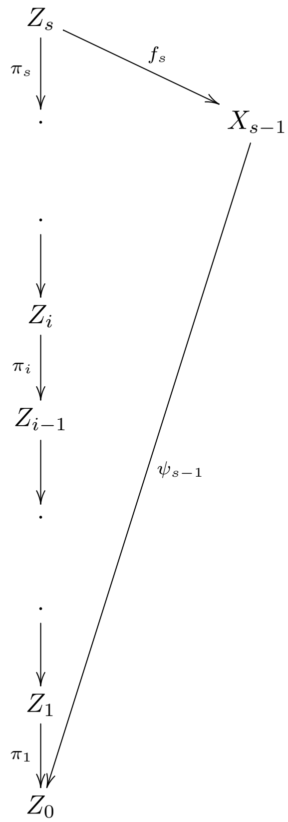
Following the notations of Definition 2.3.1, let $W_i = \pi_i(U_i)$. Then by Definition 2.3.1 $f_s|_{Z_s \setminus V_i} : Z_s \setminus V_i \to X_{s-1} \setminus f_s(V_i)$ is an isomorphism. Now we define $g : X_{s-1} \to Z_{i-1}$ as follows: $g|_{\overline{W}_i} = \pi_{s,i-1}|_{\overline{W}_i}$ and $g|_{W_i} = Id_{W_i}$. By our construction of $X_{s-1}$ $g$ is well defined, and by the definition $g : X_{s-1} \to Z_{i-1}$ is a sequence of point blow-ups.
Hence the composition $X_{s-1} \to Z_{i-1} \to Z_0$ is a sequence of point blow-ups.




□

**Theorem 3.3.3.** *Let $\pi : Z_s \to Z_0$ be a sequential morphism. Given the $n-$ary multilinear intersection form we can recover all the sequences of point blow-ups that are associated to sequential morphisms in the same algebraic equivalence class of $\pi : Z_s \to Z_0$.*

*Proof.* We will prove this result first by contracting one irreducible component of the exceptional divisor $\widetilde{E}$ each time.
Since the set formed by final divisors is not empty, let us suppose that $E_i$ is final, then by Proposition 3.3.2 there exists a regular projective contraction $(Z_s, f_s, X_{s-1})$ of $E_i$ to a point such that $X_{s-1}$ is the sky of a sequence of point blow-ups with ground $Z_0$ .




The next step in our proof refers to how to obtain the intersection form in $X_{s-1}$ associated to the simple normal crossing divisor $D_{X_{s-1}}$.

If we denote by $D_{X_{s-1},\alpha}$ to $f_s(E_\alpha)$, then by the Projection formula 1.6

$$d_{X_{s-1},i_1} \cdot d_{X_{s-1},i_2} \cdots d_{X_{s-1},i_n} = f_s^*(d_{X_{s-1},i_1}) \cdot f_s^*(d_{X_{s-1},i_2}) \cdots f_s^*(d_{X_{s-1},i_n}),$$

Applying the result of Theorem 1.2.30 then

$$d_{X_{s-1},i_1} \cdot d_{X_{s-1},i_2} \cdots d_{X_{s-1},i_n} = (e_{i_1} + \delta_{i_1,i} e_i) \cdot (e_{i_2} + \delta_{i_2,i} e_i) \cdots (e_{i_n} + \delta_{i_n,i} e_i), \quad (3.3)$$

where $\delta_{i_j,i} = 1$ if $E_i \cap E_{i_j} \neq \emptyset$ (see numerical characterization in lemma 3.2.3) and $\delta_{i_j,i} = 0$ otherwise.

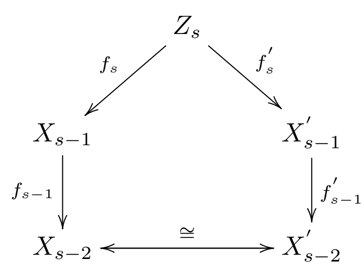
**Remark 3.3.4.** *It follows then that by iterating the above process, that is by contracting a final divisor at each step, we will obtain a sequence of point blow-ups of length s. The obtained sequence depends on the choice of final components. Below we will prove that all the sequential morphisms associated to the sequences constructed in this way are algebraically equivalent.*

## 3.4 Relations between algebraic and combinatorial equivalence classes of sequences of point blow-ups and sequential morphisms

**Proposition 3.4.1.** *Any of the sequences obtained as in 3.3.4, that is, by decomposing a regular projective contraction from a fixed sky $Z_s$ and a fixed simple normal crossing divisor $E$, are associated to sequential morphisms in the same algebraic equivalence class (see Definition 3.1.1).*

Before proving this, we need the following lemma

**Lemma 3.4.2.** *Given a fixed sky $Z_s$ and a fixed simple normal crossing divisor $E$, let us suppose that $E_i$ and $E_j$ are both finals. Then there is an isomorphism $X_{s-2} \cong X^{'}_{s-2}$ making the following diagram commutative*




*where $f_s$ is the contraction of $E_i$ and $f_{s-1}$ is the contraction of $D_{X_{s-1},j}$, whereas $f^{'}_s$ is the contraction of $E_j$ and $f^{'}_{s-1}$ is the contraction of $D_{X^{'}_{s-1},i}$.*

*Proof.* To begin with, if we denote by $O_{i,s-2} = f_{s-1} \circ f_s(E_i)$, $O_{j,s-2} = f_{s-1}(D_{X_{s-1},j})$, $O^{'}_{j,s-2} = f^{'}_{s-1} \circ f^{'}_s(E_j)$ and $O^{'}_{i,s-2} = f^{'}_{s-1}(D_{X^{'}_{s-1},i})$, then it follows that

$$X_{s-2} \setminus \{O_{i,s-2}, O_{j,s-2}\} \cong Z_s \setminus E_i \cup E_j \cong X^{'}_{s-2} \setminus \left\{O^{'}_{i,s-2}, O^{'}_{j,s-2})\right\}$$

Let $W_j$ be an open affine open neighborhood of $O_{j,s-2}$. If we denote by $V_j$ to the inverse image $f_s^{-1} \circ f_{s-1}^{-1}(W_j)$, then the image $W^{'}_j = f^{'}_s \circ f^{'}_{s-1}(V_j)$ will be an affine open neighborhood of $O^{'}_{j,s-2}$. Then since $f_{s-1} \circ f_s|_{V_j}$ and $f^{'}_{s-1} \circ f^{'}_s|_{V_j}$ are both proper morphisms, it follows by lemma 3.3.1 $W_j \cong W^{'}_j$.

If we denote by $W_i$ to an open affine neighborhood of $O_{i,s-2}$ and $W^{'}_i = f^{'}_{s-1} \circ f^{'}_s(V_i)$, where

$V_i$ is the inverse image $f_s^{-1} \circ f_{s-1}^{-1}(W_i)$, then in a similar way we can prove that $W_i \cong W_i^{'}$, so it follows $X_{s-2} \cong X_{s-2}^{'}$ since all isomorphisms are given by global sections. □

Consequently, we have the following corollary, which means that proposition 3.4.1 holds for length 2.

**Corollary 3.4.3.** *If $Z_s$ is the sky of a sequence of point blow-ups of length 2, then any of the two sequences of point blow-ups obtained following the procedure in 3.3.4 are associated to sequential morphisms in the same algebraic equivalence class .*

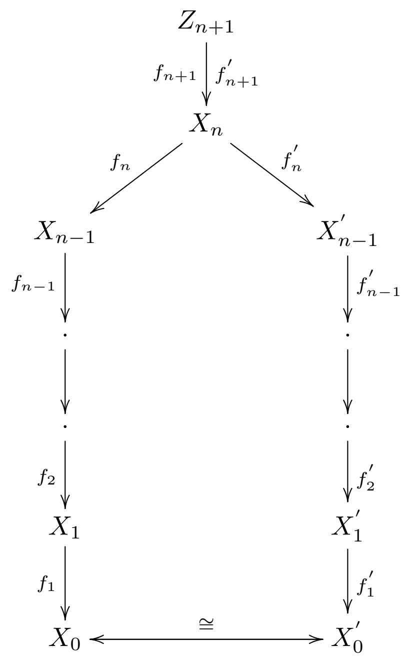
In order to prove proposition 3.4.1 we need the following definition.

**Definition 3.4.4.** *We say that two sequences of point blow-ups obtained as in remark 3.3.4, that is through the composition of regular projective contractions from a fixed sky $Z_s$ and a fixed simple normal crossing divisor $E$,*

$$\begin{array}{ccccccccccccccc}
Z_s & \xrightarrow{f_s} & X_{s-1} & \xrightarrow{f_{s-1}} & X_{s-2} & \xrightarrow{f_{s-2}} & . & \longrightarrow & . & \longrightarrow & . & \longrightarrow & X_2 & \xrightarrow{f_2} & X_1 & \xrightarrow{f_1} & X_0 \\
\updownarrow & & \updownarrow & & & & & & & & & & & & & & \updownarrow \\
Z_s & \xrightarrow{f_s^{'}} & X_{s-1}^{'} & \xrightarrow{f_{s-1}^{'}} & X_{s-2}^{'} & \xrightarrow{f_{s-2}^{'}} & . & \longrightarrow & . & \longrightarrow & . & \longrightarrow & X_2^{'} & \xrightarrow{f_2^{'}} & X_1^{'} & \xrightarrow{f_1^{'}} & X_0^{'}
\end{array}$$

*have the same end if at least the first contraction is common to both. i.e. one has $f_s = f_s^{'}$.*

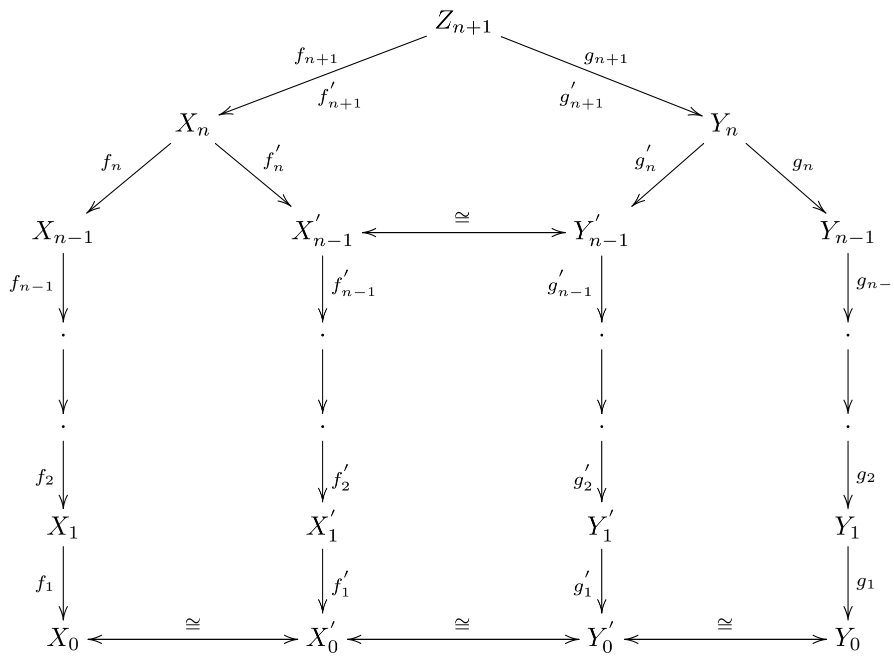
*Proof of Proposition 3.4.1.* Let us suppose then that $Z_{n+1}$ is the sky of a sequence of $n+1$ point blow ups and that proposition 3.4.1 is true for sequences of length lower or equal than $n$. If two sequences obtained as above $\rho := f_1 \circ f_2 \circ ... \circ f_n \circ f_{n+1}$ and $\rho^{'} := f_1^{'} \circ f_2^{'} \circ ... \circ f_n^{'} \circ f_{n+1}^{'}$ have the same end, then it is clear that both are associated to algebraically marked sequential morphism in the same algebraic equivalence class. It is a direct consequence of the fact that by hypothesis the assertion is true for sequences of length lower or equal than $n$.



If two sequences $\rho := f_1 \circ f_2 \circ ... \circ f_n \circ f_{n+1}$ and $\sigma := g_1 \circ g_2 \circ ... \circ g_n \circ g_{n+1}$ have not the same end, then let us suppose that $f_{n+1}$ and $g_{n+1}$ correspond to the contraction of $E_i$ and $E_j$ respectively. Consider all the sequences that belong to the tree contracting $E_j$ first, there must exist a sequence $\rho^{'} := f_1^{'} \circ f_2^{'} \circ ... \circ f_n^{'} \circ f_{n+1}^{'}$ contracting $E_{X_n,i}$ secondly. Analogously, if we consider all sequences contracting $E_i$ first, there must exist a sequence $\sigma^{'} := g_1^{'} \circ g_2^{'} \circ ... \circ g_n^{'} \circ g_{n+1}^{'}$ contracting $E_{Y_n,j}$ secondly.

By corollary 3.4.3 the sequences $f_n^{'} \circ f_{n+1}^{'}$ and $g_n^{'} \circ g_{n+1}^{'}$ of length 2 are associated to sequential morphism in the same algebraic equivalence class, so it just remain to proof that $f_1^{'} \circ f_2^{'} \circ ... \circ f_{n-2}^{'} \circ f_{n-1}^{'}$ belong to the same equivalence class that $g_1^{'} \circ g_2^{'} \circ ... \circ g_{n-2}^{'} \circ g_{n-1}^{'}$.

But this equivalence follows directly from the hypothesis, so $\rho^{'} \sim \sigma^{'}$.




Now since $\rho \overset{\text{alg}}{\sim} \rho^{'}$ and $\sigma \overset{\text{alg}}{\sim} \sigma^{'}$, then $\rho \overset{\text{alg}}{\sim} \sigma$. □

With this we conclude also the proof of Theorem 3.3.3. □

**Theorem 3.4.5.** *Two sequences of point blow-ups $(Z_s, ..., Z_0, \pi)$ and $(Z^{'}_s, ..., Z^{'}_0, \pi^{'})$, with $s = s^{'}$, are combinatorially equivalent as in Definition 3.1.7 if and only if their associated sequential morphisms $\pi : Z_s \to Z_0$ and $\pi^{'} : Z^{'}_s \to Z^{'}_0$ are combinatorially equivalent as in Definition 3.1.2, and both statements are true if and only if the associated multilinear maps $\Phi_{Z,E}$ and $\Phi_{Z^{'},E^{'}}$ are equivalent too as in Definition 3.1.2*

First we will prove that if two sequential morphisms $\pi : Z_s \to Z_0$ and $\pi^{'} : Z^{'}_s \to Z^{'}_0$ are combinatorially equivalent then the associated sequences of points blow-ups are combinatorially equivalent too. To begin with, we need a numerical characterization of proximity.

**Lemma 3.4.6.** *Let $(Z_s, ..., Z_0, \pi)_{comb}$ be a sequence of point blow-ups. Then $P_i \to P_j$ if and only if* *a* $\exists \alpha \in \{2, 3, ..., s-1, s\}$ *such that* $D_{X_\alpha,i} \cap D_{X_\alpha,j} \neq \emptyset$ *(see numerical characterization of lemma 3.2.3).*

*b* $(d_{X_\alpha,i})^n = (-1)^{n-1}$

*where* $Z_s = \rightarrow X_{s-1} \rightarrow \cdots \rightarrow X_\alpha \rightarrow \cdots \rightarrow X_0 = Z_0$ *is any sequence of contractions obtained as in remark 3.3.4.*

*Proof.* If $P_i$ is proximate to $P_j$ then $E_i^i \cap E_j^i \neq \emptyset$, con condition $i$. holds. Moreover, if $P_i \in Z_r$, then $E_i^i$ is final for the sequence $\pi_{r+1} \circ \pi_r \circ \cdots \circ \pi_1$, for some $r \geq i$ and we have condition $ii$.
Conversely, if $D_{X_\alpha,i}$ is final for the sequence $\pi_{r+1} \circ \pi_r \circ \cdots \circ \pi_1$ for some $r \geq i$ then by proposition 3.3.2 there exist a regular projective contraction $f_\alpha : X_\alpha \rightarrow X_{\alpha-1}$ of $E_{D_\alpha,i}$ such that $f_\alpha(D_{X_\alpha,i}) = O_{i,\alpha-1} \subset D_{X_{\alpha-1},j}$. □

*Proof of Theorem 3.4.5.* Assume that the sequential morphisms $\pi : Z_s \rightarrow Z_0$ and $\pi^{'} : Z^{'}_{s^{'}} \rightarrow Z^{'}_0$ are combinatorially equivalent. If $E_i$ is final, then there exists $\tau \in S_s$ such that

a $E^{'}_{\tau(i)}$ is final,

b $E_i \cap E_\beta \neq \emptyset$ if and only if $E^{'}_{\tau(i)} \cap E^{'}_{\tau(\beta)} \neq \emptyset$,

c $e_{\beta_1} \cdot e_{\beta_2} \cdots e_{\beta_n} = e^{'}_{\tau(\beta_1)} \cdot e^{'}_{\tau(\beta_2)} \cdots e^{'}_{\tau(\beta_n)}$

Furthermore, by Theorem 1.2.30

$$d_{X_{s-1},\beta_1} \cdot d_{X_{s-1},\beta_2} \cdots d_{X_{s-1},\beta_n} = (e_{\beta_1} + \delta_{\beta_1,i} e_i) \cdot (e_{\beta_2} + \delta_{\beta_2,i} e_i) \cdots (e_{\beta_d} + \delta_{\beta_d,i} e_i),$$

so it follows then that there exists $\widetilde{\tau} \in S_{s-1}$ such that

$$d_{X_{s-1},\beta_1} \cdot d_{X_{s-1},\beta_2} \cdots d_{X_{s-1},\beta_n} = d^{'}_{X^{'}_{m-1},\widetilde{\tau}(\beta_1)} \cdot d^{'}_{X^{'}_{m-1},\widetilde{\tau}(\beta_2)} \cdots d^{'}_{X^{'}_{m-1},\widetilde{\tau}(\beta_n)}$$

Consequently we have that $\Phi_{X_{s-1},D_{X_{s-1}}} \sim \Phi_{X^{'}_{s^{'}-1},D_{X^{'}_{s^{'}-1}}}$. Furthermore, by iterating the above process, then $\Phi_{X_\alpha,D_{X_\alpha}} \sim \Phi_{X^{'}_\alpha,D_{X^{'}_\alpha}}$ for $\alpha = 1, .., s-2$. So as a consequence of Lemma 3.4.6 any two sequential morphisms combinatorially equivalent preserve the proximity relations.

Conversely assume now that two sequences of point blow-ups with $s = s^{'}$ are combinatorially equivalent. We want to prove that their associated sequential morphisms $\pi : Z_s \to Z_0$ and $\pi^{'} : Z^{'}_{s'} \to Z^{'}_0$ are combinatorially equivalent. First, there exists $\sigma \in S_s$ such that by applying iteratively Theorem 1.2.30 we get

$$e_i = e^*_i - \sum_{\beta \to i} e^*_\beta$$

$$e^{'}_{\sigma(i)} = e^{'*}_{\sigma(i)} - \sum_{\sigma(\beta) \to \sigma(i)} e^{'*}_{\sigma(\beta)}$$

Moreover, as a consequence of the Projection formula 1.6

$$e^*_{\beta_1} \cdot e^*_{\beta_2} \cdots e^*_{\beta_n} \neq 0 \text{ if and only if } \beta_1 = \beta_2 = ... = \beta_n$$

and if $E_i$ is final then $E_i = E^*_i$, so it follows that there exists $\tau \in S_s$ such that

$$(e^*_i)^n = (e^{'*}_{\tau(i)})^n \ \ \forall i = 1, .., s$$

Finally, by the Theorem 1.2.30

$$e_{\beta_1} \cdot e_{\beta_2} \cdots e_{\beta_n} = (e^*_{\beta_1} - \sum_{\delta \to \beta_1} e^*_\delta) \cdot (e^*_{\beta_2} - \sum_{\delta \to \beta_2} e^*_\delta) \cdots (e^*_{\beta_n} - \sum_{\delta \to \beta_n} e^*_\delta)$$

so we have

$$e_{\beta_1} \cdot e_{\beta_2} \cdots e_{\beta_n} = e^{'}_{\tau(\beta_1)} \cdot e^{'}_{\tau(\beta_2)} \cdots e^{'}_{\tau(\beta_n)}$$

□

**Theorem 3.4.7.** *Given two sequential morphisms $\pi : Z_s \to Z_0$ and $\pi^{'} : Z^{'}_{s'} \to Z^{'}_0$ , then they are algebraically equivalent as in Definition 3.1.1 if and only if there are sequences of point blow-ups $(Z_s, ..., Z_0, \pi)$ and $(Z^{'}_{s'}, ..., Z^{'}_0, \pi^{'})$ associated to $\pi : Z_s \to Z_0$ and $\pi^{'} : Z^{'}_{s'} \to Z^{'}_0$ respectively such that they are algebraically equivalent as in Definition 3.1.5.*

*Proof.* If two sequences of point blow-ups are algebraically equivalent, then it follows directly by Definition 3.1.5 that the associated sequential morphisms are algebraically equivalent too.

Now we will prove that if two sequential morphism $\pi : Z_s \to Z_0$ and $\pi^{'} : Z^{'}_{s'} \to Z^{'}_0$ are algebraically equivalent, then there exist sequences of point blow-ups associated to them that are algebraically equivalent too. By theorem 3.3.3 given a certain sky $Z_s$ associated to a sequential morphism $\pi : Z_s \to Z_0$ , all the sequences of point blow-ups obtained by regular projective contractions are associated to sequential morphisms in the same algebraic equivalence class. Since $\pi : Z_s \to Z_0$ and $\pi^{'} : Z_s^{'} \to Z_0^{'}$ are algebraically equivalent, then there exist an isomorphism $b : Z_s \to Z^{'}_{s'}$. By applying proposition 3.3.2 and proposition 3.4.1 we conclude the result. □

## 3.5 The Chow ring of the sky $Z_s$

Note that Proposition 1.2.33 does not give a presentation of $A^\bullet(Z_{\alpha+1})$ as a $A^\bullet(Z_\alpha)-$algebra, but only states the rules of multiplication.
If we could find generators of $A^\bullet(E^{\alpha+1}_{\alpha+1})$ as a $\mathbb{Z}-$algebra, $\{\gamma_1, ..., \gamma_r\} \in A^\bullet(E^{\alpha+1}_{\alpha+1})$, then

$$A^\bullet(Z_{\alpha+1}) \cong A^\bullet(Z_\alpha)\left[j_{\alpha+1*}(\gamma_1), ..., j_{\alpha+1*}(\gamma_r)\right]$$

would be a $A^\bullet(Z_\alpha)-$algebra of finite type. One would like to have a presentation

$$A^\bullet(Z_{\alpha+1}) \cong A^\bullet(Z_\alpha)\left[w_1, ..., w_r\right]/\mathcal{J}$$

by sending $w_i$ to $j_{\alpha+1*}(\gamma_i)$, with an explicit description of the ideal $\mathcal{J}$. The ideal $\mathcal{J}$will be computed in Theorems 3.5.3 and 3.5.6. We will restrict ourselves to the case of sequences of point blow-ups, that is $C_\alpha = P_\alpha$, with the ground variety $Z_0 \cong \mathbb{P}^n$. By [15, Theorem 2.1.], $A^\bullet(Z_0) \cong \mathbb{Z}\left[u\right]/(u^{n+1})$, by sending $u$ to $h$, where $h \in A^1(Z_0)$ is the rational equivalence class of any hyperplane $[H]$ in $\mathbb{P}^n$, and $\forall\alpha$ $A^\bullet(E^\alpha_\alpha) \cong \mathbb{Z}\left[w\right]/(w^n)$ by sending $w$ to $\varsigma_\alpha$, with $\varsigma_\alpha \in A^1(E^\alpha_\alpha)$ is the rational class of any hyperplane.

**Lemma 3.5.1.** *The Chow ring of the sky $A^\bullet(Z_s)$ is generated by $\left\{h^{s*}, \{e_i^{s*}\}_{i=1}^s\right\}$ as a $\mathbb{Z}-$algebra.*

*Proof.* This follows by induction on $\alpha$. It is clear that $A^\bullet(Z_0)$ is generated by $\{h\}$. Let us suppose that $A^\bullet(Z_\alpha)$ is generated by $\left\{h^{\alpha*}, \{e_i^{\alpha*}\}_{i=1}^\alpha\right\}$. Now by Proposition 1.2.33 and due to the fact that $E^{\alpha+1}_{\alpha+1} \cong \mathbb{P}^{n-1}$, that is $A^\bullet(E^{\alpha+1}_{\alpha+1}) \cong \mathbb{Z}\left[t\right]/(t^n)$, by sending $t$ to $\varsigma_{\alpha+1}$, with $\varsigma_{\alpha+1}$ the rational equivalence class of any hyperplane in $\mathbb{P}^{n-1}$, and $e^{\alpha+1*}_{\alpha+1} \cdot e^{\alpha+1*}_{\alpha+1} = -j_{\alpha+1*}(\varsigma_{\alpha+1})$ by equation (4) then $A^\bullet(Z_{\alpha+1})$ is generated by $\left\{h^{\alpha+1*}, \{e_i^{\alpha+1*}\}_{i=1}^{\alpha+1}\right\}$ as a $\mathbb{Z}-$algebra. □

**Remark 3.5.2.** *It makes sense then to define the augmented free $\mathbb{Z}-$modules with basis $\left\{H^{k*}, \{E_i^{k*}\}_{i=1}^k\right\}$ and $\left\{H^{k*}, \{E_i^{k}\}_{i=1}^k\right\}$ and the augmented change of basis matrix $B_k^*$*

$$B_k^* = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & \vdots & \vdots \\ 0 & -p_{12} & 1 & \ddots & \vdots & \vdots \\ \vdots & -p_{13} & -p_{23} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 & \vdots \\ 0 & -p_{1k} & -p_{2k} & \cdots & -p_{k-1k} & 1 \end{pmatrix} \tag{3.4}$$

*and its inverse $B_k^{*-1}$.*

**Theorem 3.5.3.** *The Chow ring of the sky $A^\bullet(Z_s)$, when $Z_0 \cong \mathbb{P}^n$, is isomorphic to*

$$A^\bullet(Z_s) \cong \mathbb{Z}\left[x_0, x_1, ..., x_s\right]/(\{x_i \cdot x_j\}_{\substack{i,j=0 \\ i\neq j}}^s, \{(-1)^n (x_i)^n + (x_0)^n\}_{i=1}^s), \tag{3.5}$$

*by sending $x_0$ to the class $h^{s*}$ and $x_i$ to the class $e_i^{s*}$ for $i = 1, ..., s$.*

*Proof.* By lemma 3.5.1 there exist a exists a surjective morphism

$$\phi : \mathbb{Z}\left[x_0, x_1, ..., x_s\right] \to A^\bullet(Z_s),$$

such that $\phi(x_0) = h^{s*}$ and $\phi(x_i) = e_i^{s*}$ for $i = 1, ...s$. Firstly we will prove that $\mathcal{J} := \left\langle \{x_i \cdot x_j\}_{\substack{i,j=0 \\ i\neq j}}^s, \{(-1)^n (x_i)^n + (x_0)^n\}_{i=1}^s \right\rangle \subset Ker(\phi)$. To begin with, let us express the classes of the basis $\{E_i^{\alpha+1*}\}_{i=1}^{\alpha+1}$ in terms of the classes of the basis $\{E_i^{\alpha+1}\}_{i=1}^{\alpha+1}$, that is, since

$$e_i^{\alpha*} = e_i^\alpha + \sum_{j=i+1}^{\alpha} b_{j,i} e_j^\alpha,$$

then

$$e_i^{\alpha+1*} = e_i^{\alpha+1} + \sum_{j=i+1}^{\alpha} b_{j,i} e_j^{\alpha+1} + (\sum_{j=i}^{\alpha} p_{j\alpha+1} b_{j,i}) e_{\alpha+1}^{\alpha+1}$$

where $b_{j,i}$ denotes the coefficients of the augmented change of basis matrix $B_\alpha^{*-1}$.
If we denote by $\varsigma_{\alpha+1} \in A^1(E_{\alpha+1}^{\alpha+1})$ the class of any hyperplane in $E_{\alpha+1}^{\alpha+1}$ then we have the following intersection products

$$\begin{cases} e_{\alpha+1}^{\alpha+1} \cdot e_{\alpha+1}^{\alpha+1} = -j_{\alpha+1*}(\varsigma_{\alpha+1}) & \quad (3.6) \\ e_j^{\alpha+1} \cdot e_{\alpha+1}^{\alpha+1} = j_{\alpha+1*}(\varsigma_{\alpha+1}) \qquad \text{if } P_{\alpha+1} \to P_j & \quad (3.7) \\ e_j^{\alpha+1} \cdot e_{\alpha+1}^{\alpha+1} = 0 \qquad \text{otherwise} & \quad (3.8) \end{cases}$$

where Equation 3.6 follows from Proposition 1.2.33 and Equation 3.7 is a direct consequence of [17, Corollary 6.7.1], that is $\pi^*_{\alpha+1}(e^{\alpha}_j) = e^{\alpha+1}_j + e^{\alpha+1}_{\alpha+1}$, and Proposition 1.2.33 and Equation 3.6. So the following intersection product is 0

$$(e^{\alpha+1}_j + p_{j\alpha+1} e^{\alpha+1}_{\alpha+1}) \cdot e^{\alpha+1}_{\alpha+1} = 0, \tag{3.9}$$

and we can conclude that

$$e^{\alpha+1*}_i \cdot e^{\alpha+1*}_{\alpha+1} = (e^{\alpha+1}_i + \sum_{j=i+1}^{\alpha} b_{j,i} e^{\alpha+1}_j + (\sum_{j=i}^{\alpha} p_{j\alpha+1} b_{j,i}) e^{\alpha+1}_{\alpha+1}) \cdot e^{\alpha+1}_{\alpha+1} = 0.$$

On the other hand $h^{\alpha+1*} \cdot e^{\alpha+1}_{\alpha+1} = 0$ is a consequence of the moving lemma (see [17, 11.4 Moving lemma]). If we make the pull back through $\pi^*_{s,\alpha+1}$ for all $\alpha$, then it follows that $\left\langle \{x_i \cdot x_j\}^s_{\substack{i,j=0 \\ i \neq j}} \right\rangle \subset ker(\phi)$. By [17, Example 16.1.11], $A_0(Z_0)$ is a birational invariant, that is $A_0(Z_i) \cong \mathbb{Z}(h^{i*})^n$ for $i = 1, ..., s$, so since $(e^{\alpha+1}_{\alpha+1})^n = (-1)^{n-1} j_{\alpha+1*}(\varsigma^n_{\alpha+1})$ then $(e^{\alpha+1}_{\alpha+1})^n = (-1)^{n-1}(h^{\alpha+1*})^n$, and by making the pull back through $\pi^*_{s,\alpha+1}$ we conclude that $\langle \{(-1)^n (x_i)^n + (x_0)^n\}^s_{i=1} \rangle \subset Ker(\phi)$.

Now we will prove that $Ker(\phi) \subset \mathcal{J}$. Note that $\phi : \mathbb{Z}[x_0, x_1, ..., x_s] \to A^\bullet(Z_s)$ is homogenous, so $ker(\phi)$ is an homogenous ideal, and $\mathcal{J}$ is an homogenous ideal too by construction. Let us suppose that $P[x] \in Ker(\phi)/\mathcal{J}$ with $deg(P) = \eta$. Then $2 \leq \eta \leq n$, since $\left\{x^{n+1}_i\right\}^s_{i=0} \in \mathcal{J}$, and $P[x]$ must be of the form $P[x] = \sum^s_{i=0} a_i x^\eta_i mod(\mathcal{J})$, since $\{x_i \cdot x_j\}^s_{\substack{i,j=0 \\ i \neq j}} \in \mathcal{J}$. Now if $\eta < n$, then $x^{n-\eta}_i P[x]$ will be also in $Ker(\phi)$, then $\phi(x^{n-\eta}_i P[x]) = a_i (e^{s*}_i)^n = 0$, since $(e^{s*}_i)^n \neq 0$ then $a_i = 0$ for $i = 0, 1, ..., s$. If $\eta = n$, since $\{(-1)^n (x_i)^n + (x_0)^n\}^s_{i=1} \in Ker(\phi)$ then it follows that $a_0 + (-1)^{n+1} \sum^s_{i=1} a_i = 0$, so $P[x] = 0 mod(\mathcal{J})$. □

**Remark 3.5.4.** *Note that* $\langle x_0, x_1, ..., x_s \rangle Ker(\phi) = \left\langle \{x_i x_j x_k\}^s_{\substack{i,j,k=0 \\ i\neq j \\ j \neq k}}, \left\{x^{n+1}_i\right\}^s_{i=0} \right\rangle$, *so we have that*

$Ker(\phi)/\langle x_0, x_1, ..., x_s \rangle Ker(\phi)$ *is a free* $\mathbb{Z}-$*module of finite rank* $\binom{n+1}{2} + n$. *Any set of generators of the ideal* $Ker(\phi)$ *is a set of generators of* $Ker(\phi)/\langle x_0, x_1, ..., x_s \rangle Ker(\phi)$ *as* $\mathbb{Z}-$*module, so*

$\left\{ \{x_i \cdot x_j\}^s_{\substack{i,j=0 \\ i \neq j}}, \{(-1)^n (x_i)^n + (x_0)^n\}^s_{i=1} \right\}$ *is a minimal set of generators for* $Ker(\phi)$.

**Corollary 3.5.5.** *Given two sequences of point blow-ups* $(Z_0, ..., Z_s, \pi)$ *and* $(Z'_0, ..., Z'_{s'}, \pi')$, *if* $s = s'$ *then* $A^\bullet(Z_s) \cong A^\bullet(Z'_{s'})$.

*Proof.* It follows directly from Equation (3.5) in Theorem 3.5.3. □

We can use $\left\{h^{s*}, \{e_i^s\}_{i=1}^s\right\}$ as generators of the Chow ring $A^\bullet(Z_s)$ as $\mathbb{Z}-$algebra instead.

**Theorem 3.5.6.** *A presentation of $A^\bullet(Z_s)$, when $Z_0 \cong \mathbb{P}^n$, using $\left\{h^{s+}, \{e_i^s\}_{i=1}^s\right\}$ as generators is the following one:*

$$A^\bullet(Z_s) \cong \frac{\mathbb{Z}\left[y_0, y_1, ..., y_s\right]}{\mathcal{A}}, \tag{3.10}$$

*where*

$$\mathcal{A} = ((\{y_0 \cdot y_i\}_{i=1}^s, \left\{(y_i + \sum_{k=i+1}^{s} b_{k,i} y_k) \cdot (y_j + \sum_{l=j+1}^{s} b_{l,j} y_l)\right\}_{\substack{i,j=1\\ i \neq j}}^s, \\ \left\{(y_i)^n + ((-1)^n + \#\left\{j\right\}_{j \to i})(y_0)^n\right\}_{i=1}^s)),$$

*by sending $y_0$ to $h^{s*}$ and $y_i$ to $e_i^s$ for $i = 1, ..., s$.*

*Proof.* In this case there exists a surjective morphism

$$\phi^{'} : \mathbb{Z}\left[y_0, y_1, ..., y_s\right] \to A^\bullet(Z_s)$$

with $\phi^{'}(y_0) = h^{s*}$ and $\phi^{'}(y_i) = e_i^s$ for $i = 1, ..., s$. Moreover we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{Z}\left[x_0, ..., x_s\right] & & \\ \uparrow \rho & \searrow^{\phi} & \\ \mathbb{Z}\left[y_0, ..., y_s\right] & \xrightarrow{\phi^{'}} & A^\bullet(Z_s) \end{array}$$

where $\rho : \mathbb{Z}\left[y_0, ..., y_s\right] \to \mathbb{Z}\left[x_0, ..., x_s\right]$ is the isomorphism induced by the augmentated change of basis matrix $B_s^*$, that is $\rho(y_0) = x_0$ and $\rho(y_i) = x_i - \sum_{j=i+1}^s p_{ij} x_j$. Now, by considering the following images through $\rho$:

$$\begin{aligned}
\rho((y_i)^n + ((-1)^n + \#\left\{j\right\}_{j \to i})(y_0)^n) &= (x_i - \sum_{k=i+1}^{s} p_{ik} x_k)^n + ((-1)^n + \#\left\{j\right\}_{j \to i})(x_0)^n \\
&= (x_i)^n + (-1)^n \sum_{k=i+1}^{s} p_{ik}(x_k)^n + ((-1)^n + \#\left\{j\right\}_{j \to i})(x_0)^n + \\
&\sum_{\substack{n_i + n_{i+1} + ... + n_s = n\\ n_i, ..., n_s \neq n}} (-1)^{n-n_i} \binom{n}{n_i, n_{i+1} ..., n_s} \prod_{\beta=i}^{s} (p_{i\beta} x_\beta)^{n_\beta} \\
&= (-1)^n((-1)^n(x_i)^n + (x_0)^n) + \sum_{k=i+1}^{s} p_{ik}((-1)^n(x_k)^n + (x_0)^n) + \\
&\sum_{\substack{n_i + n_{i+1} + ... + n_s = n\\ n_i, ..., n_s \neq n}} (-1)^{n-n_i} \binom{n}{n_i, n_{i+1} ..., n_s} \prod_{\beta=i}^{s} (p_{i\beta} x_\beta)^{n_\beta};
\end{aligned}$$

$$\rho(y_0 \cdot y_i) = x_0 \cdot (x_i - \sum_{k=i+1}^{s} p_{ik} x_k)$$
$$= x_0 \cdot x_i - \sum_{k=i+1}^{s} p_{i,k} x_0 \cdot x_k;$$

$$\rho((y_i + \sum_{k=i+1}^{s} b_{k,i} y_k) \cdot (y_j + \sum_{l=j+1}^{s} b_{l,j} y_l)) = x_i \cdot x_j;$$

we can conclude that $\mathcal{A} \subset Ker(\phi^{'})$. The inclusion $Ker(\phi^{'}) \subset \mathcal{A}$ is straightforward by Remark 3.5.4. □

The next examples illustrate some interesting consequences of Corollary 3.5.5. In particular, the first one shows how the presentation of the Chow ring of the sky of a sequence of point blow-ups in terms of the total transforms of the exceptional components fails to detect the proximity configuration of the sequence.

**Example 3.5.7.** *Let us consider all possible proximity configurations for a sequence of point blow-ups of length* 4 *verifying that at least* $P_{i+1} \rightarrow P_i$*, that is*

*a* $P_1$*,* $P_2 \rightarrow P_1$*,* $P_3 \rightarrow P_2$ *and* $P_4 \rightarrow P_3$*,*

*b* $P_1$*,* $P_2 \rightarrow P_1$*,* $P_3 \rightarrow \{P_1, P_2\}$ *and* $P_4 \rightarrow P_3$*,*

*c* $P_1$*,* $P_2 \rightarrow P_1$*,* $P_3 \rightarrow \{P_1, P_2\}$ *and* $P_4 \rightarrow \{P_2, P_3\}$*,*

*d* $P_1$*,* $P_2 \rightarrow P_1$*,* $P_3 \rightarrow \{P_1, P_2\}$ *and* $P_4 \rightarrow \{P_1, P_3\}$*,*

*e* $P_1$*,* $P_2 \rightarrow P_1$*,* $P_3 \rightarrow \{P_1, P_2\}$ *and* $P_4 \rightarrow \{P_1, P_2, P_3\}$*,*

*f* $P_1$*,* $P_2 \rightarrow P_1$*,* $P_3 \rightarrow P_2$ *and* $P_4 \rightarrow \{P_2, P_3\}$

*We can compute a presentation of the Chow ring of the skies of these* 6 *proximity configurations using both the total transforms of the exceptional components and the strict ones as generators. Firstly, we give the presentations in terms of the strict transforms:*

*a* $A^{\bullet}(Z_4) \cong \mathbb{Z}[h^*, e_1, e_2, e_3, e_4] / \mathcal{A}_1$ *where*

*(a)* $\mathcal{A}_1 = (\{h^* \cdot e_i\}_{i=1}^4, (e_1 + e_2) \cdot (e_2 + e_3), e_1 \cdot e_3, (e_2 + e_3) \cdot (e_3 + e_4), e_1 \cdot e_4, e_2 \cdot e_4, (e_3 + e_4) \cdot e_4, (e_1)^n, (e_2)^n, (e_3)^n, (-1)(e_4)^n + (h^*)^n)$ *if* $n$ *is odd,*

(b) $\mathcal{A}_1 = (\{h^* \cdot e_i\}_{i=1}^4, (e_1+e_2)\cdot(e_2+e_3), e_1\cdot e_3, (e_2+e_3)\cdot(e_3+e_4), e_1\cdot e_4, e_2\cdot e_4, (e_3+e_4)\cdot e_4, (e_1)^n+2(h^*)^n, (e_2)^n+2(h)^n, (e_3)^n+2(h^*)^n, (e_4)^n+(h^*)^n)$ *if n is even,*

b $A^\bullet(Z_4) \cong \mathbb{Z}[h^*, e_1, e_2, e_3, e_4]/\mathcal{A}_2$ *where*

(a) $\mathcal{A}_2 = (\{h^* \cdot e_i\}_{i=1}^4, (e_1+e_2)\cdot(e_2+e_3), (e_1+e_3)\cdot(e_3+e_4), (e_2+e_3)\cdot(e_3+e_4), e_1\cdot e_4, e_2\cdot e_4, (e_3+e_4)\cdot e_4, (e_1)^n+(h^*)^n, (e_2)^n, (e_3)^n, (-1)(e_4)^n +(h^*)^n)$ *if n is odd,*

(b) $\mathcal{A}_2 = (\{h^* \cdot e_i\}_{i=1}^4, (e_1+e_2)\cdot(e_2+e_3), (e_1+e_3)\cdot(e_3+e_4), (e_2+e_3)\cdot(e_3+e_4), e_1\cdot e_4, e_2\cdot e_4, (e_3+e_4)\cdot e_4, (e_1)^n+3(h^*)^n, (e_2)^n+2(h^*)^n, (e_3)^n+2(h^*)^n, (e_4)^n+(h^*)^n)$ *if n is even,*

c $A^\bullet(Z_4) \cong \mathbb{Z}[h^*, e_1, e_2, e_3, e_4]/\mathcal{A}_3$ *where*

(a) $\mathcal{A}_3 = (\{h^* \cdot e_i\}_{i=1}^4, (e_1+e_2)\cdot(e_2+e_3+2e_4), (e_1+e_3)\cdot(e_3+e_4), (e_2+e_3)\cdot(e_3+e_4), e_1\cdot e_4, (e_2+e_4)\cdot e_4, (e_3+e_4)\cdot e_4, (e_1)^n+(h^*)^n, (e_2)^n+(h^*)^n, (e_3)^n, (-1)(e_4)^n+(h^*)^n)$ *if n is odd,*

(b) $\mathcal{A}_3 = (\{h^* \cdot e_i\}_{i=1}^4, (e_1+e_2)\cdot(e_2+e_3+2e_4), (e_1+e_3)\cdot(e_3+e_4), (e_2+e_3)\cdot(e_3+e_4), e_1\cdot e_4, (e_2+e_4)\cdot e_4, (e_3+e_4)\cdot e_4, (e_1)^n+3(h^*)^n, (e_2)^n+3(h^*)^n, (e_3)^n+2(h^*)^n, (e_4)^n+(h^*)^n)$ *if n is even,*

d $A^\bullet(Z_4) \cong \mathbb{Z}[h^*, e_1, e_2, e_3, e_4]/\mathcal{A}_4$ *where*

(a) $\mathcal{A}_4 = (\{h^* \cdot e_i\}_{i=1}^4, (e_1+e_2)\cdot(e_2+e_3+e_4), (e_1+e_3)\cdot(e_3+e_4), (e_2+e_3)\cdot(e_3+e_4), (e_1+e_4)\cdot e_4, e_2\cdot e_4, (e_3+e_4)\cdot e_4, (e_1)^n+2(h^*)^n, (e_2)^n, (e_3)^n, (-1)(e_4)^n+(h^*)^n)$ *if n is odd,*

(b) $\mathcal{A}_4 = (\{h^* \cdot e_i\}_{i=1}^4, (e_1+e_2)\cdot(e_2+e_3+e_4), (e_1+e_3)\cdot(e_3+e_4), (e_2+e_3)\cdot(e_3+e_4), (e_1+e_4)\cdot e_4, e_2\cdot e_4, (e_3+e_4)\cdot e_4, (e_1)^n+4(h^*)^n, (e_2)^n+2(h^*)^n, (e_3)^n+2(h^*)^n, (e_4)^n+(h^*)^n)$ *if n is even,*

e $A^\bullet(Z_4) \cong \mathbb{Z}[h^*, e_1, e_2, e_3, e_4]/\mathcal{A}_5$ *where*

(a) $\mathcal{A}_5 = (\{h^* \cdot e_i\}_{i=1}^4, (e_1+e_2)\cdot(e_2+e_3+2e_4), (e_1+e_3)\cdot(e_3+e_4), (e_2+e_3)\cdot(e_3+e_4), (e_1+e_4)\cdot e_4, (e_2+e_4)\cdot e_4, (e_3+e_4)\cdot e_4, (e_1)^n+2(h^*)^n, (e_2)^n+(h^*)^n, (e_3)^n, (-1)(e_4)^n+(h^*)^n)$ *if n is odd,* *(b)* $\mathcal{A}_5 = (\{h^* \cdot e_i\}_{i=1}^4, (e_1+e_2)\cdot(e_2+e_3+2e_4), (e_1+e_3)\cdot(e_3+e_4), (e_2+e_3)\cdot(e_3+e_4), (e_1+e_4)\cdot e_4, (e_2+e_4)\cdot e_4, (e_3+e_4)\cdot e_4, (e_1)^n+4(h^*)^n, (e_2)^n+3(h^*)^n, (e_3)^n+2(h^*)^n, (e_4)^n+(h^*)^n)$ *if $n$ is even,*

*f* $A^\bullet(Z_4) \cong \mathbb{Z}[h^*, e_1, e_2, e_3, e_4]/\mathcal{A}_6$ *where*

*(a)* $\mathcal{A}_6 = (\{h^* \cdot e_i\}_{i=1}^4, (e_1+e_2)\cdot(e_2+e_3+2e_4), e_1\cdot e_3, (e_2+e_3)\cdot(e_3+e_4), e_1\cdot e_4, (e_2+e_4)\cdot e_4, (e_3+e_4)\cdot e_4, (e_1)^n, (e_2)^n+(h^*)^n, (e_3)^n, (-1)(e_4)^n +(h^*)^n)$ *if $n$ is odd,*

*(b)* $\mathcal{A}_6 = (\{h^* \cdot e_i\}_{i=1}^4, (e_1+e_2)\cdot(e_2+e_3+2e_4), e_1\cdot e_3, (e_2+e_3)\cdot(e_3+e_4), e_1\cdot e_4, (e_2+e_4)\cdot e_4, (e_3+e_4)\cdot e_4, (e_1)^n+2(h^*)^n, (e_2)^n+3(h^*)^n, (e_3)^n+2(h^*)^n, (e_4)^n+(h^*)^n)$ *if $n$ is even.*

*However, the presentations of the Chow ring of the skies of these 6 different proximity configurations coincide when considering the total transforms as generators:*

$$A^\bullet(Z_4) \cong \mathbb{Z}[h^*, e_1^*, e_2^*, e_3^*, e_4^*]/\mathcal{A}, \tag{3.11}$$

*where* $\mathcal{A} = (\{h^* \cdot e_i^*\}_{i=1}^4, \{e_i^* \cdot e_j^*\}_{\substack{i,j=1\\ i\neq j}}^4, \{(-1)^n(e_i^*)^n + (h^*)^n\}_{i=1}^4)$.

Now, if we restrict ourselves to the study of sequences of point blow-ups with a fixed proximity configuration, the following example exhibits that even although the skies of two sequences may not be isomorphic, there will exist an isomorphism between their Chow rings.

**Example 3.5.8.** *Let us consider all sequences of point blow-ups of length 5 with $Z_0 \cong \mathbb{P}^2$ and the following proximity configuration: $P_1$, $P_2 \to P_1$, $P_3 \to P_1$, $P_4 \to P_1$ and $P_5 \to P_1$. Then a presentation of the Chow ring of any of the skies of these sequences using the strict transforms of the exceptional components as generators is*

$$A^\bullet(Z_5) \cong \mathbb{Z}[h^*, e_1, e_2, e_3, e_4, e_5]/\mathcal{B}, \tag{3.12}$$

*where*

$$\mathcal{B} = (\{h^* \cdot e_i\}_{i=1}^5, \{(e_1+e_j)\cdot e_j\}_{j=2}^5, \{e_j \cdot e_k\}_{\substack{j,k=2\\ j\neq k}}^5, (e_1)^2 + 5(h^*)^2, \{(e_j)^2 + (h^*)^2\}_{j=2}^5).$$

*Nonetheless, since $E_1^1 \cong \mathbb{P}^1$, it is clear that if we choose two sequences of point blow-ups as above such that the centers $\{P_2, P_3, P_4, P_5\}$ and $\left\{P_2^{'}, P_3^{'}, P_4^{'}, P_5^{'}\right\}$ have different cross ratios, then the skies of the associated sequences will not be isomorphic but their Chow rings will do.*

In Example 3.5.7 we can foresee that the proximity relations of a sequence of point blow-ups are encoded in some way in the presentation of the Chow ring of the sky when using the strict transforms of the exceptional components as generators. Now we can use Theorem 3.5.6 ir order to refine the numerical characterization of Proposition 3.2.4.

**Corollary 3.5.9.** *$E_i$ is final if and only if its class in $A^1(Z_s)$, that is $e_i^s$, satisfies the following two conditions*

$$\begin{cases} (e_i^s)^n = (-1)^r (e_i^s)^{n-r} (e_j^s)^r & (3.13) \\ (e_j^s)^{n-1} e_i^s = (h^{s*})^n & (3.14) \end{cases}$$

*for every $j$ such that $e_i^s \cdot e_j^s \neq 0$.*

*Proof.* If $E_i$ is final then $\nexists k$ such that $P_k$ is proximate to $P_i$. By Equation (3.9) $(e_j^i + e_i^i) \cdot e_i^i = 0$ if $P_i$ is proximate to $P_j$ and $e_i^i \cdot e_j^i = 0$ otherwise. Since $E_i$ is final then it follows that

$$\begin{cases} (e_j^s + e_i^s) \cdot e_i^s = 0 \quad \text{if } P_i \to P_j & (3.15) \\ e_i^s \cdot e_j^s = 0 \qquad \text{otherwise} & (3.16) \end{cases}$$

From Equation (3.15) we can deduce that $(e_i^s)^n = (-1)^r (e_i^s)^{n-r} (e_j^s)^r$. Moreover $(h^{s*})^n = (-1)^{n+1} (e_i^{s*})^n$, so $(h^{s*})^n = (-1)^{2n} e_i^s (e_j^s)^{n-1} = e_i^s (e_j^s)^{n-1}$.

Now we will prove that if $E_i$ is not final, then some of the above conditions fails. Among all the index $\{\beta\}$ satisfying $P_\beta \to P_i$ there must exist an index $j$ such that $P_j \to P_i$ but that there not exists $k$ with $P_k \to P_i$ and $P_k \to P_j$. Since $E_j^j$ is final for the sequence $(Z_0, ..., Z_j, \pi_{j,0})$, then $(e_j^j) \cdot (e_i^j)^{n-1} = (h^{j*})^n$ and $(e_i^j)^{n-1-\beta} (e_j^j)^{1+\beta} = (-1)^\beta (e_i^j)^{n-1} e_j^j$. Moreover, since $\nexists P_k$ with $P_k$ proximate to both $P_i$ and $P_j$, then we can conclude that $(e_j^s) \cdot (e_i^s)^{n-1} = (h^{s*})^n$ and $(e_i^s)^{n-1-\beta} (e_j^s)^{1+\beta} = (-1)^\beta (e_i^s)^{n-1} e_j^s$. If $n$ is even, although $(e_j^s)^{n-1} e_i^s = (h^{s*})^n$ since $n-2$ is even too, $(e_i^s)^n \neq (-1)^{n-1} (e_i^s)(e_j^s)^{n-1}$ since by Theorem

3.5.6 $(e_i^s)^n = -(1 + \#\{\beta\})(h^{s*})^n$ with $\#\{\beta\} \geq 1$ so condition (3.13) fails.

If $n$ is odd, $(e_j^s)^{n-1} e_i^s = -(h^{j*})^n$, since $n - 2$ is odd too, so condition 3.14 fails. $\square$

# Chapter 4

# Sequences of point blow-ups over a perfect field.

In this chapter we extend the results of the previous one in order to consider sequences of point blow-ups over perfect fields. This more general setting, lead us to define in the first section algebraically and combinatorially compatible partitions of the exceptional divisor. The following sections run in parallel with the ones of chapter 3, that is, the second section deals with the definition of algebraic and combinatorial equivalences of sequences of point blow-ups and sequential morphisms, the third section is devoted to the numerical characterization of final divisors and the next two sections we recover the sequences of point blow-ups from the associated sequential morphism modulo algebraic equivalence, and prove some relations between algebraic and combinatorial equivalence classes of sequences of point blow-ups and sequential morphisms.

## 4.1 Algebraically and combinatorially compatible partitions of the exceptional divisor

Fix a perfect field $k$ and chose an algebraic closure $\overline{k}$. Throughout this chapter, a variety will mean a reduced projective scheme over a perfect field $K$, with $K$ an algebraic extension of $k$, so it is also perfect, such that $K \subset \overline{k}$, and a point will mean a closed point.

In contrast to the case of sequences of point blow-ups over an algebraically closed field, now we consider sequences of point blow-ups where the centers $C_{i+1}$ (see Definition 2.1.1) could be reducible, that is $C_{i+1} = \sqcup C_{i+1,j}$ with $C_{i+1,j}$ irreducible over $K$. This difference leads us to define the concept of the length of a sequence of point blow-ups.

**Definition 4.1.1.** *The* ***length*** *$m$ over $K$ of a sequence of blow-ups is defined as $\sum_{i=1}^{s} \#C_i$, where $\#C_i$ denotes the number of irreducible components of $C_i$ over $K$. Notice that it coincides with the number of irreducible components of the exceptional divisor $E$ (over $K$ too). Therefore, the length depends on the sequential morphism $\pi : Z_s \to Z_0$ and it can be also called the length of $\pi$ over $K$, and it will be denoted by $m = lenght_K(\pi)$. Notice that $s \leq m$, and $s = m$ exactly when all the blow up centers are irreducible over $K$.*

**Remark 4.1.2.** *Note that in the case of sequences of point blow-ups if $K = \overline{k}$, with $\overline{k}$ the algebraic closure of $k$, then $m = lenght_K(\pi) = \sum_{i=1}^{s} [K(C_i) : K]$.*

**Remark 4.1.3.** *Moreover we will denote by $H_\beta$ the* ***irreducible components*** *over $K$ of the exceptional divisor $E$ of $\pi$, that is we have $E = \bigcup_\beta H_\beta$.*

In order to consider different fields $K$, with $k \subset K \subset \overline{k}$, we define the notion of compatible partition of the exceptional divisor $E$.

Combinatorial compatibility with a sequential morphism 4.1.4 will mean compatibility of the $d-$ary multilinear intersection form. Compatibility with a sequence of point blow-ups 4.1.5 will mean compatibility of proximity relations and degrees of the residue field extensions.

Also we will define the notion of algebraic compatibility, stronger than combinatorial, where the partition comes, by fiber product, from a sequential morphism 4.1.7 (resp. a sequence of blow-ups 4.1.8) defined over a smaller field $\widetilde{K}$, with $k \subset \widetilde{K} \subset K$.

**Definition 4.1.4.** *Given a sequential morphism $\pi : Z_s \to Z_0$ as in Definition 2.1.2 and a partition $E = \sqcup_{i=1}^{l} F_i$, we will say that the* ***partition is combinatorially compatible*** *with $\pi$ if for each $i = 1,..,l$, and $H_{j_1}, H_{j_2} \in F_i$ there exists $\sigma \in S_m$ such that*

*a* $\sigma(j_1) = j_2$,

*b* $\mathcal{I}_{Z_s,E}(H_{i_1}, H_{i_2}, ..., H_{i_n}) = \mathcal{I}_{Z_s,E}(H_{\sigma(i_1)}, H_{\sigma(i_2)}, ..., H_{\sigma(i_d)})\ \forall i_1, .., i_n$

Let $(Z_s, ..., Z_0, \pi)$ be a sequence of blow ups of length $m$, and $H_1, ..., H_m$ the irreducible components of the exceptional divisor over $K$ of the associated sequential morphism. For each $i$, with $i = 1, 2, .., m$, let $r(i)$ be the integer such that the image of $H_i$ at $Z_{r(i)}$ is a component of the center (codimension at least 2) whose blow-up creates $H_i$. If $j$ is different from $i$, and the image of $H_j$ at $Z_{r(i)}$ has codimension 1 and contains the image of $H_i$ at $Z_{r(i)}$, then $H_i$ is said to be proximate to $j$ and we denote it by $H_i \to H_j$. It is clear that one has $r(i) > r(j)$ when $H_i$ is proximate to $H_j$.
For sequences of point blow-ups we denote $deg(H_i) = [K(P_i) : K]$, where $P_i$ is the point in the center of $\pi_{r(i)}$ such that the image of $H_i$ in $Z_{r(i)}$ is $P_i$.

**Definition 4.1.5.** *Given a sequence of point blow-ups $(Z_0, ..., Z_s, \pi)$ and a partition of the exceptional divisor $E = \sqcup_{i=1}^{l} F_i$, we will say that the* ***partition*** *is* ***combinatorially compatible*** *with the sequence $(Z_0, ..., Z_s, \pi)$ if for each $i = 1, .., l$ and $H_{j_1}, H_{j_2} \in F_i$ there exists $\sigma \in S_m$ such that*

*a* $\sigma(j_1) = j_2$,

*b* $deg(H_{j_1}) = [K(P_{j_1}) : K] = \left[K(P_{\sigma(j_1)}) : K\right] = deg(H_{\sigma(j_1)})$,

*c if* $H_{j_1} \in F_{i_1}$, $H_{j_k} \in F_{i_k}$ *and* $H_{j_k} \to H_{j_1}$, *then* $H_{\sigma(j_k)} \to H_{\sigma(j_1)}$

**Remark 4.1.6.** *Note that it makes sense to define $F_i \to F_j$ if $\exists H_i \in F_i$, $H_j \in F_j$ with $H_i \to H_j$.*

**Definition 4.1.7.** *Given a sequential morphism $\pi : Z_s \to Z_0$ as in Definition 2.1.2 and a partition of the exceptional divisor $E = \sqcup_{i=1}^{l} F_i$, we will say that the* ***partition*** *is* ***algebraically compatible*** *with the morphism $\pi$ if there exists a smaller field $\widetilde{K} \subset K$*

*with* $k \subset \widetilde{K}$*, there are* $\widetilde{K}-$*varieties* $\widetilde{Z_0}$ *and* $\widetilde{Z}$ *and a* $\widetilde{K}-$*morphism* $\widetilde{Z} \xrightarrow{\widetilde{\pi}} \widetilde{Z_0}$

$$\begin{array}{ccc} Z \cong \widetilde{Z} \times_{Spec(\widetilde{K})} Spec(K) & \xrightarrow{\pi} & Z_0 \cong \widetilde{Z_0} \times_{Spec(\widetilde{K})} Spec(K) \\ \downarrow{\scriptstyle \beta} & & \downarrow \\ \widetilde{Z} & \xrightarrow{\widetilde{\pi}} & \widetilde{Z_0} \end{array}$$

*such that the exceptional divisor of* $\widetilde{\pi}$*,* $\widetilde{E}$*, has irreducible components* $\widetilde{H_1}, ..., \widetilde{H_l}$ *and for each* $i = 1, ..., l$ *then* $\forall H \in F_i$ $\beta(H) = \widetilde{H_i}$

**Definition 4.1.8.** *Given a sequence of point blow-ups* $(Z_0, ..., Z_s, \pi)$ *and a partition of the exceptional divisor* $E = \sqcup_{i=1}^{l} F_i$*, we sill say that the* ***partition*** *is* ***algebraically compatible*** *with the sequence* $(Z_0, ..., Z_s, \pi)$ *if there exist a smaller field* $\widetilde{K} \subset K$ *with* $k \subset \widetilde{K}$ *and there are* $\widetilde{K}-$*varieties* $\widetilde{Z_i}$ *and* $\widetilde{K}-$*morphisms* $\widetilde{Z}_{i+1} \xrightarrow{\widetilde{\pi}_{i+1}} \widetilde{Z}_i$

$$\begin{array}{ccccccccc} Z_s & \xrightarrow{\pi_s} & Z_{s-1} & \xrightarrow{\pi_{s-1}} & \cdot & \longrightarrow & Z_1 & \xrightarrow{\pi_1} & Z_0 \\ \downarrow{\scriptstyle \beta} & & \downarrow & & & & \downarrow & & \downarrow \\ \widetilde{Z}_s & \xrightarrow{\widetilde{\pi}_s} & \widetilde{Z}_{s-1} & \xrightarrow{\widetilde{\pi}_{s-1}} & \cdot & \longrightarrow & \widetilde{Z}_1 & \xrightarrow{\widetilde{\pi}_1} & \widetilde{Z}_0 \end{array}$$

*where* $Z_i \cong \widetilde{Z}_i \times_{Spec(\widetilde{K})} Spec(K)$ $\forall i = 1, ..., s$ *,such that the exceptional divisor of* $(\widetilde{Z_0}, ..., \widetilde{Z}_l, \widetilde{\pi})$ *has irreducible components* $\widetilde{H_1}, ..., \widetilde{H_l}$ *and for each* $i = 1, ..., l$ *then* $\forall H \in F_i$ $\beta(H) = \widetilde{H_i}$*.*

**Remark 4.1.9.** *Note that since* $k$ *is perfect then* $\widetilde{K} \subset K$ *is a separable algebraic extension, so* $\widetilde{K}$ *and* $K$ *are both perfect fields.*

A **combinatorially** (resp. **algebraically**) marked **sequential morphism** is denoted $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{comb}$ (resp. $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{alg}$) where $\sqcup_{i=1}^{l} F_i$ is a partition combinatorially (resp. algebraically) compatible with $\pi$. The same notation holds for sequences.

Note also that if a partition is algebraically compatible with a sequential morphism (resp. a sequence) then the partition is combinatorially compatible with the sequential morphism (resp. the sequence).

## 4.2 Algebraic and combinatorial equivalence of sequences of point blow-ups and the associated sequential morphisms

Now we define our notions of equivalence (algebraic and combinatorial) for marked sequential morphisms (Definitions 4.1.7 and 4.1.4).

**Definition 4.2.1.** *We say that two* ***algebraically marked sequential morphisms*** $(\pi : Z \to Z_0, \sqcup_{i=1}^{l} F_i)_{alg}$ *and* $(\pi^{'} : Z^{'} \to Z_0^{'}, \sqcup_{i=1}^{l^{'}} F_i^{'})_{alg}$ *over* $K$ *are* ***algebraically equivalent****, and we denote it by* $(\pi : Z \to Z_0, \sqcup_{i=1}^{l} F_i)_{alg} \overset{alg}{\sim}_K (\pi^{'} : Z^{'} \to Z_0^{'}, \sqcup_{i=1}^{l^{'}} F_i^{'})_{alg}$ *, if there exist smaller fields* $\widetilde{K}, \widetilde{K'} \subset K$ *with* $\widetilde{K} \cong_k \widetilde{K'}$ *satisfying the conditions of Definition 4.1.7*

$$\begin{array}{ccc} Z_s \cong \widetilde{Z}_s \times_{Spec(\widetilde{K})} Spec(K) & \xrightarrow{\pi} & Z_0 \cong \widetilde{Z_0} \times_{Spec(\widetilde{K})} Spec(K) \\ \downarrow{\scriptstyle \beta} & & \downarrow \\ \widetilde{Z}_s & \xrightarrow{\widetilde{\pi}} & \widetilde{Z_0} \end{array}$$

$$\begin{array}{ccc} Z_s^{'} \cong \widetilde{Z'}_s \times_{Spec(\widetilde{K'})} Spec(K) & \xrightarrow{\pi^{'}} & Z_0^{'} \cong \widetilde{Z_0^{'}} \times_{Spec(\widetilde{K'})} Spec(K) \\ \downarrow{\scriptstyle \beta^{'}} & & \downarrow \\ \widetilde{Z_s^{'}} & \xrightarrow{\widetilde{\pi^{'}}} & \widetilde{Z_0^{'}} \end{array}$$

*and there exist isomorphisms* $a$ *and* $b$ *such that the following diagram is commutative*

$$\begin{array}{ccc} \widetilde{Z} & \xleftrightarrow{b} & \widetilde{Z'} \\ \downarrow{\scriptstyle \widetilde{\pi}} & & \downarrow{\scriptstyle \widetilde{\pi^{'}}} \\ \widetilde{Z_0} & \xleftrightarrow{a} & \widetilde{Z_0^{'}} \end{array}$$

**Definition 4.2.2.** *Given a combinatorially marked sequential morphism* $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{comb}$*, we can also consider the* $n$*-ary multilinear intersection form associated to the partition*

$$\mathcal{I}_{Z, \sqcup_{i=1}^{l} F_i} : \overbrace{\mathbb{F} \times \mathbb{F} \times \cdots \times \mathbb{F}}^{n} \to \mathbb{Z},$$

*where* $\mathbb{F}$ *is the free abelian group generated by* $\{F_i\}$ *and by an abuse of notation* $F_i = \sum_{H \in F_i} H$*. The intersection form is defined by intersecting cycles in the sky* $Z_s$ *and taking degrees, that is*

$$\mathcal{I}_{Z, \sqcup_{i=1}^{l} F_i}(F_{i_1}, F_{i_2}, ..., F_{i_n}) = deg((\sum_{H \in F_{i_1}} h) \cdot (\sum_{H \in F_{i_2}} h) \cdot (\sum_{H \in F_{i_3}} h) \cdots (\sum_{H \in F_{i_n}} h),$$

*where* $(\sum_{H\in F_{i_1}} h)\cdot(\sum_{H\in F_{i_2}} h)\cdot(\sum_{H\in F_{i_3}} h)\cdots(\sum_{H\in F_{i_n}} h)$ *is a intersection class of* $0-$*cycles in the abelian group* $A_0(Z_s)$*, and deg stands for the degree.*

**Definition 4.2.3.** *Given two* ***combinatorially marked sequential morphisms*** $(\pi : Z_s \to Z_0, \sqcup_{i=1}^l F_i)_{comb}$ *and* $(\pi^{'} : Z_s^{'} \to Z_0^{'}, \sqcup_{i=1}^l F_i^{'})_{comb}$ *we say that the associated multilinear forms* $\Phi_{Z,\sqcup_{i=1}^l F_i}$ *and* $\Phi_{Z^{'},\sqcup_{i=1}^l F_i^{'}}$ *are equivalent, and we denote it by* $\Phi_{Z,\sqcup_{i=1}^l F_i} \sim \Phi_{Z^{'},\sqcup_{i=1}^l F_i^{'}}$*, if there exists* $\tau \in \mathcal{S}_l$ *such that*

$$\tau(\Phi_{Z,\sqcup_{i=1}^l F_i}) = \Phi_{Z^{'},\sqcup_{i=1}^l F_i^{'}}.$$

*Moreover, the combinatorially marked sequential morphisms* $(\pi : Z_s \to Z_0, \sqcup_{i=1}^l F_i)_{comb}$ *and* $(\pi^{'} : Z_s^{'} \to Z_0^{'}, \sqcup_{i=1}^l F_i^{'})_{comb}$ *are said to be* ***combinatorially equivalent****, and we denote it by* $(\pi : Z_s \to Z_0, \sqcup_{i=1}^l F_i)_{comb} \overset{comb}{\sim} (\pi^{'} : Z_s^{'} \to Z_0^{'}, \sqcup_{i=1}^l F_i^{'})_{comb}$*, when their associated multilinear maps* $\Phi_{Z,\sqcup_{i=1}^l F_i}$ *and* $\Phi_{Z^{'},\sqcup_{i=1}^l F_i^{'}}$ *are equivalent.*

**Definition 4.2.4.** *Given a variety* $X$ *we will call a brick blow-up with ground* $X$ *to a sequential morphism obtained as a composition of point blow-ups with disjoint centers* $\sqcup_{j=1}^l C_j \subset X$*,* $X^{'} = X_l \to X_{l-1} \to ... \to X_1 \to X$*. Note that* $Z_i \to Z_{i-1}$ *is the brick blow-up at* $C_i$*, where* $C_i$ *need not to be irreducible.*

**Definition 4.2.5.** *We say that two* ***algebraically marked sequences*** *of point blow ups,* $(Z_s, ..., Z_0, \pi, \sqcup_{i=1}^l F_i)_{alg}$ *, and* $(Z_{s^{'}}, ..., Z_0^{'}, \pi^{'}, \sqcup_{i=1}^{l^{'}} F_i^{'})_{alg}$*, are* ***algebraically equivalent*** *over* $K$*, and we denote it by* $(Z_s, ..., Z_0, \pi, \sqcup_{i=1}^l F_i)_{alg} \overset{alg}{\sim}_K (Z_{s^{'}}, ..., Z_0^{'}, \pi^{'}, \sqcup_{i=1}^{l^{'}} F_i^{'})_{alg}$*, if* $l = l^{'}$ *and there exist smaller fields* $\widetilde{K}, \widetilde{K^{'}} \subset K$ *with* $\widetilde{K} \cong_k \widetilde{K^{'}}$

$$\begin{array}{ccccccccc}
Z_s & \xrightarrow{\pi_s} & Z_{s-1} & \xrightarrow{\pi_{s-1}} & \cdot & \xrightarrow{\pi_2} & Z_1 & \xrightarrow{\pi_1} & Z_0 \\
\downarrow{\scriptstyle\beta} & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\widetilde{Z}_s & \xrightarrow{\widetilde{\pi}_s} & \widetilde{Z}_{s-1} & \xrightarrow{\widetilde{\pi}_{s-1}} & \cdot & \xrightarrow{\widetilde{\pi}_2} & \widetilde{Z}_1 & \xrightarrow{\widetilde{\pi}_1} & \widetilde{Z}_0
\end{array}$$

$$\begin{array}{ccccccccc}
Z_s^{'} & \xrightarrow{\pi_s^{'}} & Z_{s-1}^{'} & \xrightarrow{\pi_{s-1}^{'}} & \cdot & \xrightarrow{\pi_2^{'}} & Z_1^{'} & \xrightarrow{\pi_1^{'}} & Z_0^{'} \\
\downarrow{\scriptstyle\beta^{'}} & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\widetilde{Z}_s^{'} & \xrightarrow{\widetilde{\pi}_s^{'}} & \widetilde{Z}_{s-1}^{'} & \xrightarrow{\widetilde{\pi}_{s-1}^{'}} & \cdot & \xrightarrow{\widetilde{\pi}_2^{'}} & \widetilde{Z}_1^{'} & \xrightarrow{\widetilde{\pi}_1^{'}} & \widetilde{Z}_0^{'}
\end{array}$$

*with* $Z_i \cong \widetilde{Z}_i \times_{Spec(\widetilde{K})} Spec(K)$ *(resp.* $Z_i^{'} \cong \widetilde{Z}_i^{'} \times_{Spec(\widetilde{K}^{'})} Spec(K)$*) and algebraic isomorphisms* $a, b = b_t, b_{t-1}, ..., b_1$*, with* $t \leq s$*, such that there are indexes* $r_1, ..., r_t = s \in \{1, ..., l\}$ *and* $r_1^{'}, ..., r_t^{'} = s^{'} \in \left\{1, ..., s^{'}\right\}$*, where* $Z_{r_i} \to Z_{r_i - 1} \to ... \to Z_{r_{i-1}}$ *(resp.*

$Z'_{r_i} \to Z'_{r_i-1} \to ... \to Z'_{r_{i-1}}$), *with* $r_i > r_{i-1}$ *(resp* $r'_i > r'_{i-1}$*), is a brick blow-up* $\forall i = 1...t$ *as in Definition 4.2.4 and the diagram*

$$\begin{array}{ccccccccccc}
\widetilde{Z}_s & \longrightarrow & \widetilde{Z}_{r_{t-1}} & \longrightarrow & \widetilde{Z}_{r_{t-2}} & \longrightarrow & \cdot & \longrightarrow & \widetilde{Z}_{r_1} & \longrightarrow & \widetilde{Z}_0 \\
{\scriptstyle b}\downarrow & & {\scriptstyle b_{t-1}}\downarrow & & {\scriptstyle b_{t-2}}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle a} \\
\widetilde{Z}'_{s'} & \longrightarrow & \widetilde{Z}'_{r'_{t-1}} & \longrightarrow & \widetilde{Z}'_{r'_{t-2}} & \longrightarrow & \cdot & \longrightarrow & \widetilde{Z}'_{r'_1} & \longrightarrow & \widetilde{Z}'_0
\end{array}$$

*is commutative.*

**Definition 4.2.6.** *We say that two* ***combinatorially marked sequences of point blow ups****,* $(Z_s, ..., Z_0, \sqcup_{i=1}^{l} F_i, \pi)_{comb}$ *and* $(Z'_s, ..., Z'_0, \sqcup_{i=1}^{l} F'_i, \pi')_{comb}$ *as before with respective partitions* $E = \sqcup_{i=1}^{l} F_i$ *and* $E' = \sqcup_{i=1}^{l} F'_i$ *and irreducible components of the exceptional divisor* $H_1, ..., H_m$*;* $H'_1, ..., H'_m$*, with* $l = l'$*, are* ***combinatorially equivalent****, and we denote it by* $(Z_s, ..., Z_0, \sqcup_{i=1}^{l} F_i, \pi)_{comb} \overset{comb}{\sim}_K (Z'_s, ..., Z'_0, \sqcup_{i=1}^{l} F'_i, \pi')_{comb}$*, if there exits a permutation* $\tau$ *in* $S_l$ *such that for every two different indexes* $i, j$ *one has*

a $F_i$ *is proximate to* $F_j$ *if and only if* $F'_{\tau(i)}$ *is proximate to* $F'_{\tau(j)}$*,*

b $deg(F_i) = \sum_{H \in F_i} deg(H) = \sum_{H' \in F'_i} deg(H') = deg(F'_{\tau(i)})$

## 4.3 Final divisors: Numerical characterization

**Proposition 4.3.1.** $H_i$ *is final if and only if*

$$(h_i)^n = (-1)^r (h_i)^s \cdot (h_j)^r \text{ and } (h_i) \cdot (h_j)^{n-1} > 0$$

*for every* $j$ *such that* $H_i \cap H_j \neq \emptyset$ *(see Lemma 3.2.3 for a numerical characterization) and for all natural numbers* $r$ *and* $s$ *with* $r + s = n$*.*

*Proof.* We have the following commutative diagram where we denote by $D_{i,j}$ to the scheme theoretic intersection $H_i \cap H_j$ and all the morphism are regular embeddings

$$\begin{array}{ccccc} & & H_i & & \\ & \overset{i_{D_{i,j},H_i}}{\nearrow} & & \overset{j_{H_i,Z_s}}{\searrow} & \\ D_{i,j} & & \xrightarrow{\quad i_{D_{i,j},Z_s} \quad} & & Z_s \\ & \underset{i_{D_{i,j},H_j}}{\searrow} & & \underset{j_{H_j,Z_s}}{\nearrow} & \\ & & H_j & & \end{array}$$

First let us suppose that $H_i$ is final .Then $H_i \cong \mathbb{P}^{n-1}_{K(P_i)}$ and by Proposition 1.1.16 $N_{H_i/Z_s} = \mathcal{O}_{H_i}(-1)$, so it follows by Proposition 1.2.22 that

$$(h_i)^n = (-1)^{n-1} j_{H_i,Z_s *}(\varsigma^{n-1}),$$

where $\varsigma = c_1(\mathcal{O}_{H_i}(1))$.

It follows by Proposition 1.2.20 that

$$h_i \cdot h_j \cdot h_j = i_{D_{i,j},Z_s *}(c_1(N_{H_j/Z_s}|_{D_{i,j}})$$
$$h_i \cdot h_j \cdot h_i = i_{D_{i,j},Z_s *}(c_1(N_{H_i/Z_s}|_{D_{i,j}})$$

By Proposition 1.1.19 the normal bundle of $H_j$ satisfies

$$N_{H_j/Z_s} = \pi^*_{n,j}|_{H^j_j}(N_{H^{j j}/Z_j}) \otimes \bigotimes_{\alpha \to j} \pi^*_{n,\alpha}|_{H^\alpha_j}(\mathcal{O}(-H^\alpha_j \cap H^\alpha_\alpha)),$$

so since $H_i$ is final

$$\begin{aligned} N_{H_j/Z_s}|_{D_{i,j}} &\cong L \otimes \mathcal{O}(-D_{i,j})|_{D_{i,j}}, \\ &\cong L \otimes N^\vee_{D_{i,j}/H_j}, \end{aligned}$$

where $L$ denotes a trivial line bundle. As $E$ is a simple normal crossing divisor, then by Lemma 2.2.1 $N_{D_{i,j}/H_j} \cong N_{H_i/Z_s}|_{D_{i,j}}$, so it follows that

$$h_i \cdot h_j \cdot h_j = i_{D_{i,j},Z_s *}(-c_1(N_{E_i/Z_s}|_{D_{i,j}}))$$

By induction on $r$ and $s$ respectively it follows

$$h_i \cdot (h_j)^r = (-1)^{r-1} i_{D_{i,j},Z_s *}((c_1(N_{E_i/Z_s}|_{D_{i,j}}))^{r-1})$$
$$(h_i)^s \cdot h_j = i_{D_{i,j},Z_s *}((c_1(N_{E_i/Z_s}|_{D_{i,j}}))^{s-1})$$

So we can conclude that

$$(h_i)^s \cdot (h_j)^r = (-1)^{r-1} i_{D_{i,j},Z_s *}((c_1(N_{H_i/Z_s}|_{D_{i,j}}))^{r+s-2}),$$

Moreover, as a consequence of Proposition 1.2.15

$$\begin{aligned} i_{D_{i,j},Z_s *}((c_1(N_{E_i/Z_s}|_{D_{i,j}}))^{r+s-2}) &= j_{H_i,Z_s *} i_{D_{i,j},H_i *}((c_1(N_{H_i/Z_s}|_{D_{i,j}}))^{r+s-2}) \\ &= j_{H_i,Z_s *}(c_1(N_{H_i/Z_s})^{r+s-3} \cdot d_{i,j}), \end{aligned}$$

so since $d_{i,j} = \varsigma$

$$(h_i)^s \cdot (h_j)^r = (-1)^{2r+s-3} deg(j_{H_i,Z*}(\varsigma^{r+s-1})) = (-1)^{s-1} j_{E_i,Z*}(\varsigma^{r+s-1})$$

Then $(h_i)^n = (-1)^r (h_i)^s \cdot (h_j)^r$. Moreover $(h_i) \cdot (h_j)^{n-1} = j_{E_i,Z_s *}(\varsigma^{n-1}) > 0$.

Now let us suppose that $H_i$ is not final. If $P_\alpha$ is proximate to $P_i$, then we have the following commutative diagram

$$\begin{array}{ccc} H_i^\alpha \cap H_\alpha^\alpha & \xrightarrow{i_{H_i^\alpha \cap H_\alpha^\alpha, H_i^\alpha}} & H_i^\alpha \\ \downarrow & & \downarrow \pi_\alpha|_{H_i^{\alpha-1}} \\ P_\alpha & \xrightarrow{i_{P_\alpha, H_i^{\alpha-1}}} & H_i^{\alpha-1} \end{array}$$

Among all the index satisfying $\alpha \to i$ there must exist an index $j$ such that $j \to i$ but that there not exists $k$ with $k \to i$ and $k \to j$. Let $j$ be such index. Since $H_i$ is not final then by Proposition 1.1.19 its normal bundle satisfies

$$N_{H_i/Z_s} = \pi_{n,i}^*|_{H_i^i}(N_{E_i^i/Z_i}) \otimes \bigotimes_{\alpha \to i} \pi_{n,\alpha}^*|_{H_i^\alpha}(\mathcal{O}(-H_i^\alpha \cap H_\alpha^\alpha))$$

Now, by the Projection formula 1.6

$$j_{H_i,Z_s *}(\pi_{n,i}^*|_{H_i^i}(c_1(N_{H_i^i/Z_i})^{n_i})) \prod_{\alpha \to i} (\pi_{n,\alpha}^*|_{H_i^\alpha}((-1)^{n-1}(i_{H_i \cap H_\alpha^\alpha, H_i^\alpha *}(H_i^\alpha \cap H_\alpha^\alpha)^{n-1}))) = 0$$

with $n_i + \sum_{\alpha \to i} n_\alpha = n$, so

$$(h_i)^n = j_{H_i,Z_s *}(\pi_{n,i}^*|_{H_i^i}(c_1(N_{H_i^i/Z_i})^{n-1})) + \sum_{\alpha \to i} j_{H_i,Z_s *}(\pi_{n,\alpha}^*|_{H_i^\alpha}((-1)^{n-1}((d_{i,\alpha})^{n-1})))$$

Furthermore, by an analogous reasoning to the case when $H_i$ is final we have

$$h_i \cdot h_j \cdot h_j = i_{D_{i,j},Z_s *}(c_1(N_{H_j/Z_s}|_{D_{i,j}}))$$
$$h_i \cdot h_j \cdot h_i = i_{D_{i,j},Z_s *}(c_1(N_{H_i/Z_s}|_{D_{i,j}}))$$

Now $N_{H_i/Z_s}|_{D_{i,j}} \cong L \otimes N^{\vee}_{D_{i,j}/E_i}$ and by Lemma 2.2.1 $N_{E_j/Z_s}|_{D_{i,j}} \cong N_{D_{i,j}/E_i}$.
By induction on $r$ and $s$ respectively it follows

$$h_i \cdot (h_j)^r = i_{D_{i,j},Z_s *}((c_1(N_{D_{i,j}/H_i})^{r-1})$$
$$(h_i)^s \cdot h_j = (-1)^{s-1} i_{D_{i,j},Z_s *}((c_1(N_{D_{i,j}/H_i})^{s-1})$$

so it follows that

$$(h_i)^s \cdot (h_j)^r = (-1)^{s-1} i_{D_{i,j},Z_s *}((c_1(N_{D_{i,j}/H_i})^{r+s-2})$$

Moreover, by Proposition 1.2.15

$$\begin{aligned} i_{D_{i,j},Z_s *}((c_1(N_{D_{i,j}/H_i}))^{r+s-2}) &= j_{E_i,Z_s *} i_{D_{i,j},H_i *}((c_1(N_{D_{i,j}/H_i})^{r+s-2})) \\ &= j_{E_i,Z_s *}((d_{i,j})^{r+s-1}) \end{aligned}$$

If $n$ is even then

$$(-1)^r (h_i)^s \cdot (h_j)^r = (-1)^{r+s-1} j_{H_i,Z_s *}((d_{i,j})^{r+s-1}) \neq (h_i)^n$$

since

$$j_{H_i,Z_s *}(\pi^*_{n,i}|_{H^i_i}(c_1(N_{H^i_i/Z_i})^{n-1}))) + \sum_{\substack{\alpha \to i \\ \alpha \neq j}} j_{H_i,Z_s *}(\pi^*_{n,\alpha}|_{H^\alpha_i}((-1)^{n-1}((d_{i,\alpha})^{n-1}))) < 0$$

If $n$ is odd then

$$(h_i) \cdot (h_j)^{n-1} = j_{H_i,Z_s *}((d_{i,j})^{r+s-1}),$$

so $(h_i) \cdot (h_j)^{n-1} < 0$ and $H_i$.

□

**Proposition 4.3.2.** *Given an algebraically marked sequence* $(Z_s, ..., Z_0, \sqcup_{i=1}^l F_i, \pi)_{alg}$ *with* $H, H^{'} \in F_i$*, then* $H$ *is final if and only if* $H^{'}$ *is final too.*

*Proof.* If $H, H^{'} \in F_i$ then there exist a sequence $(\widetilde{Z}_s, ..., \widetilde{Z}_0, \widetilde{E}, \widetilde{\pi})$ over $\widetilde{K}$ such that $\beta(H) = \beta(H^{'})$, where $\beta : Z_s = \widetilde{Z}_s \times_{Spec(\widetilde{K})} Spec(K) \to \widetilde{Z}_s$, so it follows that if $H$ satisfies the numerical condition of proposition 4.3.1, $H^{'}$ will satisfy it too. □

**Proposition 4.3.3.** *Given an algebraically marked sequence* $(Z_s, ..., Z_0, \sqcup_{i=1}^l F_i, \pi)_{alg}$ *then* $F_i$ *is final if and only if*

$$(F_i)^n = (-1)^r (F_i)^s \cdot (F_j)^r \ \textit{and} \ (F_i) \cdot (F_j)^{n-1} > 0$$

*for every* $j$ *such that* $F_i \cap F_j \neq \emptyset$ *and for all natural numbers* $r$ *and* $s$ *with* $r + s = d$*.*

*Proof.* This is a consequence of proposition 4.3.1, since if $H, H' \in F_i$ and $H \neq H'$ then $H \cap H' = \emptyset$. □

**Corollary 4.3.4.** *Let* $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{alg}$ *and* $(\pi' : Z_s' \to Z_0', \sqcup_{i=1}^{l'} F_i')_{alg}$ *algebraically marked sequential morphisms that are algebraically equivalent. If we denote by* $b'$ *to be the isomorphism* $b' : Z_s = \widetilde{Z}_s \times_{Spec(\widetilde{K})} Spec(K) \to Z_s' = \widetilde{Z}_s' \times_{Spec(\widetilde{K}')} Spec(K)$, *that is the extension of that in Definition 4.2.1, then* $F_i$ *is final if and only if* $b'(F_i)$ *is final.*

## 4.4 Recovering of the sequence

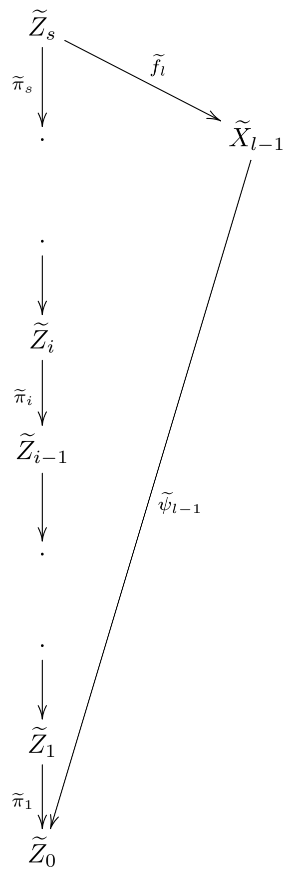
**Proposition 4.4.1.** *Let* $(Z_s, ..., Z_0, \pi)$ *be a sequence of point blow-ups (as in Definition 2.1.1) of length* $m$ *and let* $H_i \in E_i$ *be an irreducible component of the exceptional divisor of* $\pi$. *If* $H_i$ *is final, then there exists a regular projective contraction* $(Z, f_m, X_{m-1})$ *of* $H_i$ *to a point such that* $f_m(E)$ *is a simple normal crossing divisor and* $X_{m-1}$ *is the sky of a sequence of point blow-ups with ground* $Z_0$.

*Proof.* The proof is analogous to that of Proposition 3.3.2 with the exception that the alternative proof is no longer valid it makes use of the algebraic closure of the field. □

**Theorem 4.4.2.** *Let* $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{alg}$ *be an algebraically marked sequential morphism. Given the* $n-$*ary multilinear intersection form associated to the partition* $\mathcal{I}_{Z, \sqcup_{i=1}^{l} F_i}$ *(see Definition 4.2.2) we can recover all the algebraically marked sequences of point blow-ups that are associated to algebraically marked sequential morphisms in the same algebraic equivalence class of* $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{alg}$.

*Proof.* Since $\sqcup_{i=1}^{l} F_i$ is a partition algebraically compatible with $\pi$ then $\exists \widetilde{K} \subset K$ as in Definition 4.1.7. If $H \in F_i$ is final then $\widetilde{H} = \beta(H)$ is final for $\widetilde{\pi} : \widetilde{Z}_s \to \widetilde{Z}_0$. We will prove this result first by contracting one irreducible component of the exceptional divisor $\widetilde{E}$ each time.

Since the set formed by final divisors is not empty, let us suppose that $\widetilde{H}_i$ is final, then by proposition 4.4.1 there exists a regular projective contraction $(\widetilde{Z}_s, \widetilde{f}_l, \widetilde{X}_{l-1})$ of $\widetilde{H}_i$ to a point such that $\widetilde{X}_{l-1}$ is the sky of a sequence of point blow-ups with ground $\widetilde{Z}_0$ .




The next step in our proof refers to how to obtain the intersection form in $\widetilde{X}_{l-1}$ associated to the simple normal crossing divisor $\widetilde{D}_{\widetilde{X}_{l-1}}$.

If we denote by $\widetilde{H}_{\widetilde{X}_{l-1},i}$ to $\widetilde{f}_l(\widetilde{H}_i)$, then by the Projection formula 1.6

$$\widetilde{h}_{\widetilde{X}_{l-1},i_1} \cdot \widetilde{h}_{\widetilde{X}_{l-1},i_2} \cdots \widetilde{h}_{\widetilde{X}_{l-1},i_n} = \widetilde{f}_l^*(\widetilde{h}_{\widetilde{X}_{l-1},i_1}) \cdot \widetilde{f}_l^*(\widetilde{h}_{\widetilde{X}_{l-1},i_2}) \cdots \widetilde{f}_l^*(\widetilde{h}_{\widetilde{X}_{l-1},i_n}),$$

Applying the result of Theorem 1.2.30 then

$$\widetilde{h}_{\widetilde{X}_{l-1},i_1} \cdot \widetilde{h}_{\widetilde{X}_{l-1},i_2} \cdots \widetilde{h}_{\widetilde{X}_{l-1},i_n} = (\widetilde{h}_{i_1} + \delta_{i_1,i}\widetilde{h}_i) \cdot (\widetilde{h}_{i_2} + \delta_{i_2,i}\widetilde{h}_i) \cdots (\widetilde{h}_{i_n} + \delta_{i_n,i}\widetilde{h}_i), \quad (4.1)$$

where $\delta_{i_j,i} = 1$ if $\widetilde{H}_i \cap \widetilde{H}_{i_j} \neq \emptyset$ (see numerical characterization in lemma 3.2.3) and $\delta_{i_j,i} = 0$ otherwise.

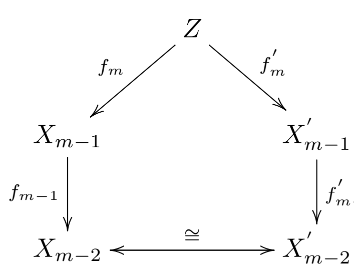
**Remark 4.4.3.** *It follows then that by iterating the above process, that is by contracting a final divisor at each step, we will obtain a sequence of point blow-ups of length $l$. The algebraically marked sequence obtained depends on the choice of final components. Below* *we will prove that all the algebraically marked sequential morphisms associated to the sequences constructed in this way are algebraically equivalent.*

## 4.5 Relations between algebraic and combinatorial equivalence classes of sequences of point blow-ups and sequential morphisms

**Proposition 4.5.1.** *Any of the algebraically marked sequences obtained as in 4.4.3, that is as composition of regular projective contractions from a fixed sky $Z_s$ and a fixed simple normal crossing divisor $E$, are associated to algebraically marked sequential morphisms in the same algebraic equivalence class (see Definition 4.2.1).*

Before proving this, we need the following lemma

**Lemma 4.5.2.** *Given a fixed sky $Z_s$ and a fixed simple normal crossing divisor $E$, let us suppose that $H_i$ and $H_j$ are both finals. Then there is an isomorphism $X_{m-2} \cong X'_{m-2}$ making the following diagram commutative*




*where $f_m$ is the contraction of $H_i$ and $f_{m-1}$ is the contraction of $H_{X_{m-1},j}$, whereas $f'_m$ is the contraction of $H_j$ and $f'_{m-1}$ is the contraction of $H_{X'_{m-1},i}$.*

*Proof.* The proof is completely analogous to that of Lemma 3.4.2.

□

Consequently, we have the following corollary, which means that Proposition 4.5.1 holds for length 2.

**Corollary 4.5.3.** *If $Z$ is the sky of a sequence of point blow-ups of length 2, then any of the two sequences of point blow-ups obtained following the procedure in 4.4.3 are associated to algebraically marked sequential morphisms in the same algebraic equivalence class .*

In order to prove proposition 4.5.1 we need the following definition.

**Definition 4.5.4.** *We say that two sequences of point blow-ups obtained as in Remark 4.4.3, that is through the composition of regular projective contractions from a fixed sky $Z_s$ and a fixed simple normal crossing divisor $E$,*

$$\begin{array}{ccccccccccccccccc}
Z_s & \xrightarrow{f_m} & X_{m-1} & \xrightarrow{f_{m-1}} & X_{m-2} & \xrightarrow{f_{s-2}} & . & \longrightarrow & . & \longrightarrow & . & \longrightarrow & X_2 & \xrightarrow{f_2} & X_1 & \xrightarrow{f_1} & X_0 \\
\updownarrow & & \updownarrow & & & & & & & & & & & & & & \updownarrow \\
Z_s & \xrightarrow{f_m^{'}} & X_{m-1}^{'} & \xrightarrow{f_{m-1}^{'}} & X_{m-2}^{'} & \xrightarrow{f_{s-2}^{'}} & . & \longrightarrow & . & \longrightarrow & . & \longrightarrow & X_2^{'} & \xrightarrow{f_2^{'}} & X_1^{'} & \xrightarrow{f_1^{'}} & X_0^{'}
\end{array}$$

*have the same end if at least the first contraction is common to both. i.e. one has $f_m = f_m^{'}$.*

*Proof of Proposition 4.5.1.* The proof is completely analogous to that of Proposition 3.4.1.

□

We can apply proposition 4.5.1 to the sequential morphism $\widetilde{Z}_s \to \widetilde{Z}_0$ and by scalar extension $\times_{Spec(\widetilde{K})} Spec(K)$ the algebraically marked sequences of point blow-ups constructed as above

$$\begin{array}{ccccccccc}
Z_s & \longrightarrow & X_{l-1} & \longrightarrow & \cdot & \longrightarrow & X_1 & \longrightarrow & X_0 \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\widetilde{Z}_s & \longrightarrow & \widetilde{X}_{l-1} & \longrightarrow & \cdot & \longrightarrow & \widetilde{X}_1 & \longrightarrow & \widetilde{X}_0
\end{array}$$

where $X_i \cong \widetilde{X}_i \times_{Spec(\widetilde{K})} Spec(K)$, so Theorem 4.4.2 is proved. □

**Theorem 4.5.5.** *Two combinatorially marked sequences of point blow-ups $(Z_s, ..., Z_0, \sqcup_{i=1}^{l} F_i, \pi)_{comb}$ and $(Z_s^{'}, ..., Z_0^{'}, \sqcup_{i=1}^{l^{'}} F_i^{'}, \pi^{'})_{comb}$, with $l = l^{'}$, are combinatorially equivalent over $K$ as in Definition 4.2.6 if and only if their associated combinatorially marked sequential morphisms $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{comb}$ and $(\pi^{'} : Z_s^{'} \to Z_0^{'}, \sqcup_{i=1}^{l^{'}} F_i^{'})_{comb}$ are combinatorially* *equivalent over $K$ as in Definition 4.2.3, and both statements are true if and only if the associated multilinear maps $\Phi_{Z,\sqcup_{i=1}^{l} F_i}$ and $\Phi_{Z',\sqcup_{i=1}^{l'} F_i'}$ are equivalent too as in Definition 4.2.3*

First we will prove that if two combinatorially marked sequential morphisms $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{comb}$ and $(\pi' : Z_s' \to Z_0', \sqcup_{i=1}^{l'} F_i')_{comb}$ are combinatorially equivalent then the associated combinatorially marked sequences of points blow-ups are combinatorially equivalent too. To begin with, we need a numerical characterization of proximity.

**Lemma 4.5.6.** *Let $(Z_s, ..., Z_0, \sqcup_{i=1}^{l} F_i, \pi)_{comb}$ be a combinatorially marked sequence. Then $P_i \to P_j$ if and only if*

a $\exists \alpha \in \{2, 3, ..., m-1, m\}$ *such that* $H_{X_\alpha,i} \cap H_{X_\alpha,j} \neq \emptyset$ *(see numerical characterization of Lemma 3.2.3).*

b $(h_{X_\alpha,i})^n = (-1)^r (h_{X_\alpha,i})^s \cdot (h_{X_\alpha,k})^r$ *and* $(h_{X_\alpha,i}) \cdot (h_{X_\alpha,k})^{n-1} > 0$ $\forall k, H_{X_\alpha,i} \cap H_{X_\alpha,k} \neq \emptyset$.

*where $Z_s = X_m \to X_{m-1} \to \cdots \to X_\alpha \to \cdots \to X_0 = Z_0$ is any sequence of contractions obtained as in remark 4.4.3.*

*Proof.* The proof is completely analogous to that of Lemma 3.4.6. □

**Remark 4.5.7.** *The result of the previous lemma also holds for characterizing numerically the proximity between elements of the combinatorially compatible partition $F_i \to F_j$.*

*Proof of Theorem 4.5.5.* Assume that the combinatorially marked sequential morphisms $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{comb}$ and $(\pi' : Z_s' \to Z_0', \sqcup_{i=1}^{l'} F_i')_{comb}$ are combinatorially equivalent. If $F_i$ is final, then there exists $\tau \in S_l$ such that

a $F'_{\tau(i)}$ is final,

b $F_i \cap F_\beta \neq \emptyset$ if and only if $F'_{\tau(i)} \cap F'_{\tau(\beta)} \neq \emptyset$,

c $F_{\beta_1} \cdot F_{\beta_2} \cdots F_{\beta_n} = F'_{\tau(\beta_1)} \cdot F'_{\tau(\beta_2)} \cdots F'_{\tau(\beta_n)}$

Furthermore, by Theorem 1.2.30

$$h_{X_{m-1},\beta_1}\cdot h_{X_{m-1},\beta_2}\cdots h_{X_{m-1},\beta_n}=(h_{\beta_1}+\delta_{\beta_1,i}h_i)\cdot(h_{\beta_2}+\delta_{\beta_2,i}h_i)\cdots(h_{\beta_d}+\delta_{\beta_d,i}h_i),$$

so it follows then that there exists $\widetilde{\tau}\in S_{l-1}$ such that

$$F_{X_{l-1},\beta_1}\cdot F_{X_{l-1},\beta_2}\cdots F_{X_{l-1},\beta_n}=F^{'}_{X^{\prime}_{l-1},\widetilde{\tau}(\beta_1)}\cdot F^{'}_{X^{\prime}_{l-1},\widetilde{\tau}(\beta_2)}\cdots F^{'}_{X^{\prime}_{l-1},\widetilde{\tau}(\beta_n)}$$

Consequently we have that $\Phi_{X_{l-1},\sqcup_{i=1}^{l-1}F_{X_{l-1},i}}\sim\Phi_{X^{\prime}_{l-1},\sqcup_{i=1}^{l-1}F_{X^{\prime}_{l-1},i}}$. Furthermore, by iterating the above process, then $\Phi_{X_{\alpha},\sqcup_{i=1}^{\alpha}F_{X_{\alpha},i}}\sim\Phi_{X^{\prime}_{\alpha},\sqcup_{i=1}^{\alpha}F_{X^{\prime}_{\alpha},i}}$ for $\alpha=1,..,l-2$. So as a consequence of Lemma 4.5.6 any two combinatorially marked sequential morphisms combinatorially equivalent preserve the proximity relations. Moreover, $deg(F_i)=deg(F^{'}_{\tau(i)})$ so combinatorially equivalent sequential morphism also preserve degrees.

Conversely assume now that the two combinatorially marked sequences of point blow-ups with $l=l^{'}$ are combinatorially equivalent. We want to prove that their associated combinatorially marked sequential morphisms $(\pi:Z_s\to Z_0,\sqcup_{i=1}^{l}F_i)_{comb}$ and $(\pi^{'}:Z^{'}_s\to Z^{'}_0,\sqcup_{i=1}^{l^{'}}F^{'}_i)_{comb}$ are combinatorially equivalent. First, there exists $\sigma\in S_m$ such that by applying iteratively Theorem 1.2.30 we get

$$H_i=H^*_i-\sum_{\beta\to i}H^*_\beta$$

$$H^{'}_{\sigma(i)}=H^{'*}_{\sigma(i)}-\sum_{\sigma(\beta)\to\sigma(i)}H^{'*}_{\sigma(\beta)}$$

Moreover, as a consequence of the Projection formula 1.6

$$h^*_{\beta_1}\cdot h^*_{\beta_2}\cdots h^*_{\beta_n}\neq 0 \text{ if and only if } \beta_1=\beta_2=...=\beta_n$$

and if $H_i$ is final then $H_i=H^*_i$, so it follows that there exists $\tau\in S_l$ such that

$$deg(F^*_i)=deg(F^{'*}_{\tau(i)})\ \ \forall i=1,..,l$$

Finally, and as a consequence of Theorem 1.2.30

$$h_{\beta_1}\cdot h_{\beta_2}\cdots h_{\beta_n}=(h^*_{\beta_1}-\sum_{\delta\to\beta_1}h^*_\delta)\cdot(h^*_{\beta_2}-\sum_{\delta\to\beta_2}h^*_\delta)\cdots(h^*_{\beta_n}-\sum_{\delta\to\beta_d}h^*_\delta)$$

so we have

$$F_{\beta_1}\cdot F_{\beta_2}\cdots F_{\beta_n}=F^{'}_{\tau(\beta_1)}\cdot F^{'}_{\tau(\beta_2)}\cdots F^{'}_{\tau(\beta_n)}$$

□

**Theorem 4.5.8.** *Given two algebraically marked sequential morphisms* $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{alg}$ *and* $(\pi^{'} : Z_s^{'} \to Z_0^{'}, \sqcup_{i=1}^{l^{'}} F_i^{'})_{alg}$ *, then they are algebraically equivalent over* $K$ *as in Definition 4.2.1 if and only if there exist algebraically marked sequences of point blow-ups* $(Z_s, ..., Z_0, \sqcup_{i=1}^{l} F_i, \pi)_{alg}$ *and* $(Z_s^{'}, ..., Z_0^{'}, \sqcup_{i=1}^{l^{'}} F_i^{'}, \pi^{'})_{alg}$ *associated to* $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{alg}$ *and* $(\pi^{'} : Z_s^{'} \to Z_0^{'}, \sqcup_{i=1}^{l^{'}} F_i^{'})_{alg}$ *respectively such that they are algebraically equivalent over* $K$ *as in Definition 4.2.5.*

*Proof.* If two algebraically marked sequences of point blow-ups are algebraically equivalent, then it follows directly by Definition 4.2.5 that the associated algebraically marked sequential morphisms are algebraically equivalent too.

Now we will prove that if two algebraically marked sequential morphism $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{alg}$ and $(\pi^{'} : Z_s^{'} \to Z_0^{'}, \sqcup_{i=1}^{l^{'}} F_i^{'})_{alg}$ are algebraically equivalent, then there exist algebraically marked sequences of point blow-ups associated to them that are algebraically equivalent too. By Theorem 4.4.2 given a certain sky $Z_s$ associated to an algebraically marked sequential morphism $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{alg}$ , all the algebraically marked sequences of point blow-ups obtained by regular projective contractions are associated to algebraically marked sequential morphisms in the same algebraic equivalence class. Since $(\pi : Z_s \to Z_0, \sqcup_{i=1}^{l} F_i)_{alg}$ and $(\pi^{'} : Z_s^{'} \to Z_0^{'}, \sqcup_{i=1}^{l^{'}} F_i^{'})_{alg}$ are algebraically equivalent, then $\exists \widetilde{K} \subset K$ such that there exist an isomorphism $\widetilde{b} : \widetilde{Z}_s \to \widetilde{Z}_s^{'}$. By applying Proposition 4.4.1 and Proposition 4.5.1 we conclude the result by scalar extension $\times_{Spec(\widetilde{K})} Spec(K)$.

□

# Chapter 5

# Hirzebruch surfaces. A basic example.

In this chapter we recall some technical results about rational ruled surfaces. The first section is devoted to the study of some general properties of vector bundles of rank 2 over curves. In the second section we review some definitions and results about ruled surfaces, and in the third section we focus on the study of rational ruled surfaces, that is Hirzebruch surfaces. Finally, in the last section we give a basic example of a Hirzebruch surface arising as the exceptional divisor of the blow-up of $\mathbb{P}^3$ with center a rational curve. The main references for this chapter are [32], [22] and [14].

## 5.1 Vector bundles of rank 2 over curves

Let $C$ be an algebraic curve of genus $g$ and let $V$ be a vector bundle of rank 2 over $C$.

**Lemma 5.1.1.** *[32, Lemma 1.1.] Degrees of subbundles of $V$ are bounded above.*

**Definition 5.1.2.** *[32, Definition 1.1.] A subbundle $L$ of $V$ is called a **maximal subbundle**[?] of $V$ if and only if $deg(L)$ is maximal. $M(V)$ denotes the maximal degree.*

We know that $V$ has at least one subbundle (see [5]). Hence there always exists a maximal subbundle by Lemma 5.1.1. The following lemma and Corollary 5.1.6 show that a maximal subbundle of $V$ is uniquely determined under some conditions.

**Lemma 5.1.3.** *[32, Lemma 1.2.] If $deg(V) - 2M(V) < 0$, where $deg(V) = \int c_1(V)$, then there is only one maximal subbundle of $V$.*

**Lemma 5.1.4.** *[32, Lemma 1.4.] If $L_1$ and $L_2$ are distinct subbundles of $V$ such that $deg(V) = deg(L_1) + deg(L_2)$, then we have that $V \cong L_1 \oplus L_2$.*

Maximal subbundles of $V$ cannot be isomorphic each other except for some special cases. In fact, we have the following result fully characterized these cases.

**Lemma 5.1.5.** *[32, Lemma 1.5.] If $L_1$ and $L_2$ are distinct maximal subbundles of $V$ and $L_1 \cong L_2$ then $V = L_1 \oplus L_1$.*

**Corollary 5.1.6.** *[32, Corollary 1.6.]*

a *If $deg(V) - 2M(V) = 0$ and if $V$ is indecomposable, then the maximal subbundle of $V$ is unique.*

b *If $deg(V) - 2M(V) = 0$, $V$ is decomposable and if $V \ncong L \oplus L$ for any subbundle $L$ of $V$, then there are only two maximal subbundles of $V$.*

**Remark 5.1.7.** *[32, Remark 1.7.] It is clear that it holds that if $V = L \oplus L$, then $V$ has infinitely many maximal subbundles. But all maximal subbundles are isomorphic to $L$ in the case.*

**Lemma 5.1.8.** *[32, Lemma 1.8.] The integer $deg(V) - 2M(V)$ is bounded above when $V$ ranges over all vector bundles of rank 2 over $X$. In fact, we have*

$$deg(V) - 2M(V) \leq \begin{cases} 2g-1 & \text{if } g \geq 1 \\ 0 & \text{if } g = 0 \end{cases}$$

Now, if we denote by $\epsilon_C$ to the set of the isomorphism classes of vector bundles of rank 2 over $C$, we define the following equivalence relation in $\epsilon_C$.

**Definition 5.1.9.** *[32, Definition 1.2.] $V_1, V_2 \in \epsilon_C$ are called equivalent if and only if there exists a line bundle $L$ such that $V_1 = V_2 \otimes L$. Then we denote this relation by $V_1 \sim V_2$.*

**Remark 5.1.10.** *It is obvious that the relation $\sim$ is an equivalence relation. Let $\mathcal{P}_C$ the quotient set $\epsilon_C/\sim$. Then $\mathcal{P}_C$ can be identified with the set of isomorphism classes of $\mathbb{P}^1-$bundles over $C$ (see Proposition 5.2.2). The class of $V$ in $\mathcal{P}_C$ is denoted by $P(V)$ and $P(V)$ is regarded as a $\mathbb{P}^1-$bundle too.*

**Definition 5.1.11.** *We define $N(V) = deg(V) - 2M(V)$ and $\mathcal{D}(V) = \{det(V) \otimes L^{-2}\}$, where $L$ ranges over all maximal subbundles of $V$, and if $L_1^{-2} \cong L_2^{-2} \cong ... \cong L_r^{-2}$ for $r$ maximal subbundles $L_1, L_2, ..., L_r$, then $det(V) \otimes L_1^{-2}$ is counted $r$ times. The degrees of elements of $\mathcal{D}(V)$ are $N(V)$.*

We have that $N(V)$ and $\mathcal{D}(V)$ verify the following statements.

**Proposition 5.1.12.** *[32, Proposition 1.9.]*

- *a* *$N(V)$ is an integer and is not greater than $g$.*
- *b* *Both $N(V)$ and $\mathcal{D}(V)$ depend only on $P(V)$ (see Remark 5.1.10).*
- *c* *$\mathcal{D}(V)$ contains only one element if one of the following conditions is satisfied:*
  - *(a)* $N(V) < 0$
  - *(b)* *$N(V) = 0$ and $V$ is indecomposable.*
- *d* *$\mathcal{D}(V)$ contains only two elements and they are dual each other if $N(V) = 0$, $V$ is decomposable and $P(V) \neq P(I \oplus I)$, where $I$ denotes a trivial line bundle.*

**Definition 5.1.13.** *[32, Definition 1.3.] A vector bundle $V$ of rank 2 is called of canonical type if $I$ is a maximal subbundle of $V$.*

**Remark 5.1.14.** *It is clear that the class $P(V)$ contains at least one vector bundle of canonical type. Thus, if $P(V)$ has only one vector bundle of canonical type, the classification of $\mathcal{P}_X$ is reduced to that of vector bundles of canonical type. In fact, under a certain condition $P(V)$ determines uniquely a vector bundle of canonical type. But the determination is not always true (see [5, Sect 5.]).*

**Lemma 5.1.15.** *[32, Lemma 1.10.]*

- *a* *Under one of the conditions of Proposition 5.1.12, iii., the class $P(V)$ contains only one vector bundle of canonical type.*

*b If $V = L_1 \oplus L_2$, $N(V) = 0$, (hence $deg(L_1) = deg(L_2)$) and $L_1 \ncong L_2$, then the vector bundles of canonical type in $P(V)$ are $I \oplus (L_2 \otimes L_1^{-1})$ and $(L_1 \otimes L_2^{-1}) \oplus I$.*

Put $\zeta_C^0 = \{(D, \xi)|\ D = \text{ divisor class on} X \text{ with } deg(D) \leq 0,\ \ \xi \in P(H^1(X, L(-D))) \cup \{0\}\}$, where $P(H^1(X, L(-D)))$ is the projective space $H^1(X, L(-D)) - \{0\}/k^*$. Let $\zeta_C$ be the quotient set of $\zeta_C^0$ by the relation such that $(D, \xi)$ and $(D^{'}, \xi^{'})$ are equivalent if and only if $i.$ $D = D^{'}$ and $\xi \cong \xi^{'}$, or $ii.$ $D^{'} = -D$ and $\xi = \xi^{'} = 0$. Then we get the following theorem.

**Theorem 5.1.16.** *[32, Theorem 1 .11.] $\mathcal{P}_X^- = \{P(V)|\ P(V) \in \mathcal{P}_C$ and $N(P(V)) \leq 0\}$ bijectively corresponds to $\zeta_C$.*

## 5.2 Ruled surfaces

In this section, the words "vector bundle" and "locally free sheaf of finite rank" are used interchangeably (see Definition 1.1.2 for the correspondence).

**Definition 5.2.1.** *[22, Definition 2.0] A geometrically ruled surface, or simply **ruled surface**, is a surface $X$, together with a surjective morphism $p : X \to C$ to a (nonsingular) curve $C$, such that the fiber $X_y$ is isomorphic to $\mathbb{P}^1$ for every point $y \in C$, and such that $p$ admits a section (i.e., a morphism $\sigma : C \to X$ such that $p \circ \sigma = id_C$).*

**Proposition 5.2.2.** *[22, Proposition 2.2.] If $p : X \to C$ is a ruled surface, then there exists a locally free sheaf $\mathcal{V}$ of rank 2 on $C$ such that $X \cong P(\mathcal{V})$ over $C$ (see Definition 1.1.6 for the definition of $P(\mathcal{V})$.) Conversely, every such $P(\mathcal{V})$ is a ruled surface over $C$. If $\mathcal{V}$ and $\mathcal{V}^{'}$ are two locally free sheaves of rank 2 on $C$, then $P(\mathcal{V})$ and $P(\mathcal{V}^{'})$ are isomorphic as ruled surfaces over $C$ if and only if there is an invertible sheaf $\mathcal{L}$ on $C$ such that $\mathcal{V}^{'} \cong \mathcal{V} \otimes \mathcal{L}$.*

Note that if $\mathcal{V}^{'} \cong \mathcal{V} \otimes \mathcal{L}$ and we denote by $V^{'}$ and $V$ to the associated vector bundles, respectively, then $V^{'} \sim V$ for the equivalence relation of Definition 5.1.9.

**Proposition 5.2.3.** *[22, Proposition 2.6.] Let $\mathcal{V}$ be a locally free sheaf of rank 2 on the curve $C$, and let $X$ be the ruled surface $P(\mathcal{V})$. Let $\mathcal{O}_X(1)$ be the invertible sheaf $\mathcal{O}_{P(\mathcal{V})}(1)$. Then there is a one-to-one correspondence between sections $\sigma : C \to X$ and* *surjections $\mathcal{V} \to \mathcal{L} \to 0$, where $\mathcal{L}$ is an invertible sheaf on $C$, given by $\mathcal{L} = \sigma^*\mathcal{O}_X(1)$. Under this correspondence, if $\mathcal{N} = ker(\mathcal{V} \to \mathcal{L})$, then $\mathcal{N}$ is an invertible sheaf on $C$, and $\mathcal{N} \cong p_*(\mathcal{O}_X(1) \otimes \mathcal{L}(-D))$, where $D = \sigma(C)$, and $p^*(\mathcal{N}) \cong \mathcal{O}_X(1) \otimes \mathcal{L}(-D)$.*

Then we have the corresponding result in terms of the associated vector bundle $V$.

**Lemma 5.2.4.** *[32, Lemma 1.14.] To give a section of $P(V)$ is equivalent to give a subbundle of $V$.*

Now, we can give the geometric meaning of the invariant $N(P(V))$ defined in the previous section (see Definition 5.1.11).

**Lemma 5.2.5.** *[32, Lemma 1.15] $N(\mathbb{P}(V))$ is the minimum of self-intersection numbers of sections of $\mathbb{P}(V)$.*

**Definition 5.2.6.** *[32, Definition 1.4.] A section $s$ of $\mathbb{P}(V)$ is called a **minimal section** of $\mathbb{P}(V)$ if $s \cot s = N(\mathbb{P}(V))$.*

**Theorem 5.2.7.** *[32, Theorem 1.16.] The set of minimal sections of $P(V)$ is bijective with the set of maximal subbundles of $V$. Moreover, if $S$ is a minimal section of $P(V)$, then $L(\pi(s \cdot s))$ is an element of $\mathcal{D}(P(V))$ and the map: $s \to L(\pi(s \cdot s))$ of the set of minimal sections of $P(V)$ into $\mathcal{D}(P(V))$ is bijective.*

**Corollary 5.2.8.** *[32, Corollary 1.17.] If $N(P(V)) < 0$ or if $N(P(V)) = 0$ and $V$ is indecomposable, then $P(V)$ has only one minimal section. On the other hand, if $N(P(V)) = 0$, $V$ is decomposable and if $P(V)$ is not the trivial bundle, then $P(V)$ has only two minimal sections.*

For the self-intersection number of an arbitrary section, we have the following result.

**Proposition 5.2.9.** *[32, Proposition 1.18.] Let $s$ be a section of $\mathbb{P}(V)$ which is not a minimal section.*

*a If $N(\mathbb{P}(V)) < 0$, then $s \cdot s \geq -N(\mathbb{P}(V))$.*

*b If $N(\mathbb{P}(V)) > 0$, then $s \cdot s \geq 2 + N(\mathbb{P}(V))$.*

*Moreover, if $N(\mathbb{P}(V))$ is even, then $s \cdot s$ is even and if $N(\mathbb{P}(V))$ is odd, then $s \cdot s$ is odd.*

**Remark 5.2.10.** *[32, Remark 1.19.] Let $S$ and $S^{'}$ be distinct sections of $P(V)$ and let $L(S)$ and $L(S^{'})$, be subbundles of $V$ corresponding to $S$ and $S^{'}$ respectively. Then we have that $p(s \cdot s^{'}) \in \left|(detV) \otimes L(S)^{-1} \otimes L(S^{'})^{-1}\right|$, where $p$ is the projection morphism of $P(V)$ and $|L|$ is the complete linear system of a divisor defined by $L$. Thus, $p(s \cdot s) + p(s^{'} \cdot s^{'}) = 2p(s \cdot s^{'})$, if one regards $p(s \cdot s), p(s^{'} \cdot s^{'})$ and $p(s \cdot s^{'})$ as the divisor classes on $X$.*

**Proposition 5.2.11.** *[22, Proposition 2.8.] If $p : X \to C$ is a ruled surface, it is possible to write $X \cong \mathbb{P}(\mathcal{V})$ where $\mathcal{V}$ is a locally free sheaf on $C$ with the property that $H^0(\mathcal{V}) \neq 0$ but for all invertible sheaves $\mathcal{L}$ on $C$ with $deg(\mathcal{L}) < 0$, we have $H^0(\mathcal{V} \otimes \mathcal{L}) = 0$. In this case the integer $\delta = -deg(\mathcal{V})$ is an invariant of $X$. Furthermore in this case there is a section $\sigma_0 : C \to X$ with image $S_0$, such that $\mathcal{L}(S_0) \cong \mathcal{O}_X(1)$.*

The translation of Proposition 5.2.11 in terms of the associated vector bundle $V$ corresponds to Remark 5.1.14, that is, every class $P(V)$ contains at least one element of canonical type.

We write $X \cong P(\mathcal{V})$, where $\mathcal{V}$ satisfies the conditions of Proposition 5.2.11, in which case we say $\mathcal{V}$ is normalized. This does not necessarily determine $\mathcal{V}$ uniquely, but it does determine $deg(\mathcal{V})$. We let $D$ be the divisor on $C$ corresponding to the invertible sheaf $\bigwedge 2\mathcal{V}$, so that $\delta = -deg(D)$. We fix a section $S_0$ of $X$ with $\mathcal{L}(S_0) \cong \mathcal{O}_{P(\mathcal{V})}(1)$. If $B$ is any divisor on $C$, then we denote the divisor $p^*(B)$ on $X$ by $BF$, by abuse of notation. Thus any element of $Pic(X)$ can be written $aS_0 + BF$ with $a \in \mathbb{Z}$ and $B \in Pic(C)$. Any element of $Num(X)$ can be written $aS_0 + bF$ with $a, b \in \mathbb{Z}$.

**Proposition 5.2.12.** *[22, Proposition 2.9.] If $S$ is any section of $X$, corresponding to a surjection $\mathcal{V} \to \mathcal{L} \to 0$ , and if $\mathcal{L} = \mathcal{L}(B)$ for some divisor $B$ on $C$, then $deg(B) = s_0 \cdot s$, and*

$$S = S_0 + (B - D)F$$

*In particular, we have $(s_0)^2 = deg(D) = -\delta$.*

We can rewrite Proposition 5.2.12 in terms of the associated vector bundle $V$. Since any section $S$ is isomorphic to $P(L)$, where $L$ is a line subbundle of $V$, then by Proposition 1.2.27

$$s = \varsigma + c_1(\frac{V}{L})f. \tag{5.1}$$

In particular, if $V$ is of canonical type, then $V = I \oplus L_2$, with $deg(L_2) < 0$. Moreover, by Theorem 5.2.7, minimal sections correspond to maximal subbundles, so

$$s_0 = \varsigma + c_1(\frac{V}{I})f, \tag{5.2}$$

$$= \varsigma + c_1(L_2)f, \tag{5.3}$$

and $s_0 \cdot s_0 = -c_1(L_2) + 2c_1(L_2) = c_1(L_2)$.

**Theorem 5.2.13.** *[22, Theorem 2.12.] Let $X$ be a ruled surface over the curve $C$ of genus $g$, determined by a normalized locally free sheaf $\mathcal{V}$.*

*a If $\mathcal{V}$ is decomposable (i.e., a direct sum of two invertible sheaves) then $\mathcal{V} \cong \mathcal{O}_C \oplus \mathcal{L}$ for some $\mathcal{L}$ with $deg(\mathcal{L}) \leq 0$. Therefore $\delta \geq 0$. All values of $\delta \geq 0$ are possible.*

*b If $\mathcal{V}$ is indecomposable, then $-2g \leq \delta \leq 2g - 2$.*

**Lemma 5.2.14.** *[22, Lemma 2.10.] The canonical divisor $K$ on $X$ is given by*

$$K = -2S_0 + (\mathfrak{K} + D)F,$$

*where $\mathfrak{K}$ is the canonical divisor on $C$.*

**Corollary 5.2.15.** *[22, Corollary 2.11.] For numerical equivalence, we have*

$$K \equiv -2S_0 + (2g - 2 - \delta)F,$$

*and therefore $(k)^2 = 8(1 - g)$.*

**Proposition 5.2.16.** *[22, Proposition 2.20.] Let $X$ be a ruled surface over a curve $C$, with invariant $\delta \geq 0$.*

*a If $Y = aS_0 + bF$ is an irreducible curve, with $Y \neq S_0, F$, then $a > 0$, $b \geq a\delta$.*

*b A divisor $D = aS_0 + bF$ is ample if and only if $a > 0, b > a\delta$.*

**Proposition 5.2.17.** *[22, Proposition 2.21.] Let $X$ be a ruled surface over a curve $C$ of genus $g$, with invariant $\delta < 0$, and assume furthermore either char $k = 0$ or $g \leq 1$.*

*a If $Y = aS0 + bF$ is an irreducible curve with $Y \neq S_0, F$, then either $a = 1$, $b \geq 0$ or $a \geq 2$, $b \geq \frac{1}{2}a\delta$.*

*b A divisor $D = aS_0 + bF$ is ample if and only if $a > 0$, $b > \frac{1}{2}a\delta$.*

## 5.3 Rational ruled surfaces

In this section we study the particular case when $C$ is an algebraic curve of genus 0.

**Lemma 5.3.1.** *[22, Corollary 2.14.] Every locally free sheaf $\mathcal{V}$ of rank 2 on $\mathbb{P}^1$ is decomposable.*

So in particular, there exists just one element of canonical type in $P(V)$.

**Corollary 5.3.2.** *[22, Corollary 2.13.] If $g = 0$, then $\delta \geq 0$, and for each $\delta \geq 0$ there is exactly one rational ruled surface with invariant $\delta$, $\mathbb{F}_\delta$, given by $\mathcal{V} = \mathcal{O} \oplus \mathcal{O}(-\delta)$ over $C \cong \mathbb{P}^1$.*

Moreover, we can particularize Proposition 5.2.12 and Proposition 5.2.16 to the case of rational ruled surfaces.

**Theorem 5.3.3.** *[22, Theorem 2.17.] Let $\mathbb{F}_\delta$, for any $\delta \geq 0$, be the rational ruled surface defined by $\mathcal{V} = \mathcal{O} \oplus \mathcal{O}(-\delta)$ on $C \cong \mathbb{P}^1$. Then:*

*a there is a section $S = S_0 + nF$ if and only if $n = 0$ or $n \geq \delta$. In particular, there is a section $S_1 = S_0 + \delta f$ with $S_0 \cap S_1 = \emptyset$ and $s_1 \cdot s_1 = \delta$;*

*b the linear system $|S_0 + nF|$ is base-point-free if and only if $n \geq \delta$;*

*c the linear system $|S_0 + nF|$ is very ample if and only if $n > \delta$.*

**Corollary 5.3.4.** *[22, Corollary 2.18.] Let $D$ be the divisor $aS_0 + bF$ on the rational ruled surface $\mathbb{F}_\delta$, with $\delta \geq 0$. Then:*

*a $D$ is very ample $\Leftrightarrow$ $D$ is ample $\Leftrightarrow$ $a > 0$ and $b > a\delta$;*

*b the linear system $|D|$ contains an irreducible nonsingular curve $\Leftrightarrow$ it contains an irreducible curve $\Leftrightarrow$ $a = 0, b = 1$ (namely $F$); or $a = 1, b = 0$ (namely $S_0$); or $a > 0, b > a\delta$; or $e > 0, a > 0, b = a\delta$.*

A natural question that arises is: Given a rational ruled surface $\mathbb{F}_\delta$, with $\delta \geq 0$, can we characterize the classes of the irreducible non-singular rational curves on it?

**Proposition 5.3.5.** *Given a Hirzebruch surface $\mathbb{F}_\delta$, then any irreducible non-singular rational curve $C \subset \mathbb{F}_\delta$ is of one of the following types*

*A. either a section of class $S_0 + bF$ with $b = 0$ or $b \geq \delta$,*

*B. or a fiber $F$,*

*C. or a curve of class $2S_0 + 2F$ if $\delta = 1$,*

*D. or a curve of class $aS_0 + F$ with $a > 0$ if $\delta = 0$.*

*Proof.* By the Adjunction formula (see [22, Proposition 1.5.]) and Corollary 5.2.15 we have

$$\begin{aligned} g(C) &= \frac{C \cdot C + K_{\mathbb{F}_\delta} \cdot C}{2} + 1, \\ &= \frac{(as_0 + bf)^2 + (-2s_0 + (-2-\delta)f) \cdot (as_0 + bf)}{2} + 1, \\ &= \frac{-a^2\delta + 2ab + 2a\delta - 2b - 2a - a\delta}{2} + 1, \end{aligned}$$

so if $g(C) = 0$, then the coefficients $a$ and $b$ must be integer solutions of the equation

$$-a^2\delta + 2ab + 2a\delta - 2b - 2a - a\delta = -2. \tag{5.4}$$

Moreover, since $C$ is irreducible and non-singular, then by Corollary 5.3.4 $C$ must be of one of the following types

$$C = \begin{cases} F & \text{if } a = 0 \ b = 1 & (5.5) \\ S_0 & \text{if } a = 1 \ b = 0 & (5.6) \\ aS_0 + bF & \text{if } a > 0 \ b > a\delta & (5.7) \\ aS_0 + a\delta F & \text{if } a > 0 \text{ and } \delta > 0 & (5.8) \end{cases}$$

In the cases 5.5 and 5.6 the equation 5.4 is satisfied for any $\delta$.
In the case 5.6 the equation 5.4 has two types of integer solutions: $a = 1$, $b > \delta$ for any $\delta$, and $a > 0$, $b = 1$ for $\delta = 0$.
In the case 5.8 the equation 5.4 has as integer solutions: $a = 1$, $b = \delta$ for any $\delta > 0$, and $a = 2$, $b = 2$ for $\delta = 1$. □

**Proposition 5.3.6.** *[34, Proposition 5.] If $X$ is a rational ruled surface, then $X$ is an $F_\delta$ with a $\delta$. If, furthermore, $X$ has another structure as a ruled surface, then $\delta = 0$ and has no more such structure.*

## 5.4 A basic example

Let $C \subset \mathbb{P}^3$ be a smooth rational curve of degree $\gamma$ , and let $\pi : Z_1 \to Z_0 = \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ with center $C$, so that we have the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{j} & Z_1 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \pi} \\ C & \xrightarrow{i} & Z_0 \end{array}$$

Thus $E_1 \cong P(N_{C/\mathbb{P}^3})$, that is $E_1$ is a rational ruled surface. Some natural question that arise are: the value of $\delta$ such that $E_1 \cong \mathbb{F}_\delta$, is it uniquely determined? Otherwise, which values of $\delta$ are admissible?

In this basic example the image of the $n-$ary intersection form $\mathcal{I}_{Z_s,E}$ (see Definition 2.4.1) consists of just one value, that of $(e_1)^3$. Since $N_{E_1/Z_1} \cong \mathcal{O}(-1)$, then by Proposition 1.2.33

$$\begin{aligned} e_1 \cdot e_1 \cdot e_1 &= j_*(1) \cdot j_*(1) \cdot j_*(1) \\ &= j_*(\varsigma^2) \end{aligned}$$

As a consequence of Theorem 1.2.25 $j_*(\varsigma^2) = j_*(-\varsigma \cdot c_1(N_{C/\mathbb{P}^3}))$, so we have

$$e_1 \cdot e_1 \cdot e_1 = -deg(c_1(N_{C/\mathbb{P}^3}). \tag{5.9}$$

Finally, $deg(c_1(N_{C/\mathbb{P}^3}) = 4\gamma - 2$, so the value $(e_1)^3$ just give us information about the degree $\gamma$ of the curve $C$.

We can then reformulate the questions above. Given a smooth rational curve $C \subset \mathbb{P}^3$ of a certain degree $\gamma$, let $\pi : Z_1 \to \mathbb{P}^3$ be the blow-up with center $C$ and let $E_1$ be the exceptional divisor. the value of $\delta$ such that $E_1 \cong \mathbb{F}_\delta$, is it uniquely determined? Otherwise, which values of $\delta$ are admissible? The answer to these questions can be found in the following results

**Theorem 5.4.1.** *[14, Theorem 4.] Given any integer $\gamma \geq 4$, there exist smooth rational curves $C$ of degree $n$ in $\mathbb{P}^3$ with normal bundle isomorphic to $\mathcal{O}_C(2\gamma - 1 - a) \oplus \mathcal{O}_C(2\gamma - 1 + a)$ if and only if $|a| \leq \gamma - 4$.*

It is a well know result that any smooth rational space curve $C$ of degree $1 < n \leq 3$ is contained in a smooth quadric.

**Theorem 5.4.2.** *[26, Theorem 1.] If $C$ is a smooth curve on a smooth quadric $Q$ then one of the following possibilities holds:*

- *a $C$ has bidegree $(a, a)$. Then it is the complete intersection with a surface of degree $a$. Its normal bundle splits as $N_{C/\mathbb{P}^3} \cong \mathcal{O}_C(2) \oplus \mathcal{O}_C(a)$.*
- *b $C$ is rational but not a hyperplane section. Then $C$ has bidegree $(1, a)$ or $(a, 1)$ with $a \neq 1$. Its normal bundle is $N_{C/\mathbb{P}^3} \cong \mathcal{O}(2a + 1) \oplus \mathcal{O}(2a + 1)$.*

Finally, any smooth rational space curve $C$ of degree $n = 1$ is contained in the intersection of two hyperplane sections of $\mathbb{P}^3$, so in this particular case $N_{C/\mathbb{P}^3} \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$.

We sum up the previous results in the following proposition.

**Proposition 5.4.3.** *Let $\mathcal{C} \subset \mathbb{P}^3$ be an irreducible rational smooth curve of degree $\gamma$. Then its normal bundle $N_{\mathcal{C}/\mathbb{P}^3}$ satisfies*

$$N_{\mathcal{C}/\mathbb{P}^3} \cong \begin{cases} \mathcal{O}(1) \oplus \mathcal{O}(1) & \text{if } \gamma = 1, \quad (5.10) \\ \mathcal{O}(4) \oplus \mathcal{O}(2) & \text{if } \gamma = 2, \quad (5.11) \\ \mathcal{O}(5) \oplus \mathcal{O}(5) & \text{if } \gamma = 3, \quad (5.12) \\ \mathcal{O}(2\gamma - 1 - a) \oplus \mathcal{O}(2\gamma - 1 + a) & \text{if } \gamma \geq 4, \quad (5.13) \end{cases}$$

*where $|a| \leq \gamma - 4$.*

So even in this basic example, if $deg(C) \geq 5$, then the value of $\delta$ such that $E_1 \cong \mathbb{F}_\delta$ is not uniquely determined, and it depends on the embedding $i_C : C \to \mathbb{P}^3$.

# Chapter 6

# Sequences of point and rational curve blow-ups in dimension 3.

In this chapter we will focus on the study of sequences of blow-ups at either points or rational curves, with $Z_0 \cong \mathbb{P}^3$. The first section is devoted to establish some numerical properties of rational curves when considered as centers of blow-ups. In the second section we establish a numerical criterion that characterizes final divisors in terms of some relations defined over the Chow group of zero-cycles of its sky $A_0(Z_s)$. Finally, in the last section of this chapter we give a presentation of the Chow ring of the sky of a sequence of point and rational curve blow-ups $A^\bullet(Z_s)$ considering the total transforms of the exceptional components as generators.

## 6.1 Some algebraic and numerical properties when rational curves are allowed as centers of blow-ups

Let $Z_s \xrightarrow{\pi_s} Z_{s-1} \xrightarrow{\pi_{s-1}} \cdots \xrightarrow{\pi_2} Z_1 \xrightarrow{\pi_1} Z_0$ be a sequence of blow-ups as in Definition 2.1.1 with $Z_0 \cong \mathbb{P}^3$ and such that the centers $C_i$ are either points o rational curves.

**Definition 6.1.1.** *We will say that a curve $\mathcal{C}^\alpha \subset Z_\alpha$ is an* ***"old" curve*** *if there exists* *a curve $\mathcal{C} \subset Z_0$ such that $\mathcal{C}^\alpha$ is the strict transform of $\mathcal{C}$ by the sequential morphism $\pi_{\alpha,0} : Z_\alpha \to Z_0$. Otherwise, we say that $\mathcal{C}^\alpha$ is a **"new" curve**.*

*We will say that an "old" curve $\mathcal{C}^\alpha \subset Z_\alpha$ is unmodified with respect to the sequential morphism $\pi_{\alpha,0} : Z_\alpha \to Z_0$ if the following condition holds:*

$$C_\beta \cap \mathcal{C}^\beta = \emptyset, \tag{6.1}$$

*for $\beta = 1, ..., \alpha$.*

*On the other hand, we will say that an "old" curve $\mathcal{C}^\alpha \subset Z_\alpha$ is modified by the blow-up $\pi_{\alpha+1} : Z_{\alpha+1} \to Z_\alpha$ if*

- a *either $C_{\alpha+1} \in \mathcal{C}^\alpha$, with dim$(C_{\alpha+1}) = 0$, and in this case we will say that $\pi_{\alpha+1}$ is a modification of type $I$,*
- b *or $C_{\alpha+1} \cap \mathcal{C}^\alpha \neq \emptyset$, with dim$(C_{\alpha+1}) = 1$, and in this case we will refer to $\pi_{\alpha+1}$ as a modification of type $II$.*

**Lemma 6.1.2.** *Let $\mathcal{C}^\alpha \subset Z_\alpha$ be an "old" curve and let $\pi_{\alpha+1} : Z_{\alpha+1} \to Z_\alpha$ be a blow-up verifying $C_{\alpha+1} \cap \mathcal{C}^\alpha \neq \emptyset$. Then one of the following conditions is satisfied:*

- A *either dim$(C_{\alpha+1}) = 0$, $C_{\alpha+1} \in \mathcal{C}^\alpha$, and in this case we have*

$$N_{\mathcal{C}^{\alpha+1}/Z_{\alpha+1}} \cong \pi^*_{\alpha+1}(N_{\mathcal{C}^\alpha/Z_\alpha}) \otimes \mathcal{O}(-E^{\alpha+1}_{\alpha+1} \cap \mathcal{C}^{\alpha+1}), \tag{6.2}$$

- B *or dim$(C_{\alpha+1}) = 1$, $C_{\alpha+1}$ intersects $\mathcal{C}^\alpha$ improperly, and in this case*

$$N_{\mathcal{C}^{\alpha+1}/Z_{\alpha+1}} \cong \pi^*_{\alpha+1}(N_{\mathcal{C}^\alpha/T}) \oplus \pi^*_{\alpha+1}(N_{T/Z_\alpha}|_{\mathcal{C}^\alpha}) \otimes \mathcal{O}(-E^{\alpha+1}_{\alpha+1} \cap T^{\alpha+1} \cap \mathcal{C}^{\alpha+1}), \tag{6.3}$$

  *where $T \subset Z_\alpha$ denotes a smooth surface such that $C_{\alpha+1}, \mathcal{C}^\alpha \subset T$ and booth are regularly embedded.*

*Proof.* In the first case A, the expression for the normal bundle of the strict transform $\mathcal{C}^{\alpha+1}$ in $Z_{\alpha+1}$ follows directly from Proposition 1.1.19.

Let us now consider the case arising when $C_{\alpha+1}$ and $\mathcal{C}^\alpha$ intersect improperly. First of all, let $T \subset Z_\alpha$ be a smooth surface such that $C_{\alpha+1}, \mathcal{C}^\alpha \subset T$ are both regularly embedded. As a consequence of Proposition [20, Proposition 19.1.5]), we have the following exact sequence

$$0 \to N_{\mathcal{C}^\alpha/T} \to N_{\mathcal{C}^\alpha/Z_\alpha} \to N_{T/Z_\alpha}|_{\mathcal{C}^\alpha} \to 0, \tag{6.4}$$

and the splitting

$$N_{\mathcal{C}^\alpha/Z_\alpha} \cong N_{\mathcal{C}^\alpha/T} \oplus N_{T/Z_\alpha}|_{\mathcal{C}^\alpha}. \tag{6.5}$$

Moreover, it follows from proposition 1.1.19 that the normal bundle of the strict transform of $T$, that we denote by $T^{\alpha+1}$, satisfies

$$N_{T^{\alpha+1}/Z_{\alpha+1}} \cong \pi^*_{\alpha+1}(N_{T/Z_{\alpha+}}) \otimes \mathcal{O}(-T^{\alpha+1} \cap E^{\alpha+1}_{\alpha+1}), \tag{6.6}$$

and, again, it follows from Proposition [20, Proposition 19.1.5]) that

$$N_{\mathcal{C}^{\alpha+1}/Z_{\alpha+1}} \cong N_{\mathcal{C}^{\alpha+1}/T^{\alpha+1}} \oplus N_{T^{\alpha+1}/Z_{\alpha+1}}|_{\mathcal{C}^{\alpha+1}}. \tag{6.7}$$

so, since $N_{\mathcal{C}^{\alpha+1}/T^{\alpha+1}} \cong \pi^*_{\alpha+1}(N_{\mathcal{C}^\alpha/T})$ we can conclude that

$$N_{\mathcal{C}^{\alpha+1}/Z_{\alpha+1}} \cong \pi^*_{\alpha+1}(N_{\mathcal{C}^\alpha/T}) \oplus \pi^*_{\alpha+1}(N_{T/Z_\alpha}|_{\mathcal{C}^\alpha}) \otimes \mathcal{O}(-E^{\alpha+1}_{\alpha+1} \cap T^{\alpha+1} \cap \mathcal{C}^{\alpha+1}), \tag{6.8}$$

□

**Definition 6.1.3.** *We will say that the blow-up $\pi_\alpha : Z_\alpha \to Z_{\alpha-1}$ corresponding at the $\alpha-$level of a sequence of blow-ups is*

*a an* ***extrinsic elementary modification*** *with respect to an irreducible exceptional component $E_i^{\alpha-1}$ if*

$$dim(C_\alpha) = 1 \;\; and \;\; C_\alpha \to C_i,$$

*b or an* ***intrinsic elementary modification*** *with respect to an irreducible exceptional component $E_i^{\alpha-1}$ if*

$$dim(C_\alpha) = 1 \;\; and \;\; C_\alpha \xrightarrow{t} C_i,$$

*c or a* ***mixed elementary modification*** *with respect to an irreducible exceptional component $E_i^{\alpha-1}$ if*

$$dim(C_\alpha) = 0 \;\; and \;\; C_\alpha \to C_i.$$

**Remark 6.1.4.** *Note that an extrinsic elementary modification just varies the normal bundle* $N_{E_i^{\alpha-1}/Z_{\alpha-1}}$ *of the irreducible exceptional component* $E_i^{\alpha-1}$*, whereas an intrinsic elementary modification varies only the normal bundle* $N_{W/E_i^{\alpha-1}}$ *of the subvarieties* $W \subset E_i^{\alpha-1}$ *satisfying* $W \cap C_\alpha \neq \emptyset$*. A mixed elementary modification produces a variation on both,* $N_{E_i^{\alpha-1}/Z_{\alpha-1}}$ *and* $N_{W/E_i^{\alpha-1}}$*.*

**Theorem 6.1.5.** *Let* $E_i, E_j \subset Z_s$ *be both final divisors for the sequential morphism* $\pi : Z_s \to Z_0$*. Then* $E_i \cap E_j \neq \emptyset$ *if and only if* $E_i$ *is proximate to* $E_j$ *and* $E_j$ *is* $t-$*proximate to* $E_i$*, or vice versa.*

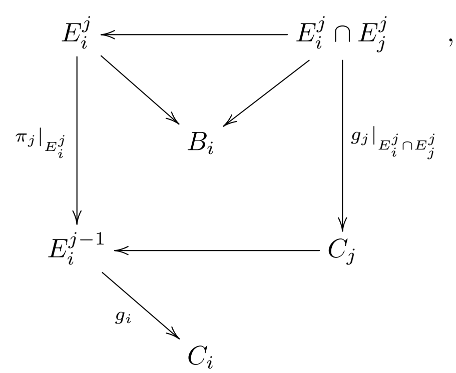
*Proof.* Let us suppose that $E_i \cap E_j \neq \emptyset$. Then one of the following conditions must be satisfied:

A either $E_i \to E_j$ and $E_j \to E_i$,

B or $E_i \xrightarrow{t} E_j$ and $E_j \xrightarrow{t} E_i$,

C or $E_i \to E_j$ and $E_j \xrightarrow{t} E_i$ (or vice versa).

In the case A, let us consider a sequence of blow-ups associated to the sequential morphism $\pi : Z_s \to Z_0$ realizing $E_j$ as a final divisor. If we focus on the blow-up corresponding at the $j-$level, that is $\pi_j : Z_j \to Z_{j-1}$, and we restrict it to $E_i^{j-1}$, then we have the following diagram:




(6.9)

where $E_i^j \cap E_j^j$ must be a projective subbundle of $E_i^j$, since $E_i$ is final too. In order to continue with the proof, we need to distinguish between the two following cases:

A.i $dim(C_i) = 0$,

A.ii $dim(C_i) = 1$.

In the particular case A.i, let us suppose that the morphism $\pi_j|_{E_i^j} : E_i^j \to E_i^{j-1}$ is a divisorial contraction, that is $dim(C_j) = 0$. Then, as a consequence of [21, Theorem 1.1.], $\pi_j|_{E_i^j} : E_i^j \to E_i^{j-1}$ is a Sarkisov link of type $I$, so there must exists a morphism $h_i : B_i \to C_i$, giving $B_i$ a projective bundle structure over $C_i$. Moreover, by Proposition 1.1.17 $E_i^j$ is isomorphic to a projective bundle over $\mathbb{P}^1$, and the pull back of the hyperplane class $\varsigma_i \in A^1(E_i^i)$ satisfies (see Corollary 1.2.26):

$$\pi_j|_{E_i^j}^*(\varsigma_i) = \left[E_i^j \cap E_j^j\right] + f, \tag{6.10}$$

where $f$ denotes the class of a fiber $F \subset E_i^j$. Finally, as a consequence of Proposition 1.1.19, we have that

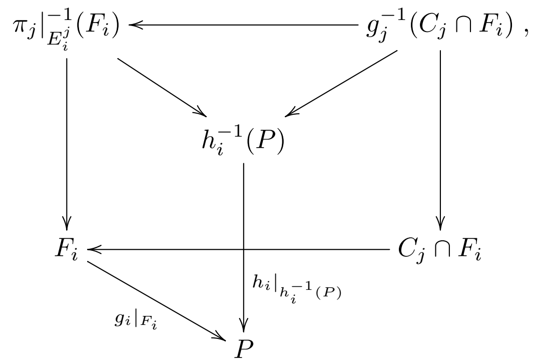
$$N_{E_i^j/Z_j} \cong \pi_j|_{E_i^j}^*(N_{E_i^{j-1}/Z_{j-1}}) \otimes \mathcal{O}(-E_i^j \cap E_j^j), \tag{6.11}$$

so the necessary condition to be final $N_{E_i^j/Z_j} \cong \mathcal{O}_{E_i^j}(-1)$ (see Proposition 2.3.7) fails to be true.
Now, if $dim(C_j)$ is 1, then the class $[C_j] \in A^1(E_i^{j-1})$ must be an integer multiple of the hyperplane section $\varsigma_i$, so applying Proposition 1.1.19 we have

$$N_{E_i^j/Z_j} \cong \pi_j|_{E_i^j}^*(N_{E_i^{j-1}/Z_{j-1}}) \otimes \mathcal{O}(-E_i^j \cap E_j^j), \tag{6.12}$$

and the necessary condition to be final $N_{E_i^j/Z_j} \cong \mathcal{O}_{E_i^j}(-1)$ does not hold.

In the particular case A.ii, let us suppose that the morphism $\pi_j|_{E_i^j} : E_i^j \to E_i^{j-1}$ is a divisorial contraction. Then, as a consequence of [21, Theorem 1.1.], $\pi_j|_{E_i^j} : E_i^j \to E_i^{j-1}$ should be a Sarkisov link of type $I$, so there must exists a morphism $h_i : B_i \to C_i$ giving $B_i$ a projective bundle structure over $C_i$. Now, if we denote by $F_i$ to $g_i^{-1}(P)$, where $P \in C_i$ is a point, then we have the following diagram, obtained just by restriction of the previous one:




(6.13)

Note that it must be verified $dim(C_j \cap F_i) = 0$, so $dim(C_j) = 1$ and $\pi_j|_{E_i^j} : E_i^j \to E_i^{j-1}$ is not a divisorial contraction any more. Moreover, since we are supposing that $E_i$ is final too, then $C_j$ must be isomorphic to a projective line subbundle of $E_i^{j-1}$, that is $[C_j] = \varsigma_i + c_1(N_{C_i/Z_{i-1}}/L_{i+1})f$, where $L_{i+1} \subset N_{C_i/Z_{i-1}}$ denotes a line subbundle. As a consequence of Proposition 1.1.19 it holds

$$N_{E_i^j/Z_j} \cong \pi_j|^*_{E_i^j}(N_{E_i^{j-1}/Z_{j-1}}) \otimes \mathcal{O}(-E_i^j \cap E_j^j), \tag{6.14}$$

so the necessary condition to be final $N_{E_i^j/Z_j} \cong \mathcal{O}_{E_i^j}(-1)$ is not satisfied in this case too.

Now, let us consider the case B. Since both $E_i \xrightarrow{t} E_j$ and $E_j \xrightarrow{t} E_i$, then $E_i \cap E_j$ must be isomorphic to a fiber of both $E_i$ and $E_j$. Let us suppose that both are finals and let $(Z_0, ..., Z_s, \pi)$ be a sequence realizing $E_i$ as a final divisor. Then there must exists a regular projective contraction $f : Z_s \to X_{s-1}$ such that $f(E_i) = C_i$. However, if we consider the restriction $f|_{E_j}$ then it can not be a regular projective contraction any more since it defines a contraction of $E_i \cap E_j$ whereas $N_{E_i \cap E_j/E_j} \ncong \mathcal{O}_{E_i \cap E_j}(-1)$.

Finally, in the case C, that is $E_i \to E_j$ and $E_j \xrightarrow{t} E_i$ (or vice versa), it follows from Theorem 2.3.9 that both are finals. □

**Corollary 6.1.6.** *Let $E_i \subset Z_s$ be a final divisor that is not a minimal surface, that is such that there exists a rational curve $\mathcal{C} \in E_i$ verifying $[\mathcal{C}] \cdot [\mathcal{C}] = -1$. Let us suppose that there exist just one index $\beta$ such that $E_i \cap E_\beta \neq \emptyset$, verifying $e_i \cdot (e_\beta)^2 = -1$. Then the following conditions are satisfied:*

*a* $E_i \cong \mathbb{F}_1$,

*b* *there exists a sequence of blow-ups associated to the sequential morphism* $\pi : Z_s \to Z_0$ *such that:*

*b.i* $E_i^i \cong \mathbb{P}^2$, *that is* $dim(C_i) = 0$,

*b.ii* *there exists a center* $C_\beta$, *with* $dim(C_\beta) = 1$, *such that* $C_\beta \xrightarrow{t} C_i$ *and* $E_i^\alpha \cap E_\beta^\alpha$ *is an irreducible curve,*

*b.iii* *and* $E_i \cong E_i^\alpha$.

*Proof.* Firstly, since $E_i$ is final and $e_i \cdot (e_\beta)^2 = -1$, then by our hypothesis about the centers of the sequence of blow-ups we have $E_i \cong \mathbb{F}_1$. Moreover, it follows directly from Theorem 6.1.5 that if $E_i$ is final and there exists an index $\beta$ verifying $e_i \cdot (e_\beta)^2 = -1$, then there must exist a sequence of blow-ups associated to the sequential morphism $\pi : Z_s \to Z_0$ that satisfies conditions b.i, b.ii and b.iii. □

**Proposition 6.1.7.** *Let* $E_i \subset Z_s$ *be a final divisor for the sequential morphism* $\pi : Z_s \to Z_0$, *and let* $j, k$ *be two indices such that* $E_i \cap E_j \neq \emptyset$ *and* $E_i \cap E_k \neq \emptyset$. *Then one of the following characterizations is satisfied, where* $\eta_j, \eta_k \in \mathbb{Z}_+$*:*

*I* *either* $dim(C_i) = 1$, *with* $C_i$ *proximate to* $C_j$ *and* $t-$*proximate to* $C_k$ *(or vice versa), and in this case we have that:*

$$(e_i + e_j)^2 \cdot e_i = 0$$
$$(e_i)^2 \cdot e_k = -\eta_k$$
$$e_i \cdot (e_k)^2 = 0$$
$$e_i \cdot e_j \cdot e_k = \eta_k$$

*II* *or* $dim(C_i) = 1$, *with* $C_i$ $t-$*proximate to both* $C_j$ *and* $C_k$, *and then the following* *relations are verified:*

$$e_i \cdot (e_j)^2 = 0$$
$$(e_i)^2 \cdot e_j = -\eta_j$$
$$e_i \cdot (e_k)^2 = 0$$
$$(e_i)^2 \cdot e_j = -\eta_k$$
$$e_i \cdot e_j \cdot e_k = 0$$

III *or $dim(C_i) = 1$, with $C_i$ proximate to both $C_j$ and $C_k$, and in this case we have that:*

$$(e_i + e_j)^2 \cdot e_i = 0$$
$$(e_i + e_k)^2 \cdot e_i = 0$$
$$e_i \cdot (e_k)^2 = -e_i \cdot (e_j)^2,$$
$$(e_i)^2 \cdot e_k = (e_i)^2 \cdot e_j + e_i \cdot (e_j)^2,$$

IV *or $dim(C_i) = 0$, with $C_i$ proximate to both $C_j$ and $C_k$, and tehn the following relations are verified:*

$$(e_i + e_j) \cdot e_i = (e_i + e_k) \cdot e_i = 0$$
$$(e_i)^2 \cdot e_j = (e_i)^2 \cdot e_k = -1$$
$$e_i \cdot (e_j)^2 = e_i \cdot (e_k)^2 = 1$$
$$e_i \cdot e_j \cdot e_k = 1$$

*Proof.* Let $E_i$ be a final divisor, with $dim(C_i) = 1$, and let $\alpha$ be an index such that $C_i \to C_\alpha$, that is $E^i_\alpha = \pi_i^*(E^{i-1}_\alpha) - E^i_i$. Then, as a consequence of Proposition 1.2.33 we have that

$$\pi_i^*(e^{i-1}_\alpha) \cdot e^i_i = j_{E^i_i *}(m_\alpha f),$$

so

$$(\pi_i^*(e^{i-1}_\alpha))^2 \cdot e^i_i = j_{E^i_i *}(0\varsigma_i \cdot f),$$

and the relation $(e_i + e_\alpha)^2 \cdot e_i = 0$ is satisfied.
Let us consider now an index $\lambda$ such that $C_i \xrightarrow{t} C_\lambda$, that is $E^i_\lambda = \pi_i^*(E^{i-1}_\lambda)$ . Then, as a consequence of Proposition 1.2.33 we have

$$\pi_i^*(e^{i-1}_\lambda) \cdot e^i_i = j_{E^i_i *}(\eta_\lambda f),$$

so

$$
\begin{aligned}
(\pi_i^*(e_\lambda^{i-1}))^2 \cdot e_i^i &= j_{E_i^i*}(0\varsigma_i \cdot f),\\
\pi_i^*(e_\lambda^{i-1}) \cdot (e_i^i)^2 &= j_{E_i^i*}(-\eta_\lambda \varsigma_i \cdot f)
\end{aligned}
$$

and the relations $(e_\lambda)^2 \cdot e_i = 0$ and $e_\lambda \cdot (e_i^2) = -\eta_\lambda$ are verified.

In the case I, since $C_i \to C_j$, and $C_i \xrightarrow{t} C_k$ we have that

$$
\begin{aligned}
e_i^i \cdot e_j^i \cdot e_k^i &= e_i^i \cdot (\pi_i^*(e_j^{i-1}) - e_i^i) \cdot \pi_i(e_k^{i-1}),\\
&= e_i^i \cdot (\pi_i^*(e_j^{i-1}) \cdot \pi_i(e_k^{i-1}) - (e_i^i)^2 \cdot \pi_i(e_k^{i-1}),\\
&= j_{E_i^{i*}}(\eta_k \varsigma_i \cdot f),
\end{aligned}
$$

so the relation $e_i \cdot e_j \cdot e_k = \eta_k$ also holds.

In the case II, since $C_i \xrightarrow{t} C_j$ and $C_i \xrightarrow{t} C_k$, then it follows that

$$
\begin{aligned}
e_i^i \cdot e_j^i \cdot e_k^i &= e_i^i \cdot \pi_i^*(e_j^{i-1}) \cdot \pi_i(e_k^{i-1}),\\
&= j_{E_i^*}(\eta_j \eta_k f \cdot f),\\
&= j_{E_i^*}(0\varsigma_i \cdot f),
\end{aligned}
$$

so the relation $e_i \cdot e_j \cdot e_k = 0$ is also satisfied.

In the case III, since $C_i \to C_j$ and $C_i \to C_k$, then as a consequence of Lemma 2.2.1 we have that the normal bundle $N_{C_i/Z_{i-1}}$ has the following splitting:

$$
N_{C_i/Z_{i-1}} \cong N_{C_i/E_j^{i-1}} \oplus N_{C_i/E_k^{i-1}}.
$$

Moreover, as a consequence of Lemma 2.3.8, the classes of $E_i \cap E_j$ and $E_i \cap E_k$ in $A^1(E_i)$ satisfy

$$
\begin{aligned}
[E_i \cap E_j] &= \varsigma_i + c_1\left(\frac{N_{C_i/E_j^{i-1}} \oplus N_{C_i/E_k^{i-1}}}{N_{C_i/E_j^{i-1}}}\right) f,\\
[E_i \cap E_k] &= \varsigma_i + c_1\left(\frac{N_{C_i/E_j^{i-1}} \oplus N_{C_i/E_k^{i-1}}}{N_{C_i/E_k^{i-1}}}\right) f.
\end{aligned}
$$

so

$$\begin{aligned}
e_i \cdot (e_j)^2 &= j_{E_i*}((\varsigma_i + c_1(N_{C_i/E_k^{i-1}})f)^2),\\
&= j_{E_i*}((c_1(N_{C_i/E_k^{i-1}}) - c_1(N_{C_i/E_j^{i-1}}))\varsigma_i \cdot f);\\
(e_i)^2 \cdot e_j &= j_{E_i*}(-\varsigma_i \cdot (\varsigma_i + c_1(N_{C_i/E_k^{i-1}})f)),\\
&= j_{E_i*}((c_1(N_{C_i/E_j^{i-1}}))\varsigma_i \cdot f);\\
e_i \cdot (e_k)^2 &= j_{E_i*}((\varsigma_i + c_1(N_{C_i/E_j^{i-1}})f)^2),\\
&= j_{E_i*}((-c_1(N_{C_i/E_k^{i-1}}) + c_1(N_{C_i/E_j^{i-1}}))\varsigma_i \cdot f);\\
(e_i)^2 \cdot e_k &= j_{E_i*}(-\varsigma_i \cdot (\varsigma_i + c_1(N_{C_i/E_j^{i-1}})f)),\\
&= j_{E_i*}((c_1(N_{C_i/E_k^{i-1}})\varsigma_i \cdot f).
\end{aligned}$$

We can conclude then that the following relations are verified:

$$\begin{aligned}
(e_i + e_j)^2 \cdot e_i &= 0\\
(e_i + e_k)^2 \cdot e_i &= 0\\
e_i \cdot (e_k)^2 &= -e_i \cdot (e_j)^2,\\
(e_i)^2 \cdot e_k &= (e_i)^2 \cdot e_j + e_i \cdot (e_j)^2.
\end{aligned}$$

In the case IV, since $C_i \to C_j$ and $C_i \to C_k$, with $dim(C_i) = 0$, then the classes of $E_i \cap E_j$ and $E_i \cap E_k$ in $A^1(E_i)$ satisfy

$$\begin{aligned}
[E_i \cap E_j] &= \varsigma_i,\\
[E_i \cap E_k] &= \varsigma_i,
\end{aligned}$$

so we have that

$$\begin{aligned}
(e_i^2) \cdot e_j &= j_{E_i*}(\varsigma_i \cdot -\varsigma_i),\\
&= j_{E_i*}((-1)(\varsigma_i)^2);\\
e_i \cdot e_j^2 &= j_{E_i*}((\varsigma_i)^2),\\
&= j_{E_i*}(1(\varsigma_i)^2);\\
(e_i^2) \cdot e_k &= j_{E_i*}(\varsigma_i \cdot -\varsigma_i),\\
&= j_{E_i*}((-1)(\varsigma_i)^2);\\
e_i \cdot e_k^2 &= j_{E_i*}((\varsigma_i)^2),\\
&= j_{E_i*}(1(\varsigma_i)^2);
\end{aligned}$$

and

$$e_i \cdot e_j \cdot e_k = j_{E_i *}(\varsigma_i \cdot \varsigma_i),$$
$$= j_{E_i *}(1(\varsigma_i)^2).$$

It follows then that the following relations are satisfied:

$$(e_i + e_j) \cdot e_i = (e_i + e_k) \cdot e_i = 0,$$
$$(e_i)^2 \cdot e_j = (e_i)^2 \cdot e_k = -1,$$
$$e_i \cdot (e_j)^2 = e_i \cdot (e_k)^2 = 1,$$
$$e_i \cdot e_j \cdot e_k = 1.$$

□

In order to motivate the following results, let us suppose that $E_i$ is an irreducible exceptional component that is not final with respect to the sequential morphism $\pi : Z_s \to Z_0$, and let $j$ be an index such that either $E_j \to E_i$ or $E_j \xrightarrow{t} E_i$. Now, if we consider the blow-up corresponding to the $(j-1)-$level of a sequence realizing the sequential morphism $\pi : Z_s \to Z_0$, that is $\pi_j : Z_j \to Z_{j-1}$, since $E_j^j$ is final for the sequential morphism $\pi_{j,0} : Z_j \to Z_0$, then one of the following characterizations is verified:

a either $E_j^j$ is proximate to $E_i^j$, and then it is satisfied

$$(e_i^j + e_j^j)^2 \cdot e_j^j = 0. \tag{6.15}$$

b or $E_j^j$ is $t-$proximate to $E_i^j$ and then the following relations hold

$$(e_j^j)^2 \cdot e_i^j = -\eta_j, \tag{6.16}$$
$$e_j^j \cdot (e_i^j)^2 = 0, \tag{6.17}$$

with $\eta \in \mathbb{Z}_+$.

We are interested in studying the very special configurations where

a either the following relation is verified:

$$(e_i^j + e_j^j)^2 \cdot e_i^j = 0, \tag{6.18}$$

b or the following relations hold:

$$(e_i^j)^2 \cdot e_j^j = -\eta_i, \tag{6.19}$$

$$e_i^j \cdot (e_j^j)^2 = 0. \tag{6.20}$$

By adding Equations 6.15 and 6.18 we have that

$$(e_i^j + e_j^j)^3 = 0, \tag{6.21}$$

and as a consequence of the Projection formula 1.6, it follows that

$$(e_i^{j-1})^3 = 0, \tag{6.22}$$

so we will be interested in characterizing configurations leading to $(e_i^{j-1})^3 = 0$ (Proposition 6.1.8).

On the other hand, if relations (6.16) and (6.18) are both satisfied, then this implies that $(e_i^{j-1})^3 = \eta > 0$, so we will be also interested in characterizing configurations leading to $(e_i^{j-1})^3 = \eta > 0$ (Propositions 6.1.9 and 6.1.10).

Finally we will focus our attention to configurations where the relations $(e_i^j)^2 \cdot e_j^j = -\eta_i$ and $e_i^j \cdot (e_j^j)^2 = 0$ hold (Proposition 6.1.13).

**Proposition 6.1.8.** *Let $E_i^\alpha$ be an irreducible exceptional component of a sequence of point and rational curve blow-ups $(Z_\alpha, ..., Z_0, \pi)$, such that:*

*a* $(e_i^\alpha)^3 = 0$,

*b and there exits at most one index $\beta$, with $E_i^\alpha \cap E_\beta^\alpha \neq \emptyset$, such that $e_i^\alpha \cdot (e_\beta^\alpha)^2 < 0$.*

*Then one of the following charaterizations holds:*

*A either $E_i^\alpha$ is not final, $dim(C_i) = 0$ and there exists just an index $\beta$ such that $C_\beta \to C_i$, with $dim(C_\beta) = 0$,*

*B or $E_i^\alpha$ is not final, and $C_i$ is an unmodified "old" curve, that is there exists at least one index $\gamma$ such that $C_\gamma \to C_i$, with $C_\gamma$ non isomorphic to a generic fiber $F$ of $E_i^i$,*

*C or $E_i^\alpha$ is final, $C_i$ is an "old" curve, and there exists at least one index $\beta$ such that*

*C.i either $C_\beta \to C_i$, with $C_\beta$ isomorphic to a generic fiber of $E_i^i$,*

*C.ii or $dim(C_\beta) = 1$, and $C_i \xrightarrow{t} C_\beta$,*

*that is, $C_i$ is a modified "old" curve,*

*D or $E_i^\alpha$ is final or not final, and $C_i$ is a "new" curve.*

*Moreover, in the case D, if $E_i^\alpha$ is final, that is $E_i^\alpha \cong E_i^i$, and there exists just one index $\beta$ such that $C_i \to C_\beta$, then one of the following set of relations is satisfied: either*

$$\begin{aligned} (e_\beta^\alpha)^2 \cdot e_i^\alpha &= -2a, \\ (e_i^\alpha)^2 \cdot e_\beta^\alpha &= a, \end{aligned} \tag{6.23}$$

*or*

$$\begin{aligned} (e_\beta^\alpha)^2 \cdot e_i^\alpha &= 2a, \\ (e_i^\alpha)^2 \cdot e_\beta^\alpha &= -a, \end{aligned} \tag{6.24}$$

*Proof.* Firstly, we consider the case where $dim(C_i) = 1$. Let us suppose that $C_i$ is an unmodified "old curve" and $E_i^\alpha$ is a final divisor. Since $C_i$ is unmodified then, as a consequence of Theorem 6.1.5 it does not exist any index $\beta$, with $C_\beta \to C_i$, with $C_\beta$ isomorphic to a generic fiber of $F \subset E_i^i$. Now, by Proposition 2.3.7 $N_{E_i^\alpha/Z_\alpha} \cong \mathcal{O}_{E_i^i}(-1)$, so we have that:

$$\begin{aligned} (e_i^\alpha)^3 &= \pi_{\alpha,i}^*(j_{i*}(\varsigma^2)), \\ &= -c_1(N_{C_i/Z_{i-1}}), \end{aligned}$$

but according to Proposition 5.4.3, since $C_i$ is an unmodified "old" curve, it follows that $(e_i^\alpha) < 0$, so condition a does not hold.
Now, we study the case where $dim(C_i) = 0$. Let us suppose that there exists an index $\lambda$ such that $C_\lambda \to C_i$, with $dim(C_\lambda) = 1$. Then $C_\lambda$ must a rational curve and $[C_\lambda] = \gamma_\lambda \varsigma_i$ in $A(E_i^i)$, where $\varsigma_i$ denotes the hyperplane class and $\gamma$ the degree of $C_\lambda$. By Proposition 1.1.19 we have that

$$\begin{aligned} N_{E_i^\lambda/Z_\lambda} &\cong \pi_{i,\lambda}^*(N_{E_i^i/Z_i}) \otimes \mathcal{O}(-E_i^\lambda \cap E_\lambda^\lambda), \\ &\cong \mathcal{O}(-1-\gamma_\lambda), \end{aligned}$$

Now, as a consequence of Proposition 1.2.22, it follows that

$$(e_i^{\lambda}*)^3 = (-1-\gamma_{\lambda})^2 \neq 0.$$

Finally, let us suppose that $E_i^{\alpha}$ is final and there exists just one index $\beta$ such that $C_i \to C_{\beta}$. Firstly, since $(e_i^{\alpha})^3 = 0$, then we know that $E_i^{\alpha} = P(\mathcal{O}(a) \oplus \mathcal{O}(-a))$, with $a \in \mathbb{Z}$. Now, as a consequence of [20, Proposition 19.1.5] we have the following splitting of the normal bundle $N_{C_i/Z_{i-1}}$:

$$N_{C_i/Z_{i-1}} = N_{C_i/E_{\beta}^{i-1}} \oplus N_{E_{\beta}^{i-1}/Z_{i-1}}|_{C_i}.$$

Moreover, since $E_i^{\alpha}$ is final, then we know that the class $\left[E_i^{\alpha} \cap E_{\beta}^{\alpha}\right] \in A^1(E_i^{\alpha})$ corresponds to the section associated to the line subbundle $N_{C_i/E_{\beta}^{i-1}}$, so by Proposition 1.2.22 and Lemma 2.2.1

$$\begin{aligned}
(e_{\beta}^{\alpha})^2 \cdot e_i^{\alpha} &= j_{E_{\beta}^{\alpha}*}(c_1(N_{E_{\beta}^{\alpha}/Z_{\alpha}}) \cdot \left[E_i^{\alpha} \cap E_{\beta}^{\alpha}\right]_{\beta}), \\
&= j_{E_i^{\alpha}*}((\varsigma_i + c_1(N_{E_{\beta}^{i-1}/Z_{i-1}}|_{C_i})f)^2), \\
&= j_{E_i^{\alpha}*}((-c_1(N_{C_i/E_{\beta}^{i-1}}) + c_1(N_{E_{\beta}^{i-1}/Z_{i-1}}|_{C_i}))\varsigma_i \cdot f),
\end{aligned}$$

and

$$\begin{aligned}
(e_i^{\alpha})^2 \cdot e_{\beta}^{\alpha} &= j_{E_i^{\alpha}*}((\varsigma_i + c_1(N_{E_{\beta}^{i-1}/Z_{i-1}}|_{C_i})f) \cdot (-\varsigma_i)), \\
&= j_{E_i^{\alpha}*}(c_1(N_{C_i/E_{\beta}^{i-1}})\varsigma_i \cdot f).
\end{aligned}$$

Thus, we can conclude that either relations 6.23 or relations 6.24 hold. □

**Proposition 6.1.9.** *Let $E_i^{\alpha}$ be an irreducible exceptional component of the sequence of point and rational curve blow-ups $(Z_s, ..., Z_0, \pi)$, such that:*

*a* $(e_i^{\alpha})^3 = 1$,

*b and there exists at most one index $\beta$, with $E_i^{\alpha} \cap E_{\beta}^{\alpha} \neq \emptyset$, such that $e_i^{\alpha} \cdot (e_{\beta}^{\alpha})^2 < 0$.*

*Then one of the following characterizations is verified:*

*A either $E_i^{\alpha}$ is final, with $dim(C_i) = 0$,*

*B or $E_i^{\alpha}$ is not final, with $dim(C_i) = 0$, and there exists at least an index $\beta$ such that $C_{\beta} \xrightarrow{t} C_i$, where the number of connected components $\#\left\{E_{\beta}^{\alpha} \cap E_i^{\alpha}\right\} > 1$,* *C or $E_i^\alpha$ is final, $C_i$ is a modified "old" curve with a modification of type $II$, that is, there exists at least one index $\beta$ such that $dim(C_\beta) = 1$, and $C_i \xrightarrow{t} C_\beta$,*

*D or $E_i^\alpha$ is not final, with $C_i$ a modified "old" curve,*

*E or $E_i^\alpha$ is final or not final, and $C_i$ is a "new" curve.*

*Proof.* Firstly, let us suppose that $E_i^i \cong \mathbb{P}^2$, that is $dim(C_i) = 0$, and there exists an index $\lambda$, such that $C_\lambda \to C_i$, with $dim(C_\lambda) = 1$. Then, since $E_i^i \cong \mathbb{P}^2$, we have that $[C_\lambda] = \gamma_\lambda \varsigma_i$ in $A^1(E_i^i)$, where $\gamma \in \mathbb{Z}_+$. It follows now from Proposition 1.1.19 that

$$\begin{aligned} N_{E_i^\lambda/Z_\lambda} &\cong \pi_{\lambda,i}^*(N_{E_i^i/Z_i}) \otimes \mathcal{O}(-E_i^\lambda \cap E_\lambda^\lambda), \\ &\cong \mathcal{O}_{E_i^\lambda}(-1-\gamma_\lambda) \end{aligned}$$

so by Proposition 1.2.22

$$(e_i^\lambda)^3 = j_{E_i^\lambda *}((c_1(N_{E_i^\lambda/Z_\lambda}))^2) \neq 1.$$

Now, let us suppose that $E_i^\alpha$ is final, with $dim(C_i) = 1$, and $C_i$ is either an unmodified "old" curve or a modified "old" curve with modifications just of type $I$. Then, by Proposition 2.3.7 we have that $N_{E_i^\alpha/Z_\alpha} \cong \mathcal{O}_{E_i^\alpha}(-1)$, so it follows that

$$\begin{aligned} (e_i^\alpha)^3 &= j_{E_i^\alpha *}((c_1(N_{E_i^\alpha/Z_\alpha}))^2), \\ &= j_{E_i^\alpha *}((-\varsigma_i)^2), \\ &= j_{E_i^\alpha *}((-c_1(N_{C_i/Z_{i-1}})\varsigma_i \cdot f). \end{aligned}$$

In the former case, that is $C_i$ is unmodified, it follows from Proposition 5.4.3 that $(e_i^\alpha)^3$ is even, so condition a is not satisfied. In the latter case, that is $C_i$ has been modified with modification of type $I$, as a consequence of Proposition 1.1.19 we have that $c_1(N_{C_i/Z_{i-1}})$ is even too, so condition a is not satisfied either. □

**Proposition 6.1.10.** *Let $E_i^\alpha$ be an irreducible exceptional component of a sequence of point and rational curve blow-ups $(Z_\alpha, ..., Z_0, \pi)$, such that:*

*a $(e_i^\alpha)^3 = \eta$, with $\eta > 1$,*

*b and there exits at most one index $\beta$, with $E_i^\alpha \cap E_\beta^\alpha \neq \emptyset$, such that $e_i^\alpha \cdot (e_\beta^\alpha)^2 < 0$.*

*Then one of the following charaterizations holds:*

A *either $E_i^\alpha$ is not final, where $dim(C_i) = 0$, and there exists at least one index $\beta$, with $dim(C_\beta) = 1$, such that $C_\beta \to C_i$,*

B *or $E_i^\alpha$ is final, with $dim(C_i) = 1$, where $C_i$ is a modified "old" curve, and there exists at least one index $\beta$ such that*

B.i *either $C_\beta \to C_i$, with $C_\beta$ isomorphic to a generic fiber of $E_i^i$ if $\eta$ is even,*

B.ii *or $dim(C_\beta) = 1$, and $C_i \xrightarrow{t} C_\beta$,*

C *or $E_i^\alpha$ is not final, where $C_i$ is an"old" curve, that is there exists at least one index $\gamma$ such that $C_\gamma \to C_i$,*

D *or $E_i^\alpha$ is final or not final, where $C_i$ is a "new" curve.*

*Proof.* Firstly, we consider the case where $dim(C_i) = 0$, that is $E_i^i \cong \mathbb{P}^2$. Let us suppose that there exists just one index $\beta$ such that $C_\beta \to C_i$, with $dim(C_\beta) = 0$ . Then, as a consequence of Proposition 1.1.17 and Corollary 1.2.26, $E_i^\beta \cong \mathbb{F}_1$. Moreover, if we denote by $\widetilde{\varsigma}_i$ to $c_1(\mathcal{O}_{E_i^\beta}(1))$, it is verified that $c_1(N_{E_i^\beta/Z_\beta}) = -2\widetilde{\varsigma}_i + f$ in $A^1(E_i^\beta)$, so by Proposition 1.2.22

$$\begin{aligned}(e_i^\beta)^3 &= j_{E_i^\beta *}((-2\widetilde{\varsigma}_i + f)^2),\\ &= j_{E_i^\beta *}(0\widetilde{\varsigma}_i \cdot f).\end{aligned}$$

Thus, we can conclude that condition a does not hold. Now let us suppose that $dim(C_i) = 1$. If $E_i^\alpha$ is final, then it follows from Proposition 2.3.7 that $N_{E_i^\alpha/Z_\alpha} \cong \mathcal{O}_{E_i^\alpha}(-1)$, so we have that

$$\begin{aligned}(e_i^\alpha)^3 &= j_{E_i^\alpha *}((-\varsigma_i)^2),\\ &= j_{E_i^\alpha *}(-c_1(N_{C_i/Z_{i-1}})\varsigma_i \cdot f).\end{aligned}$$

However, as a consequence of Proposition 5.4.3, if $C_i$ is an unmodified "old curve" then $c_1(N_{C_i/Z_{i-1}} > 0$, so $(e_i^\alpha)^3 < 0$ and condition a is not satisfied. □

**Proposition 6.1.11.** *Let $E_i^\alpha$ be an irreducible exceptional component of a sequence of point and rational curve blow-ups $(Z_0, ..., Z_\alpha, \pi)$, verifying the following conditions:*

*a* $E_i^\alpha$ *is final, with base a "new" curve* $C_i$*,*

*b* $E_i^\alpha \cong \mathbb{F}_0$*,*

*c and* $(e_i^\alpha)^3 > 0$

*Then, the cardinal of the set of indexes* $\gamma$ *such that such that* $E_i^\alpha \cap E_\gamma^\alpha \neq \emptyset$ *must be at least two. Moreover, if* $\#\{\gamma\} = 2$*, that is, this set just contains two indexes* $\eta$ *and* $\kappa$*, and* $C_i \to C_\eta$ *then* $C_\kappa$ *can not be a point* $P$*, that is* $dim(C_\kappa) = 1$*.*

*Proof.* Firstly, since $E_i^\alpha$ is final, then by Proposition 2.3.7 we have that

$$(e_i^\alpha)^3 = j_{E_i^\alpha *}((-\varsigma_i)^2), \tag{6.25}$$

$$= j_{E_i^\alpha *}(-c_1(N_{C_i/Z_{i-1}})\varsigma_i \cdot f). \tag{6.26}$$

Moreover, as $E_i^\alpha \cong \mathbb{F}_0$ by our hypothesis, then there exists an integer $a \in \mathbb{Z}$ such that $N_{C_i/Z_{i-1}} \cong \mathcal{O}(a) \oplus \mathcal{O}(a)$. Now, it follows from 6.25 that

$$(e_i^\alpha)^3 = -2a,$$

so condition c implies that $a < 0$. Let us suppose that the cardinal set of indexes $\#\{\gamma\}$ such that $E_i^\alpha \cap E_\gamma^\alpha \neq \emptyset$ is 1. Since $C_i$ is a "new" curve, then there must exist a curve $C_\lambda$ such that $C_i \to C_\lambda$. Moreover, since $N_{C_i/Z_{i-1}} \cong \mathcal{O}(a) \oplus \mathcal{O}(a)$, with $a < 0$, it follows that $C_i$ must be isomorphic to the unique section of $E_\lambda^{i-1}$ with negative self-intersection. Since we are just considering as centers of blow-ups rational curves, we have that $E_\lambda^\lambda \cong \mathbb{F}_{\delta_\lambda}$, so in particular there is and integer $b \in \mathbb{Z}$ such that $N_{C_\lambda/Z_{\lambda-1}} \cong \mathcal{O}(b) \oplus \mathcal{O}(b - \delta_\lambda)$. Then, $C_i$ is associated with the line subbundle of maximal degree and its class $[C_i]$ in $A^1(E_\lambda^\lambda)$ satisfies:

$$[C_i] = \varsigma_\lambda + (b - \delta_\lambda)f,$$

so

$$c_1(N_{E_\lambda^{i-1}/Z_{i-1}}|_{C_i} = b$$

$$[C_i] \cdot [C_i] = -\delta_\lambda.$$

This lead us to conclude that $b < 0$. As a consequence of Proposition 5.4.3, $C_\lambda$ must be a "new curve", that is there must exists a curve $C_\mu$ such that $C_\lambda \to C_\mu$ . As we are supposing that the set of indexes $\{\gamma\}$ such that $E_i^\alpha \cap E_\gamma^\alpha \neq \emptyset$ just contains $\lambda$, then $C_\lambda$

must be isomorphic to the section of $E_\mu^\mu$ with negative self-intersection, and reasoning in an analogous manner as above, we conclude that $N_{C_\mu/Z_{\mu-1}} \cong \mathcal{O}(c) \oplus \mathcal{O}(c-\delta_\mu)$, with $c < 0$. The hypothesis considering $\gamma$ as the only index such that $E_i^\alpha \cap E_\gamma^\alpha \neq \emptyset$ led us to a sequence of centers $\{C_\nu\}$ verifying $N_{C_\nu/Z_{\nu-1}} \cong \mathcal{O}(n_\nu) \oplus \mathcal{O}(n_\nu - \delta_\nu)$, with $n_\nu < 0$, which is an absurd as a consequence of Proposition 5.4.3. $\square$

**Remark 6.1.12.** *Sometimes, given a rational curve $C$ contained in an irreducible exceptional component $E_i^\alpha$, for some technical reasons our interest is focus on the self-intersection number $[C]\cdot[C]$, where by $[C]$ we denote its equivalence class in $A^1(E_i^\alpha)$. As a consequence, when applying Proposition 1.1.19 to compute $N_{E_i^{\alpha+m}}$ in the sequence $Z_{\alpha+m} \xrightarrow{\pi_{\alpha+m}} Z_{\alpha+m-1} \xrightarrow{\pi_{\alpha+m-1}} \cdots \xrightarrow{\pi_{\alpha+2}} Z_{\alpha+1} \xrightarrow{\pi_{\alpha+1}} Z_\alpha$, where $C_{\alpha+1} = C$ and $C_{\alpha+j} = E_i^{\alpha+j-1} \cap E_{\alpha+j-1}^{\alpha+j-1}$ for $j = 2, \ldots, m$, with a slight abuse of notation, we write:*

$$N_{E_i^{\alpha+m}} \cong \pi^*_{\alpha+m,\alpha}(N_{E_i^\alpha/Z_\alpha}) \otimes \mathcal{O}(-C)^{\otimes m}.$$

**Proposition 6.1.13.** *Let $E_i^\alpha$ be an irreducible exceptional component of a sequence of point and rational curve blow-ups $(Z_0, ..., Z_\alpha, \pi)$, verifying the following conditions:*

*a* $C_\alpha \to C_i$,

*b* *the following relations are satisfied*

$$(e_i^\alpha)^2 \cdot e_\alpha^\alpha = -\eta, \tag{6.27}$$

$$e_i^\alpha \cdot (e_\alpha^\alpha)^2 = 0, \tag{6.28}$$

*where $\eta \in \mathbb{Z}_+$, with $\eta > 1$,*

*c* *and there exists at most one index $\beta$, with $E_i^\alpha \cap E_\beta^\alpha \neq \emptyset$, such that $e_i^\alpha \cdot (e_\beta^\alpha)^2 < 0$.*

*Then there must exist at least one index $\gamma \neq \alpha$ such that $E_i^\alpha \cap E_\gamma^\alpha \neq \emptyset$. In fact, one of the following characterizations is verified:*

*A.i* *either $E_i^{\alpha-1} \cong \mathbb{F}_0$ is final, with base a modified "old" curve $C_i$ with a modification of type $I$, that is, such that there exists at least one index $\beta$, with $C_\beta \to C_i$, where $C_\beta$ is isomorphic to a generic fiber of $E_i^i$, and $C_\alpha \cong S_0$,*

*A.ii* *or $E_i^{\alpha-1} \cong \mathbb{F}_0$ is final, with base an "new" curve $C_i$, and $C_\alpha \cong S_0$,* B.i *or* $E_i^{\alpha-1} \cong \mathbb{F}_1$ *is not final, with* $dim(C_i) = 0$*, and* $C_\alpha$ *is isomorphic to a generic fiber* $F$ *of* $E_i^{\alpha-1}$,

B.ii *or* $E_i^{\alpha-1}$ *is not final, it is a birational model of* $\mathbb{F}_1$*, and there exists just one index* $\beta$ *such that*

i. *either* $dim(C_\beta) = 0$, $C_\beta \to C_i$*, with* $C_\beta \in S_1$,

ii. *or* $dim(C_\beta) = 1$, $C_\beta \xrightarrow{t} C_i$*, with* $E_i^{\beta-1} \cap C_\beta \in S_1$,

*and* $C_\alpha \cong S_1^\beta$,

C.i *or* $E_i^{\alpha-1}$ *is not final, it is a birational model of* $\mathbb{F}_\delta$*,and there exists just one index* $\beta$*, with* $dim(C_\beta) = 1$, $C_\beta \xrightarrow{t} C_i$*, verifying* $\#\left\{C_\beta \cap E_i^{\beta-1}\right\} = \delta + 2n$, $E_i^{\beta-1} \cap C_\beta \in S_{\delta+2n}$*, and* $C_\alpha \cong S_{\delta+2n}^\beta$,

C.ii *or* $E_i^{\alpha-1} \cong \mathbb{F}_\delta$ *is not final, there exists at least one index* $\beta$ *such that* $C_\beta \to C_i$*, and* $C_\alpha$ *is isomorphic to a fiber* $F$ *of* $E_i^i$.

*Proof.* By our hypothesis $E_i^\alpha$ is a birational model of either $\mathbb{P}^2$ or $\mathbb{F}_\delta$, with $\delta \in \mathbb{Z}_+$. Moreover, if we denote by $C_{i,\alpha} = E_i^\alpha \cap E_\alpha^\alpha$, and $[C_{i,\alpha}] \in A^1(E_i^\alpha$ to its corresponding class, then condition 6.28 implies that:

$$[C_{i,\alpha}] \cdot [C_{i,\alpha}] = 0.$$

By considering condition c, it follows from Proposition 5.3.5 that one of the following characterizations holds:

A either $E_i^{\alpha-1}$ is a birational model of $\mathbb{F}_0$ and $C_\alpha \cong S_0$,

B or $E_i^{\alpha-1}$ is a birational model of $\mathbb{F}_1$, and there exists just one index $\beta$ such that

B.i either $dim(C_\beta) = 0$, $C_\beta \to C_i$ and $C_\beta \in S_1$,

B.ii or $dim(C_\beta) = 1$, $C_\beta \xrightarrow{t} C_i$ and $E_i^{\beta-1} \cap C_\beta \in S_1$,

with $C_\alpha \cong S_1^\beta$,

C or $E_i^{\alpha-1}$ is a birational model of $\mathbb{F}_\delta$, with $\delta \in \mathbb{Z}_+$, and

C.i either there exists just one index $\beta$, with $dim(C_\beta) = 1$, $C_\beta \xrightarrow{t} C_i$, verifying $\#\left\{C_\beta \cap E_i^{\beta-1}\right\} = \delta + 2n$, $E_i^{\beta-1} \cap C_\beta \in S_{\delta+2n}$, and $C_\alpha \cong S_{\delta+2n}^\beta$, C.ii or $C_\alpha$ is isomorphic to a fiber $F$ of $E_i^i$,

Now, let us suppose that the set of indexes $\{\gamma\}$, such that $E_i^\alpha \cap E_\gamma^\alpha \neq \emptyset$, just contains the index $\alpha$. Then, one of the following characterizations is verified:

A.iii either $E_i^{\alpha-1} \cong \mathbb{F}_0$ is final, with base $C_i$ an unmodified "old" curve, and $C_\alpha \cong S_0$,

A.iv or $E_i^{\alpha-1} \cong \mathbb{F}_0$ is not final, with base $C_i$ an unmodified "old" curve, there exists just one index $\beta$ such that $C_\beta \cong S_0$ and $C_\alpha \cong E_i^{\alpha-1} \cap E_\beta^{\alpha-1}$.

C.iii or $E_i^\alpha \cong \mathbb{F}_\delta$ is final, with base $C_i$ an unmodified "old" curve, and $C_\alpha$ is isomorphic to a fiber of $E_i^i$,

C.iv or $E_i^\alpha \cong \mathbb{F}_\delta$ is final, with base $C_i$ a modified curve, with a simply modification of type $I$, that is there exists just one index $\beta$ such that $C_\beta$ is isomorphic to a fiber of $E_i^i$, and $C_\alpha \cong E_i^{\alpha-1} \cap E_\beta^{\alpha-1}$,

Note that we are not considering the case where $E_i^\alpha \cong \mathbb{F}_0$ is not final, with base a "new curve" $C_i$, since as a consequence of Proposition 6.1.11 the cardinal of the set of indexes $\#\{\mu\}$ verifying $E_i^{\alpha-1} \cap E_\mu^{\alpha-1} \neq \emptyset$ is at least two.
In the case A.iii, since $E_i^{\alpha-1}$ is final, then by Proposition 2.3.7 we have that $N_{E_i^{\alpha-1}/Z_{\alpha-1}} \cong \mathcal{O}_{E_i^{\alpha-1}}(-1)$. Moreover, as $C_i$ is an unmodified "old" curve, and $E_i^{\alpha-1} \cong \mathbb{F}_0$, then $N_{C_i/Z_{i-1}} \cong \mathcal{O}(a) \otimes \mathcal{O}(a)$, for some $a \in \mathbb{Z}_+$. Now, by Proposition 1.1.19, we know that $N_{E_i^\alpha/Z_\alpha} \cong \pi_\alpha^*(N_{E_i^{\alpha-1}/Z_{\alpha-1}}) \otimes \mathcal{O}(-E_i^\alpha \cap E_\alpha^\alpha)$, and since $C_\alpha \cong S_0$, then $[E_i^\alpha \cap E_\alpha^\alpha] = \varsigma_i + af$ in $A^1(E_i^\alpha)$, then by Proposition 1.2.22 we have that

$$\begin{aligned}(e_i^\alpha)^2 \cdot e_\alpha^\alpha &= j_{E_i^\alpha *}((\varsigma_i + af) \cdot (-2\varsigma_i - af)),\\ &= j_{E_i^\alpha *}(a\varsigma_i \cdot f).\end{aligned}$$

As a result, $(e_i^\alpha)^2 \cdot e_\alpha^\alpha = a > 0$, so condition 6.27 fails to be true.
In the case A.iv, we know from the previous case that $c_1(N_{E_i^{\alpha-1}/Z_{\alpha-1}}) = -2\varsigma_i - af$. If we apply again Proposition 1.1.19 we have that $N_{E_i^\alpha/Z_\alpha} \cong \pi_\alpha^*(N_{E_i^{\alpha-1}/Z_{\alpha-1}}) \otimes \mathcal{O}(-E_i^\alpha \cap E_\alpha^\alpha)$, and since $C_\alpha \cong S_0$ too, then reasoning in an analogous manner we have that

$$\begin{aligned}(e_i^\alpha)^2 \cdot e_\alpha^\alpha &= j_{E_i^\alpha *}((\varsigma_i + af) \cdot (-3\varsigma_i - 2af)),\\ &= j_{E_i^\alpha *}(a\varsigma_i \cdot f).\end{aligned}$$

It follows that $(e_i^\alpha)^2 \cdot e_\alpha^\alpha = a > 0$, so in this case condition 6.27 does not hold either.
In the case C.iii, since $E_i^\alpha$ is final, then from Proposition 2.3.7 we know that $N_{E_i^\alpha/Z_\alpha} \cong \mathcal{O}_{E_i^\alpha}(-1)$, and since $C_\alpha \cong F_i$, by Proposirion 1.2.22 we have that

$$(e_i^\alpha)^2 \cdot e_\alpha^\alpha = j_{E_i^\alpha *}(f \cdot (-\varsigma_i)). \tag{6.29}$$

As a result, $(e_i^\alpha)^2 \cdot e_\alpha^\alpha = -1 > -\eta$, so condition 6.27 does not hold. Finally, in the case C.iv, since $E_i^\alpha$ is final too, then we can proceed in an analogous manner to the previous case. we can conclude then that in this case $(e_i^\alpha)^2 \cdot e_\alpha^\alpha = -1 > -\eta$ too, so condition 6.27 is not satisfied either. □

## 6.2 Final divisors: Numerical characterization

This section is devoted to give a numerical characterization of final divisors for sequential morphisms associated to sequences of point and rational curve blow-ups. Before proving the main results, that is Propositions 6.2.6, 6.2.7 and Theorem 6.2.11, we introduce some auxiliary technical results that will be used on their corresponding proofs.

**Proposition 6.2.1.** *Let $E_i^\alpha$ be an irreducible exceptional component of a sequence of point and rational curve blow-ups $(Z_0, ..., Z_\alpha, \pi)$, verifying the following conditions:*

*a $E_i^\alpha$ is final, with base a "new" curve $C_i$,*

*b there exists two indexes $j$ and $k$, such that $C_i \to C_j$ and $C_i \to C_k$,*

*c $E_i^\alpha \cong \mathbb{F}_0$,*

*d and $(e_i^\alpha)^3 > 0$.*

*Then there exists another index $\gamma$, verifying $E_i^\alpha \cap E_\gamma^\alpha, E_j^\alpha \cap E_\gamma^\alpha, E_k^\alpha \cap E_\gamma^\alpha \neq \emptyset$.*

*Proof.* To begin with, let us suppose that $C_i = E_j^{i-1} \cap E_k^{i-1}$ and there not exists any other index $\gamma$ such that $E_j^{i-1} \cap E_\gamma^{i-1}, E_k^{i-1} \cap E_\gamma^{i-1}, E_j^{i-1} \cap E_k^{i-1} \cap E_\gamma^{i-1} \neq \emptyset$. Since $E_j^{i-1} \cap E_k^{i-1} \neq \emptyset$, then some of the following characterizations holds:

A either $C_j \to C_k$ (or vice versa), and

A.i either $dim(C_j) = 0$ $(dim(C_k) = 0)$,

A.ii or $C_j$ $(C_k)$ is an exceptional curve,

A.iii or $dim(C_j) = 1$ $(dim(C_k) = 1)$ but it is not an exceptional curve, that is (see Proposition 5.3.5)

A.iii.i if $\delta \neq 0, 1$, then $C_j$ $(C_k)$ is isomorphic to either a section or a generic fiber of $E_k^k \cong P(\mathcal{O}(a) \oplus \mathcal{O}(a - \delta))$,

A.iii.ii if $\delta = 1$, then $C_j$ $(C_k)$ is isomorphic to either a section, or a generic fiber of $E_k^k \cong P(\mathcal{O}(a) \oplus \mathcal{O}(a-1))$ or to the pull-back of a conic by the morphism $\pi : \mathbb{F}_1 \to \mathbb{P}^2$

A.iii.iii if $\delta = 0$, then $C_j$ $(C_k)$ is isomorphic to either a section, or a generic fiber of $E_k^k \cong P(\mathcal{O}(a) \oplus \mathcal{O}(a))$ or to a rational curve $\mathcal{C}$ whose class $[\mathcal{C}] = n\varsigma_k + (na + 1)f \in A^1(E_k^k)$.

B or $C_j \xrightarrow{t} C_k$ (or viceversa).

In all cases, it follow from Lemma 2.2.1, that the normal bundle of $C_i$, $N_{C_i/Z_{i-1}}$, verifies the following splitting:

$$N_{C_i/Z_{i-1}} = N_{C_i/E_j^{i-1}} \oplus N_{C_i/E_k^{i-1}}.$$

In the subcase A.i, it follows directly that

$$N_{C_i/Z_{i-1}} = \mathcal{O}(1) \oplus \mathcal{O}(-1),$$

so condition c does not hold.

In the subcase A.ii, as a consequence of Proposition 1.1.19 we have

$$N_{E_k^j/Z_j} \cong \pi_{j,k}^*(N_{E_k^k/Z_k}) \otimes \mathcal{O}(-\mathcal{C}_e)^{\otimes m_j},$$

where $m_j \in \mathbb{Z}_+$, with $m_j \geq 1$, so

$$N_{C_i/E_j^{i-1}} \cong N_{E_k^j/Z_j}|_{C_i} \cong \mathcal{O}(m_j + 1).$$

Since $N_{C_i/E_k^{i-1}} \cong \mathcal{O}(-1)$, we can conclude that $E_i^i \ncong \mathbb{F}_0$, so condition c is not satisfied either.

In the subcase A.iii, as a consequence of Proposition 1.1.19 we have that

$$N_{E_k^j/Z_j} \cong \pi_{j,k}^*(N_{E_k^k/Z_k}) \otimes \mathcal{O}(-E_k^j \cap E_j^j)^{\otimes m_j},$$

where $m_j \in \mathbb{Z}_+$, with $m_j \geq 1$.
The case where $C_j$ is isomorphic to a generic fiber is equivalent to the case where $C_k \xrightarrow{t} C_j$ by Theorem 6.1.5, so we will focus on the remaining cases. Firstly, let us suppose that $C_j$ is isomorphic to a section of $E_k^{j-1}$. Since $E_k^{j-1} = P(\mathcal{O}(a) \oplus \mathcal{O}(a - \delta_k))$, then either

$$[C_j] = \varsigma_k + (a - \delta_k)f \in A^1(E_{j-1}^k),$$

or

$$[C_j] = \varsigma_k + (a + n_j)f \in A^1(E_{j-1}^k),$$

where $n_j \in \mathbb{Z}_+$. As a consequence, by Proposition 1.2.22 and Lemma 2.2.1 we have that either

$$\begin{aligned}
(e_k^j)^2 \cdot e_j^j &= j_{E_k^j *}((\varsigma_k + (a - \delta_k)f) \cdot (-(1 + m_j)\varsigma_k - m_j(a - \delta_k)f)), \\
&= j_{E_k^j *}((a + m_j\delta_k)\varsigma_k \cdot f); \\
e_k^j \cdot (e_j^j)^2 &= j_{E_j^j *}(c_1(N_{E_j^j/Z_j}) \cdot \left[E_k^j \cap E_j^j\right]_j), \\
&= j_{E_k^j *}((\varsigma_k + m_j(a - \delta_k)f)^2), \\
&= j_{E_k^j *}((-\delta_k)\varsigma_k \cdot f);
\end{aligned}$$

or

$$\begin{aligned}
(e_k^j)^2 \cdot e_j^j &= j_{E_k^j *}((\varsigma_k + (a + n_j)f) \cdot (-(1 + m_j)\varsigma_k - m_j(a + n_j)f)), \\
&= j_{E_k^j *}((a - (1 + m_j)\delta_k - (1 + 2m_j)n_j)\varsigma_k \cdot f); \\
e_k^j \cdot (e_j^j)^2 &= j_{E_j^j *}(c_1(N_{E_j^j/Z_j}) \cdot \left[E_k^j \cap E_j^j\right]_j), \\
&= j_{E_k^j *}((\varsigma_k + m_j(a + n_j)f)^2), \\
&= j_{E_k^j *}((\delta_k + 2n_j)\varsigma_k \cdot f).
\end{aligned}$$

Thus, it follows that either $N_{C_i/Z_{i-1}} \cong \mathcal{O}(a + m_j\delta) \oplus \mathcal{O}(-\delta)$ or $N_{C_i/Z_{i-1}} \cong \mathcal{O}((a - (1 + m_j)\delta - (1 + 2m_j)n) \oplus (\delta + 2n)$. In the former case $E_i^i \cong \mathbb{F}_0$ only if $a = -(m_j + 1)\delta$, but by an analogous reasoning to that of Proposition 6.1.11 this would lead to a sequence of centers $\{C_\mu\}$ with negative normal bundle, which is an absurd. In the latter case, even if there exists values of $a > 0$ such that $E_i^i \cong \mathbb{F}_0$, condition d fails to be true.

Now, if $C_j$ is isomorphic to the pull-back of a conic by the morphism $\pi : \mathbb{F}_1 \to \mathbb{P}^2$, so $E_k^{j-1} = P(\mathcal{O}(a) \oplus \mathcal{O}(a - 1))$, then

$$[C_j] = 2\varsigma_k + 2af \in A^1(E_{j-1}^k),$$

and as a consequence of Proposition 1.1.19 we have

$$N_{E_k^j/Z_j} \cong \pi_{j,k}^*(N_{E_k^k/Z_k}) \otimes \mathcal{O}(-E_k^j \cap E_j^j)^{\otimes m_j},$$

where $m_j \in \mathbb{Z}_+$, with $m_j \geq 1$. Now, by Proposition 1.2.22 and Lemma 2.2.1 we have that

$$\begin{aligned}
(e_k^j)^2 \cdot e_j^j &= j_{E_k^j *}((2\varsigma_k + 2af) \cdot (-(1+2m_j)\varsigma_k - 2am_j f)),\\
&= j_{E_k^j *}((2a - 2 - 4m_j)\varsigma_k \cdot f);\\
e_k^j \cdot (e_j^j)^2 &= j_{E_j^j *}(c_1(N_{E_j^j/Z_j}) \cdot \left[E_k^j \cap E_j^j\right]_j),\\
&= j_{E_k^j *}((2\varsigma_k + 2af)^2),\\
&= j_{E_k^j *}(4\varsigma_k \cdot f);
\end{aligned}$$

so we can conclude that even if there exists some values of $a > 0$ such that $E_i^i \cong \mathbb{F}_0$, condition d does not hold for these ones.

Finally, if $\delta_k = 0$ and $C_j = \mathcal{C}$, then by Proposition 1.2.22 and Lemma 2.2.1 we have that

$$\begin{aligned}
(e_k^j)^2 \cdot e_j^j &= j_{E_k^j *}((n_j\varsigma_k + (n_j a + 1)f) \cdot (-(1+n_j m_j)\varsigma_k - m_j(n_j a + 1)f)),\\
&= j_{E_k^j *}((n_j a - 2n_j m_j - 1)\varsigma_k \cdot f);\\
e_k^j \cdot (e_j^j)^2 &= j_{E_j^j *}(c_1(N_{E_j^j/Z_j}) \cdot \left[E_k^j \cap E_j^j\right]_j),\\
&= j_{E_k^j *}((n_j\varsigma_k + (n_j a + 1)f)^2),\\
&= j_{E_k^j *}((2n_j)\varsigma_k \cdot f),
\end{aligned}$$

so we can conclude, as in the previous case, that even if there exists some values of $a > 0$ such that $E_i^i \cong \mathbb{F}_0$, condition d fails to be true for these ones either.

In the case B, since $C_j \xrightarrow{t} C_k$ then it follows directly that

$$N_{C_i/Z_{i-1}} \cong \mathcal{O} \oplus \mathcal{O}(-1),$$

so condition c, that is $E_i^i \cong \mathcal{F}_0$, does not hold. □

**Corollary 6.2.2.** *Let $E_i^\alpha$ be an irreducible exceptional component of a sequence of point and rational curve blow-ups $(Z_0, ..., Z_\alpha, \pi)$, verifying the conditions of Proposition 6.2.1, that is:*

*a* $E_i^\alpha$ *is final, with base a "new" curve* $C_i$*,*

*b there exists two indexes* $j$ *and* $k$*, such that* $C_i \to C_j$ *and* $C_i \to C_k$*,*

*c* $E_i^\alpha \cong \mathbb{F}_0$*,*

*d and* $(e_i^\alpha)^3 > 0$*.*

*If* $(e_i^\alpha)^2 \cdot e_j^\alpha = (e_i^\alpha)^2 \cdot e_k^\alpha = -\lambda$*, for* $\lambda \in \mathbb{Z}_+$*, and all the centers associated to the set of indexes* $\{\gamma\}$ *in Proposition 6.2.1 verify* $dim(C_\gamma) = 0$*, then the cardinal* $\#\{\gamma\}$ *verifies* $\#\{\gamma\} \geq \lambda + 1$*.*

*Proof.* It follows from the proof of Proposition 6.2.1 that if $E_i^\alpha$ is final, it satisfies condition c, and it does not exists any index $\gamma$ such that $E_j^{i-1} \cap E_\gamma^{i-1}, E_k^{i-1} \cap E_\gamma^{i-1}, E_j^{i-1} \cap E_k^{i-1} \cap E_\gamma^{i-1} \neq \emptyset$, then $E_i^\alpha = P(\mathcal{O}(a) \oplus \mathcal{O}(a))$, with $a \geq 1$. Let us consider $C_\gamma \in E_j^j \cap E_k^j$, where $dim(C_\gamma) = 0$. Now, as a consequence of Proposition 1.1.19 we have that the normal bundle $N_{C_i^\gamma/Z_\gamma}$ verifies

$$N_{\widetilde{C}_i^\gamma/Z_\gamma} \cong \pi_{\gamma,j}^*(N_{C_i/Z_j}) \otimes \mathcal{O}(-C_i^\gamma \cap E_\gamma^\gamma),$$

so

$$N_{\widetilde{C}_i^\gamma/Z_\gamma} \cong \mathcal{O}(a-1) \oplus \mathcal{O}(a-1).$$

By induction, if we denote by $N$ to the cardinal $\#\{\gamma\}$, it follows that

$$N_{\widetilde{C}_i^{\gamma+N-1}/Z_{\gamma+N-1}} \cong \mathcal{O}(a-N) \oplus \mathcal{O}(a-N),$$

so $N > a$ in order to satisfy condition d. Moreover, by Proposition 1.2.22 we have that

$$\begin{aligned}
(e_i^\alpha)^2 \cdot e_j^\alpha &= j_{E_i^\alpha *}((\varsigma_i + (a-N)f) \cdot -\varsigma_i), \\
&= j_{E_i^\alpha *}((a-N)\varsigma_i \cdot f); \\
(e_i^\alpha)^2 \cdot e_k^\alpha &= j_{E_i^\alpha *}((\varsigma_i + (a-N)f) \cdot -\varsigma_i), \\
&= j_{E_i^\alpha *}((a-N)\varsigma_i \cdot f).
\end{aligned}$$

Since $a \geq 1$, if $(e_i^\alpha)^2 \cdot e_j^\alpha = (e_i^\alpha)^2 \cdot e_k^\alpha = -\lambda$ then it follows that $N \geq \lambda + 1$. □

**Lemma 6.2.3.** *Let* $E_i^\alpha$ *be an irreducible exceptional component of a sequence of point and rational curve blow-ups* $(Z_0, ..., Z_\alpha, \pi)$*, verifying the following conditions:*

a $E_i^\alpha$ *is final, with base a "new" curve* $C_i$,

b *there exists just two indexes* $j$ *and* $k$, *such that* $E_i^\alpha \cap E_j^\alpha, E_i^\alpha \cap E_k^\alpha \neq \emptyset$, *and* $C_i \to C_j$ *and* $C_i \xrightarrow{t} C_k$,

c $E_i^\alpha \cong \mathbb{F}_0$,

d *and the following conditions are satisfied:* $(e_i^\alpha)^3 > 0$, $(e_i^\alpha)^2 \cdot e_j^\alpha = (e_i^\alpha)^2 \cdot e_k^\alpha$.

*Then* $C_i$ *is isomorphic to a fiber of* $E_j^j$, $C_k \xrightarrow{t} C_j$ *and* $(e_i^\alpha)^3 = 2$.

*Proof.* Firstly, since $E_i^\alpha \cong \mathbb{F}_0$, in particular $E_i^\alpha = P(\mathcal{O}(a) \oplus \mathcal{O}(a))$ for some $a \in \mathbb{Z}$. As by the hypothesis $E_i^\alpha$ is final, then $N_{E_i^\alpha/Z_\alpha} \cong \mathcal{O}_{E_i^\alpha}(-1)$ so we have that

$$\begin{aligned}(e_i^\alpha)^3 &= j_{E_i^\alpha *}((-\varsigma_i)^2),\\ &= j_{E_i^\alpha *}((-2a)\varsigma_i \cdot f).\end{aligned}$$

Since $(e_i^\alpha)^3 > 0$, it follows that $a < 0$. Moreover, as a consequence of Lemma 2.2.1, the normal bundle of $C_i$, $N_{C_i/Z_{i-1}}$, verifies the following splitting:

$$N_{C_i/Z_{i-1}} \cong N_{C_i/E_j^{i-1}} \oplus N_{E_j^{i-1}/Z_{i-1}}|_{C_i},$$

Since $E_j^{i-1} \cap E_k^{i-1} \neq \emptyset$, then one of the following conditions holds:

A either $C_k \to C_j$, with $dim(C_k) = 1$ (or vice versa),

B or $C_k \xrightarrow{t} C_j$ (or vice versa).

In the case A, as a consequence of Proposition 1.1.19, we have that $N_{E_j^k/Z_k} = \pi_k^*(N_{E_j^{k-1}/Z_{k-1}}) \otimes \mathcal{O}(-E_j^k \cap E_k^k)$, so

$$c_1(N_{E_j^{i-1}/Z_{i-1}}|_{C_i}) = \pi_{i-1,k-1}^*(c_1(N_{E_j^{k-1}/Z_{k-1}}) \cdot [C_i]) + \pi_{i-1,k}^*(-\left[E_j^k \cap E_k^k\right] \cdot [C_i]).$$

Now, by Propositions 1.2.33 and 1.2.22 we have that

$$\begin{aligned}(e_i^i)^2 \cdot e_k^i &= j_{E_i^i *}(\left[C_i^{i-1}\right] \cdot E_k^{i-1} \cdot c_1(N_{E_i^i/Z_i})),\\ &= \pi_{i-1,k}^*(-\left[E_j^k \cap E_k^k\right] \cdot \left[C_i^k\right]),\\ &= -a,\end{aligned}$$

so it follows that $\pi^*_{i-1,k-1}(c_1(N_{E_j^{k-1}/Z_{k-1}}) \cdot \left[C_i^{k-1}\right]) = 0$. Thus, we have that $E_j^{i-1} = P(\mathcal{O} \oplus \mathcal{O}(-a))$. From condition b, $C_j$ is an unmodified "old" curve, but then we get to a contradiction by Proposition 5.4.3.

In the case B, since the blow-up $\pi_k : Z_k \to Z_{k-1}$ is an intrinsic elementary modification of $E_j^{k-1}$, then we have that

$$N_{C_i^{i-1}/E_j^{i-1}} \cong \pi^*_{i,k-1}(N_{C_i^{k-1}/E_j^{k-1}}) \otimes \mathcal{O}(-E_j^k \cap E_k^k \cap C_i),$$

and since $c_1(N_{C_i^{i-1}/E_j^{i-1}}) = -a$, then it follows that $\pi^*_{i-1,k-1}(c_1(N_{C_i^{k-1}/E_j^{k-1}})) = 0$. As a result, we have that either $C_i$ is isomorphic to a fiber of $E_j^{k-1}$ or $E_j^{k-1} \cong \mathbb{F}_0$ and $C_i^{k-1}$ is isomorphic to a section $S_0$. In the latter case, we have that $E_j^k = P(\mathcal{O}(-a) \oplus \mathcal{O}(-a))$, so by Proposition 6.1.11 there must exists some other index $\gamma$ verifying $E_j^k \cap E_k^k \cap E_\gamma^k \neq \emptyset$. In the former case, we have that $N_{E_j^k/Z_k}|_{C_i^k} \cong \mathcal{O}(-1)$, so it follows

$$N_{C_i^k/Z_k} = \mathcal{O}(-1) \oplus \mathcal{O}(-1),$$

and condition $(e_i^\alpha)^3 = 2$ is satisfied. □

**Proposition 6.2.4.** *Let $E_i^\alpha$ be the irreducible exceptional component of a sequence of point and rational curve blow-ups $(Z_0, ..., Z_\alpha, \pi)$. If $E_i^\alpha$ is final and there exists an index $\gamma$ such that the following relations hold*

*a* $(e_i^\alpha)^3 = c$,

*b* $(e_i^\alpha)^2 \cdot e_\gamma^\alpha = -c$,

*c* $e_i^\alpha \cdot (e_\gamma^\alpha)^2 = c$,

*with $c \in \mathbb{Z}_+$, then there must exist some other index $\lambda$, with $\lambda \neq \gamma$, verifying $E_i^\alpha \cap E_\lambda^\alpha \neq \emptyset$.*

*Proof.* Let us suppose that $\gamma$ is the only index such that $E_i^\alpha \cap E_\gamma^\alpha \neq \emptyset$. There must exist an integer $a \in \mathbb{Z}_+$ such that $N_{C_i/Z_{i-1}} \cong \mathcal{O}(a) \oplus \mathcal{O}(a - \delta_i)$, so $E_i^\alpha \cong \mathbb{F}_{\delta_i}$. Moreover, since by the hypothesis $E_i^\alpha$ is final, then $E_i^\alpha \cap E_\gamma^\alpha$ must be isomorphic to a section of $E_i^\alpha$. As $e_i^\alpha \cdot (e_\gamma^\alpha)^2 = c > 0$, then the class of $E_i^\alpha \cap E_\gamma^\alpha$ in $A^1(E_i^\alpha)$ must be of the form $\left[E_i^\alpha \cap E_\gamma^\alpha\right] = \varsigma_i + (a + n_\gamma)f$. Now, as a consequence of Proposition 2.3.7 we have that

$N_{E_i^\alpha/Z_\alpha} \cong \mathcal{O}_{E_i^\alpha}(-1)$, so

$$\begin{aligned}
(e_i^\alpha)^3 &= j_{E_i^\alpha *}((-\varsigma_i)^2), \\
&= j_{E_i^\alpha *}(-2a+\delta_i)\varsigma_i \cdot f); \\
(e_i^\alpha)^2 \cdot e_\gamma^\alpha &= j_{E_i^\alpha *}(-\varsigma_i \cdot (\varsigma_i + (a+n_\gamma)f)), \\
&= j_{E_i^\alpha *}((a-\delta_i-n_\gamma)\varsigma_i \cdot f); \\
e_i^\alpha \cdot (e_\gamma^\alpha)^2 &= j_{E_i^\alpha *}((\varsigma_i + (a+n_\gamma)f)^2), \\
&= j_{E_i^\alpha *}((\delta_i + 2n_\gamma)\varsigma_i \cdot f),
\end{aligned}$$

Then, we have that

$$\begin{aligned}
-2a+\delta_i &= c \\
a-\delta_i-n_\gamma &= -c \\
\delta_i + 2n_\gamma &= c,
\end{aligned}$$

and by solving the following linear system we get to $a = 0$, $\delta_i = c$, and $n_\gamma = 0$, so $E_i^\alpha \cong \mathbb{F}_c$. Moreover, $e_i^\alpha)^2 \cdot e_\gamma^\alpha = -c < 0$, so $C_i$ must be isomorphic to the unique section of $E_\gamma^\gamma$ with negative self-intersection. Since we are just considering points and rational curves as centers of the blow-ups, we have that $E_\gamma^\gamma \cong \mathbb{F}_{\delta_\gamma}$, so in particular there exits an integer $b \in \mathbb{Z}$ such that $N_{C_\gamma/Z_{\gamma-1}} \cong \mathcal{O}(b) \oplus \mathcal{O}(b-\delta_\gamma)$. Then, $C_i$ is associated with the line subbundle of maximal degree and its class $[C_i]$ in $A^1(E_\gamma^{i-1})$ satisfies:

$$[C_i] = \varsigma_\gamma + (b-\delta_\gamma)f,$$

so we have that

$$\begin{aligned}
c_1(N_{E_\gamma^{i-1}/Z_{i-1}}|_{C_i} &= b, \\
[C_i]\cdot[C_i] &= -\delta_\gamma.
\end{aligned}$$

This fact lead us to conclude that it must be verified $b = 0$ and $\delta_\gamma = c$. As a consequence of Proposition 5.4.3, $C_\gamma$ must be a "new curve", that is, there must exists a curve $C_\mu$ such that $C_\gamma \to C_\mu$ . As we are supposing that that $\gamma$ is the only index such that $E_i^\alpha \cap E_\gamma^\alpha \neq \emptyset$, then it must be verified that $e_\gamma^\gamma \cdot (e_\mu^\gamma)^2 = c$, so $C_\gamma$ must be isomorphic to the unique section of $E_\mu^{\gamma-1}$ with negative self-intersection. Reasoning in an analogous manner as above, we can conclude that $N_{C_\mu/Z_{\mu-1}} \cong \mathcal{O} \oplus \mathcal{O}(-c)$, so the hypothesis considering $\gamma$ as the only index such that $E_i^\alpha \cap E_\gamma^\alpha \neq \emptyset$ led us to a sequence of centers $\{C_\nu\}$ verifying $N_{C_\nu/Z_{\nu-1}} \cong \mathcal{O} \oplus \mathcal{O}(-c)$, which has non sense as a consequence of Proposition 5.4.3. $\square$

**Corollary 6.2.5.** *Let $E_i^\alpha = P(\mathcal{O} \oplus \mathcal{O}(-1))$ be an irreducible exceptional component of a sequence of point and rational curve blow-ups $(Z_0, ..., Z_\alpha, \pi)$. If $E_i^\alpha$ is final and there exists an index $\gamma$ such that $C_i \to C_\gamma$, with $C_i$ non isomorphic to a fiber of $E_\gamma^{i-1}$, and $(e_i^\alpha)^2 \cdot e_\gamma^\alpha = -1$, then there exists some other index $\lambda$, with $\lambda \neq \gamma$, verifying $E_i^\alpha \cap E_\lambda^\alpha \neq \emptyset$.*

**Proposition 6.2.6.** *Given a sequence of point and rational curve blow-ups $(Z_\alpha, ..., Z_0, \pi)$, let $E_i^\alpha \subset Z_\alpha$ be an irreducible exceptional component . Furthermore, let us suppose that the following conditions are satisfied:*

*a There exists just two indexes $j, k$, with $E_i^\alpha \cap E_j^\alpha \neq \emptyset$ and $E_i^\alpha \cap E_k^\alpha \neq \emptyset$, that verify the following conditions:*

  *a.i* $(e_k^\alpha)^2 \cdot e_i^\alpha = -(e_j^\alpha)^2 \cdot e_i^\alpha$,

  *a.ii* $e_k^\alpha \cdot (e_i^\alpha)^2 = e_j^\alpha \cdot (e_i^\alpha)^2 + (e_j^\alpha)^2 \cdot e_i^\alpha$,

  *a.iii and* $(e_j^\alpha + e_i^\alpha)^2 \cdot e_i^\alpha = (e_k^\alpha + e_i^\alpha)^2 \cdot e_i^\alpha = 0$,

  *a.iv* $e_i^\alpha \cdot e_j^\alpha \cdot e_k^\alpha = 0$.

*b there exists at most one index $\beta$, with $E_i^\alpha \cap E_\beta^\alpha \neq \emptyset$, such that $e_i^\alpha \cdot (e_\beta^\alpha)^2 < 0$, if $(e_j^\alpha)^2 \cdot e_i^\alpha \neq 0$, $(e_k^\alpha)^2 \cdot e_i^\alpha \neq 0$, otherwise such an index does not exist,*

*c if there exists any other index $\gamma$, with $\gamma \neq j, k$, such that $E_i^\alpha \cap E_\gamma^\alpha \neq \emptyset$, then the following relations are satisfied, where $\eta \in \mathbb{Z}_+$:*

$$(e_i^\alpha)^2 \cdot e_\gamma^\alpha = -\eta$$
$$e_i^\alpha \cdot (e_\gamma^\alpha)^2 = 0$$
$$e_i^\alpha \cdot e_j^\alpha \cdot e_\gamma^\alpha = e_i^\alpha \cdot e_k^\alpha \cdot e_\gamma^\alpha = \eta.$$

*d and in the particular case where $(e_k^\alpha)^2 \cdot e_i^\alpha = (e_j^\alpha)^2 \cdot e_i^\alpha = 0$, with $e_k^\alpha \cdot (e_i^\alpha)^2 = e_j^\alpha \cdot (e_i^\alpha)^2 = -\lambda$, for some $\lambda \in \mathbb{Z}_+$, if the following relations hold:*

$$(e_\gamma^\alpha)^2 \cdot e_j^\alpha = -1,$$
$$e_\gamma^\alpha \cdot (e_j^\alpha)^2 = 0,$$
$$(e_\gamma^\alpha)^2 \cdot e_k^\alpha = -1,$$
$$e_\gamma^\alpha \cdot (e_k^\alpha)^2 = 0,$$

*thus $\# \{\gamma\} \geq \lambda + 1$.*

*Then* $E_i^\alpha \cong \mathbb{F}_\delta$*, with* $\delta = \left|(e_j^\alpha)^2 \cdot e_i^\alpha\right| = \left|(e_k^\alpha)^2 \cdot e_i^\alpha\right|$*, and* $N_{E_i^\alpha/Z_\alpha} \cong \mathcal{O}_{E_i^\alpha}(-1)$.

*Proof.* By our hypotheses about the centers of the sequences of blow-ups we are working with, if we denote by $\mathcal{C}_{i,j} = E_j^\alpha \cap E_i^\alpha$ and $\mathcal{C}_{i,k} = E_k^\alpha \cap E_i^\alpha$, then both $\mathcal{C}_{i,j}, \mathcal{C}_{i,k}$ must be rational curves. Moreover, if we denote by $[\mathcal{C}_{i,j}], [\mathcal{C}_{i,k}]$ to the classes of $\mathcal{C}_{i,j}, \mathcal{C}_{i,k}$ in $A^1(E_i^\alpha)$, then condition a.i implies that $([\mathcal{C}_{i,k}])^2 = -([\mathcal{C}_{i,j}])^2$, so it follows from condition a.iv that $[\mathcal{C}_{i,j}] \cdot [\mathcal{C}_{i,k}] = 0$. Let us suppose that $e_i^\alpha \cdot (e_j^\alpha)^2 \geq 0$. Note that if $(e_j^\alpha)^2 \cdot e_i^\alpha > 0$, then by condition b there can not exists any other index $\gamma$, with $\gamma \neq k$, such that $E_i^\alpha \cap E_\gamma^\alpha \neq \emptyset$ and verifying $e_i^\alpha \cdot (e_\gamma^\alpha)^2 < 0$. This fact led us to distinguish between the three following cases:

- A $(e_j^\alpha)^2 \cdot e_i^\alpha = (e_k^\alpha)^2 \cdot e_i^\alpha = 0$.
- B $(e_j^\alpha)^2 \cdot e_i^\alpha, (e_k^\alpha)^2 \cdot e_i^\alpha = \pm 1$.
- C $(e_j^\alpha)^2 \cdot e_i^\alpha, (e_k^\alpha)^2 \cdot e_i^\alpha \neq 0, \pm 1$,

Firstly, we consider the case C. Let us suppose then that $(e_j^\alpha)^2 \cdot e_i^\alpha = \lambda > 1$. As conditions a.i and a.iv are both satisfied, then it follows from Proposition 5.3.5 and Theorem 5.3.3 that some of the following characterizations hold:

C.i either $E_i^\alpha \cong \mathbb{F}_{\delta_i}$, with $\delta_i = \lambda$, in particular $E_i^\alpha \cong P(\mathcal{O}(a) \oplus \mathcal{O}(a - \delta_i))$, and $C_{i,j} \subset E_i^\alpha$ is isomorphic to the section corresponding to the line subbundle $\mathcal{O}(a - \delta_i)$ so

$$[C_{i,j}] = \varsigma_i + af \in A^1(E_i^\alpha),$$

and $C_{i,k} \subset E_i^\alpha$ is isomorphic to the section corresponding to the maximal line subbundle $\mathcal{O}(a)$, that is

$$[C_{i,k}] = \varsigma_i + (a - \delta_i)f \in A^1(E_i^\alpha).$$

C.ii or $E_i^\alpha \cong \mathbb{F}_{\delta_i}$, with $\lambda = \delta_i + 2n$ for some $n \in \mathbb{Z}_+$, in particular $E_i^\alpha \cong P(\mathcal{O}(a) \oplus \mathcal{O}(a - \delta_i))$, and $C_{i,j} \subset E_i^\alpha$ must be isomorphic to the section corresponding to the line subbundle $\mathcal{O}(a - \delta_i - n)$ so

$$[C_{i,j}] = \varsigma_i + (a + n)f \in A^1(E_i^\alpha),$$

and $C_k \xrightarrow{t} C_i$, so $C_{i,k} \subset E_i^\alpha$ is isomorphic to $\lambda$ fibers of $E_k^\alpha$, and

$$[C_{i,k}] = \lambda f \in A^1(E_k^\alpha),$$

C.iii or if $\sqrt{\lambda} \in \mathbb{Z}_+$, $E_i^\alpha$ is a birational model of $\mathbb{P}^2$, and $C_{i,j} \subset E_i^\alpha$ is isomorphic to a rational curve of degree $\sqrt{\lambda}$ and $C_k \xrightarrow{t} C_i$, so $C_{i,k} \subset E_i^\alpha$ is isomorphic to $\lambda$ fibers of $E_k^\alpha$, and

$$[C_{i,k}] = \lambda f \in A^1(E_k^\alpha),$$

In the subcase C.i, let us suppose that $N_{E_i^\alpha/Z_\alpha} \ncong \mathcal{O}_{E_i^\alpha}(-1)$. Then, there must exists an extrinsic modification of $E_i^i$, that is, either $C_j \to C_i$, or $C_k \to C_i$, or $C_j, C_k \to C_i$. Now, it follows from Proposition 1.1.19 that

$$N_{E_i^\alpha/Z_\alpha} \cong \pi_{\alpha,i}^*(N_{E_i^i/Z_i}) \otimes (\mathcal{O}(\chi_{i,j}(-E_i^\alpha \cap E_j^\alpha)))^{\otimes m_j} \otimes (\mathcal{O}(\chi_{i,k}(-E_i^\alpha \cap E_k^\alpha)))^{\otimes m_k},$$

where $m_j, m_k \in \mathbb{Z}_+$, with $m_j, m_k \geq 1$, and either $\chi_{i,j} = 1$ or $\chi_{i,k} = 1$ or $\chi_{i,j} = \chi_{i,k} = 1$, so by Proposition 1.2.22

$$\begin{aligned}(e_i^\alpha)^2 \cdot e_j^\alpha &= j_{E_i^\alpha *}((\varsigma_i + af) \cdot (-(1 + \chi_{i,j}m_j + \chi_{i,k}m_k)\varsigma_i - (\chi_{i,j}m_j a + \chi_{i,k}m_k(a - \delta_i))f)),\\ &= j_{E_i^\alpha *}((a - \delta_i(1 + \chi_{i,j}m_j + \chi_{i,k}2m_k))\varsigma_i \cdot f),\end{aligned}$$

and

$$\begin{aligned}(e_i^\alpha)^2 \cdot e_k^\alpha &= j_{E_i^\alpha *}((\varsigma_i + (a - \delta_i)f) \cdot (-(1 + \chi_{i,j}m_j + \chi_{i,k}m_k)\varsigma_i - (\chi_{i,j}m_j a + \chi_{i,k}m_k(a - \delta_i))f))\\ &= j_{E_i^\alpha *}((a - \delta_i(\chi_{i,k}m_k))\varsigma_i \cdot f).\end{aligned}$$

As a result, condition a.ii does not hold since by our hypothesis either $\chi_{i,j}$ or $\chi_{i,k}$ are not 0.

In the subcase C.ii, since $C_k \xrightarrow{t} C_i$, by Proposition 1.2.22 we have that

$$\begin{aligned}(e_i^\alpha)^2 \cdot e_k^\alpha &= j_{E_i^\alpha *}(c_1(N_{E_i^\alpha/Z_\alpha}|_{E_i^\alpha \cap E_k^\alpha})\varsigma_i \cdot f),\\ &= j_{E_i^\alpha *}(0\varsigma_i \cdot f),\end{aligned}$$

so in order to satisfy condition a.ii $(e_i^\alpha)^2 \cdot e_j^\alpha = -\lambda$. Moreover, in order to satisfy condition a.iii then $(e_i^\alpha)^3 = \lambda$. It then follows, as a consequence of Proposition 6.2.4, that there must exist some index $\gamma$, with $\gamma \neq j$, such that $E_i^\alpha \cap E_\gamma^\alpha \neq \emptyset$, and verifying either $C_i \to C_\gamma$ or $C_i \xrightarrow{t} C_\gamma$. In the former case, since $C_i \to C_j$ and $C_i \to C_\gamma$, then by Proposition 6.1.7

we have that $e_i^\alpha \cdot (e_\gamma^\alpha)^2 = -\lambda$, so condition b fails to be true. In the latter case, that is $C_i \xrightarrow{t} C_\gamma$, in order to satisfy condition c, then $C_k \to C_\gamma$, so by Proposition 1.2.33

$$\begin{aligned}(e_\gamma^k)^2 \cdot e_i^k &= (\pi_k^*(e_\gamma^{k-1}) - e_k^k)^2 \cdot \pi_k^*(e_i^{k-1}),\\ &= (e_k^k)^2 \cdot e_i^k.\end{aligned}$$

We can conclude then that $(e_\gamma^\alpha)^2 \cdot e_i^\alpha \neq 0$, so condition c does not hold.
In the subcase C.iii, since $C_k \xrightarrow{t} C_i$, then by Proposition 1.2.22 we have that

$$\begin{aligned}(e_i^\alpha)^2 \cdot e_k^\alpha &= j_{E_i^\alpha *}(c_1(N_{E_i^\alpha/Z_\alpha}|_{E_i^\alpha \cap E_k^\alpha})\varsigma_i \cdot f),\\ &= j_{E_i^\alpha *}(0\varsigma_i \cdot f),\end{aligned}$$

so in order to satisfy condition a.ii $(e_i^\alpha)^2 \cdot e_j^\alpha = -\lambda$. Moreover, in order to satisfy condition a.iii then $(e_i^\alpha)^3 = \lambda$. Now, as a consequence of Proposition 6.2.4 there must exist some index $\gamma$, with $\gamma \neq j$, such that $E_i^\alpha \cap E_\gamma^\alpha \neq \emptyset$, and verifying either $C_i \to C_\gamma$ or $C_i \xrightarrow{t} C_\gamma$. In the former case, since $C_i \to C_j$ and $C_i \to C_\gamma$, then by Proposition 6.1.7 we have that $e_i^\alpha \cdot (e_\gamma^\alpha)^2 = -\lambda$, so condition b fails to be true. In the latter case, that is $C_i \xrightarrow{t} C_\gamma$, in order to satisfy condition c, then $C_k \to C_\gamma$, so by Proposition 1.2.33

$$\begin{aligned}(e_\gamma^k)^2 \cdot e_i^k &= (\pi_k^*(e_\gamma^{k-1}) - e_k^k)^2 \cdot \pi_\alpha^*(e_i^{k-1}),\\ &= (e_k^k)^2 \cdot e_i^k.\end{aligned}$$

As a result, we have that $(e_\gamma^\alpha)^2 \cdot e_i^\alpha \neq 0$, so condition c is not satisfied.

Now, let us consider the case B. To begin with, let us suppose that $(e_j^\alpha)^2 \cdot e_i^\alpha = 1$. Then, as a consequence of Proposition 5.3.5 and Theorem 5.3.3, conditions a.i and a.iv imply that $E_i^\alpha$ must be either a birational model of $\mathbb{P}^2$ or a birational model of $\mathbb{F}_1$, so one of the following characterizations holds:

B.i either $E_i^\alpha \cong \mathbb{F}_1$, in particular $E_i^\alpha \cong P(\mathcal{O}(a) \oplus \mathcal{O}(a-1))$, and $\mathcal{C}_{i,j} \subset E_i^\alpha$ is isomorphic to the section corresponding to the line subbundle $\mathcal{O}(a-1)$ so

$$[\mathcal{C}_{i,j}] = \varsigma_i + af \in A^1(E_i^\alpha),$$

and $\mathcal{C}_{i,k} \subset E_i^\alpha$ is isomorphic to the section corresponding to the maximal line subbundle $\mathcal{O}(a)$, that is

$$[\mathcal{C}_{i,k}] = \varsigma_i + (a-1)f \in A^1(E_i^\alpha),$$

B.ii or $E_i^\alpha \cong \mathbb{F}_1$, in particular $E_i^\alpha \cong P(\mathcal{O}(a) \oplus \mathcal{O}(a-1))$, and $\mathcal{C}_{i,j} \subset E_i^\alpha$ is isomorphic to the section corresponding to the line subbundle $\mathcal{O}(a-1)$ so

$$[\mathcal{C}_{i,j}] = \varsigma_i + af \in A^1(E_i^\alpha),$$

and $C_k \xrightarrow{t} C_i$, so $\mathcal{C}_{i,k} \subset E_i^\alpha$ is isomorphic to an exceptional curve, and

$$[\mathcal{C}_{i,k}] = e_{i,k} \in A^1(E_i^\alpha),$$

B.iii or $E_i^\alpha \cong \mathbb{F}_1$, in particular $E_i^\alpha \cong P(\mathcal{O}(a) \oplus \mathcal{O}(a-1))$, and $\mathcal{C}_{i,j} \subset E_i^\alpha$ must be isomorphic to the section corresponding to the line subbundle $\mathcal{O}(a-1)$ so

$$[\mathcal{C}_{i,j}] = \varsigma_i + af \in A^1(E_i^\alpha),$$

and $C_k \to C_i$, with $dim(C_k) = 0$, so $\mathcal{C}_{i,k} \subset E_i^\alpha$ is isomorphic to an exceptional curve, and

$$[\mathcal{C}_{i,k}] = e_{i,k} \in A^1(E_i^\alpha),$$

B.iv or $E_i^\alpha \cong \mathbb{F}_1$, in particular $E_i^\alpha \cong P(\mathcal{O}(a) \oplus \mathcal{O}(a-1))$, and $\mathcal{C}_{i,j} \subset E_i^\alpha$ is isomorphic to the section corresponding to the line subbundle $\mathcal{O}(a-1)$, so

$$[\mathcal{C}_{i,j}] = \varsigma_i + af \in A^1(E_i^\alpha),$$

and $C_k \to C_i$, with $dim(C_k) = 1$, so $\mathcal{C}_{i,k} \subset E_i^\alpha$ is isomorphic to the exceptional curve $C_k$, and

$$[\mathcal{C}_{i,k}] = [C_k] \in A^1(E_i^\alpha).$$

In the subcase B.i, let us suppose that $N_{E_i^\alpha/Z_\alpha} \ncong \mathcal{O}_{E_i^\alpha}(-1)$. Then, there must exist an extrinsic elementary modification of $E_i^i$, that is, either $C_j \to C_i$, or $C_k \to C_i$, or $C_j, C_k \to C_i$. Now, it follows from Proposition 1.1.19 that

$$N_{E_i^\alpha/Z_\alpha} \cong \pi_{\alpha,i}^*(N_{E_i^i/Z_i}) \otimes (\mathcal{O}(\chi_{i,j}(-E_i^\alpha \cap E_j^\alpha)))^{\otimes m_j} \otimes (\mathcal{O}(\chi_{i,k}(-E_i^\alpha \cap E_k^\alpha)))^{\otimes m_k},$$

where $m_j, m_k \in \mathbb{Z}_+$ and $m_j, m_k \geq 1$ and either $\chi_{i,j} = 1$ or $\chi_{i,k} = 1$ or $\chi_{i,j} = \chi_{i,k} = 1$. Now, by Proposition 1.2.22 we have that

$$\begin{aligned}(e_i^\alpha)^2 \cdot e_j^\alpha &= j_{E_i^\alpha *}((\varsigma_i + af) \cdot (-(1 + \chi_{i,j}m_j + \chi_{i,k}m_k)\varsigma_i - (\chi_{i,j}m_j a + \chi_{i,k}m_k(a-1))f)), \\ &= j_{E_i^\alpha *}((a - 1(1 + \chi_{i,j}m_j + \chi_{i,k}2m_k))\varsigma_i \cdot f),\end{aligned}$$

and

$$
\begin{aligned}
(e_i^\alpha)^2 \cdot e_k^\alpha &= j_{E_i^\alpha *}((\varsigma_i + (a-\delta)f) \cdot (-(1+\chi_{i,j}m_j + \chi_{i,k}m_k)\varsigma_i - (\chi_{i,j}m_j a + \chi_{i,k}m_k(a-1))f)) \\
&= j_{E_i^\alpha *}((a-1(\chi_{i,k}m_k))\varsigma_i \cdot f).
\end{aligned}
$$

We can conclude then that condition a.ii does not hold since by our hypothesis either $\chi_{i,j}$ or $\chi_{i,k}$ are not 0 .

In the subcase B.ii, since $C_k \xrightarrow{t} C_i$, then by Proposition 1.2.22 we have that

$$
\begin{aligned}
(e_i^\alpha)^2 \cdot e_k^\alpha &= j_{E_i^\alpha *}(c_1(N_{E_i^\alpha/Z_\alpha}|_{E_i^\alpha \cap E_k^\alpha})\varsigma_i \cdot f), \\
&= j_{E_i^\alpha *}(0\varsigma_i \cdot f),
\end{aligned}
$$

so in order to satisfy condition a.ii then $(e_i^\alpha)^2 \cdot e_j^\alpha = -1$. Moreover, in order to satisfy condition a.iii t$(e_i^\alpha)^3 = 1$ must be satisfied. It then follows, as a consequence of Proposition 6.2.4 there must exist some index $\gamma$, with $\gamma \neq j$, such that $E_i^\alpha \cap E_\gamma^\alpha \neq \emptyset$, verifying either $C_i \to C_\gamma$ or $C_i \xrightarrow{t} C_\gamma$. In the former case, since $C_i \to C_j$ and $C_i \to C_\gamma$, then by Proposition 6.1.7 we have that $e_i^\alpha \cdot (e_\gamma^\alpha)^2 = -1$, so condition b fails to be true. In the latter case, that is $C_i \xrightarrow{t} C_\gamma$, in order to satisfy condition c, then $C_k \to C_\gamma$, so by Proposition 1.2.33

$$
\begin{aligned}
(e_\gamma^k)^2 \cdot e_i^k &= (\pi_k^*(e_\gamma^{k-1}) - e_k^k)^2 \cdot \pi_\alpha^*(e_i^{k-1}), \\
&= (e_k^k)^2 \cdot e_i^k.
\end{aligned}
$$

We can conclude then that $(e_\gamma^\alpha)^2 \cdot e_i^\alpha \neq 0$, so condition c does not hold.

In the subcase B.iii, since $C_k \to C_i$, with $dim(C_k) = 0$, we have by Proposition 1.2.33

$$
\begin{aligned}
(e_i^k)^2 \cdot e_k^k &= (\pi_k^*(e_i^{k-1}) - e_k^k)^2 \cdot e_k^k, \\
&= (e_k^k)^3; \\
e_i^k \cdot (e_k^k)^2 &= (\pi_k^*(e_i^{k-1}) - e_k^k) \cdot (e_k^k)^2, \\
&= -(e_k^k)^3,
\end{aligned}
$$

and

$$
\begin{aligned}
(e_i^k)^3 &= (\pi_k^*(e_i^{k-1}) - e_k^k)^3, \\
&= (\pi_k^*(e_i^{k-1}))^3 - (e_k^k)^3;
\end{aligned}
$$

so in order to satisfy condition a.ii $(e_i^\alpha)^2 \cdot e_j^\alpha = 0$. Moreover, in order to satisfy condition a.iii then $(e_i^\alpha)^3 = -1$ must hold, so $(e_i^{k-1})^3 = 0$. Since $[C_{i,j}] = \varsigma_i + af \in A^1(E_i^\alpha)$, as a consequence of Proposition 1.1.19 we have that

$$N_{E_i^\alpha/Z_\alpha} \cong \pi_{\alpha,i}^*(N_{E_i^i/Z_i}) \otimes (\mathcal{O}(\chi_{i,j}(-E_i^\alpha \cap E_j^\alpha)))^{\otimes m_j} \otimes (\mathcal{O}(-E_i^\alpha \cap E_k^\alpha)),$$

where $m_j \in \mathbb{Z}_+$, with $m_j \geq 1$, and $\chi_{i,j} \in \{0,1\}$, so by Proposition 1.2.22

$$\begin{aligned}(e_i^\alpha)^2 \cdot e_j^\alpha &= j_{E_i^\alpha *}((\varsigma_i + af) \cdot (-(1+\chi_{i,j}m_j)\varsigma_i - (\chi_{i,j}m_j a)f - [C_{i,k}])), \\ &= j_{E_i^\alpha *}((a - 1 - \chi_{i,j}m_j)\varsigma_i \cdot f),\end{aligned}$$

and

$$\begin{aligned}(e_i^\alpha)^3 &= j_{E_i^\alpha *}((-(1+\chi_{i,j}m_j)\varsigma_i - (\chi_{i,j}m_j a)f - [C_{i,k}])^2), \\ &= j_{E_i^\alpha *}((-2a(\chi_{i,j}m_j + 1) + (1+\chi_{i,j}m_j)^2 - 1)\varsigma_i \cdot f).\end{aligned}$$

It then follows that either condition a.ii or condition a.iii does not hold.
In the subcase B.iv, since $C_k \to C_i$, with $C_k$ isomorphic to an exceptional curve, then as a consequence of Proposition 1.1.19 we have that

$$N_{E_i^\alpha/Z_\alpha} \cong \pi_{\alpha,i}^*(N_{E_i^i/Z_i}) \otimes (\mathcal{O}(\chi_{i,j}(-E_i^\alpha \cap E_j^\alpha)))^{\otimes m_j} \otimes (\mathcal{O}(-E_i^\alpha \cap E_k^\alpha))^{\otimes m_k},$$

where where $m_j, m_k \in \mathbb{Z}_+$, with $m_j, m_k \geq 1$, and $\chi_{i,j} \in \{0,1\}$, so by Propostion 1.2.22

$$\begin{aligned}(e_i^\alpha)^2 \cdot e_k^\alpha &= j_{E_i^\alpha *}([\mathcal{C}_{i,k}] \cdot (-(1+\chi_{i,j}m_j)\varsigma_i - (\chi_{i,j}m_j a)f - m_k\,[\mathcal{C}_{i,k}])), \\ &= j_{E_i^\alpha *}(-m_k\,[\mathcal{C}_{i,k}]^2\,\varsigma_i \cdot f); \\ (e_i^\alpha)^2 \cdot e_j^\alpha &= j_{E_i^\alpha *}((\varsigma_i + af) \cdot (-(1+\chi_{i,j}m_j)\varsigma_i - (\chi_{i,j}m_j a)f - m_k\,[\mathcal{C}_{i,k}])), \\ &= j_{E_i^\alpha *}((a - 1 - \chi_{i,j}m_j)\varsigma_i \cdot f),\end{aligned}$$

and

$$\begin{aligned}(e_i^\alpha)^3 &= j_{E_i^\alpha *}((-(1+\chi_{i,j}m_j)\varsigma_i - (\chi_{i,j}m_j a)f - m_k\,[\mathcal{C}_{i,k}])^2), \\ &= j_{E_i^\alpha *}(((-2a(\chi_{i,j}m_j + 1) + (1+\chi_{i,j}m_j)^2) - (m_k)^2)\varsigma_i \cdot f).\end{aligned}$$

As a consequence, either condition a.ii or condition a.iii does not hold. Finally, let us consider the case A. Since $[\mathcal{C}_{i,j}] \cdot [\mathcal{C}_{i,j}] = [\mathcal{C}_{i,k}] \cdot [\mathcal{C}_{i,k}] = 0$ in $A^0(E_i^\alpha)$, and conditions b and a.iv are satisfied, then it follows from Proposition 5.3.5 and Theorem 5.3.3 that some of the following characterizations holds:

A.i either $E_i^\alpha$ is a birational model of $\mathbb{F}_\delta$ and $[\mathcal{C}_{i,j}], [\mathcal{C}_{i,k}]$ are isomorphic to two fibers of $E_i^i$, so

$$[\mathcal{C}_{i,j}] = f \in A^1(E_i^\alpha), \tag{6.30}$$

and

$$[\mathcal{C}_{i,k}] = f \in A^1(E_i^\alpha), \tag{6.31}$$

A.ii or $E_i^\alpha$ is a birational model of $\mathbb{F}_0$, in particular $E_i^\alpha \cong P(\mathcal{O}(a) \oplus \mathcal{O}(a))$, and $C_{i,j}, C_{i,k} \subset E_i^\alpha$ are isomorphic to two sections corresponding to the line subbundle $\mathcal{O}(a)$ so

$$[\mathcal{C}_{i,j}] = \varsigma_i + af \in A^1(E_i^\alpha),$$

and

$$[\mathcal{C}_{i,k}] = \varsigma_i + af \in A^1(E_i^\alpha).$$

In the subcase A.i, if $N_{E_i^\alpha/Z_\alpha} \ncong \mathcal{O}(-1)$, then by Proposition 1.1.19 there must exist some index $\gamma$, such that $C_\gamma \to C_i$. In order to satisfy condition c, then $\delta_i = 0$, in particular $E_i^\alpha \cong P(\mathcal{O}(a) \oplus \mathcal{O}(a))$, and $\mathcal{C}_{i,\gamma}$ must be isomorphic to the section corresponding to the line subbundle $\mathcal{O}(a)$, so

$$[C_{i,\gamma}] = \varsigma_i + af \in A^1(E_i^\alpha), \tag{6.32}$$

. Now, we should distingish between the two following cases:

A.i.i either $C_\gamma \xrightarrow{t} C_j$ and $C_\gamma \xrightarrow{t} C_k$,

A.i.ii or $C_j \xrightarrow{t} C_\gamma$ and $C_k \xrightarrow{t} C_\gamma$.

In the subcase A.i.i, as a consequence of Theorem 6.1.5 $N_{E_i^{\gamma-1}/Z_{\gamma-1}} \cong \mathcal{O}_{E_i^{\gamma-1}/Z_{\gamma-1}}(-1)$, so it follows from Proposition 1.1.19 that

$$N_{E_i^\alpha/Z_\alpha} \cong \pi^*_{\alpha,\gamma-1}(N_{E_i^{\gamma-1}/Z_{\gamma-1}}) \otimes \mathcal{O}(-E_i^\alpha \cap E_\gamma^\alpha)^{\otimes m_\gamma},$$

where $m_\gamma \in \mathbb{Z}_+$, with $m_\gamma \geq 1$. Thus, by Proposition 1.2.22 we have that

$$\begin{aligned}
(e_i^\alpha)^2 \cdot e_\gamma^\alpha &= j_{E_i^\alpha *}((\varsigma_i + af) \cdot (-(1+m_\gamma)\varsigma_i - (m_\gamma a)f)),\\
&= j_{E_i^\alpha *}(a\varsigma_i \cdot f);\\
(e_i^\alpha)^2 \cdot e_j^\alpha &= j_{E_i^\alpha *}(f \cdot (-(1+m_\gamma)\varsigma_i - (m_\gamma a)f)),\\
&= j_{E_i^\alpha *}((-1-m_\gamma)\varsigma_i \cdot f);\\
(e_i^\alpha)^2 \cdot e_k^\alpha &= j_{E_i^\alpha *}(f \cdot (-(1+m_\gamma)\varsigma_i - (m_\gamma a)f)),\\
&= j_{E_i^\alpha *}((-1-m_\gamma)\varsigma_i \cdot f);
\end{aligned}$$

and

$$
\begin{aligned}
(e_i^\alpha)^3 &= j_{E_i^\alpha *}((-(1+m_\gamma)\varsigma_i - (m_\gamma a)f)^2), \\
&= j_{E_i^\alpha *}((-2a)(m_\gamma+1)\varsigma_i \cdot f).
\end{aligned}
$$

. Moreover, we know that

$$
\begin{aligned}
e_i^\alpha \cdot e_j^\alpha \cdot e_\gamma^\alpha &= j_{E_i^\alpha *}((\varsigma_i + af) \cdot f), \\
&= j_{E_i^\alpha *}(1\varsigma_i \cdot f), \\
e_i^\alpha \cdot e_k^\alpha \cdot e_\gamma^\alpha &= j_{E_i^\alpha *}((\varsigma_i + af) \cdot f), \\
&= j_{E_i^\alpha *}(1\varsigma_i \cdot f).
\end{aligned}
$$

As a result, conditions a.i, a.ii, a.iii, a.iv and b hold if $m_\gamma = 1$ and $a = -1$. However, as $C_\gamma$ is $t-$proximate to both $C_j$ and $C_k$, then we have that

$$
\begin{aligned}
(e_\gamma^\alpha)^2 \cdot e_j^\alpha &= j_{E_\gamma^\alpha *}(f \cdot -\varsigma_\gamma), \\
&= j_{E_\gamma^\alpha *}((-1)\varsigma_\gamma \cdot f), \\
e_\gamma^\alpha \cdot (e_j^\alpha)^2 &= j_{E_\gamma^\alpha *}(f \cdot f), \\
&= j_{E_\gamma^\alpha *}(0\varsigma_\gamma \cdot f), \\
(e_\gamma^\alpha)^2 \cdot e_k^\alpha &= j_{E_\gamma^\alpha *}(f \cdot -\varsigma_\gamma), \\
&= j_{E_\gamma^\alpha *}((-1)\varsigma_\gamma \cdot f), \\
e_\gamma^\alpha \cdot (e_k^\alpha)^2 &= j_{E_\gamma^\alpha *}(f \cdot f), \\
&= j_{E_\gamma^\alpha *}(0\varsigma_\gamma \cdot f).
\end{aligned}
$$

Thus, condition d does not hold since $\#\{\gamma\} = 1 < \left|(e_i^\alpha)^2 \cdot e_j^\alpha\right|$.
In the subcase A.i.ii, since $C_j$, $C_k$ and $C_\gamma$ are all proximate to $C_i$, as a consequence of Proposition 1.1.19 we have that

$$
N_{E_i^\alpha/Z_\alpha} \cong \pi_{\alpha,i}^*(N_{E_i^i/Z_i}) \otimes \mathcal{O}(-E_i^\alpha \cap E_j^\alpha) \otimes \mathcal{O}(-E_i^\alpha \cap E_k^\alpha) \otimes \mathcal{O}(-E_i^\alpha \cap E_\gamma^\alpha),
$$

so by Proposition 1.2.22

$$\begin{aligned}
(e_i^\alpha)^2 \cdot e_j^\alpha &= j_{E_i^\alpha *}(f \cdot -(1+m_\gamma)\varsigma_i - (m_\gamma a + 2)f), \\
&= j_{E_i^\alpha *}(-(1+m_\gamma)\varsigma_i \cdot f), \\
(e_i^\alpha)^2 \cdot e_k^\alpha &= j_{E_i^\alpha *}(f \cdot -(1+m_\gamma)\varsigma_i - (m_\gamma a + 2)f), \\
&= j_{E_i^\alpha *}(-(1+m_\gamma)\varsigma_i \cdot f), \\
(e_i^\alpha)^2 \cdot e_\gamma^\alpha &= j_{E_i^\alpha *}((\varsigma_i + af) \cdot -(1+m_\gamma)\varsigma_i - (m_\gamma a + 2)f), \\
&= j_{E_i^\alpha *}(a\varsigma_i \cdot f);
\end{aligned}$$

and

$$\begin{aligned}
(e_i^\alpha)^3 &= j_{E_i^\alpha *}((-(1+m_\gamma)\varsigma_i - (m_\gamma a + 2)f)^2), \\
&= j_{E_i^\alpha *}((-2a(1+m_\gamma) + 4(1+m_\gamma))\varsigma_i \cdot f).
\end{aligned}$$

As a result, condition a.iii holds only if $a = 1$, but in this case condition c fails to be true.

In the subcase A.ii, let us suppose that $N_{E_i^\alpha/Z_\alpha} \ncong \mathcal{O}_{E_i^\alpha}(-1)$. Then, there must exist an extrinsic elementary modification of $E_i^i$, that is, either $C_j \to C_i$, or $C_k \to C_i$, or $C_j, C_k \to C_i$. Now, it follows from Proposition 1.1.19 that

$$N_{E_i^\alpha/Z_\alpha} \cong \pi_{\alpha,i}^*(N_{E_i^i/Z_i}) \otimes (\mathcal{O}(\chi_{i,j}(-E_i^\alpha \cap E_j^\alpha)))^{\otimes m_j} \otimes (\mathcal{O}(\chi_{i,k}(-E_i^\alpha \cap E_k^\alpha)))^{\otimes m_k},$$

where $m_j, m_k \in \mathbb{Z}_+$, with $m_j, m_k \geq 1$, and either $\chi_{i,j} = 1$ or $\chi_{i,k} = 1$ or $\chi_{i,j} = \chi_{i,k} = 1$. Now, by Proposition 1.2.22

$$\begin{aligned}
(e_i^\alpha)^2 \cdot e_j^\alpha &= j_{E_i^\alpha *}((\varsigma_i + af) \cdot (-(1 + \chi_{i,j}m_j + \chi_{i,k}m_k)\varsigma_i - (a(\chi_{i,j}m_j + \chi_{i,k}m_k)f)), \\
&= j_{E_i^\alpha *}(a\varsigma_i \cdot f); \\
(e_i^\alpha)^2 \cdot e_k^\alpha &= j_{E_i^\alpha *}((\varsigma_i + (a-\delta)f) \cdot (-(1 + \chi_{i,j}m_j + \chi_{i,k}m_k)\varsigma_i - (a(\chi_{i,j}m_j + \chi_{i,k}m_k)f)) \\
&= j_{E_i^\alpha *}(a\varsigma_i \cdot f);
\end{aligned}$$

and

$$\begin{aligned}
(e_i^\alpha)^3 &= j_{E_i^\alpha *}((-(1 + \chi_{i,j}m_j + \chi_{i,k}m_k)\varsigma_i - (a(\chi_{i,j}m_j + \chi_{i,k}m_k)f))^2), \\
&= j_{E_i^\alpha *}((-2a(1 + \chi_{i,j}m_j + \chi_{i,k}m_k))\varsigma_i \cdot f),
\end{aligned}$$

We can conclude then that condition a.iii does not hold since by our hypothesis either $\chi_{i,j}$ or $\chi_{i,k}$ are not 0 .

□

As we will see later in this section, the conditions of Proposition 6.2.6 imply that $E_i^\alpha$ has an admissible final configuration for the sequential morphism $\pi_\alpha : Z_\alpha \to Z_0$.

**Proposition 6.2.7.** *Let $E_i^\alpha \subset Z_\alpha$ be the strict transform of the exceptional irreducible component $E_i^i$. Let us suppose that the following conditions hold:*

*a there exists two indexes $j, k$, with $E_i^\alpha \cap E_j^\alpha \neq \emptyset$ and $E_i^\alpha \cap E_k^\alpha \neq \emptyset$, verifying*

- *a.i* $(e_j^\alpha)^2 \cdot e_i^\alpha = (e_k^\alpha)^2 \cdot e_i^\alpha = 1,$
- *a.ii* $e_j^\alpha \cdot (e_i^\alpha)^2 = e_k^\alpha \cdot (e_i^\alpha)^2 = -1,$
- *a.iii* $e_i \cdot e_j \cdot e_k = 1,$
- *a.iv and* $(e_i^\alpha + e_j^\alpha)^2 \cdot e_i^\alpha = (e_i^\alpha + e_k^\alpha)^2 \cdot e_i^\alpha = 0.$

*b if there exists any other index $\gamma$, with $\gamma \neq j, k$, such that $E_i^\alpha \cap E_\gamma^\alpha \neq \emptyset$, the following relations are satisfied:*

$$(e_i^\alpha)^2 \cdot e_\gamma^\alpha = -1, \tag{6.33}$$

$$e_i^\alpha \cdot (e_\gamma^\alpha)^2 = 1, \tag{6.34}$$

$$e_i^\alpha \cdot e_j^\alpha \cdot e_\gamma^\alpha = e_i^\alpha \cdot e_k^\alpha \cdot e_\gamma^\alpha = 1. \tag{6.35}$$

*Then $E_i^\alpha \cong \mathbb{P}^2$ and $N_{E_i^\alpha/Z_\alpha} \cong \mathcal{O}_{E_i^\alpha}(-1)$.*

*Proof.* Firstly, by our hypothesis about the centers of the sequence of blow-ups, $E_i^\alpha$ must be a birational model of either $\mathbb{P}^2$ or $\mathbb{F}_\delta$. As a consequence of Proposition 5.3.5 and Theorem 5.3.3, conditions a.i and b imply that $E_i^\alpha \cong \mathbb{P}^2$ and $\left[E_i^\alpha \cap E_j^\alpha\right], \left[E_i^\alpha \cap E_k^\alpha\right], \left[E_i^\alpha \cap E_\gamma^\alpha\right] = \varsigma_i$ in $A^1(E_i^\alpha)$. Let us suppose then that there exists at least one index $\lambda \in \{j, k, \gamma\}$ such that $C_\lambda \to C_i$. Then, by Proposition 1.1.19 we have

$$N_{E_i^\alpha/Z_\alpha} \cong \pi_{\alpha,i}^*|_{E_i^i}(N_{E_i^i/Z_i}) \otimes \mathcal{O}(\chi_{i,j}(-E_i^\alpha \cap E_j^\alpha))^{\otimes m_j} \otimes \mathcal{O}(\chi_{i,k}(-E_i^\alpha \cap E_k^\alpha))^{\otimes m_k} \otimes \mathcal{O}(\chi_{i,\gamma}(-E_i^\alpha \cap E_\gamma^\alpha))^{\otimes m_\gamma},$$

where $m_j, m_k, m_\gamma \in \mathbb{Z}_+$, with $m_j, m_k, m_\gamma \geq 1$, and either $\chi_{i,j}$ or $\chi_{i,k}$ or $\chi_{i,\gamma}$ are not equal to 0. By applying Proposition 1.2.22 we have that

$$\begin{aligned}(e_i^\alpha)^2 \cdot e_j^\alpha &= j_{E_i^\alpha *}(\varsigma_i \cdot (-(1 + \chi_{i,j} m_j + \chi_{i,k} m_k + \chi_{i,\gamma} m_\gamma)\varsigma_i)),\\ &= j_{E_i^\alpha *}(-(1 + \chi_{i,j} m_j + \chi_{i,k} m_k + \chi_{i,\gamma} m_\gamma)(\varsigma_i)^2),\\ (e_i^\alpha)^2 \cdot e_k^\alpha &= j_{E_i^\alpha *}(\varsigma_i \cdot (-(1 + \chi_{i,j} m_j + \chi_{i,k} m_k + \chi_{i,\gamma} m_\gamma)\varsigma_i)),\\ &= j_{E_i^\alpha *}(-(1 + \chi_{i,j} m_j + \chi_{i,k} m_k + \chi_{i,\gamma} m_\gamma)(\varsigma_i)^2),\\ (e_i^\alpha)^2 \cdot e_\gamma^\alpha &= j_{E_i^\alpha *}(\varsigma_i \cdot (-(1 + \chi_{i,j} m_j + \chi_{i,k} m_k + \chi_{i,\gamma} m_\gamma)\varsigma_i)),\\ &= j_{E_i^\alpha *}(-(1 + \chi_{i,j} m_j + \chi_{i,k} m_k + \chi_{i,\gamma} m_\gamma)(\varsigma_i)^2),\end{aligned}$$

so condition a.ii fails to be true since either $\chi_{i,j}$ or $\chi_{i,k}$ or $\chi_{i,\gamma}$ are not equal to 0 by our hypothesis. □

**Definition 6.2.8.** *Given an irreducible exceptional component $E_i^\alpha$, such that $E_i^\alpha \cap E_j^\alpha, E_i^\alpha \cap E_k^\alpha \neq \emptyset$ then we will say that*

- *a $E_i^\alpha$ has an admissible proximity configuration of type I with respect to $E_j^\alpha$ and $E_k^\alpha$ if it satisfies the relations I in Proposition 6.1.7,*
- *b $E_i^\alpha$ has an admissible proximity configuration of type II with respect to $E_j^\alpha$ and $E_k^\alpha$ if it satisfies the relations II in Proposition 6.1.7.*
- *c $E_i^\alpha$ has an admissible proximity configuration of type III with respect to $E_j^\alpha$ and $E_k^\alpha$ if if it satisfies the relations III in Proposition 6.1.7,*
- *d $E_i^\alpha$ has an admissible proximity configuration of type IV with respect to $E_j^\alpha$ and $E_k^\alpha$ if it satisfies the relations IV in Proposition 6.1.7.*

**Proposition 6.2.9.** *Let $\pi : Z_s \to Z_0$ be a sequential morphism, and $E_i$ a final divisor verifying that there exists just one index $j$ such that $E_i \cap E_j \neq \emptyset$. Then the classes $e_i$ and $e_j$ of $E_i$ and $E_j$ (respectively) do not satisfy any of the following relations:*

- *a $(e_i)^3 = 3e_i \cdot (e_j)^2$, $(e_i)^2 \cdot e_j = -2e_i \cdot (e_j)^2$, and $(e_j)^3 = 0$,*
- *b $(e_i)^3 = 2e_i \cdot (e_j)^2$, $(e_i)^2 \cdot e_j = -\frac{3}{2}e_i \cdot (e_j)^2$, and $(e_j)^3 = -\frac{1}{2}e_i \cdot (e_j)^2$.*

*Proof.* Since $E_i \cap E_j \neq \emptyset$ and $E_i$ is final then either $C_i \to C_j$ or $C_i \xrightarrow{t} C_j$. In the latter case, as $e_i \cdot (e_j)^2 = 0$, if condition a or condition b hold, it would imply that

$(e_i)^3 = (e_i)^2 \cdot e_j = 0$. In the former case, that is $C_i \to C_j$ we have to distinguish between the following cases:

- A either $dim(C_j) = 0$,
- B or $C_j$ is a rational curve and
  - B.i either $C_j$ is a "new" curve, so $E_j^{i-1} = P(\mathcal{O}(a) \oplus \mathcal{O}(a - \delta_j))$,
    - B.i.i and either $[C_i] = \varsigma_j + (a - \delta_j)f \in A^1(E_j^{i-1})$,
    - B.i.ii or $[C_i] = \varsigma_j + af \in A^1(E_j^{i-1})$;
  - B.ii or $C_j$ is an unmodified "old" curve,
    - B.ii.i and either there exists an index $k \neq i$ such that $E_k \cap E_j \neq \emptyset$, so
      - B.ii.i.i either $[C_i] = \varsigma_j + (a - \delta_j)f \in A^1(E_j^{i-1})$,
      - B.ii.i.ii or $[C_i] = \varsigma_j + af \in A^1(E_j^{i-1})$;
    - B.ii.ii or there not exists an index $k$ such that $E_k \cap E_j \neq \emptyset$ with $k \neq i$ so
      - B.ii.ii.i either $[C_i] = \varsigma_j + (a - \delta_j)f \in A^1(E_j^{i-1})$,
      - B.ii.ii.ii or $[C_i] = \varsigma_j + af \in A^1(E_j^{i-1})$,
      - B.ii.ii.iii or $[C_i] = \varsigma_j + (a + m)f \in A^1(E_j^{i-1})$,
      - B.ii.ii.iv or $[C_i] = f$,
      - B.ii.ii.v or $[C_i] = 2\varsigma_j + 2af \in A^1(E_j^{i-1})$, if $\delta_j = 1$,
      - B.ii.ii.vi or $[C_i] = b\varsigma_j + (ba - b\delta_j + 1)f \in A^1(E_j^{i-1})$, with $b > 0$, if $\delta_j = 0$.

To begin with, let us consider the case A. Since $j$ is the only index verifying that $E_i \cap E_j \neq \emptyset$, then there can not exists any other index $k$ such that $C_j \to C_k$. Moreover, we have that $E_j^{i-1} \cong \mathbb{P}^2$, so $[C_i] = \gamma_i \varsigma_j \in A^1(E_j^{i-1})$ and it follows that

$$\begin{aligned}
(e_i)^3 &= -\gamma_i^2 + \gamma_i, \\
(e_i)^2 \cdot e_j &= \gamma_i^2, \\
e_i \cdot (e_j)^2 &= -\gamma_i^2 - \gamma_i, \\
(e_j)^3 &= (1 + \gamma_i)^2.
\end{aligned}$$

As a result, if either condition a or condition b holds, this would imply that $\gamma_i < 0$.

Now let us suppose that $C_j$ is a rational curve. Then $E_j^{i-1} = P(\mathcal{O}(a) \oplus \mathcal{O}(a - \delta_j))$. Furthermore, let us suppose that $C_i$verifies that $[C_i] = \varsigma_j + (a - \delta)f \in A^1(E_j^{i-1})$, Thus, we have that:

$$\begin{aligned} (e_i)^3 &= -a + \delta_j, \\ (e_i)^2 \cdot e_j &= -\delta_j, \\ e_i \cdot (e_j)^2 &= a + \delta_j. \end{aligned}$$

Moreover, in the subcases B.i.i and B.ii.iii it is verified that

$$(e_j)^3 = -4a,$$

so if either condition a or condition b holds, this would imply that $a = \delta_j = 0$.
On the other hand, that is in the subcase B.ii.ii, we have that

$$(e_j)^3 = -6a + 3\delta_j,$$

so if either condition a or condition b holds, this would imply that $a = \delta_j = 0$.

Now, let us suppose that the class of $C_i$ verifies that $[C_i] = \varsigma_j + (a)f \in A^1(E_j^{i-1})$, Thus, we have that:

$$\begin{aligned} (e_i)^3 &= -a, \\ (e_i)^2 \cdot e_j &= \delta_j, \\ e_i \cdot (e_j)^2 &= a - 2\delta_j. \end{aligned}$$

Moreover, in the subcases B.i.ii and B.ii.iiii it is verified that

$$(e_j)^3 = -4a + 4\delta_j,$$

so if either condition a or condition b holds, this would imply that $a = \delta_j = 0$.
On the other hand, that is in the subcase B.ii.iii, we have that

$$(e_j)^3 = -6a + 3\delta_j,$$

so if either condition a or condition b holds, this would imply that $a = \delta_j = 0$.

In the subcase B.ii.iiiv, we have that $(e_i)^2 \cdot e_j = 0$, so if either condition a or condition b holds, this would imply that $a = \delta_j = 0$.

In the subcase B.ii.iiv, we have that $E_j^{i-1} \cong P(\mathcal{O}(a) \oplus \mathcal{O}(a-1))$, and

$$\begin{aligned} (e_i)^3 &= -2a - 2, \\ (e_i)^2 \cdot e_j &= 4, \\ e_i \cdot (e_j)^2 &= 2a - 6, \\ (e_j)^3 &= -6a + 9, \end{aligned}$$

so neither condition a nor condition b is satisfied.

Finally, in the subcase B.ii.iivi, we have that $E_j^{i-1} \cong P(\mathcal{O}(a) \oplus \mathcal{O}(a))$, and the following relations hold

$$\begin{aligned} (e_i)^3 &= -2b - ba + 1, \\ (e_i)^2 \cdot e_j &= 2b, \\ e_i \cdot (e_j)^2 &= -2b + ba - 1, \\ (e_j)^3 &= -2ab - 2a + 2b + 2, \end{aligned}$$

so if either condition a or condition b holds, then $a = 0$. □

**Definition 6.2.10.** *We will say that an irreducible exceptional component $E_i$ has an admissible final configuration whenever it satisfies:*

- *a If there exists just one index $j$ such that $E_i \cap E_j \neq \emptyset$, then*
  - *a.i either $(e_j + e_i)^2 \cdot e_i = 0$ with the following exceptions:*
    - *i. $(e_i)^3 = 3e_i \cdot (e_j)^2$, $(e_i)^2 \cdot e_j = -2e_i \cdot (e_j)^2$, and $(e_j)^3 = 0$,*
    - *ii. $(e_i)^3 = 2e_i \cdot (e_j)^2$, $(e_i)^2 \cdot e_j = -\frac{3}{2}e_i \cdot (e_j)^2$, and $(e_j)^3 = -\frac{1}{2}e_i \cdot (e_j)^2$,*
  - *a.ii or $(e_j)^2 \cdot e_i = 0$ and $e_j \cdot (e_i)^2 = -\eta$.*
- *b If the cardinal set of indexes $\{\gamma\}$ such that $E_i \cap E_\gamma \neq \emptyset$ is greater or equal to 2, $\#\{\gamma\} \geq 2$, then it verifies one of the conditions stated in Proposition 6.1.7 with respect to any pair $\{j, k\} \subset \{\gamma\}$, that is, $E_i$ has an admissible proximity configuration with respect to $E_j$ and $E_k$. Moreover, in case the irreducible exceptional component $E_i$ has an admissible proximity configuration of type III, then it is with respect to at most two irreducible exceptional components, and if it has an admissible proximity configuration of type IV then it is with respect to at most three irreducible exceptional components.*

c *There exists at most one index* $\gamma$ *such that* $(e_\gamma)^2 \cdot e_i < 0$.

d *If there exists some index* $\beta$ *such that* $(e_i+e_\beta)^2 \cdot e_i = 0$, *with* $(e_i)^3 > 0$ *and* $e_i \cdot (e_\beta)^2 = 0$, *then* $E_i$ *verifies the conditions of Proposition 6.1.11 about the cardinality of the set of index* $\{\gamma\}$ *verifying* $E_i \cap E_\gamma \neq \emptyset$. *Moreover, if it has an admissible proximity configuration of type III with respect to* $E_j$ *and* $E_\beta$ *then* $E_i$ *verifies Proposition 6.2.1 and Corollary 6.2.2 (if the other hypothesis also hold), or if it has an admissible proximity configuration of type I with respect to* $E_\beta$ *and* $E_j$ *then it verifies Lemma 6.2.3 (if the other hypothesis are verified too).*

e *If there exists some index* $\lambda$ *such that* $(e_i)^2 \cdot e_\lambda = -1$, $e_i \cdot (e_\lambda)^2 = 0$ *that also verifies the above conditions then*

e.i *if there exists some index* $\mu$ *such that* $E_\lambda$ *has an admissible proximity configuration of type III with respect to* $E_i$ *and* $E_\mu$, *then* $E_i$ *already verifies the above conditions and the same relations with respect to all the same indexes but* $E_\lambda$ *just by replacing* $e_i$ *by* $\bar{e}_i = (e_i + e_\lambda)$ *and* $e_\mu$ *by* $\bar{e}_\mu = (e_\mu + e_\lambda)$ *in the computations, and it also satisfies* $(\bar{e}_i)^2 \cdot \bar{e}_\mu = -1$ *and* $\bar{e}_i \cdot (\bar{e}_\mu)^2 = 0$,

e.ii *otherwise,* $E_i$ *already verifies the above conditions and the same relations with respect to all the same indexes but* $E_\lambda$ *just by replacing* $e_i$ *by* $\bar{e}_i = (e_i + e_\lambda)$ *in the computations.*

**Theorem 6.2.11.** *An irreducible exceptional component* $E_i \subset Z_s$ *is a final divisor for the sequential morphism* $\pi : Z_s \to Z_0$ *if and only if* $E_i$ *has an admissible final configuration.*

*Proof.* To begin with, let us suppose that $E_i$ is final. Then, by Definition 2.3.1 $E_i$ is isomorphic to either $\mathbb{F}_{\delta_i}$ or $\mathbb{P}^2$, and its normal bundle $N_{E_i/Z_s}$ verifies $N_{E_i/Z_s} \cong \mathcal{O}(-1)$. As a consequence of Propositions 6.1.7 and 6.2.9 and Theorem 6.1.5, $E_i$ verifies either condition a or b on Definition 6.2.10. Moreover, since $(e_\gamma)^2 \cdot e_i = j_{E_i *}(([E_i \cap E_\gamma])^2)$, where $[E_i \cap E_\gamma] \in A^1(E_i)$, then it follows from Theorem 5.3.3 that condition c is also satisfied. Finally, if there exists some index $\beta$ such that $(e_i + e_\beta)^2 \cdot e_i = 0$, with $(e_i)^3 > 0$ and $e_i \cdot (e_\beta)^2 = 0$, then $C_i \to C_\beta$ and $E_i \cong \mathbb{F}_0$, so condition d is verified too, and condition e follows directly from Theorem 6.1.5 and the Projection formula 1.6.

Let us now suppose that $E_i$ is not final with respect to the sequential morphism $\pi : Z_s \to Z_0$, that is there not exists any sequence of point and rational curve blow-ups

$(Z_0, ..., Z_s, \pi)$ realizing it, for which $E_i$ is final. Firstly, let $j$ be an index such that either $C_j \to C_i$ or $C_j \xrightarrow{t} C_i$, but such that there not exists any other index $\gamma$ with $C_\gamma$ proximate to both $C_i$ and $C_j$, or proximate and $t-$proximate to $C_i$ and $C_j$ respectively (or vice versa). Now, if we consider the blow-up corresponding to the $(j-1)-$level of a sequence realizing the sequential morphism $\pi : Z_s \to Z_0$, that is $\pi_j : Z_j \to Z_{j-1}$, since $E_j^j$ is final for the sequential morphism $\pi_{j,0} : Z_j \to Z_0$, then one of the following conditions is verified:

A either $C_j$ is proximate to $C_i$, and then it is satisfied that:

$$(e_i^j + e_j^j)^2 \cdot e_j^j = 0. \tag{6.36}$$

B or $C_j$ is $t-$proximate to $C_i$ and then the following relations hold:

$$(e_j^j)^2 \cdot e_i^j = -\eta_j, \tag{6.37}$$

$$e_j^j \cdot (e_i^j)^2 = 0, \tag{6.38}$$

with $\eta_j \in \mathbb{Z}_+$.

To begin with, let us consider the particular case A, that is $C_j \to C_i$. Then, we need to distinguish between the three following cases:

A.i either the following relation is satisfied:

$$(e_i^j + e_j^j)^2 \cdot e_i^j = 0, \tag{6.39}$$

A.ii or the following relations are verified:

$$(e_i^j)^2 \cdot e_j^j = -\eta_i, \tag{6.40}$$

$$e_i^j \cdot (e_j^j)^2 = 0, \tag{6.41}$$

A.iii otherwise.

Firstly let us consider the case A.i. By adding Equations (6.36) and (6.39) we get to the following relation

$$(e_i^j + e_j^j)^3 = 0, \tag{6.42}$$

and as a consequence of the Projection formula 1.6, we have that

$$(e_i^{j-1})^3 = 0. \tag{6.43}$$

Now, by Proposition 6.1.8, the set of indexes $\{\gamma\}$ verifying $E_i^{j-1} \cap E_\gamma^{j-1} \neq \emptyset$ is non empty. In fact, one of the following characterizations is satisfied:

- A.i.i either $E_i^{j-1}$ is not final for the sequential morphism $\pi_{j-1,0} : Z_{j-1} \to Z_0$, with $dim(C_i) = 0$, and there exists just one index $k$ such that $C_k \to C_i$, with $dim(C_k) = 0$,
- A.i.ii or $E_i^{j-1}$ is not final for the sequential morphism $\pi_{j-1,0} : Z_{j-1} \to Z_0$, where $C_i$ is an unmodified "old" curve, that is, there exists at least one index $\gamma$ such that $C_\gamma \to C_i$, with $C_\gamma$ non isomorphic to a generic fiber $F_i$ of $E_i^i$,
- A.i.iii or $E_i^{j-1}$ is final for the sequential morphism $\pi_{j-1,0} : Z_{j-1} \to Z_0$, where $C_i$ is a modified "old" curve, that is, there exists at least one index $\gamma$ such that
  - A.i.iii.i either $C_\gamma \to C_i$, with $C_\gamma$ isomorphic to a generic fiber $F_i$ of $E_i^i$,
  - A.i.iii.ii or $C_i \xrightarrow{t} C_\gamma$,
- A.i.iv or $E_i^{j-1}$ is final or not final for the sequential morphism $\pi_{j-1,0} : Z_{j-1} \to Z_0$, where $C_i$ is a "new" curve.

To begin with, let us suppose that the cardinal of the set of indexes $\gamma$ such that $E_i^{j-1} \cap E_\gamma^{j-1}$, $\#\{\gamma\}$, is equal to 1, and $C_j = E_i^{j-1} \cap E_\gamma^{j-1}$. Then either $C_i \to C_\gamma$ or $C_\gamma \to C_i$. In the former case, it follows from Proposition 6.1.8 that either relations

$$(e_\gamma^{j-1})^2 \cdot e_i^{j-1} = -2a,$$
$$(e_i^{j-1})^2 \cdot e_\gamma^{j-1} = a,$$

or relations

$$(e_\gamma^{j-1})^2 \cdot e_i^{j-1} = 2a,$$
$$(e_i^{j-1})^2 \cdot e_\gamma^{j-1} = -a,$$

are satisfied. Thus, $(e_j^j)^3 = \pm a$, and it follows from Proposition 1.2.33 that

$$\begin{aligned}(e_i^j)^2 \cdot e_j^j &= (\pi_j^*(e_i^{j-1}) - e_j^j)^2 \cdot e_j^j,\\ &= -2\pi_j^*(e_i^{j-1}) \cdot (e_j^j)^2 + (e_j^j)^3;\\ e_i^j \cdot (e_j^j)^2 &= (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot (e_j^j)^2,\\ &= (\pi_j^*(e_i^{j-1})) \cdot (e_j^j)^2 - (e_j^j)^3,\end{aligned}$$

and

$$\begin{aligned}(e_i^j)^3 &= (\pi_j^*(e_i^{j-1}) - e_j^j)^3,\\ &= 3(\pi_j^*(e_i^{j-1}) \cdot (e_j^j)^2 - (e_j^j)^3.\end{aligned}$$

As a result, if $(e_j^j)^3 = a$ then we have that:

$$\begin{aligned}(e_i^j)^2 \cdot e_j^j &= 3a,\\ e_i^j \cdot (e_j^j)^2 &= -2a,\\ (e_i^j)^3 &= -4a,\end{aligned}$$

and if $(e_j^j)^3 = -a$ the following relations are satisfied:

$$\begin{aligned}(e_i^j)^2 \cdot e_j^j &= -3a,\\ e_i^j \cdot (e_j^j)^2 &= 2a,\\ (e_i^j)^3 &= 4a.\end{aligned}$$

In both cases, we can conclude that $E_i^j$ does not have an admissible final configuration, as the previous relations correspond to the exceptions to condition a in Definition 6.2.10. In the latter case, that is $C_\gamma \to C_i$, firstly, let us suppose that $E_i^{\gamma-1} = P(\mathcal{O}(a) \oplus \mathcal{O}(a-\delta_i))$ and $C_\gamma$ is isomorphic to a section, that is, $[C_\gamma] = \varsigma_i + c_1(\frac{\mathcal{O}(a)\oplus\mathcal{O}(a-\delta_i)}{L}f \in A^1(E_i^{\gamma-1}$. It follows from Proposition 1.1.19 that $N_{E_i^\gamma/Z_\gamma} \cong \pi_\gamma^*(N_{E_i^{\gamma-1}}) \otimes \mathcal{O}(-E_i^\gamma \cap E_\gamma^\gamma)^{\otimes m_\gamma}$. Now, by Proposition 1.2.22 and Lemma 2.2.1 we have that:

$$(e_i^\gamma)^3 = j_{E_i^\gamma *}((c_1(\pi_\gamma^*(N_{E_i^{\gamma-1}}) \otimes \mathcal{O}(-E_i^\gamma \cap E_\gamma^\gamma)^{\otimes m_\gamma}))^2), \tag{6.44}$$

$$= j_{E_i^\gamma *}((-\varsigma_i - m_\gamma(\varsigma_i + (2a - \delta_i - c_1(L))f))^2), \tag{6.45}$$

$$= j_{E_i^\gamma *}((1 + m_\gamma)((-2a + \delta_i)(1 - m_\gamma) - 2m_\gamma c_1(L))\varsigma_i \cdot f), \tag{6.46}$$

and

$$
\begin{aligned}
(e_i^\gamma)^2 \cdot e_\gamma^\gamma &= j_{E_i^\gamma *}((c_1(pi_\gamma^*(N_{E_i^{\gamma-1}/Z_{\gamma-1}}) \otimes \mathcal{O}(-E_i^\gamma \cap E_\gamma^\gamma)^{\otimes m_\gamma})) \cdot \left[E_i^\gamma \cap E_\gamma^\gamma\right]), \\
&= j_{E_i^\gamma *}((c_1(pi_\gamma^*(N_{E_i^{\gamma-1}/Z_{\gamma-1}}))) \cdot \left[E_i^\gamma \cap E_\gamma^\gamma\right] - m_\gamma(\left[E_i^\gamma \cap E_\gamma^\gamma\right])^2); \\
e_i^\gamma \cdot (e_\gamma^\gamma)^2 &= j_{E_\gamma^\gamma *}(c_1(N_{E_\gamma^\gamma/Z_\gamma} \cdot \left[E_i^\gamma \cap E_\gamma^\gamma\right]), \\
&= j_{E_i^\gamma *}((\left[E_i^\gamma \cap E_\gamma^\gamma\right])^2).
\end{aligned}
$$

Since $(e_i^\gamma)^3 = 0$, then it follows from Equation (6.46) that $m_\gamma = 1$ and $c_1(L) = 0$. As a result, $(c_1(pi_\gamma^*(N_{E_i^{\gamma-1}}))) \cdot \left[E_i^\gamma \cap E_\gamma^\gamma\right] = 0$ and we can conclude that $(e_i^\gamma)^2 \cdot e_\gamma^\gamma = -e_i^\gamma \cdot (e_\gamma^\gamma)^2$. Now, let us suppose that $e_i^\gamma \cdot (e_\gamma^\gamma)^2 = a$. Then we have that $(e_j^j)^3 = 0$, and by applying Proposition 1.2.33 we get that:

$$
\begin{aligned}
(e_i^j)^2 \cdot e_j^j &= (\pi_j^*(e_i^{j-1}) - e_j^j)^2 \cdot e_j^j, \\
&= -2(\pi_j^*(e_i^{j-1})) \cdot (e_j^j)^2 + (e_j^j)^3, \\
&= -2a; \\
e_i^j \cdot (e_j^j)^2 &= (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot (e_j^j)^2, \\
&= (\pi_j^*(e_i^{j-1})) \cdot (e_j^j)^2 - (e_j^j)^3, \\
&= a,
\end{aligned}
$$

and

$$
\begin{aligned}
(e_i^j)^3 &= (\pi_j^*(e_i^{j-1}) - e_j^j)^3, \\
&= (\pi_j^*(e_i^{j-1}))^3 + 3(\pi_j^*(e_i^{j-1})) \cdot (e_j^j)^2 - (e_j^j)^3, \\
&= 3a.
\end{aligned}
$$

As a result, we can conclude that $E_i^j$ does not have an admissible final configuration, as the previous relations correspond to the exceptions to condition a in Definition 6.2.10. Now, if $E_i^{\gamma-1} \cong \mathbb{P}^2$, then $C_\gamma$ is a point and $E_i^\gamma \cong \mathbb{F}_1$ by Proposition 1.1.17. Let us denote by $\widetilde{\varsigma}_i$ to $c_1(\mathcal{O}_{E_i^\gamma}(1))$. It follows from Proposition 1.1.19 that $N_{E_i^\gamma/Z_\gamma} \cong \pi_\gamma^*(N_{E_i^{\gamma-1}/Z_{\gamma-1}}) \otimes \mathcal{O}(-E_i^\gamma \cap E_\gamma^\gamma)$. By applying Proposition 1.2.22 and Lemma 2.2.1 we have that:

$$
\begin{aligned}
(e_i^\gamma)^3 &= j_{E_i^\gamma *}((c_1(pi_\gamma^*(N_{E_i^{\gamma-1}}) \otimes \mathcal{O}(-E_i^\gamma \cap E_\gamma^\gamma)))^2), && (6.47) \\
&= j_{E_i^\gamma *}((-\pi_\gamma^*(\varsigma_i) - (\left[E_i^\gamma \cap E_\gamma^\gamma\right])^2), && (6.48) \\
&= j_{E_i^\gamma *}((-\widetilde{\varsigma}_i - \widetilde{\varsigma}_i + f)^2), && (6.49)
\end{aligned}
$$

and

$$
\begin{aligned}
(e_i^\gamma)^2 \cdot e_\gamma^\gamma &= j_{E_i^\gamma *}((c_1(pi_\gamma^*(N_{E_i^{\gamma-1}}) \otimes \mathcal{O}(-E_i^\gamma \cap E_\gamma^\gamma))) \cdot \left[E_i^\gamma \cap E_\gamma^\gamma\right]),\\
&= j_{E_i^\gamma *}(-(\left[E_i^\gamma \cap E_\gamma^\gamma\right])^2),\\
&= j_{E_i^\gamma *}(-(\widetilde{\varsigma}_i - f)^2),\\
&= j_{E_i^\gamma *}(\widetilde{\varsigma}_i \cdot f);\\
e_i^\gamma \cdot (e_\gamma^\gamma)^2 &= j_{E_\gamma^\gamma *}(c_1(N_{E_\gamma^\gamma/Z_\gamma}) \cdot \left[E_i^\gamma \cap E_\gamma^\gamma\right]),\\
&= j_{E_i^\gamma *}((\left[E_i^\gamma \cap E_\gamma^\gamma\right])^2),\\
&= j_{E_i^\gamma *}((\widetilde{\varsigma}_i - f)^2),\\
&= j_{E_i^\gamma *}((-1)\widetilde{\varsigma}_i \cdot f).
\end{aligned}
$$

As a result, $(e_j^j)^3 = 0$, and by applying Proposition 1.2.33 we get that:

$$
\begin{aligned}
(e_i^j)^2 \cdot e_j^j &= (\pi_j^*(e_i^{j-1}) - e_j^j)^2 \cdot e_j^j,\\
&= -2(\pi_j^*(e_i^{j-1})) \cdot (e_j^j)^2 + (e_j^j)^3,\\
&= 2;\\
e_i^j \cdot (e_j^j)^2 &= (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot (e_j^j)^2,\\
&= (\pi_j^*(e_i^{j-1})) \cdot (e_j^j)^2 - (e_j^j)^3,\\
&= -1,
\end{aligned}
$$

and

$$
\begin{aligned}
(e_i^j)^3 &= (\pi_j^*(e_i^{j-1}) - e_j^j)^3,\\
&= (\pi_j^*(e_i^{j-1}))^3 + 3(\pi_j^*(e_i^{j-1})) \cdot (e_j^j)^2 - (e_j^j)^3,\\
&= -3.
\end{aligned}
$$

Thus, we can conclude that $E_i^j$ does not have an admissible final configuration, as the previous relations correspond to the exceptions to condition a in Definition 6.2.10.

If there are more than one index $\gamma$ verifying $E_i^{j-1} \cap E_\gamma^{j-1} \neq \emptyset$ or $C_j \neq E_i^\gamma \cap E_\gamma^\gamma$, then we have to distinguish between the different subcases.
In the particular subcase A.i.i, since $C_k$ and $C_j$ are both proximate to $C_i$, then it follows from Proposition 1.1.19 that $N_{E_i^j/Z_j} \ncong \mathcal{O}_{E_i^j}(-1)$. By applying Propositions 6.2.6 and 6.2.7 we can conclude that $E_i^j$ does not have an admissible proximity configuration neither

of type III nor of type IV with respect to $E_k^j$ and $E_j^j$. Now, we study the only remaining case corresponding to an admissible proximity configuration of type I. In this case either $C_j \to C_i, C_k$, with $dim(C_j) = 0$, or $C_j \xrightarrow{t} C_k$. In the former case, by Proposition 1.2.33 we have that

$$\begin{aligned} e_i^j \cdot (e_k^j)^2 &= (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot (\pi_j^*(e_k^{j-1}) - e_j^j)^2, \\ &= \pi_j^*(e_i^{j-1}) \cdot (\pi_j^*(e_k^{j-1}))^2 - (e_j^j)^3, \end{aligned}$$

so $e_i^j \cdot (e_k^j)^2 = -2 \neq 0$, and then $E_i^j$ fails to have an admissible proximity configuration of type I with respect to $E_k^j$ and $E_j^j$. With respect to the latter case, that is $C_j \xrightarrow{t} C_k$, since the blow-up $\pi_j : Z_j \to Z_{j-1}$ gives rive to an extrinsic elementary modification with respect to $E_i^{j-1}$, then we have that

$$\begin{aligned} e_i^j \cdot (e_k^j)^2 &= (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot (\pi_j^*(e_k^{j-1}))^2, \\ &= \pi_j^*(e_i^{j-1}) \cdot (\pi_j^*(e_k^{j-1}))^2, \\ &= \pi_j^*(e_i^{j-1} \cdot (e_k^{j-1})^2), \end{aligned}$$

so $e_i^j \cdot (e_k^j)^2 = -1 \neq 0$, and in this case $E_i^j$ fails also to have an admissible proximity configuration of type I with respect to $E_k^j$ and $E_j^j$.

In the subcase A.i.ii, let $k \in \{\gamma\}$ be an index such that $C_k \to C_i$, but that there not exists any index $\gamma \neq j$ such that $C_\gamma$ is proximate to both $C_i$ and $C_k$, or proximate and $t-$proximate to $C_i$ and $C_k$ respectively (or vice versa). Since $C_k$ and $C_j$ are both proximate to $C_i$, then $N_{E_i^j/Z_j} \ncong \mathcal{O}_{E_i^j}(-1)$ by Proposition 1.1.19. By considering Propositions 6.2.6 and 6.2.7 we can conclude that $E_i^j$ has not an admissible proximity configuration neither of type III nor of type IV with respect to $E_k^j$ and $E_j^j$. The only remaining case, that is corresponding to have an admissible proximity configuration of type I, it implies that either $C_j \to C_i, C_k$, with $dim(C_j) = 0$, or $C_j \xrightarrow{t} C_k$. In the former case, that is

$C_j \to C_i, C_k$, as a consequence of Proposition 1.2.33 we have that

$$\begin{aligned} e_i^j \cdot (e_k^j)^2 &= (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot (\pi_j^*(e_k^{j-1}) - e_j^j)^2, \\ &= \pi_j^*(e_i^{j-1}) \cdot (\pi_j^*(e_k^{j-1}))^2 - (e_j^j)^3, \\ &= \pi_j^*(e_i^{j-1} \cdot (e_k^{j-1})^2) - (e_j^j)^3; \\ (e_i^j)^2 \cdot (e_k^j) &= (\pi_j^*(e_i^{j-1}) - e_j^j)^2 \cdot (\pi_j^*(e_k^{j-1}) - e_j^j), \\ &= (\pi_j^*(e_i^{j-1}))^2 \cdot \pi_j^*(e_k^{j-1}) - (e_j^j)^3, \\ &= (\pi_j^*((e_i^{j-1})^2 \cdot e_k^{j-1}) - (e_j^j)^3, \end{aligned}$$

and

$$\begin{aligned} e_i^j \cdot e_j^j \cdot e_k^j &= (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot e_j^j \cdot (\pi_j^*(e_k^{j-1}) - e_j^j), \\ &= (e_j^j)^3. \end{aligned}$$

In order to satisfy condition c, then the following relations must be satisfied:

$$\pi_j^*((e_i^{j-1})^2 \cdot e_k^{j-1}) = 0, \tag{6.50}$$

$$\pi_j^*(e_i^{j-1} \cdot (e_k^{j-1})^2) = 1, \tag{6.51}$$

From equation 6.51, it follows that $E_i^{j-1} \cong \mathbb{F}_1$, in particular $E_i^{j-1} \cong P(\mathcal{O}(a) \oplus \mathcal{O}(a-1))$ . Moreover, we know that $E_i^{k-1}$ must be final for the sequential morphism $\pi_{k-1,0} : Z_{k-1} \to Z_0$ (otherwise $E_i^j$ could not have an admissible proximity configuration), so

$$\left[E_i^k \cap E_k^k\right] = \varsigma_i + af \in A^1(E_i^k).$$

Now, as a consequence of Proposition 1.1.19 we have that $N_{E_i^k/Z_k} \cong \pi_k^*(N_{E_i^{k-1}/Z_{k-1}}) \otimes \mathcal{O}(-E_i^k \cap E_k^k)$, so by Proposition 1.2.22

$$\begin{aligned} (e_i^k)^2 \cdot e_k^k &= j_{E_i^k *}((\varsigma_i + af) \cdot (-(1+m_k)\varsigma - m_k af), \\ &= j_{E_i^k *}((a - 1 - m_k)\varsigma_i \cdot f); \\ (e_i^k)^3 &= j_{E_i^k *}((-(1+m_k)\varsigma_i - m_k af)^2), \\ &= j_{E_i^k *}((-2a(1+m_k) + (1+m_k)^2)\varsigma_i \cdot f). \end{aligned}$$

In order to satisfy Equation (6.50) then $a = 1 + m_k$, but since $(e_i^k)^3 = 0$ then $a = \frac{1+m_k}{2}$, so we get to a contradiction.

With respect to the latter case, that is $C_j \xrightarrow{t} C_k$, since the blow-up $\pi : Z_j \to Z_{j-1}$ defines an extrinsic elementary modification with respect to $E_i^{j-1}$, then by Proposition 1.2.33 we have

$$\begin{aligned} e_i^j \cdot (e_k^j)^2 &= (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot \pi_j^*(e_k^{j-1})^2, \\ &= \pi_j^*(e_i^{j-1} \cdot \pi_j^*(e_k^{j-1})^2, \\ &= \pi_j^*(e_i^{j-1} \cdot (e_k^{j-1})^2); \\ (e_i^j)^2 \cdot e_k^j) &= (\pi_j^*(e_i^{j-1}) - e_j^j)^2 \cdot \pi_j^*(e_k^{j-1}), \\ &= (\pi_j^*(e_i^{j-1}))^2 \cdot \pi_j^*(e_k^{j-1}) + (e_j^j)^2 \cdot \pi_j^*(e_k^{j-1}), \\ &= \pi_j^*((e_i^{j-1})^2 \cdot (e_k^{j-1})) + (e_j^j)^2 \cdot \pi_j^*(e_k^{j-1}). \end{aligned}$$

Moreover, it is satisfied that

$$\begin{aligned} e_i^j \cdot e_j^j \cdot e_k^j &= (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot e_j^j \cdot \pi_j^*(e_k^{j-1}), \\ &= -(e_j^j)^2 \cdot \pi_j^*(e_k^{j-1}), \end{aligned}$$

so $(e_i^j)^2 \cdot e_k^j) \neq -e_i^j \cdot e_j^j \cdot e_k^j$ unless $\pi_j^*((e_i^{j-1})^2 \cdot (e_k^{j-1})) = 0$, but in order to get an admissible proximity configuration of type IV the relation $e_i^j \cdot (e_k^j)^2 = \pi_j^*(e_i^{j-1} \cdot (e_k^{j-1})^2) = 0$ must be satisfied too. Thus $(e_i^{k-1})^3 = 0$, and by Proposition 1.1.19 this implies $N_{C_k/Z_{k-1}} \cong \mathcal{O} \oplus \mathcal{O}$, which is an absurd.

In the case A.i.iii.i, let $k \in \{\gamma\}$ be an index such that $C_k \to C_i$, with $C_k$ isomorphic to a fiber $F_i$ of $E_i^i$, but there not exists any index $\gamma \neq j$ such that $C_\gamma$ is proximate to both $C_i$ and $C_k$, or proximate and $t-$proximate to $C_i$ and $C_k$ respectively (or vice versa). Now, since $C_j$ and $C_k$ are both proximate to $C_i$, with $C_j$ non isomorphic to a fiber $F_i$ of $E_i^i$, by applying Propositions 6.2.6 and 6.2.7 we can conclude that $E_i^j$ has not an admissible proximity configuration neither of type III nor of type IV with respect to $E_k^j$ and $E_j^j$. It may have an admissible proximity configuration of type I, but in this case we have that either $dim(C_j) = 0$ and $C_j$ is proximate to both $C_i$ and $C_k$, or $dim(C_j) = 1$ and $C_j$ is $t-$proximate to $C_k$. In the former case, we have

$$\begin{aligned} e_i^j \cdot (e_k^j)^2 &= (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot (\pi_j^*(e_k^{j-1}) - e_j^j)^2, \\ &= \pi_j^*(e_i^{j-1}) \cdot \pi_j^*(e_k^{j-1})^2 - (e_j^j)^3; \\ (e_i^j)^2 \cdot (e_k^j) &= (\pi_j^*(e_i^{j-1}) - e_j^j)^2 \cdot (\pi_j^*(e_k^{j-1}) - e_j^j), \\ &= \pi_j^*(e_i^{j-1})^2 \cdot \pi_j^*(e_k^{j-1}) - (e_j^j)^3, \end{aligned}$$

and

$$e_i^j \cdot e_j^j \cdot e_k^j = (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot e_j^j \cdot (\pi_j^*(e_k^{j-1}) - e_j^j),$$
$$= (e_j^j)^3,$$

so $e_i^j \cdot (e_k^j)^2 = -1 \neq 0$ and $E_i^j$ fails to have an admissible proximity configuration of type I with respect to $E_k^j$ and $E_j^j$.

With respect to the latter case, since the blow-up $\pi : Z_j \to Z_{j-1}$ defines an extrinsic elementary modification with respect to $E_i^{j-1}$ we have that

$$(e_i^j)^2 \cdot e_k^j = (\pi_j^*(e_i^{j-1}) - e_j^j)^2 \cdot \pi_j^*(e_k^{j-1}),$$
$$= \pi_j^*((e_i^{j-1})^2 \cdot \pi_j^*(e_k^{j-1}) + (e_j^j)^2 \cdot \pi_j^*(e_k^{j-1}),$$

and

$$e_i^j \cdot e_j^j \cdot e_k^j = (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot e_j^j \cdot \pi_j^*(e_k^{j-1}),$$
$$= -(e_j^j)^2 \cdot \pi_j^*(e_k^{j-1}),$$

so $(e_i^j)^2 \cdot e_k^j \neq -e_i^j \cdot e_j^j \cdot e_k^j$ and $E_i^j$ fails also to have an admissible proximity configuration of type I with respect to $E_j^j$ and $E_k^j$.

In the case A.i.iii.ii, let $k \in \{\gamma\}$ be an index such that $C_i \xrightarrow{t} C_k$ but that there not exists any index $\gamma \neq j$ such that $C_\gamma$ is proximate to both $C_i$ and $C_k$, or proximate and $t-$proximate (or vice versa). Now, since $C_j$ is proximate to $C_i$, as a consequence of Propositions 6.2.6 and 6.2.7 we can conclude that $E_i^j$ has not an admissible proximity configuration neither of type III nor of type IV with respect to $E_j^j$ and $E_k^j$. In order to get an admissible proximity configuration of type I, then either $dim(C_j) = 0$ and $C_j$ is proximate to both $C_i$ and $C_k$ or $dim(C_j) = 1$ and it is $t-$proximate to $C_k$. In the former case, we have

$$e_i^j \cdot (e_k^j)^2 = (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot (\pi_j^*(e_k^{j-1}) - e_j^j)^2,$$
$$= \pi_j^*(e_i^{j-1}) \cdot \pi_j^*(e_k^{j-1})^2 - (e_j^j)^3,$$

so $e_i^j \cdot (e_k^j)^2 = -1 \neq 0$ and $E_i^j$ fails to have an admissible proximity configuration of type I with respect to $E_j^j$ and $E_k^j$. In the latter case, since the blow-up $\pi : Z_j \to Z_{j-1}$ defines an extrinsic elementary modification with respect to $E_i^{j-1}$, by Proposition 1.2.33 we have that

$$(e_i^j)^2 \cdot e_k^j = (\pi_j^*(e_i^{j-1}) - e_j^j)^2 \cdot \pi_j^*(e_k^{j-1}),$$
$$= \pi_j^*((e_i^{j-1})^2 \cdot \pi_j^*(e_k^{j-1}) + (e_j^j)^2 \cdot \pi_j^*(e_k^{j-1}),$$

and

$$e_i^j \cdot e_j^j \cdot e_k^j = (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot e_j^j \cdot \pi_j^*(e_k^{j-1}),$$
$$= -(e_j^j)^2 \cdot \pi_j^*(e_k^{j-1}),$$

so $(e_i^j)^2 \cdot e_k^j \neq -e_i^j \cdot e_j^j \cdot e_k^j$ and $E_i^j$ also fails to have an admissible proximity configuration of type I with respect to $E_j^j$ and $E_k^j$.

In the subcase A.i.iv let us suppose that $E_i^{j-1}$ is final, and let $k$ the index such that $C_i \to C_k$, but that there not exists any index $\gamma \neq j$ such that $C_\gamma$ is proximate to both $C_i$ and $C_k$, or proximate and $t-$proximate. Since $C_j \to C_i$, then by Proposition 1.1.19 we have that $N_{E_i^j/Z_j} \not\cong \mathcal{O}(-1)$. It follows then from Propositions 6.2.6 and 6.2.7 that $E_i^j$ has not an admissible proximity configuration neither of type III nor of type IV with respect to $E_k^j$ and $E_j^j$. In order to get an admissible proximity configuration of type I, then either $dim(C_j) = 0$ and $C_j$ is proximate to both $C_i$ and $C_k$ or $dim(C_j) = 1$ and it is $t-$proximate to $C_k$. In the former case, we have

$$e_i^j \cdot (e_k^j)^2 = (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot (\pi_j^*(e_k^{j-1}) - e_j^j)^2,$$
$$= \pi_j^*(e_i^{j-1}) \cdot \pi_j^*(e_k^{j-1})^2 - (e_j^j)^3;$$
$$(e_i^j)^2 \cdot (e_k^j) = (\pi_j^*(e_i^{j-1}) - e_j^j)^2 \cdot (\pi_j^*(e_k^{j-1}) - e_j^j),$$
$$= \pi_j^*(e_i^{j-1})^2 \cdot \pi_j^*(e_k^{j-1}) - (e_j^j)^3,$$

and

$$e_i^j \cdot e_j^j \cdot e_k^j = (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot e_j^j \cdot (\pi_j^*(e_k^{j-1}) - e_j^j),$$
$$= (e_j^j)^3,$$

so $e_i^j \cdot (e_k^j)^2 = \pm 2a - 1 \neq 0$ and $E_i^j$ fails to have an admissible proximity configuration of type I with respect to $E_j^j$ and $E_k^j$. In the latter case, since the blow-up $\pi : Z_j \to Z_{j-1}$ defines an extrinsic elementary modification with respect to $E_i^{j-1}$ we have that

$$(e_i^j)^2 \cdot e_k^j = (\pi_j^*(e_i^{j-1}) - e_j^j)^2 \cdot \pi_j^*(e_k^{j-1}),$$
$$= \pi_j^*((e_i^{j-1})^2 \cdot \pi_j^*(e_k^{j-1}) + (e_j^j)^2 \cdot \pi_j^*(e_k^{j-1}),$$

and

$$e_i^j \cdot e_j^j \cdot e_k^j = (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot e_j^j \cdot \pi_j^*(e_k^{j-1}),$$
$$= -(e_j^j)^2 \cdot \pi_j^*(e_k^{j-1}),$$

so $E_i$ has not an admissible proximity configuration of type I with respect to $E_j$ and $E_{j-1}$ either.

Let us now consider the case A.ii, that is, we have that

$$(e_i^j)^2 \cdot e_j^j = -\eta_i,$$
$$e_i^j \cdot (e_j^j)^2 = 0.$$

We shall now distinguish between two different cases:

A.ii.i either $\eta_i = 1$,

A.ii.ii or $\eta_i > 1$.

In the subcase A.ii.i, we should distingish two cases. If the index $i$ is the only one verifying that $E_j^j \cap E_i^j \neq \emptyset$, then since $E_j^j$ is final for the sequential morphism $\pi_{j,0} : Z_j \to Z_0$, then by Corollary 6.1.6 $E_j^j \cap E_i^j \cong C_j$ must be isomorphic to a fiber $F_i$ of $E_i^i$. We should then consider the next index $k < j$ satisfying that $C_k \to C_i$ or $C_k \xrightarrow{t} C_i$ but such that there not exists any index $\gamma$ such that $C_\gamma$ is proximate to both $C_i$ and $C_k$, or proximate and $t-$proximate to $C_i$ and $C_k$ respectively (or vice versa).
Now, let us suppose that there exists other indexes $\{\beta\}$, with $\beta \neq i$, such that $E_j^j \cap E_\beta^j \neq \emptyset$ and $E_i^j \cap E_\beta^j \neq \emptyset$. Let $k \in \{\beta\}$ be an index such that $C_k \to C_i$ or $C_k \xrightarrow{t} C_i$ (or vice versa), but that there not exists any index $\lambda \neq j$ such that $C_\lambda$ is proximate to both $C_i$ and $C_k$, or proximate and $t-$proximate (or vice versa). Now, since $C_j$ is proximate to $C_i$, it follows from Proposition 1.1.19 that $N_{E_i^j/Z_j} \ncong \mathcal{O}(-1)$. As a consequence of Propositions 6.2.6 and 6.2.7 we can conclude that $E_i^j$ has not an admissible proximity configuration neither of type III nor of type IV with respect to $E_j^j$ and $E_k^j$. In all possible cases $C_j$ must be $t-$proximate to $C_k$. Now, we should distinguish between the following subcases:

A.ii.i.i either $C_k$ is $t-$proximate to $C_i$,

A.ii.i.ii or $C_k$ is proximate to $C_i$,

A.ii.i.iii or $C_i$ is $t-$proximate to $C_k$,

A.ii.i.iv or $C_i$ is proximate to $C_k$.

In the subcase A.ii.i.i, let us suppose that $E_i^{k-1} \cong P(\mathcal{O} \oplus \mathcal{O}(a - \delta_i))$. Since $E_j^j$ is final for the sequential morphism $\pi_{j,0} : Z_j \to Z_0$, in particular $(e_i^j + e_j^j)^2 \cdot e_j^j = 0$, so $(e_j^j)^3 = 1$. In order to get an admissible proximity configuration of type I for $E_i^j$, then $e_i^j \cdot e_j^j \cdot e_k^j = \eta_i = 1$. As $E_j^j$ is final for $\pi_{j,0}$ so it must be verified that $(e_j^j)^2 \cdot e_k^j = -\eta_i = -1$. Now, since the blow-up $\pi_k : Z_k \to Z_{k-1}$ defines an intrinsic elementary modification of $E_i^{k-1}$ then by Proposition 1.2.22 and Lemma 2.2.1 we have that

$$\begin{aligned}(e_j^j)^2 \cdot e_i^j &= j_{E_j^j *}(c_1(N_{E_j^j/Z_j}) \cdot \left[S_{\delta_i+2n_j}\right] - \left[E_i^j \cap E_k^j\right]),\\ &= j_{E_i^j *}((\left[S_{\delta_i+2n_j}\right] - \left[E_i^j \cap E_k^j\right])^2),\\ &= j_{E_i^j *}((\left[S_{\delta_i+2n_j}\right])^2) + j_{E_i^j *}((\left[E_i^j \cap E_k^j\right])^2),\end{aligned}$$

so $j_{E_i^j *}((\left[S_{delta_i+2n_j}\right])^2) = 1$ as $(e_j^j)^2 \cdot e_i^j = 0$. It follows then that $\delta_i = 1$ and $n_j = 0$, so in particular, $[C_j] = \left[S_1^{j-1}\right] \in A^1(E_i^{j-1})$, where $[S_1] = \varsigma_i + af \in A^1(E_i^{k-1})$. In addition, as a consequence of Proposition 1.1.19 we have that $N_{E_i^j/Z_j} \cong \pi_{j,k}^*(N_{E_i^j/Z_j}) \otimes \mathcal{O}(-E_i^j \cap E_j^j)^{\otimes m_j}$, so by Proposition 1.2.22 we have that

$$\begin{aligned}(e_i^j)^2 \cdot e_j^j &= j_{E_i^j *}(c_1(\pi_{j,k}^*(N_{E_i^{j-1}/Z_{j-1}}) \otimes \mathcal{O}(-E_i^j \cap E_j^j)^{\otimes m_j}) \cdot \left[E_i^j \cap E_j^j\right]),\\ &= j_{E_i^j *}(c_1(\pi_{j,k}^*(N_{E_i^j/Z_j})) \cdot \left[\widetilde{S}_1\right]),\\ &= j_{E_i^j *}(-\varsigma_i \cdot (\varsigma_i + af - \left[E_i^j \cap E_k^j\right])),\\ &= j_{E_i^j *}((a-1)\varsigma_i \cdot f).\end{aligned}$$

As a result, since $(e_i^j)^2 \cdot e_j^j = -1$, then $a = 0$, that is, $E_i^{k-1} \cong P(\mathcal{O} \oplus \mathcal{O}(-1))$. Now, as a consequence of Proposition 6.1.9 there exists at least one index $\gamma$ such that either $C_i \xrightarrow{t} C_\gamma$ or $C_i \to C_\gamma$. In the former case $E_j^j \cap E_i^j \cap E_\gamma^j \neq \emptyset$ so $E_i^j$ fails to have an admissible proximity configuration. In the latter case, either $E_k^j \cap E_\gamma^j = \emptyset$ so $E_i^j$ fails also to have an admissible proximity configuration, or there exists another index $\lambda$ such that $C_i \to C_\lambda$.

In the subcase A.ii.i.ii, we consider first $dim(C_k) = 0$. Let us suppose that $E_i^{k-1} = P(\mathcal{O} \oplus \mathcal{O}(a - \delta_i))$. Since $E_j^j$ is final for the sequential morphism $\pi_{j,0} : Z_j \to Z_0$, then $(e_i^j + e_j^j)^2 \cdot e_j^j = 0$, so in particular $(e_j^j)^3 = 1$. In order to get an admissible proximity configuration of type I for $E_i^j$, then $e_i^j \cdot e_j^j \cdot e_k^j = \eta_i = 1$, so since $E_j^j$ is final it must be verified that $(e_j^j)^2 \cdot e_k^j = -\eta_i = -1$. Since the blow-up $\pi_k : Z_k \to Z_{k-1}$ defines a mixed elementary modification of $E_i^{k-1}$ then by Proposition 1.2.22 and Lemma 2.2.1

$$\begin{aligned}(e_j^j)^2 \cdot e_i^j &= j_{E_j^j *}(c_1(N_{E_j^j/Z_j}) \cdot \left[S_{\delta_i+2n_j}\right] - \left[E_i^j \cap E_k^j\right]), \\ &= j_{E_i^j *}((\left[S_{\delta_i+2n_j}\right] - \left[E_i^j \cap E_k^j\right])^2), \\ &= j_{E_i^j *}((\left[S_{\delta_i+2n_j}\right])^2) + j_{E_i^j *}((\left[E_i^j \cap E_k^j\right])^2).\end{aligned}$$

Since $(e_j^j)^2 \cdot e_i^j = 0$, so $j_{E_i^j *}((\left[S_{delta_i+2n_j}\right])^2) = 1$, and it follows that $\delta_i = 1$ and $n_j = 0$. In particular, $[C_j] = \left[S_1^{j-1}\right] \in A^1(E_i^{j-1})$, where $[S_1] = \varsigma_i + af \in A^1(E_i^{k-1})$. In addition, as a consequence of Proposition 1.1.19 we have that $N_{E_i^j/Z_j} \cong \pi_{j,k}^*(N_{E_i^j/Z_j} \otimes \mathcal{O}(-E_i^k \cap E_k^k)^{\otimes m_k}) \otimes \mathcal{O}(-E_i^j \cap E_j^j)$, so by Proposition 1.2.22 we have that

$$\begin{aligned}(e_i^j)^2 \cdot e_j^j &= j_{E_i^j *}(c_1(\pi_{j,k}^*(N_{E_i^j/Z_j} \otimes \mathcal{O}(-E_i^k \cap E_k^k)^{\otimes m_k}) \otimes \mathcal{O}(-E_i^j \cap E_j^j)) \cdot \left[E_i^j \cap E_j^j\right]), \\ &= j_{E_i^j *}(c_1(\pi_{j,k}^*(N_{E_i^j/Z_j} \otimes \mathcal{O}(-E_i^k \cap E_k^k)^{\otimes m_k})) \cdot \left[E_i^j \cap E_j^j\right]), \\ &= j_{E_i^j *}((-\varsigma_i - m_k \left[E_i^j \cap E_k^j\right]) \cdot (\varsigma_i + af - \left[E_i^j \cap E_k^j\right])), \\ &= j_{E_i^j *}((a - m_k - 1)\varsigma_i \cdot f).\end{aligned}$$

As a result, since $(e_i^j)^2 \cdot e_j^j = -1$, then $a = m_k$, that is, $E_i^{k-1} \cong P(\mathcal{O}(m_k) \oplus \mathcal{O}(m_k - 1))$. Now, by Proposition 1.2.33 we have that

$$\begin{aligned}(e_i^j)^3 &= (\pi_{j,k-1}^* e_i^{k-1} - e_k^j - e_j^j)^3, \\ &= (\pi_j^* e_i^{k-1})^3 + 3(\pi_j^* e_i^{j-1}) \cdot (e_k^j + e_j^j)^2 - (e_j^j)^3,\end{aligned}$$

so $(e_i^j)^3 = -(m_k)^2 - 2m_k + 3$, but we have that

$$\begin{aligned}(e_i^j)^2 \cdot e_k^j &= (\pi_j^*(e_i^{j-1}) - e_j^j)^2 \cdot \pi_j^*(e_k^{j-1}), \\ &= (e_j^j)^2 \cdot e_k^j.\end{aligned}$$

As a result, $(e_i^j)^2 \cdot e_k^j = m_k - 1$, and $E_i^j$ fails to have an admissible proximity configuration of type I with respect to $E_j^j$ and $E_k^j$.

Let us now suppose that $dim(C_k) = 1$. Then, since both $C_j$ and $C_k$ are proximate to $C_i$, its corresponding blow-ups $\pi_k : Z_k \to Z_{k-1}$ and $\pi_j : Z_j \to Z_{j-1}$ define an extrinsic elementary modification of $E_i^{k-1}$ and $E_i^{j-1}$, respectively. Now, it follows from the relation $(e_j^j)^2 \cdot e_i^j = 0$ that $E_i^{k-1} \cong E_i^{j-1} \cong E_i^j = P(\mathcal{O}(a) \oplus \mathcal{O}(a)) \cong \mathbb{F}_0$. Moreover, in order to get an admissible proximity configuration of type I then $e_i^j \cdot e_j^j \cdot e_k^j = \eta_i = 1$. If we denote by $[C_j], \left[E_i^{j-1} \cap E_k^{j-1}\right] \in A^1(E_i^{j-1})$ to its classes, then $[C_j] = \varsigma_i + af$ and

$\left[E_i^{j-1} \cap E_k^{j-1}\right] = f$. As a consequence of Proposition 1.1.19 we have that

$$N_{E_i^j/Z_j} \cong \pi_j^*(\pi_{j-1,k-1}^*(N_{E_i^{k-1}}) \otimes \mathcal{O}(-E_i^k \cap E_k^k)^{\otimes m_k}) \otimes \mathcal{O}(-E_i^j \cap E_j^j),$$

where $m_k, m_j \in \mathbb{Z}_+$, so by Proposition 1.2.22

$$\begin{aligned}
(e_i^j)^2 \cdot e_j^j &= j_{E_i^j *}(c_1(N_{E_i^j/Z_j}) \cdot \left[E_i^j \cap E_j^j\right]),\\
&= j_{E_i^j *}((a - m_k)\varsigma_i \cdot f);\\
(e_i^j)^2 \cdot e_k^j &= j_{E_i^j *}(c_1(N_{E_i^j/Z_j}) \cdot \left[E_i^j \cap E_k^j\right]),\\
&= j_{E_i^j *}((-1 - m_j)\varsigma_i \cdot f).
\end{aligned}$$

Moreover, we have that

$$\begin{aligned}
(e_i^j)^3 &== j_{E_i^j *}((c_1(N_{E_i^j/Z_j}))^2),\\
&== j_{E_i^j *}((2a(-1 - m_j) + 2m_k(1 + m_j))\varsigma_i \cdot f),
\end{aligned}$$

so in order to get an admissible proximity configuration of type I, $m_j, m_k$ satisfy:

$$\begin{aligned}
a - m_k &= -1,\\
2a(-1 - m_j) + 2m_k(1 + m_j) + 2(-1 - m_j) &= 0.
\end{aligned}$$

Moreover, since the following relations hold

$$\begin{aligned}
(e_j^j)^3 &= 1,\\
(e_j^j)^2 \cdot e_i^j &= 0,\\
e_j^j \cdot (e_i^j)^2 &= -1,\\
(e_j^j)^2 \cdot e_k^j &= -1,\\
e_j^j \cdot (e_k^j)^2 &= 0,
\end{aligned}$$

then it follows from condition b than the following relation must be verified $(e_i^j + e_j^j + e_k^j)^2 \cdot (e_i^j + e_j^j + e_k^j) = 0$, that is, $(e_i^{j-1} + e_k^{j-1})^2 \cdot e_i^{j-1} = 0$, but this implies $(e_i^{j-1})^3 > 0$ so by Proposition 6.1.11 that there exists another index $\gamma \neq k$, such that $E_\gamma^{j-1} \cap E_i^{j-1} \cap E_k^{-1} \neq \emptyset$, and $E_i^j$ fails to have an admissible final configuration.

In the subcase A.ii.i.iii, as $\pi_j : Z_j \to Z_{j-1}$ defines an extrinsic elementary modification of $E_i^{j-1}$ and we know that $(e_j^j)^2 \cdot e_i^j = 0$, then $E_i^{j-1} \cong P(\mathcal{O}(a) \oplus \mathcal{O}(a))$, and in particular $[C_j] = \varsigma_i + af \in A^1(E_i^{j-1})$. Moreover, since $E_j^j$ is final for the sequential morphism

$\pi_{j,0} : Z_j \to Z_0$, then $(e_i^j + e_j^j)^2 \cdot e_j^j = 0$ so $(e_j^j)^3 = 1$. Now by Proposition 1.2.33 we have that:

$$\begin{aligned}(e_i^j)^2 \cdot e_j^j &= (\pi_j^*(e_i^{j-1}) - e_j^j)^2 \cdot e_j^j,\\ &= -2\pi_j^*(e_i^{j-1}) \cdot (e_j^j)^2 + (e_j^j)^3,\\ &= a,\end{aligned}$$

so since $(e_i^j)^2 \cdot e_j^j = -1$, then $a = -1$, that is $E_i^{j-1} \cong P(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$. As a consequence of Proposition 6.1.11 there must exist at least another index $\gamma \neq k$ such that $E_i^{j-1} \cap E_\gamma^{j-1} \neq \emptyset$, and it follows from Corollary 6.2.5 that if $(e_i^{j-1})^2 \cdot e_\gamma^{j-1} = -1$ then there must exist another index $\lambda \neq \gamma$ such that $E_i^{j-1} \cap E_\lambda^{j-1} \neq \emptyset$. We can conclude then that $E_i^j$ fails to have an admissible final configuration.
In the subcase A.ii.i.iv, since the blow-up $\pi_j : Z_j \to Z_{j-1}$ defines an extrinsic elementary modification of $E_i^{j-1}$ and $(e_j^j)^2 \cdot e_i^j = 0$, then $E_i^{j-1} \cong P(\mathcal{O}(a) \oplus \mathcal{O}(a))$. In order to get an admissible proximity configuration of type I then $e_i^j \cdot e_j^j \cdot e_k^j = \eta_i = 1$. If we denote by $[C_j], \left[E_i^{j-1} \cap E_k^{j-1}\right] \in A^1(E_i^{j-1})$ to its classes, then $[C_j] = f$ and $\left[E_i^{j-1} \cap E_k^{j-1}\right] = \varsigma_i + (a + n_k)f$. As a consequence of Theorem 6.1.5 $E_i^j$ is final, so we get to a contradiction with our initial hypothesis ($E_i$ not final).

In the subcase A.ii.ii, as a consequence of Proposition 6.1.13, $E_i^{j-1}$ verifies one of the following characterizations:

- A.ii.ii.i either $E_i^{j-1} \cong \mathbb{F}_1$ is not final, with $dim(C_i) = 0$, and $C_j$ is isomorphic to a fiber of $E_i^{j-1}$;
- A.ii.ii.ii or $E_i^{j-1} \cong \mathbb{F}_0$ is final, with base a modified "old" curve $C_i$ with a modification of type $I$, that is, such that there exists at least one index $\beta$, with $C_\beta \to C_i$ and $C_\beta$ isomorphic to a fiber of $E_i^i$, and $C_j \cong S_0$;
- A.ii.ii.iii or $E_i^{j-1} \cong \mathbb{F}_0$ is final, with base an "new" curve $C_i$, and $C_j \cong S_0$,
- A.ii.ii.iv or $E_i^{j-1}$ is not final, it is a birational model of $\mathbb{F}_1$ and there exists just one index $\beta$ such that either $dim(C_\beta) = 0$, $C_\beta \to C_i$, with $C_\beta \in S_1$, or $dim(C_\beta) = 1$, $C_\beta \xrightarrow{t} C_i$, with $E_i^{\beta-1} \cap C_\beta \in S_1$, and $C_j \cong S_1^\beta$,
- A.ii.ii.v or $E_i^{j-1}$ is not final, it is a birational model of $\mathbb{F}_\delta$,and there exists just one index $\beta$ such that $dim(C_\beta) = 1$, $C_\beta \xrightarrow{t} C_i$, verifying

$$\#\left\{C_\beta \cap E_i^{\beta-1}\right\} = \delta + 2n,\ E_i^{\beta-1} \cap C_\beta \in S_{\delta+2n},\ \text{and } C_j \cong S^{\beta}_{\delta+2n},$$

A.ii.ii.vi or $E_i^{j-1} \cong \mathbb{F}_\delta$ is not final, there exists at least one index $\beta$ such that $C_\beta \to C_i$, and $C_j$ is isomorphic to a fiber $F_i$ of $E_i^i$.

In the subcase A.ii.ii.i, let $k$ the index such that $C_k \to C_i$, but that there not exists any index $\gamma \neq j$ such that $C_\gamma$ is proximate to both $C_i$ and $C_k$, or proximate and $t-$proximate (or vice versa). Now, since $C_j$ and $C_k$ are proximate to $C_i$, then $N_{E_i^j} \not\cong \mathcal{O}(-1)$ by Proposition 1.1.19, and as a consequence of Propositions 6.2.6 and 6.2.7 we can conclude that $E_i^j$ has not an admissible proximity configuration neither of type III nor of type IV with respect to $E_j^j$ and $E_k^j$. It may have an admissible proximity configuration of type I, and then $C_j$ must be $t-$proximate to $C_k$, so by Proposition 1.2.33 we have that

$$\begin{aligned} e_i^j \cdot (e_j^j)^2 &= (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot (e_j^j)^2, \\ &= (\pi_j^*(e_i^{j-1}) \cdot (e_j^j)^2 - (e_j^j)^3; \\ (e_i^j)^2 \cdot e_j^j &= (\pi_j^*(e_i^{j-1}) - e_j^j)^2 \cdot e_j^j, \\ &= -2(\pi_j^*(e_i^{j-1}) \cdot (e_j^j)^2 + (e_j^j)^3, \\ &= -(e_j^j)^3; \end{aligned}$$

and

$$\begin{aligned} e_i^j \cdot e_j^j \cdot e_k^j &= (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot e_j^j \cdot \pi_j^*(e_k^{j-1}), \\ &= -(\pi_j^*(e_k^{j-1})) \cdot (e_j^j)^2. \end{aligned}$$

By our hypotheses $e_i^j \cdot (e_j^j)^2 = 0$, so since $e_i^j \cdot e_j^j \cdot e_k^j = 1$ and $(e_i^j)^2 \cdot e_j^j = -\eta_i$, with $\eta_i > 1$, then $E_i^\alpha$ fails to have an admissible proximity configuration of type I with respecto to $E_k^j$ and $E_j^j$.

In the subcase A.ii.ii.ii, let $k$ the index such that $C_k \to C_i$, with $C_k$ isomorphic to a fiber of $E_i^i$, but that there not exists any index $\gamma \neq j$ such that $C_\gamma$ is proximate to both $C_i$ and $C_k$, or proximate and $t-$proximate (or vice versa). Now, since $C_j$ is proximate to $C_i$ and it is not isomorphic to a fiber, then as a consequence of Propositions 6.2.6 and 6.2.7 we can conclude that $E_i^j$ has not an admissible proximity configuration neither of type III nor of type IV with respect to $E_j^j$ and $E_k^j$. Since $C_j$ is isomorphic to a section, then it is $t-$proximate to $C_k$, so by Proposition 1.2.22 and Proposition 1.2.33 we have that

$$(e_j^j)^2 \cdot e_k^j = j_{E_j^j *}(f \cdot -\varsigma_j),$$
$$= j_{E_j^j *}((-1)\varsigma_j \cdot f),$$

and

$$e_i^j \cdot e_j^j \cdot e_k^j = (\pi j^*(e_i^{j-1}) - e_j^j) \cdot e_j^j \cdot \pi j^*(e_k^{j-1}),$$
$$= -(e_j^j)^2 \cdot \pi j^*(e_k^{j-1}),$$
$$= -(e_j^j)^2 \cdot e_k^j,$$

As a result, $e_i^j \cdot e_j^j \cdot e_k^j = 1 \neq -(e_i^j)^2 \cdot e_j^j = -\eta$, and $E_i^\alpha$ fails to have an admissible proximity configuration of type I with respecto to $E_k^j$ and $E_j^j$.

In the subcase A.ii.ii.iii, let $k$ be an index such that $C_i \to C_k$. We have proved in Proposition 6.1.11 that there exists at least one index $\gamma$, with $\gamma \neq k$, such that

$$E_i^{j-1} \cap E_\gamma^{j-1} \cap E_k^{j-1} \neq \emptyset. \tag{6.52}$$

We should distinguish between two cases: either $C_j \cong E_i^{j-1} \cap E_k^{j-1}$, or $C_j$ is isomorphic to a section of $E_i^{j-1}$ and then $E_j^j \cap E_k^j = \emptyset$. In both subcases, since $C_j$ is proximate to $C_i$ and it is not isomorphic to a fiber of $E_i^i$, it follows from Propositions 6.2.6 and 6.2.7 that $E_i^j$ has not an admissible proximity configuration neither of type III nor of type IV with respect to $E_j^j$ and $E_k^j$. Now, let us consider the subcase $C_j \cong E_i^{j-1} \cap E_k^{j-1}$. Since $E_i^{j-1}$ is final, then $N_{E_i^{j-1}/Z_{j-1}} \cong \mathcal{O}_{E_i^{j-1}}(-1)$. Moreover, $E_i^{j-1} \cong \mathbb{F}_0$, in particular $E_i^{j-1} = P(\mathcal{O}(-\eta_i) \oplus \mathcal{O}(-\eta_i))$, so by Proposition 1.2.33 we have that

$$(e_i^{j-1})^3 = j_{E_i^{j-1} *}((-\varsigma_i)^2),$$
$$= j_{E_i^{j-1} *}(2\eta_i\varsigma_i \cdot f).$$

Firstly, let us suppose that $C_i \xrightarrow{t} C_\gamma$. Then by Proposition 1.2.33 we have that

$$(e_i^{j-1})^2 \cdot e_\gamma^{j-1} = j_{E_i^{j-1} *}(\eta_\gamma f \cdot (-\varsigma_i)),$$
$$= j_{E_i^{j-1} *}((-\eta_\gamma)\varsigma_i \cdot f),$$
$$e_i^{j-1} \cdot (e_\gamma^{j-1})^2 = j_{E_i^{j-1} *}((-\eta_\gamma f)^2),$$
$$= j_{E_i^{j-1} *}(0\varsigma_i \cdot f).$$

Moreover, the class of $E_j^j \cap E_i^j$ verifies $\left[E_j^j \cap E_i^j\right] = \varsigma_i - \eta f \in A^1(E_i^j)$, so by Propositions 1.1.19 and 1.2.22

$$\begin{aligned}
(e_i^j)^3 &= j_{E_i^j *}((-(1+m_j)\varsigma_i + m_j\eta_j f)^2),\\
&= j_{E_i^j *}(2\eta(1+m_j)\varsigma_i \cdot f);\\
(e_i^j)^2 \cdot e_\gamma^j &= j_{E_i^j *}(\eta_\gamma f \cdot (-(1+m_j)\varsigma_i + m_j\eta_j f)),\\
&= j_{E_i^j *}(-\eta_\gamma(1+m_j)\varsigma_i \cdot f),
\end{aligned}$$

Now, since $E_j^j$ is final for the sequential morphism $\pi_{j,0} : Z_j \to Z_0$, and $C_j \xrightarrow{t} C_\gamma$ then

$$\begin{aligned}
(e_j^j)^2 \cdot e_\gamma^j &= j_{E_j^j *}(\eta_\gamma f \cdot (-\varsigma_j)),\\
&= j_{E_j^j *}((-\eta_\gamma)\varsigma_j \cdot f),
\end{aligned}$$

and

$$\begin{aligned}
e_i^j \cdot e_j^j \cdot e_\gamma^j &= (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot e_j^j \cdot \pi_j^*(e_\gamma^{j-1}),\\
&= -(e_j^j)^2 \cdot \pi_j^*(e_\gamma^{j-1}),\\
&= -(e_j^j)^2 \cdot e_\gamma^j.
\end{aligned}$$

In order to get an admissible proximity configuration of type I, with respect to $E_j^j$ and $E_\gamma^j$, then the following relations must hold:

$$\begin{aligned}
(e_i^j)^3 + 2(e_i^j)^2 \cdot e_\gamma^j &= 0,\\
e_i^j \cdot e_j^j \cdot e_\gamma^j &= -(e_i^j)^2 \cdot e_j^j,
\end{aligned}$$

so $\eta_i = \eta_\gamma$. It follows then by Lemma 6.2.3 that $C_i$ is isomorphic to a fiber of $E_k^k$ and $(e_i^{j-1})^3 = 2$, that is $\eta_i = 1$. According to this, we have then that

$$\begin{aligned}
(e_i^j)^3 &= 2(1+m_j)\\
(e_i^j)^2 \cdot e_\gamma^j &= -(1+m_j),\\
e_i^j \cdot (e_\gamma^j)^2 &= 0,\\
(e_i^j)^2 \cdot e_j^j &= -1,\\
e_i^j \cdot (e_j^j)^2 &= 0,
\end{aligned}$$

where $m_j \geq 1$. Applying again Lemma 6.2.3, we can conclude that $E_i^\alpha$ does not have an admissible proximity configuration of type I, with respect to $E_j^j$ and $E_\gamma^j$.

If we consider now the case where $C_i \to C_\gamma$, then as a consequence of Proposition 6.2.1 there exists another index $\lambda \neq k, \gamma$ such that $C_i \xrightarrow{t} C_\lambda$, so we can proceed as above.
In the subcase corresponding to $C_j$ isomorphic to a section of $E_i^{j-1}$, since $C_j$ is isomorphic to a section, then $\pi_j : Z_j \to Z_{j-1}$ is an extrinsic elementary modification with respect to $E_i^{j-1}$ and by Proposition 1.2.33 we have that

$$\begin{aligned} e_i^j \cdot e_j^j \cdot e_k^j &= (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot e_j^j \cdot \pi_j^*(e_k^{j-1}), \\ &= -(e_j^j)^2 \cdot \pi_j^*(e_k^{j-1}), \\ &= -(e_j^j)^2 \cdot e_k^j, \end{aligned}$$

but $(e_j^j)^2 \cdot e_k^j = 0$, so $E_i^j$ fails to have an admissible proximity configuration of type I with respecto to $E_k^j$ and $E_j^j$.
In the subcase A.ii.ii.iv, since $E_j^j$ is final for the sequential morphism $\pi_{j,0} : Z_j \to Z_0$ and $C_j \xrightarrow{t} C_\beta$ then

$$\begin{aligned} (e_j^j)^2 \cdot e_\beta^j &= j_{E_j^j *}(f \cdot (-\varsigma_j)), \\ &= j_{E_j^j *}((-1)\varsigma_j \cdot f). \end{aligned}$$

Moreover,

$$\begin{aligned} e_i^j \cdot e_j^j \cdot e_\beta^j &= (\pi_j^*(e_i^{j-1}) - e_j^j) \cdot e_j^j \cdot \pi_j^*(e_\beta^{j-1}), \\ &= -(e_j^j)^2 \cdot \pi_j^*(e_\beta^{j-1}), \\ &= (e_j^j)^2 \cdot e_\beta^j, \end{aligned}$$

so in order to have an admissible proximity configuration of type I then $(e_i^j)^2 \cdot e_j^j = -1$. Moreover, we know that $E_i^{\beta-1} \cong \mathbb{F}_1$, in particular $E_i^{\beta-1} \cong P(\mathcal{O}(a) \oplus \mathcal{O}(a-1))$, and $\left[E_j^j \cap E_i^j\right] = \left[S_1^\beta\right]$, where $[S_1] = \varsigma_i + af \in A^1(E_i^{\beta-1})$. Since $\pi_\beta : Z_\beta \to Z_{\beta-1}$ is an intrinsic modification for $E_i^{\beta-1}$, then we have $(e_i^j)^2 \cdot e_j^j = (e_i^{\beta-1})^2 \cdot [S_1]$, and by Proposition 1.2.22

$$\begin{aligned} (e_i^{\beta-1})^2 \cdot [S_1] &= j_{E_i^{\beta-1} *}((\varsigma_i + af) \cdot (-\varsigma_i)), \\ &= j_{E_i^{\beta-1} *}((a-1)\varsigma_i \cdot f)), \end{aligned}$$

so $a = 0$, and then $E_i^{\beta-1} \cong P(\mathcal{O} \oplus \mathcal{O}(-1))$. It follows from Proposition 5.4.3 that $C_i$ can not be an unmodified "old" curve, so either $C_i$ is an "old" curve with a modifications of type $I$ and $II$ or it is a "new curve". In both cases, there exists and index $\gamma$ such that $E^\beta_\gamma \cap E^\beta_i \neq \emptyset$. In order to get an admissible proximity configuration of type I, then $C_\beta \to C_\gamma$ and

$$(e^i_i)^2 \cdot e^i_\gamma = -1,$$
$$e^i_i \cdot (e^i_\gamma)^2 = 1.$$

Since $(e^i_i)^3 = 1$, then as a consequence of Proposition 6.2.4, there must exist some other index $\lambda$, with $\lambda \neq \gamma$, such that $E^i_i \cap E^i_\lambda \neq \emptyset$.

In the subcase A.ii.ii.v, since $E^j_j$ is final for the sequential morphism $\pi_{j,0} : Z_j \to Z_0$, $C_j \cong S^\beta_{\delta+2n}$, and $C_j \xrightarrow{t} C_\beta$ then

$$\begin{aligned}(e^j_j)^2 \cdot e^j_\beta &= j_{E^j_j *}(\delta + 2nf \cdot (-\varsigma_j)),\\ &= j_{E^j_j *}(-(\delta+2n)\varsigma_j \cdot f).\end{aligned}$$

Moreover,

$$\begin{aligned}e^j_i \cdot e^j_j \cdot e^j_\beta &= (\pi^*_j(e^{j-1}_i) - e^j_j) \cdot e^j_j \cdot \pi^*_j(e^{j-1}_\beta),\\ &= -(e^j_j)^2 \cdot \pi^*_j(e^{j-1}_\beta),\\ &= (e^j_j)^2 \cdot e^j_\beta,\end{aligned}$$

so in order to have an admissible proximity configuration of type I then $(e^j_i)^2 \cdot e^j_j = -(\delta + 2n)$. Moreover, we know that $E^{\beta-1}_i \cong \mathbb{F}_\delta$, in particular $E^{\beta-1}_i = P(\mathcal{O}(a) \oplus \mathcal{O}(a-\delta_i))$, and $\left[E^j_j \cap E^j_i\right] = \left[S^\beta_{\delta_i+2n}\right]$, where $[S_{\delta_i+2n}] = \varsigma_i + (a+n)f \in A^1(E^{\beta-1}_i)$. Since $\pi_\beta : Z_\beta \to Z_{\beta-1}$ is an intrinsic elementary modification of $E^{\beta-1}_i$, then we have $(e^j_i)^2 \cdot e^j_j = (e^{\beta-1}_i)^2 \cdot [S_{\delta_i+2n}]$, and by Proposition 1.2.22 we know

$$\begin{aligned}(e^{\beta-1}_i)^2 \cdot [S_1] &= j_{E^{\beta-1}_i *}((\varsigma_i + (a+n)f) \cdot (-\varsigma_i)),\\ &= j_{E^{\beta-1}_i *}((a-\delta-n)\varsigma_i \cdot f)),\end{aligned}$$

so $a = -n$, and $E^{\beta-1}_i = P(\mathcal{O}(-n) \oplus \mathcal{O}(-n-\delta))$. Moreover, we have

$$\begin{aligned}(e^j_i)^2 \cdot e^j_\beta &= (\pi^*_j(e^{j-1}_i) - e^j_j)^2 \cdot \pi^*_j(e^{j-1}_\beta),\\ &= (\pi^*_j(e^{j-1}_i))^2 \cdot \pi^*_j(e^{j-1}_\beta) + (e^j_j)^2 \cdot \pi^*_j(e^{j-1}_\beta),\\ &= (e^j_j)^2 \cdot e^j_\beta,\end{aligned}$$

and

$$\begin{aligned}(e^j_i)^3 &= (\pi^*_j(e^{j-1}_i) - e^j_j)^3,\\ &= (\pi^*_j(e^{j-1}_i))^3 - 3(\pi^*_j(e^{j-1}_i))^2 \cdot e^j_j + 3\pi^*_j(e^{j-1}_i) \cdot (e^j_j)^2 - (e^j_j)^3\end{aligned}$$

It follows from Proposition 5.4.3 that $C_i$ can not be an unmodified "old" curve, so either $C_i$ is an "old" curve with a modifications of type $I$ and $II$ or it is a "new curve". In both cases, there exists and index $\gamma$ such that $E^\beta_\gamma \cap E^\beta_i \neq \emptyset$. In order to get an admissible proximity configuration of type I, then $C_\beta \to C_\gamma$, $C_i \to C_\gamma$, so $n = 0$ as a consequence of Lemma 2.2.1, and

$$\begin{aligned} (e^i_i)^2 \cdot e^i_\gamma &= -\delta_i, \\ e^i_i \cdot (e^i_\gamma)^2 &= \delta_i. \end{aligned}$$

Since $(e^i_i)^3 = \delta_i$ then, as a consequence of Proposition 6.2.4, there must exist some other index $\lambda$, with $\lambda \neq \gamma$, such that $E^i_i \cap E^i_\lambda \neq \emptyset$.
In the subcase A.ii.ii.vi, since $C_k \to C_i$ and $C_k$ is non isomorphic to a fiber of $E^i_i$, then it follows from Propositions 6.2.6 and Proposition 6.2.7 we can conclude that $E^j_i$ has not an admissible proximity configuration neither of type III nor of type IV with respect to $E^j_j$ and $E^j_k$. It may have an admissible configuration of type I. Since $C_j$ is final for the sequential morphism $\pi_{j,0} : Z_j \to Z_0$, and $C_j \xrightarrow{t} C_k$, then

$$\begin{aligned} (e^j_j)^2 \cdot e^j_k &= j_{E^j_j *}(-\eta_j f \cdot (-\varsigma_j)), \\ &= j_{E^j_j *}((-\eta_j)\varsigma_j \cdot f). \end{aligned}$$

Moreover, it follows from Proposition 1.2.33 that

$$\begin{aligned} e^j_i \cdot e^j_j \cdot e^j_k &= (\pi^*_j(e^{j-1}_i) - e^j_j) \cdot e^j_j \cdot \pi^*_j(e^{j-1}_k), \\ &= -(e^j_j)^2 \cdot \pi^*_j(e^{j-1}_k), \\ &= -(e^j_j)^2 \cdot e^j_k, \end{aligned}$$

so in order to have an admissible proximity configuration of type I, then $(e^j_i)^2 \cdot e^j_j = -\eta_j$. Now, as a consequence of Proposition 1.1.19, the normal bundle $N_{E^j_i/Z_j}$ verifies

$$N_{E^j_i/Z_j} \cong \pi^*_{j,k-1}(N_{E^{k-1}_i/Z_{k-1}}) \otimes \mathcal{O}(-E^j_i \cap E^j_k) \otimes \mathcal{O}(-E^j_i \cap E^j_j),$$

so by Proposition 1.2.22

$$\begin{aligned} (e^j_i)^2 \cdot e^j_j &= j_{E^j_i *}(f \cdot c_1(N_{E^j_i/Z_j})), \\ &= j_{E^j_i *}(f \cdot g^*_{j,k-1} c_1(N_{E^{k-1}_i/Z_{k-1}}) - \left[E^j_i \cap E^j_k\right] - f), \\ &= j_{E^j_i *}(f \cdot g^*_{j,k-1} c_1(N_{E^{k-1}_i/Z_{k-1}}) - (\eta_j)\varsigma_i \cdot f). \end{aligned}$$

Since $f \cdot g^*_{j,k-1} c_1(N_{E_i^{k-1}/Z_{k-1}}) \neq 0$, then $E_i^j$ fails to have an admissible proximity configuration of type I with respect to $E_j^j$ and $E_k^j$.

Let us now consider the case B, that is, the case where the following conditions hold:

$$(e_j^j)^2 \cdot e_i^j = -\eta_j,$$
$$e_j^j \cdot (e_i^j)^2 = 0.$$

We shall distinguish between the following two cases:

B.i the case where the following relation is satisfied:

$$(e_i^j + e_j^j)^2 \cdot e_i^j = 0, \tag{6.53}$$

B.ii otherwise.

In the case B.i, we shall distinguish between the following subcases:

B.i.i $\eta_j = 1$,

B.i.ii $\eta_j > 1$.

In the subcase B.i.i, as a consequence of Proposition 6.1.9, $E_i^{j-1}$ must satisfy one of the following characterizations:

B.i.i.i either $E_i^{j-1} \cong \mathbb{P}^2$ is final, with $dim(C_i) = 0$,

B.i.i.ii or $E_i^{j-1}$ is not final, with $dim(C_i) = 0$, and there exists at least an index $\beta$ such that $C_\beta \xrightarrow{t} C_i$, with the cardinal $\eta_k = \#\left\{C_k \cap E_i^{k-1}\right\} > 1$,

B.i.i.iii or $E_i^{j-1}$ is final, $C_i$ is a modified "old" curve with a modification of type $II$, that is, there exists at least one index $\beta$ such that $dim(C_\beta) = 1$, and $C_i \xrightarrow{t} C_\beta$,

B.i.i.iv or $E_i^{j-1}$ is not final, with $C_i$ a modified "old" curve, that is there exists at least one index $\beta$ such that $C_\beta \to C_i$, with $C_\beta$ non isomorphic to a fiber of $E_i^i$,

B.i.i.v or $E_i^{j-1}$ is final or not final, and $C_i$ is a "new" curve.

In the subcase B.i.i.i, it follows from Theorem 6.1.5 that $E_i^j$ is final, so our hypothesis, $E_i$ non final, fails to be true.

In the subcase B.i.i.ii, let $k \in \{\beta\}$ be an index such that $C_k \to C_i$, but that there not exists any index $\gamma \neq j$ such that $C_\gamma$ is proximate to both $C_i$ and $C_k$, or proximate and $t-$proximate to $C_i$ and $C_k$ respectively (or vice versa) Since $C_k \xrightarrow{t} C_i$, with $\eta_k = \#\left\{C_k \cap E_i^{k-1}\right\} > 1$, and $C_j \xrightarrow{t} C_i$, it follows from Propositions 6.2.6 and 6.2.7 that $E_i^j$ can not have an admissible proximity configuration, neither o type III nor of type IV with respect to $E_j^j$ and $E_k^j$. It may have an admissible configuration of type I, and in this case $C_j \to C_k$. However, by Proposition 1.2.33 have that

$$\begin{aligned} e_i^j \cdot (e_k^j)^2 &= \pi_j^*(e_i^{j-1}) \cdot (\pi_j^*(e_k^{j-1}) - e_j^j)^2, \\ &= \pi_j^*(e_i^{j-1}) \cdot (\pi_j^*(e_k^{j-1}))^2 + \pi_j^*(e_i^{j-1}) \cdot (e_j^j)^2, \\ &= \pi_j^*(e_i^{j-1}) \cdot (\pi_j^*(e_k^{j-1}))^2 + e_i^j \cdot (e_j^j)^2; (e_i^j)^2 \cdot e_k^j \quad = (\pi_j^*(e_i^{j-1}))^2 \cdot (\pi_j^*(e_k^{j-1}) - e_j^j), \\ &= \pi_j^*((e_i^{j-1})^2 \cdot e_k^{j-1}); \end{aligned}$$

so $(e_i^j)^2 \cdot e_k^j = 0$ and $e_i^j \cdot (e_k^j)^2 < 0$. Thus, $E_i^j$ fails to have an admissible proximity configuration of type I with respect to $E_j^j$ and $E_k^j$.

In the subcase B.i.i.iii, let $k \in \{\beta\}$ be an index such that $C_k \to C_i$, but that there not exists any index $\gamma \neq j$ such that $C_\gamma$ is proximate to both $C_i$ and $C_k$, or proximate and $t-$proximate to $C_i$ and $C_k$ respectively (or vice versa). Since $C_j \xrightarrow{t} C_i$, then it follows from Propositions 6.2.6 and 6.2.7 that $E_i^j$ can not have an admissible proximity configuration, neither o type III nor of type IV with respect to $E_j^j$ and $E_k^j$. In order to have an admissible proximity configuration of type I, then $C_j \to C_k$, that is $\pi_j : Z_j \to Z_{j-1}$ is an extrinsic elementary modification of $E_k^{j-1}$, so by Proposition 1.2.33 we have that

$$\begin{aligned} (e_k^j)^2 \cdot e_i^j &= (\pi_j^*(e_k^{j-1}) - e_j^j)^2 \cdot \pi_j^*(e_i^{j-1}), \\ &= \pi_j^*((e_k^{j-1})^2 \cdot e_i^{j-1}) + (e_j^j)^2 \cdot e_i^k, \end{aligned}$$

so $(e_k^j)^2 \cdot e_i^j = -1 \neq 0$ and $E_i^j$ fails to have an admissible proximity configuration of type I with respect to $E_j^j$ and $E_k^j$. In the remaining subcases, let $k \in \{\beta\}$ be an index such that $C_k \to C_i$, but that there not exists any index $\gamma \neq j$ such that $C_\gamma$ is proximate to both $C_i$ and $C_k$, or proximate and $t-$proximate to $C_i$ and $C_k$ respectively (or vice versa). Since $C_j \xrightarrow{t} C_i$, then it follows from Propositions 6.2.6 and 6.2.7 that $E_i^j$ can not have an admissible proximity configuration, neither o type III nor of type IV with respect to $E_j^j$ and $E_k^j$. It may have an admissible configuration of type I, and in this case $C_j \to C_k$. Since $\pi_j : Z_j \to Z_{j-1}$ is an intrinsic elementary modification with respect to $E_i^{j-1}$ then by Proposition 1.2.33 we have that

$$\begin{aligned}
(e_i^j)^2 \cdot e_k^j &= (\pi_j(e_i^{j-1}))^2 \cdot (\pi_j(e_k^{j-1}) - e_j^j), \\
&= (\pi_j(e_i^{j-1}))^2 \cdot \pi_j(e_k^{j-1}); \\
e_i^j \cdot (e_k^j)^2 &= \pi_j(e_i^{j-1}) \cdot (\pi_j(e_k^{j-1}) - e_j^j)^2, \\
&= \pi_j(e_i^{j-1}) \cdot (\pi_j(e_k^{j-1}))^2 + \pi_j(e_i^{j-1}) \cdot (e_j^j)^2, \\
&= \pi_j(e_i^{j-1}) \cdot (\pi_j(e_k^{j-1}))^2 + e_i^j \cdot (e_j^j)^2,
\end{aligned}$$

and

$$\begin{aligned}
e_i^j \cdot e_j^j \cdot e_k^j &= \pi_j(e_i^{j-1}) \cdot e_j^j \cdot (\pi_j(e_k^{j-1}) - e_j^j), \\
&= -e_i^j \cdot (e_j^j)^2,
\end{aligned}$$

so in order to get an admissible proximity configuration of type IV with respect to $E_j^j$ and $E_k^j$, the following conditions must hold:

$$\begin{aligned}
(e_i^{j-1})^2 \cdot e_k^{j-1} &= -1, \\
e_i^{j-1} \cdot (e_k^{j-1})^2 &= 1.
\end{aligned} \tag{6.54}$$

Since $(e_j^j)^2 \cdot e_i^j < 0$, there can not exist any other indexes $\beta$ such that $(e_\beta^j)^2 \cdot e_i^j < 0$, so $E_i^{j-1} \cong \mathbb{F}_1$. In particular, it follows from relation 6.54 that $E_i^{k-1} \cong P(\mathcal{O}(a) \oplus \mathcal{O}(a-1))$, so $\left[E_i^{j-1} \cap E_k^{j-1}\right] = \varsigma_i + af \in A^1(E_i^{j-1})$. Let us suppose that $C_k \to C_i$. Then it follows from Proposition 1.1.19 that $N_{E_i^{j-1}/Z_{j-1}} \cong \pi_{j-1,k-1}^*(N_{E_i^{k-1}/Z_{k-1}}) \otimes \mathcal{O}(-E_i^k \cap E_k^k)^{\otimes m_k}$, with $m_k \in \mathbb{Z}_+$, so by Proposition 1.2.22 and Lemma 2.2.1

$$\begin{aligned}
(e_i^{j-1})^2 \cdot e_k^{j-1} &= j_{E_i^{j-1}*}((\varsigma_i + af) \cdot (-(1+m_k)\varsigma_i - am_k f)), \\
&= j_{E_i^{j-1}*}((a - (1+m_k))\varsigma_i \cdot f); \\
e_i^{j-1} \cdot (e_k^{j-1})^2 &= j_{E_k^{j-1}*}(c_1(N_{E_k^{j-1}}) \cdot \left[E_i^{j-1} \cap E_k^{j-1}\right]), \\
&= j_{E_i^{j-1}*}((\left[E_i^{j-1} \cap E_k^{j-1}\right])^2), \\
&= j_{E_i^{j-1}*}(1\varsigma_i \cdot f);
\end{aligned}$$

and

$$\begin{aligned}
(e_i^{j-1})^3 &= j_{E_i^{j-1}*}((-(1+m_k)\varsigma_i - am_k f)^2), \\
&= j_{E_i^{j-1}*}((-2a(1+m_k) + (1+m_k)^2)\varsigma_i \cdot f),
\end{aligned}$$

The system defined by the previous relations, that is,

$$a - (1 + m_k) = -1,$$
$$-2a(1 + m_k) + (1 + m_k)^2 = 1,$$

has only the trivial solution $a = 0$, $m_k = 0$, so $C_k$ can not be the only center proximate to $C_i$. Then, the only possibilities are either there exists another index $\gamma \neq k$, such that $C_\gamma \to C_i$ or $E_i^{j-1}$ is final, with base a "new" curve $C_i$. In the latter case, it follows from Proposition 6.2.4 that there exists another index $\lambda$ such that $E_i^{j-1} \cap E_k^{j-1} \cap E_\lambda^{j-1} \neq \emptyset$, and $E_i^j$ fails to have an admissible final configuration.

In the case B.i.ii, as a consequence of Proposition 6.1.10, one of the following characterizations is satisfied:

- B.i.ii.i either $E_i^{j-1} \cong \mathbb{P}^2$ is not final, and there exists at least one index $\beta$ verifying $C_\beta \to C_i$, with $dim(C_\beta) = 1$,
- B.i.ii.ii or $E_i^{j-1}$ is final, with base a modified "old" curve, with modifications of type $I$, that is there exists at least one index $\beta$ such that $C_\beta \to C_i$, with $C_\beta$ isomorphic to a fiber of $E_i^i$,
- B.i.ii.iii or $E_i^{j-1}$ is final, with base a modified "old" curve with a modification of type $II$, that is, such that there exists at least one index $\beta$ verifying $C_i \xrightarrow{t} C_\beta$,
- B.i.ii.iv or $E_i^{j-1}$ is not final, with base an "old" curve, that is, there exists at least one index $\beta$ verifying $C_\beta \to C_i$, with $C_\beta$ non isomorphic to a fiber of $E_i^i$,
- B.i.ii.v or $E_i^{j-1}$ is final or not final, with base a "new" curve.

In the case B.i.ii.i, let $k \in \{\beta\}$ be an index such that $C_k \to C_i$, but that there not exists any index $\gamma \neq j$ such that $C_\gamma$ is proximate to both $C_i$ and $C_k$, or proximate and $t-$proximate to $C_i$ and $C_k$ respectively (or vice versa). Now, since $C_j \xrightarrow{t} C_i$ and $C_k \to C_i$, it follows from Proposition 6.2.6 and Proposition 6.2.7 we can conclude that $E_i^j$ has not an admissible proximity configuration neither of type III nor of type IV with respect to $E_j^j$ and $E_k^j$. Moreover, since $C_j$ is $t-$ proximate to $C_i$, then in order to get an admissible proximity configuration of type I, it must be proximate to $C_k$, that is $C_j \to C_k$. Now, by Proposition 1.2.33 we have that

$$
\begin{aligned}
-e_i^j \cdot e_k^j \cdot e_j^j &= -\pi_j^*(e_i^{j-1}) \cdot (\pi_j^*(e_k^{j-1}) - e_j^j) \cdot e_j^j, \\
&= (e_j^j)^2 \cdot e_i^j,
\end{aligned}
$$

but $(e_i^j)^2 \cdot e_k^j \neq -e_i^j \cdot e_k^j \cdot e_j^j$, so $E_i^j$ fails to have an admissible proximity configuration of type I with respect to $E_k^j$ and $E_j^j$.

In the case B.i.ii.iv, let $k \in \{\beta\}$ be an index such that $C_k \to C_i$, but that there not exists any index $\gamma \neq j$ such that $C_\gamma$ is proximate to both $C_i$ and $C_k$, or proximate and $t-$proximate to $C_i$ and $C_k$ respectively (or vice versa). Now, as a consequence of Proposition 6.2.6 and Proposition 6.2.7 we can conclude that $E_i$ has not an admissible proximity configuration neither of type III nor of type IV with respect to $E_j^j$ and $E_k^j$. Moreover, since $C_j$ is $t-$proximate to $C_i$, then it must be proximate to $C_k$ in order to get an admissible proximity configuration of type I, and by Proposition 1.2.33 we have that

$$
\begin{aligned}
e_i^j \cdot (e_k^j)^2 &= \pi_j^*(e_i^{j-1}) \cdot (\pi_j^*(e_k^{j-1}) - e_j^j)^2, \\
&= -\eta_j \neq 0.
\end{aligned}
$$

We can conclude then that $E_i^j$ has not an admissible proximity configuration of type I with respect to $E_k^j$ and $E_j^j$ either.

In the subcase B.i.ii.iii, let $k \in \{\beta\}$ be an index such that $C_k \to C_i$, but that there not exists any index $\gamma \neq j$ such that $C_\gamma$ is proximate to both $C_i$ and $C_k$, or proximate and $t-$proximate to $C_i$ and $C_k$ respectively (or vice versa). Since $C - j \xrightarrow{t} C_i$, it follows from Proposition 6.2.6 and Proposition 6.2.7 we can conclude that $E_i$ has not an admissible proximity configuration neither of type III nor of type IV with respect to $E_j^j$ and $E_k^j$. Moreover, in order to get an admissible configuration of type I, then, since $C_j$ is $t-$proximate to $C_i$, it must be proximate to $C_k$, and by Proposition 1.2.33 we have that

$$
\begin{aligned}
e_i^j \cdot (e_k^j)^2 &= \pi_j^*(e_i^{j-1}) \cdot (\pi_j^*(e_k^{j-1}) - e_j^j)^2, \\
&= -\eta_j \neq 0.
\end{aligned}
$$

We can conclude then that $E_i$ has not an admissible proximity configuration of type I with respect to $E_k^j$ and $E_j^j$.

In the remaining subcases, let $k \in \{\beta\}$ be an index such that $C_k \to C_i$, but that there not exists any index $\gamma \neq j$ such that $C_\gamma$ is proximate to both $C_i$ and $C_k$, or proximate and

$t-$proximate to $C_i$ and $C_k$ respectively (or vice versa). As $C_j \xrightarrow{t} C_i$, by Propositions 6.2.6 and 6.2.7 we can conclude that $E_i^j$ can not have an admissible proximity configuration, neither o type III nor of type IV with respect to $E_j^j$ and $E_k^j$. It may have an admissible configuration of type I, and in this case $C_j \to C_k$. Since $\pi_j : Z_j \to Z_{j-1}$ is an intrinsic elementary modification with respect to $E_i^{j-1}$, then by Proposition 1.2.33 we have that

$$\begin{aligned}
(e_i^j)^2 \cdot e_k^j &= (\pi_j(e_i^{j-1}))^2 \cdot (\pi_j(e_k^{j-1}) - e_j^j), \\
&= (\pi_j(e_i^{j-1}))^2 \cdot \pi_j(e_k^{j-1}); \\
e_i^j \cdot (e_k^j)^2 &= \pi_j(e_i^{j-1}) \cdot (\pi_j(e_k^{j-1}) - e_j^j)^2, \\
&= \pi_j(e_i^{j-1}) \cdot (\pi_j(e_k^{j-1}))^2 + \pi_j(e_i^{j-1}) \cdot (e_j^j)^2, \\
&= \pi_j(e_i^{j-1}) \cdot (\pi_j(e_k^{j-1}))^2 + e_i^j \cdot (e_j^j)^2;
\end{aligned}$$

and

$$\begin{aligned}
e_i^j \cdot e_j^j \cdot e_k^j &= \pi_j(e_i^{j-1}) \cdot e_j^j \cdot (\pi_j(e_k^{j-1}) - e_j^j), \\
&= -e_i^j \cdot (e_j^j)^2,
\end{aligned}$$

so in order to have an admissible proximity configuration of type I with respect to $E_j^j$ and $E_k^j$ the following conditions must hold:

$$(e_i^{j-1})^2 \cdot e_k^{j-1} = -\eta_j, \tag{6.55}$$

$$e_i^{j-1} \cdot (e_k^{j-1})^2 = \eta_j. \tag{6.56}$$

Since $(e_j^j)^2 \cdot e_i^j < 0$, there can not exists any other indexes $\beta$ such that $(e_\beta^j)^2 \cdot e_i^j < 0$, so $E_i^{j-1} \cong \mathbb{F}_\delta$. Moreover, as $E_i^{j-1} = P(\mathcal{O}(a) \oplus \mathcal{O}(a - \delta_i))$, it follows that $\left[E_i^{j-1} \cap E_k^{j-1}\right] = \varsigma_i + (a + n_k)f \in A^1(E_i^{j-1})$, with $\eta_j = \delta_i + 2n_k$. Firstly, let us suppose that $C_k \to C_i$ and that there not exists anyother index $\gamma \neq k$ such that $C_\gamma \to C_i$. Then, it follows from Proposition 1.1.19 that $N_{E_i^{j-1}/Z_{j-1}} \cong \pi_{j-1,k-1}^*(N_{E_i^{k-1}/Z_{k-1}}) \otimes \mathcal{O}(-E_i^k \cap E_k^k)^{\otimes m_k}$, with $m_k \in \mathbb{Z}_+$, so by Proposition 1.2.22 and Lemma 2.2.1 we have that

$$\begin{aligned}
(e_i^{j-1})^2 \cdot e_k^{j-1} &= j_{E_i^j *}((\varsigma_i + (a + n_k)f) \cdot (-(1 + m_k)\varsigma_i - m_k(a + n_k)f)), \\
&= j_{E_i^j *}((a - \delta_i(1 + m_k) - n_k(1 + 2m_k))\varsigma_i \cdot f); \\
e_i^{j-1} \cdot (e_k^{j-1})^2 &= j_{E_k^{j-1} *}(c_1(N_{E_k^{j-1}}) \cdot \left[E_i^{j-1} \cap E_k^{j-1}\right]), \\
&= j_{E_i^{j-1} *}((\left[E_i^{j-1} \cap E_k^{j-1}\right])^2), \\
&= j_{E_i^{j-1} *}((\delta_i + 2n_k)\varsigma_i \cdot f);
\end{aligned}$$

and

$$(e_i^{j-1})^3 = j_{E_i^{j-1}*}((-(1+m_k)\varsigma_i - m_k(a+n_k)f)^2),$$
$$= j_{E_i^{j-1}*}((-2a(1+m_k) + \delta_i(1+m_k)^2 + 2n_k m_k(1+m_k))\varsigma_i \cdot f).$$

The system defined by the previous relations, that is,

$$a - \delta_i(1+m_k) - n_k(1+2m_k) = -\delta_i - 2n_k,$$
$$-2a(1+m_k) + \delta_i(1+m_k)^2 + 2n_k m_k(1+m_k) = \delta_i + 2n_k,$$

has no non-trivial solutions verifying $\delta_i, n_k \geq 0$, so $C_k$ can not be the only center proximate to $C_i$. Then, the only possibilities are either there exists another index $\gamma \neq k$ such that $C_\gamma \to C_i$, or $E_i^{j-1}$ is final for the sequential morphism $\pi_{j-1,0} : Z_{j-1} \to Z_0$, with base a “new” curve $C_i$, that is, there exists an index $\gamma$ such that $C_i \to C_\gamma$. However, in the latter case, it follows from Proposition 6.2.4 that there exists another index $\lambda$ such that $E_i^{j-1} \cap E_\gamma^{j-1} \cap E_\lambda^{j-1} \neq \emptyset$, and $E_i^j$ fails to have an admissible final configuration.

□

## 6.3 The Chow ring of the sky $Z_s$

As for the whole of this chapter, we will restrict ourselves to the case of sequences of point and rational curve blow-ups, that is, either $C_\alpha = P$ or $C_\alpha = \mathcal{C}$, with the ground variety $Z_0 \cong \mathbb{P}^3$.

As a consequence of Theorem 1.2.24, we have that the Chow ring of the ground variety $A^\bullet(Z_0)$ is isomorphic to

$$A^\bullet(Z_0) \cong \mathbb{Z}[u]/(u^4), \tag{6.57}$$

by sending $u$ to $h$, where $h \in A^1(Z_0)$ is the rational equivalence class of any hyperplane $[H]$ in $\mathbb{P}^3$. Moreover, since $\forall\alpha$ it is satisfied that either $E_\alpha^\alpha \cong \mathbb{P}^2$ or $E_\alpha^\alpha \cong \mathbb{F}_\delta$, then it follows from Theorems 1.2.24 and 1.2.25 that:

$$A^\bullet(E_\alpha^\alpha) \cong \begin{cases} \mathbb{Z}[s]/(s^3) \text{ by sending } s \text{ to } \varsigma_\alpha \text{ if } C_\alpha = P_\alpha, & (6.58) \\ \mathbb{Z}[t,u]/(t^2 + c_1(N_{\mathcal{C}_\alpha/Z_{\alpha-1}})t\cdot u, u^2) \text{ by sending } t,u \text{ to } \varsigma_\alpha, f \text{ respectively} & (6.59) \\ \text{if } C_\alpha = \mathcal{C}_\alpha, & (6.60) \end{cases}$$

where $\varsigma_\alpha \in A^1(E_\alpha^\alpha)$ is the rational class of any hyperplane and $f \in A^1(E_\alpha^\alpha)$ is the rational class of a fiber.

In these sequences, we are able to give generators of the Chow ring of the sky $A^\bullet(Z_s)$ as a $\mathbb{Z}-$algebra. To begin with, let us consider the following partition of the centers of the sequence of blow-ups:

$$\{C_i\}_{i=1}^s = \{C_i\}_{i\in\mathcal{I}_1} \sqcup \{C_i\}_{i\in\mathcal{I}_2}, \tag{6.61}$$

where $i \in \mathcal{I}_1$ if $dim(C_i) = 0$ and $i \in \mathcal{I}_2$ otherwise.

**Lemma 6.3.1.** *The Chow ring of the sky of the sequence $A^\bullet(Z_s)$ is generated by $\left\{h^{s*}, \{e_\alpha^{s*}\}_{\alpha\in\mathcal{I}_1}, \{e_\alpha^{s*}, w_\alpha^{s*}\}_{\alpha\in\mathcal{I}_2}\right\}$ as a $\mathbb{Z}-$algebra.*

*Proof.* The result follows by induction on $\alpha$. It is clear that $A^\bullet(Z_0)$ is generated by $\{h\}$. Let us suppose that $A^\bullet(Z_\alpha)$ is generated by

$$\left\{h^{\alpha*}, \{e_i^{\alpha*}\}_{\substack{i\in\mathcal{I}_1\\ i\leq\alpha}}, \left\{e_i^{\alpha*}, w_i^{\beta*}\right\}_{\substack{i\in\mathcal{I}_2\\ i\leq\alpha}}\right\}. \tag{6.62}$$

Now we have to consider the two following settings: either $dim(C_{\alpha+1}) = 0$ or $dim(C_{\alpha+1}) = 1$. In the former case, since $E_{\alpha+1}^{\alpha+1} \cong \mathbb{P}^2$, that is $A^\bullet(E_{\alpha+1}^{\alpha+1}) \cong \mathbb{Z}[s]/(s^3)$, and $e_{\alpha+1}^{\alpha+1*} \cdot e_{\alpha+1}^{\alpha+1*} = -j_{\alpha+1*}(\varsigma_{\alpha+1})$ by Proposition 1.2.33, then by Proposition 1.2.29 and Theorem 1.2.24 we have that $A^\bullet(Z_{\alpha+1})$ is generated by

$$\left\{h^{\alpha+1*}, \{e_i^{\alpha+1*}\}_{\substack{i\in\mathcal{I}_1\\ i\leq\alpha+1}}, \{e_i^{\alpha+1*}, w_i^{\alpha+1*}\}_{\substack{i\in\mathcal{I}_2\\ i\leq\alpha}}\right\} \tag{6.63}$$

as a $\mathbb{Z}-$algebra. In the latter case, that is $dim(C_{\alpha+1}) = 1$, we have that $E_{\alpha+1}^{\alpha+1} \cong \mathbb{F}_\delta$, that is $A^\bullet(E_{\alpha+1}^{\alpha+1}) \cong \mathbb{Z}[t,u]/(t^2 + c_1(N_{\mathcal{C}_\alpha/Z_{\alpha-1}})t \cdot u, u^2)$, and $e_{\alpha+1}^{\alpha+1*} \cdot e_{\alpha+1}^{\alpha+1*} = -j_{\alpha+1*}(\varsigma_{\alpha+1})$ by Proposition 1.2.33. Then as a consequence of Proposition 1.2.29 and Theorem 1.2.25 we have that $A^\bullet(Z_{\alpha+1})$ is generated by

$$\left\{h^{\alpha+1*}, \{e_i^{\alpha+1*}\}_{\substack{i\in\mathcal{I}_1\\ i\leq\alpha}}, \{e_i^{\alpha+1*}, w_i^{\alpha+1*}\}_{\substack{i\in\mathcal{I}_2\\ i\leq\alpha+1}}\right\}. \tag{6.64}$$

□

Now, in order to compute the relations between the generators, let us restrict firstly to the blow-up at the $\alpha+1-$level, that is $\pi_{\alpha+1} : Z_{\alpha+1} \to Z_\alpha$, where $C_{\alpha+1}$ is a rational curve, and $Z_\alpha$ is the sky of a sequence of $\#\mathcal{I}_1^\alpha$ point blow-ups and $\#\mathcal{I}_2^\alpha$ rational curve blow-ups.

From Proposition 1.2.29 we know that $A^\bullet(Z_{\alpha+1})$ is generated by $\pi_{\alpha+1}^* A^\bullet(Z_\alpha)$ and $j_{\alpha+1*}A^\bullet E_{\alpha+1}^{\alpha+1}$. We can thus define a ring homomorphism

$$f_{\alpha+1} : A^\bullet(Z_\alpha)\left[e_{\alpha+1}^{\alpha+1}, w_{\alpha+1}^{\alpha+1}\right] \to A^\bullet(Z_{\alpha+1}),$$

such that,

$$f_{\alpha+1}(x) = \begin{cases} \pi_{\alpha+1}^*(x) & \text{if } x \in A^\bullet(Z_\alpha), \quad (6.65) \\ j_{\alpha+1*}(1) & \text{if } x = e_{\alpha+1}^{\alpha+1}, \quad (6.66) \\ j_{\alpha+1*}(g_{\alpha+1}^*(P)) & \text{if } x = w_{\alpha+1}^{\alpha+1}, \quad (6.67) \end{cases}$$

where the class of $P$, $[P] \in A^1(C_{\alpha+1})$ is a generator of $A^1(C_{\alpha+1})$. Consequently, we have that

$$A^\bullet(Z_{\alpha+1}) \cong A^\bullet(Z_\alpha)\left[e_{\alpha+1}^{\alpha+1}, w_{\alpha+1}^{\alpha+1}\right]/ker f_{\alpha+1}. \quad (6.68)$$

**Theorem 6.3.2.** *The Chow ring of $Z_{\alpha+1}$, $A^\bullet(Z_{\alpha+1})$, is isomorphic to*

$$A^\bullet(Z_{\alpha+1}) \cong \frac{A^\bullet(Z_\alpha)\left[e_{\alpha+1}^{\alpha+1}, w_{\alpha+1}^{\alpha+1}\right]}{\mathcal{J}_{\alpha+1}}, \quad (6.69)$$

*where*

$$\mathcal{J}_{\alpha+1} = (ker i_{\alpha+1}^* \cdot e_{\alpha+1}^{\alpha+1}, h^{\alpha+1*} \cdot e_{\alpha+1}^{\alpha+1} - \mu_0 w_{\alpha+1}^{\alpha+1}, \left\{e_\beta^{\alpha*} \cdot e_{\alpha+1}^{\alpha+1} - \mu_\beta w_{\alpha+1}^{\alpha+1}\right\}, (w_{\alpha+1}^{\alpha+1})^2, h^{\alpha+1*} \cdot w_{\alpha+1}^{\alpha+1},$$
$$\left\{e_\beta^{\alpha*} \cdot w_{\alpha+1}^{\alpha+1}\right\}_{\beta=1}^{\alpha}, (e_{\alpha+1}^{\alpha+1})^2 - c_1(N_{C_{\alpha+1}/Z_\alpha}) w_{\alpha+1}^{\alpha+1} + [C_{\alpha+1}], e_{\alpha+1}^{\alpha+1} \cdot w_{\alpha+1}^{\alpha+1} + (h^{\alpha+1*})^3) \quad (6.70)$$

*with $\mu_\beta = e_\beta^{\alpha*} \cdot [C_{\alpha+1}]$.*

*Proof.* In order to compute the relations between the generators of $A^\bullet(Z_{\alpha+1})$, let us now consider the ring homomorphism induced by the inclusion $i_{\alpha+1} : C_{\alpha+1} \to Z_\alpha$, that is

$$i_{\alpha+1}^* : A^\bullet(Z_\alpha) \to A^\bullet(C_{\alpha+1}),$$

and let us denote by $\mathcal{I}_{\alpha+1}$ to

$$\mathcal{I}_{\alpha+1} := (\left\{a_0 h^{\alpha*} + \sum_{\beta=1}^{\alpha} a_\beta e_\beta^{\alpha*}\right\}, (h^{\alpha*})^2, \left\{h^{\alpha*} \cdot e_\beta^{\alpha*}\right\}_{\beta=1}^{\alpha},$$
$$\left\{e_\beta^{\alpha*} \cdot e_\delta^{\alpha*}\right\}_{\substack{\beta,\delta<\alpha+1 \\ \beta\neq\delta}}, \left\{w_\beta^{\alpha*}\right\}_{\substack{\beta\in\mathcal{I}_2 \\ \beta<\alpha+1}}), \quad (6.71)$$

where $\left\{a_0\mu_0 + \sum_{\beta=1}^{\alpha} a_\beta \mu_\beta\right\}$ denotes the minimum set of relations of the finitely generated free abelian group $\mathcal{S}_{\alpha+1}$ generated by

$$\left\{\mu_0 = deg(i_{\alpha+1}^* h^{\alpha *}), \left\{\mu_i = deg(i_{\alpha+1}^* e_\beta^{\alpha *})\right\}_{\beta=1}^{\alpha}\right\}. \tag{6.72}$$

Then, it can be proved that $Ker(i_{\alpha+1}^*) = \mathcal{I}_{\alpha+1}$. Firstly, we will prove that $\mathcal{I}_{\alpha+1} \subset Ker(i_{\alpha+1}^*)$. Since $C_{\alpha+1}$ is a rational curve, then $A^1(C_{\alpha+1})$ is generated by the class $[P] \in A^1(C_{\alpha+1})$ so we can conclude that $\left\{a_0 m_0 + \sum_{\beta=1}^{\alpha} a_\beta m_\beta\right\} \subset Ker(i_{\alpha+1}^*)$. Moreover, we know that the pull-back morphism $i_{\alpha+1}^* : A^\bullet(Z_\alpha) \to A^\bullet(C_{\alpha+1})$ is graded on codimension, so it follows that

$$((h^{\alpha *})^2, \left\{h^{\alpha *} \cdot e_\beta^{\alpha *}\right\}_{\beta=1}^{\alpha}, \left\{e_\beta^{\alpha *} \cdot e_\delta^{\alpha *}\right\}_{\substack{\beta,\delta<\alpha+1 \\ \beta\neq\delta}}, \left\{w_\beta^{\alpha *}\right\}_{\substack{\beta\in\mathcal{I}_2 \\ \beta<\alpha+1}}) \subset Ker(i_{\alpha+1}^*). \tag{6.73}$$

Now, we will prove that $Ker(i_{\alpha+1}^*) \subset \mathcal{I}_{\alpha+1}$. Note that $i_{\alpha+1}^* : A^\bullet(Z_\alpha) \to A^\bullet(C_{\alpha+1})$ is homogenous, so $ker(i_{\alpha+1}^*)$ is an homogenous ideal, and $\mathcal{I}_{\alpha+1}$ is an homogenous ideal too by construction. Let us suppose that

$$Q\left[h^{\alpha *}, e_1^{\alpha *}, ..., e_\alpha^{\alpha *}, w_1^{\alpha *}, ..., w_\alpha^{\alpha *}\right] \in Ker(i_{\alpha+1}^*)/\mathcal{I}_{\alpha+1}, \tag{6.74}$$

with $deg(Q) = \eta$. Then $\eta \leq 1$, since all polynomials of weighted degree 2 are all in $\mathcal{I}_{\alpha+1}$, and $Q\left[h^{\alpha *}, e_1^{\alpha *}, ..., e_\alpha^{\alpha *}, w_1^{\alpha *}, ..., w_\alpha^{\alpha *}\right]$ must be of the form $Q\left[h^{\alpha *}, e_1^{\alpha *}, ..., e_\alpha^{\alpha *}, w_1^{\alpha *}, ..., w_\alpha^{\alpha *}\right] = b_0 h^{\alpha *} + \sum_{i=1}^{\alpha} b_i e_i^{\alpha *} mod(\mathcal{I}_{\alpha+1})$. But then $b_i = 0$ for $i = 0, ..., \alpha$ since $\left\{a_0\mu_0 + \sum_{\beta=1}^{\alpha} a_\beta \mu_\beta\right\}$ is the minimum set of relations of the finitely generated free abelian group $\mathcal{S}_{\alpha+1}$.
Before going on, we should distinguish between two possible cases, that is:

A either $i_{\alpha+1}^*$ is surjective,

B or $i_{\alpha+1}^*$ is not surjective.

In case A we have that $A^\bullet(C_{\alpha+1}) \cong A^\bullet(Z_\alpha)/ker i_{\alpha+1}^*$. Moreover, there must exist a relation of the form

$$a_0 i_{\alpha+1}^* h^{\alpha *} + \sum a_\beta i_{\alpha+1}^* e_\beta^{\alpha *} = [P], \tag{6.75}$$

where $[P] \in A^1(C_{\alpha+1})$ is a generator of $A^1(C_{\alpha+1})$, so in this case we can conclude that:

$$A^\bullet(C_{\alpha+1}) \cong A^\bullet(Z_\alpha)[P]/(ker i_{\alpha+1}^*, a_0 h^{\alpha *} + \sum a_\beta e_\beta^{\alpha *} - [P]). \tag{6.76}$$

However, the case B is a bit more tricky. In particular, we have that $A^\bullet(C_{\alpha+1})$ is isomorphic to

$$A^\bullet(C_{\alpha+1}) \cong A^\bullet(Z_\alpha)[P]/(ker i_{\alpha+1}^*, h^{\alpha *} - \mu_0 [P], \left\{e_\beta^{\alpha *} - \mu_\beta [P]\right\}_{\beta=1}^{\alpha}, ([P])^2). \tag{6.77}$$

Now, since $E_{\alpha+1}^{\alpha+1}$ is isomorphic to the projective bundle $P(N_{C_{\alpha+1}/Z_\alpha})$ over $C_{\alpha+1}$, then by Theorem 1.2.25 it follows that:

$$A^\bullet(E_{\alpha+1}^{\alpha+1}) \cong A^\bullet(C_{\alpha+1})\left[\varsigma_{\alpha+1}\right]/(\varsigma_{\alpha+1}^2 + c_1(N_{C_{\alpha+1}/Z_\alpha})\varsigma_{\alpha+1}\cdot p). \tag{6.78}$$

It can be proved that in both cases A and B it is satisfied the following inclusion $\mathcal{J}_{\alpha+1} \subset Kerf_{\alpha+1}$. Recall that

$$\begin{aligned}\mathcal{J}_{\alpha+1} = (keri_{\alpha+1}^*\cdot e_{\alpha+1}^{\alpha+1}, h^{\alpha+1*}\cdot e_{\alpha+1}^{\alpha+1} - \mu_0 w_{\alpha+1}^{\alpha+1}, \left\{e_\beta^{\alpha*}\cdot e_{\alpha+1}^{\alpha+1} - \mu_\beta w_{\alpha+1}^{\alpha+1}\right\}, (w_{\alpha+1}^{\alpha+1})^2, h^{\alpha+1*}\cdot w_{\alpha+1}^{\alpha+1},\\ \left\{e_\beta^{\alpha*}\cdot w_{\alpha+1}^{\alpha+1}\right\}_{\beta=1}^{\alpha}, (e_{\alpha+1}^{\alpha+1})^2 - c_1(N_{C_{\alpha+1}/Z_\alpha})w_{\alpha+1}^{\alpha+1} + [C_{\alpha+1}], e_{\alpha+1}^{\alpha+1}\cdot w_{\alpha+1}^{\alpha+1} + (h^{\alpha+1*})^3)\end{aligned} \tag{6.79}$$

Firstly, since $f_{\alpha+1}$ is a ring homomorphism then we have that $f_{\alpha+1}(x\cdot y) = f_{\alpha+1}(x)\cdot f_{\alpha+1}(y)$, so the inclusion

$$\begin{aligned}(keri_{\alpha+1}^*\cdot e_{\alpha+1}^{\alpha+1}, h^{\alpha+1*}\cdot e_{\alpha+1}^{\alpha+1} - \mu_0 w_{\alpha+1}^{\alpha+1}, \left\{e_\beta^{\alpha*}\cdot e_{\alpha+1}^{\alpha+1} - \mu_\beta w_{\alpha+1}^{\alpha+1}\right\}, (w_{\alpha+1}^{\alpha+1})^2, h^{\alpha+1*}\cdot w_{\alpha+1}^{\alpha+1},\\ \left\{e_\beta^{\alpha*}\cdot w_{\alpha+1}^{\alpha+1}\right\}_{\beta=1}^{\alpha}) \subset kerf_{\alpha+1},\end{aligned} \tag{6.80}$$

follows directly from Proposition 1.2.33. Moreover, the inclusion

$$((e_{\alpha+1}^{\alpha+1})^2 - c_1(N_{C_{\alpha+1}/Z_\alpha})w_{\alpha+1}^{\alpha+1} + [C_{\alpha+1}]) \subset kerf_{\alpha+1}, \tag{6.81}$$

is a direct consequence of the key formula (see Proposition 1.2.29). Finally, the inclusion $(e_{\alpha+1}^{\alpha+1}\cdot w_{\alpha+1}^{\alpha+1} + (h^{\alpha+1*})^3) \subset kerf_{\alpha+1}$ follows from the key formula and the birational invariance of $A_0(Z_i)$ (see [17, Example 16.1.11]).

In order to continue with the proof, let us recall that by Proposition 1.2.29 we have the following exact sequence:

$$0 \to A^\bullet(C_{\alpha+1}) \xrightarrow{l} A^\bullet(E_{\alpha+1}^{\alpha+1}) \oplus A^\bullet(Z_\alpha) \xrightarrow{m} A^\bullet(Z_{\alpha+1}) \to 0 \tag{6.82}$$

where $l(x) = ((g_{\alpha+1}^* c_1(N_{C_{\alpha+1}/Z_\alpha}) + \varsigma_{\alpha+1}) \cap g_{\alpha+1}^*(x), i_{\alpha+1*}(x))$ and $m(y) = (-j_{\alpha+1*}(y), \pi_{\alpha+1}^*(y))$. Let us now define

$$R_{\alpha+1} := A^\bullet(Z_\alpha)\left[e_{\alpha+1}^{\alpha+1}, w_{\alpha+1}^{\alpha+1}\right]/\mathcal{J}_{\alpha+1}, \tag{6.83}$$

and a group homorphism $\gamma : A^\bullet(E_{\alpha+1}^{\alpha+1}) \oplus A(Z_\alpha) \to R_{\alpha+1}$ such that $\gamma(x, y) = -h_{\alpha+1}(x) +$

$\pi^*_{\alpha+1}(y)$, where

$$h_{\alpha+1}(x) = \begin{cases} (e^{\alpha+1}_{\alpha+1})^{\eta+1} & \text{if } x = (-1)^\eta(\varsigma_{\alpha+1})^\eta \text{ for } \eta \geq 1 \\ w^{\alpha+1}_{\alpha+1} & \text{if } x = p \\ (d_0 h^{\alpha *} + \sum_{\beta=1}^{\alpha} d_\beta e^{\alpha *}_\beta)^{\lambda-1} \cdot w^{\alpha+1}_{\alpha+1} & \text{if } x = (p)^\lambda \text{ for } \lambda \geq 2 \\ h_{\alpha+1}((p)^\lambda) \cdot (e^{\alpha+1}_{\alpha+1})^\eta & \text{if } x = (p)^\lambda \cdot (\varsigma_{\alpha+1})^\eta \text{ for } \lambda \geq 2 \text{ and } \eta \geq 1 \end{cases}$$

, giving factorization of $m : A^\bullet(E^{\alpha+1}_{\alpha+1}) \oplus A^\bullet(Z_\alpha) \to A^\bullet(Z_{\alpha+1})$, that is,

$$\begin{array}{ccc} A^\bullet(E^{\alpha+1}_{\alpha+1}) \oplus A^\bullet(Z_\alpha) & \xrightarrow{m} & A^\bullet(Z_{\alpha+1}) \\ \downarrow \gamma & & \updownarrow \\ R_{\alpha+1} & \xrightarrow{\varphi_{\alpha+1}} & A^\bullet(Z_\alpha)\left[e^{\alpha+1}_{\alpha+1}, w^{\alpha+1}_{\alpha+1}\right]/ker f_{\alpha+1} \end{array} \tag{6.84}$$

In order to prove that $R_{\alpha+1} \cong A^\bullet(Z_{\alpha+1})$, that is $\varphi_{\alpha+1}$ is an isomorphism, it suffices to verify that $\gamma \circ l = 0$. Choose $[C_{\alpha+1}] \in A^0(C_{\alpha+1})$. Then $l([C_{\alpha+1}]) = (\varsigma_\alpha + c_1(N_{C_\alpha/Z_{\alpha-1}})p, [C_{\alpha+1}])$, and

$$\gamma(l([C_{\alpha+1}])) = (e^{\alpha+1}_{\alpha+1})^2 - c_1(N_{C_{\alpha+1}/Z_\alpha})w_\alpha + [C_{\alpha+1}] = 0. \tag{6.85}$$

Choose now $[P_{\alpha+1}] \in A^1(C_{\alpha+1})$. Then $l([P_\alpha]) = (\varsigma_{\alpha+1} \cdot p, (h^{\alpha *})^3)$ and

$$\gamma(l([P_{\alpha+1}])) = e^{\alpha+1}_{\alpha+1} \cdot w^{\alpha+1}_{\alpha+1} + (h^{\alpha *})^3 = 0. \tag{6.86}$$

□

**Corollary 6.3.3.** *The Chow ring of the sky $A^\bullet(Z_s)$ is isomorphic to*

$$A^\bullet(Z_s) \cong \frac{\mathbb{Z}\left[h^{s*}, \{e^{s*}_\alpha\}_{\alpha \in \mathcal{I}_1}, \left\{e^{s*}_\beta, w^{s*}_\beta\right\}_{\beta \in \mathcal{I}_2}\right]}{\mathcal{A}}, \tag{6.87}$$

*where*

$$\mathcal{A} = ((h^{s*})^4, \left\{\{h^{s*} \cdot e^{s*}_\alpha\}, \left\{e^{s*}_\alpha \cdot e^{s*}_\beta\right\}_{\alpha \neq \beta}, \left\{-(e^{s*}_\alpha)^3 + (h^{s*})^n\right\}\right\}_{\alpha,\beta \in \mathcal{I}_1},$$
$$\left\{keri^{s*}_\alpha \cdot e^{s*}_\alpha, h^{s*} \cdot e^{s*}_\alpha - \mu_0 w^{s*}_\alpha, \left\{e^{s*}_\beta \cdot e^{s*}_\alpha - \mu_\beta w^{s*}_\alpha\right\}_{\beta<\alpha}, (w^{s*}_\alpha)^2, h^{s*} \cdot w^{s*}_\alpha, \left\{e^{s*}_\beta \cdot w^{s*}_\alpha\right\}_{\beta<\alpha},\right.$$
$$\left.(e^{s*}_\alpha)^2 - c_1(N_{C_\alpha/Z_{\alpha-1}})w^{s*}_\alpha + [C_\alpha]^{s*}, e^{s*}_\alpha \cdot w^{s*}_\alpha + (h^{s*})^3\right\}_{\alpha,\beta \in \mathcal{I}_2}). \tag{6.88}$$

*Proof.* By Theorems 3.5.3 and 6.3.2 we know that

A if $dim(C_{\alpha+1}) = 0$, that is $C_{\alpha+1} = P_{\alpha+1}$, then

$$A^{\bullet}(Z_{\alpha+1}) \cong \frac{A^{\bullet}(Z_{\alpha})\left[e_{\alpha+1}^{\alpha+1*}\right]}{(h^{\alpha+1*}\cdot e_{\alpha+1}^{\alpha+1*}, \left\{e_i^{\alpha+1*}\cdot e_{\alpha+1}^{\alpha+1*}\right\}_{i=1}^{\alpha}, -(e_{\alpha+1}^{\alpha+1*})^3 + (h^{\alpha+1*})^3)},$$

B and if $dim(C_{\alpha+1}) = 1$, that is $C_{\alpha+1} = \mathcal{C}_{\alpha+1}$, then

$$A^{\bullet}(Z_{\alpha+1}) \cong \frac{A^{\bullet}(Z_{\alpha})\left[e_{\alpha+1}^{\alpha+1}, w_{\alpha+1}^{\alpha+1}\right]}{\mathcal{J}_{\alpha+1}}.$$

So, since $A^{\bullet}(Z_0) \cong \frac{\mathbb{Z}[h]}{(h)^4}$ by Theorem 1.2.24, the result follows directly by induction. □

**Corollary 6.3.4.** *Given two sequences of blow-ups* $(Z_s, ..., Z_0, \pi)$, $(Z_s^{'}, ..., Z_0^{'}, \pi^{'})$, *such that* $Z_0 \cong Z_0^{'}$, *with the same length and proximity relations, then* $A^{\bullet}(Z_s)$ *and* $A^{\bullet}(Z_s^{'})$ *may be non-isomorphic.*

*Proof.* Let $\pi_1 : Z_1 \to Z_0$ be the blow-up with center $C_1$ a rational curve of degree $\gamma_1$, with $\gamma_1 \geq 4$, let $\pi_2 : Z_2 \to Z_1$ be the blow-up with center $C_2$ the section corresponding to the line subbundle $\mathcal{O}_{C_1}(2\gamma_1 - 1 - a - n)$, and let $\pi_2^{'} : Z_2^{'} \to Z_1$ be the blow-up with center $C_2^{'}$ the section corresponding to the line subbundle $\mathcal{O}_{C_1}(2\gamma_1 - 1 - a - m)$ with $m \neq n$. Then, it follows from Corollary 6.3.3 that the Chow ring $A^{\bullet}(Z_2)$ is isomorphic to

$$A^{\bullet}(Z_2) \cong \frac{\mathbb{Z}\left[h^{2*}, e_1^{2*}, w_1^{2*}, e_2^{2*}, w_2^{2*}\right]}{\mathcal{A}}, \tag{6.89}$$

with

$$\begin{gathered}\mathcal{A} = ((h^{2*})^4, (h^{2*})^2 \cdot e_1^{2*}, h^{2*} \cdot e_1^{2*} - \gamma_1 w_1^{2*}, h^{2*} \cdot w_1^{2*}, (w_1^{2*})^2, (e_1^{2*})^2 - (4\gamma_1 - 2)w_1^{2*} + \gamma_1(h^{2*})^2, \\ e_1^{2*} \cdot w_1^{2*} + (h^{2*})^3, (h^{2*})^2 \cdot e_2^{2*}, (e_1^{2*})^2 \cdot e_2^{2*}, h^{2*} \cdot e_1^{2*} \cdot e_2^{2*}, w_1^{2*} \cdot e_2^{2*}, h^{2*} \cdot e_2^{2*} - \gamma_1 w_2^{2*}, \\ e_1^{2*} \cdot e_2^{2*} - (2\gamma_1 - 1 - a - n)w_2^{2*}, h^{2*} \cdot w_2^{2*}, e_1^{2*} \cdot w_2^{2}, w_1^{2*} \cdot w_2^{2*}, (w_2^{2*})^2, \\ (e_2^{2*})^2 - (2\gamma_1 - 1 + a + n)w_2^{2*} + (-(e_1^{2*})^2 + (2\gamma_1 - 1 + a + n)w_1^{2*}), e_2^{2*} \cdot w_2^{2*} + (h^{2*})^3),\end{gathered} \tag{6.90}$$

and the Chow ring $A^{\bullet}(Z_2^{'})$ is isomorphic to

$$A^{\bullet}(Z_2) \cong \frac{\mathbb{Z}\left[h^{'2*}, e_1^{2'*}, w_1^{2'*}, e_2^{2'*}, w_2^{2'*}\right]}{\mathcal{A}^{'}}, \tag{6.91}$$

with

$$\mathcal{A}' = ((h^{2'*})^4, (h^{'2*})^2 \cdot e_1^{2'*}, h^{2'*} \cdot e_{1'}^{2*} - \gamma_1 w_1^{2'*}, h^{2'*} \cdot w_1^{2'*}, (w_1^{2'*})^2, (e_1^{2'*})^2 - (4\gamma_1 - 2) w_1^{2'*} + \gamma_1 (h^{2'*})^2,$$
$$e_1^{2'*} \cdot w_1^{2'*} + (h^{2'*})^3, (h^{2'*})^2 \cdot e_2^{2'*}, (e_1^{2'*})^2 \cdot e_2^{2'*}, h^{2'*} \cdot e_1^{2'*} \cdot e_2^{2'*}, w_1^{2'*} \cdot e_2^{2'*}, h^{2'*} \cdot e_2^{2'*} - \gamma_1 w_2^{2'*},$$
$$e_1^{2*} \cdot e_2^{2*} - (2\gamma_1 - 1 - a - m) w_2^{2'*}, h^{2'*} \cdot w_2^{2'*}, e_1^{2'*} \cdot w_2^{2'*}, w_1^{2'*} \cdot w_2^{2'*}, (w_2^{2'*})^2,$$
$$(e_2^{2'*})^2 - (2\gamma_1 - 1 + a + m) w_2^{2'*} + (-(e_1^{2'*})^2 + (2\gamma_1 - 1 + a + m) w_1^{2'*}), e_2^{2'*} \cdot w_2^{2'*} + (h^{2'*})^3). \tag{6.92}$$

Now we will prove that $A^\bullet(Z_2)$ and $A^\bullet(Z_2^{'})$ are not isomorphic. To begin with, we know that $A^\bullet(Z_2)$ is generated by

$$\begin{cases} \{h^{2*}, e_1^{2*}, e_2^{2*}\} & \text{in codimension } 1, \\ \{(h^{2*})^2, w_1^{2*}, w_2^{2*}\} & \text{in codimension } 2, \\ \{(h^{2*})^3\} & \text{in codimension } 3, \end{cases}$$

and $A^\bullet(Z_2^{'})$ is generated by

$$\begin{cases} \left\{h^{2'*}, e_1^{2'*}, e_2^{2'*}\right\} & \text{in codimension } 1, \\ \left\{(h^{2'*})^2, w_1^{2'*}, w_2^{2'*}\right\} & \text{in codimension } 2, \\ \left\{(h^{2'*})^3\right\} & \text{in codimension } 3. \end{cases}$$

Let us define a graded ring homomorphism $\phi : A^\bullet(Z_2) \to A^\bullet(Z_2^{'})$, so

$$\begin{aligned} \phi(h^{2*}) &= a_0 h^{2'*} + a_1 e_1^{2'*} + a_2 e_2^{2'*}, \\ \phi(e_1^{2*}) &= b_0 h^{2'*} + b_1 e_1^{2'*} + b_2 e_2^{2'*}, \\ \phi(e_2^{2*}) &= c_0 h^{2'*} + c_1 e_1^{2'*} + c_2 e_2^{2'*}, \\ \phi(w_1^{2*}) &= d_0 (h^{2'*})^2 + d_1 w_1^{2'*} + d_2 w_2^{2'*}, \\ \phi(w_2^{2*}) &= f_0 (h^{2'*})^2 + f_1 w_1^{2'*} + f_2 w_2^{2'*}. \end{aligned}$$

Since $A^\bullet(Z_2^{'})$ is generated by $\left\{(h^{2'*})^3\right\}$ in codimension 3, then we have that

$$\phi((h^{2*})^3) = (\phi(h^{2*}))^3 = (h^{2'*})^3, \tag{6.93}$$

so we can conclude that $\phi(h^{2*}) = h^{2'*}$, that is, $a_0 = 1, a_1 = a_2 = 0$. If $\phi$ would be a graded isomorphism, then the following relations will hold

$$\phi((h^{2*})^2 \cdot e_1^{2*}) = \phi(h^{2*})^2 \cdot \phi(e_1^{2*}) \qquad = 0, \tag{6.94}$$

$$\phi((h^{2*})^2 \cdot e_2^{2*}) = \phi(h^{2*})^2 \cdot \phi(e_2^{2*}) \qquad = 0, \tag{6.95}$$

$$\phi(h^{2*} \cdot e_1^{2*} - \gamma_1 w_1^{2*}) = \phi(h^{2*}) \cdot \phi(e_1^{2*}) - \gamma_1 \phi(w_1^{2*}) \qquad = 0, \tag{6.96}$$

$$\phi(h^{2*} \cdot e_2^{2*} - \gamma_1 w_2^{2*}) = \phi(h^{2*}) \cdot \phi(e_2^{2*}) - \gamma_1 \phi(w_2^{2*}) \qquad = 0, \tag{6.97}$$

$$\phi(e_1^{2*} \cdot w_2^{2*}) = \phi(e_1^{2*}) \cdot \phi(w_2^{2*}) \qquad = 0, \tag{6.98}$$

$$\phi(e_2^{2*} \cdot w_1^{2*}) = \phi(e_2^{2*}) \cdot \phi(w_1^{2*}) \qquad = 0, \tag{6.99}$$

$$\phi(e_1^{2*} \cdot w_1^{2*} + (h^{2*})^3) = \phi(e_1^{2*}) \cdot \phi(w_1^{2*}) + \phi(h^{2*})^3 \qquad = 0, \tag{6.100}$$

$$\phi(e_2^{2*} \cdot w_2^{2*} + (h^{2*})^3) = \phi(e_2^{2*}) \cdot \phi(w_2^{2*}) + \phi(h^{2*})^3 \qquad = 0, \tag{6.101}$$

$$\phi(e_2^{2*} \cdot e_2^{2*} - (2\gamma_1 - 1 - a - n) w_2^{2*}) = \phi(e_2^{2*}) \cdot \phi(e_2^{2*}) - (2\gamma_1 - 1 - a - n)\phi(w_2^{2*}) \qquad = 0. \tag{6.102}$$

From equations 6.94 and 6.94, we have that $b_0 = c_0 = 0$. Moreover, equations 6.96 and 6.97 implies that $b_1 = d_1$, $b_2 = d_2$, $c_1 = f_1$ and $c_2 = f_2$. Now, as a consequence of equations 6.98 an6.99 the following relations hold $-b_1 f_1 - b_2 f_2 = -c_1 d_1 - c_2 d_2 = 0$. Finally, it follows from equations 6.100 and 6.101 that $-(b_1)^2 - (b_2)^2 = -(c_1)^2 - (c_2)^2 = -1$, so we can conclude that if $\phi$ would be a graded isomorphism, then

A either $b_1 = d_1 = c_2 = f_2 = 1$ and $b_2 = d_2 = c_1 = f_1 = 0$, that is, $\phi(e_1^{2*}) = e_1^{2'*}, \phi(w_1^{2*}) = w_1^{2'*}, \phi(e_2^{2*}) = e_2^{2'*}, \phi(w_2^{2*}) = w_2^{2'*}$,

B or $b_1 = d_1 = c_2 = f_2 = 0$ and $b_2 = d_2 = c_1 = f_1 = 1$, that is, $\phi(e_1^{2*}) = e_2^{2'*}, \phi(w_1^{2*}) = w_2^{2'*}, \phi(e_2^{2*}) = e_1^{2'*}, \phi(w_2^{2*}) = w_1^{2'*}$.

Let us now consider the ideal $I = (e_1^{2*}) \leq A^\bullet(Z_2)$. If $\phi$ would be a graded isomorphism, then $I \leq A^\bullet(Z_2)$ and $\phi(I) \leq A^\bullet(Z_2^{'})$ would have the same Hilbert-Poincaré series, but in the particular case B this equality is not true

$$P_{(e_1^{2*})}(t_0,t_1) = 1 - t_0 - t_0^2 - 5t_0^3 - 5t_0^2t_1 + 9t_0^4 + 15t_0^3t_1 + t_0^2t_1^2 + 9t_0^5 - 5t_0^4t_1 - 3t_0^3t_1^2 - 19t_0^6 - 25t_0^5t_1 + t_0^4t_1^2 + t_0^7 + 25t_0^6t_1 + 5t_0^5t_1^2 + 10t_0^8 + 5t_0^7t_1 - 5t_0^6t_1^2 - 4t_0^9 - 15t_0^8t_1 - t_0^7t_1^2 + 5t_0^9t_1 + 3t_0^8t_1^2 - t_0^9t_1^2, \quad (6.103)$$

$$P_{(e_2^{2'*})}(t_0,t_1) = 1 - t_0 - 2t_0^2 - 2t_0^3 - 3t_0^2t_1 + 8t_0^4 + 9t_0^3t_1 + t_0^2t_1^2 + 4t_0^5 - 3t_0^4t_1 - 3t_0^3t_1^2 - 14t_0^6 - 15t_0^5t_1 + t_0^4t_1^2 + 2t_0^7 + 15t_0^6t_1 + 5t_0^5t_1^2 + 7t_0^8 + 3t_0^7t_1 - 5t_0^6t_1^2 - 3t_0^8 - 9t_0^8t_1 - t_0^7t_1^2 + 3t_0^9t_1 + 3t_0^8t_1^2 - t_0^9t_1^2, \quad (6.104)$$

so we have that $\phi(e_1^{2*}) = e_1^{2'*}, \phi(w_1^{2*}) = w_1^{2'*}, \phi(e_2^{2*}) = e_2^{2'*}, \phi(w_2^{2*}) = w_2^{2'*}$ in order to have a graded isomorphism. However, even in the case A, the relation of equation 6.102 is not satisfied since

$$\phi(e_2^{2*} \cdot e_2^{2*} - (2\gamma_1 - 1 - a - n)w_2^{2*}) = (-m+n)w_2^{2'*} \neq 0, \quad (6.105)$$

so we can conclude that the graded homomorphism $\phi : A^\bullet(Z_2) \to A^\bullet(Z_2^{'})$ is not a graded isomorphism.

□

We finish this section by giving some interesting and non-trivial examples of sequences of point and rational curve blow-ups, where we explicitly compute the Chow ring of their corresponding skies.

**Example 6.3.5.** *Let $P \in \mathbb{P}^3$ be a point, and let $\pi_1 : Z_1 \to Z_0 = \mathbb{P}^3$ be the blow-up of $Z_0$ with center $C_1 = P$. Consider now a rational curve of degree one $\mathcal{C} \in E_1^1$, and let $\pi_2 : Z_2 \to Z_1$ be the blow-up of $Z_1$ with center $C_2 = \mathcal{C}$. Finally, If we denote by $\mathcal{D}$ to $E_1^2 \cap E_2^2$, let $\pi_3 : Z_3 \to Z_2$ be the blow-up of $Z_2$ with center $C_3 = \mathcal{D}$.*

$$Z_3 \xrightarrow{\pi_3} Z_2 \xrightarrow{\pi_2} Z_1 \xrightarrow{\pi_1} Z_0$$

*The Chow ring of $Z_3$, $A^\bullet(Z_3)$, is isomorphic to*

$$A^\bullet(Z_3) \cong \frac{\mathbb{Z}\left[y_0, y_1, w_1, y_2, y_3, w_3\right]}{\mathcal{J}_3},$$

*where*

$$\mathcal{J}_3 = (y_0 \cdot y_1, (y_1)^3 + (y_0)^3, y_0 \cdot y_2, (y_1 + y_2 + 2y_3)^2 \cdot (y_2 + y_3), (y_1 + y_2 + 2y_3) \cdot y_2 + w_2,$$
$$(y_1 + y_2 + 2y_3) \cdot w_2, (y_2 + y_3)^2 - (y_1 + y_2 + 2y_3)^2, (y_2 + y_3) \cdot (w_2 + w_3) + (y_0)^3,$$
$$y_0 \cdot y_3, (y_1 + y_2 + 2y_3)^2 \cdot y_3, (y_2 + y_3)^2 \cdot y_3, (y_1 + y_3) \cdot w_3, (y_2 + y_3) \cdot w_3,$$
$$(y_1 + y_2 + 2y_3) \cdot (y_2 + y_3) \cdot y_3, (y_1 + y_3) \cdot y_3 + 2w_3, (y_2 + y_3) \cdot y_3 - w_3,$$
$$(y_3)^2 + w_3 + (y_1 + y_3) \cdot (y_2 + y_3), y_3 \cdot w_3 + (y_0)^3),$$

*by sending* $y_0$, $y_1$, $y_2$, $w_2$, $y_3$ *and* $w_3$ *to* $h^*$, $e_1^3$, $e_2^3$, $r_2^3$, $e_3^3$ *and* $r_3^3$ *respectively.*

*We are going to verify that only* $E_3$ *is a final divisor, even if for* $E_2$ *the relation* $(e_2 + e_3)^2 \cdot e_2 = 0$ *also holds. Let us start with* $E_3$*. To begin with, we already know that relation* $(e_2 + e_3)^2 \cdot e_3 = 0$ *is satisfied. Moreover, from relations* $(e_1 + e_2 + 2e_3)^2 \cdot e_3 = 0$, $(e_2+e_3)\cdot e_3 - r_3 = 0$ *and* $(e_1+e_3)\cdot r_3 = 0$ *it follows that* $(e_1+e_3)^2 \cdot e_3 = 0$ *also holds. Finally, we can conclude from relations* $(e_1 + e_3) \cdot e_3 + 2r_3$, $(e_2 + e_3) \cdot e_3 - r_3$, $(e_1 + e_3) \cdot r_3 = 0$, $(e_2 + e_3) \cdot r_3 = 0$ *and* $e_3 \cdot r_3 + (h^*)^3 = 0$ *that the following* $e_3 \cdot (e_2)^2 = -e_3 \cdot (e_1)^2$, $(e_3)^2 \cdot e_2 = (e_3)^2 \cdot e_1 + e_3 \cdot (e_1)^2$ *and* $e_1 \cdot e_2 \cdot e_3 = 0$ *are satisfied too.*
*Now, we will see that* $E_2$ *has a non-admissible final configuration. To begin with, it follows from* $(e_2 + e_3)^2 - (e_1 + e_2 + 2e_3)^2 = 0$ *and* $(e_1 + e_2 + 2e_3)^2 \cdot (e_2 + e_3) = 0$ *that relation* $(e_2+e_3)^3 = 0$ *holds and consequently* $(e_3+e_3)^2 \cdot e_2 = 0$ *is satisfied too. Moreover, it follows from relations* $(e_2+e_3)\cdot e_3 - r_3$, $e_3 \cdot r_3 + (h^*)^3$ *and* $(e_3)^2 + r_3 + (e_1+e_3)\cdot(e_2+e_3)$ *that* $(e_3)^3 + \frac{1}{2}(e_3)^2 \cdot e_2 = 0$ *also holds. Finally, from relations* $(e_2 + e_3) \cdot e_3 - r_3 = 0$, $(e_2 + e_3) \cdot r_3 = 0$ *and* $e_3 \cdot r_3 + (h^*)^3 = 0$*, we can conclude that* $(e_2)^2 \cdot e_3 + \frac{3}{2} e_2 \cdot (e_3)^2 = 0$ *is verified.*

**Example 6.3.6.** *Let* $P \in \mathbb{P}^3$ *be a point, and let* $\pi_1 : Z_1 \to Z_0 = \mathbb{P}^3$ *be the blow-up of* $Z_0$ *with center* $C_1 = P$*. Consider now a point* $Q \in E_1^1$*, and let* $\pi_2 : Z_2 \to Z_1$ *be the blow-up of* $Z_1$ *with center* $C_2 = Q$*. Finally, If we denote by* $\mathcal{C}$ *to* $E_1^2 \cap E_2^2$*, let* $\pi_3 : Z_3 \to Z_2$ *be the blow-up of* $Z_2$ *with center* $C_3 = \mathcal{C}$*.*

$$Z_3 \xrightarrow{\pi_3} Z_2 \xrightarrow{\pi_2} Z_1 \xrightarrow{\pi_1} Z_0$$

*The Chow ring of* $Z_3$*,* $A^\bullet(Z_3)$*, is isomorphic to*

$$A^\bullet(Z_3) \cong \frac{\mathbb{Z}[y_0, y_1, y_2, y_3, w_3]}{\mathcal{J}_3},$$

*where*

$$\mathcal{J}_3 = (y_0 \cdot y_1, (y_1 + y_2 + 2y_3)^2 \cdot y_1 + (y_0)^3, y_0 \cdot y_2, (y_1 + y_2 + 2y_3) \cdot (y_2 + y_3), (y_2 + y_3)^3 + (y_0)^3, y_0 \cdot y_3, (y_1 + y_2 + 2y_3)^2 \cdot y_3, (y_2 + y_3)^2 \cdot y_3, (y_1 + y_3) \cdot w_3, (y_2 + y_3) \cdot w_3, (y_1 + y_2 + 2y_3) \cdot (y_2 + y_3) \cdot y_3, (y_1 + y_3) \cdot y_3 - w_3, (y_2 + y_3) \cdot y_3 + w_3, (y_3)^2 + (y_1 + y_3) \cdot (y_2 + y_3), y_3 \cdot w_3 + (y_0)^3),$$

*by sending $y_0$, $y_1$, $y_2$, $y_3$ and $w_3$ to $h^*$, $e_1^3$, $e_2^3$, $e_3^3$ and $r_3^3$ respectively.*

*We are going to verify that only $E_3$ is a final divisor, even if for $E_1$ the relation $(e_1 + e_3)^2 \cdot e_1 = 0$ also holds. Let us start with $E_3$. To begin with, we already know that relation $(e_2 + e_3)^2 \cdot e_3 = 0$ is satisfied. Moreover, it follows from relations $(e_1 + e_2 + 2e_3)^2 \cdot e_3$, $(e_2+e_3) \cdot e_3 + r_3 = 0$ and $(e_1+e_3) \cdot r_3 = 0$ that $(e_1+e_3)^2 \cdot e_3 = 0$ is satisfied too. Now, we can conclude that relations $e_3 \cdot (e_2)^2 + e_3 \cdot (e_1)^2 = 0$ and $(e_3)^2 \cdot e_2 = (e_3)^2 \cdot e_1 + e_3 \cdot (e_1)^2$ are deduced from relations $(e_1 + e_3) \cdot e_3 - r_3 = 0$, $(e_2 + e_3) \cdot e_3 + r_3 = 0$, $e_3 \cdot r_3 + (h^*)^3 = 0$ and $(e_3)^2 + (e_1 + e_3) \cdot (e_2 + e_3) = 0$.*
*Now, we will see that $E_1$ has a non-admissible final configuration. To begin with, it follows from relations $(e_1+e_2+2e_3)^3+(h^*)^3 = 0$, $(e_2+e_3)^3+(h^*)^3 = 0$, $(e_1+e_2+2e_3)\cdot(e_2+e_3) = 0$, so in particular $(e_1 + e_3)^2 \cdot e_1 = 0$ that $(e_1 + e_3)^2 \cdot e_1 = 0$ i also satisfied.Now, we get $(e_3)^3 = 0$ by considering the relations $(e_3)^2 + (e_1 + e_3) \cdot (e_2 + e_3) = 0$, $(e_2 + e_3) \cdot e_3 + r_3$ and $(e_1 + e_3) \cdot r_3 = 0$. Moreover, it follows from relation $(e_1 + e_3) \cdot e_3 - r_3 = 0$ that $(e_1)^2 \cdot e_3 + 2e_1 \cdot (e_3)^2 = 0$ holds. Now, we can verify from $(e_1 + e_3)^2 \cdot e_1 = 0$ that relation $(e_1)^3 - 3e_1 \cdot (e_3)^2 = 0$ holds too.*

**Example 6.3.7.** *Let $\mathcal{C} \in \mathbb{P}^3$ be a rational curve of degree $\gamma > 4$, and let $\pi_1 : Z_1 \to Z_0 = \mathbb{P}^3$ be the blow-up of $Z_0$ with center $C_1 = \mathcal{C}$, so $E_1^1 = P(\mathcal{O}(2\gamma - 1 + a) \oplus \mathcal{O}(2\gamma - 1 - a))$. Consider now a section $\mathcal{S} \in E_1^1$ such that $[\mathcal{S}] = \varsigma_1 + (2\gamma - 1 + a + m)f$ in $A^1(E_1^1)$. If $P \in \mathcal{S}$ is a closed point, let $\pi_2 : Z_2 \to Z_1$ be the blow-up of $Z_1$ with center $C_2 = P$. Finally, If we denote by $\widetilde{\mathcal{S}}$ the strict transform of $\mathcal{S}$, let $\pi_3 : Z_3 \to Z_2$ be the blow-up of $Z_2$ with center $C_3 = \widetilde{\mathcal{S}}$.*

$$Z_3 \xrightarrow{\pi_3} Z_2 \xrightarrow{\pi_2} Z_1 \xrightarrow{\pi_1} Z_0$$

*The Chow ring of $Z_3$, $A^\bullet(Z_3)$ is isomorphic to*

$$A^\bullet(Z_3) \cong \frac{\mathbb{Z}\left[y_0, y_1, w_1, y_2, y_3, w_3\right]}{\mathcal{J}_3}$$

*where*

$$\mathcal{J}_3 = ((y_0)^2 \cdot y_1, y_0 \cdot w_1, y_0 \cdot (y_1 + y_2 + y_3) - \gamma(w_1 + w_3), (w_1 + w_3)^2,$$
$$(y_1 + y_2 + y_3)^2 + (-4\gamma + 2)(w_1 + w_3) + \gamma(y_0)^2, (y_1 + y_2 + y_3) \cdot (w_1 + w_3) + (y_0)^3, y_0 \cdot y_2,$$
$$(y_1 + y_2 + y_3) \cdot y_2, w_1 \cdot y_2, (y_2)^3 - (y_0)^3, (y_0)^2 \cdot y_3, (y_1 + y_2 + y_3)^2 \cdot y_3, (w_1 + w_3) \cdot y_3,$$
$$y_2 \cdot y_3, y_0 \cdot w_3, (y_1 + y_2 + y_3) \cdot w_3, y_2 \cdot w_3, y_0 \cdot y_3 - \gamma w_3, (y_1 + y_3) \cdot y_3 - (2\gamma - a - m - 2)w_3,$$
$$y_2 \cdot y_3 - w_3, (y_3)^2 - (2\gamma - 3 + a + m)w_3 - (y_1 + y_2 + y_3)^2 + (2\gamma - 1 + a + m)(w_1 + w_3)-$$
$$- y_2, y_3 \cdot w_3 + (y_0)^3),$$

*by sending* $y_0$*,* $y_1$*,* $w_1$*,* $y_2$*,* $y_3$ *and* $w_3$ *to* $h^*$*,* $e_1^3$*,* $r_1^3$*,* $e_2^3$*,* $e_3^3$ *and* $r_3^3$ *respectively.*

*We are going to verify that both* $E_2$ *and* $E_3$ *are final divisors. Let us start with* $E_3$*. We already know that relation* $e_3 \cdot (e_2)^2 = 0$ *is satisfied. Now, from relations* $e_2 \cdot e_3 - w_3 = 0$ *and* $e_3 \cdot r_3 + (h^*)^3 = 0$*, we get* $(e_3)^2 \cdot e_2 = -(h^*)^3$*. Moreover, by combaning relations* $e_2 \cdot e_3 - r_3 = 0$*,* $(e_1 + e_2 + e_3) \cdot r_3$*,* $e_2 \cdot r_3$ *and* $e_3 \cdot r_3 + (h^*)^3 = 0$ *we can conclude that* $e_1 \cdot e_2 \cdot e_3 - (h^*)^3 = 0$*. Finally, it follows from relation* $(e_1 + e_2 + e_3)^2 \cdot e_3 = 0$ *that relation* $(e_1 + e_3)^2 \cdot e_3 = 0$*.*
*In the following, we verify that* $E_2$ *is final too. From relations* $(e_1 + e_2 + e_3) \cdot e_2 = 0$*,* $(e_2)^2 \cdot e_3 = 0$*,* $(e_2)^3 - (h^*)^3 = 0$*, we obtain* $(e_2)^2 \cdot e_1 + (h^*)^3 = 0$*. Now, it follows from relation* $(e_1 + e_2 + e_3) \cdot e_2 = 0$ *that* $e_2 \cdot (e_1)^2 = 0$ *also holds. Finally, we obtain from relations* $(e_2)^3 - (h^*)^3 = 0$ *and* $e_3 \cdot (e_2)^2 = 0$ *that relation* $(e_2 + e_3)^2 \cdot e_2 = 0$ *is satisfied too.*

# Conclusions

Let $Z_s \xrightarrow{\pi_s} Z_{s-1} \xrightarrow{\pi_{s-1}} \cdots \xrightarrow{\pi_2} Z_1 \xrightarrow{\pi_1} Z_0$ be a sequence of blow-ups at smooth centers (see Definition 2.1.1), with $Z_0 \cong \mathbb{P}^n$. As higher dimensional centers are allowed, not just 0 dimensional ones, we introduce the concept of $t-$proximity. We say that $C_j$ is $t-$proximate to $C_i$, and write $C_j \xrightarrow{t} C_i$ if $C_j \cap E_i^{j-1} \neq \emptyset$ but $C_j \not\subset E_i^{j-1}$. Moreover, we define sequential morphisms as those which can be expressed, in at least one way, as a composition of blow-ups verifying the conditions of Definition 2.1.1. In order to study both sequences of blow-ups and its associated sequential morphisms, we introduce the key concept of final divisor. Roughly speaking, an irreducible exceptional component $E_i$ is final with respect to a sequence of blow-ups $(Z_s, ..., Z_0, \pi)$ if there exists an open set $U_i$ on $Z_i$, with $E_i^i \subset U_i$, such that the restriction of the composition $\pi_{i+1} \circ \pi_{i+2} \circ ... \circ \pi_{s-1} \circ \pi_s|_{U_i}$ is an isomorphism ($E_i$ is final with respect to a sequential morphism if it is final for one of the sequences of blow-ups that realize it). Some quite natural question arises when dealing with final divisors: Given a sequential morphism $\pi : Z \to Z_0$ is it possible for two irreducible exceptional components $E_i$ and $E_j$ to be final with $E_i \cap E_j \neq \emptyset$? And in this case, which type of proximity relation could exist between them? Moreover, what is the geometric structure of $E_i \cap E_j$ when $E_i$ is final?, and is it possible to exploit this structure and give a characterization of final divisor in terms of some relations defined over the Chow group of zero-cycles of its sky $A_0(Z_s)$?

We answer the first three ones in the general setting, that is considering general smooth centers $C_i$, with $dim(C_i) \geq 0$. In Theorem 2.3.10 we prove that it can exists two final divisors $E_i$ and $E_j$, with $E_i \cap E_j \neq \emptyset$, but in this case proximity relations are quite restrictive, that is either $E_i \to E_j$ and $E_j \xrightarrow{t} E_i$ or vice versa. Moreover, regarding the geometric structure of $E_i \cap E_j$ when $E_i$ is final, we prove in Proposition 2.3.12 that if

$dim(C_i) \geq 1$ then either $(e_j + e_i) \cdot e_i$ or $e_j \cdot e_i$ equals the pull-back of a Weil divisor in $A^1(C_i)$, and $e_j \cdot e_i$ is equivalent to the hyperplane class $\varsigma_i \in A^1(E_i)$ otherwise.

In the case of sequences of point blow-ups defined over both algebraically closed fields and perfect fields, we introduce two equivalence relations with classification purposes: the algebraic equivalence and the combinatorial one, for both sequences of blow-ups and its associated sequential morphisms. Previous to this, in the particular case where the base field $k$ is perfect, then in order to consider different fields $K$, with $k \subset K \subset \overline{k}$, we define combinatorially and algebraically compatible partitions of the exceptional divisors (see Definitions 4.1.4, 4.1.5, 4.1.7 and 4.1.8). Whereas the algebraic equivalence has to deal with the existence of certain isomorphism, the combinatorial one is related to the existences of certain permutations preserving the proximity relations an the intersection numbers.
Moreover, we give a positive answer to the fourth question proposed above, that is in Propositions 3.2.4 and 4.3.1 we characterize final divisor in terms of some relations defined over the Chow group of zero-cycles of its sky $A_0(Z_s)$. By using these results, we are able to recover the sequence of point blow-ups, modulo algrebaic equivalence, from the associated sequential morphism, and prove Theorems 3.4.7, 3.4.5, 4.5.8, 4.5.5 which relate the algebraic and combinatorial equivalence classes of sequences of blow-ups with the corresponding ones of sequential morphisms.
Finally, we give two explicit presentations of the Chow ring of the sky of a sequence of point blow-ups using the strict and the total transforms of the irreducible exceptional components in Theorems 3.5.3 and 3.5.6, and come to a surprising result, that is two sequences of point blow-ups of the same length have isomorphic Chow rings (see Corollary 3.5.5).

In the case of sequences of point and rational curve blow-ups with $dim(Z_i) = 3$, we also give a positive answer to the question related to characterize final divisors in terms of some relations defined over the Chow group of zero-cycles of its sky in Theorem 6.2.11. Moreover, we give an explicit presentation of the Chow ring of its sky $A^\bullet(Z_s)$ by considering the total transform of the irreducible components of the exceptional divisor as generators, and to which we have to add the total transforms of a generic fiber of the associated projective bundles (see Corollary 6.3.3). As a result, we prove in Corollary 6.3.4 that there exists an important difference with respect to the sequences of point blow-ups, that is, two sequences of point and rational curve blow-ups of the same length and even with the same proximity relations may not have isomorphic Chow rings.

As a general conclusion, the surprising result of independency on the geometry of the point centers of a sequential morphism, seems to be not easy to extend for cases of centers of higher dimensions. Even for blow-ups of few rational curves or points in dimension 3, the geometry of the centers is influent for the Chow ring as shown in Chapter 6. The basic example in Chapter 5 shows how, in fact, the geometry of the exceptional divisor of the blow-up of a smooth rational curve in $\mathbb{P}^3$ of degree $\gamma \geq 4$ depends on its embedding in $\mathbb{P}^3$ and not only on the numerical value of $\gamma$. Thus, the surprising result looks as a special result of its kind.

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