Universidad de Valladolid

Escuela de Doctorado Universidad de Valladolid

PROGRAMA DE DOCTORADO EN MATEMÁTICAS

TESIS DOCTORAL:

# Stability properties and Borel-Ritt theorems in ultraholomorphic classes. An application to a generalized moment problem

Presentada por Ignacio Miguel Cantero para optar al grado de
Doctor por la Universidad de Valladolid

Dirigida por:
Dr. D. Javier Sanz Gil

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# Contents

# Introducción

La presente memoria trata principalmente de varios problemas en el contexto de las clases ultraholomorfas en sectores no acotados de la superficie de Riemann del logaritmo, a saber: resultados de estabilidad en estas clases bajo algunas operaciones estándar; nuevos resultados de sobreyectividad para la aplicación de Borel asintótica en el caso de las clases ultraholomorfas de Carleman, definidas en términos de una sucesión peso; y la existencia de operadores de extensión lineal, inversas por la derecha para la aplicación de Borel. Como subproducto, que surge de la detección de una nueva condición que garantiza la sobreyectividad de la aplicación de Borel en sectores adecuadamente estrechos, se puede abordar un problema de momentos de Stieltjes modificado en espacios generales de Gelfand-Shilov definidos por sucesiones peso.

El primer capítulo consta de toda la información preliminar necesaria para introducir los espacios y los problemas que se estudiarán. En particular, se recopila la información básica estándar sobre sucesiones peso, funciones peso y matrices peso para su uso posterior, junto con algunos métodos, ya clásicos, para pasar de una sucesión peso a una función peso, y de esta última a una matriz peso. Además, se analizan varios índices de O-variación regular, asociados con sucesiones peso o funciones peso, ya que jugarán un papel destacado al determinar la apertura límite, para sectores en la superficie de Riemann del logaritmo, por debajo de la cual nuestros resultados de sobreyectividad y de extensión serán válidos.

Describimos ahora con más detalle los resultados obtenidos en la tesis, y comenzamos con los contenidos del segundo capítulo. En la literatura se pueden encontrar con frecuencia las llamadas clases ultradiferenciables, tanto en el sentido de Carleman como de Braun-Meise-Taylor, cuyos elementos son funciones indefinidamente derivables definidas sobre subconjuntos abiertos de $\mathbb{R}^n$ (o posiblemente gérmenes en un punto) de manera que el ritmo de crecimiento de sus derivadas sucesivas esté controlado (excepto por un factor geométrico) en términos de una sucesión dada de números reales positivos en el primer caso, o en términos de (valores obtenidos a partir de) una función peso dada en el segundo uno. Además, dependiendo de la elección de un cuantificador universal o existencial para el factor geométrico en las estimaciones, se pueden considerar clases tipo Beurling o tipo Roumieu en ambas situaciones. El estudio de la estabilidad bajo inversas (o división) en este contexto tiene una larga historia, véanse los trabajos de W. Rudin [66], J. Bruna [8] y J. A. Siddiqi [75], y también se ha estudiado la composición en el trabajo de C. Fernández y A. Galbis [21]. Recientemente, la introducción por parte de G. Schindl [69, 70] de clases asociadas a una matriz peso, que engloban estrictamente las clases mencionadas anteriormente, les ha llevado a él y a A. Rainer [59, 60] a la caracterización de la estabilidad bajo diferentes operaciones en términos de condiciones para la matriz peso considerada, dando así una solución general satisfactoria a estos problemas.

En relación con la teoría asintótica de soluciones para ecuaciones diferenciales y en diferencias alrededor de puntos singulares en el dominio complejo, es natural considerar el análogo en el dominio complejo de tales clases, generalmente llamadas clases ultraholomorfas. Estas contienen funciones holomorfas en regiones sectoriales en la superficie de Riemann del logaritmo (se supone que el punto singular está en 0, el vértice de la región) cuyas derivadas admiten nuevamente estimaciones adecuadas de tipo Roumieu en términos de una sucesión de números reales positivos, que en las aplicaciones suele ser una sucesión de Gevrey $(p!^a)_{p\in\mathbb{N}_0}$ para algún $a > 1$ ($\mathbb{N}_0 = \{0, 1, 2, \dots\}$). El estudio de las propiedades de estabilidad en tales clases es bien conocido en el caso Gevrey, ver el libro de W. Balser [1], pero ya en 1987 M. Ider y J. A. Siddiqi [76] estudiaron la estabilidad bajo composición con funciones analíticas y bajo inversión para clases generales de Carleman-Roumieu en sectores no acotados que no sean más amplios que un semiplano. El primer objetivo de esta disertación es ampliar sus resultados en varios sentidos: (1) consideramos clases de Roumieu definidas por matrices peso, incluyendo en nuestras consideraciones aquellas de tipo Carleman y aquellas definidas por una función peso, como en el entorno ultradiferenciable; (2) podemos tratar con clases definidas en sectores de apertura arbitraria en la superficie de Riemann del logaritmo, y (3) ampliamos la lista de propiedades de estabilidad, incluida la de cierre por composición. Es importante señalar que, en el caso de clases dadas por una función peso, la condición de que esta función sea equivalente a una función peso cóncava, lo que equivale a la propiedad de casi crecimiento de las raíces para la matriz peso asociada, juega un papel fundamental en las propiedades de estabilidad.

Las principales novedades surgen de dos fuentes diferentes. Por un lado, las técnicas propias del trabajo con matrices peso permiten una mejor comprensión de las condiciones que suelen aparecer en dichos resultados de estabilidad, y proporcionan una manera clara de establecer resultados para el caso de sucesiones peso y funciones eso. De hecho, nuestros resultados amplían los conocidos para las clases de Carleman y coinciden, en el límite cuando la apertura del sector tiende a 0, con los de las clases ultradiferenciables en un semieje. Por otro lado, las principales afirmaciones se basan en gran medida en la construcción de las llamadas funciones características en las clases ultraholomorfas de Carleman-Roumieu en sectores de apertura arbitraria. Estas funciones son aquellas de una clase que no pueden pertenecer a una clase estrictamente contenida en la original y, por lo tanto, son en cierto sentido maximales dentro de la clase. Mientras que Ider y Siddiqi sólo obtuvieron tales funciones en sectores adecuadamente estrechos, el trabajo de B. Rodríguez Salinas [65] proporciona de hecho los datos clave para trabajar en sectores generales, y esto a su vez es crucial para nuestros propósitos.

Para las clases ultraholomorfas introducidas en la Sección 2.1, mostramos cómo construir funciones características en la Sección 2.2. Los resultados de estabilidad para clases asociadas con matrices peso se dan en la Sección 2.3, y la Sección 2.4 está dedicada a su particularización para el caso de clases inducidas por una función peso. Presentamos también en la Sección 2.5 algunos ejemplos, incluidos los de las clases Gevrey y $q$-Gevrey, para ilustrar los resultados obtenidos. Cabe mencionar que los resultados sobre estabilidad discutidos hasta ahora han sido publicados en un trabajo conjunto con J. Jiménez-Garrido, J. Sanz y G. Schindl [29].

En la última sección de este capítulo nos centramos en problemas similares para las clases correspondientes tipo Beurling. La propiedad de estabilidad bajo composición debe adaptarse adecuadamente, y establecerse convenientemente la condición que caracterizará la conservación de la estabilidad. Sin embargo, la principal diferencia en las técnicas y los resultados se debe a la falta en este marco de funciones que puedan desempeñar un papel similar al que desempeñan las funciones características en el caso Roumieu. Dado que sólo para sectores no más amplios que un semiplano tenemos una familia conveniente (de hecho, compuesta de exponenciales) disponible para nuestros argumentos, nuestros resultados sólo considerarán tales sectores. Además, esta familia es perfectamente apropiada para la aplicación de resultados de la teoría de álgebras de Fréchet multiplicativamente convexas que llevan a la solución.

El tercer capítulo contiene nuestros resultados sobre la sobreyectividad de la aplicación de Borel, y la existencia de inversas por la derecha para la misma, en clases ultraholomorfas de Carleman en sectores no acotados de la superficie de Riemann del logaritmo. La aplicación de Borel asintótica envía una función, que admite un desarrollo asintótico en una región sectorial, a la serie de potencias formal que proporciona dicho desarrollo. En muchos problemas dentro de la teoría asintótica para ecuaciones diferenciales ordinarias meromorfas en puntos singulares irregulares en el dominio complejo, es importante decidir sobre la inyectividad y sobreyectividad de esta aplicación cuando se considera entre las clases ultraholomorfas de Carleman-Roumieu y la clase correspondiente de series formales, definidas restringiendo el crecimiento de algunos de los datos característicos de sus elementos (las derivadas de las funciones, los restos del desarrollo o los coeficientes de la serie) en términos de una sucesión peso dada $\boldsymbol{M} = (M_p)_{p\in\mathbb{N}_0}$ de números reales positivos (ver Subsección 3.1 para la definición de tales clases). La inyectividad ha sido completamente caracterizada para regiones sectoriales y sucesiónes peso generales, ver los trabajos de S. Mandelbrojt [46], B. Rodríguez-Salinas [64] y J. Jiménez-Garrido, J. Sanz y G. Schindl [33]. Sin embargo, el problema de la sobrejetividad aún está en estudio.

El teorema clásico de Borel-Ritt-Gevrey de B. Malgrange y J.-P. Ramis [63], que resuelve el problema para el caso de los desarrollos de tipo Gevrey (para el cual $\boldsymbol{M} = (p!^{\alpha})_{p\in\mathbb{N}_0}$, $\alpha > 0$), fue parcialmente extendido a diferentes situaciones más generales por J. Schmets y M. Valdivia [74], V. Thilliez [79, 80], J. Sanz [68], J. Jiménez-Garrido, J. Sanz y G. Schindl [33, 37] y A. Debrouwere [14, 15]. Resumiendo, cuando empezamos a estudiar este problema se sabía que la condición de no casianaliticidad fuerte, abreviada (snq), para $\boldsymbol{M}$, y que equivale al hecho de que el índice $\gamma(\boldsymbol{M})$ introducido por V. Thilliez sea positivo, es de hecho necesaria para la sobreyectividad. Además, para un sector no acotado $S_\gamma$ de apertura $\pi\gamma$ ($\gamma > 0$) en la superficie de Riemann del logaritmo y para sucesiones peso regulares en el sentido de E. M. Dyn'kin [20] –aquellas que satisfacen la condición de cierre por derivación, es decir, $M_{p+1} \leq C_0 H^{p+1} M_p$ para todo $p \in \mathbb{N}_0$ y ciertos $C_0 > 0$ y $H \geq 1$–, la aplicación de Borel es sobreyectiva siempre que $\gamma < \gamma(\boldsymbol{M})$, mientras que no lo es para $\gamma > \gamma(\boldsymbol{M})$ (la situación para $\gamma = \gamma(\boldsymbol{M})$ todavía no está clara en general). Aquí, $\gamma(\boldsymbol{M})$ es un índice de crecimiento para la sucesión $\boldsymbol{M}$ introducido por V. Thilliez [80] para sucesiones fuertemente regulares y posteriormente estudiado por J. Jiménez-Garrido, J. Sanz y G. Schindl [34] para cualquier sucesión peso. Es importante señalar que la prueba conocida de la sobreyectividad en esta situación no era constructiva, sino que se basaba en la caracterización, mediante técnicas abstractas de análisis funcional, de la sobreyectividad de la aplicación de momentos de Stieltjes en espacios de Gelfand-Shilov definidos por sucesiones regulares debida a A. Debrouwere [14]. Esta información se transfirió al contexto asintótico en un semiplano mediante la transformada de Fourier, y las transformadas analíticas de Laplace y Borel de orden arbitrario permitieron concluir para sectores generales, ver [37]. Sin embargo, en el caso particular de clases dadas por sucesiones fuertemente regulares en el sentido de V. Thilliez, la prueba de sobreyectividad de la aplicación de Borel [80] descansa en la construcción de funciones planas óptimas en sectores adecuados y una doble aplicación de resultados de extensión tipo Whitney. Posteriormente, A. Lastra, S. Malek y J.Sanz [42] probaron de nuevo la sobreyectividad de una manera más explícita mediante una transformada formal de Borel y otra de Laplace truncadas, definidas a partir de funciones núcleo adecuadas obtenidas a partir de funciones planas óptimas.

El primer objetivo de este capítulo es construir funciones planas óptimas para clases ultraholomorfas de Carleman-Roumieu definidas por sucesiones peso generales (no solo para las fuertemente regulares) y en sectores $S_\gamma$ con $\gamma < \gamma(\boldsymbol{M})$.

La idea clave proviene de un trabajo reciente de D. N. Nenning, A. Rainer y G. Schindl [51], donde estudiaron el problema mixto de Borel en clases ultradiferenciables de Beurling. Estos autores consideran una condición mixta inspirada en otra relacionada (ver (3.15)) y que aparece en un artículo de M. Langenbruch [41]. Resulta que la condición de Langenbruch es, bajo hipótesis naturales, equivalente al hecho de que $\gamma(\boldsymbol{M}) > 1$, y es crucial para construir funciones planas óptimas en un semiplano mediante la clásica extensión armónica de la función asociada $\omega_{\boldsymbol{M}}$. Un proceso de ramificación proporciona entonces funciones planas óptimas en la situación general.

En segundo lugar, para clases ultraholomorfas definidas por sucesiones regulares, obtenemos la sobreyectividad de la aplicación de Borel proporcionando una técnica constructiva para los operadores de extensión locales correspondientes, inversos por la derecha lineales y continuos para la aplicación de Borel cuando actúa sobre espacios de Banach adecuados dentro de nuestras clases, en la misma línea que en [42]. En aras de la exhaustividad, en el caso de sucesiones fuertemente regulares damos también un enfoque alternativo, basado en el trabajo de J. Bruna [9].

Para resaltar la potencia de esta técnica en situaciones concretas, también presentamos una familia de sucesiones (no fuertemente) regulares para las cuales se pueden proporcionar funciones planas óptimas en cualquier sector de la superficie de Riemann del logaritmo (lo que concuerda con la hecho de que el índice $\gamma(\boldsymbol{M})$ es en este caso igual a $\infty$), basándose en estimaciones precisas para la función asociada $\omega_{\boldsymbol{M}}$ en lugar de apelar a su extensión armónica. Observamos que las sucesiones clásicas $q$-Gevrey se encuentran entre estos ejemplos. Terminamos mostrando cómo se pueden obtener funciones planas óptimas y resultados de extensión para sucesiones convolucionadas, en caso de que las sucesiones factores admitan tales construcciones por separado. Se comentan algunos ejemplos al respecto de esta técnica. Los resultados presentados hasta este punto en este capítulo han aparecido en un trabajo conjunto con J. Jiménez-Garrido, J. Sanz y G. Schindl [28].

El objetivo principal de la Sección 3.5 es proponer una nueva condición para la sucesión peso, mucho más débil que la condición de cierre de derivación incluida en la definición de sucesión peso regular, y que aún permita la obtención de teoremas de Borel-Ritt en nuestro marco de manera constructiva. Decimos que $\boldsymbol{M} = (M_p)_p$ tiene momentos desplazados, (sm) para abreviar, si existen $C_0 > 0$ y $H > 1$ tales que

$$\log\left(\frac{m_{p+1}}{m_p}\right) \le C_0 H^{p+1}, \quad p \in \{0, 1, 2, \dots\},$$

donde $m_p = M_{p+1}/M_p$. Resulta que, siempre que $\gamma(\boldsymbol{M}) > 0$, (sm) equivale a la equivalencia de $\boldsymbol{M}_{+1} := (M_{p+1})_p$ y la sucesión de momentos de Stieltjes para un núcleo $e(z) = G(1/z)$ definido a partir de una función plana óptima $G$ en la clase definida por $\boldsymbol{M}$. Bajo esta condición débil, es posible adaptar las transformadas formal de Borel y de Laplace truncada para hacer que nuestra técnica funcione y obtener operadores de extensión locales y, por lo tanto, la sobreyectividad, de la aplicación de Borel para clases de Roumieu.

Respecto al caso Beurling, A. Debrouwere [14, Th. 7.4] caracterizó por primera vez la sobreyectividad de la aplicación asintótica de Borel en un semiplano para sucesiones regulares, y más tarde resolvió completamente el problema para desarrollos asintóticos no uniformes, y proporcionó operadores de extensión globales para $\gamma < \gamma(\boldsymbol{M})$ en el caso de estimaciones uniformes, véase [15]. Presentaremos en la Sección 3.6.1 una técnica diferente para tratar el problema para clases con estimaciones uniformes, siguiendo las mismas ideas que en el caso de Roumieu [37], que se basan en el uso de transformadas integrales de Borel y de Laplace ramificadas. Para hacer esto, necesitamos probar el Teorema 3.6.2, que mejora ligeramente un resultado de J. Schmets y M. Valdivia [74] (Teorema 3.6.1 en este trabajo) y la implicación $(i) \Rightarrow (iii)$ del resultado antes mencionado de A. Debrouwere (Teorema 3.6.4 en este trabajo). Finalmente, la nueva condición (sm) también es válida para demostrar la sobreyectividad para clases de Beurling siempre que $0 < \gamma < \gamma(\boldsymbol{M})$, gracias a una técnica de J. Chaumat y A. M. Chollet [11] ya aplicada por V. Thilliez [80, Th. 3.4.1] para sucesiones fuertemente regulares. Es importante mencionar que, bajo la condición (sm) y tanto en el caso Roumieu como en el Beurling, no podemos determinar la longitud del intervalo de sobreyectividad, formado por los valores $\gamma > 0$ para los que la aplicación de Borel es sobreyectiva para la clase definida en $S_\gamma$, a diferencia de lo que ocurre cuando se supone satisfecha (dc). En otras palabras, a partir de la sobreyectividad para $S_\gamma$ y suponiendo (sm) no somos capaces de deducir que $\gamma \leq \gamma(\boldsymbol{M})$.

El último capítulo contiene una nueva contribución al estudio del problema de momentos de Stieltjes en el contexto de los espacios de Gelfand-Shilov de tipo Roumieu definidos por sucesiones peso, presentados por primera vez en su libro [24]. El problema de momentos tiene una larga tradición que se remonta al trabajo fundamental de T. J. Stieltjes [77]. En 1939, R. P. Boas [5] y G. Pólya [57] demostraron de forma independiente que, para cada sucesión $(c_p)_{p=0}^{\infty}$ de números complejos, existe una función $F$ de variación acotada tal que

$$\int_0^\infty x^p \mathrm{d}F(x) = c_p, \qquad p \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

Este resultado fue mejorado por A. J. Durán [18] en 1989, quien demostró constructivamente que, para cada sucesión $(c_p)_{p\in\mathbb{N}_0}$ de números complejos, el sistema infinito de ecuaciones lineales

$$\mu_p(\varphi) := \int_0^\infty x^p \varphi(x) \mathrm{d}x = c_p, \qquad p \in \mathbb{N}_0, \tag{1}$$

admite una solución $\varphi \in \mathcal{S}(0,\infty)$, el subespacio del espacio de Schwartz de funciones complejas indefinidamente derivables y de decrecimiento rápido en $\mathbb{R}$ y con soporte en $[0,\infty)$ (este resultado también se puede deducir mediante un breve argumento no constructivo a través del teorema de Eidelheit [49, Thm. 26.27]).

Dadas dos sucesiones de números reales positivos $\boldsymbol{M} = (M_p)_{p\in\mathbb{N}_0}$ y $\boldsymbol{A} = (A_p)_{p\in\mathbb{N}_0}$, consideramos los espacios de Gelfand-Shilov de tipo Roumieu $\mathcal{S}^{\{\boldsymbol{A}\}}_{\{\boldsymbol{M}\}}(0,\infty)$ y $\mathcal{S}_{\{\boldsymbol{M}\}}(0,\infty)$, que constan de todas las funciones $\varphi \in \mathcal{S}(0,\infty)$ tales que existe $h>0$ con

$$\sup_{p,q\in\mathbb{N}_0}\sup_{x\in\mathbb{R}}\frac{|x^p\varphi^{(q)}(x)|}{h^{p+q}M_pA_q}<\infty$$

y

$$\sup_{p\in\mathbb{N}_0}\sup_{x\in\mathbb{R}}\frac{|x^p\varphi^{(q)}(x)|}{h^pM_p}<\infty \qquad \text{por cada } q\in\mathbb{N}_0,$$

respectivamente. Está claro que $\mathcal{S}^{\{\boldsymbol{A}\}}_{\{\boldsymbol{M}\}}(0,\infty) \subset \mathcal{S}_{\{\boldsymbol{M}\}}(0,\infty)$, y que para cada $\varphi \in \mathcal{S}_{\{\boldsymbol{M}\}}(0,\infty)$ la sucesión de momentos de Stieltjes $(\mu_p(\varphi))_{p\in\mathbb{N}_0}$ está bien definida y tiene un crecimiento restringido. En caso de que $\boldsymbol{M}$ sea cerrada por derivación, es fácil comprobar que la sucesión de momentos pertenece a la clase $\Lambda_{\{\boldsymbol{M}\}} = \{(c_p)_{p\in\mathbb{N}_0} \colon \sup_{p\in\mathbb{N}_0}\frac{|c_p|}{h^pM_p}<\infty \text{ para algún } h>0\}$. El problema de momentos de Stieltjes estándar en este contexto consiste entonces en el estudio de la sobreyectividad e inyectividad de la aplicación de momentos de Stieltjes $\mathcal{M}$, que envía $\varphi$ en $(\mu_p(\varphi))_{p\in\mathbb{N}_0}$, cuando se define en $\mathcal{S}^{\{\boldsymbol{A}\}}_{\{\boldsymbol{M}\}}(0,\infty)$ o en $\mathcal{S}_{\{\boldsymbol{M}\}}(0,\infty)$ y con rango $\Lambda_{\{\boldsymbol{M}\}}$.

La sobreyectividad se ha estudiado en una serie de artículos, siempre basándose en ideas de A. L. Durán y R. Estrada [19] que combinan la transformada de Fourier con teoremas tipo Borel-Ritt propios del análisis asintótico, ver S.-Y. Chung, D. Kim y Y. Yeom [13, Thm. 3.1] para $\boldsymbol{M} = (p!^{\alpha})_{p\in\mathbb{N}_0}$ (las sucesiones de Gevrey) siempre que $\alpha > 2$, y A. Lastra y J. Sanz [43, 44] para $\mathcal{S}_{\{(p!M_p)_p\}}(0,\infty)$ y sucesiones fuertemente regulares $\boldsymbol{M}$ cuyo índice de crecimiento $\gamma(\boldsymbol{M})$ es estrictamente mayor que 1. Posteriormente, A. Debrouwere, J. Jiménez-Garrido y J. Sanz [16] mejoraron y completaron estos resultados incluyendo los espacios $\mathcal{S}^{\{(p!A_p)_p\}}_{\{(p!M_p)_p\}}(0,\infty)$ en sus consideraciones, suprimiendo algunas hipótesis sobre $\boldsymbol{M}$ (especialmente la de crecimiento moderado, más fuerte que el cierre por derivación), y estudiando también la inyectividad de la aplicación de momentos de Stieltjes. Las nuevas herramientas clave fueron una mejor comprensión del significado de las diferentes condiciones de crecimiento generalmente impuestas a la sucesión $\boldsymbol{M}$ y su expresión en términos de índices de O-variación regular, como se desarrolló en [34], y la información mejorada obtenida en [33] acerca de la inyectividad y sobreyectividad de la aplicación asintótica de Borel en clases ultraholomorfas de Carleman-Roumieu en sectores y definidas por sucesiones $\boldsymbol{M}$ sujetas a condiciones mínimas. Finalmente, A. Debrouwere [14] caracterizó completamente la sobreyectividad y la existencia de inversas por la derecha globales para la aplicación de momentos en espacios de Gelfand-Shilov de tipo Roumieu o Beurling bajo cierre por derivación. Su técnica no se basa en teoremas tipo Borel-Ritt, sino que relaciona el problema con la sobreyectividad y existencia de inversas por la derecha globales para la aplicación de Borel en clases ultradiferenciables de Carleman, ya caracterizadas por H.-J. Petzsche [52].

El objetivo principal de este último capítulo es el estudio del problema del momento de Stieltjes en un nuevo marco, que permite considerar de forma natural un espacio de llegada más grande para la aplicación de momentos. La motivación proviene de la introducción de la condición (sm), mucho más débil que el cierre por derivación, en los resultados anteriores de tipo Borel-Ritt. El hecho clave es que (sm) caracteriza la equivalencia de $\boldsymbol{M}_{+1} := (M_{p+1})_p$ y la sucesión de momentos de Stieltjes de la función núcleo $e(z)$ (que aparece en una transformada de Laplace truncada), lo que hace que el procedimiento funcione. Entonces, se vuelve natural (ver Proposiciones 4.2.2 y 4.2.3) cambiar el espacio de llegada a uno más grande, $\Lambda_{\{\boldsymbol{M}_{+1}\}}$, y estudiar nuevamente la inyectividad y la sobreyectividad en este nuevo escenario. Como la técnica in [14] no parece ser de aplicación, hemos recuperado la técnica in [44], apoyándonos en la construcción de inversas por la derecha locales para la aplicación de momentos. Esto requiere un estudio cuidadoso de la acción de la transformada de Fourier bajo esta nueva condición (sm) (Proposición 4.1.10), y la adaptación de algunos resultados auxiliares que ya fueron útiles en situaciones anteriores.

Hemos podido caracterizar la inyectividad de la aplicación de momentos de Stieltjes bajo la condición (sm) en el Teorema 4.2.6, mientras que el Teorema 4.2.7 estudia el problema de sobreyectividad y su conexión con la existencia de inversas por la derecha locales para $\mathcal{M}$ con un escalado uniforme del parámetro que define los espacios de Banach bajo consideración.

# Introduction

The present dissertation deals mainly with several problems in the framework of ultraholomorphic classes in unbounded sectors of the Riemann surface of the logarithm, namely: stability results of these classes under some standard operations; new surjectivity results for the asymptotic Borel mapping in the case of Carleman ultraholomorphic classes, defined in terms of a weight sequence; and the existence of linear extension operators, right inverses for the Borel mapping. As a by-product, emanating from the detection of a new condition guaranteeing the surjectivity of the Borel mapping in suitably narrow sectors, a modified Stieltjes moment problem can be dealt with in general Gelfand-Shilov spaces defined by weight sequences.

The first chapter consists of all the preliminary information needed in order to introduce the spaces and problems under study. In particular, the standard, basic information concerning weight sequences, weight functions and weight matrices is gathered for later use, together with some, by now classical, methods to go from a weight sequence to a weight function, and from this latter to a weight matrix. Also, several indices of O-regular variation, associated with either weight sequences or weight functions, are discussed, since they will play a prominent role when determining the limiting opening, for sectors in the Riemann surface of the logarithm, below which our surjectivity and extension results will be valid.

We describe now in more detail the results obtained in the dissertation, and start with the contents of the second chapter. In the literature one can frequently find the so-called ultradifferentiable classes, both in the Carleman and the Braun-Meise-Taylor sense, whose elements are smooth functions defined on open subsets of $\mathbb{R}^n$ (or possibly germs at a point) such that the rate of growth of their successive derivatives is controlled (except for a geometric factor) in terms of a given sequence of positive real numbers in the first case, or in terms of (values obtained from) a given weight function in the second one. Moreover, depending on the choice of a universal or existential quantifier for the geometric factor in the estimates, one can consider Beurling- or Roumieu-like classes in both situations. The study of stability under inversion (or division) in these frameworks has a long history, see the works of W. Rudin [66], J. Bruna [8] and J. A. Siddiqi [75], and also composition has been studied in the work of C. Fernández and A. Galbis [21]. Recently, the introduction by G. Schindl [69, 70] of classes associated with a weight matrix, which strictly encompass those classes mentioned before, has led him and A. Rainer [59, 60] to the characterization of stability under different operations in terms of conditions for the weight matrix under consideration, so giving a satisfactory general solution to these problems.

In connection with the asymptotic theory of solutions for differential and difference equations around singular points in the complex domain, it is natural to consider the complex analogue of such classes, usually called ultraholomorphic classes. They consist of holomorphic functions in sectorial regions in the Riemann surface of the logarithm (the singular point is assumed to be at 0, the vertex of the region) whose derivatives admit again suitable estimates of Roumieu type in terms of a sequence of positive real numbers, which in the applications is typically a Gevrey sequence $(p!^a)_{p\in\mathbb{N}_0}$ for some $a > 1$ ($\mathbb{N}_0 = \{0, 1, 2, \dots\}$). The study of stability properties in such classes is well-known for the Gevrey ones, see the book of W. Balser [1], but already in 1987 M. Ider and J. A. Siddiqi [76] studied stability under composition with analytic functions and under inversion for general Carleman-Roumieu classes in unbounded sectors not wider than a half-plane. The first aim of this dissertation is to extend their results in several senses: (1) we consider Roumieu classes defined by weight matrices, so including in our considerations those of Carleman type and those defined by a weight function, as in the ultradifferentiable setting; (2) we are able to deal with classes defined in sectors of arbitrary opening in the Riemann surface of the logarithm, and (3) we extend the list of stability properties, including that of composition closedness. It is important to note that, in the case of classes given by a weight function, a fundamental role in the stability properties is played by the condition that this function is equivalent to a concave weight function, what amounts to the root almost increasing property for the associated weight matrix.

The main novelties arise from two different sources. On the one hand, the techniques coming with the weight matrix structure allow for a better understanding of the conditions usually appearing in such stability results, and provide a clear way to establish results for the weight sequence and weight function approach. Indeed, our results extend the known ones for Carleman classes, and they match, in the limit when the opening of the sector tends to 0, with the ones for ultradifferentiable classes on a half-line. On the other hand, the main statements heavily rest on the construction of so-called characteristic functions in Carleman-Roumieu ultraholomorphic classes in sectors of arbitrary opening. These functions are those in a class which cannot belong to a class strictly contained in the original one, and so are in a sense maximal within the class. While Ider and Siddiqi only got such functions in suitably narrow sectors, the work of B. Rodríguez Salinas [65] provides indeed the key facts for working in general sectors, and this is in turn crucial for our purposes.

For the ultraholomorphic classes introduced in Section 2.1 we show how to construct characteristic functions in Section 2.2. The stability results for classes associated with weight matrices are given in Section 2.3, and Section 2.4 is devoted to their particularization to the case of classes induced by a weight function. We present also in Section 2.5 some examples, including those of Gevrey and $q$-Gevrey classes, in order to illustrate the obtained results. It may be mentioned that the results about stability discussed so far have been published in a joint work with J. Jiménez-Garrido, J. Sanz and G. Schindl [29].

In the last section of this chapter we focus on similar problems for the Beurling-like corresponding classes. The property of stability under composition has to be suitably adapted, and the characterizing condition for this stability to hold conveniently stated. However, the main difference in the techniques and the results is due to the lack in this framework of functions that can play a similar role as the one played by characteristic functions in the Roumieu setting. Since only for sectors not wider than a half-plane we have a convenient family (indeed, consisting of exponentials) available for our arguments, our results will only consider such narrow sectors. Moreover, this family is perfectly appropriate for the application of results from the theory of multiplicatively convex Fréchet algebras which yield the solution.

The third chapter contains our results about the surjectivity of the Borel mapping, and the existence of right inverses for it, in Carleman ultraholomorphic classes in unbounded sectors of the Riemann surface of the logarithm. The asymptotic Borel mapping sends a function, admitting an asymptotic expansion in a sectorial region, into the formal power series providing such expansion. In many problems within the asymptotic theory for meromorphic ordinary differential equations at irregular singular points in the complex domain, it is important to decide about the injectivity and surjectivity of this map when considered between Carleman-Roumieu ultraholomorphic classes and the corresponding class of formal series, defined by restricting the growth of some of the characteristic data of their elements (the derivatives of the functions, the remainders in the expansion, or the coefficients of the series) in terms of a given weight sequence $\boldsymbol{M} = (M_p)_{p\in\mathbb{N}_0}$ of positive real numbers (see Subsection 3.1 for the definition of such classes). The injectivity has been fully characterized for sectorial regions and general weight sequences, see the works of S. Mandelbrojt [46], B. Rodríguez-Salinas [64] and J. Jiménez-Garrido, J. Sanz and G. Schindl [33]. However, the surjectivity problem is still under study. The classical Borel-Ritt-Gevrey theorem of B. Malgrange and J.-P. Ramis [63], solving the case of Gevrey asymptotics (for which $\boldsymbol{M} = (p!^{\alpha})_{p\in\mathbb{N}_0}$, $\alpha > 0$), was partially extended to different more general situations by J. Schmets and M. Valdivia [74], V. Thilliez [79, 80], J. Sanz [68], J. Jiménez-Garrido, J. Sanz and G. Schindl [33, 37] and A. Debrouwere [14, 15]. Summing up, when we started to study this problem it was known that the strong nonquasianalyticity condition (snq) for $\boldsymbol{M}$, equivalent to the fact that the index $\gamma(\boldsymbol{M})$ introduced by V. Thilliez is positive, is indeed necessary for surjectivity. Moreover, for an unbounded sector $S_\gamma$ of opening $\pi\gamma$ ($\gamma > 0$) in the Riemann surface of the logarithm and for regular weight sequences in the sense of E. M. Dyn'kin [20] – those satisfying derivation closedness, that is, $M_{p+1} \leq C_0 H^{p+1} M_p$ for every $p \in \mathbb{N}_0$ and some $C_0 > 0$ and $H \geq 1$–, the Borel map is surjective whenever $\gamma < \gamma(\boldsymbol{M})$, while it is not for $\gamma > \gamma(\boldsymbol{M})$ (the situation for $\gamma = \gamma(\boldsymbol{M})$ is still unclear in general). Here, $\gamma(\boldsymbol{M})$ is a growth index for the sequence $\boldsymbol{M}$ introduced by V. Thilliez [80] for strongly regular sequences and later studied by J. Jiménez-Garrido, J. Sanz and G. Schindl [34] for any weight sequence. It is important to note that the known proof of surjectivity in this situation was not constructive, but rested on the characterization, by abstract functional-analytic techniques, of the surjectivity of the Stieltjes moment mapping in Gelfand-Shilov spaces defined by regular sequences due to A. Debrouwere [14]. This information had been transferred into the asymptotic framework in a halfplane by means of the Fourier transform, and in [37] Laplace and Borel analytic transforms of arbitrary order allowed to conclude for general sectors. However, in the particular case of classes given by strongly regular sequences in the sense of V. Thilliez, his proof of surjectivity of the Borel map [80] rests on the construction of optimal flat functions in suitable sectors and a double application of Whitney extension results. Subsequently, A. Lastra, S. Malek and J. Sanz [42] reproved surjectivity in a more explicit way by means of formal Borel- and truncated Laplace-like transforms, defined from suitable kernel functions obtained from those optimal flat functions. The first aim in this chapter is to construct such optimal flat functions for Carleman-Roumieu ultraholomorphic classes defined by general weight sequences (not just strongly regular ones) and in sectors $S_\gamma$ with $\gamma < \gamma(\boldsymbol{M})$. The key idea comes from a recent work by D. N. Nenning, A. Rainer and G. Schindl [51], where they have studied the mixed Borel problem in Beurling ultradifferentiable classes. They consider a mixed condition inspired by a related one (see (3.15)) appearing in a paper of M. Langenbruch [41]. It turns out that the condition of Langenbruch is, under natural hypotheses, equivalent to the fact that $\gamma(\boldsymbol{M}) > 1$, and it is crucial in order to construct optimal flat functions in a halfplane by means of the classical harmonic extension of the associated function $\omega_{\boldsymbol{M}}$. A ramification process provides then optimal flat functions in the general situation.

Secondly, for ultraholomorphic classes defined by regular sequences, we obtain the surjectivity of the Borel map by providing a constructive technique for the corresponding local extension operators, linear and continuous right inverses for the Borel map when acting on suitable Banach spaces within our classes, in the same vein as in [42]. For the sake of completeness, in the case of strongly regular sequences we also give an alternative approach, based on the work of J. Bruna [9].

In order to highlight the power of the technique in concrete situations, we will also present a family of (non strongly) regular sequences for which such optimal flat functions can be provided in any sector of the Riemann surface of the logarithm (what agrees with the fact that the index $\gamma(\boldsymbol{M})$ is in this case equal to $\infty$), resting on precise estimates for the associated function $\omega_{\boldsymbol{M}}$ instead of appealing to its harmonic extension. We note that the classical $q$-Gevrey sequences are found among these examples. We end by showing how optimal flat functions and extension results can be obtained for convolved sequences, in case the factor sequences admit such constructions separately. Some examples are commented on in regard with this technique. The results presented up to this point in this chapter have appeared in a joint work with J. Jiménez-Garrido, J. Sanz and G. Schindl [28].

The main aim of Section 3.5 is to put forward a new condition for weight sequences, much weaker than the condition of derivation closedness included in the definition of regular weight sequence, and still allowing for the obtention of Borel-Ritt theorems in our setting in a constructive way. We say $\boldsymbol{M} = (M_p)_p$ has shifted moments, (sm) for short, if there exist $C_0 > 0$ and $H > 1$ such that

$$\log\left(\frac{m_{p+1}}{m_p}\right) \leq C_0 H^{p+1}, \quad p \in \{0, 1, 2, \dots\},$$

where $m_p = M_{p+1}/M_p$. It turns out that, whenever $\gamma(\boldsymbol{M}) > 0$, (sm) amounts to the equivalence of $\boldsymbol{M}_{+1} := (M_{p+1})_p$ and the sequence of Stieltjes moments for a kernel $e(z) = G(1/z)$ defined from an optimal flat function $G$ in the class defined by $\boldsymbol{M}$. Under this weak condition it is possible to adapt the Borel- and truncated Laplace-transforms in order to make our technique work and obtain local extension operators, and so the surjectivity, of the Borel map for Roumieu classes.

Regarding the Beurling case, A. Debrouwere [14, Th. 7.4] first characterized the surjectivity of the asymptotic Borel map in the right half-plane for regular sequences, and later on he completely solved the problem for non-uniform asymptotics, and provided global extension operators for $\gamma < \gamma(\boldsymbol{M})$ in the case with uniform estimates, see [15]. We will present in Section 3.6.1 a different technique in order to treat the problem for classes with uniform estimates, following the same ideas as in the Roumieu case [37], which rest on the use of ramified Borel and Laplace integral transforms. In order to do this, we need to prove Theorem 3.6.2, which slightly improves both a result of J. Schmets and M. Valdivia [74], Theorem 3.6.1 in this paper, and the implication $(i) \Rightarrow (iii)$ of the aforementioned result of A. Debrouwere (Theorem 3.6.4 in this work). Finally, the new condition (sm) is also valid in order to prove surjectivity for Beurling classes as long as $0 < \gamma < \gamma(\boldsymbol{M})$, thanks to a technique of J. Chaumat and A. M. Chollet [11] already applied by V. Thilliez [80, Th. 3.4.1] for strongly regular sequences. It is important to mention that, under condition (sm) and both in the Roumieu and the Beurling cases, we cannot determine the length of the surjectivity interval, consisting of the values $\gamma > 0$ such that the Borel mapping is surjective for the class defined on $S_\gamma$, unlike the situation when (dc) is assumed. In other words, from the surjectivity for $S_\gamma$ and under (sm) we are not able to deduce that $\gamma \leq \gamma(\boldsymbol{M})$.

The last chapter contains a new contribution to the study of the Stieltjes moment problem in the context of Gelfand-Shilov spaces of Roumieu type defined by weight sequences, first introduced in their book [24]. The moment problem has a long tradition that goes back to the seminal work of T. J. Stieltjes [77]. In 1939, R. P. Boas [5] and G. Pólya [57] independently showed that, for every sequence $(c_p)_{p=0}^{\infty}$ of complex numbers, there is a function $F$ of bounded variation such that

$$\int_0^\infty x^p \mathrm{d}F(x) = c_p, \qquad p \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

This result was improved by A. J. Durán [18] in 1989, who constructively showed that, for every sequence $(c_p)_{p\in\mathbb{N}_0}$ of complex numbers, the infinite system of linear equations

$$\mu_p(\varphi) := \int_0^\infty x^p \varphi(x) \mathrm{d}x = c_p, \qquad p \in \mathbb{N}_0, \tag{2}$$

admits a solution $\varphi \in \mathcal{S}(0,\infty)$, the subspace of the Schwartz space of rapidly decreasing complex smooth functions in $\mathbb{R}$ with support in $[0,\infty)$ (this result can also be deduced by a short non-constructive argument via Eidelheit's theorem [49, Thm. 26.27]). Given two sequences of positive real numbers $\boldsymbol{M} = (M_p)_{p\in\mathbb{N}_0}$ and $\boldsymbol{A} = (A_p)_{p\in\mathbb{N}_0}$, we consider the Gelfand-Shilov spaces of Roumieu type $\mathcal{S}^{\{\boldsymbol{A}\}}_{\{\boldsymbol{M}\}}(0,\infty)$ and $\mathcal{S}_{\{\boldsymbol{M}\}}(0,\infty)$, consisting of all $\varphi \in \mathcal{S}(0,\infty)$ such that there exists $h > 0$ with

$$\sup_{p,q\in\mathbb{N}_0} \sup_{x\in\mathbb{R}} \frac{|x^p \varphi^{(q)}(x)|}{h^{p+q} M_p A_q} < \infty$$

and

$$\sup_{p\in\mathbb{N}_0} \sup_{x\in\mathbb{R}} \frac{|x^p \varphi^{(q)}(x)|}{h^p M_p} < \infty \qquad \text{for every } q \in \mathbb{N}_0,$$

respectively. It is clear that $\mathcal{S}^{\{\boldsymbol{A}\}}_{\{\boldsymbol{M}\}}(0,\infty) \subset \mathcal{S}_{\{\boldsymbol{M}\}}(0,\infty)$, and that for every $\varphi \in \mathcal{S}_{\{\boldsymbol{M}\}}(0,\infty)$ the sequence of Stieltjes moments $(\mu_p(\varphi))_{p\in\mathbb{N}_0}$ is well defined and has a restricted growth. In case $\boldsymbol{M}$ is derivation closed, it is easy to check that the moment sequence belongs to $\Lambda_{\{\boldsymbol{M}\}} = \{(c_p)_{p\in\mathbb{N}_0} \colon \sup_{p\in\mathbb{N}_0} \frac{|c_p|}{h^p M_p} < \infty \text{ for some } h > 0\}$. The standard Stieltjes moment problem in this context consists then in the study of the surjectivity and injectivity of the Stieltjes moment mapping $\mathcal{M}$, sending $\varphi$ to $(\mu_p(\varphi))_{p\in\mathbb{N}_0}$, when defined on either $\mathcal{S}^{\{\boldsymbol{A}\}}_{\{\boldsymbol{M}\}}(0,\infty)$ or $\mathcal{S}_{\{\boldsymbol{M}\}}(0,\infty)$ and with range $\Lambda_{\{\boldsymbol{M}\}}$. Surjectivity has been studied in a series of papers, always resting on ideas of A. L. Durán and R. Estrada [19] that combine the Fourier transform with Borel-Ritt-like theorems from asymptotic analysis, see S.-Y. Chung, D. Kim and Y. Yeom [13, Thm. 3.1] for $\boldsymbol{M} = (p!^{\alpha})_{p\in\mathbb{N}_0}$ (the Gevrey sequences) whenever $\alpha > 2$, and A. Lastra and J. Sanz [43, 44] for $\mathcal{S}_{\{(p!M_p)_p\}}(0,\infty)$ and general strongly regular sequences $\boldsymbol{M}$ whose growth index $\gamma(\boldsymbol{M})$ is strictly greater than 1. Subsequently, A. Debrouwere, J. Jiménez-Garrido and J. Sanz [16] improved and completed these results by including the spaces $\mathcal{S}^{\{(p!A_p)_p\}}_{\{(p!M_p)_p\}}(0,\infty)$ in their considerations, by dropping some hypotheses on $\boldsymbol{M}$ (specially moderate growth, stronger than derivation closedness), and by also studying the injectivity of the Stieltjes moment mapping. The new key tools were a better understanding of the meaning of the different growth conditions usually imposed on the sequence $\boldsymbol{M}$ and their expression in terms of indices of O-regular variation, as developed in [34], and the enhanced information obtained in [33] about the injectivity and surjectivity of the asymptotic Borel mapping on Carleman-Roumieu ultraholomorphic classes in sectors defined by sequences $\boldsymbol{M}$ subject to minimal conditions. Finally, A. Debrouwere [14] completely characterized the surjectivity and the existence of global right inverses for the moment mapping in Gelfand-Shilov spaces of both Roumieu and Beurling type under derivation closedness. His technique does not rest on Borel-Ritt-like theorems, but relates the problem to the surjectivity and existence of global right inverses for the Borel mapping in Carleman ultradifferentiable classes, already characterized by H.-J. Petzsche [52].

The main aim of this final chapter is the study of the Stieltjes moment problem in a new framework, allowing for a naturally larger target space for the moment mapping. The motivation comes from the introduction of the condition (sm), much weaker than derivation closedness, in the previous results of Borel-Ritt type. The key fact is that (sm) characterizes the equivalence of $\boldsymbol{M}_{+1} := (M_{p+1})_p$ and the Stieltjes moment sequence of the kernel function $e(z)$ (appearing in a truncated Laplace transform), what makes the procedure work. So, it becomes natural (see Propositions 4.2.2 and 4.2.3) to change the target space into the larger one $\Lambda_{\{\boldsymbol{M}_{+1}\}}$, and study again the injectivity and surjectivity in this new setting. As the technique in [14] does not seem to apply, we have recovered the technique in [44], resting on the construction of local right inverses for the moment mapping. This requires a careful study of the action of the Fourier transform under this new condition (sm) (Proposition 4.1.10), and the adaptation of some auxiliary results which were already useful in previous frameworks. We are able to characterize the injectivity of the Stieltjes moment mapping under condition (sm) in Theorem 4.2.6, while Theorem 4.2.7 studies the surjectivity problem and its connection to the existence of local right inverses for $\mathcal{M}$ with a uniform scaling of the parameter defining the Banach spaces under consideration.

# Chapter 1

# Preliminaries

The classes of functions that will appear throughout this PhD dissertation are described by means of a precise control on the growth of their derivatives, when ultraholomorphic classes of functions are dealt with, or on the growth of the remainders of their asymptotic expansions when those are available. Accordingly, classes of formal complex power series or of sequences of complex numbers will be defined by a suitable control on the growth of their coefficients, respectively of their elements. This control can be established in terms of three kinds of weight structures: weight sequences, weight functions, or weight matrices. This chapter consists mainly of the definitions and main properties of all these objects, and collects many of the well-known and useful facts needed in the sequel. Moreover, two growth indices, appearing in the literature within the general theory of O-regular variation and playing an important role in some of our results, will be described.

## 1.1 Weight sequences

In this section, we will treat the notion of weight sequence and describe some of the properties that can be assigned to these sequences and which will be particularly relevant during this work. These properties have mainly appeared in several classical works such as those of S. Mandelbrojt [46] and H. Komatsu [38]. The study of the gamma index for strongly regular weight sequences was initiated by V. Thilliez [80], and we collect here the main facts concerning it for general sequences.

### 1.1.1 Definition and properties

We write $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ and $\mathbb{N} := \{1, 2, 3, \dots\}$. In what follows, we always denote by $\boldsymbol{M} = (M_j)_{j\in\mathbb{N}_0}$ a sequence of positive real numbers with $M_0 = 1$. We also denote by $\widetilde{\boldsymbol{M}} = (\widetilde{M}_j)_j$ (resp. $\widehat{\boldsymbol{M}} = (\widehat{M}_j)_j$) the sequence defined by $\widetilde{M}_j := \frac{M_j}{j!}$

(resp. $\widehat{M}_j := j!M_j$). Now, let us start with some properties of these sequences.

**Definition 1.1.1.** We say that:

(i) $\boldsymbol{M}$ is called *normalized* if $1 = M_0 \leq M_1$ holds true.

(ii) $\boldsymbol{M}$ is *logarithmically convex* (for short, (lc)) if

$$M_j^2 \leq M_{j-1}M_{j+1}, \qquad j \in \mathbb{N}.$$

(iii) $\boldsymbol{M}$ is *stable under differential operators* or satisfies the *derivation closedness condition* (briefly, (dc)) if there exists $D > 0$ such that

$$M_{j+1} \leq D^{j+1}M_j, \qquad j \in \mathbb{N}_0.$$

(iv) $\boldsymbol{M}$ is of, or has, *moderate growth* (for the sake of brevity, (mg)) if there exists $A > 0$ such that

$$M_{j+k} \leq A^{j+k}M_jM_k, \qquad j,k \in \mathbb{N}_0.$$

(v) $\boldsymbol{M}$ satisfies the condition (nq) of *non-quasianalyticity* if

$$\sum_{j=0}^{\infty} \frac{M_j}{(j+1)M_{j+1}} < +\infty.$$

(vi) $\boldsymbol{M}$ satisfies the condition (snq) of *strong non-quasianalyticity* if there exists $B > 0$ such that

$$\sum_{j=k}^{\infty} \frac{M_j}{(j+1)M_{j+1}} \leq B\frac{M_k}{M_{k+1}}, \qquad k \in \mathbb{N}_0.$$

According to V. Thilliez [80], if $\boldsymbol{M}$ satisfies (lc), (mg) and (snq), we say $\boldsymbol{M}$ is *strongly regular*.

**Remark 1.1.2.** In the classical work of H. Komatsu [38], the properties (lc), (dc) and (mg) are denoted by $(M.1)$, $(M.2)'$ and $(M.2)$, respectively, while (nq) and (snq) for $\boldsymbol{M}$ are the same as properties $(M.3)'$ and $(M.3)$ for $\widehat{\boldsymbol{M}}$, respectively.

All these properties are preserved when passing from $\boldsymbol{M}$ to $\widehat{\boldsymbol{M}}$, but not conversely, i .e., when passing from $\boldsymbol{M}$ to $\widecheck{\boldsymbol{M}}$. For example, it is straightforward to check that both conditions (mg) and (dc) hold simultaneously true or false for $\boldsymbol{M}$ and $\widecheck{\boldsymbol{M}}$, but if $\boldsymbol{M}$ satisfies (lc), or (nq), or (snq), then $\widecheck{\boldsymbol{M}}$ does not necessarily satisfy any of them.

**Remark 1.1.3.** The form of the estimates in some of these properties admits slight modifications. For example, the property (dc) can be clearly rephrased as

$$\exists C_0 > 0,\ D \geq 1 \colon M_{j+1} \leq C_0 D^{j+1} M_j, \quad j \in \mathbb{N}_0.$$

This alternative expression can provide some flexibility in some of our arguments, and we will use one or another without further comment.

For a given sequence $\boldsymbol{M}$ we can associate a new one, defined by the quotients between two consecutive terms of $\boldsymbol{M}$.

**Definition 1.1.4.** Let $\boldsymbol{M}$ be a sequence, we define its associated *sequence of quotients* $\boldsymbol{m} = (m_j)_{j\in\mathbb{N}_0}$ by

$$m_j := \frac{M_{j+1}}{M_j}, \qquad j \in \mathbb{N}_0.$$

**Remark 1.1.5.** Let us note that if $M_0 = 1$, there is a one-to-one correspondence between $\boldsymbol{M}$ and $\boldsymbol{m}$ by observing that

$$M_j := \frac{M_j}{M_{j-1}} \frac{M_{j-1}}{M_{j-2}} \cdots \frac{M_2}{M_1} \frac{M_1}{1} = \prod_{i=0}^{j-1} m_i, \qquad j \in \mathbb{N}. \tag{1.1}$$

Sometimes, we will express some properties for $\boldsymbol{M}$ in terms of $\boldsymbol{m}$ without further mentioning this relation. In general, if $\boldsymbol{L}$ is a sequence we denote by a lowercase letter $\boldsymbol{\ell}$ the corresponding sequence of quotients.

We can obtain the following properties as a consequence of the previous definitions.

**Lemma 1.1.6.** *(see, for example, [26, Lemmas 1.1.6 and 1.1.7]) For every sequence $\boldsymbol{M}$ we have that:*

*(i) If $\boldsymbol{M}$ has moderate growth then $\boldsymbol{M}$ satisfies the derivation closedness condition.*

*(ii) If $\boldsymbol{M}$ satisfies the condition of strong non-quasianalyticity then $\boldsymbol{M}$ has the condition of non-quasianalyticity.*

*(iii) $\boldsymbol{M}$ is logarithmically convex if and only if $\boldsymbol{m}$ is nondecreasing.*

*(iv) If $\boldsymbol{M}$ is logarithmically convex, then $(M_j)^{1/j} \leq m_{j-1}$ for every $j \in \mathbb{N}$, $((M_j)^{1/j})_{j\in\mathbb{N}}$ is nondecreasing, and $\lim_{j\to\infty}(M_j)^{1/j} = \infty$ if, and only if, $\lim_{j\to\infty} m_j = \infty$.*

(v) *If $\boldsymbol{M}$ is logarithmically convex and nonquasianalytic, then* $\lim_{j\to\infty} m_j = \infty$.

(vi) *If $\boldsymbol{M}$ is logarithmically convex, then $M_j M_p \le M_{j+p}$ for every $j, p \in \mathbb{N}_0$.*

**Example 1.1.7.** We mention some interesting examples. In particular, those in (i), (iii) and (iv) appear in the applications of summability theory to the study of formal power series solutions for different kinds of equations.

(i) The sequences $\boldsymbol{M}_{\alpha,\beta} := \big(j!^{\alpha} \prod_{i=0}^{j} \log^{\beta}(e+i)\big)_{j\in\mathbb{N}_0}$, where $\alpha > 0$ and $\beta \in \mathbb{R}$, are strongly regular (in case $\beta < 0$, the first terms of the sequence have to be suitably modified in order to ensure (lc)).

(ii) The sequence $\boldsymbol{M}_{0,\beta} := (\prod_{i=0}^{j} \log^{\beta}(e+i))_{j\in\mathbb{N}_0}$, with $\beta > 0$, satisfies (lc) and (mg), and $\boldsymbol{m}$ tends to infinity; (nq) is satisfied only if $\beta > 1$, and (snq) is never satisfied.

(iii) For $a \in \mathbb{R}$ we set

$$\boldsymbol{G}^{a} := (j!^{a})_{j\in\mathbb{N}_0}, \qquad \overline{\boldsymbol{G}}^{a} := (j^{ja})_{j\in\mathbb{N}_0} \qquad \boldsymbol{\Gamma}_a := (\Gamma(1+ja))_{j\in\mathbb{N}_0}.$$

For $a > 0$ the sequence $\boldsymbol{G}^a$ is the *Gevrey sequence of index $a$* (note that it is $\boldsymbol{M}_{a,0}$ in example (i)). Clearly $\boldsymbol{G}^a$ and $\overline{\boldsymbol{G}}^a$ are the best known example of a strongly regular sequence for any $a > 0$ (by the convention $0^0 := 1$).

(iv) For $q > 1$ and $\sigma > 1$, $\boldsymbol{M}_{q,\sigma} := (q^{j^{\sigma}})_{j\in\mathbb{N}_0}$ satisfies (lc) and (snq), but not (mg). The condition (dc) is satisfied if, and only if, $1 < \sigma \le 2$. In case $\sigma = 2$, we get the well-known *$q$-Gevrey sequence.*

The next properties have a key role in the study of the stability properties in some classes, as we are going to see in the following chapter.

**Definition 1.1.8.** We say that:

(i) $\boldsymbol{M}$ has the *root almost increasing property*, denoted by (rai), if the sequence of roots $(\widetilde{M}_j^{1/j})_{j\in\mathbb{N}}$ is almost increasing, that is, if there exists $C > 0$ such that

$$\widetilde{M}_j^{1/j} \le C\widetilde{M}_k^{1/k}, \qquad j,k \in \mathbb{N}_0,\ 1 \le j \le k.$$

(ii) $\boldsymbol{M}$ has the *Faà-di-Bruno property*, denoted by (FdB), if there exist $C, h \ge 1$ such that

$$\widetilde{M}_j^{\circ} \le C h^j \widetilde{M}_j, \qquad j \in \mathbb{N}_0,$$

where $\widetilde{\boldsymbol{M}}^{\circ} := (\widetilde{M}_j^{\circ})_{j\in\mathbb{N}_0}$ is the sequence defined by

$$\widetilde{M}_k^{\circ} := \max\left\{\widetilde{M}_{\ell} \cdot \widetilde{M}_{j_1} \cdots \widetilde{M}_{j_\ell} : j_i \in \mathbb{N}, \sum_{i=1}^{\ell} j_i = k\right\},\ k \in \mathbb{N}; \quad \widetilde{M}_0^{\circ} := 1. \tag{1.2}$$

**Definition 1.1.9.** We say that a sequence $\boldsymbol{M}$ is a *weight sequence* if it is logarithmically convex and $\lim_{j\to\infty} m_j = \infty$.

In the third chapter, we will present several Borel-Ritt-like theorems, stating the surjectivity of the Borel map in ultraholomorphic classes, which will be valid for more general sequences than covered previously in the literature. In particular, we will deal with regular sequences in the sense of E. M. Dyn'kin [20].

**Definition 1.1.10.** We say $\widehat{\boldsymbol{M}}$ is *regular* if $\boldsymbol{M}$ is a weight sequence and satisfies (dc).

**Lemma 1.1.11.** *Let $\boldsymbol{M}$ be a sequence. If $\boldsymbol{M}$ is strongly regular, then $\widehat{\boldsymbol{M}}$ is regular.*

*Proof.* If $\boldsymbol{M}$ is strongly regular, then $\boldsymbol{M}$ has (lc), (mg) and (snq). Thanks to 1.1.6 we know that $\boldsymbol{M}$ has (nq), (dc) and $\lim_{j\to\infty} m_j = \infty$. Therefore, $\widehat{\boldsymbol{M}}$ is regular. □

## 1.1.2 Comparable and equivalent sequences

The notions of comparable and equivalent sequences are naturally present in the consideration of our classes of functions, as we will see in the next sections.

**Definition 1.1.12.** Let $\boldsymbol{M} = (M_p)_{p\in\mathbb{N}_0}$ and $\boldsymbol{L} = (L_p)_{p\in\mathbb{N}_0}$ be sequences with arbitrary $M_0, L_0 > 0$, we say that $\boldsymbol{M}$ is *smaller than* $\boldsymbol{L}$, and we write $\boldsymbol{M} \precsim \boldsymbol{L}$, if there exist $A, B > 0$ such that

$$M_j \le AB^j L_j, \qquad j \in \mathbb{N}_0,$$

or, equivalently, if

$$\sup_{j\in\mathbb{N}_0} \left(\frac{M_j}{L_j}\right)^{1/j} < +\infty.$$

If $\boldsymbol{M} \precsim \boldsymbol{L}$ and $\boldsymbol{L} \precsim \boldsymbol{M}$, we say that $\boldsymbol{M}$ is *equivalent to* $\boldsymbol{L}$, and we write $\boldsymbol{M} \approx \boldsymbol{L}$. Note that, in case $M_0 = L_0 = 1$, equivalence amounts to $B^j M_j \le L_j \le C^j M_j$ for every $j \in \mathbb{N}_0$ and suitable $B, C > 0$.

**Remark 1.1.13.** Some properties like (mg) and (dc) are clearly preserved under equivalence for the relation $\approx$.

**Example 1.1.14.** We recall some useful elementary estimates,

$$\forall\, j \in \mathbb{N}: \quad \frac{j^j}{e^j} \le j! \le j^j, \tag{1.3}$$

which immediately imply that $\boldsymbol{G}^a \approx \overline{\boldsymbol{G}}^a$ for any $a \in \mathbb{R}$. Stirling's formula for Euler's Gamma function implies that also $\boldsymbol{G}^a \approx \boldsymbol{\Gamma}_a$.

We can also establish an equivalence relation at the level of the sequence of quotients.

**Definition 1.1.15.** Let $\boldsymbol{m}$ and $\boldsymbol{\ell}$ be the sequences of quotients associated with $\boldsymbol{M}$ and $\boldsymbol{L}$, respectively. We say that $\boldsymbol{m}$ is *strongly smaller than* $\boldsymbol{\ell}$, and we write $\boldsymbol{m} \preceq \boldsymbol{\ell}$, if it exists $A > 0$ such that

$$m_j \leq A\ell_j, \qquad j \in \mathbb{N}_0,$$

or, equivalently, if

$$\sup_{j\in\mathbb{N}_0} \frac{m_j}{\ell_j} < +\infty.$$

If $\boldsymbol{m} \preceq \boldsymbol{\ell}$ and $\boldsymbol{\ell} \preceq \boldsymbol{m}$, we say that $\boldsymbol{m}$ is *strongly equivalent to* $\boldsymbol{\ell}$, and we write $\boldsymbol{m} \simeq \boldsymbol{\ell}$.

**Remark 1.1.16.** Whenever $\boldsymbol{m} \simeq \boldsymbol{\ell}$ we have $\boldsymbol{M} \approx \boldsymbol{L}$ (see (1.1)), but in general the converse does not hold.

Finally, we can compare sequences term by term. For this, we use the following notation.

**Definition 1.1.17.** Let $\boldsymbol{M} = (M_p)_{p\in\mathbb{N}_0}$ and $\boldsymbol{L} = (L_p)_{p\in\mathbb{N}_0}$ be sequences with arbitrary $M_0, L_0 > 0$, we write $\boldsymbol{M} \leq \boldsymbol{L}$ if

$$M_j \leq L_j, \qquad j \in \mathbb{N}_0.$$

### 1.1.3 The growth index $\gamma(\boldsymbol{M})$

The index $\gamma(\boldsymbol{M})$, introduced by V. Thilliez [80, Sect. 1.3] for strongly regular sequences $\boldsymbol{M}$, can be equally defined for (lc) sequences, or even any sequence. For a comprehensive study of this index we refer to the work of J. Jiménez-Garrido, J. Sanz and G. Schindl [34, Sect. 3], especially to the characterizing result [34, Thm. 3.11].

**Definition 1.1.18.** Let $\boldsymbol{M}$ be a sequence and $\gamma \in \mathbb{R}$. We say $\boldsymbol{M}$ satisfies property $(P_\gamma)$ if there exists a sequence of real numbers $\boldsymbol{\ell} = (\ell_j)_{j\in\mathbb{N}_0}$ such that:

(i) $\boldsymbol{m} \simeq \boldsymbol{\ell}$ (and therefore $\boldsymbol{M} \approx \boldsymbol{L}$), that is,

$$\exists\, a \geq 1\, \forall\, j \in \mathbb{N}_0 : \quad a^{-1}m_j \leq \ell_j \leq am_j,$$

(ii) $((j+1)^{-\gamma}\ell_j)_{j\in\mathbb{N}_0}$ is nondecreasing.

If $(P_\gamma)$ holds true for $\boldsymbol{M}$, then $(P_{\gamma'})$ also holds for any $\gamma' \leq \gamma$. It is then natural to define the *growth index* $\gamma(\boldsymbol{M})$ by

$$\gamma(\boldsymbol{M}) := \sup\{\gamma \in \mathbb{R} : \ (P_\gamma) \text{ is fulfilled}\}.$$

**Remark 1.1.19.** We use the conventions $\inf \emptyset = \sup \mathbb{R} = +\infty$ and $\inf \mathbb{R} = \sup \emptyset = -\infty$.

We can also introduce the $\gamma(\boldsymbol{M})$ index for an (lc) sequence $\boldsymbol{M}$ by using the condition $(\gamma_\beta)$ for $\boldsymbol{m}$, given by J. Schmets and M. Valdivia [74]. We refer to the PhD dissertation of J. Jiménez-Garrido [26] and [34] for more details.

**Definition 1.1.20.** Let $\boldsymbol{M}$ be an (lc) sequence. For any $\beta > 0$ we say that $\boldsymbol{m}$ satisfies the condition $(\gamma_\beta)$ if there exists $A > 0$ such that

$$\sum_{\ell=p}^{\infty} \frac{1}{(m_\ell)^{1/\beta}} \leq \frac{A(p+1)}{(m_p)^{1/\beta}}, \qquad p \in \mathbb{N}_0. \tag{$\gamma_\beta$}$$

It turns out that

$$\gamma(\boldsymbol{M}) = \sup\{\beta > 0; \ \boldsymbol{m} \text{ satisfies } (\gamma_\beta)\,\},$$

and moreover

$$\gamma(\boldsymbol{M}) > \beta \iff \boldsymbol{m} \text{ satisfies } (\gamma_\beta). \tag{1.4}$$

There exists also a third approach by using the theory of O-regular variation. In this context, the index $\gamma(\boldsymbol{M})$ is precisely the lower Matuszewska index $\beta(\boldsymbol{m})$. Again, we refer to [26, 34].

**Definition 1.1.21.** A sequence $(c_p)_{p\in\mathbb{N}_0}$ is *almost increasing* if there exists $a > 0$ such that for every $p \in \mathbb{N}_0$ we have that $c_p \leq a c_q$ for every $q \geq p$. Then for any sequence $\boldsymbol{M}$ one has

$$\gamma(\boldsymbol{M}) = \sup\{\gamma > 0 : (m_p/(p+1)^\gamma)_{p\in\mathbb{N}_0} \text{ is almost increasing}\}. \tag{1.5}$$

A straightforward verification shows the following properties for the gamma index.

**Proposition 1.1.22.** *Let $\boldsymbol{M}$ be a sequence and $s > 0$, one has*

$$\gamma((p!^s M_p)_{p\in\mathbb{N}_0}) = \gamma((\Gamma(1+sp)M_p)_{p\in\mathbb{N}_0}) = \gamma(\boldsymbol{M}) + s, \tag{1.6}$$
$$\gamma((M_p/p!^s)_{p\in\mathbb{N}_0}) = \gamma((M_p/\Gamma(1+sp))_{p\in\mathbb{N}_0}) = \gamma(\boldsymbol{M}) - s. \tag{1.7}$$

In this dissertation, the termwise product, respectively quotient, of two sequences $\boldsymbol{M}$ and $\boldsymbol{L}$ will be denoted, resp., by

$$\boldsymbol{M}\cdot\boldsymbol{L}=\boldsymbol{M}\boldsymbol{L}:=(M_pL_p)_{p\in\mathbb{N}_0}\qquad\text{and}\qquad \boldsymbol{M}/\boldsymbol{L}:=(M_p/L_p)_{p\in\mathbb{N}_0}.$$

We can characterize the property of strong non quasianalyticity in terms of the gamma index associated to $\boldsymbol{M}$.

**Proposition 1.1.23.** *Let $\boldsymbol{M}$ be a sequence. Then, $\boldsymbol{M}$ satisfies (snq) if, and only if, $\gamma(\boldsymbol{M})>0$.*

*Proof.* The condition (snq) for $\boldsymbol{M}$ is precisely $(\gamma_1)$ for $\widehat{m}$, the sequence of quotients for $\widehat{\boldsymbol{M}}$. Thanks to the fact that $\gamma(\widehat{\boldsymbol{M}})=\gamma(\boldsymbol{M})+1$ (this is clear from (1.6)), we deduce from (1.4) the desired result. □

Moreover, from the very initial definition of the gamma index we can establish the following comparison with Gevrey sequences.

**Lemma 1.1.24.** *Let $\boldsymbol{M}$ be a weight sequence and $\beta>0$. If $\gamma(\boldsymbol{M})>\beta$, then we have that $(p!^{\beta})_p\precsim\boldsymbol{M}$.*

We recall also the following result for later use.

**Lemma 1.1.25** ([34], Remark 3.15)**.** *For an arbitrary sequence $\boldsymbol{M}$ such that $\gamma(\boldsymbol{M})>1$, there exists a weight sequence $\boldsymbol{L}$ such that $\widehat{\boldsymbol{\ell}}\simeq\boldsymbol{m}$, and so $\widehat{\boldsymbol{L}}\approx\boldsymbol{M}$ and $\gamma(\widehat{\boldsymbol{L}})=\gamma(\boldsymbol{M})$.*

Finally, the gamma index is stable under $\simeq$. However, in general it is not possible to extend the equality between the gamma index of two sequences under the weaker equivalence $\approx$. In this sense, we present a partial result

**Proposition 1.1.26** ([34], Corollary 3.14)**.** *Let $\boldsymbol{M}$ and $\boldsymbol{L}$ be sequences with $\boldsymbol{M}\approx\boldsymbol{L}$. Assume that there exists $r\geq 0$ such that $\boldsymbol{G}^r\boldsymbol{M}$ and $\boldsymbol{G}^r\boldsymbol{L}$ are weight sequences. Then $\gamma(\boldsymbol{M})=\gamma(\boldsymbol{L})$ holds true.*

*In particular, if $\boldsymbol{M}$ and $\boldsymbol{L}$ are weight sequences with $\boldsymbol{M}\approx\boldsymbol{L}$, the last equality holds.*

### 1.1.4 Auxiliary functions

In this subsection, we are going to introduce some auxiliary functions which have a key role in the study of the properties of classes of functions defined in terms of a sequence. For instance, see the works of H. Komatsu [38], J. Chaumat and A.-M. Chollet [11] and V. Thilliez [80].

**Definition 1.1.27.** For an arbitrary sequence $\boldsymbol{M}$ we consider the function $h_{\boldsymbol{M}}:[0,\infty)\to\mathbb{R}$ defined as

$$h_{\boldsymbol{M}}(t) := \inf_{p\in\mathbb{N}_0} M_p t^p, \quad t>0; \qquad h_{\boldsymbol{M}}(0)=0.$$

In fact, if $\boldsymbol{M}$ is a weight sequence we have that

$$h_{\boldsymbol{M}}(t) = \begin{cases} M_p t^p, & \text{if } t\in[\frac{1}{m_p},\frac{1}{m_{p-1}}),\ p\in\mathbb{N},\\ 1, & \text{if } t\geq \frac{1}{m_0}. \end{cases} \tag{1.8}$$

Now, the following elementary facts about $h_{\boldsymbol{M}}$ are straightforward.

**Lemma 1.1.28.** *Let $\boldsymbol{M}=(M_p)_{p\in\mathbb{N}_0}$ be a weight sequence, then:*

*(i)* $h_{\boldsymbol{M}}(t)$ *is nondecreasing and continuous.*

*(ii)* $h_{\boldsymbol{M}}(t)\leq 1$ *for all* $t>0$*, and* $h_{\boldsymbol{M}}(t)=1$ *for all* $t$ *sufficiently large.*

*(iii)* $\lim_{t\to 0} h_{\boldsymbol{M}}(t)=0$.

We can consider the logarithmically convex minorant sequence $\boldsymbol{M}^{\mathrm{lc}}$ of a sequence $\boldsymbol{M}$, that is, the (lc) sequence such that $\boldsymbol{M}^{\mathrm{lc}}\leq\boldsymbol{M}$, and for every other (lc) sequence $\boldsymbol{L}$ with $\boldsymbol{L}\leq\boldsymbol{M}$ we have that $\boldsymbol{L}\leq\boldsymbol{M}^{\mathrm{lc}}$. In particular, $\boldsymbol{M}$ is (lc) if and only if $\boldsymbol{M}=\boldsymbol{M}^{\mathrm{lc}}$.

Indeed, we can recover the logarithmically convex minorant $\boldsymbol{M}^{\mathrm{lc}}$ from the knowledge of $h_{\boldsymbol{M}}$. More precisely,

**Proposition 1.1.29** (H. Komatsu [38], G. Schindl [73])**.** *Let $\boldsymbol{M}$ be a sequence with $\lim_{p\to\infty}(M_p)^{1/p}=\infty$. Then, one has that*

$$M_p^{lc} = \sup_{t>0} t^p h_{\boldsymbol{M}}(1/t), \quad p\in\mathbb{N}_0, \tag{1.9}$$

*In particular, we can compute the terms of a weight sequence by using the previous expression.*

If we consider two sequences which are equivalent, then there exists a relation between the associated functions.

**Lemma 1.1.30.** *Let $\boldsymbol{M}$ and $\boldsymbol{L}$ be two equivalent weight sequences. Then, there exist $A,B>0$ such that*

$$h_{\boldsymbol{M}}(At)\leq h_{\boldsymbol{L}}(t)\leq h_{\boldsymbol{M}}(Bt), \qquad t\geq 0.$$

We also introduce a second associated function, namely the counting function $\nu_{\boldsymbol{m}}$ for the sequence $\boldsymbol{m}$.

**Definition 1.1.31.** For an arbitrary weight sequence $\boldsymbol{M}$, we define the counting function $\nu_{\boldsymbol{m}} : (0, \infty) \to \mathbb{N}_0$ for the sequence $\boldsymbol{m}$, as

$$\nu_{\boldsymbol{m}}(t) := \#\{p \in \mathbb{N}_0 : m_p \leq t\} = \max\{p \in \mathbb{N} : m_{p-1} \leq t\}. \tag{1.10}$$

The counting function $\nu_{\boldsymbol{m}}$ is obviously nondrecreasing, and $\lim_{t\to\infty} \nu_{\boldsymbol{m}}(t) = \infty$.

In [34], the relation between the indices of O-regular variation of $\boldsymbol{m}$ and $\nu_{\boldsymbol{m}}$ is clarified, and from this connection we can characterize some properties of $\nu_{\boldsymbol{m}}$ that will be important for our aim.

**Lemma 1.1.32.** *([34]) Let $\boldsymbol{M} = (M_p)_{p\in\mathbb{N}_0}$ be a weight sequence, then:*

*(i)* $\gamma(\boldsymbol{M}) > 0$ *if and only if* $\nu_{\boldsymbol{m}}$ *satisfies the condition* $\nu_{\boldsymbol{m}}(2t) = O(\nu_{\boldsymbol{m}}(t))$ *as* $t$ *tends to* $\infty$.

*(ii)* $\gamma(\boldsymbol{M}) > 1$ *if and only if* $\nu_{\boldsymbol{m}}$ *satisfies the condition* $(\omega_{snq})$, *i. e., there exists* $D > 0$ *such that*

$$\int_1^\infty \frac{\nu_{\boldsymbol{m}}(ys)}{s^2}\, ds \leq D\nu_{\boldsymbol{m}}(y) + D, \qquad y \geq 0.$$

*Proof.* $(i)$ follows by Proposition 1.1.23 and [34, Corollary 4.2.(ii)].
$(ii)$ holds true by combining [34, Lemma 2.10], [34, Corollary 2.13], [34, Theorem 3.10] and [34, Proposition 4.1]. □

# 1.2 Weight functions in the sense of Braun-Meise-Taylor

In the second chapter of this dissertation, we will deal with classes of functions associated with a weight function; for this reason, we introduce the definition and main properties of these functions below. For this section, we refer to the PhD dissertation of G. Schindl [70] and the references therein.

Let us start with the definition of weight function.

**Definition 1.2.1.** A *weight function* is a continuous, nondecreasing function $\omega : [0, +\infty) \to [0, +\infty)$ with $\omega(0) = 0$ and $\lim_{t\to+\infty} \omega(t) = +\infty$.

If $\omega$ satisfies in addition $\omega(t) = 0$ for all $t \in [0, 1]$, then we call $\omega$ a *normalized weight function.* For convenience we will write that $\omega$ has $(\omega_0)$ if it is a normalized weight function.

**Remark 1.2.2.** Sometimes in the literature the weight function $\omega$ is extended to $t \in \mathbb{R}$ or $t \in \mathbb{C}$ in a radial-symmetric way, i. e., we replace $t$ by $|t|$.

Now, we present some examples of weight functions.

**Example 1.2.3.** The following are easily checked to be weight functions:

(i) For $a > 0$, $\omega(t) = t^a$ (Gevrey weight function of index $a$).

(ii) For $a > 1$, $\omega(t) = (\log(1+t))^a$.

(iii) For $a > 1$, $\omega(t) = t(\log(e+t))^{-a}$.

(iv) For $0 < a < 1$ and $b > 0$, $\omega(t) = \exp(b(\log(1+t))^a)$.

**Remark 1.2.4.** For any $a > 0$ we put $\omega^a$ for the function given by $\omega^a(t) := \omega(t^a)$, i. e., the result of composing $\omega$ with the Gevrey weight $t \mapsto t^a$.

In a similar way as one does with sequences, we can compare two weight functions.

**Definition 1.2.5.** Let $\sigma, \tau$ be weight functions, we write $\sigma \preceq \tau$ if

$$\tau(t) = O(\sigma(t)), \qquad t \to +\infty,$$

and call them equivalent, denoted by $\sigma \sim \tau$, if

$$\sigma \preceq \tau \quad \text{and} \quad \tau \preceq \sigma.$$

**Remark 1.2.6.** Note that if we take a weight function $\omega$, we can always consider a new weight function $\sigma$ which is equal to $\omega$ for all large $t > 0$ and $\sigma(t) = 0$ for all $t \in [0,1]$. In other words, we can always take a normalized weight function which is equivalent to the original one, and so the property $(\omega_0)$ can be assumed without loss of generality.

Now, we consider the following (standard) conditions on weight functions. This terminology has already been used in [70] and some subsequent works.

$(\omega_1)$ $\omega(2t) = O(\omega(t))$ as $t \to +\infty$, i. e. $\exists\, L \geq 1\ \forall\, t \geq 0: \quad \omega(2t) \leq L(\omega(t)+1)$.

$(\omega_2)$ $\omega(t) = O(t)$ as $t \to +\infty$.

$(\omega_3)$ $\ln(t) = o(\omega(t))$ as $t \to +\infty$.

$(\omega_4)$ $\varphi_\omega : t \mapsto \omega(e^t)$ is a convex function on $\mathbb{R}$.

$(\omega_5)$ $\omega(t) = o(t)$ as $t \to +\infty$.

$(\omega_6)$ $\exists\, H \geq 1\ \forall\, t \geq 0:\ 2\omega(t) \leq \omega(Ht) + H$.

**Example 1.2.7.** The Gevrey weight of index $a > 0$, satisfies $(\omega_1)$, $(\omega_3)$, $(\omega_4)$ and $(\omega_6)$ for all $a > 0$. However, $(\omega_2)$ holds only if $a \leq 1$, and $(\omega_5)$ does only if $a < 1$.

Some of these properties on $\omega$ are stable with respect to the relation $\sim$.

**Proposition 1.2.8.** *([70, Lemma 3.2.2]) The properties $(\omega_1)$, $(\omega_2)$, $(\omega_3)$, $(\omega_5)$ and $(\omega_6)$ are stable under the relation $\sim$ for weight functions.*

The following sets of weight functions will be important in Chapter 2.

**Definition 1.2.9.** We define the set $\mathcal{W}_0$ as

$$\mathcal{W}_0 := \{\omega : [0,\infty) \to [0,\infty) : \omega \text{ has } (\omega_0), (\omega_3), (\omega_4)\},$$

and the set $\mathcal{W}$ as

$$\mathcal{W} := \{\omega \in \mathcal{W}_0 : \omega \text{ has } (\omega_1)\}.$$

Finally, let us recall the following crucial growth assumption.

**Definition 1.2.10.** We say that a weight function $\omega$ satisfies the property $(\alpha_0)$ if the following holds:

$$\exists\, C \geq 1\ \exists\, t_0 \geq 0\ \forall\, \lambda \geq 1\ \forall\, t \geq t_0 : \quad \omega(\lambda t) \leq C\lambda\omega(t). \tag{1.11}$$

**Remark 1.2.11.** This property $(\alpha_0)$ has a key-role. Following a recent paper of G. Schindl [72, Sect. 4.1] and the citations therein, we can establish an equivalence between a weight function $\omega$ and a subadditive weight function $\sigma$ (i. e., such that $\sigma(s+t) \leq \sigma(s) + \sigma(t)$ for every $s, t \geq 0$), or even to a concave weight function, if and only if (1.11) holds true.

It is also known that $(\alpha_0)$ characterizes some desired stability properties for ultradifferentiable classes $\mathcal{E}_{[\omega]}$, e. g. closedness under composition, inverse closedness and closedness under solving ODE's. The definition of such classes (which will not be used in this dissertation) and these results can be found in the works of A. Rainer and G. Schindl [59], [60, Thm. 1, Thm. 3], of these two authors with S. Fürdös and D. N. Nenning [22, Thm. 4.8], and of C. Fernández and A. Galbis [21].

## 1.2.1 Associated weight function

In this subsection, we introduce an auxiliary weight function $\omega_{\boldsymbol{M}}$ constructed from a weight sequence $\boldsymbol{M}$ and even determining it, and already appearing in the classical works of S. Mandelbrojt [46] and H. Komatsu [38]. We note that G. Schindl [73] has recently added more information about the construction of the function $\omega_{\boldsymbol{M}}$.

**Definition 1.2.12.** For an arbitrary sequence $\boldsymbol{M}$ we consider the *associated function* $\omega_{\boldsymbol{M}} : [0,\infty) \to \mathbb{R} \cup \{+\infty\}$ defined as

$$\omega_{\boldsymbol{M}}(t) := \sup_{j\in\mathbb{N}_0} \ln\left(\frac{t^j}{M_j}\right), \quad t > 0, \qquad \omega_{\boldsymbol{M}}(0) := 0.$$

Note that $\omega_{\boldsymbol{M}}(t) \geq 0$ for any $t \geq 0$, since $t^0/M_0 = 1$ for all $t > 0$.

If $\liminf_{j\to+\infty}(M_j)^{1/j} > 0$, then $\omega_{\boldsymbol{M}}(t) = 0$ for sufficiently small positive $t$, since $t^0/M_0 = 1$ and $\ln(t^j/M_j) < 0$ precisely if $t < (M_j)^{1/j}$, for all $j \in \mathbb{N}$. In particular, if $M_j \geq 1$ for all $j \in \mathbb{N}$, then $\omega_{\boldsymbol{M}}$ vanishes on $[0,1]$. Moreover, under this assumption $t \mapsto \omega_{\boldsymbol{M}}(t)$ is a continuous nondecreasing function, which is convex in the variable $\ln(t)$ and tends faster to infinity than any $\ln(t^j)$, $j \geq 1$, as $t \to +\infty$. However, we cannot guarantee, in general, that the function $\omega_{\boldsymbol{M}}$ will be finite for all $t > 0$, in particular, $\omega_{\boldsymbol{M}}$ would not be a weight function as defined in Definition 1.2.1.

**Example 1.2.13.** If $\boldsymbol{M} = \{1/2^n\}_{n=0}^{\infty}$, then $\omega_{\boldsymbol{M}}(t) = 0$ for any $t \leq 1/2$, and $\omega_{\boldsymbol{M}}(t) = \infty$ for any $t > 1/2$.

It is not difficult to check that $\lim_{j\to+\infty}(M_j)^{1/j} = +\infty$ if, and only if, $\omega_{\boldsymbol{M}}(t) < +\infty$ for each finite $t$, so this will be a basic assumption for defining $\omega_{\boldsymbol{M}}$. In this case, the function $\omega_{\boldsymbol{M}}$ is indeed a weight function and it satisfies $(\omega_3)$ and $(\omega_4)$. Note that weight sequences $\boldsymbol{M}$ satisfy this condition, see Lemma 1.1.6.(iv). In fact, for a weight sequence $\boldsymbol{M}$ we have that

$$\omega_{\boldsymbol{M}}(t) = \begin{cases} p\ln(t) - \ln(M_p) & \text{if } t \in [m_p, m_{p+1}),\ p = 0, 1, \dots, \\ 0 & \text{if } t < m_0. \end{cases} \tag{1.12}$$

In particular, we deduce that

$$\omega_{\boldsymbol{M}}(m_p) = \ln\left(\frac{m_p^p}{M_p}\right), \qquad p \in \mathbb{N}_0.$$

Most of the properties for the weight function $\omega_{\boldsymbol{M}}$ can be characterized in terms of the weight sequence $\boldsymbol{M}$. For example, G. Schindl [72, Thm. 4.5] did so for the condition $(\alpha_0)$. We recall the following facts, which can be found, e. g., in H. Komatsu [38, Lemma 4.1], A. Rainer and G. Schindl [59, Sect. 5] and J. Jiménez-Garrido, J. Sanz and G. Schindl [35, Lemma 2.4].

**Lemma 1.2.14.** *Let $\boldsymbol{M}$ be a normalized weight sequence, then $\omega_{\boldsymbol{M}} \in \mathcal{W}_0$ holds true. Moreover,*

(*i*) $\liminf_{j\to\infty}(\widecheck{M}_j)^{1/j} > 0$ *if and only if* $(\omega_2)$ *holds for* $\omega_{\boldsymbol{M}}$*,* (*ii*) $\lim_{j\to\infty}(\widecheck{M}_j)^{1/j} = +\infty$ *if and only if* $(\omega_5)$ *holds for* $\omega_{\boldsymbol{M}}$,

(*iii*) $(\omega_6)$ *holds for* $\omega_{\boldsymbol{M}}$ *if and only if* $\boldsymbol{M}$ *does have* (mg).

In addition, if we compose some of these weight functions $\omega_{\boldsymbol{M}}$ with the Gevrey weight function of index $\beta > 0$ then we obtain, except for a constant, the weight function associated with the same power of the original sequence $\boldsymbol{M}$. More precisely, for $\beta > 0$ we write $\boldsymbol{M}^{1/\beta} := (M_j^{1/\beta})_{j\in\mathbb{N}_0}$, then (see [35, (2.7)])

$$\omega^{\beta}_{\boldsymbol{M}}(t) := \omega_{\boldsymbol{M}}(t^{\beta}) = \beta\omega_{\boldsymbol{M}^{1/\beta}}(t), \qquad t \geq 0.$$

For any sequence $\boldsymbol{M}$, the functions $h_{\boldsymbol{M}}$ and $\omega_{\boldsymbol{M}}$ are related by

$$h_{\boldsymbol{M}}(t) = \exp(-\omega_{\boldsymbol{M}}(1/t)), \quad t > 0 \tag{1.13}$$

(where we adopt the convention $e^{-\infty} = 0$). Hence, as it occurs with the function $h_{\boldsymbol{M}}$ (compare with (1.9)), we can compute the log convex minorant associated with a sequence $\boldsymbol{M}$ such that $\lim_{p\to\infty}(M_p)^{1/p} = \infty$ as follows (see S. Mandelbrojt [46, Chapitre I, 1.4, 1.8] and H. Komatsu [38, Prop. 3.2]):

$$M_p^{\mathrm{lc}} = \sup_{t\geq 0} \frac{t^p}{\exp(\omega_{\boldsymbol{M}}(t))} \quad p \in \mathbb{N}_0. \tag{1.14}$$

It also follows that $\omega_{\boldsymbol{M}} \equiv \omega_{\boldsymbol{M}^{\mathrm{lc}}}$, and that if $\boldsymbol{M}$ is log-convex, the right-hand side of formula (1.14) yields $M_p$.

Also, from Lemma 1.1.30 or by a direct computation one gets that, if $\boldsymbol{M}$ and $\boldsymbol{L}$ are two equivalent weight sequences, then there exist $A, B > 0$ such that

$$\omega_{\boldsymbol{M}}(At) \leq \omega_{\boldsymbol{L}}(t) \leq \omega_{\boldsymbol{M}}(Bt), \qquad t \geq 0.$$

Finally, if $\boldsymbol{M}$ is a weight sequence, we can also establish a relation between the functions $\nu_{\boldsymbol{m}}$ and $\omega_{\boldsymbol{M}}$ through the following integral representation, see [38, (3.11)]:

$$\omega_{\boldsymbol{M}}(x) = \int_0^x \frac{\nu_{\boldsymbol{m}}(\lambda)}{\lambda} d\lambda = \int_{m_0}^x \frac{\nu_{\boldsymbol{m}}(\lambda)}{\lambda} d\lambda, \quad x > 0. \tag{1.15}$$

### 1.2.2 The growth index $\gamma(\omega)$

In [35] and [36], J. Jiménez-Garrido, J. Sanz and G. Schindl introduced the index $\gamma(\omega)$ associated with a weight function $\omega$, in order to study the maximal opening of a sector in the Riemann surface of the logarithm for which the Borel map, in the corresponding ultraholomorphic class associated with $\omega$, is surjective. In this dissertation we are only going to consider such index for weight functions, but a more general treatment is possible, see [34, Sect. 2.3] and the references therein.

**Definition 1.2.15.** Let $\omega$ be a weight function and $\gamma > 0$. We say that $\omega$ has property $(P_{\omega,\gamma})$ if there exists $K > 1$ such that

$$\limsup_{t\to+\infty} \frac{\omega(K^\gamma t)}{\omega(t)} < K.$$

If $(P_{\omega,\gamma})$ holds for some $K > 1$, then also $(P_{\omega,\gamma'})$ is satisfied for all $\gamma' \leq \gamma$ with the same $K$. It is then natural to define the *growth index* $\gamma(\omega)$ by

$$\gamma(\omega) := \sup\{\gamma > 0 : (P_{\omega,\gamma}) \text{ is satisfied}\}.$$

Note that in the previous definition we can restrict to $\gamma > 0$, because for $\gamma \leq 0$ condition $(P_{\omega,\gamma})$ is satisfied for all weights $\omega$, since $\omega$ is nondecreasing and $K > 1$.

We recall some facts about this index:

*(i)* If $\omega \sim \sigma$ then $\gamma(\omega) = \gamma(\sigma)$, see [34, Rem. 2.12].

*(ii)* $\gamma(\omega) > 0$ holds if and only if $(\omega_1)$, see [34, Cor. 2.14].

*(iii)* By definition one has $\gamma(\omega^a) = \frac{1}{a}\gamma(\omega)$ for any $a > 0$.

*(iv)* If $\boldsymbol{M}$ is a weight sequence, in general we can only establish the inequality $\gamma(\boldsymbol{M}) \leq \gamma(\omega_{\boldsymbol{W}})$ between gamma indices [34, Cor. 4.6 (*i*)]. Moreover, if $\boldsymbol{M}$ has in addition $(mg)$, then $\gamma(\boldsymbol{M}) = \gamma(\omega_{\boldsymbol{M}})$, see [34, Cor. 4.6 (*iii*)].

## 1.3 Weight matrices

A. Rainer and G. Schindl [59, 70] introduced the new notion of weight matrix. Their goal when considering this object was to give a unified treatment of the spaces defined by a single weight sequence and by a single weight function, by considering both cases as spaces associated with suitable weight matrices. For the following definitions and conditions see [59, Sect. 4].

**Definition 1.3.1.** A *weight matrix* $\mathcal{M}$ is a (one parameter) family of sequences $\mathcal{M} := \{\boldsymbol{M}^{(\alpha)} \in \mathbb{R}_{>0}^{\mathbb{N}_0} : \alpha > 0\}$, such that

$$\boldsymbol{M}^{(\alpha)} \leq \boldsymbol{M}^{(\beta)} \text{ for } \alpha \leq \beta; \quad M_0^{(\alpha)} = 1, \quad \forall\, \alpha > 0.$$

We use the same notation as for the case of weight sequences. For example, for each $\alpha > 0$ we denote by $\widecheck{M}_j^{(\alpha)} := M_j^{(\alpha)}/j!$ for $j \in \mathbb{N}_0$, and $m_j^{(\alpha)} := M_{j+1}^{(\alpha)}/M_j^{(\alpha)}$ for $j \in \mathbb{N}_0$.

Moreover, we can stablish properties for weight matrices, by taking the corresponding ones for each sequences.

**Definition 1.3.2.** We say that:

(i) A weight matrix $\mathcal{M}$ is *log-convex,* denoted by $(\mathcal{M}_{\mathrm{lc}})$, if $\boldsymbol{M}^{(\alpha)}$ is a log-convex sequence for all $\alpha > 0$.

(ii) A weight matrix $\mathcal{M}$ is *standard log-convex,* abbreviated by $(\mathcal{M}_{\mathrm{sc}})$, if $\boldsymbol{M}^{(\alpha)}$ is a normalized weight sequence for all $\alpha > 0$.

If $\mathcal{M}$ is a weight matrix with $\lim_{j\to\infty}(M_j^{(\alpha)})^{1/j} = +\infty$ for all $\alpha > 0$, then we can compute the log-convex minorant, or log-convex regularization, of the weight matrix $\mathcal{M}$, defined as

$$\mathcal{M}^{\mathrm{lc}} := \{(\boldsymbol{M}^{(\alpha)})^{\mathrm{lc}} : \alpha > 0\}.$$

Let us observe that for $0 < \alpha \leq \beta$, since $\boldsymbol{M}^{(\alpha)} \leq \boldsymbol{M}^{(\beta)}$ we have $(\boldsymbol{M}^{(\alpha)})^{\mathrm{lc}} \leq (\boldsymbol{M}^{(\beta)})^{\mathrm{lc}}$. Moreover, $(M^{(\alpha)})_0^{\mathrm{lc}} = M_0^{(\alpha)} = 1$.

Let us consider the following crucial assumptions (of Roumieu-type) on a given weight matrix $\mathcal{M}$, see [59, Sect. 4.1] and [70, Sect. 7.2]:

**Definition 1.3.3.** We say that:

(i) $\mathcal{M}$ has the $\mathrm{C}^\omega$ property, denoted by $(\mathcal{M}_{\{\mathrm{C}^\omega\}})$, if there exists some $\alpha > 0$ such that

$$\liminf_{j\to\infty}(\widecheck{M}_j^{(\alpha)})^{1/j} > 0.$$

(ii) $\mathcal{M}$ has the $\mathcal{H}$ property, denoted by $(\mathcal{M}_{\mathcal{H}})$, if for all $\alpha > 0$ we have that

$$\liminf_{j\to\infty}(\widecheck{M}_j^{(\alpha)})^{1/j} > 0.$$

(iii) $\mathcal{M}$ has the *root almost increasing property*, denoted by $(\mathcal{M}_{\{\mathrm{rai}\}})$, if

$$\forall\, \alpha > 0\; \exists\, C > 0\; \exists\, \beta > 0\; \forall\, 1 \leq j \leq k : \quad (\widecheck{M}_j^{(\alpha)})^{1/j} \leq C(\widecheck{M}_k^{(\beta)})^{1/k}.$$

(iv) $\mathcal{M}$ has the *Faà-di-Bruno property*, denoted by $(\mathcal{M}_{\{\mathrm{FdB}\}})$, if

$$\forall\, \alpha > 0\; \exists\, \beta > 0 : \quad (\widecheck{\boldsymbol{M}}^{(\alpha)})^{\circ} \preceq \widecheck{\boldsymbol{M}}^{(\beta)},$$

where $(\widecheck{\boldsymbol{M}}^{(\alpha)})^{\circ}$ is the sequence defined by (1.2).

(v) $\mathcal{M}$ is of, or has, *moderate growth*, denoted by $(\mathcal{M}_{\{\mathrm{mg}\}})$, if

$$\forall\, \alpha > 0\; \exists\, C > 0\; \exists\, \beta > 0\; \forall\, j,k \in \mathbb{N}_0 : M_{j+k}^{(\alpha)} \leq C^{j+k} M_j^{(\beta)} M_k^{(\beta)}.$$

(vi) $\mathcal{M}$ satisfies the *derivation closedness condition*, denoted by $(\mathcal{M}_{\{\mathrm{dc}\}})$, if

$$\forall\, \alpha > 0\; \exists\, C > 0\; \exists\, \beta > 0\; \forall\, j \in \mathbb{N}_0 : M^{(\alpha)}_{j+1} \le C^{j+1} M^{(\beta)}_j.$$

We can compare two matrices, as we can see in the following definition.

**Definition 1.3.4.** Let $\mathcal{M} = \{\boldsymbol{M}^{(\alpha)} : \alpha > 0\}$ and $\mathcal{L} = \{\boldsymbol{L}^{(\alpha)} : \alpha > 0\}$ be given. We write $\mathcal{M}\{\preceq\}\mathcal{L}$ if

$$\forall\, \alpha > 0\; \exists\, \beta > 0 : \quad \boldsymbol{M}^{(\alpha)} \preceq \boldsymbol{L}^{(\beta)},$$

and call $\mathcal{M}$ and $\mathcal{L}$ $R$-equivalent, if $\mathcal{M}\{\preceq\}\mathcal{L}$ and $\mathcal{L}\{\preceq\}\mathcal{M}$ ($R$ stands for Roumieu).

**Remark 1.3.5.** A matrix is called *constant* if $\boldsymbol{M}^{(\alpha)} \approx \boldsymbol{M}^{(\beta)}$ for all $\alpha, \beta > 0$.

Let us gather now some relevant information needed in the forthcoming sections.

**Lemma 1.3.6.** *Let $\mathcal{M} = \{\boldsymbol{M}^{(\alpha)} : \alpha > 0\}$ be a weight matrix. If $\mathcal{M}$ has $(\mathcal{M}_{\{\mathrm{rai}\}})$, then*

$$\begin{aligned} &\forall\, \alpha > 0\; \exists\, H \ge 1\; \exists\, \alpha'(\ge \alpha)\; \forall\, k \in \mathbb{N}\; \forall\, j_1, \dots, j_k \in \mathbb{N}_0 : \\ &\qquad\qquad \breve{M}^{(\alpha)}_{j_1} \cdots \breve{M}^{(\alpha)}_{j_k} \le H^{j_1+\dots+j_k} \breve{M}^{(\alpha')}_{j_1+\dots+j_k}. \end{aligned} \tag{1.16}$$

*Note that the indices $\alpha$ and $\alpha'$ are related by property $(\mathcal{M}_{\{\mathrm{rai}\}})$.*

*Proof.* If $j_1, \dots, j_k \ge 1$ we estimate by

$$\begin{aligned} \breve{M}^{(\alpha)}_{j_1} \cdots \breve{M}^{(\alpha)}_{j_k} &\le H^{j_1} \big(\breve{M}^{(\alpha')}_{j_1+\dots+j_k}\big)^{\frac{j_1}{j_1+\dots+j_k}} \cdots H^{j_k} \big(\breve{M}^{(\alpha')}_{j_1+\dots+j_k}\big)^{\frac{j_k}{j_1+\dots+j_k}} \\ &= H^{j_1+\dots+j_k} \breve{M}^{(\alpha')}_{j_1+\dots+j_k}, \end{aligned}$$

and the remaining cases follow by $\breve{M}^{(\alpha)}_0 = M^{(\alpha)}_0 = 1$. $\square$

There exist some connections between the different properties of weight matrices.

**Lemma 1.3.7.** *Let $\mathcal{M} = \{\boldsymbol{M}^{(\alpha)} : \alpha > 0\}$ be a weight matrix. Then we have the following:*

*(i)* $(\mathcal{M}_{\{\mathrm{rai}\}})$ *implies* $(\mathcal{M}_{\mathcal{H}})$ *up to equivalence of matrices, i. e., there exists a weight matrix $\mathcal{N}$ which is $R$-equivalent to $\mathcal{M}$ and has $(\mathcal{M}_{\mathcal{H}})$.*

*(ii)* $(\mathcal{M}_{\{\mathrm{dc}\}})$ *and* $(\mathcal{M}_{\{\mathrm{rai}\}})$ *imply* $(\mathcal{M}_{\{\mathrm{FdB}\}})$.

*(iii)* *If*

$$\forall\,\alpha>0\;\exists\,H\geq 1\;\forall\,1\leq j\leq k:\quad (M_j^{(\alpha)})^{1/j}\leq H(M_k^{(\alpha)})^{1/k}, \tag{1.17}$$

*i. e., if each sequence* $((M_j^{(\alpha)})^{1/j})_j$ *is almost increasing, then* $(\mathcal{M}_{\mathcal{H}})$ *and* $(\mathcal{M}_{\{\mathrm{FdB}\}})$ *imply* $(\mathcal{M}_{\{\mathrm{rai}\}})$.

*In particular,* (1.17) *holds true (with* $H=1$ *for any* $\alpha$*) provided that* $\mathcal{M}$ *is log-convex.*

*Proof.* (i) By the order of the sequences we can assume without loss of generality $\beta\geq\alpha$ and for each $\alpha>0$ there exists a minimal $\beta=\beta(\alpha)\geq\alpha$ such that $\widecheck{\boldsymbol{M}}^{(\alpha)}$ and $\widecheck{\boldsymbol{M}}^{(\beta)}$ are related by $(\mathcal{M}_{\{\mathrm{rai}\}})$. Then $(\widecheck{M}_j^{(\beta)})^{1/j}\geq\frac{\widecheck{M}_1^{(\alpha)}}{C}>0$ for some $C\geq 1$ and all $j\geq 1$ (see also [72, Lemma 3.6 (*ii*)]). Since without loss of generality we can restrict in the Roumieu case to all $\beta(\alpha)$ (yielding an $R$-equivalent matrix) we are done.

(ii) See the proofs of [59, Thm. 4.9 (3) $\Rightarrow$ (4)] and [70, Lemma 8.2.3 (2)].

(iii) See the proofs of [60, Lemma 1 (2)] and [70, Lemma 8.2.3 (4)].

□

### 1.3.1 Weight matrices associated with weight functions

In this subsection, we associate a weight matrix $\mathcal{M}_\omega$ with a given weight function $\omega$. The idea is to transfer properties from the weight function $\omega$ to the associated weight matrix.

For a given weight function $\omega$, let us start introducing the Legendre-Fenchel-Young-conjugate of the function $\omega\circ\exp$.

**Definition 1.3.8.** For any $\omega\in\mathcal{W}_0$ we define the *Legendre-Fenchel-Young-conjugate* of $\varphi_\omega: t\mapsto\omega(e^t)$ by

$$\varphi_\omega^*(x):=\sup\{xy-\varphi_\omega(y): y\geq 0\},\quad x\geq 0. \tag{1.18}$$

Note that by normalization we can extend the supremum in (1.18) from $y\geq 0$ to $y\in\mathbb{R}$ without changing the value of $\varphi_\omega^*(x)$ for given $x\geq 0$.

In the work of R. W. Braun, R. Meise and B. A. Taylor [7, Remark 1.3, Lemma 1.5] one finds that, for a given weight function $\omega\in\mathcal{W}_0$, the Legendre-Fenchel-Young-conjugate of $\varphi_\omega$ has the following properties:

(i) $\varphi_\omega^*$ is a convex, nondecreasing function. Moreover, $\varphi_\omega^*(0)=0$ and $\varphi_\omega^{**}=\varphi_\omega$.

(ii) The Legendre-Fenchel-Young-conjugate $\varphi_\omega^*$ tends to infinity faster than $x$, and therefore $\lim_{x\to+\infty} x/\varphi_\omega^*(x)$ is equal to zero.

(iii) The functions

$$x \mapsto \frac{\varphi_\omega(x)}{x}, \qquad \text{and} \qquad x \mapsto \frac{\varphi^*_\omega(x)}{x}, \qquad x \in [0, +\infty),$$

are nondecreasing.

Thanks to this conjugate, we can associate a weight matrix with a given weight function $\omega \in \mathcal{W}_0$.

**Definition 1.3.9.** Let $\omega$ be a weight function in $\mathcal{W}_0$, then we can associate a weight matrix $\mathcal{M}_\omega$ defined as

$$\mathcal{M}_\omega := \{\boldsymbol{W}^{(\ell)} = (W_j^{(\ell)})_{j\in\mathbb{N}_0} : \ell > 0\},$$

where

$$W_j^{(\ell)} := \exp\left(\frac{1}{\ell}\varphi^*_\omega(\ell j)\right). \tag{1.19}$$

We summarize some facts which are shown in [59, Section 5] and are needed in this work. Observe that we obtain strong properties for $\mathcal{M}_\omega$ automatically by considering general weight functions $\omega$ with $(\omega_0)$, $(\omega_3)$ and $(\omega_4)$. Of course, extra conditions on $\omega$ provide new properties for the associated weight matrix.

**Remark 1.3.10.** Let $\omega$ be a weight function in $\mathcal{W}_0$, then we have that:

(i) The weight matrix $\mathcal{M}_\omega$ is standard log-convex $(\mathcal{M}_{\text{sc}})$, see [70, Lemma 5.1.1].

(ii) The weight matrix $\mathcal{M}_\omega$ satisfies

$$\forall\, \ell > 0\; \forall\, j,k \in \mathbb{N}_0 : \quad W_{j+k}^{(\ell)} \le W_j^{(2\ell)} W_k^{(2\ell)}, \tag{1.20}$$

so both $(\mathcal{M}_{\{\text{mg}\}})$ and $(\mathcal{M}_{\{\text{dc}\}})$ are satisfied, see [70, Lemma 5.1.2].

(iii) There exists some connection between the weight function $\omega$ and the weight function associated to each weight sequence $\boldsymbol{W}^{(\ell)}$. More precisely, we have $\omega \sim \omega_{\boldsymbol{W}^{(\ell)}}$ for each $\ell > 0$, in fact

$$\forall\, \ell > 0\; \exists\, D_\ell > 0\; \forall\, t \ge 0 : \quad \ell\omega_{\boldsymbol{W}^{(\ell)}}(t) \le \omega(t) \le 2\ell\omega_{\boldsymbol{W}^{(\ell)}}(t) + D_\ell, \tag{1.21}$$

see [70, Theorem. 4.0.3, Lemma 5.1.3] and also [35, Lemma 2.5].

(iv) In case $\omega$ has in addition $(\omega_1)$, then $\mathcal{M}_\omega$ has also

$$\forall\, h \ge 1\; \exists\, A \ge 1\; \forall\, \ell > 0\; \exists\, D \ge 1\; \forall\, j \in \mathbb{N}_0 : \quad h^j W_j^{(\ell)} \le D W_j^{(A\ell)}, \tag{1.22}$$

see [59, Lemma 5.9 (5.10)].

(v) Condition $(\omega_6)$ holds if and only if some/each $\boldsymbol{W}^{(\ell)}$ satisfies (mg) if and only if $\boldsymbol{W}^{(\ell)} \approx \boldsymbol{W}^{(\ell_1)}$ for each $\ell, \ell_1 > 0$, see [70, Proposition 5.2.2]. Consequently $(\omega_6)$ characterizes the situation when $\mathcal{M}_\omega$ is constant.

**Proposition 1.3.11.** *Let $\omega$ be a weight function in $\mathcal{W}_0$, then $\mathcal{M}_\omega$ satisfies $(\mathcal{M}_\mathcal{H})$ if and only if $\omega$ has in addition $(\omega_2)$.*

*Proof.* The condition $(\omega_2)$ is stable with respect to the relation $\sim$, see Proposition 1.2.8. Moreover, the previous Remark (see (1.21)), and $(i)$ in Lemma 1.2.14 show the equivalence. □

**Proposition 1.3.12.** *Let $\omega$ be a weight function in $\mathcal{W}_0$ with $(\omega_2)$, then properties $(\mathcal{M}_{\{\mathrm{rai}\}})$ and $(\mathcal{M}_{\{\mathrm{FdB}\}})$ for $\mathcal{M}_\omega$ are simultaneously satisfied or violated.*

*Proof.* In view of $(i)$, $(ii)$ in the previous Remark, and Proposition 1.3.11, we can apply Lemma 1.3.7 ([60, Lemma 1]) to $\mathcal{M}_\omega$ in order to show the equivalence between both properties. □

Finally, despite a gamma index associated with a weight matrix has not been introduced by now, we can establish some relation between the gamma index associated with the weight function $\omega$, and the gamma index of each associated weight function $\omega_{\boldsymbol{W}^{(\ell)}}$.

**Remark 1.3.13.** If $\omega \in \mathcal{W}_0$ is given with associated weight matrix $\mathcal{M}_\omega := \{\boldsymbol{W}^{(\ell)} : \ell > 0\}$ and $\gamma(\omega) > \beta$, then (1.21) implies $\gamma(\omega_{\boldsymbol{W}^{(\ell)}}) > \beta$. However, in general we can only obtain the following inequality $\gamma(\boldsymbol{W}^{(\ell)}) \leq \gamma(\omega_{\boldsymbol{W}^{(\ell)}})$ (see the comments at the end of the Subsection 1.2.2). Here $\gamma(\boldsymbol{W}^{(\ell)})$ is the index in Subsection 1.1.3.

# Chapter 2

# Stability properties in ultraholomorphic classes

When dealing with function spaces (usually called classes) it is very interesting to decide whether the usual operations (pointwise product, composition, algebraic inversion, differentiation, integration, etc.) on the functions of the space provide new functions inside it. These stability properties play a crucial role in the setting and the solution of, for example, algebraic, differential or integro-differential equations in the class. This chapter is devoted to the study of several stability properties, such as inverse or composition closedness, for ultraholomorphic function classes of both Roumieu and Beurling type defined in terms of a weight matrix.

Firstly, we will transfer and extend known results for Roumieu classes in the work of J. Siddiqi and M. Ider [76], from the weight sequence setting and in sectors not wider than a half-plane, to the weight matrix framework and for sectors in the Riemann surface of the logarithm with arbitrary opening. The key argument rests on the construction, under suitable hypotheses, of characteristic functions in these classes for unrestricted sectors. As a by-product, we obtain new stability results when the growth control in these classes is expressed in terms of a weight sequence, or of a weight function in the sense of Braun-Meise-Taylor.

Secondly, in the Beurling setting and in the weight matrix framework, we are only able to treat the case of sectors not wider than a half-plane, as the construction of characteristic functions is no longer available. The technique now is completely different, and rests on the work of W. Żelazko [85] on multiplicatively convex Fréchet algebras.

## 2.1 Roumieu ultraholomorphic classes

In this section, we present the ultraholomorphic classes of functions we will deal with. While those associated to a weight sequence are classical and already appeared in the works of S. Mandelbrojt [46], B. Rodríguez-Salinas [64, 65] and J. Schmets and M. Valdivia [74], to cite but a few, the ones associated with a weight function or a weight matrix were first introduced in the works of J. Jiménez-Garrido, J. Sanz and G. Schindl [35, Sect. 2.5] and [36, Sect. 2.5].

The functions under consideration are defined in regions of the Riemann surface of the logarithm, $\mathcal{R}$. In fact, we wish to work in general unbounded sectors in $\mathcal{R}$ with vertex at 0. Since the problems under study will be rotation-invariant, we will always suppose that they are bisected by the positive real line, and we just record their opening $\alpha\pi$, for $\alpha > 0$, in the notation. So, we set

$$S_\alpha := \left\{ z \in \mathcal{R} : |\arg(z)| < \frac{\alpha\pi}{2} \right\}.$$

In what follows, given an open set $U \subset \mathcal{R}$, the set of all holomorphic functions in $U$ will be denoted by $\mathcal{H}(U)$.

**Definition 2.1.1.** Let $\boldsymbol{M}$ be a sequence, $S \subseteq \mathcal{R}$ an (unbounded) sector and $h > 0$. We define

$$\mathcal{A}_{\boldsymbol{M},h}(S) := \{ f \in \mathcal{H}(S) : \|f\|_{\boldsymbol{M},h} := \sup_{z\in S, j\in\mathbb{N}_0} \frac{|f^{(j)}(z)|}{h^j M_j} < +\infty \}.$$

$(\mathcal{A}_{\boldsymbol{M},h}(S), \|\cdot\|_{\boldsymbol{M},h})$ is a Banach space and

$$\mathcal{A}_{\{\boldsymbol{M}\}}(S) := \bigcup_{h>0} \mathcal{A}_{\boldsymbol{M},h}(S),$$

is called the Denjoy-Carleman ultraholomorphic class of Roumieu type associated with $\boldsymbol{M}$ in the sector $S$. It has a natural structure of an $(LB)$ space.

**Remark 2.1.2.** Note that, by definition, it is immediate that $\boldsymbol{M} \approx \boldsymbol{L}$ implies $\mathcal{A}_{\{\boldsymbol{M}\}}(S) = \mathcal{A}_{\{\boldsymbol{L}\}}(S)$ (as locally convex vector spaces) for any sector $S$. Moreover, if the sequence $\boldsymbol{M}$ has additional properties, then the ultraholomorphic class has more structure. In this sense, it is straightforward to check that, if the sequence $\boldsymbol{M}$ is (lc), then the ultraholomorphic class is an algebra; if the sequence has (dc) then the class is closed under taking derivatives.

Similarly as for the ultradifferentiable case, we now define ultraholomorphic classes associated with a weight function $\omega$.

**Definition 2.1.3.** Let $\omega$ be a weight function in $\mathcal{W}_0$, $S \subseteq \mathcal{R}$ an (unbounded) sector and $\ell > 0$, we first define

$$\mathcal{A}_{\omega,\ell}(S) := \{f \in \mathcal{H}(S) : \|f\|_{\omega,\ell} := \sup_{z\in S, j\in\mathbb{N}_0} \frac{|f^{(j)}(z)|}{\exp(\frac{1}{\ell}\varphi^*_\omega(\ell j))} < +\infty\}.$$

$(\mathcal{A}_{\omega,\ell}(S), \|\cdot\|_{\omega,\ell})$ is a Banach space and we put

$$\mathcal{A}_{\{\omega\}}(S) := \bigcup_{\ell>0} \mathcal{A}_{\omega,\ell}(S),$$

which is called the ultraholomorphic class of Roumieu type associated with $\omega$ in the sector $S$ and it is again an $(LB)$ space.

Again, equivalent weight functions provide equal associated ultraholomorphic classes.

Finally, we define ultraholomorphic classes of Roumieu type defined by a weight matrix $\mathcal{M}$, analogously as for the ultradifferentiable counterparts introduced by A. Rainer and G. Schindl in [70, Section 7] and [59, Section 4.2].

**Definition 2.1.4.** Let $\mathcal{M} = \{\boldsymbol{M}^{(\alpha)} \in \mathbb{R}_{>0}^{\mathbb{N}_0} : \alpha > 0\}$ be a weight matrix and $S$ be an unbounded sector in $\mathcal{R}$. We introduce the ultraholomorphic class of Roumieu type associated with $\mathcal{M}$, denoted by $\mathcal{A}_{\{\mathcal{M}\}}(S)$, as

$$\mathcal{A}_{\{\mathcal{M}\}}(S) := \bigcup_{\alpha>0} \mathcal{A}_{\{\boldsymbol{M}^{(\alpha)}\}}(S).$$

As in the previous cases, R-equivalent weight matrices yield (as locally convex vector spaces) the same function class on each sector $S$.

For a given weight function, we can compute the associated weight matrix, as we see in Subsection 1.3.1. It is natural to ask about the relation between the corresponding ultraholomorphic classes.

**Remark 2.1.5.** Let $\omega \in \mathcal{W}$ and let $\mathcal{M}_\omega$ be the associated weight matrix given in Definition 1.3.9, then

$$\mathcal{A}_{\{\omega\}}(S) = \mathcal{A}_{\{\mathcal{M}_\omega\}}(S) \tag{2.1}$$

holds as locally convex vector spaces. This equality is an easy consequence of (1.22) and the way the seminorms are defined in these spaces.

On the other hand, by Remark 1.3.10 $(v)$ we get the following result.

**Lemma 2.1.6.** *Let $\omega \in \mathcal{W}$ and assume that $\omega$ has $(\omega_6)$. Then, for all sectors $S$ we get that*

$$\forall\, \ell > 0 : \quad \mathcal{A}_{\{\omega\}}(S) = \mathcal{A}_{\{\boldsymbol{W}^{(\ell)}\}}(S),$$

*as locally convex vector spaces.*

Finally, for a given sector $S$, if $f$ belongs to any of the previous classes, we may define the complex numbers

$$f^{(p)}(0) := \lim_{z\in S, z\to 0} f^{(p)}(z) \in \mathbb{C} \qquad p \in \mathbb{N}_0. \tag{2.2}$$

thanks to the fact that all the derivatives of $f$ are Lipschitz.

## 2.2 Characteristic functions in Roumieu ultraholomorphic classes

In this section we introduce characteristic functions. This concept has a major role in the stability properties for the previous classes. The aim of this section is to construct, under suitable assumptions, characteristic functions in $\mathcal{A}_{\{\boldsymbol{M}\}}(S_\alpha)$. We start with the following definition.

**Definition 2.2.1.** Let $\boldsymbol{L} \in \mathbb{R}_{>0}^{\mathbb{N}_0}$ and $S$ be a given sector. A function $f \in \mathcal{A}_{\{\boldsymbol{L}\}}(S)$ is said to be *characteristic* in the class $\mathcal{A}_{\{\boldsymbol{L}\}}(S)$ if, whenever $f \in \mathcal{A}_{\{\boldsymbol{M}\}}(S) \subseteq \mathcal{A}_{\{\boldsymbol{L}\}}(S)$ for some $\boldsymbol{M} \in \mathbb{R}_{>0}^{\mathbb{N}_0}$, we have that $\mathcal{A}_{\{\boldsymbol{M}\}}(S) = \mathcal{A}_{\{\boldsymbol{L}\}}(S)$.

For $f \in \mathcal{A}_{\{\boldsymbol{L}\}}(S)$ we consider the sequence defined by

$$C_n(f) := \sup_{z\in S} |f^{(n)}(z)|, \qquad n \in \mathbb{N}_0.$$

The next statement provides conditions on $f$ which imply it is characteristic.

**Theorem 2.2.2.** *Let $\boldsymbol{L} \in \mathbb{R}_{>0}^{\mathbb{N}_0}$, $S$ be a given sector and $f \in \mathcal{A}_{\{\boldsymbol{L}\}}(S)$. Then, each of the following conditions implies the next one:*

(1) *The sequence $(|f^{(j)}(0)|)_{j\in\mathbb{N}_0}$ is equivalent to $\boldsymbol{L}$.*

(2) *The sequence $(C_j(f))_{j\in\mathbb{N}_0}$ is equivalent to $\boldsymbol{L}$.*

(3) *$f$ is characteristic in the class $\mathcal{A}_{\{\boldsymbol{L}\}}(S)$.*

**Proof.** (1) $\Rightarrow$ (2) As $f \in \mathcal{A}_{\{\boldsymbol{L}\}}(S)$, there exist $A, B > 0$ such that $C_n(f) \le AB^n L_n$ for every $n \in \mathbb{N}_0$. On the other hand, it is clear that $C_n(f) \ge |f^{(n)}(0)|$, and the hypothesis allows us to conclude the other estimate.

(2) $\Rightarrow$ (3) By assumption, there exist $A, B > 0$ such that $L_n \le AB^n C_n(f)$ for every $n \in \mathbb{N}_0$. If for some $\boldsymbol{M} = (M_n)_{n\in\mathbb{N}_0} \in \mathbb{R}_{>0}^{\mathbb{N}_0}$ we have $f \in \mathcal{A}_{\{\boldsymbol{M}\}}(S) \subseteq \mathcal{A}_{\{\boldsymbol{L}\}}(S)$, there exist $C, D > 0$ such that $C_n(f) \le CD^n M_n$ for every $n \in \mathbb{N}_0$. The two deduced inequalities show that $L_n \le AC(BD)^n M_n$ for every $n \in \mathbb{N}_0$, what easily implies that $\mathcal{A}_{\{\boldsymbol{L}\}}(S) \subseteq \mathcal{A}_{\{\boldsymbol{M}\}}(S)$, and we are done. □

### 2.2.1 Basic functions

Recall the notations $\boldsymbol{G}^s := (j!^s)_{j\in\mathbb{N}_0}$ and $\overline{\boldsymbol{G}}^s := (j^{js})_{j\in\mathbb{N}_0}$, $s \in \mathbb{R}$, and that $\overline{\boldsymbol{G}}^s \approx \boldsymbol{G}^s$, see (1.3). We introduce in this section two examples of characteristic functions in the ultraholomorphic class $\mathcal{A}_{\{\overline{\boldsymbol{G}}^{\alpha-1}\}}(S_\alpha)$ if $0 < \alpha \leq 1$, and in $\mathcal{A}_{\{\overline{\boldsymbol{G}}^{\alpha'-1}\}}(S_\alpha)$ if $1 < \alpha \leq \alpha'$.

**Definition 2.2.3.** The *two-parametric Mittag-Leffler function* is defined for all complex parameters $A, B$ with $\Re(A) > 0$ (where $\Re$ denotes the real part) by

$$E_{A,B}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(Aj + B)}, \quad z \in \mathbb{C},$$

where $\Gamma$ denotes Euler's Gamma function.

For the construction of characteristic functions in sectors $S_\alpha$ for $\alpha \in (0, 1]$ we will take $A = 2 - \alpha$ and $B = 4 - \alpha$ and we set

$$\widetilde{E}_\alpha(z) := E_{2-\alpha,4-\alpha}(-z) = \sum_{j=0}^{\infty} \frac{(-1)^j z^j}{\Gamma((2-\alpha)(j+1)+2)}, \quad z \in \mathbb{C}.$$

We recall the following statements from the work of B. Rodríguez-Salinas [65], where use is made of the implication (1)⇒(3) in Theorem 2.2.2.

**Theorem 2.2.4.** *([65, Thm. 5, Thm. 20]) Let $\alpha \in (0, 1]$, then*

$$\forall\, z \in S_\alpha\ \forall\, n \in \mathbb{N}_0 : \quad \left|\widetilde{E}_\alpha^{(n)}(z)\right| \leq 2\frac{n!e^n}{n^{(2-\alpha)n}}. \tag{2.3}$$

*Consequently, $\widetilde{E}_\alpha \in \mathcal{A}_{\{\overline{\boldsymbol{G}}^{\alpha-1}\}}(S_\alpha)$. Moreover,*

$$\widetilde{E}_\alpha^{(n)}(0) = \frac{(-1)^n n!}{\Gamma((2-\alpha)(n+1)+2)}, \quad n \in \mathbb{N}_0,$$

*and so $\widetilde{E}_\alpha$ is a characteristic function in the class $\mathcal{A}_{\{\overline{\boldsymbol{G}}^{\alpha-1}\}}(S_\alpha)$.*

**Definition 2.2.5.** Let $\alpha > 1$ and take $\alpha' > \alpha$. For all $z \in S_\alpha$ we define

$$g_{\alpha,\alpha'}(z) := \int_0^{\infty(-\phi)} e^{-zv^{\alpha'-1}} e^{-v} dv, \tag{2.4}$$

where we choose $\phi \in (-\frac{(\alpha-1)}{(\alpha'-1)}\frac{\pi}{2}, \frac{(\alpha-1)}{(\alpha'-1)}\frac{\pi}{2})$ with $|\arg(z) - (\alpha' - 1)\phi| < \pi/2$.

**Theorem 2.2.6.** *([65, Thm. 28]) Let $\alpha > 1$, $\alpha' > \alpha$ and $g_{\alpha,\alpha'}$ be the function from* (2.4)*.*

*Then,*

$$\exists\, C, A \geq 1\ \forall\, z \in S_\alpha\ \forall\, n \in \mathbb{N}_0: \quad \left| g_{\alpha,\alpha'}^{(n)}(z) \right| \leq CA^n \Gamma((\alpha'-1)n+1). \tag{2.5}$$

*Consequently, $g_{\alpha,\alpha'} \in \mathcal{A}_{\{\overline{\boldsymbol{G}}^{\alpha'-1}\}}(S_\alpha)$. Moreover,*

$$g_{\alpha,\alpha'}^{(n)}(0) = (-1)^n \Gamma((\alpha'-1)n+1), \qquad n \in \mathbb{N}_0,$$

*and so $g_{\alpha,\alpha'}$ is a characteristic function of the class $\mathcal{A}_{\{\overline{\boldsymbol{G}}^{\alpha'-1}\}}(S_\alpha)$.*

### 2.2.2 Characteristic transform

Following again the work of B. Rodríguez Salinas [65], we present a functional transform that modifies the derivatives at 0 of a function in a ultraholomorphic class with a precise control, which allows for the construction of characteristic functions in more general classes than the Gevrey ones, considered previously.

**Definition 2.2.7.** Let $\boldsymbol{M}$ be an (lc) sequence, $\boldsymbol{L} \in \mathbb{R}_{>0}^{\mathbb{N}_0}$, $S$ a sector and $f \in \mathcal{A}_{\{\boldsymbol{L}\}}(S)$. Then we define the $\mathcal{T}_{\boldsymbol{M}}-$transform of $f$ by

$$\mathcal{T}_{\boldsymbol{M}}(f)(z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \frac{M_j}{m_j^j} f(m_j z), \qquad z \in S.$$

This expression should be compared with the characteristic functions obtained in the ultradifferentiable setting by V. Thilliez [81, Thm. 1] and A. Rainer and G. Schindl [59, Lemma 2.9], and originating in the classical work of T. Bang [2].

For every $j \in \mathbb{N}_0$ let us set

$$R_j := \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{M_n}{m_n^n} m_n^j.$$

The following result provides estimates for this sequence in terms of the general sequence $\boldsymbol{M}$ we depart from.

**Lemma 2.2.8.** *Let $\boldsymbol{M} \in \mathbb{R}_{>0}^{\mathbb{N}_0}$, then*

$$\forall\, j \in \mathbb{N}_0: \quad R_j \geq \frac{1}{2^j} M_j.$$

*If $\boldsymbol{M}$ is (lc), then also*

$$\forall\, j \in \mathbb{N}_0: \quad R_j \leq 2M_j,$$

*and so $(R_j)_{j\in\mathbb{N}_0}$ is equivalent to $\boldsymbol{M}$.*

**Proof.** For any $j \in \mathbb{N}_0$ we choose $n = j$ in the sum and get $R_j \geq \frac{1}{2^j}\frac{M_j}{m_j^j}m_j^j = \frac{1}{2^j}M_j$.

For the converse we recall that since $\boldsymbol{M}$ is (lc) we have $m_0 \leq m_1 \leq \dots$ and so

$$\forall\, j,n \in \mathbb{N}_0: \quad (m_n)^{j-n} \leq \frac{M_j}{M_n},$$

see [81, Thm. 1] and the detailed proof in [69, (3.1.2)]. Thus

$$R_j = \sum_{n=0}^{\infty} \frac{1}{2^n} M_n m_n^{j-n} \leq \sum_{n=0}^{\infty} \frac{1}{2^n} M_n \frac{M_j}{M_n} = 2M_j$$

for all $j \in \mathbb{N}_0$. □

Recall that $\boldsymbol{LM}$ denotes the sequence obtained by the termwise product of two sequences $\boldsymbol{L}$ and $\boldsymbol{M}$.

**Theorem 2.2.9.** *Let $\boldsymbol{M}$ be an (lc) sequence, $\boldsymbol{L} \in \mathbb{R}_{>0}^{\mathbb{N}_0}$ and for a given sector $S$ take $f \in \mathcal{A}_{\{\boldsymbol{L}\}}(S)$. Then, $\mathcal{T}_{\boldsymbol{M}}(f) \in \mathcal{A}_{\{\boldsymbol{LM}\}}(S)$ with*

$$\mathcal{T}_{\boldsymbol{M}}(f)^{(j)}(0) = R_j f^{(j)}(0), \qquad j \in \mathbb{N}_0. \tag{2.6}$$

*Moreover, for any $A > 0$, $\mathcal{T}_{\boldsymbol{M}} : \mathcal{A}_{\boldsymbol{L},A}(S) \to \mathcal{A}_{\boldsymbol{LM},A}(S)$ is a continuous linear operator.*

*Proof.* By definition of $\mathcal{A}_{\{\boldsymbol{L}\}}(S)$ we have that $f$ is bounded in $S$ by some constant $C > 0$. Since $\boldsymbol{M}$ is log-convex, we have that $M_j \leq m_j^j$ for all $j \in \mathbb{N}_0$ and then

$$\sum_{j=0}^{\infty} \frac{1}{2^j} \frac{M_j}{m_j^j} |f(m_j z)| \leq C \sum_{j=0}^{\infty} \frac{1}{2^j} = 2C, \qquad z \in S.$$

Consequently, the series defining $\mathcal{T}_{\boldsymbol{M}}(f)$ normally converges in the whole of $S$, it provides a function holomorphic in $S$, and differentiation and limits can be interchanged with summation. For each $z \in S$ and every $j \in \mathbb{N}_0$ we observe then that

$$(\mathcal{T}_{\boldsymbol{M}}(f))^{(j)}(z) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{M_n}{m_n^n} m_n^j f^{(j)}(m_n z),$$

and so

$$\mathcal{T}_{\boldsymbol{M}}(f)^{(j)}(0) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{M_n}{m_n^n} m_n^j f^{(j)}(0) = R_j f^{(j)}(0), \qquad j \in \mathbb{N}_0,$$

as desired.

Suppose $f \in \mathcal{A}_{\boldsymbol{L},A}(S)$ for some $A > 0$, then for all $j \in \mathbb{N}_0$ we can estimate

$$\begin{aligned} |(\mathcal{T}_{\boldsymbol{M}}(f))^{(j)}(z)| &\leq \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{M_n}{m_n^n} m_n^j |f^{(j)}(m_n z)| \\ &\leq \|f\|_{\boldsymbol{M},A} A^j L_j \sum_{n=0}^{\infty} \frac{1}{2^n} M_n m_n^{j-n} = \|f\|_{\boldsymbol{M},A} A^j L_j R_j. \end{aligned}$$

By Lemma 2.2.8 we know that $R_j \leq 2M_j$, so $\mathcal{T}_{\boldsymbol{M}}(f) \in \mathcal{A}_{\boldsymbol{LM},A}(S)$, and moreover

$$\|\mathcal{T}_{\boldsymbol{M}}(f)\|_{\boldsymbol{LM},A} = \sup_{z \in S} \frac{|(\mathcal{T}_{\boldsymbol{M}}(f))^{(j)}(z)|}{A^j L_j M_j} \leq 2\|f\|_{\boldsymbol{M},A}.$$

It follows that $\mathcal{T}_{\boldsymbol{M}} : \mathcal{A}_{\boldsymbol{L},A}(S) \to \mathcal{A}_{\boldsymbol{LM},A}(S)$ is a well-defined continuous linear operator for any $A > 0$. □

**Theorem 2.2.10.** *Let $\boldsymbol{M}$ be an (lc) sequence, $\boldsymbol{L} \in \mathbb{R}_{>0}^{\mathbb{N}_0}$ and for a given sector $S$ take $f \in \mathcal{A}_{\{\boldsymbol{L}\}}(S)$. If $(|f^{(j)}(0)|)_{j \in \mathbb{N}_0}$ is equivalent to $\boldsymbol{L}$, then $(|\mathcal{T}_{\boldsymbol{M}}(f)^{(j)}(0)|)_{j \in \mathbb{N}_0}$ is equivalent to $\boldsymbol{LM}$. Consequently, $\mathcal{T}_{\boldsymbol{M}}(f)$ is characteristic in the class $\mathcal{A}_{\{\boldsymbol{LM}\}}(S)$.*

**Proof.** The first assertion is clear from Lemma 2.2.8 and (2.6). The second one stems from Theorem 2.2.2. □

### 2.2.3 Construction of characteristic functions

Given a sequence $\boldsymbol{M} \in \mathbb{R}_{>0}^{\mathbb{N}_0}$ and $\alpha > 0$ we construct now, under suitable assumptions, characteristic functions in $\mathcal{A}_{\{\boldsymbol{M}\}}(S_\alpha)$. For this we are using the basic functions from Subsection 2.2.1 and the characteristic transform from Subsection 2.2.2.

**Theorem 2.2.11.** *Let $\boldsymbol{M} \in \mathbb{R}_{>0}^{\mathbb{N}_0}$ and $\alpha > 0$.*

1. *If $\alpha \leq 1$, we assume that $\overline{\boldsymbol{G}}^{1-\alpha}\boldsymbol{M} := (j^{(1-\alpha)j} M_j)_{j \in \mathbb{N}_0}$ is equivalent to an (lc) sequence $\boldsymbol{L}$. Then, $\mathcal{T}_{\boldsymbol{L}}(\widetilde{E}_\alpha)$ is characteristic in the class $\mathcal{A}_{\{\boldsymbol{M}\}}(S_\alpha)$.*

2. *If $\alpha > 1$, we assume that there exists $\alpha' > \alpha$ such that $\overline{\boldsymbol{G}}^{1-\alpha'}\boldsymbol{M}$ is equivalent to an (lc) sequence $\boldsymbol{L}$. Then, $\mathcal{T}_{\boldsymbol{L}}(g_{\alpha,\alpha'})$ is characteristic in the class $\mathcal{A}_{\{\boldsymbol{M}\}}(S_\alpha)$.*

*Proof.* This follows by Theorems 2.2.4, 2.2.6, 2.2.9 and 2.2.10, and from the fact that $\overline{\boldsymbol{G}}^{\alpha-1}\boldsymbol{L}$ in case 1, resp. $\overline{\boldsymbol{G}}^{\alpha'-1}\boldsymbol{L}$ in case 2, is equivalent to $\boldsymbol{M}$. □

**Remark 2.2.12.** In order to guarantee that the hypotheses in the previous theorem are satisfied, one can compute the index $\gamma(\boldsymbol{M})$ and check whether it is greater than $\alpha - 1$. If this is the case, the very definition of this index implies that for any $\beta$ such that $\gamma(\boldsymbol{M}) > \beta > \alpha - 1$ the property $(P_\beta)$ (see Subsection 1.1.3) is satisfied, and so there exists a suitable (lc) sequence $\boldsymbol{L}$ with the desired conditions.

## 2.3 Stability properties for Roumieu ultraholomorphic classes defined by weight matrices

The aim of this section is to generalize and extend the stability result of Ider and Siddiqi [76, Thm. 1], valid for Carleman-Roumieu ultraholomorphic classes in sectors not wider than a half-plane. We give the proof in the general weight matrix setting, we get rid of the restriction on the opening of the sector (thanks to the construction of characteristic functions in arbitrary sectors), and we extend the list of stability properties.

Our main result is concerned with several stability properties which will be defined next.

**Definition 2.3.1.** Let $\boldsymbol{M} \in \mathbb{R}_{>0}^{\mathbb{N}_0}$ be a sequence and $U \subseteq \mathbb{C}$ be an open set. Given a compact set $K \subset U$, we define

$$\mathcal{H}_{\boldsymbol{M},h}(K) := \{f \in \mathcal{H}(U) : \|f\|_{\boldsymbol{M},K,h} := \sup_{z\in K, j\in\mathbb{N}_0} \frac{|f^{(j)}(z)|}{h^j M_j} < +\infty\}.$$

We put

$$\mathcal{H}_{\{\boldsymbol{M}\}}(K) := \bigcup_{h>0} \mathcal{H}_{\boldsymbol{M},h}(K).$$

Moreover, given a weight matrix $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$, we may introduce the class $\mathcal{H}_{\{\mathcal{M}\}}(U)$ as

$$\mathcal{H}_{\{\mathcal{M}\}}(U) := \bigcap_{K\subset U} \bigcup_{p>0} \mathcal{H}_{\{\boldsymbol{M}^{(p)}\}}(K).$$

**Definition 2.3.2.** Let $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$ be a weight matrix and $\alpha > 0$. The class $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ is said to be:

- $(i)$ holomorphically closed, if for all $f \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ and $g \in \mathcal{H}(U)$, where $U \subseteq \mathbb{C}$ is an open set containing the closure of the range of $f$, we have $g \circ f \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$.
- $(ii)$ inverse-closed, if for all $f \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ such that $\inf_{z\in S_\alpha} |f(z)| > 0$, we have $1/f \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$.

*(iii)* closed under composition, if for all $f \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ and for all $g \in \mathcal{H}_{\{\mathcal{M}\}}(U)$, where $U \subseteq \mathbb{C}$ is an open set containing the closure of the range of $f$, we have $g \circ f \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$.

**Remark 2.3.3.** We wish to highlight that it is important to state these definitions in a clear way. We cannot relax the condition $\inf_{z\in S_\alpha} |f(z)| > 0$ in the definition of inverse-closed by considering, for example, the weaker requirement:

$$f(z) \neq 0 \qquad \text{for all } z \in S_\alpha.$$

While this is enough when working with ultradifferentiable classes on compact intervals, as done by P. Malliavin [45], our situation is different as $S_\alpha$ is not compact. This is easily seen by considering the function $z \mapsto \exp(-1/z)$, which belongs to the class $\mathcal{A}_{\{\boldsymbol{G}^2\}}(S_\alpha)$ for every $\alpha \in (0,1)$ (as a consequence of Cauchy's integral formula for the derivatives) and never vanishes in $S_\alpha$. However, observe that its multiplicative inverse $z \mapsto \exp(1/z)$ is not bounded, and hence it does not belong to any of the ultraholomorphic classes under consideration.

In the same vein, the open set $U$ in (i) and (iii) has to contain the closure of the range of $f$, and not just the range. This is clearly seen in the forthcoming arguments involving the function $z \mapsto 1/z$, whose derivatives admit global analytic bounds in closed subsets of $\mathbb{C} \setminus \{0\}$, but not in the whole of it.

Our first statement will consider classes in sectors $S_\alpha$ contained in a half-plane and defined by a weight matrix $\mathcal{M}$. In this case, the matrix can be changed, without altering the class, into a new matrix $\mathcal{M}^\alpha$ which we define now.

**Definition 2.3.4.** Let $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$ be a weight matrix (not necessarily satisfying $(\mathcal{M}_{\mathrm{sc}})$). Given $\alpha > 0$ we assume that $\lim_{j\to+\infty}(j^{(1-\alpha)j} M_j^{(p)})^{1/j} = \infty$ for all $p > 0$. The matrix

$$\mathcal{M}^\alpha := \{\boldsymbol{M}^{(p,\alpha)} : p > 0\}$$

is defined as

$$\boldsymbol{M}^{(p,\alpha)} = \overline{\boldsymbol{G}}^{\alpha-1}\left(\overline{\boldsymbol{G}}^{1-\alpha}\boldsymbol{M}^{(p)}\right)^{\mathrm{lc}}, \quad M_j^{(p,\alpha)} = j^{(\alpha-1)j}\left(\overline{\boldsymbol{G}}^{1-\alpha}\boldsymbol{M}^{(p)}\right)^{\mathrm{lc}}_j, \quad j \in \mathbb{N}_0. \tag{2.7}$$

So, every sequence in the original matrix is termwise multiplied by the Gevrey-like sequence $\overline{\boldsymbol{G}}^{1-\alpha}$ (recall that $\overline{\boldsymbol{G}}^{1-\alpha} \approx \boldsymbol{G}^{1-\alpha}$), this sequence is changed into its log-convex regularization, and finally one termwise divides by $\overline{\boldsymbol{G}}^{1-\alpha}$ again. It is clear that $M_0^{(p,\alpha)} = M_0^{(p)} = 1$ (recall the convention $0^0 := 1$) for all $\alpha > 0$ and $p > 0$, and that the map $p \mapsto M_j^{(p,\alpha)}$ is non-decreasing for any $j \in \mathbb{N}_0$ fixed. So, $\boldsymbol{M}^{(p,\alpha)} \leq \boldsymbol{M}^{(p',\alpha)}$ for all $0 < p < p'$, i. e., $\mathcal{M}^\alpha$ is a weight matrix according to the definition given in Subsection 1.3. However, in general $\mathcal{M}^\alpha$ is not log-convex.

**Remark 2.3.5.** Let us observe that if there exists some positive value $p$ such that $\lim_{j\to+\infty}(j^{(1-\alpha)j}M_j^{(p)})^{1/j}=\infty$, then the same is valid for all $p'>p$, thanks to the fact that the $\boldsymbol{M}^{(p)}\leq\boldsymbol{M}^{(p')}$. In this situation, since we also have $\mathcal{A}_{\{\boldsymbol{M}^{(p)}\}}(S_\alpha)\subseteq\mathcal{A}_{\{\boldsymbol{M}^{(p')}\}}(S_\alpha)$ and the class associated to the weight matrix $\mathcal{M}$ is the increasing union of such classes, in order to study stability properties in it we can restrict our attention to the case described in the previous definition.

In case $\lim_{j\to+\infty}(j^{(1-\alpha)j}M_j^{(p)})^{1/j}$ exists but is not infinity for any $p>0$, then there are some possibilities:

(i) If $\alpha>1$ and $\liminf_{j\to+\infty}(j^{(1-\alpha)j}M_j^{(p)})^{1/j}<\infty$ for all $p>0$, the class $\mathcal{A}_{\{\boldsymbol{M}^{(p)}\}}(S_\alpha)$ only contains constant functions, see [65, Thm. 21, and p. 8], and the same holds for the class $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$. So, the stability results turn out to be trivial.

(ii) If $0<\alpha\leq 1$ and $\liminf_{j\to+\infty}(j^{(1-\alpha)j}M_j^{(p)})^{1/j}=0$ for all $p>0$, the class $\mathcal{A}_{\{\boldsymbol{M}^{(p)}\}}(S_\alpha)$ only contains constant functions, see [65, Thm. 20], and again we are done.

(iii) If $0<\alpha\leq 1$ and $\liminf_{j\to+\infty}(j^{(1-\alpha)j}M_j^{(p)})^{1/j}\in(0,\infty)$ for all $p>0$ (or from some $p_0>0$ on), taking into account [65, Cor. 8] we have that the class $\mathcal{A}_{\{\boldsymbol{M}^{(p)}\}}(S_\alpha)$ coincides with $\mathcal{A}_{\{\overline{\boldsymbol{G}}^{\alpha-1}\}}(S_\alpha)$ for all $p>0$ (or for $p\geq p_0$), and so $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)=\mathcal{A}_{\{\overline{\mathcal{G}}^{\alpha-1}\}}(S_\alpha)$, where $\overline{\mathcal{G}}^{\alpha-1}$ is the matrix with all the rows equal to the sequence $\overline{\boldsymbol{G}}^{\alpha-1}$. We will study the stability properties for this class in Section 2.5.

In order to prove the aforementioned equality of the classes associated with $\mathcal{M}$ and $\mathcal{M}^\alpha$, it is convenient to recall the following result of B. Rodríguez-Salinas, which provides Gorny-Cartan like inequalities for holomorphic functions in sectors.

**Theorem 2.3.6.** *([65, Thm. 23]) Let $0<\alpha\leq 1$ and $f\in\mathcal{H}(S_\alpha)$. If $C_n(f)=\sup_{z\in S_\alpha}|f^{(n)}(z)|$, $n\in\mathbb{N}_0$, then the sequence $B_n=n^{(1-\alpha)n}C_n(f)$ verifies*

$$B_n\leq Aq^{(1-\alpha)n}B_{n_1}^{\frac{n_2-n}{n_2-n_1}}B_{n_2}^{\frac{n-n_1}{n_2-n_1}},\qquad n_1<n<n_2,$$

*where $A=4$ and $q=1$ if $\alpha=1$, or $A=8\pi$ and $q=2e(2-\alpha)/(1-\alpha)$ for the remaining cases.*

**Theorem 2.3.7.** *Let $\mathcal{M}=\{\boldsymbol{M}^{(p)}:p>0\}$ be a weight matrix and $0<\alpha\leq 1$ be given such that $\lim_{j\to+\infty}(j^{(1-\alpha)j}M_j^{(p)})^{1/j}=\infty$ for all $p>0$. Let $\mathcal{M}^\alpha=\{\boldsymbol{M}^{(p,\alpha)}:p>0\}$ be the matrix given in (2.7). Then, we have that*

$$\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)=\mathcal{A}_{\{\mathcal{M}^\alpha\}}(S_\alpha).$$

**Proof.** Given $f \in \mathcal{A}_{\{\mathcal{M}^\alpha\}}(S_\alpha)$, there exists some $p > 0$ such that $f \in \mathcal{A}_{\{\boldsymbol{M}^{(p,\alpha)}\}}(S_\alpha)$. Since $\overline{\boldsymbol{G}}^{1-\alpha}\boldsymbol{M}^{(p,\alpha)}$ is the log convex minorant of $\overline{\boldsymbol{G}}^{1-\alpha}\boldsymbol{M}^{(p)}$, we obviously have that $\overline{\boldsymbol{G}}^{1-\alpha}\boldsymbol{M}^{(p,\alpha)} \leq \overline{\boldsymbol{G}}^{1-\alpha}\boldsymbol{M}^{(p)}$, and therefore $\boldsymbol{M}^{(p,\alpha)} \leq \boldsymbol{M}^{(p)}$. We conclude that $f \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$.
For the converse inclusion, let us consider $f \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$. There exist some $C, D \in \mathbb{R}_{>0}$ and $p > 0$ such that $C_n(f) = \sup_{z\in S_\alpha} |f^{(n)}(z)| \leq CD^n M_n^{(p)}$, for all $n \in \mathbb{N}_0$.

Let us fix $n \in \mathbb{N}_0$ and distinguish two cases:

i) If $M_n^{(p,\alpha)} = M_n^{(p)}$ then $\sup_{z\in S_\alpha} |f^{(n)}(z)| \leq CD^n M_n^{(p,\alpha)}$.

ii) If not, by the construction of the log convex minorant, there exist so-called principal indices $n_1, n_2 \in \mathbb{N}_0$, with $n_1 < n < n_2$, such that $M_{n_i}^{(p,\alpha)} = M_{n_i}^{(p)}$ for $i = 1, 2$ (see [46, Chapitre I] and, for a detailed discussion of the regularization process and its intricacies, [73]). So, we have

$$\begin{aligned}
\ln(n^{(1-\alpha)n} M_n^{(p,\alpha)}) &= \frac{n_2 - n}{n_2 - n_1} \ln(n_1^{(1-\alpha)n_1} M_{n_1}^{(p,\alpha)}) + \frac{n - n_1}{n_2 - n_1} \ln(n_2^{(1-\alpha)n_2} M_{n_2}^{(p,\alpha)}) \\
&\geq \frac{n_2 - n}{n_2 - n_1} \ln(\frac{1}{CD^{n_1}} n_1^{(1-\alpha)n_1} C_{n_1}(f)) \\
&+ \frac{n - n_1}{n_2 - n_1} \ln(\frac{1}{CD^{n_2}} n_2^{(1-\alpha)n_2} C_{n_2}(f)).
\end{aligned}$$

Therefore, with the notation of the previous theorem, we deduce from above:

$$B_{n_1}^{\frac{n_2-n}{n_2-n_1}} B_{n_2}^{\frac{n-n_1}{n_2-n_1}} \leq (CD^{n_1})^{\frac{n_2-n}{n_2-n_1}} (CD^{n_2})^{\frac{n-n_1}{n_2-n_1}} n^{(1-\alpha)n} M_n^{(p,\alpha)} = CD^n n^{(1-\alpha)n} M_n^{(p,\alpha)}.$$

Now, from the previous estimate and by applying Theorem 2.3.6, there exist some $A, q > 0$ such that

$$C_n(f) \leq n^{(\alpha-1)n} A q^{(1-\alpha)n} B_{n_1}^{\frac{n_2-n}{n_2-n_1}} B_{n_2}^{\frac{n-n_1}{n_2-n_1}} \leq AC(q^{(1-\alpha)}D)^n M_n^{(p,\alpha)}.$$

We conclude that $f \in \mathcal{A}_{\{\mathcal{M}^\alpha\}}(S_\alpha)$. □

Now, we need to establish a suitable condition under which the equality of the classes $\mathcal{H}_{\{\mathcal{M}\}}(U)$ and $\mathcal{H}_{\{\mathcal{M}^\alpha\}}(U)$ can be stated for a general open set $U$ in $\mathbb{C}$. This will be necessary in the proof of the implication $(e) \Rightarrow (d)$ in the forthcoming Theorem 2.3.14, as noted by A. Rainer [58], who has also provided us with a proof for the aforementioned equality. Let us start with some preliminary definitions and results in the ultradifferentiable framework.

**Definition 2.3.8.** Let $\boldsymbol{M}$ be a sequence of positive real numbers and $U \subseteq \mathbb{C}$ be an open set. Given a compact set $K \subset U$, we define

$$\mathcal{E}_{\boldsymbol{M},h}(K) := \{f \in \mathrm{C}^\infty(U) : \|f\|_{\boldsymbol{M},K,h} := \sup_{z\in K, j\in\mathbb{N}_0} \frac{|f^{(j)}(z)|}{h^j M_j} = \sup_{j\in\mathbb{N}_0} \frac{\|f^{(j)}\|_K}{h^j M_j} < +\infty\},$$

where $\|\cdot\|_K$ is the supremum norm. We put

$$\mathcal{E}_{\{\boldsymbol{M}\}}(K) := \bigcup_{h>0} \mathcal{E}_{\boldsymbol{M},h}(K).$$

Moreover, given a weight matrix $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$, we may introduce the Denjoy-Carleman class of Roumieu type $\mathcal{E}_{\{\mathcal{M}\}}(U)$ as

$$\mathcal{E}_{\{\mathcal{M}\}}(U) := \bigcap_{K\subset U} \bigcup_{p>0} \mathcal{E}_{\{\boldsymbol{M}^{(p)}\}}(K).$$

We first recall a classical result of H. Cartan.

**Lemma 2.3.9.** *([10, Lemme 2]) Let $f : [-r, r] \to \mathbb{C}$ be a $C^p$ function satisfying*

$$\|f\|_{[-r,r]} \leq A_0, \qquad \|f^{(p)}\|_{[-r,r]} \leq A_p.$$

*Then*

$$|f^{(k)}(0)| < 2e^k A_0^{1-k/p} \max\{A_p, A_0 p!/r^p\}^{k/p}.$$

In order to establish the aforementioned equality for the classes associated to $\mathcal{M}$ and $\mathcal{M}^\alpha$, we introduce the following property.

**Definition 2.3.10.** We say that $\mathcal{M}$ has the property $(\mathcal{M}_{\{\mathbf{G}\}})$ if

$$\frac{M_k^{(p)}}{M_j^{(p)}} \geq (k-j)! \qquad \text{whenever } k \geq j$$

for all $p > 0$.

Observe that, taking $j = 0$ in the previous condition, one deduces that

$$M_k^{(p)} \geq k!, \qquad k \geq 0, \tag{2.8}$$

what guarantees that, given $\alpha \in (0, 1]$, $\lim_{j\to+\infty}(j^{(1-\alpha)j} M_j^{(p)})^{1/j} = \infty$ for all $p > 0$, and so the matrix $\mathcal{M}^\alpha$ can be considered. We are ready to prove a first result in the one-dimensional case.

**Lemma 2.3.11.** *Let $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$ be a weight matrix that has $(\mathcal{M}_{\{\mathbf{G}\}})$. Fix $\alpha \in (0,1]$. Consider $\mathcal{M}^\alpha = \{\boldsymbol{M}^{(p,\alpha)} : p > 0\}$, the matrix given in (2.7). Then, for each open interval $I \subseteq \mathbb{R}$ we have that*

$$\mathcal{E}_{\{\mathcal{M}\}}(I) = \mathcal{E}_{\{\mathcal{M}^\alpha\}}(I).$$

*Proof.* Since $\boldsymbol{M}^{(p,\alpha)} \le \boldsymbol{M}^{(p)}$, only the inclusion $\mathcal{E}_{\{\mathcal{M}\}}(I) \subseteq \mathcal{E}_{\{\mathcal{M}^\alpha\}}(I)$ must be checked. Let $f \in \mathcal{E}_{\{\mathcal{M}\}}(I)$. Then for each compact interval $J \subseteq I$ there exist $C, \rho, p > 0$ such that

$$||f^{(k)}||_J \le C\rho^k M_k^{(p)}, \qquad k \in \mathbb{N}.$$

Let $\delta = \operatorname{dist}(J, \mathbb{R} \setminus I)$. We may assume that $\rho \ge 1/\delta$. There is a strictly increasing sequence $k_n \in \mathbb{N}_0$ such that

$$\boldsymbol{M}_{k_n}^{(p,\alpha)} = \boldsymbol{M}_{k_n}^{(p)}, \qquad n \in \mathbb{N}_0.$$

Let $k_n < k < k_{n+1}$. Note that, by $(\mathcal{M}_{\{\mathbf{G}\}})$ and since $\rho \ge 1/\delta$,

$$\rho^{k_n+1} M_{k_n+1}^{(p)} \ge \rho^{k_n} M_{k_n}^{(p)} \frac{(k_{n+1} - k_n)!}{\delta^{k_{n+1}-k_n}}.$$

By Lemma 2.3.9,

$$\begin{aligned} \|f^{(k)}\|_J &\le 2e^{k-k_n} (C\rho^{k_n} M_{k_n}^{(p)})^{1-\frac{k-k_n}{k_{n+1}-k_n}} (C\rho^{k_{n+1}} M_{k_{n+1}}^{(p)})^{\frac{k-k_n}{k_{n+1}-k_n}} \\ &= 2Ce^{k-k_n} \rho^k (k_n^{(\alpha-1)k_n} k_n^{(1-\alpha)k_n} M_{k_n}^{(p,\alpha)})^{1-\frac{k-k_n}{k_{n+1}-k_n}} \\ &\quad \cdot (k_{n+1}^{(\alpha-1)k_{n+1}} k_{n+1}^{(1-\alpha)k_{n+1}} M_{k_{n+1}}^{(p,\alpha)})^{\frac{k-k_n}{k_{n+1}-k_n}}. \end{aligned}$$

By the construction of the log-convex minorant,

$$(k_n^{(1-\alpha)k_n} M_{k_n}^{(p,\alpha)})^{1-\frac{k-k_n}{k_{n+1}-k_n}} (k_{n+1}^{(1-\alpha)k_{n+1}} M_{k_{n+1}}^{(p,\alpha)})^{\frac{k-k_n}{k_{n+1}-k_n}} = k^{(1-\alpha)k} M_k^{(p,\alpha)}.$$

Furthermore

$$(k_n^{(\alpha-1)k_n})^{1-\frac{k-k_n}{k_{n+1}-k_n}} (k_{n+1}^{(\alpha-1)k_{n+1}})^{\frac{k-k_n}{k_{n+1}-k_n}} \le k^{(\alpha-1)k}$$

thanks to the log-convexity of the map $k \mapsto k^k$. We conclude that

$$\|f^{(k)}\|_J \le 2C(e\rho)^k M_k^{(p,\alpha)}.$$

This implies the assertion. □

**Corollary 2.3.12.** *In the setting of Lemma 2.3.11, for each open subset $U \subseteq \mathbb{C}$,*

$$\mathcal{E}_{\{\mathcal{M}\}}(U) = \mathcal{E}_{\{\mathcal{M}^\alpha\}}(U).$$

*More precisely, if $p > 0$, $\delta > 0$, $K \subseteq \mathbb{C}$ is compact and $K_\delta := \{x \in \mathbb{C} : \operatorname{dist}(x, K) \le \delta\}$, then (by restriction)*

$$\mathcal{E}_{\{\boldsymbol{M}^{(p)}\}}(K_\delta) = \mathcal{E}_{\{\boldsymbol{M}^{(p,\alpha)}\}}(K).$$

*Proof.* This follows from Lemma 2.3.11, using directional derivatives and the polarization inequality. Let $f \in \mathcal{E}_{\{\boldsymbol{M}^{(p)}\}}(K_\delta)$. For all $x \in K$ and $v \in \mathbb{S}^1$, the function $f_{x,v}(t) = f(x+tv)$ satisfies $\|f_{x,v}^{(k)}\|_{[-\delta,\delta]} \leq C\rho^k M_k^{(p)}$ for some $C, \rho > 0$ (independently of $x$ and $v$), where we may assume that $\rho \geq 1/\delta$. The proof of Lemma 2.3.11 yields that $|d_v^k f(x)| = |f_{x,v}^{(k)}(0)| \leq 2C(e\rho)^k M_k^{(p,\alpha)}$. In view of the polarization inequality [40, (7.13.1)], the assertion follows. $\square$

Finally, we deduce the desired equality.

**Proposition 2.3.13.** *Let $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$ be a weight matrix such that $\mathcal{M}$ has $(\mathcal{M}_{\{\mathbf{G}\}})$. Fix $\alpha \in (0,1]$. Consider $\mathcal{M}^\alpha = \{\boldsymbol{M}^{(p,\alpha)} : p > 0\}$ given in (2.7). Then, for each open subset $U \subset \mathbb{C}$,*

$$\mathcal{H}_{\{\mathcal{M}\}}(U) = \mathcal{H}_{\{\mathcal{M}^\alpha\}}(U).$$

*Proof.* This follows from Corollary 2.3.12 and the fact that for a holomorphic function $f \in \mathcal{H}(U)$ we have (by the Cauchy-Riemann equations)

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}.$$

$\square$

We are ready to state our first main result.

**Theorem 2.3.14.** *Let $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$ be a weight matrix (not necessarily $(\mathcal{M}_{\mathrm{sc}})$) and $0 < \alpha \leq 1$ be given such that $\lim_{j\to+\infty}(j^{(1-\alpha)j} M_j^{(p)})^{1/j} = \infty$ for all $p > 0$. Let $\mathcal{M}^\alpha = \{\boldsymbol{M}^{(p,\alpha)} : p > 0\}$ be the matrix according to (2.7). Then:*
*(I) The following assertions are equivalent:*

- $(a)$ *The matrix $\mathcal{M}^\alpha$ satisfies the property $(\mathcal{M}_{\{\mathrm{rai}\}})$.*
- $(b)$ *The class $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ is holomorphically closed.*
- $(c)$ *The class $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ is inverse-closed.*

*(II) If $\mathcal{M}^\alpha$ has $(\mathcal{M}_{\{\mathrm{dc}\}})$, then any of the previous statements implies:*

- $(e)$ *The matrix $\mathcal{M}^\alpha$ satisfies the property $(\mathcal{M}_{\{\mathrm{FdB}\}})$.*

*(III) If $\mathcal{M}$ has $(\mathcal{M}_{\{\mathbf{G}\}})$, then $(e)$ implies:*

- $(d)$ *The class $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ is closed under composition.*

*(IV) If $\mathcal{M}$ has $(\mathcal{M}_{\{C^\omega\}})$, then (d) implies (b).*
*(V) If $\mathcal{M}$ has $(\mathcal{M}_{\{\mathbf{G}\}})$ and $\mathcal{M}^\alpha$ has $(\mathcal{M}_{\{\mathrm{dc}\}})$, then all the statements from (a) to (e) are equivalent.*

**Proof.** (I) $(a) \Rightarrow (b)$ First recall that by the so-called *Faà-di-Bruno formula* for the composition we get

$$(g \circ f)^{(n)}(z) = \sum_{\substack{\sum_{i=1}^n k_i = k \\ \sum_{i=1}^n ik_i = n}} \frac{n!}{k_1! \cdots k_n!} g^{(k)}(f(z)) \prod_{i=1}^n \left( \frac{f^{(i)}(z)}{i!} \right)^{k_i}, \quad z \in S_\alpha,\ n \in \mathbb{N}.$$

Let now $f \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ be given. By Theorem 2.3.7 we know that the classes $\mathcal{A}_{\{\mathcal{M}^\alpha\}}(S_\alpha)$ and $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ are equal, therefore $f \in \mathcal{A}_{\{\mathcal{M}^\alpha\}}(S_\alpha)$. In particular, $f$ is bounded and thus any function $g$ which is analytic in a domain containing the (compact) closure of the range of $f$ satisfies

$$\exists\, C_1, h_1 \geq 1\ \forall\, k \in \mathbb{N}_0\ \forall\, z \in S_\alpha : \quad |g^{(k)}(f(z))| \leq C_1 h_1^k k!. \tag{2.9}$$

By applying this and the fact that $f \in \mathcal{A}_{\{\mathcal{M}^\alpha\}}(S_\alpha)$, we estimate as follows for all $n \in \mathbb{N}$ and $z \in S_\alpha$:

$$\begin{aligned}
|(g \circ f)^{(n)}(z)| &\leq \sum_{\sum_{i=1}^n k_i = k, \sum_{i=1}^n ik_i = n} \frac{n!}{k_1! \cdots k_n!} |g^{(k)}(f(z))| \prod_{i=1}^n \left| \frac{f^{(i)}(z)}{i!} \right|^{k_i} \\
&\leq \sum_{\sum_{i=1}^n k_i = k, \sum_{i=1}^n ik_i = n} \frac{n!}{k_1! \cdots k_n!} C_1 h_1^k k! \prod_{i=1}^n \left( C_2 h_2^i \widetilde{M}_i^{(p,\alpha)} \right)^{k_i} \\
&\leq C_1 \sum_{\substack{\sum_{i=1}^n k_i = k \\ \sum_{i=1}^n ik_i = n}} \frac{n!}{k_1! \cdots k_n!} h_1^k k! C_2^{k_1 + \cdots + k_n} h_2^{k_1 + \cdots + nk_n} \prod_{i=1}^n (\widetilde{M}_i^{(p,\alpha)})^{k_i} \\
&\underbrace{\leq}_{(1.16)} C_1 (C_2 h_1 h_2)^n \sum_{\sum_{i=1}^n k_i = k, \sum_{i=1}^n ik_i = n} \frac{n!}{k_1! \cdots k_n!} k! \prod_{i=1}^n H_1^{ik_i} \widetilde{M}_{ik_i}^{(p',\alpha)} \\
&\underbrace{\leq}_{(1.16)} C_1 (H_1 C_2 h_1 h_2)^n \sum_{\substack{\sum_{i=1}^n k_i = k \\ \sum_{i=1}^n ik_i = n}} \frac{n!}{k_1! \cdots k_n!} k! H_2^{k_1 + \cdots + nk_n} \widetilde{M}_{k_1 + \cdots + nk_n}^{(p'',\alpha)} \\
&= C_1 (H_1 H_2 C_2 h_1 h_2)^n M_n^{(p'',\alpha)} \sum_{\sum_{i=1}^n k_i = k, \sum_{i=1}^n ik_i = n} \frac{k!}{k_1! \cdots k_n!} \\
&\leq C_1 C_3 (H_1 H_2 C_2 C_4 h_1 h_2)^n M_n^{(p'',\alpha)}.
\end{aligned}$$

For the estimates also note that $k \leq n$ and w.l.o.g. $C_2, h_1, h_2, H_1 \geq 1$. Moreover, we have that

$$\sum_{\sum_{i=1}^n k_i = k, \sum_{i=1}^n ik_i = n} \frac{k!}{k_1! \cdots k_n!} = 2^{n-1},$$

see the book of S. G. Krantz and H. R. Parks [39, Lemma 1.4.1] or C. Fernández and A. Galbis [21, Prop. 2.1]. Finally, by taking into account that the classes $\mathcal{A}_{\{\mathcal{M}^\alpha\}}(S_\alpha)$ and $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ are equal, then $g \circ f \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ is verified.

$(b) \Rightarrow (c)$ This is obvious by taking $g : z \mapsto \frac{1}{z}$ since $g \in \mathcal{H}(\mathbb{C}\backslash\{0\})$ and $\mathbb{C}\backslash\{0\}$ contains the (compact) closure of the image of any element $f \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ such that $\inf_{z \in S_\alpha} |f(z)| > 0$.

$(c) \Rightarrow (a)$ We follow the ideas from J. A. Siddiqi and M. Ider [76, Thm. 1] and apply the constructions from the previous section. First, recall that $\boldsymbol{L}^{(p)} := \overline{\boldsymbol{G}}^{1-\alpha}\boldsymbol{M}^{(p,\alpha)} = (\overline{\boldsymbol{G}}^{1-\alpha}\boldsymbol{M}^{(p)})^{\mathrm{lc}}$ is log-convex for any $p > 0$, see (2.7). Let $p > 0$ be arbitrary but from now on fixed. According to Theorem 2.2.11 we put

$$f_p(z) := \mathcal{T}_{\boldsymbol{L}^{(p)}}(\widetilde{E}_\alpha)(z).$$

By using (2.3) and Lemma 2.2.8 we estimate as follows:

$$\begin{aligned} |f_p^{(n)}(z)| &\leq \sum_{k=0}^{\infty} \frac{1}{2^k} L_k^{(p)} \frac{(\ell_k^{(p)})^n}{(\ell_k^{(p)})^k} |\widetilde{E}_\alpha^{(n)}(\ell_k^{(p)} z)| \leq 4 L_n^{(p)} \frac{n! e^n}{n^{(2-\alpha)n}} \\ &= 4 M_n^{(p,\alpha)} \frac{n! e^n}{n^n} \leq 4 e^n M_n^{(p,\alpha)}, \end{aligned}$$

for all $n \in \mathbb{N}_0$ and $z \in S_\alpha$. This estimate shows that $f_p \in \mathcal{A}_{\{\mathcal{M}^\alpha\}}(S_\alpha)$ and, in particular when being applied to $n = 0$, it yields $\sup_{z \in S_\alpha} |f_p(z)| \leq 4 < +\infty$.

Set $R_n^{(p)} := \sum_{k=0}^{\infty} \frac{1}{2^k} L_k^{(p)} (\ell_k^{(p)})^{n-k}$ and so we get

$$\forall\, n \in \mathbb{N}_0 : \quad f_p^{(n)}(0) = R_n^{(p)} \frac{n!(-1)^n}{\Gamma((2-\alpha)(n+1)+2)}, \tag{2.10}$$

and from Lemma 2.2.8

$$\forall\, n \in \mathbb{N}_0 : \quad R_n^{(p)} \geq \frac{n^{(1-\alpha)n} M_n^{(p,\alpha)}}{2^n}. \tag{2.11}$$

Take $\lambda > 4$ (note that in [75, p. 349, line 5] there is a mistake, one should write $\lambda > C_0(f) M_0^\alpha$). Then, if we put $\widetilde{f}_p := \lambda - f_p$, we have that $\widetilde{f}_p \in \mathcal{A}_{\{\mathcal{M}^\alpha\}}(S_\alpha)$. Moreover, since $\inf_{z \in S_\alpha} |\widetilde{f}_p(z)| > 0$ and $\mathcal{A}_{\{\mathcal{M}^\alpha\}}(S_\alpha)$, which coincides with $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$, is assumed to be inverse-closed, we get that $z \mapsto \frac{1}{\widetilde{f}_p(z)} = \frac{1}{\lambda - f_p(z)} \in \mathcal{A}_{\{\mathcal{M}^\alpha\}}(S_\alpha)$. We write $g : z \mapsto \frac{1}{\lambda - z}$, then by applying again the Faà-di-Bruno-formula to the composition $g \circ f_p \in \mathcal{A}_{\{\mathcal{M}^\alpha\}}(S_\alpha)$ and thanks to the fact that $g^{(k)}(z) = \frac{k!}{(\lambda-z)^{k+1}}$ for all $k \in \mathbb{N}_0$, yields: For some $C, h > 0$ and some index $p' > 0$ (large) we get for all $n \in \mathbb{N}_0$ that

$$\begin{aligned} |(g \circ f_p)^{(n)}(0)| &= \left| \sum_{\sum_{i=1}^n k_i = k, \sum_{i=1}^n i k_i = n} \frac{n!}{k_1! \cdots k_n!} \frac{k!}{(\lambda - f_p(0))^{k+1}} \prod_{i=1}^n \left( \frac{f_p^{(i)}(0)}{i!} \right)^{k_i} \right| \\ &\le C h^n M_n^{(p',\alpha)}. \end{aligned}$$

By (2.10) we see

$$\left( \frac{f_p^{(i)}(0)}{i!} \right)^{k_i} = \left( \frac{(-1)^i R_i^{(p)}}{\Gamma((2-\alpha)(i+1)+2)} \right)^{k_i}, \quad 1 \le i \le n,$$

and by taking into account that $\prod_{i=1}^n (-1)^{i k_i} = (-1)^n$, we deduce that for every $n \in \mathbb{N}_0$,

$$\sum_{\substack{\sum_{i=1}^n k_i = k \\ \sum_{i=1}^n i k_i = n}} \frac{n!}{k_1! \cdots k_n!} \frac{k!}{(\lambda - f_p(0))^{k+1}} \prod_{i=1}^n \left( \frac{R_i^{(p)}}{\Gamma((2-\alpha)(i+1)+2)} \right)^{k_i} \le C h^n M_n^{(p',\alpha)}.$$

Each summand in this sum is strictly positive and we focus now on the one given by the choices $k_j = k$, $k_i = 0$ for $i \neq j$ and $n = j k_j = jk$ with $j, k \in \mathbb{N}$. Thus, there exist $C, h, p' > 0$ such that

$$\forall\, j, k \in \mathbb{N} : \quad \frac{(jk)!}{(\lambda - f_p(0))^{k+1}} \left( \frac{R_j^{(p)}}{\Gamma((2-\alpha)(j+1)+2)} \right)^k \le C h^{jk} M_{jk}^{(p',\alpha)},$$

is valid and clearly $(\lambda - f_p(0))^{k+1} \le h_1^{jk+1}$ for some $h_1 > 0$ (large) and all $k \in \mathbb{N}_0$. Hence

$$\exists\, C, h, h_1, p' > 0 \; \forall\, j, k \in \mathbb{N} : \quad \left( \frac{R_j^{(p)}}{\Gamma((2-\alpha)(j+1)+2)} \right)^k \le C h_1 (h h_1)^{jk} \frac{M_{jk}^{(p',\alpha)}}{(jk)!}. \tag{2.12}$$

By involving (2.11) we estimate the left-hand side of (2.12) as follows:

$$\begin{aligned} \frac{R_j^{(p)}}{\Gamma((2-\alpha)(j+1)+2)} &\ge \frac{j!^{1-\alpha} M_j^{(p,\alpha)}}{2^j((2-\alpha)(j+1)+1)\Gamma((2-\alpha)(j+1)+1)} \\ &\ge \frac{M_j^{(p,\alpha)}}{C_1 12^j h_3^{j+1} j!}. \end{aligned}$$

The last estimate is valid since $(2-\alpha)(j+1)+1 \leq 2(j+1)+(j+1) = 3(j+1) \leq 6^j$ for all $j \in \mathbb{N}$, and $\Gamma((2-\alpha)(j+1)+1) \leq C_1 h_2^{(2-\alpha)(j+1)} j!^{2-\alpha}$ for some $C_1, h_2 \geq 1$ and all $j \geq 1$ (by the properties of the Gamma function), where we have put $h_3 := h_2^{2-\alpha}$. Consequently, by (2.12) we get

$$\exists\, C, C_1, h, h_1, h_3, p' > 0\ \forall\, j,k \in \mathbb{N}: \quad \left(\frac{M_j^{(p,\alpha)}}{j!}\right)^k \leq C h_1 (12hC_1h_1h_3^2)^{jk} \frac{M_{jk}^{(p',\alpha)}}{(jk)!},$$

and so

$$\exists\, H \geq 1\ \exists\, p'(\geq p) > 0\ \forall\, j,k \in \mathbb{N}: \quad \left(\frac{M_j^{(p,\alpha)}}{j!}\right)^{1/j} \leq H \left(\frac{M_{jk}^{(p',\alpha)}}{(jk)!}\right)^{1/(jk)}. \tag{2.13}$$

(2.13) establishes $(\mathcal{M}_{\{\mathrm{rai}\}})$ for indices $p$ and $p'$ for all choices $j,k \in \mathbb{N}$ and so for all multiplies $n = jk$ of $j \in \mathbb{N}$. For the remaining cases let now $n \geq 1$ such that $jk < n < j(k+1)$ for some $j,k \in \mathbb{N}$. Then, by using (2.13) (with appearing constant $H$), (1.3) and the fact that $j \mapsto (j^{(1-\alpha)j} M_j^{(p',\alpha)})^{1/j}$ is non-decreasing for each index $p' > 0$ (by log-convexity), we estimate as follows:

$$\begin{aligned}
\left(\frac{M_n^{(p',\alpha)}}{n!}\right)^{1/n} &= \frac{(n^{(1-\alpha)n} M_n^{(p',\alpha)})^{1/n}}{n^{1-\alpha}(n!)^{1/n}} \geq \frac{((jk)^{(1-\alpha)jk} M_{jk}^{(p',\alpha)})^{1/(jk)}}{n^{1-\alpha}(n!)^{1/n}} \\
&= \frac{(jk)^{1-\alpha}}{n!^{1/n} n^{1-\alpha}} \left(\frac{M_{jk}^{(p',\alpha)}}{(jk)!}\right)^{1/(jk)} (jk)!^{1/(jk)} \\
&\geq \frac{1}{H} \left(\frac{M_j^{(p,\alpha)}}{j!}\right)^{1/j} \frac{(jk)!^{1/(jk)}}{n!^{1/n}} \left(\frac{jk}{n}\right)^{1-\alpha} \\
&\geq \frac{1}{H} \left(\frac{M_j^{(p,\alpha)}}{j!}\right)^{1/j} \frac{e^{-1} jk}{n} \left(\frac{jk}{j(k+1)}\right)^{1-\alpha} \\
&\geq \frac{1}{H} \left(\frac{M_j^{(p,\alpha)}}{j!}\right)^{1/j} \frac{jk}{ej(k+1)} \left(\frac{1}{2}\right)^{1-\alpha} \geq \frac{1}{He2^{2-\alpha}} \left(\frac{M_j^{(p,\alpha)}}{j!}\right)^{1/j}.
\end{aligned}$$

Summarizing, property $(\mathcal{M}_{\{\mathrm{rai}\}})$ is verified for the matrix $\mathcal{M}^\alpha$ between the indices $p$ and $p'$ and when choosing the constant $C := He2^{2-\alpha} (> H)$.

(II) $(a) \Rightarrow (e)$ follows by $(ii)$ in Lemma 1.3.7.

(III) $(e) \Rightarrow (d)$ follows by Theorem 2.3.7, Proposition 2.3.13 and by repeating the arguments in the proof of $(a) \Rightarrow (b)$ above (a word-by-word repetition of the proof in the ultradifferentiable setting), see [70, Thm. 8.3.1].

(IV) The property $(\mathcal{M}_{\{\mathrm{C}^\omega\}})$ implies that for all open set $U \subseteq \mathbb{C}$, the class $\mathcal{H}(U)$ is contained in $\mathcal{H}_{\{\mathcal{M}\}}(U)$, so $(d) \Rightarrow (b)$.

(V) It suffices to observe that, according to (2.8), the property $(\mathcal{M}_{\{\mathbf{G}\}})$ for $\mathcal{M}$ implies that $\mathcal{M}$ has $(\mathcal{M}_{\{\mathrm{C}^\omega\}})$. So, all the previous implications are valid. □

**Remark 2.3.15.** (i) If $\mathcal{M}$ has $(\mathcal{M}_{\{\mathrm{dc}\}})$ then $\mathcal{M}^\alpha$ has it too (the converse is not clear in general).

(ii) As said before, the property $(\mathcal{M}_{\{\mathbf{G}\}})$ implies $(\mathcal{M}_{\{\mathrm{C}^\omega\}})$. The converse does not hold, as shown by the constant matrix given by the sequence $\boldsymbol{M}$ defined as $M_{2j} = (2j)!$, $M_{2j+1} = (2j+1)!^2$, $j \in \mathbb{N}_0$.

(iii) Regardless the property $(\mathcal{M}_{\{\mathbf{G}\}})$, the implication $(e) \Rightarrow (d)$ is valid provided that $\mathcal{M}$ and $\mathcal{M}^\alpha$ are $R$-equivalent, since then $\mathcal{H}_{\{\mathcal{M}\}}(U)$ and $\mathcal{H}_{\{\mathcal{M}^\alpha\}}(U)$ are equal for every open $U \subseteq \mathbb{C}$. This trivially happens when $\mathcal{M}$ is $(\mathcal{M}_{\mathrm{lc}})$ (i. e. $\boldsymbol{M}^{(p)}$ is a log-convex sequence for all $p > 0$), or if $\overline{\boldsymbol{G}}^{1-\alpha}\boldsymbol{M}^{(p)}$ is log-convex for every $p > 0$, since in any case we would indeed have $\mathcal{M} = \mathcal{M}^\alpha$.

(iv) The condition that $\lim_{j\to+\infty}(j^{(1-\alpha)j}M_j^{(p)})^{1/j} = \infty$ for all $p > 0$ can be weakened as long as the log-convex regularization of $\overline{\boldsymbol{G}}^{1-\alpha}\boldsymbol{M}^{(p)}$ makes sense (for example, in case $\boldsymbol{M}^{(p)} = \overline{\boldsymbol{G}}^{\alpha-1}$). In this situation, the proof of Theorem 2.3.7 is still valid, Theorem 2.2.11 can be applied and the availability of characteristic functions (needed in the previous proof of the implication $(c) \Rightarrow (a)$) is guaranteed. A similar comment can be made regarding the next corollary.

For a sequence $\boldsymbol{M} \in \mathbb{R}_{>0}^{\mathbb{N}_0}$ such that $\lim_{j\to+\infty}(j^{(1-\alpha)j}M_j^{(p)})^{1/j} = \infty$, we can extend [76, Thm. 1] by considering the constant weight matrix $\mathcal{M} = \{\boldsymbol{M}^{(p)} = \boldsymbol{M} : p > 0\}$ and applying to it the previous result.

**Corollary 2.3.16.** *Let $\boldsymbol{M} \in \mathbb{R}_{>0}^{\mathbb{N}_0}$ be a sequence, and $0 < \alpha \leq 1$ be given such that $\lim_{j\to+\infty}(j^{(1-\alpha)j}M_j)^{1/j} = \infty$. Let $\boldsymbol{M}^{(\alpha)} := \overline{\boldsymbol{G}}^{\alpha-1}\left(\overline{\boldsymbol{G}}^{1-\alpha}\boldsymbol{M}\right)^{\mathrm{lc}}$. Then:*
*(I) The following assertions are equivalent:*

($a$) *The sequence $\boldsymbol{M}^{(\alpha)}$ has the property* (rai).

($b$) *The class $\mathcal{A}_{\{\boldsymbol{M}\}}(S_\alpha)$ is holomorphically closed.*

($c$) *The class $\mathcal{A}_{\{\boldsymbol{M}\}}(S_\alpha)$ is inverse-closed.*

*(II) If $\boldsymbol{M}^{(\alpha)}$ is (dc), then any of the previous statements implies:*

($e$) *The sequence $\boldsymbol{M}^{(\alpha)}$ has the property* (FdB).

*(III) If the constant matrix $\mathcal{M} = \{\boldsymbol{M}^{(p)} = \boldsymbol{M} : p > 0\}$ has $(\mathcal{M}_{\{\mathbf{G}\}})$, then $(e)$ implies:*

- $(d)$ *The class $\mathcal{A}_{\{\boldsymbol{M}\}}(S_\alpha)$ is closed under composition.*

*(IV) If $\mathcal{M} = \{\boldsymbol{M}^{(p)} = \boldsymbol{M} : p > 0\}$ has $(\mathcal{M}_{\{C^\omega\}})$, then (d) implies (b).*
*(V) If $\mathcal{M} = \{\boldsymbol{M}^{(p)} = \boldsymbol{M} : p > 0\}$ has $(\mathcal{M}_{\{\mathbf{G}\}})$ and $\boldsymbol{M}^{(\alpha)}$ is (dc), then all the statements from $(a)$ to $(e)$ are equivalent.*

**Remark 2.3.17.** Although the ultradifferentiable classes $\mathcal{E}_{\{\boldsymbol{M}\}}(0, +\infty)$ are defined by local estimates, instead of the global ones defining our ultraholomorphic classes, it is interesting to mention that the main stability result of A. Rainer and G. Schindl [60, Thm. 1] (see also [59, Thm. 3.2]) for the former ones can be partially seen as the limiting case when taking $\alpha = 0$ in the previous result, i. e. when the sector $S_\alpha$ "collapses" to the ray $(0, +\infty)$.

Thanks to the construction of characteristic functions in classes defined in sectors of arbitrary opening, undertaken in Subsection 2.2.3, we study now the stability properties for classes defined in sectors wider than a half-plane.

**Theorem 2.3.18.** *Let $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$ be a weight matrix and consider $\alpha > 1$. For each $p > 0$, we suppose that there exists some $\alpha_p > \alpha$ such that $\overline{\boldsymbol{G}}^{1-\alpha_p}\boldsymbol{M}^{(p)}$ is equivalent to an (lc) sequence $\boldsymbol{L}^{(p)}$ depending on $\alpha_p$. Then the following assertions are equivalent:*

- $(a)$ *The matrix $\mathcal{M}$ satisfies the property $(\mathcal{M}_{\{\mathrm{rai}\}})$.*
- $(b)$ *The class $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ is holomorphically closed.*
- $(c)$ *The class $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ is inverse-closed.*

*If $\mathcal{M}$ has in addition $(\mathcal{M}_{\{C^\omega\}})$ and $(\mathcal{M}_{\{\mathrm{dc}\}})$, then the list of equivalences can be extended by*

- $(d)$ *The class $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ is closed under composition.*
- $(e)$ *The matrix $\mathcal{M}$ satisfies the property $(\mathcal{M}_{\{\mathrm{FdB}\}})$.*

**Proof.** The proof of $(a) \Rightarrow (b) \Rightarrow (c)$ is similar to the one in Theorem 2.3.14.

$(c) \Rightarrow (a)$ Although the arguments are similar to those developed in the same implication in Theorem 2.3.14, we consider it worthy to complete the details because now we will work with the original weight matrix (instead of $\mathcal{M}^\alpha$), and the characteristic functions are different in this framework. Let $p > 0$ be arbitrary but from now on fixed. There exist $\alpha_p > \alpha$ and $\boldsymbol{L}^{(p)}$ log-convex such that

$\overline{\boldsymbol{G}}^{1-\alpha_p}\boldsymbol{M}^{(p)} \approx \boldsymbol{L}^{(p)}$. Then, there exist $A_p, B_p > 0$ such that $A_p^n n^{(1-\alpha_p)n} M_n^{(p)} \leq L_n^{(p)} \leq B_p^n n^{(1-\alpha_p)n} M_n^{(p)}$ for all $n \in \mathbb{N}_0$. According to Theorem 2.2.11 we put

$$f_p(z) := \mathcal{T}_{\boldsymbol{L}^{(p)}}(g_{\alpha,\alpha_p})(z).$$

By using (2.5), Lemma 2.2.8 and the above inequality we have

$$\begin{aligned}|f_p^{(n)}(z)| &\leq \sum_{k=0}^{\infty} \frac{1}{2^k} L_k^{(p)} \frac{(\ell_k^{(p)})^n}{(\ell_k^{(p)})^k} |g_{\alpha,\alpha_p}^{(n)}(\ell_k^{(p)} z)| \leq 2CD^n L_n^{(p)} \Gamma((\alpha_p - 1)n + 1) \\ &\leq E\widetilde{B}_p^n n^{(1-\alpha_p)n} M_n^{(p)} n^{(\alpha_p-1)n} = E\widetilde{B}_p^n M_n^{(p)},\end{aligned}$$

for suitable constant $\widetilde{B}_p, C, D, E > 1$ and for all $n \in \mathbb{N}_0$ and $z \in S_\alpha$. This estimate shows that $f_p \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ and, in particular, it yields $\sup_{z\in S_\alpha} |f_p(z)| \leq E < +\infty$.

Set $R_n^{(p)} := \sum_{k=0}^{\infty} \frac{1}{2^k} L_k^{(p)} (\ell_k^{(p)})^{n-k}$, so that

$$\forall\, n \in \mathbb{N}_0: \quad f_p^{(n)}(0) = (-1)^n \Gamma((\alpha_p - 1)n + 1) R_n^{(p)}, \tag{2.14}$$

and from Lemma 2.2.8,

$$\forall\, n \in \mathbb{N}_0: \quad R_n^{(p)} \geq \frac{L_n^{(p)}}{2^n} \geq \frac{A_p^n n^{(1-\alpha_p)n} M_n^{(p)}}{2^n}. \tag{2.15}$$

Now take $\lambda > E$ and put $\widetilde{f}_p := \lambda - f_p$. Thus we get $\widetilde{f}_p \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$, and moreover $\inf_{z\in S_\alpha} |\widetilde{f}_p(z)| > 0$. Since $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ is assumed to be inverse-closed, we get that $z \mapsto \frac{1}{\lambda - f_p(z)} \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$. When writing $g_p : z \mapsto \frac{1}{\lambda - z}$, the dependence on $p$ is justified because $\lambda$ is clearly depending on this chosen index. By applying the Faà-di-Bruno-formula to the composition $g_p \circ f_p$ we get that for some $F, h > 0$ and some index $p' > 0$ (large) and for all $n \in \mathbb{N}_0$,

$$\begin{aligned}|(g_p \circ f_p)^{(n)}(0)| &= \left| \sum_{\sum_{i=1}^n k_i = k, \sum_{i=1}^n i k_i = n} \frac{n!}{k_1! \cdots k_n!} \frac{k!}{(\lambda - f_p(0))^{k+1}} \prod_{i=1}^n \left( \frac{f_p^{(i)}(0)}{i!} \right)^{k_i} \right| \\ &\leq F h^n M_n^{(p')}.\end{aligned}$$

Using (2.14) and since $\prod_{i=1}^n (-1)^{ik_i} = (-1)^n$, we deduce that for every $n \in \mathbb{N}_0$

$$\sum_{\substack{\sum_{i=1}^n k_i = k \\ \sum_{i=1}^n i k_i = n}} \frac{n!}{k_1! \cdots k_n!} \frac{k!}{(\lambda - f_p(0))^{k+1}} \prod_{i=1}^n \left( \frac{\Gamma((\alpha_p - 1)i + 1) R_i^{(p)}}{i!} \right)^{k_i} \leq F h^n M_n^{(p')}.$$

Given $j,k\in\mathbb{N}$, we focus on the summand for $k_j=k$, $k_i=0$ for $i\neq j$ and $n=jk_j=jk$, so we get that

$$\exists\, F,h,p'>0\;\forall\, j,k\in\mathbb{N}:\quad \frac{(jk)!}{(\lambda-f_p(0))^{k+1}}\left(\frac{\Gamma((\alpha_p-1)j+1)R_j^{(p)}}{j!}\right)^k\leq Fh^{jk}M_{jk}^{(p')}.$$

Clearly, $(\lambda-f_p(0))^{k+1}\leq h_1^{jk+1}$ for some $h_1>0$ (large) and all $k\in\mathbb{N}_0$. Hence, for all $j,k\in\mathbb{N}$ we have

$$\exists\, F,h,h_1,p'>0\;\forall\, j,k\in\mathbb{N}:\quad \left(\frac{\Gamma((\alpha_p-1)j+1)R_j^{(p)}}{j!}\right)^k\leq Fh_1(hh_1)^{jk}\frac{M_{jk}^{(p')}}{(jk)!}. \tag{2.16}$$

By involving (2.15) we estimate the left-hand side of (2.16) as follows:

$$\begin{aligned}\frac{\Gamma((\alpha_p-1)j+1)R_j^{(p)}}{j!}&\geq\frac{A_p^jj^{(1-\alpha_p)j}\Gamma((\alpha_p-1)j+1)M_j^{(p)}}{2^jj!}\\&\geq\frac{\widetilde{A}_p^jj^{(1-\alpha_p)j}j^{(\alpha_p-1)j}M_j^{(p)}}{2^jj!}=\frac{M_j^{(p)}}{\overline{A}_p^jj!}.\end{aligned}$$

The last inequality is a consequence of the properties of the Gamma function for a suitable constant $\widetilde{A}_p>0$, and we have put $\overline{A}_p=2/\widetilde{A}_p$. Consequently, by (2.16) we get

$$\exists\, F,h,h_1,\overline{A}_p,p'>0\;\forall\, j,k\in\mathbb{N}:\quad \left(\frac{M_j^{(p)}}{j!}\right)^k\leq Fh_1(hh_1\overline{A}_p)^{jk}\frac{M_{jk}^{(p')}}{(jk)!},$$

and so there exists $H\geq 1$ such that

$$\left(\frac{M_j^{(p)}}{j!}\right)^{1/j}\leq H\left(\frac{M_{jk}^{(p')}}{(jk)!}\right)^{1/(jk)}. \tag{2.17}$$

Equation (2.17) establishes $(\mathcal{M}_{\{\mathrm{rai}\}})$ for indices $p$ and $p'$ for all choices $j,k\in\mathbb{N}$ and so for all multiples $n=jk$ of $j\in\mathbb{N}$. For the remaining cases let now $n\geq 1$ such that $jk<n<j(k+1)$ for some $j,k\in\mathbb{N}$. Then, by using (2.17), (1.3), the equivalence $\overline{\boldsymbol{G}}^{1-\alpha_{p'}}\boldsymbol{M}^{(p')}\approx\boldsymbol{L}^{(p')}$ and the fact that $j\mapsto(L_j^{(p')})^{1/j}$ is non-decreasing

for each index $p' > 0$, we estimate

$$\begin{aligned}
\left(\frac{M_n^{(p')}}{n!}\right)^{1/n} &= \frac{(B_{p'}^n n^{(1-\alpha_{p'})n} M_n^{(p')})^{1/n}}{B_{p'} n^{1-\alpha_{p'}} (n!)^{1/n}} \geq \frac{(L_n^{(p')})^{1/n}}{B_{p'} n^{1-\alpha_{p'}} (n!)^{1/n}} \geq \frac{(L_{jk}^{(p')})^{1/(jk)}}{B_{p'} n^{1-\alpha_{p'}} (n!)^{1/n}} \\
&\geq \frac{(A_{p'}^{jk} (jk)^{(1-\alpha_{p'})jk} M_{jk}^{(p')})^{1/(jk)}}{B_{p'} n^{1-\alpha_{p'}} (n!)^{1/n}} = \frac{A_{p'} (jk)^{1-\alpha_{p'}}}{B_{p'} n!^{1/n} n^{1-\alpha_{p'}}} \left(\frac{M_{jk}^{(p')}}{(jk)!}\right)^{1/(jk)} (jk)!^{1/(jk)} \\
&\geq \frac{A_{p'}}{B_{p'} H} \left(\frac{M_j^{(p)}}{j!}\right)^{1/j} \frac{(jk)!^{1/(jk)}}{n!^{1/n}} \left(\frac{jk}{n}\right)^{1-\alpha_{p'}} \\
&\geq \frac{A_{p'}}{B_{p'} H} \left(\frac{M_j^{(p)}}{j!}\right)^{1/j} \frac{e^{-1} jk}{n} \\
&\geq \frac{A_{p'}}{B_{p'} H} \left(\frac{M_j^{(p)}}{j!}\right)^{1/j} \frac{jk}{ej(k+1)} \geq \frac{A_{p'}}{2B_{p'} He} \left(\frac{M_j^{(p)}}{j!}\right)^{1/j}.
\end{aligned}$$

Summarizing, property $(\mathcal{M}_{\{\mathrm{rai}\}})$ is verified for the matrix $\mathcal{M}$ between the indices $p$ and $p'$ and when the constant $C := 2B_{p'}He/A_{p'}$ is chosen.

$(a) \Rightarrow (e)$ and $(d) \Rightarrow (b)$ are as in Theorem 2.3.14.

$(e) \Rightarrow (d)$ One can repeat the proof in the ultradifferentiable setting, see [70, Thm. 8.3.1]. $\square$

**Remark 2.3.19.** In the same line of Remark 2.2.12, if for a weight matrix $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$ we know that $\gamma(\boldsymbol{M}^{(p)}) > \alpha - 1$ for all $p > 0$, then there exists some $\alpha_p > \alpha$ such that $\overline{\boldsymbol{G}}^{1-\alpha_p} \boldsymbol{M}^{(p)}$ is equivalent to an (lc) sequence $\boldsymbol{L}^{(p)}$ depending on $\alpha_p$.

Note that there exist some differences between the statements of the Theorems 2.3.14 and 2.3.18, concerning the fact that the conditions for stability are imposed on different weight matrices, $\mathcal{M}$ or $\mathcal{M}^\alpha$. In general, if $\alpha > 1$ we only know that $\mathcal{A}_{\{\mathcal{M}^\alpha\}}(S_\alpha) \subset \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$. However, the hypotheses of the second theorem have strong implications and, under an additional assumption, these results perfectly match, as the next proposition shows.

**Proposition 2.3.20.** *Let $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$ be a given weight matrix. Suppose that for every $p > 0$ there exists $\alpha_p > 0$ such that $\overline{\boldsymbol{G}}^{1-\alpha_p} \boldsymbol{M}^{(p)}$ is equivalent to an (lc) sequence $\boldsymbol{L}^{(p)}$, and that there exists $\beta > 0$ such that $\beta < \alpha_p$ for all $p > 0$. Then, for every $p > 0$ one has $\lim_{j\to+\infty} (j^{(1-\beta)j} M_j^{(p)})^{1/j} = \infty$, $\mathcal{M}$ and*

*$\mathcal{M}^\beta$ (defined as in* (2.7)*) are $R$-equivalent, and therefore $\mathcal{M}$ satisfies the property $(\mathcal{M}_{\{\mathrm{rai}\}})$ (resp.$(\mathcal{M}_{\{\mathrm{FdB}\}})$) if and only if the matrix $\mathcal{M}^\beta$ satisfies this condition too. Moreover, $\mathcal{A}_{\{\mathcal{M}^\beta\}}(S_\gamma) = \mathcal{A}_{\{\mathcal{M}\}}(S_\gamma)$, for all $\gamma > 0$.*

**Proof.** Let $p > 0$ be arbitrary but fixed. First, note that

$$\overline{\boldsymbol{G}}^{1-\beta}\boldsymbol{M}^{(p)} = \overline{\boldsymbol{G}}^{\alpha_p-\beta}(\overline{\boldsymbol{G}}^{1-\alpha_p}\boldsymbol{M}^{(p)}) \approx \overline{\boldsymbol{G}}^{\alpha_p-\beta}\boldsymbol{L}^{(p)} =: \widetilde{\boldsymbol{L}}^{(p)},$$

where the sequence $\widetilde{\boldsymbol{L}}^{(p)}$ is log-convex (as the product of two such sequences).

On the one hand, the condition $\boldsymbol{L}^{(p)} \approx \overline{\boldsymbol{G}}^{1-\alpha_p}\boldsymbol{M}^{(p)}$ guarantees that there exists some $A > 0$ such that $A^j L_j^{(p)} \leq j^{(1-\alpha_p)j} M_j^{(p)}$, for all $j \in \mathbb{N}_0$. Moreover, for all $j > 0$ we can estimate $(j^{(1-\beta)j} M_j^{(p)})^{1/j} = j^{(\alpha_p-\beta)}(j^{(1-\alpha_p)j} M_j^{(p)})^{1/j} \geq j^{(\alpha_p-\beta)}(A^j L_j^{(p)})^{1/j}$, and thanks to the fact that $\boldsymbol{L}^{(p)}$ is (lc) and $\alpha_p > \beta$, we deduce that $\lim_{j\to+\infty}(j^{(1-\beta)j} M_j^{(p)})^{1/j} = \infty$. Moreover, there exists some $\widetilde{A} > 0$ such that the (lc) sequence $\mathbb{B}^{(p)} := (\widetilde{A}^j \widetilde{L}_j^{(p)})_j$ satisfies $\mathbb{B}^{(p)} \leq \overline{\boldsymbol{G}}^{1-\beta}\boldsymbol{M}^{(p)}$. Then, we have that $\mathbb{B}^{(p)} = (\mathbb{B}^{(p)})^{\mathrm{lc}} \leq (\overline{\boldsymbol{G}}^{1-\beta}\boldsymbol{M}^{(p)})^{\mathrm{lc}}$, which implies that $\widetilde{\boldsymbol{L}}^{(p)} \preceq (\overline{\boldsymbol{G}}^{1-\beta}\boldsymbol{M}^{(p)})^{\mathrm{lc}}$.

On the other hand, we observe that $\overline{\boldsymbol{G}}^{1-\beta}\boldsymbol{M}^{(p)} \preceq \widetilde{\boldsymbol{L}}^{(p)}$, and therefore, we have $(\overline{\boldsymbol{G}}^{1-\beta}\boldsymbol{M}^{(p)})^{\mathrm{lc}} \preceq \widetilde{\boldsymbol{L}}^{(p)}$. Finally, we conclude that $\widetilde{\boldsymbol{L}}^{(p)} \approx (\overline{\boldsymbol{G}}^{1-\beta}\boldsymbol{M}^{(p)})^{\mathrm{lc}}$.

The previous equivalence ensures that $\boldsymbol{M}^{(p,\beta)}$ is equivalent to $\overline{\boldsymbol{G}}^{\beta-1}\widetilde{\boldsymbol{L}}^{(p)}$, and therefore $\boldsymbol{M}^{(p)} \approx \boldsymbol{M}^{(p,\beta)}$. Finally, the two matrices $\mathcal{M}$ and $\mathcal{M}^\beta$ are $R$-equivalent, and the property $(\mathcal{M}_{\{\mathrm{rai}\}})$ (resp.$(\mathcal{M}_{\{\mathrm{FdB}\}})$) is stable under $R$-equivalence, see [70, Remark 8.2.2]. □

Under the assumptions of the previous proposition, we can prove a weaker variant of Theorem 2.3.18 using a similar technique to the one used in the proof of Theorem 2.3.14.

**Corollary 2.3.21.** *Let $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$ be a weight matrix and consider $\alpha > 1$. For each $p > 0$, we suppose that there exist some $\alpha_p > \alpha$ such that $\overline{\boldsymbol{G}}^{1-\alpha_p}\boldsymbol{M}^{(p)}$ is equivalent to an (lc) sequence $\boldsymbol{L}^{(p)}$ depending on $\alpha_p$, and that there exists $\beta > \alpha$ such that $\beta < \alpha_p$ for all $p > 0$. Then the following assertions are equivalent:*

(*a*) *The matrix $\mathcal{M}$, or equivalently $\mathcal{M}^\beta$, satisfies property $(\mathcal{M}_{\{\mathrm{rai}\}})$.*

(*b*) *The class $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ is holomorphically closed.*

(*c*) *The class $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ is inverse-closed.*

*If $\mathcal{M}$ has in addition $(\mathcal{M}_{\{C^\omega\}})$ and $(\mathcal{M}_{\{\mathrm{dc}\}})$, then the list of equivalences can be extended by* (*d*) *The class* $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ *is closed under composition.*

(*e*) *The matrix* $\mathcal{M}$*, or equivalently* $\mathcal{M}^\beta$*, satisfies property* $(\mathcal{M}_{\{\mathrm{FdB}\}})$.

We end this section by providing the version of Corollary 2.3.16 for wide sectors, which can be again deduced as a straightforward consequence of the corresponding result for weight matrices, Theorem 2.3.18.

**Corollary 2.3.22.** *Let* $\boldsymbol{M} \in \mathbb{R}_{>0}^{\mathbb{N}_0}$ *and* $\alpha > 1$*. Suppose there exists* $\alpha' > \alpha$ *such that* $\overline{\boldsymbol{G}}^{1-\alpha'}\boldsymbol{M}$ *is equivalent to an (lc) sequence* $\boldsymbol{L}$ *(depending on* $\alpha'$*). Then the following assertions are equivalent:*

(*a*) *The sequence* $\boldsymbol{M}$ *has the property* (rai).

(*b*) *The class* $\mathcal{A}_{\{\boldsymbol{M}\}}(S_\alpha)$ *is holomorphically closed.*

(*c*) *The class* $\mathcal{A}_{\{\boldsymbol{M}\}}(S_\alpha)$ *is inverse-closed.*

*If* $\liminf_{j\to\infty}(\breve{M}_j)^{1/j} > 0$ *and* $\boldsymbol{M}$ *is (dc), then the list of equivalences can be extended by*

(*d*) *The class* $\mathcal{A}_{\{\boldsymbol{M}\}}(S_\alpha)$ *is closed under composition.*

(*e*) *The sequence* $\boldsymbol{M}$ *has the property* (FdB).

## 2.4 The weight function case in the Roumieu setting

We start proving, for the reader's convenience, how the condition $(\mathcal{M}_{\{\mathrm{rai}\}})$ for a weight matrix associated to a weight function $\omega$ translates into a condition on $\omega$. Note that this matrix has $(\mathcal{M}_{\mathrm{lc}})$ and therefore $(\mathcal{M}_\omega)^\alpha \equiv \mathcal{M}_\omega$ for all $\alpha \in (0,1]$.

**Lemma 2.4.1.** *Let* $\omega \in \mathcal{W}_0$ *be given with associated weight matrix* $\mathcal{M}_\omega := \{\boldsymbol{W}^{(\ell)} : \ell > 0\}$*. Then the following are equivalent:*

(*a*) *The matrix* $\mathcal{M}_\omega$ *has* $(\mathcal{M}_{\{\mathrm{rai}\}})$*, i. e. (recall* $\breve{W}_j^{(\ell)} = W_j^{(\ell)}/j!$*)*

$$\forall\, \ell > 0\; \exists\, \ell' > 0\; \exists\, H \geq 1\; \forall\, 1 \leq j \leq k: \quad (\breve{W}_j^{(\ell)})^{1/j} \leq H(\breve{W}_k^{(\ell')})^{1/k}.$$

(*b*) $\omega$ *satisfies the condition* $(\alpha_0)$ *(see* (1.11)*), i. e.*

$$\exists\, C \geq 1\; \exists\, t_0 \geq 0\; \forall\, \lambda \geq 1\; \forall\, t \geq t_0: \quad \omega(\lambda t) \leq C\lambda\omega(t).$$

**Proof.**

$(a) \Rightarrow (b)$ The property $(\mathcal{M}_{\{\text{rai}\}})$ is preserved under equivalence of matrices, then $\mathcal{M}_{\omega_{\boldsymbol{W}^{(\ell)}}}$ has $(\mathcal{M}_{\{\text{rai}\}})$ for some/any $l > 0$. By a result of G. Schindl [72, Thm. 4.5 $(iv) \Leftrightarrow (i)$], $\omega_{\boldsymbol{W}^{(\ell)}}$ satisfies the condition $(\alpha_0)$, and therefore $\omega$ satisfies it too, because $\omega \sim \omega_{\boldsymbol{W}^{(\ell)}}$ (see (1.21)) and the condition $(\alpha_0)$ is preserved under equivalence of weight functions.

$(b) \Rightarrow (a)$ If $\omega$ satisfies the condition $(\alpha_0)$, then $\omega_{\boldsymbol{W}^{(\ell)}}$ satisfies it too (arguing as before). By [72, Thm. 4.5 $(i) \Leftrightarrow (iv)$], the matrix $\mathcal{M}_{\omega_{\boldsymbol{W}^{(\ell)}}}$ has $(\mathcal{M}_{\{\text{rai}\}})$ for some/any $l > 0$. Finally, by [70, Lemma 5.3.1] the matrices $\mathcal{M}_{\omega_{\boldsymbol{W}^{(\ell)}}}$ and $\mathcal{M}_\omega$ are equivalent, and $(\mathcal{M}_{\{\text{rai}\}})$ is preserved under equivalence of matrices. □

We can provide now a statement about stability properties for classes associated to a weight function in small sectors.

**Theorem 2.4.2.** *Let $\omega \in \mathcal{W}$ be given with associated weight matrix $\mathcal{M}_\omega := \{\boldsymbol{W}^{(\ell)} : \ell > 0\}$ and let $0 < \alpha \leq 1$. Then the following are equivalent:*

- $(a)$ *The matrix $\mathcal{M}_\omega$ has $(\mathcal{M}_{\{\text{rai}\}})$.*
- $(b)$ *$\omega$ satisfies the condition $(\alpha_0)$ (see (1.11)).*
- $(c)$ *The class $\mathcal{A}_{\{\omega\}}(S_\alpha)$ is holomorphically closed.*
- $(d)$ *The class $\mathcal{A}_{\{\omega\}}(S_\alpha)$ is inverse-closed.*

*If $\omega$ has in addition $(\omega_2)$, then the list of equivalences can be extended by:*

- $(e)$ *The class $\mathcal{A}_{\{\omega\}}(S_\alpha)$ is closed under composition.*
- $(f)$ *The matrix $\mathcal{M}_\omega$ satisfies the property $(\mathcal{M}_{\{\text{FdB}\}})$.*

**Proof.** The equivalence $(a) \Leftrightarrow (b)$ is a consequence of the Lemma 2.4.1. Moreover, the equivalences $(a) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f)$ follow by applying Theorem 2.3.14 to $\mathcal{M} \equiv \mathcal{M}_\omega$. Let us observe that $\mathcal{M}^\alpha \equiv \mathcal{M}_\omega$, thanks to the fact that $\boldsymbol{W}^{(\ell)}$ is (lc) for all $\ell > 0$. Moreover, $\omega$ has $(\omega_1)$ and therefore $\mathcal{A}_{\{\omega\}}(S_\alpha) = \mathcal{A}_{\{\mathcal{M}_\omega\}}(S_\alpha)$, see (2.1). In addition, note that $\mathcal{M}_\omega$ has automatically $(\mathcal{M}_{\{\text{dc}\}})$ by (1.20).

□

**Remark 2.4.3.** With the same cautions as in Remark 2.3.17, related to the different nature (local versus global) of the imposed estimates, we mention that, when taking $\alpha = 0$ in the previous result, i. e., when the sector $S_\alpha$ "collapses" to the ray $(0, +\infty)$, then we partially get the main stability result [60, Thm. 3] for the ultradifferentiable class $\mathcal{E}_{\{\omega\}}((0, +\infty))$, see also [59, Thm. 6.3].

The next lemma will be necessary for stating a similar result for wide sectors.

**Lemma 2.4.4.** *Let $\omega \in \mathcal{W}_0$ be given with associated weight matrix $\mathcal{M}_\omega := \{\boldsymbol{W}^{(\ell)} : \ell > 0\}$. Suppose there exists $s > 0$ such that, for $\omega^s(t) := \omega(t^s)$, one has:*

*(i) $\omega^s(t) = o(t)$ as $t \to \infty$, (i. e., $\omega^s(t)$ has $(\omega_5)$.)*

*(ii) $\omega^s$ satisfies the condition $(\alpha_0)$, i. e., it is equivalent to a concave weight function.*

*Then there exists a weight matrix $\mathcal{U} = \{\mathbf{U}^{(\ell)} : \ell > 0\}$, $R$-equivalent to $\mathcal{M}_\omega$, and such that for each $\ell > 0$, the sequence $\overline{\boldsymbol{G}}^{-s}\mathbf{U}^{(\ell)}$ is equivalent to an (lc) sequence $\boldsymbol{L}^{(\ell)}$ depending on $s$.*

**Proof.** First, let us consider the matrix $\mathcal{M}_{\omega^s} := \{\mathbf{V}^{(\ell,s)} : \ell > 0\}$. There exists a relation between both matrices (see [36]), more precisely, for all $\ell > 0$ we have that $\mathbf{V}^{(\ell,s)} = (\boldsymbol{W}^{(\ell/s)})^{1/s}$. So, we can write

$$\boldsymbol{W}^{(\ell)} = (\mathbf{V}^{(\ell s,s)})^s = \boldsymbol{G}^s(\check{\mathbf{V}}^{(\ell s,s)})^s \qquad \ell > 0.$$

Now, by taking into account that $\omega^s$ satisfies the condition $(\alpha_0)$ and $(\omega_5)$ we deduce from [62, Prop 3] that the matrices $\widecheck{\mathcal{M}}_{\omega^s} := \{\check{\mathbf{V}}^{(\ell,s)} : \ell > 0\}$ and $\widecheck{\mathcal{M}}_{\omega^s}^{\mathrm{lc}}$ are $R$-equivalent. Finally, since taking the power $s$ in each sequence of these two matrices respects $R$-equivalence for the resulting matrices, we deduce that $\mathcal{U} := \{\boldsymbol{G}^s[(\check{\mathbf{V}}^{(\ell,s)})^{\mathrm{lc}}]^s : \ell > 0\}$ and $\mathcal{M}_\omega$ are $R$-equivalent. □

**Theorem 2.4.5.** *Let $\omega \in \mathcal{W}_0$ be given with associated weight matrix $\mathcal{M}_\omega := \{\boldsymbol{W}^{(\ell)} : \ell > 0\}$ and let $\alpha > 1$. Suppose there exists $s > \alpha - 1$ such that, for $\omega^s(t) := \omega(t^s)$, one has:*

*(i) $\omega^s(t) = o(t)$ as $t \to \infty$, (i.e $\omega^s(t)$ has $(\omega_5)$).*

*(ii) $\omega^s$ satisfies the condition $(\alpha_0)$, i. e., it is equivalent to a concave weight function.*

*Then the following are equivalent:*

*(a) The matrix $\mathcal{M}_\omega$ has $(\mathcal{M}_{\{\mathrm{rai}\}})$.*

*(b) $\omega$ satisfies the condition $(\alpha_0)$.*

*(c) The class $\mathcal{A}_{\{\omega\}}(S_\alpha)$ is holomorphically closed.*

*(d) The class $\mathcal{A}_{\{\omega\}}(S_\alpha)$ is inverse-closed.*

*If $\omega$ has in addition $(\omega_2)$, then the list of equivalences can be extended by:*

(e) *The class $\mathcal{A}_{\{\omega\}}(S_\alpha)$ is closed under composition.*

(f) *The matrix $\mathcal{M}_\omega$ satisfies the condition $(\mathcal{M}_{\{\mathrm{FdB}\}})$.*

**Proof.** The equivalence $(a) \Leftrightarrow (b)$ is a consequence of Lemma 2.4.1. Lemma 2.4.4 ensures that there exists a weight matrix $\mathcal{U} := \{\mathbf{U}^{(\ell)} : \ell > 0\}$, $R$-equivalent to $\mathcal{M}_\omega$ (and therefore $\mathcal{A}_{\{\mathcal{U}\}}(S_\alpha) = \mathcal{A}_{\{\mathcal{M}_\omega\}}(S_\alpha)$), such that for each $\ell > 0$ the sequence $\overline{\boldsymbol{G}}^{-s}\mathbf{U}^{(\ell)}$ is equivalent to an (lc) sequence $\boldsymbol{L}^{(\ell)}$ depending on $s$. Then, the equivalences $(a) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f)$ follow by applying Theorem 2.3.18 to $\mathcal{M} \equiv \mathcal{U}$, and taking $\alpha_\ell = s+1$. Finally, thanks to the fact that $\omega^s$ has $(\alpha_0)$, then $\omega$ satisfies $(\omega_1)$ and therefore $\mathcal{A}_{\{\omega\}}(S_\alpha) = \mathcal{A}_{\{\mathcal{M}_\omega\}}(S_\alpha)$, see (2.1). In addition, note that $\mathcal{M}_\omega$ has automatically $(\mathcal{M}_{\{\mathrm{dc}\}})$ by (1.20). And $(\omega_2)$ for $\omega$ implies that $\mathcal{M}_\omega$ has $(\mathcal{M}_{\mathcal{H}})$. Finally, the conditions $(\mathcal{M}_{\{\mathrm{dc}\}})$ and $(\mathcal{M}_{\mathcal{H}})$ are stable under $R$-equivalence, and therefore $\mathcal{U}$ satisfies both too. □

**Remark 2.4.6.** The hypotheses (i) and (ii) on $\omega$ in Theorem 2.4.5 can be quickly guaranteed by the condition $\gamma(\omega) > \alpha - 1$, in terms of the index described in Subsection 1.2.2. Note that, by choosing $s$ such that $\gamma(\omega) > s > \alpha - 1$, we have $\gamma(\omega^s) = \gamma(\omega)/s > 1$ (see property *(iii)* in that subsection), and this fact implies:

(a) By [34, Remark 2.15 $(i) \Rightarrow (v)$], we have property $(\omega_5)$ for $\omega^s$.

(b) By [34, Thm. 2.11 $(v) \Rightarrow (ii)$], we deduce that $\omega^s$ is equivalent to a concave weight function, and so $(\alpha_0)$ is satisfied by $\omega^s$.

**Remark 2.4.7.** In some situations it is straightforward that all the conditions on the weight function $\omega$ in the previous result are satisfied, and so all the statements (a) through (f) are equivalent. We comment on two special cases:

(i) If $2 > \alpha > 1$, suppose that $\omega(t) = O(t)$ as $t \to \infty$, (i.e $\omega(t)$ has $(\omega_2)$), and that there exists some $s > \alpha - 1$ such that $\omega^s$ satisfies the condition $(\alpha_0)$. Let us observe that we can take $s' < s$ such that $1 > s' > \alpha - 1$, and it is then easy to show that $\omega^{s'}$ satisfies the conditions $(\omega_5)$ and $(\alpha_0)$.

(ii) If $\alpha \geq 2$, suppose there exists $s$ according to the assumptions in the theorem. Then, we will have $s > 1$, and since $\omega^s$ satisfies the condition $(\omega_5)$, we can check immediately that $\omega$ has $(\omega_2)$.

## 2.5 Examples

In this section, we apply the previous results to some well-known examples of ultraholomorphic classes. Let us fix $\alpha > 0$.

### 2.5.1 Gevrey-related classes

Consider the sequence $\overline{\boldsymbol{G}}^{\beta} := (j^{j\beta})_{j\in\mathbb{N}_0}$ of index $\beta \in \mathbb{R}$. Note that this sequence has the (rai) property if and only if $\beta \geq 1$. We are going to study the stability of the class $\mathcal{A}_{\{\overline{\boldsymbol{G}}^{\beta}\}}(S_\alpha)$ in terms of the values of $\alpha$ and $\beta$. Let us distinguish some cases:

(a) Let $\alpha \in (0,1]$:

  (i) If $\beta < \alpha - 1$ then $\lim_{j\to+\infty}(j^{(1-\alpha)j}j^{j\beta})^{1/j} = 0$, and therefore the class is stable because it is trivial, i. e., it only contains constant functions (see Remark 2.3.5).

  (ii) If $\beta \in (\alpha-1, 1)$ Corollary 2.3.16, together with the fact that $\overline{\boldsymbol{G}}^{\beta}$ has not the (rai) property, ensure that the class is non stable.

  (iii) If $\beta = \alpha - 1$, the sequence $\boldsymbol{M}^{\alpha}$ is $\overline{\boldsymbol{G}}^{\beta}$, which does not satisfy (rai). So, by Remark 2.3.15 and Corollary 2.3.16 the class is not stable.

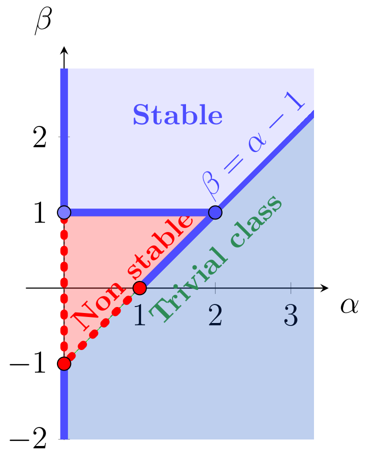
  (iv) If $\beta \geq 1$ we deduce from the Corollary 2.3.16 that the class is stable.

(b) Let $\alpha > 1$:

  (i) If $\beta \leq \alpha - 1$ then the $\liminf_{j\to+\infty}(j^{(1-\alpha)j}j^{j\beta})^{1/j} < \infty$, and therefore the class is stable because it only contains constant functions (see Remark 2.3.5).

  (ii) If $\beta > \alpha - 1$, we have stability provided that $\beta \geq 1$, thanks to the Corollary 2.3.22.

We include a graphic in order to see the stability (resp. non stability) regions:



We consider now a second example. Let us fix $\alpha > 1$, take some $\beta > \alpha$ and consider the weight matrix $\mathcal{L}^{(\beta)} = \{\overline{\boldsymbol{G}}^{\beta-\frac{1}{p+1}} : p > 0\}$. Note that the ultraholomorphic class associated with $\mathcal{L}^{(\beta)}$ is strictly smaller than the class associated with the constant matrix $\mathcal{G}^{\beta} = \{\overline{\boldsymbol{G}}^{\beta} : p > 0\}$. Under these assumptions, let us observe that $\overline{\boldsymbol{G}}^{\beta-\frac{1}{p+1}}$ is an (lc) sequence for all $p > 0$. Then Theorem 2.3.18 guarantees that the class $\mathcal{A}_{\{\mathcal{L}^{(\beta)}\}}(S_\alpha)$ is stable, thanks to the fact that $\beta - \frac{1}{p+1} > 1$ for large $p$, and we can ensure that the corresponding matrix has $(\mathcal{M}_{\{\mathrm{rai}\}})$.

### 2.5.2 q-Gevrey case

In this subsection, we will work, for $q > 1$, with the *q-Gevrey sequence*, i.e $\boldsymbol{M}_q = (q^{j^2})_{j\geq 0}$. First, thanks to the fact that the sequence $\boldsymbol{M}_q$ has (lc) and (dc), and moreover $\widecheck{M}_q$ is also (lc), we can easily prove the stability properties for the class $\mathcal{A}_{\{\boldsymbol{M}_q\}}(S_\alpha)$. For $\alpha \in (0, 1]$, Corollary 2.3.16 ensures that the class $\mathcal{A}_{\{\boldsymbol{M}_q\}}(S_\alpha)$ is stable. On the other hand, for $\alpha > 1$ and for any $\beta > \alpha$ the sequence $\overline{\boldsymbol{G}}^{1-\beta}\boldsymbol{M}_q$ is equivalent to an (lc) sequence, because the gamma index of $\boldsymbol{M}_q$ is infinity. So, Corollary 2.3.22 again ensures the stability.

Now, we want to study the stability properties for the class $\mathcal{A}_{\{\omega_{\boldsymbol{M}_q}\}}(S_\alpha)$. For this purpose, let us observe that we can estimate the normalized weight function $\omega_{\boldsymbol{M}_q}$,

$$\omega_{\boldsymbol{M}_q}(t) = \sup_{j\in\mathbb{N}_0} \ln\left(\frac{t^j}{q^{j^2}}\right) = \sup_{j\in\mathbb{N}_0}(j\ln(t) - j^2\ln(q)), \quad t > 1.$$

Obviously, $\omega_{\boldsymbol{M}_q}(t)$ is bounded above by the supremum of $x\ln(t) - x^2\ln(q)$ when $x$ runs over $(0, \infty)$, which is easily obtained by elementary calculus and occurs at the point

$$\left(\frac{\ln(t)}{2\ln(q)}, \frac{\ln^2(t)}{4\ln(q)}\right).$$

In particular, it is easy to check that $\omega(t) := \ln^2(t)/(4\ln(q))$ verifies (after normalization in the interval $[0, 1]$) that $\omega \in \mathcal{W}$, $\omega$ has $(\omega_5)$ (and therefore $(\omega_2)$) and $\omega \sim \omega_{\boldsymbol{M}_q}$, so the corresponding matrices $\mathcal{M}_\omega$ and $\mathcal{M}_{\omega_{\boldsymbol{M}_q}}$ are $R$-equivalent. In order to compute the matrix associated with $\omega$, the Legendre-Fenchel-Young-conjugate of $\varphi_\omega$ is

$$\varphi^*_\omega(x) := \sup_{y\geq 0}\{xy - \omega(\exp(y))\} = x^2\ln(q) = \ln(q^{x^2}), \qquad x \geq 0.$$

So, we have that

$$W^{(\ell)}_j = \exp(\frac{1}{\ell}\varphi^*_\omega(\ell j)) = q^{\ell j^2}, \qquad j \geq 0,$$

and therefore
$$\boldsymbol{W}^{(\ell)} = (q^{\ell j^2})_{j\geq 0}, \qquad \ell > 0.$$
Note that each sequence $\boldsymbol{W}^{(\ell)}$ is (lc), (dc) and has the property (rai) for all $\ell > 0$, in this situation Theorem 2.4.2 ensures that the class $\mathcal{A}_{\{\omega\}}(S_\alpha)$ (resp. $\mathcal{A}_{\{\omega_{\boldsymbol{M}_q}\}}(S_\alpha)$) is stable for $\alpha \in (0, 1]$. On the other hand, note that $\gamma(\omega) = \infty$, since $\gamma(\omega) \geq \gamma(\boldsymbol{W}^{(\ell)})$ for all $\ell > 0$ (see Subsection 1.2.2) and $\gamma(\boldsymbol{W}^{(\ell)})$ is also infinity. In this case, Remark 2.4.6 ensures that we can apply Theorem 2.4.5 in order to deduce that the class $\mathcal{A}_{\{\omega\}}(S_\alpha)$ (resp. $\mathcal{A}_{\{\omega_{\boldsymbol{M}_q}\}}(S_\alpha)$) is stable for $\alpha > 1$.

## 2.6 The Beurling case

We turn now our attention to the Beurling-like ultraholomorphic classes, and try to obtain similar results. However, due to the lack of characteristic functions in wide sectors in this situation, we have been able to reason only for sectors contained in a half-plane.

### 2.6.1 Beurling ultraholomorphic classes

We introduce now the classes under consideration in this section analogously as we did for the Roumieu setting in Section 2.1; these Beurling type spaces have been already considered by J. Jiménez-Garrido, J. Sanz and G. Schindl [36, Sect. 2.5], A. Debrouwere [15] and A. Rainer and G. Schindl[50].

**Definition 2.6.1.** Let $\boldsymbol{M}$ be a sequence of positive real numbers and $S \subseteq \mathcal{R}$ an unbounded sector. We define the Denjoy-Carleman ultraholomorphic class of Beurling type associated with $\boldsymbol{M}$ in the sector $S$, denoted by $\mathcal{A}_{(\boldsymbol{M})}(S)$, as
$$\mathcal{A}_{(\boldsymbol{M})}(S) := \bigcap_{h>0} \mathcal{A}_{\boldsymbol{M},h}(S).$$
It has a natural structure of Fréchet space.

As it occurs in the Roumieu case, it is straightforward from the definition that $\boldsymbol{M} \approx \boldsymbol{L}$ implies $\mathcal{A}_{(\boldsymbol{M})}(S) = \mathcal{A}_{(\boldsymbol{L})}(S)$ (as locally convex vector spaces) for any sector $S$.

Now, we define ultraholomorphic classes of Beurling type defined by a weight matrix $\mathcal{M}$ analogously as the ultradifferentiable counterparts introduced in [70, Sect. 7] and also in [59, Sect. 4.2].

**Definition 2.6.2.** Let $\mathcal{M} = \{\boldsymbol{M}^{(\alpha)} \in \mathbb{R}_{>0}^{\mathbb{N}_0} : \alpha > 0\}$ be a weight matrix and $S$ be an unbounded sector. The ultraholomorphic class of Beurling type associated with $\mathcal{M}$ in $S$, denoted by $\mathcal{A}_{(\mathcal{M})}(S)$, is

$$\mathcal{A}_{(\mathcal{M})}(S) := \bigcap_{\alpha>0} \mathcal{A}_{(\boldsymbol{M}^{(\alpha)})}(S).$$

For the Beurling context, we define a different notion of equivalence between weight matrices.

**Definition 2.6.3.** Let $\mathcal{M} = \{\boldsymbol{M}^{(\alpha)} : \alpha > 0\}$ and $\mathcal{L} = \{\boldsymbol{L}^{(\alpha)} : \alpha > 0\}$ be given. We write $\boldsymbol{M}(\preceq)\boldsymbol{L}$ if

$$\forall\, \alpha > 0\; \exists\, \beta > 0 : \quad \boldsymbol{M}^{(\beta)} \preceq \boldsymbol{L}^{(\alpha)},$$

and call $\mathcal{M}$ and $\mathcal{L}$ B-equivalent, if $\mathcal{M}(\preceq)\mathcal{L}$ and $\mathcal{L}(\preceq)\mathcal{M}$ (B stands for Beurling).

By definition, B-equivalent weight matrices yield (as locally convex vector spaces) the same function classes of Beurling type on each sector $S$.

Similarly as for the ultradifferentiable case, we now define ultraholomorphic classes of Beurling type associated with a weight function $\omega \in \mathcal{W}_0$.

**Definition 2.6.4.** Let $\omega$ be a weight function in $\mathcal{W}_0$, the ultraholomorphic class of Beurling type associated with $\omega$ in the sector $S$, denoted by $\mathcal{A}_{(\omega)}(S)$, is

$$\mathcal{A}_{(\omega)}(S) := \bigcap_{\ell>0} \mathcal{A}_{\omega,\ell}(S).$$

It is again a Fréchet space.

Of course, equivalent weight functions provide equal associated ultraholomorphic classes of Beurling type.

Moreover, let $\omega \in \mathcal{W}$ be given and let $\mathcal{M}_\omega$ be the associated weight matrix defined in Subsection 1.3.1. Then, analogously as (2.1) we get that

$$\mathcal{A}_{(\omega)}(S) = \mathcal{A}_{(\mathcal{M}_\omega)}(S) \tag{2.18}$$

holds as locally convex vector spaces. This equality is an easy consequence of (1.22) and the way the seminorms are defined in these spaces.

On the other hand, by $(v)$ in Remark 1.3.10 we get the following result, which is analogous to Lemma 2.1.6.

**Lemma 2.6.5.** *Let $\omega \in \mathcal{W}$ be given and assume that $\omega$ has $(\omega_6)$. Then, for all sectors $S$ we get that*

$$\forall\, \ell > 0 : \quad \mathcal{A}_{(\omega)}(S) = \mathcal{A}_{(\boldsymbol{W}^{(\ell)})}(S),$$

*as locally convex vector spaces.*

Finally, as in the Roumieu case, if $f$ belongs to any of such classes, we may define the complex numbers

$$f^{(j)}(0) := \lim_{z\in S, z\to 0} f^{(j)}(z), \quad j \in \mathbb{N}_0. \tag{2.19}$$

### 2.6.2 Stability properties for Beurling ultraholomorphic classes defined by weight matrices

The aim of this section is to transfer the stability results from Section 2.3 to the Beurling setting. In order to proceed, analogously to the Roumieu setting, we begin by introducing some auxiliary spaces, and defining the stability properties under study. First, we recall the definition 2.3.1, and we adapt it to this new framework.

**Definition 2.6.6.** Let $\boldsymbol{M}$ be a sequence of positive real numbers and $U \subseteq \mathbb{C}$ be an open set. Given a compact set $K \subset U$, we put

$$\mathcal{H}_{(\boldsymbol{M})}(K) := \bigcap_{h>0} \mathcal{H}_{\boldsymbol{M},h}(K).$$

Moreover, given a weight matrix $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$, we may introduce the class $\mathcal{H}_{(\mathcal{M})}(U)$ as

$$\mathcal{H}_{(\mathcal{M})}(U) := \bigcap_{K \subset U} \bigcap_{p>0} \mathcal{H}_{(\boldsymbol{M}^{(p)})}(K).$$

We continue with the analogue of the definition 2.3.2.

**Definition 2.6.7.** Let $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$ be a weight matrix and $\alpha > 0$. The class $\mathcal{A}_{(\mathcal{M})}(S_\alpha)$ is said to be:

($i$) *closed under (composition with) Beurling-analytic functions,* if for all functions $f \in \mathcal{A}_{(\mathcal{M})}(S_\alpha)$ and every $g \in \mathcal{H}_{(\mathcal{G}^1)}(U)$, where $U$ is an open set containing the closure of the range of $f$, we have $g \circ f \in \mathcal{A}_{(\mathcal{M})}(S_\alpha)$. We recall that $\mathcal{G}^1 = \{\overline{\boldsymbol{G}} : p > 0\}$.

($ii$) *inverse-closed,* if for all $f \in \mathcal{A}_{(\mathcal{M})}(S_\alpha)$ such that $\inf_{z \in S_\alpha} |f(z)| > 0$, we have $1/f \in \mathcal{A}_{(\mathcal{M})}(S_\alpha)$.

($iii$) *closed under composition,* if for all $f \in \mathcal{A}_{(\mathcal{M})}(S_\alpha)$ and for all $g \in \mathcal{H}_{(\mathcal{M})}(U)$, where $U \subseteq \mathbb{C}$ is an open set containing the closure of the range of $f$, we have $g \circ f \in \mathcal{A}_{(\mathcal{M})}(S_\alpha)$.

**Remark 2.6.8.** As occurs in the Roumieu case, see Remark 2.3.3, we cannot relax the condition $\inf_{z \in S_\alpha} |f(z)| > 0$ in the definition of inverse-closedness. Note that the function $z \mapsto \exp(-1/z)$ belongs to the class $\mathcal{A}_{(\boldsymbol{G}^\beta)}(S_\alpha)$ for every $\alpha \in (0,1)$ and $\beta > 2$ (as a consequence of Cauchy's integral formula for the derivatives) and never vanishes in $S_\alpha$. However, its multiplicative inverse $z \mapsto \exp(1/z)$ is not bounded, and hence it does not belong to any of the Beurling ultraholomorphic classes under consideration.

Also, the open set $U$ in (i) and (iii) has to contain the closure of the range of $f$, and not just the range. This is clearly seen in the forthcoming arguments involving the function $z \mapsto 1/z$, whose derivatives admit global Beurling-analytic bounds in closed subsets of $\mathbb{C} \setminus \{0\}$, but not in the whole of it.

We will consider classes in sectors $S_\alpha$ contained in a half-plane and defined by a weight matrix $\mathcal{M}$. As it occurs in the Roumieu case, we are going to prove that the weight matrix $\mathcal{M}^\alpha$ (see definition 2.3.4) induces the same Beurling class as the original matrix.

**Remark 2.6.9.** For all $p' \geq p$ we have $\boldsymbol{M}^{(p)} \leq \boldsymbol{M}^{(p')}$ and so $\mathcal{A}_{(\boldsymbol{M}^{(p)})}(S_\alpha) \subseteq \mathcal{A}_{(\boldsymbol{M}^{(p')})}(S_\alpha)$. Moreover, $\mathcal{A}_{(\mathcal{M})}(S_\alpha)$ is the intersection of all classes $\mathcal{A}_{(\boldsymbol{M}^{(p)})}(S_\alpha)$. Therefore, any (small) index is relevant and we cannot consider here the situation described in Remark 2.3.5.

Next, we state the Beurling variant of Theorem 2.3.7. Although the idea of its proof is the same but by taking into account the Beurling type estimates, we include it for the sake of completeness.

**Theorem 2.6.10.** *Let $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$ be a weight matrix and $0 < \alpha \leq 1$ be given such that $\lim_{j\to+\infty}(j^{(1-\alpha)j} M_j^{(p)})^{1/j} = \infty$ for all $p > 0$. Let $\mathcal{M}^\alpha = \{\boldsymbol{M}^{(p,\alpha)} : p > 0\}$ be the matrix given in (2.7). Then, we have that*

$$\mathcal{A}_{(\mathcal{M})}(S_\alpha) = \mathcal{A}_{(\mathcal{M}^\alpha)}(S_\alpha).$$

*Proof.* Since $\overline{\boldsymbol{G}}^{1-\alpha}\boldsymbol{M}^{(p,\alpha)}$ is the log convex minorant of $\overline{\boldsymbol{G}}^{1-\alpha}\boldsymbol{M}^{(p)}$, we have that $\overline{\boldsymbol{G}}^{1-\alpha}\boldsymbol{M}^{(p,\alpha)} \leq \overline{\boldsymbol{G}}^{1-\alpha}\boldsymbol{M}^{(p)}$, and therefore $\boldsymbol{M}^{(p,\alpha)} \leq \boldsymbol{M}^{(p)}$ for all $p > 0$. Consequently, by the definition of the classes, we also get $\mathcal{A}_{(\mathcal{M}^\alpha)}(S_\alpha) \subseteq \mathcal{A}_{(\mathcal{M})}(S_\alpha)$.

For the converse inclusion, let us consider $f \in \mathcal{A}_{(\mathcal{M})}(S_\alpha)$. Then for all $p > 0$ and $h > 0$ there exist some $D = D(p,h) \in \mathbb{R}_{>0}$ (large) such that $C_n(f) := \sup_{z\in S_\alpha} |f^{(n)}(z)| \leq Dh^n M_n^{(p)}$, for all $n \in \mathbb{N}_0$.

Let us fix now an arbitrary index $p$ and $h > 0$. Consider $n \in \mathbb{N}_0$ and distinguish two cases:

i) If $M_n^{(p,\alpha)} = M_n^{(p)}$ then $\sup_{z\in S_\alpha} |f^{(n)}(z)| \leq Dh^n M_n^{(p,\alpha)}$.

ii) If not, there exist principal indices $n_1, n_2 \in \mathbb{N}_0$, with $n_1 < n < n_2$, such that $M_{n_i}^{(p,\alpha)} = M_{n_i}^{(p)}$ for $i = 1, 2$. Note that these indices may also depend on the sequence, i. e. on the given but fixed index $p$. So, we have

$$\begin{aligned}\ln(n^{(1-\alpha)n}M_n^{(p,\alpha)}) &= \frac{n_2-n}{n_2-n_1}\ln(n_1^{(1-\alpha)n_1}M_{n_1}^{(p,\alpha)}) + \frac{n-n_1}{n_2-n_1}\ln(n_2^{(1-\alpha)n_2}M_{n_2}^{(p,\alpha)})\\ &\geq \frac{n_2-n}{n_2-n_1}\ln\left(\frac{1}{Dh^{n_1}}n_1^{(1-\alpha)n_1}C_{n_1}(f)\right)\\ &+\frac{n-n_1}{n_2-n_1}\ln\left(\frac{1}{Dh^{n_2}}n_2^{(1-\alpha)n_2}C_{n_2}(f)\right).\end{aligned}$$

Therefore, with the notation of Theorem 2.3.6, we deduce from above:

$$B_{n_1}^{\frac{n_2-n}{n_2-n_1}}B_{n_2}^{\frac{n-n_1}{n_2-n_1}} \leq (Dh^{n_1})^{\frac{n_2-n}{n_2-n_1}}(Dj^{n_2})^{\frac{n-n_1}{n_2-n_1}}n^{(1-\alpha)n}M_n^{(p,\alpha)} = Dh^n n^{(1-\alpha)n}M_n^{(p,\alpha)}.$$

Now, from the previous estimate and by applying Theorem 2.3.6, there exist some $A, q > 0$ only depending on the opening $\alpha$, such that

$$C_n(f) \leq n^{(\alpha-1)n}Aq^{(1-\alpha)n}B_{n_1}^{\frac{n_2-n}{n_2-n_1}}B_{n_2}^{\frac{n-n_1}{n_2-n_1}} \leq AD(q^{(1-\alpha)}h)^n M_n^{(p,\alpha)}. \tag{2.20}$$

Since $q$ is only depending on the opening $\alpha$ and since the above choice for the principal indices is only depending on the sequence/index but not on $h$, as $h \to 0$ we conclude that $f \in \mathcal{A}_{(\boldsymbol{M}^{(p,\alpha)})}(S_\alpha)$. Finally, since $p$ was arbitrary and (2.20) holds then for any index $p$ we have verified $f \in \mathcal{A}_{(\mathcal{M}^\alpha)}(S_\alpha)$. □

As in the Roumieu case, we need to establish a suitable condition ensuring the equality of the classes $\mathcal{H}_{(\mathcal{M})}(U)$ and $\mathcal{H}_{(\mathcal{M}^\alpha)}(U)$ for any open set $U$. Let us start with some preliminary results in the ultradifferentiable framework.

**Definition 2.6.11.** Let $\boldsymbol{M} \in \mathbb{R}_{>0}^{\mathbb{N}_0}$ be a sequence and $U \subseteq \mathbb{C}$ be an open set. Given a compact set $K \subset U$, we define

$$\mathcal{E}_{(\boldsymbol{M})}(K) := \bigcap_{h>0}\mathcal{E}_{\boldsymbol{M},h}(K).$$

Moreover, given a weight matrix $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$, we may introduce the Denjoy-Carleman class of Beurling type $\mathcal{E}_{(\mathcal{M})}(U)$ as

$$\mathcal{E}_{(\mathcal{M})}(U) := \bigcap_{K\subset U}\bigcap_{p>0}\mathcal{E}_{(\boldsymbol{M}^{(p)})}(K).$$

We consider now a Beurling-type condition for the matrix $\mathcal{M}$.

**Definition 2.6.12.** We say that $\mathcal{M}$ has the property $(\mathcal{M}_{(\mathbf{G})})$ (of Beurling type) if for all $p, A > 0$ there is $B > 0$ ($B$ may depend on $p$ and $A$) such that

$$B\frac{M_k^{(p)}}{M_j^{(p)}} \geq A^{k-j}(k-j)! \qquad \text{whenever } k \geq j.$$

We note that the previous condition guarantees the possibility of constructing the matrix $\mathcal{M}^\alpha$ for $\alpha \in (0,1]$.

**Proposition 2.6.13.** *Let $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$ be a weight matrix which has $(\mathcal{M}_{(\mathbf{G})})$. Fix $\alpha \in (0,1]$ and consider the matrix $\mathcal{M}^\alpha = \{\boldsymbol{M}^{(p,\alpha)} : p > 0\}$ given in* (2.7)*. Then, for each open $U \subseteq \mathbb{C}$ we have that*

$$\mathcal{E}_{(\mathcal{M})}(U) = \mathcal{E}_{(\mathcal{M}^\alpha)}(U).$$

*Proof.* We have to show the inclusion $\mathcal{E}_{(\mathcal{M})}(U) \subseteq \mathcal{E}_{(\mathcal{M}^\alpha)}(U)$. By reasoning as in proof of Corollary 2.3.12, it suffices to prove the equality in the one-dimensional situation. Fix $p > 0$, and assume $I$ is an open interval in $\mathbb{R}$. Let $f \in \mathcal{E}_{(\boldsymbol{M}^{(p)})}(I)$. Then for each compact interval $J \subset I$ and each $\rho > 0$ there is $C > 0$ such that

$$\|f^{(k)}\|_J \leq C\rho^k M_k^{(p)}, \qquad k \in \mathbb{N}_0.$$

Let $\delta = \operatorname{dist}(J, \mathbb{R} \setminus I)$. Let $(k_n)$ be the sequence from the proof of Lemma 2.3.11. Thanks to $(\mathcal{M}_{(\mathbf{G})})$ (for $A := 1/(\rho\delta)$), there is $B \geq 1$ such that

$$B\rho^{k_n+1} M_{k_n+1}^{(p)} \geq \rho^{k_n} M_{k_n}^{(p)} \frac{(k_{n+1} - k_n)!}{\delta^{k_{n+1}-k_n}}.$$

Then the proof of Lemma 2.3.11 implies that $f \in \mathcal{E}_{(\boldsymbol{M}^{(p,\alpha)})}(I)$, and we are done. □

Finally, we deduce the proposed equality as before.

**Corollary 2.6.14.** *Let $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$ be a weight matrix satisfying $(\mathcal{M}_{(\mathbf{G})})$. Fix $\alpha \in (0,1]$ and onsider $\mathcal{M}^\alpha = \{\boldsymbol{M}^{(p,\alpha)} : p > 0\}$ given in* (2.7)*. Then, for each open $U \subseteq \mathbb{C}$ we have that*

$$\mathcal{H}_{(\mathcal{M})}(U) = \mathcal{H}_{(\mathcal{M}^\alpha)}(U).$$

### 2.6.3 On m-convexity for Beurling ultraholomorphic classes

According to the definitions given in Section 2.6.1, the topology of $\mathcal{A}_{(\mathcal{M})}(S)$ is given by the family of seminorms $\{\|\cdot\|_{\boldsymbol{M}^{(p)},h} : p, h > 0\}$ (we may only consider the values $p = h = \frac{1}{n}$ with $n \in \mathbb{N}$ arbitrary), which make it a Fréchet space. In our regards, it is interesting to have a structure of algebra.

**Lemma 2.6.15.** *Let $\mathcal{M}$ be a log-convex weight matrix, i. e., such that $(\mathcal{M}_{\mathrm{lc}})$ is valid. Then for any sector $S$ the space $\mathcal{A}_{(\mathcal{M})}(S)$ is a commutative Fréchet algebra with respect to the point-wise multiplication of functions.*

*Proof.* Since each $\boldsymbol{M}^{(\alpha)}$ is a log-convex sequence we get

$$\forall\, \alpha > 0\, \forall\, j,k \in \mathbb{N}_0 : \quad M_j^{(\alpha)} M_k^{(\alpha)} \le M_{j+k}^{(\alpha)}.$$

Combining this estimate with Leibniz's product rule yields closedness under pointwise multiplication and that multiplication is continuous. □

**Remark 2.6.16.** Note that for $\mathcal{A}_{(\mathcal{M})}(S)$ being a commutative Fréchet-algebra it suffices to assume for the matrix $\mathcal{M}$ that

$$\forall\, \alpha > 0\, \exists\, \beta > 0\, \exists\, C \ge 1\, \forall\, j,k \in \mathbb{N}_0 : \quad M_j^{(\beta)} M_k^{(\beta)} \le C^{j+k} M_{j+k}^{(\alpha)}.$$

In order to formulate and prove the main result in this section, first we have to recall some crucial abstract results by W. Żelazko [85].

**Definition 2.6.17.** ([85, Def. 7.7, Def. 9.1 and Def. 10.1]) We recall:

(∗) A topological algebra $\mathcal{B}$ in which the set of invertible elements is open is called a *Q-algebra.*

(∗) A Fréchet algebra $\mathcal{B}$ is called *multiplicatively convex*, or *m-convex* for short, if there exists an equivalent system of (countably many) seminorms $\{\|\cdot\|_i : i \in \mathbb{N}\}$ satisfying the submultiplicativity condition

$$\forall\, i \in \mathbb{N}\, \forall\, x,y \in \mathcal{B} : \quad \|xy\|_i \le \|x\|_i \|y\|_i;$$

see [85, (9.6.1)].

**Theorem 2.6.18.** *[85, Thm. 13.17] Let $\mathcal{B}$ be a commutative Fréchet algebra and assume that it is also a Q-algebra. Then $\mathcal{B}$ is m-convex.*

In order to characterize the stability properties of Beurling ultraholomorphic classes, we need to adapt the root almost increasing property.

**Definition 2.6.19.** We say that a weight matrix $\mathcal{M}$ has the *root almost increasing property* of Beurling type, denoted by $(\mathcal{M}_{(\mathrm{rai})})$, if

$$\forall\, \alpha > 0\, \exists\, H > 0\, \exists\, \beta > 0\, \forall\, 1 \le j \le k : \quad (\breve{M}_j^{(\beta)})^{1/j} \le H (\breve{M}_k^{(\alpha)})^{1/k}.$$

Under suitable conditions, stability properties for the Beurling ultraholomorphic class guarantee the root almost increasing property of Beurling type for the associated weight matrix.

**Theorem 2.6.20.** *Let $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$ be a log-convex weight matrix, and $0 < \alpha \le 1$ be given. Consider the following assertions:*

(*a*) $\mathcal{A}_{(\mathcal{M})}(S_\alpha)$ *is inverse-closed.*

(*b*) $\mathcal{A}_{(\mathcal{M})}(S_\alpha)$ *is a locally m-convex Fréchet-algebra (w.r.t. the point-wise multiplication of functions).*

(*c*) *The matrix* $\mathcal{M}$ *satisfies the property* $(\mathcal{M}_{(\mathrm{rai})})$.

*Then* (*a*) $\Rightarrow$ (*b*) *is valid. Moreover, if in addition* $\lim_{j\to\infty}(M_j^{(p)})^{1/j}=\infty$ *for all* $p>0$, *then* (*b*) $\Rightarrow$ (*c*) *is valid, too.*

*Proof.* We follow the ideas and techniques given in a paper of J. Bruna [8, Thm. 5.2]; see also the works of A. Rainer and G. Schindl [60, Lemma 3] and [59, Thm. 4.11 (2) $\Rightarrow$ (3)].

(*a*) $\Rightarrow$ (*b*) Since $\mathcal{M}$ is log-convex, by Lemma 2.6.15 we have that $\mathcal{A}_{(\mathcal{M})}(S_\alpha)$ is a commutative Fréchet algebra, and we show now that it is also a $Q$-algebra.

By assumption, those functions in $\mathcal{A}_{(\mathcal{M})}(S_\alpha)$ whose modulus is bounded away from 0 uniformly on $S_\alpha$ are precisely the invertible elements in $\mathcal{A}_{(\mathcal{M})}(S_\alpha)$. If $f\in\mathcal{A}_{(\mathcal{M})}(S_\alpha)$ is given with $|f(z)|\geq c>0$ for some $c>0$ and all $z\in S_\alpha$, consider $h>0$ and a function $g\in\mathcal{A}_{(\mathcal{M})}(S_\alpha)$ such that $\|f-g\|_{\boldsymbol{M}^{(p)},h}<c/2$ for some $p>0$. Then, $\sup_{z\in S_\alpha}|f(z)-g(z)|<c/2$, and we necessarily get $|g(z)|\geq c/2>0$ for all $z\in S_\alpha$ and so $1/g\in\mathcal{A}_{(\mathcal{M})}(S_\alpha)$. Consequently, the set $\{f\in\mathcal{A}_{(\mathcal{M})}(S_\alpha):1/f\in\mathcal{A}_{(\mathcal{M})}(S_\alpha)\}$ is open (in the topology generated by the seminorms $\{\|\cdot\|_{\boldsymbol{M}^{(p)},h}:p,h>0\}$).

Thus, we get m-convexity by Theorem 2.6.18.

(*b*) $\Rightarrow$ (*c*) By m-convexity we get that the canonical system of seminorms $\{\|\cdot\|_{\boldsymbol{M}^{(p)},h}:p,h>0\})$ is equivalent to a submultiplicative system $\{\|\cdot\|_i:i\in\mathbb{N}\}$ and this implies the following crucial estimate:

$$\begin{aligned}&\forall\,p,h>0\;\exists\,C,D\geq 1\;\exists\,i_0\in\mathbb{N}\;\exists\,p_1,h_1>0\;\forall\,m\in\mathbb{N}\;\forall\,f\in\mathcal{A}_{(\mathcal{M})}(S_\alpha):\\&\|f^m\|_{\boldsymbol{M}^{(p)},h}\leq C\|f^m\|_{i_0}\leq C(\|f\|_{i_0})^m\leq CD^m(\|f\|_{\boldsymbol{M}^{(p_1)},h_1})^m.\end{aligned}$$

The aim is to apply this estimate to a "convenient" one-parameter family of functions $f_t\in\mathcal{A}_{(\mathcal{M})}(S_\alpha)$, for every $t\geq 0$. Then let us set $f_t(z):=e^{-tz}$ for $t\geq 0$ and $z\in S_\alpha$. We get $f_t^{(j)}(z)=(-t)^je^{-tz}$ for all $j\in\mathbb{N}_0$ and $|e^{-tz}|=e^{-t\Re(z)}\leq 1$ for all $z\in S_\alpha$. Since, by assumption, $\lim_{j\to\infty}(M_j^{(p)})^{1/j}=\infty$ for any $p>0$, we get

$$\forall\,t\geq 0\;\forall\,h>0\;\exists\,C\geq 1\;\forall\,j\in\mathbb{N}_0\;\forall\,z\in S_\alpha:\quad |f_t^{(j)}(z)|\leq t^j\leq Ch^jM_j^{(p)},$$

which proves $f_t\in\mathcal{A}_{(\mathcal{M})}(S_\alpha)$ for any $t\geq 0$. Now fix $p>0$, choose above for simplicity $h:=1$ and let $C,D\geq 1$ and $p_1,h_1>0$ be the parameters depending on the index $p>0$ and $h=1$. Then $(f_t^m)^{(j)}(z)=(-tm)^je^{-tmz}$ for all $j\in\mathbb{N}_0$, $m\in\mathbb{N}$, $z\in S_\alpha$ and $t\geq 0$ and therefore there exist $C,D\geq 1$ and $p_1,h_1>0$ such that

$$\forall\,m\in\mathbb{N}\;\forall\,t\geq 0:\qquad \sup_{z\in S_\alpha,j\in\mathbb{N}_0}\frac{(tm)^je^{-tm\Re(z)}}{M_j^{(p)}}\leq CD^m\left(\sup_{z\in S_\alpha,j\in\mathbb{N}_0}\frac{t^je^{-t\Re(z)}}{M_j^{(p_1)}h_1^j}\right)^m.$$

It follows from above, that

$$\sup_{z\in S_\alpha, j\in\mathbb{N}_0} \frac{t^j e^{-t\Re(z)}}{M_j^{(p_1)} h_1^j} = \sup_{j\in\mathbb{N}_0} \frac{t^j}{M_j^{(p_1)} h_1^j}, \quad \text{and} \quad \sup_{z\in S_\alpha, j\in\mathbb{N}_0} \frac{(tm)^j e^{-tm\Re(z)}}{M_j^{(p)}} = \sup_{j\in\mathbb{N}_0} \frac{(tm)^j}{M_j^{(p)}},$$

because $\sup_{z\in S_\alpha} |e^{-tz}| = \sup_{z\in S_\alpha} e^{-t\Re(z)} = 1$. Thus the definition (1.2.12) provides that there exist $C, D \geq 1$ and $p_1, h_1 > 0$ such that

$$\forall\, m \in \mathbb{N}\ \forall\, t \geq 0 : \quad \exp(\omega_{\boldsymbol{M}^{(p)}}(mt) \leq CD^m \exp(m\omega_{\boldsymbol{M}^{(p_1)}}(t/h_1)). \tag{2.21}$$

Now let $1 \leq j \leq k$ and first we assume that $k = \ell j$ for some $\ell \in \mathbb{N}$. Then applying (2.21) to $m := \ell$ gives

$$\begin{aligned} \forall\, t \geq 0 : \quad \exp(k^{-1}\omega_{\boldsymbol{M}^{(p)}}(t)) &= \exp(k^{-1}\omega_{\boldsymbol{M}^{(p)}}(\ell t/\ell)) \\ &\leq C^{1/k} D^{1/j} \exp(j^{-1}\omega_{\boldsymbol{M}^{(p_1)}}(t/(\ell h_1)). \end{aligned} \tag{2.22}$$

Set $D_1 := CD(\geq 1)$ and by combining now (2.22) with (1.14) and recalling the assumption that each sequence is log-convex, we get

$$\begin{aligned} (M_k^{(p)})^{1/k} &= \sup_{t\geq 0} \frac{t}{\exp(k^{-1}\omega_{\boldsymbol{M}^{(p)}}(t))} \geq \frac{1}{D_1} \sup_{t\geq 0} \frac{t}{\exp(j^{-1}\omega_{\boldsymbol{M}^{(p_1)}}(t/(\ell h_1)))} \\ &= \frac{1}{D_1} \sup_{s\geq 0} \frac{(s\ell h_1)}{\exp(j^{-1}\omega_{\boldsymbol{M}^{(p_1)}}(s))} = \frac{\ell h_1}{D_1} \left( \sup_{s\geq 0} \frac{s^j}{\exp(\omega_{\boldsymbol{M}^{(p_1)}}(s))} \right)^{1/j} \\ &= \frac{\ell h_1}{D_1} (M_j^{(p_1)})^{1/j}. \end{aligned}$$

Moreover, by (1.3) we continue the estimate as follows:

$$\begin{aligned} (\widecheck{M}_k^{(p)})^{1/k} &= \left( \frac{M_k^{(p)}}{k!} \right)^{1/k} \geq \frac{(M_k^{(p)})^{1/k}}{k} \geq \frac{\ell h_1}{k D_1} (M_j^{(p_1)})^{1/j} \\ &= \frac{\ell h_1}{k D_1} j!^{1/j} (\widecheck{M}_j^{(p_1)})^{1/j} \geq \frac{\ell j h_1}{k e D_1} (\widecheck{M}_j^{(p_1)})^{1/j} = \frac{h_1}{e D_1} (\widecheck{M}_j^{(p_1)})^{1/j}. \end{aligned}$$

So far we have verified $(\mathcal{M}_{(\mathrm{rai})})$ between the sequences $\boldsymbol{M}^{(p)}$ and $\boldsymbol{M}^{(p_1)}$ with $H := eD_1/h_1$ and for all $1 \leq j \leq k$ such that $k = \ell j$ for some $\ell \in \mathbb{N}$.

Now let $1 \leq j \leq k$ such that $\ell j < k < (\ell+1)j$ for some $\ell \in \mathbb{N}$. Then, by taking into account the fact that $j \mapsto (M_j^{(q)})^{1/j}$ is non-decreasing for each index $q > 0$, which follows by log-convexity and $M_0^{(q)} = 1$, we have

$$(M_k^{(p)})^{1/k} \geq (M_{\ell j}^{(p)})^{1/(\ell j)} \geq \frac{\ell h_1}{D_1} (M_j^{(p_1)})^{1/j} \geq \frac{(\ell+1) h_1}{2 D_1} (M_j^{(p_1)})^{1/j},$$

which gives similarly as before:

$$
\begin{aligned}
(\breve{M}_k^{(p)})^{1/k} &\geq \frac{(M_k^{(p)})^{1/k}}{k} \geq \frac{(\ell+1)h_1}{2kD_1}(M_j^{(p_1)})^{1/j} \\
&\geq \frac{j(\ell+1)h_1}{2ekD_1}(\breve{M}_j^{(p_1)})^{1/j} \geq \frac{h_1}{2eD_1}(\breve{M}_j^{(p_1)})^{1/j}.
\end{aligned}
$$

Summarizing, $(\mathcal{M}_{(\mathrm{rai})})$ between the sequences $\boldsymbol{M}^{(p)}$ and $\boldsymbol{M}^{(p_1)}$ is verified with $H := 2eD_1/h_1$. □

### 2.6.4 Main characterizing results

In order to establish our first stability result for the ultraholomorphic classes of Beurling type, we shall consider the following crucial assumptions of Beurling-type on a given weight matrix $\mathcal{M}$, see [59, Sect. 4.1] and [70, Sect. 7.2].

**Definition 2.6.21.** We say that:

(i) $\mathcal{M}$ has the $\mathrm{C}^\omega$ property of Beurling type, denoted by $(\mathcal{M}_{(\mathrm{C}^\omega)})$, if for all $\alpha > 0$ we have that

$$\lim_{j\to\infty} (\breve{M}_j^{(\alpha)})^{1/j} = +\infty.$$

(ii) $\mathcal{M}$ has the *Faà-di-Bruno property* of Beurling type, denoted by $(\mathcal{M}_{(\mathrm{FdB})})$, if

$$\forall\, \alpha > 0\, \exists\, \beta > 0 : \quad (\breve{\boldsymbol{M}}^{(\beta)})^\circ \precsim \breve{\boldsymbol{M}}^{(\alpha)},$$

where $(\breve{\boldsymbol{M}}^{(\alpha)})^\circ$ is the sequence defined by (1.2).

(iii) $\mathcal{M}$ satisfies the *derivation closedness condition* of Beurling type, denoted by $(\mathcal{M}_{(\mathrm{dc})})$, if

$$\forall\, \alpha > 0\, \exists\, C > 0\, \exists\, \beta > 0\, \forall\, j \in \mathbb{N}_0 : M_{j+1}^{(\beta)} \leq C^{j+1} M_j^{(\alpha)}.$$

Using these Beurling-type conditions we immediately get the following analogue of Lemma 1.3.7, which is needed in the forthcoming arguments.

**Lemma 2.6.22.** *Let $\mathcal{M} = \{\boldsymbol{M}^{(\alpha)} : \alpha > 0\}$ be a weight matrix. Then we have the following:*

*(i)* $(\mathcal{M}_{(\mathrm{rai})})$ *implies* $(\mathcal{M}_{\mathcal{H}})$ *up to equivalence of matrices.*

*(ii)* $(\mathcal{M}_{(\mathrm{dc})})$ *and* $(\mathcal{M}_{(\mathrm{rai})})$ *imply* $(\mathcal{M}_{(\mathrm{FdB})})$.

(*iii*) *If*
$$\forall\, \alpha > 0\, \exists\, H \geq 1\, \forall\, 1 \leq j \leq k: \quad (M_j^{(\alpha)})^{1/j} \leq H (M_k^{(\alpha)})^{1/k}, \tag{2.23}$$
*i. e. if each sequence* $((M_j^{(\alpha)})^{1/j})_j$ *is almost increasing, then* $(\mathcal{M}_{\mathcal{H}})$ *and* $(\mathcal{M}_{(\mathrm{FdB})})$ *imply* $(\mathcal{M}_{(\mathrm{rai})})$.

*In particular,* (2.23) *holds true (with* $H = 1$ *for any* $\alpha$*) provided that* $\mathcal{M}$ *is log-convex.*

*Proof.* (i) By $(\mathcal{M}_{(\mathrm{rai})})$ we get that for each $\alpha > 0$ there exist some $C \geq 1$ and $\beta = \beta(\alpha) \leq \alpha$ such that $(\breve{M}_j^{(\alpha)})^{1/j} \geq \breve{M}_1^{(\beta)}/C > 0$ for all $j \geq 1$ (see also [72, Lemma 3.6 (*ii*)]).

(ii) See the proofs of [59, Thm. 4.11 (3) $\Rightarrow$ (4)] and [70, Lemma 8.2.3 (2)].

(iii) See the proofs of [60, Lemma 1 (2)] and [70, Lemma 8.2.3 (4)].

□

We are ready to state our first main result for small openings, which is analogous to Theorem 2.3.14. Observe that if $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$ is a weight matrix which satisfies $(\mathcal{M}_{(\mathrm{C}^\omega)})$ and $0 < \alpha \leq 1$, then $\lim_{j\to+\infty}(j^{(1-\alpha)j} M_j^{(p)})^{1/j} = \infty$ for all $p > 0$, and therefore we can consider the matrix $\mathcal{M}^\alpha = \{\boldsymbol{M}^{(p,\alpha)} : p > 0\}$ defined in (2.7).

**Theorem 2.6.23.** *Let* $\mathcal{M} = \{\boldsymbol{M}^{(p)} : p > 0\}$ *be a weight matrix which satisfies* $(\mathcal{M}_{(\mathbf{G})})$ *and* $(\mathcal{M}_{(dc)})$*, and let* $0 < \alpha \leq 1$ *be given. Then, the following assertions are equivalent:*

(*a*) *The class* $\mathcal{A}_{(\mathcal{M})}(S_\alpha)$ *is closed under composition.*

(*b*) *The class* $\mathcal{A}_{(\mathcal{M})}(S_\alpha)$ *is closed under Beurling-analytic functions.*

(*c*) *The class* $\mathcal{A}_{(\mathcal{M})}(S_\alpha)$ *is inverse-closed.*

(*d*) *The matrix* $\mathcal{M}^\alpha$ *satisfies the property* $(\mathcal{M}_{(\mathrm{rai})})$.

(*e*) *The matrix* $\mathcal{M}^\alpha$ *satisfies the property* $(\mathcal{M}_{(\mathrm{FdB})})$.

*Proof.* First, note that if $\mathcal{M}$ has $(\mathcal{M}_{(\mathbf{G})})$, then for all $A > 0$ there exist some $B > 0$ such that $BM_k^{(p)} \geq A^k k!$ for all $k \geq 0$ and $p > 0$. Thanks to the fact that $A$ is arbitrary, we can deduce that $\mathcal{M}$ has $(\mathcal{M}_{(\mathrm{C}^\omega)})$ and therefore the matrix $\mathcal{M}^\alpha$ is well defined.

(*a*) $\Rightarrow$ (*b*) If $\mathcal{M}$ has the property $(\mathcal{M}_{(\mathrm{C}^\omega)})$, then we can establish the relation $\mathcal{G}^1(\preceq)\mathcal{M}$, which implies that $\mathcal{H}_{(\mathcal{G}^1)}(U) \subseteq \mathcal{H}_{(\mathcal{M})}(U)$, where $U \subseteq \mathbb{C}$ is an open set.

Consequently, if the class $\mathcal{A}_{(\mathcal{M})}(S_\alpha)$ is closed under composition, then it is closed under Beurling-analytic functions.

$(b) \Rightarrow (c)$ Again, property $(\mathcal{M}_{(\mathrm{C}^\omega)})$ implies that

$$\forall\, p > 0\, \forall\, h > 0\, \exists\, C = C_{p,h} \geq 1\, \forall\, j \in \mathbb{N}_0 : \quad j! \leq C h^j M_j^{(p)}, \tag{2.24}$$

and therefore the map $g : z \mapsto \frac{1}{z}$ belongs to the class $\mathcal{H}_{(\mathcal{M})}((\mathbb{C}\backslash\{0\}))$ and $\mathbb{C}\backslash\{0\}$ contains the (compact) closure of the image of any element $f \in \mathcal{A}_{(\mathcal{M})}(S_\alpha)$ such that $\inf_{z\in S_\alpha} |f(z)| > 0$.

$(c) \Rightarrow (d)$ The previous estimate (2.24) applied with $h = 1$, ensures that $\lim_{j\to+\infty}(j^{(1-\alpha)j} M_j^{(p)})^{1/j} = \infty$ for all $p > 0$ and for a given $0 < \alpha \leq 1$. Thanks to Theorem 2.6.10, we deduce that both classes, $\mathcal{A}_{(\mathcal{M})}(S_\alpha)$ and $\mathcal{A}_{(\mathcal{M}^\alpha)}(S_\alpha)$, coincide and therefore the class $\mathcal{A}_{(\mathcal{M}^\alpha)}(S_\alpha)$ is inverse-closed. Finally, Theorem 2.6.20 ensures that $\mathcal{M}^\alpha$ satisfies the property $(\mathcal{M}_{(\mathrm{rai})})$.

$(d) \Rightarrow (e)$ It is straightforward from Lemma 2.6.22.

$(e) \Rightarrow (a)$ First recall that by the so-called *Faà-di-Bruno formula* for the composition (see [83, pp. 124–126]) we get

$$(g \circ f)^{(n)}(z) = n! \sum_{i=1}^{n} \sum_{\sum_{j=1}^{i} k_j = n,\, k_h \geq 1} \frac{g^{(i)}(f(z))}{i!} \prod_{j=1}^{i} \frac{f^{(k_j)}(z)}{k_j!} \quad z \in S_\alpha,\ n \in \mathbb{N}.$$

Let $f \in \mathcal{A}_{(\mathcal{M})}(S_\alpha)$ be given. By Theorem 2.6.10 we know that $\mathcal{A}_{(\mathcal{M}^\alpha)}(S_\alpha) = \mathcal{A}_{(\mathcal{M})}(S_\alpha)$, therefore $f \in \mathcal{A}_{(\mathcal{M}^\alpha)}(S_\alpha)$. Also, for any function $g \in \mathcal{H}_{(\mathcal{M})}(U)$, where $U \subseteq \mathbb{C}$ is an open set containing the closure of the range of $f$, we have that $g \in \mathcal{H}_{(\mathcal{M}^\alpha)}(U)$ thanks to Corollary 2.6.14, and therefore

$$\forall p > 0 \forall\, h_1 > 0\, \exists\, C_1 \geq 1\, \forall\, k \in \mathbb{N}_0\, \forall\, z \in S_\alpha : \quad |g^{(k)}(f(z))| \leq C_1 h_1^k M_k^{(p,\alpha)}. \tag{2.25}$$

Let now $p > 0$ be a given index and $\widetilde{h} > 0$, both arbitrary and small but from now on fixed. By applying $(\mathcal{M}_{(\mathrm{FdB})})$ once we get an index $p' > 0$ and a constant $H > 0$ such that

$$(\widetilde{\boldsymbol{M}}^{(p')})^\circ \precsim \widetilde{\boldsymbol{M}}^{(p)}, \quad \text{and therefore} \quad (\widetilde{M}_j^{(p')})^\circ \leq H^j \widetilde{M}_j^{(p)}, \qquad j \in \mathbb{N}_0. \tag{2.26}$$

By assumption $f \in \mathcal{A}_{(\mathcal{M}^\alpha)}(S_\alpha)$ we have that

$$\forall\, h_2 > 0\, \exists\, C_2 \geq 1\, \forall\, k \in \mathbb{N}_0\, \forall\, z \in S_\alpha : \quad |f^{(k)}(z)| \leq C_2 h_2^k M_k^{(p',\alpha)}. \tag{2.27}$$

We choose now $h_2 := \frac{\widetilde{h}}{2H}$ in (2.27) and $h_1$ in (2.25) small enough to ensure $h_1 C_2 \leq 1$. Then we can estimate as follows for all $n \in \mathbb{N}$ and $z \in S_\alpha$:

$$\begin{aligned}
|(g\circ f)^{(n)}(z)| &\leq n! \sum_{i=1}^{n} \sum_{\sum_{j=1}^{i} k_j = n,\, k_h \geq 1} \frac{|g^{(i)}(f(z))|}{i!} \prod_{j=1}^{i} \frac{|f^{(k_j)}(z)|}{k_j!} \\
&\leq n! \sum_{i=1}^{n} \sum_{\sum_{j=1}^{i} k_j = n,\, k_h \geq 1} C_1 h_1^i \widetilde{M}_i^{(p',\alpha)} \prod_{j=1}^{i} \left( C_2 h_2^{k_j} \widetilde{M}_{k_j}^{(p',\alpha)} \right) \\
&\leq C_1 n! \sum_{i=1}^{n} \sum_{\sum_{j=1}^{i} k_j = n,\, k_h \geq 1} h_1^i C_2^i h_2^{k_1+\cdots+k_i} \widetilde{M}_i^{(p',\alpha)} \prod_{j=1}^{i} \left( \widetilde{M}_{k_j}^{(p',\alpha)} \right) \\
&\leq C_1 h_2^n n! \sum_{i=1}^{n} \sum_{\sum_{j=1}^{i} k_j = n,\, k_h \geq 1} (h_1 C_2)^i \widetilde{M}_i^{(p',\alpha)} \prod_{j=1}^{i} \left( \widetilde{M}_{k_j}^{(p',\alpha)} \right) \\
&\underbrace{\leq}_{(\mathcal{M}_{(\mathrm{FdB})})} C_1 h_2^n n! \sum_{i=1}^{n} \sum_{\sum_{j=1}^{i} k_j = n,\, k_h \geq 1} (h_1 C_2)^i (\widetilde{M}_n^{(p')})^{\circ} \\
&\underbrace{\leq}_{(2.26)} C_1 h_2^n n! \sum_{i=1}^{n} \sum_{\sum_{j=1}^{i} k_j = n,\, k_h \geq 1} (h_1 C_2)^i H^n \widetilde{M}_n^{(p)} \\
&\leq C_1 (H h_2)^n M_n^{(p)} \sum_{i=1}^{n} \sum_{\sum_{j=1}^{i} k_j = n,\, k_h \geq 1} (h_1 C_2)^i \\
&= C_1 (H h_2)^n M_n^{(p)} \sum_{i=1}^{n} \binom{n-1}{i-1} (h_1 C_2)^i \\
&= h_1 C_1 C_2 (H h_2)^n M_n^{(p)} (1 + (h_1 C_2))^{n-1} \\
&\leq h_1 C_1 C_2 (2 H h_2)^n M_n^{(p)} = h_1 C_1 C_2 \widetilde{h}^n M_n^{(p)}.
\end{aligned}$$

Note that $H$ depends only on $p$ (via $(\mathcal{M}_{(\mathrm{FdB})})$); $C_2$ depends on $p'$ (and therefore $p$) and on $\widetilde{h}$ via $h_2$ and the choice for the constant $H$. Thus, finally $C_1$ depends on $p$ and on $\widetilde{h}$ as well since $h_1$ depends on $C_2$.

Summarizing, by taking into account that $\mathcal{A}_{(\mathcal{M}^{\alpha})}(S_\alpha) = \mathcal{A}_{(\mathcal{M})}(S_\alpha)$ and since both $p$ and $\widetilde{h}$ are arbitrary we have verified $g \circ f \in \mathcal{A}_{(\mathcal{M})}(S_\alpha)$. $\square$

For a sequence $\boldsymbol{M}$ with (dc) and such that for all $A > 0$ there exist some $B > 0$ such that $BM_k/M_j \geq A^{k-j}(k-j)!$ whenever $k \geq j$, we can study the stability of the class $\mathcal{A}_{(\boldsymbol{M})}(S_\alpha)$, for $0 < \alpha \leq 1$, by considering the constant weight matrix $\mathcal{M} = \{\boldsymbol{M}^{(p)} = \boldsymbol{M} : p > 0\}$ and applying to it the previous result.

**Corollary 2.6.24.** *Let $\boldsymbol{M}$ be a sequence, and $0 < \alpha \leq 1$ be given such that $\boldsymbol{M}$ has (dc) and the weight matrix $\mathcal{M} = \{\boldsymbol{M}^{(p)} = \boldsymbol{M} : p > 0\}$ has $(\mathcal{M}_{(\mathbf{G})})$. Then, $\lim_{j\to+\infty}(j^{(1-\alpha)j}M_j)^{1/j} = \infty$, and therefore we can consider the sequence $\boldsymbol{M}^{(\alpha)} := \overline{\boldsymbol{G}}^{\alpha-1}\left(\overline{\boldsymbol{G}}^{1-\alpha}\boldsymbol{M}\right)^{\mathrm{lc}}$. Moreover, the following assertions are equivalent:*

*(a) The class $\mathcal{A}_{(\boldsymbol{M})}(S_\alpha)$ is closed under composition.*

*(b) The class $\mathcal{A}_{(\boldsymbol{M})}(S_\alpha)$ is closed under Beurling-analytic functions.*

*(c) The class $\mathcal{A}_{(\boldsymbol{M})}(S_\alpha)$ is inverse-closed.*

*(d) The sequence $\boldsymbol{M}^{(\alpha)}$ has the property* (FdB).

*(e) The sequence $\boldsymbol{M}^{(\alpha)}$ has the property* (rai).

Now, we can adapt theorem 2.6.23 to the weight function case. First, let us observe how the condition $(\mathcal{M}_{(\mathrm{FdB})})$ for a weight matrix associated to a weight function $\omega$ translates into a condition on $\omega$. As occurs in the Roumieu case, note that this matrix has $(\mathcal{M}_{\mathrm{lc}})$ and therefore $(\mathcal{M}_\omega)^\alpha \equiv \mathcal{M}_\omega$ for all $\alpha \in (0, 1]$.

**Lemma 2.6.25.** *([59, Theorem 6.5]) Let $\omega \in \mathcal{W}$ be given with associated weight matrix $\mathcal{M}_\omega := \{\boldsymbol{W}^{(\ell)} : \ell > 0\}$. Suppose that $\omega$ has also $(\omega_2)$. Then the following are equivalent:*

*(a) The matrix $\mathcal{M}_\omega$ has $(\mathcal{M}_{(\mathrm{FdB})})$.*

*(b) $\omega$ satisfies the condition $(\alpha_0)$ (see* (1.11)*).*

We can provide now a statement about stability properties for classes associated to a weight function in small sectors.

**Theorem 2.6.26.** *Let $\omega \in \mathcal{W}$ be given with associated weight matrix $\mathcal{M}_\omega := \{\boldsymbol{W}^{(\ell)} : \ell > 0\}$ and let $0 < \alpha \leq 1$. Suppose that $\omega$ has also $(\omega_5)$. Then the following are equivalent:*

*(a) $\omega$ satisfies the condition $(\alpha_0)$ (see* (1.11)*).*

*(b) The matrix $\mathcal{M}_\omega$ has $(\mathcal{M}_{(\mathrm{FdB})})$.*

*(c) The class $\mathcal{A}_{(\omega)}(S_\alpha)$ is closed under composition.*

*(d) The class $\mathcal{A}_{(\omega)}(S_\alpha)$ is closed under Beurling-analytic functions.*

*(e) The class $\mathcal{A}_{(\omega)}(S_\alpha)$ is inverse-closed.*

*(f) The matrix $\mathcal{M}_\omega$ satisfies the property $(\mathcal{M}_{(\mathrm{rai})})$.*

*Proof.* The equivalence $(a) \Leftrightarrow (b)$ is a consequence of the Lemma 2.6.25.

Moreover, the equivalences $(b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f)$ follow by applying Theorem 2.6.23 to $\mathcal{M} \equiv \mathcal{M}_\omega$. Let us observe that $\mathcal{M}^\alpha \equiv \mathcal{M}_\omega$, thanks to the fact that $\boldsymbol{W}^{(\ell)}$ is (lc) for all $\ell > 0$. Moreover, $\omega$ has $(\omega_1)$ and therefore $\mathcal{A}_{(\omega)}(S_\alpha) = \mathcal{A}_{(\mathcal{M}_\omega)}(S_\alpha)$, see (2.18). In addition, note that $\mathcal{M}_\omega$ has automatically $(\mathcal{M}_{(\mathrm{dc})})$ by (1.20) and $(\mathcal{M}_{(\mathrm{C}^\omega)})$ thanks to $(\omega_5)$ (see [70, Lemma 5.3.2.] and the comments below or [59, Corollary 5.15]). $\square$

# Chapter 3

# Borel-Ritt theorems and extension operators

This chapter is devoted to present several results of Borel-Ritt type, stating the surjectivity of the asymptotic Borel mapping in Carleman ultraholomorphic classes in unbounded sectors. Closely related to these classes are the ones consisting of functions admitting a uniform asymptotic expansion at the vertex of the sector, and in some situations one or the other classes are preferable. In most cases such results come with extension operators, i. e., linear and continuous right inverses for the Borel mapping. Both the Roumieu case, predominant in the literature, and the Beurling case will be addressed.

## 3.1 Asymptotic expansions and the asymptotic Borel map

We introduce now some new classes under consideration in this chapter, i.e., the classes of functions that admit a uniform asymptotic expansion at the vertex of the sector where they are defined. We define classes of both Roumieu and Beurling type, analogously as it was done for ultraholomorphic classes.

Recall that $\mathcal{R}$ stands for the Riemann surface of the logarithm. Let $T$ and $S$ be sectors in $\mathcal{R}$ with vertex at 0. We say that $T$ is a *proper subsector* of $S$ if $\overline{T} \subset S$ (where the closure of $T$ is taken in $\mathcal{R}$, and so the vertex of the sector is not under consideration).

We denote by $\mathbb{C}[[z]]$ the space of formal power series in $z$ with complex coefficients. We start by recalling the concept of uniform asymptotic expansion.

**Definition 3.1.1.** Let $S$ be an unbounded sector and $\boldsymbol{M}$ be a sequence. We say a holomorphic function $f\colon S \to \mathbb{C}$ admits $\widehat{f} = \sum_{n\geq 0} a_n z^n \in \mathbb{C}[[z]]$ as its *uniform*

$\boldsymbol{M}$*-asymptotic expansion in* $S$ *(of type* $1/h$ *for some* $h > 0$*)* if there exists $C > 0$ such that for every $p \in \mathbb{N}_0$, one has

$$\left| f(z) - \sum_{n=0}^{p-1} a_n z^n \right| \leq C h^p M_p |z|^p, \qquad z \in S. \tag{3.1}$$

In this case we write $f \sim^u_{\boldsymbol{M},h} \widehat{f}$ in $S$, and $\widetilde{\mathcal{A}}^u_{\boldsymbol{M},h}(S)$ denotes the space of functions admitting uniform $\boldsymbol{M}$-asymptotic expansion of type $1/h$ in $S$, endowed with the norm

$$\|f\|_{\boldsymbol{M},h,\widetilde{u}} := \sup_{z \in S, p \in \mathbb{N}_0} \frac{|f(z) - \sum_{k=0}^{p-1} a_k z^k|}{h^p M_p |z|^p}, \tag{3.2}$$

which makes it a Banach space.

Now, we define the classes of uniform asymptotic expansion of Roumieu and Beurling type

**Definition 3.1.2.** Let $S$ be an unbounded sector and $\boldsymbol{M}$ be a sequence. We define the $(LB)$ space of functions admitting a uniform $\{\boldsymbol{M}\}$-asymptotic expansion in $S$ (of Roumieu type), denoted by $\widetilde{\mathcal{A}}^u_{\{\boldsymbol{M}\}}(S)$, as

$$\widetilde{\mathcal{A}}^u_{\{\boldsymbol{M}\}}(S) = \bigcup_{h>0} \widetilde{\mathcal{A}}^u_{\boldsymbol{M},}(S).$$

When the type needs not be specified, we simply write $f \sim^u_{\{\boldsymbol{M}\}} \widehat{f}$ in $S$.

Moreover, we can consider the space of Beurling-type, denoted by $\widetilde{\mathcal{A}}^u_{(\boldsymbol{M})}(S)$, and defined as

$$\widetilde{\mathcal{A}}^u_{(\boldsymbol{M})}(S) := \bigcap_{h>0} \widetilde{\mathcal{A}}^u_{\boldsymbol{M},h}(S),$$

which becomes a Fréchet space when endowed with the topology generated by the family of seminorms $(\|\cdot\|_{\boldsymbol{M},h,\widetilde{u}})_{h>0}$.

**Remark 3.1.3.** First, note that, taking $p = 0$ in (3.1), we deduce that every function in $\widetilde{\mathcal{A}}^u_{\{\boldsymbol{M}\}}(S)$ or $\widetilde{\mathcal{A}}^u_{(\boldsymbol{M})}(S)$ is a bounded function.

Secondly, when a statement is valid for both Roumieu and Beurling classes, we will use the notation $\mathcal{A}_{[\boldsymbol{M}]}(S)$, $\widetilde{\mathcal{A}}^u_{[\boldsymbol{M}]}(S)$ and so on (substituting every square bracket by either of them, curly brackets or parentheses, but the same all through the statement). For example, if $\boldsymbol{M}$ is (lc), the spaces $\mathcal{A}_{[\boldsymbol{M}]}(S)$ and $\widetilde{\mathcal{A}}^u_{[\boldsymbol{M}]}(S)$ are algebras, and if $\boldsymbol{M}$ is (dc) they are stable under taking derivatives. Moreover, if $\boldsymbol{M} \approx \boldsymbol{L}$ the corresponding classes coincide.

As a consequence of Taylor's formula and Cauchy's integral formula for the derivatives, there is a close relation between Carleman ultraholomorphic classes and the concept of asymptotic expansion (this can be proved similarly as [1, Prop. 8]).

**Proposition 3.1.4.** *Let $\boldsymbol{M}$ be a sequence and $S$ be a sector. Then,*

*(i) If $f \in \mathcal{A}_{\widehat{\boldsymbol{M}},h}(S)$ then $f$ admits $\widehat{f} := \sum_{p\in\mathbb{N}_0} \frac{1}{p!} f^{(p)}(0) z^p$ as its uniform $\boldsymbol{M}$-asymptotic expansion in $S$ of type $1/h$, where $(f^{(p)}(0))_{p\in\mathbb{N}_0}$ is given by (2.2). Moreover, $\|f\|_{\boldsymbol{M},h,\widetilde{u}} \leq \|f\|_{\widehat{\boldsymbol{M}},h}$, and so the identity map $\mathcal{A}_{\widehat{\boldsymbol{M}},h}(S) \hookrightarrow \widetilde{\mathcal{A}}^u_{\boldsymbol{M},h}(S)$ is continuous. Consequently, we also have that $\mathcal{A}_{[\widehat{\boldsymbol{M}}]}(S) \subseteq \widetilde{\mathcal{A}}^u_{[\boldsymbol{M}]}(S)$ and $\mathcal{A}_{[\widehat{\boldsymbol{M}}]}(S) \hookrightarrow \widetilde{\mathcal{A}}^u_{[\boldsymbol{M}]}(S)$ is continuous.*

*(ii) If $S$ is unbounded and $T$ is a proper subsector of $S$, then there exists a constant $c = c(T, S) > 0$ such that the restriction to $T$, $f|_T$, of functions $f$ defined on $S$ and admitting a uniform $\boldsymbol{M}$-asymptotic expansion in $S$ of type $1/h > 0$, belongs to $\mathcal{A}_{\widehat{\boldsymbol{M}},ch}(T)$, and $\|f|_T\|_{\widehat{\boldsymbol{M}},ch} \leq \|f\|_{\boldsymbol{M},h,\widetilde{u}}$. So, the restriction map from $\widetilde{\mathcal{A}}^u_{\boldsymbol{M},h}(S)$ to $\mathcal{A}_{\widehat{\boldsymbol{M}},ch}(T)$ is continuous, and it is also continuous from $\widetilde{\mathcal{A}}^u_{[\boldsymbol{M}]}(S)$ to $\mathcal{A}_{[\widehat{\boldsymbol{M}}]}(T)$.*

One may similarly define classes of formal power series. More precisely:

**Definition 3.1.5.** Let $\boldsymbol{M}$ be a sequence, and $h$ be a positive number. We define the class of formal power series as

$$\mathbb{C}[[z]]_{\boldsymbol{M},h} = \Big\{ \widehat{f} = \sum_{p=0}^{\infty} a_p z^p \in \mathbb{C}[[z]] : \left| \widehat{f} \right|_{\boldsymbol{M},h} := \sup_{p\in\mathbb{N}_0} \frac{|a_p|}{h^p M_p} < \infty \Big\}. \tag{3.3}$$

Moreover, by taking into account that $(\mathbb{C}[[z]]_{\boldsymbol{M},h}, |\cdot|_{\boldsymbol{M},h})$ is a Banach space, we can consider the Carleman-Roumieu $(LB)$ space of formal power series, defined as

$$\mathbb{C}[[z]]_{\{\boldsymbol{M}\}} := \bigcup_{h>0} \mathbb{C}[[z]]_{\boldsymbol{M},h},$$

and the Carleman-Beurling Fréchet space, defined as

$$\mathbb{C}[[z]]_{(\boldsymbol{M})} := \bigcap_{h>0} \mathbb{C}[[z]]_{\boldsymbol{M},h}.$$

After we have introduced the previous spaces, it is natural to consider the Borel map, more precisely.

**Definition 3.1.6.** Let $S$ be a sector, and $\boldsymbol{M}$ be a sequence. We define the *asymptotic Borel map*, denoted by $\widetilde{\mathcal{B}}$, as the map sending a function $f \in \widetilde{\mathcal{A}}^u_{\boldsymbol{M},h}(S)$ into its $\boldsymbol{M}$-asymptotic expansion $\widehat{f} \in \mathbb{C}[[z]]_{\boldsymbol{M},h}$, i.e. $\widetilde{\mathcal{B}}(f) := \widehat{f}$.

**Remark 3.1.7.** Note that, by Proposition 3.1.4.(i) the asymptotic Borel map may be defined from $\widetilde{\mathcal{A}}^u_{[\boldsymbol{M}]}(S)$ or $\mathcal{A}_{[\widehat{\boldsymbol{M}}]}(S)$ into $\mathbb{C}[[z]]_{[\boldsymbol{M}]}$ (with the aforementioned meaning), and from $\mathcal{A}_{\widehat{\boldsymbol{M}},h}(S)$ into $\mathbb{C}[[z]]_{\boldsymbol{M},h}$, and it is continuous when considered between the corresponding $(LB)$, Fréchet or Banach spaces.

Moreover, if $\boldsymbol{M}$ is (lc), $\widetilde{\mathcal{B}}$ is a homomorphism of algebras; if $\boldsymbol{M}$ is also (dc), differentiation commutes with $\widetilde{\mathcal{B}}$. Finally, $\boldsymbol{M} \approx \boldsymbol{L}$ implies $\mathbb{C}[[z]]_{[\boldsymbol{M}]} = \mathbb{C}[[z]]_{[\boldsymbol{L}]}$, and the corresponding Borel maps are in all cases identical.

We will focus on the surjectivity of the Borel map in unbounded sectors $S_\gamma$ bisected by direction 0, as this problem is invariant under rotation. Note that the value $\gamma$ can be any positive real number, since we work in the Riemann surface of the logarithm; in case $\gamma$ is greater than 2, multivalued functions (i. e. whose values depend on the considered sheet within the Riemann surface) naturally occur. We define

$$\begin{aligned} S_{[\widehat{\boldsymbol{M}}]} :=& \{\gamma > 0; \quad \widetilde{\mathcal{B}} : \mathcal{A}_{[\widehat{\boldsymbol{M}}]}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{[\boldsymbol{M}]} \text{ is surjective}\}, \\ \widetilde{S}^u_{[\boldsymbol{M}]} :=& \{\gamma > 0; \quad \widetilde{\mathcal{B}} : \widetilde{\mathcal{A}}^u_{[\boldsymbol{M}]}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{[\boldsymbol{M}]} \text{ is surjective}\}. \end{aligned}$$

Thanks to the fact that if $\gamma > 0$ is in any of those sets then every $0 < \gamma' < \gamma$ also is, we deduce that $S_{[\widehat{\boldsymbol{M}}]}$ and $\widetilde{S}^u_{[\boldsymbol{M}]}$ are either empty or left-open intervals having 0 as endpoint, called *surjectivity intervals*. Moreover, by Proposition 3.1.4, we see that

$$(\widetilde{S}^u_{[\boldsymbol{M}]})^\circ \subseteq S_{[\widehat{\boldsymbol{M}}]} \subseteq \widetilde{S}^u_{[\boldsymbol{M}]}, \tag{3.4}$$

where $I^\circ$ is the interior of $I$. The determination of these intervals is closely related to the existence of right inverses for the asymptotic Borel map.

**Definition 3.1.8.** We say that $T$ is a *extension operator* for $\widetilde{\mathcal{B}}$ if it is linear and continuous, and such that $\widetilde{\mathcal{B}} \circ T$ is the identity map on a class of formal power series. It can be *global*, defined from $\mathbb{C}[[z]]_{[\boldsymbol{M}]}$ into $\widetilde{\mathcal{A}}^u_{[\boldsymbol{M}]}(S)$ or $\mathcal{A}_{[\widehat{\boldsymbol{M}}]}(S)$ (with its respective $(LB)$ or Fréchet space structures), and *local*, at the level of Banach spaces, defined from $\mathbb{C}[[z]]_{\boldsymbol{M},h}$ into some $\widetilde{\mathcal{A}}^u_{\boldsymbol{M},h'}(S)$ or $\mathcal{A}_{\widehat{\boldsymbol{M}},h'}(S)$ for suitable $h'$ depending on $h$. In this latter case, it is common that a scaling of the type occurs, that is, $h' = ch$ for a universal constant $c > 0$ independent from $h$.

## 3.2 Optimal flat functions and a Borel-Ritt result for Roumieu classes under (dc)

The following result for the Roumieu case, already hinted at in the work of V. Thilliez [80, Subsect. 3.3] and resting on a result of H.-J. Petzsche [52, Th. 3.5], appeared, in a slightly different form, in [33, Lemma 4.5]. Since the result of Petzsche is equally valid for the Beurling case [52, Th. 3.4], one can state the following.

**Lemma 3.2.1.** *Let $\boldsymbol{M}$ be a weight sequence. If $\widetilde{S}^u_{[\boldsymbol{M}]} \neq \emptyset$, then $\boldsymbol{M}$ satisfies* (snq) *or, equivalently, $\gamma(\boldsymbol{M}) > 0$.*

Regarding the precise determination of the surjectivity intervals, the first seminal results appeared in a work of J. Schmets and M. Valdivia [74], whose results prove that

$$(0, \lceil \gamma(\boldsymbol{M}) \rceil - 1) \subset S_{[\widehat{\boldsymbol{M}}]},$$

where $\lceil x \rceil$ is the least integer greater than or equal to a real number $x$. Moreover, for such openings surjectivity comes with local extension operators with scaling of the type, and with global extension operators in the Beurling case, while global extension operators in the Roumieu case need the extra condition $(\beta_2)$ of H.-J. Petzsche [52]. In the case of strongly regular sequences, V.Thilliez [80] showed that $(0, \gamma(\boldsymbol{M})) \subset S_{[\widehat{\boldsymbol{M}}]}$, again with local extension operators with scaling of the type. Several improvements followed in the Roumieu case [68, 33, 14, 37], trying firstly to determine the surjectivity intervals, or at least their length, for (certain classes of) strongly regular sequences, and afterwards trying to weaken the condition of moderate growth. These efforts have lead to the following precise statement that appeared in [37, Th. 3.7] under the condition (dc). It shows that the length of the surjectivity intervals is precisely given by $\gamma(\boldsymbol{M})$.

**Theorem 3.2.2.** *Let $\widehat{\boldsymbol{M}}$ be a regular sequence such that $\gamma(\boldsymbol{M}) > 0$. Then,*

$$(0, \gamma(\boldsymbol{M})) \subseteq S_{\{\widehat{\boldsymbol{M}}\}} \subseteq \widetilde{S}^u_{\{\boldsymbol{M}\}} \subseteq (0, \gamma(\boldsymbol{M})].$$

*In particular, if $\gamma(\boldsymbol{M}) = \infty$, we have that $S_{\{\widehat{\boldsymbol{M}}\}} = \widetilde{S}^u_{\{\boldsymbol{M}\}} = (0, \infty)$.*

So, the surjectivity of the Borel map for regular sequences is governed by the value of the index $\gamma(\boldsymbol{M})$.

### 3.2.1 Construction of optimal flat functions

Our aim is to relate the surjectivity of the Borel map in a sector to the existence of optimal flat functions in it, which we now define and construct in this subsection.

**Definition 3.2.3.** Let $\boldsymbol{M}$ be a weight sequence, $S$ an unbounded sector bisected by direction $d = 0$, i.e., by the positive real line $(0, +\infty) \subset \mathcal{R}$. A holomorphic function $G\colon S \to \mathbb{C}$ is called an *optimal* $\{\boldsymbol{M}\}$*-flat function* in $S$ if:

($i$) There exist $K_1, K_2 > 0$ such that for all $x > 0$,

$$K_1 h_{\boldsymbol{M}}(K_2 x) \leq G(x). \tag{3.5}$$

($ii$) There exist $K_3, K_4 > 0$ such that for all $z \in S$, one has

$$|G(z)| \leq K_3 h_{\boldsymbol{M}}(K_4 |z|). \tag{3.6}$$

Besides the symmetry imposed by condition ($i$) (observe that $G(x) > 0$ for $x > 0$, and so $G(\overline{z}) = \overline{G(z)}$, $z \in S$), we note that the estimates in (3.6) amount to the fact that

$$|G(z)| \leq K_3 K_4^p M_p |z|^p, \qquad p \in \mathbb{N}_0,\ z \in S,$$

which exactly means that $G \in \widetilde{\mathcal{A}}^u_{\{\boldsymbol{M}\}}(S)$ and is $\{\boldsymbol{M}\}$*-flat*, i.e., its uniform $\{\boldsymbol{M}\}$-asymptotic expansion is given by the null series. The inequality imposed in (3.5) makes the function optimal in a sense, as its rate of decrease on the positive real axis when $t$ tends to 0 is accurately specified by the function $h_{\boldsymbol{M}}$. Note that, in previous instances where such optimal flat functions appear [80, 42, 27], the estimates from below in (3.5) are imposed and/or obtained in the whole sector $S$, and not just on its bisecting direction. We think the present definition is more convenient, since it is easier to check for concrete functions, and for our purposes it provides all the necessary information in order to work with such functions.

In order to construct such optimal flat functions, we need to start by introducing the harmonic extension and a particular majorant of a nondecreasing nonquasianalytic function.

**Definition 3.2.4.** A nondecreasing (or even just measurable) function $\sigma : [0, \infty) \to [0, \infty)$ satisfies the *nonquasianalyticity* property $(\omega_{nq})$, and we say $\sigma$ is *nonquasianalytic*, if

$$\int_1^\infty \frac{\sigma(t)}{t^2}\, dt < \infty.$$

For a nondecreasing nonquasianalytic function, we can consider the harmonic extension of such function.

**Definition 3.2.5.** Let $\sigma : [0, \infty) \to [0, \infty)$ be a nondecreasing nonquasianalytic function. The *harmonic extension* $P_\sigma$ of $\sigma$ to the open upper and lower halfplanes of $\mathbb{C}$ is defined by

$$P_\sigma(x + iy) = \begin{cases} \sigma(|x|) & \text{if } x \in \mathbb{R},\ y = 0, \\ \dfrac{|y|}{\pi} \displaystyle\int_{-\infty}^{\infty} \frac{\sigma(|t|)}{(t-x)^2 + y^2} dt & \text{if } x \in \mathbb{R},\ y \neq 0. \end{cases} \tag{3.7}$$

There exist some relation between $\sigma$ and the harmonic extension $P_\sigma$ of $\sigma$. More precisely.

**Remark 3.2.6.** For every $z \in \mathbb{C}$ one has (see, for example, [6, Remark 3.2] or [51, Prop. 5.5]):

$$\sigma(|z|) \leq P_\sigma(z). \tag{3.8}$$

Moreover, the harmonic extension $P_\sigma$ has the following properties.

**Proposition 3.2.7.** *Let $\sigma : [0, \infty) \to [0, \infty)$ be a nondecreasing nonquasianalytic function. Then, we have the following properties*

*(1) $\sigma_1 \leq \sigma_2$ implies $P_{\sigma_1} \leq P_{\sigma_2}$.*

*(2) $\lambda P_{\sigma_1}(z) + \mu P_{\sigma_2}(z) = P_{\lambda\sigma_1+\mu\sigma_2}(z)$, $\lambda, \mu \in \mathbb{R}$.*

*(3) $P_{t\mapsto\sigma(Ct)}(z) = P_\sigma(Cz)$, $C > 0$.*

Another important auxiliary function appears in the study of extension results in Braun-Meise-Taylor ultradifferentiable classes, defined in terms of weight functions (see, for example, [48, 6] and the references therein).

**Definition 3.2.8.** Let $\sigma : [0, \infty) \to [0, \infty)$ be a nondecreasing and nonquasianalytic function. Then, the function $\kappa_\sigma$ is defined by

$$\kappa_\sigma(y) = \int_1^\infty \frac{\sigma(ys)}{s^2}\, ds, \qquad y \geq 0,$$

**Remark 3.2.9.** Thanks to the fact that $\sigma$ is nondecreasing, we have the following estimate

$$\sigma(y) \leq \kappa_\sigma(y), \quad y \geq 0. \tag{3.9}$$

Moreover, if $\sigma$ is also continuous, then $\kappa_\sigma$ is concave, cf. the proof of (3) $\Rightarrow$ (4) in [48, Proposition 1.3].

In particular, consider a weight sequence $\boldsymbol{M}$ such that $\sum_{p=0}^\infty 1/m_p < \infty$ (this is condition $(M3)'$ in [38]); in other words, the sequence $\widetilde{\boldsymbol{M}} := (M_p/p!)_{p\in\mathbb{N}_0}$ satisfies (nq). According to [38, Lemma 4.1], this property amounts to $\nu_{\boldsymbol{m}}$ and/or $\omega_{\boldsymbol{M}}$ being nonquasianalytic. So, it makes sense to consider the concave function $\kappa_{\omega_{\boldsymbol{M}}}$ associated with $\omega_{\boldsymbol{M}}$, and $\kappa_{\nu_{\boldsymbol{m}}}$ associated with the counting function $\nu_{\boldsymbol{m}}$. Moreover, we can establish the following equality

**Proposition 3.2.10** ([38], Proposition 4.4)**.** *Let $\boldsymbol{M}$ be a weight sequence such that $\widetilde{\boldsymbol{M}}$ satisfies (nq). Then, we have that*

$$\kappa_{\omega_{\boldsymbol{M}}}(y) = \omega_{\boldsymbol{M}}(y) + \kappa_{\nu_{\boldsymbol{m}}}(y), \qquad y \geq 0. \tag{3.10}$$

As a first step for the construction of such flat functions, we need to estimate the harmonic extension $P_\sigma$ in terms of the majorant $\kappa_\sigma$. The right-hand side estimate in the next result is a slight refinement of the one in [6, Lemma 3.3], which was not precise enough for our purposes. We include the whole proof for the sake of completeness.

**Proposition 3.2.11.** *Let $\sigma : [0,\infty) \to [0,\infty)$ be a nondecreasing nonquasianalytic function. Then, we have*

$$\frac{1}{\pi}\kappa_\sigma(y) \leq P_\sigma(iy) \leq \kappa_\sigma(y), \qquad y \geq 0. \tag{3.11}$$

*Proof.* If $y = 0$ all the values are equal to $\sigma(0)$ and so the inequalities hold true. Now, for $y > 0$ we have

$$\begin{aligned} P_\sigma(iy) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\sigma(|t|)}{t^2+y^2}dt = \frac{2y}{\pi}\int_0^\infty \frac{\sigma(t)}{t^2+y^2}dt &= \frac{2}{\pi}\int_0^\infty \frac{\sigma(ys)}{s^2+1}ds \\ &\geq \frac{2}{\pi}\int_1^\infty \frac{\sigma(ys)}{s^2+1}ds. \end{aligned}$$

Since $s^2+1 \leq 2s^2$ for $s \geq 1$, we deduce that

$$P_\sigma(iy) \geq \frac{1}{\pi}\int_1^\infty \frac{\sigma(ys)}{s^2}ds = \frac{1}{\pi}\kappa_\sigma(y).$$

In order to prove the right inequality, we start by splitting the integral into two parts:

$$P_\sigma(iy) = \frac{2}{\pi}\int_0^\infty \frac{\sigma(ys)}{s^2+1}ds = \frac{2}{\pi}\left(\int_0^1 \frac{\sigma(ys)}{s^2+1}ds + \int_1^\infty \frac{\sigma(ys)}{s^2+1}ds\right). \tag{3.12}$$

As $\sigma$ is nondecreasing, we may write

$$\int_0^1 \frac{\sigma(ys)}{s^2+1}ds \leq \sigma(y)\int_0^1 \frac{1}{s^2+1}ds = \frac{\pi}{4}\sigma(y), \tag{3.13}$$

and

$$\int_1^\infty \frac{\sigma(ys)}{s^2+1}ds = \kappa_\sigma(y) - \int_1^\infty \left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)\sigma(ys)ds \leq \kappa_\sigma(y) - \sigma(y)\left(1-\frac{\pi}{4}\right). \tag{3.14}$$

From (3.12), (3.13), (3.14) and (3.9) we deduce that

$$P_\sigma(iy) \leq \frac{2}{\pi}\left(\frac{\pi}{2}\sigma(y) + \kappa_\sigma(y) - \sigma(y)\right) \leq \kappa_\sigma(y).$$

□

The key condition for weight sequences that will allow us to construct optimal flat functions appeared in a work of M. Langenbruch [41].

**Definition 3.2.12.** Let $\boldsymbol{M}$ be a weight sequence such that $\breve{\boldsymbol{M}}$ satisfies (nq), so that $P_{\omega_{\boldsymbol{M}}}$ is well-defined. We say that the sequence satisfies the *Langenbruch's condition* if there exists a constant $C>0$ such that for all $y\geq 0$ we have

$$P_{\omega_{\boldsymbol{M}}}(iy)\leq \omega_{\boldsymbol{M}}(Cy)+C. \tag{3.15}$$

We can characterize the previous condition in terms of the index $\gamma(\boldsymbol{M})$. This connection has very recently appeared for the first time in a work of D. N. Nenning, A. Rainer and G. Schindl [51]. Although the additional hypothesis of (dc) appears in their (indirect) arguments, it can be removed as long as the sequence satisfies (snq), as we now show. Observe that, by Lemma 3.2.1, the condition (snq) (equivalently, $\gamma(\boldsymbol{M})>0$) is necessary for surjectivity, so it is not a restriction for our aim.

**Proposition 3.2.13.** *Let $\boldsymbol{M}$ be a weight sequence. The following are equivalent:*

*(i) $\gamma(\boldsymbol{M})>0$, $\breve{\boldsymbol{M}}$ satisfies (nq) and $\boldsymbol{M}$ satisfies Langenbruch's condition.*

*(ii) $\gamma(\boldsymbol{M})>1$.*

*Proof.* First, from (1.15) we deduce that for all $r\geq 0$ and $B\geq 0$,

$$\omega_{\boldsymbol{M}}(e^{B}r)=\int_0^{e^{B}r}\frac{\nu_{\boldsymbol{m}}(u)}{u}du=\omega_{\boldsymbol{M}}(r)+\int_r^{e^{B}r}\frac{\nu_{\boldsymbol{m}}(u)}{u}du\geq \omega_{\boldsymbol{M}}(r)+B\nu_{\boldsymbol{m}}(r). \tag{3.16}$$

The last inequality is a consequence of the monotonicity of $\nu_{\boldsymbol{m}}$.

(i)$\Rightarrow$(ii) By taking into account (3.8) and (3.11), we deduce

$$\omega_{\boldsymbol{M}}(y)+\kappa_{\nu_{\boldsymbol{m}}}(y)\leq P_{\omega_{\boldsymbol{M}}}(iy)+\pi P_{\nu_{\boldsymbol{m}}}(iy)=P_{\omega_{\boldsymbol{M}}+\pi\nu_{\boldsymbol{m}}}(iy),\qquad y\geq 0.$$

Thanks to (3.16) and the monotonicity of the harmonic extension with respect to the argument function we get from above

$$\omega_{\boldsymbol{M}}(y)+\kappa_{\nu_{\boldsymbol{m}}}(y)\leq P_{\omega_{\boldsymbol{M}}(e^{\pi}\cdot)}(y)=P_{\omega_{\boldsymbol{M}}}(ie^{\pi}y)\leq \omega_{\boldsymbol{M}}(Ce^{\pi}y)+C,\qquad y\geq 0.$$

Next, by using the integral expression (1.15) and the monotonicity of $\nu_{\boldsymbol{m}}$ we have that

$$\begin{aligned}\kappa_{\nu_{\boldsymbol{m}}}(y)&\leq \omega_{\boldsymbol{M}}(Ce^{\pi}y)-\omega_{\boldsymbol{M}}(y)+C\\ &=\int_y^{Ce^{\pi}y}\frac{\nu_{\boldsymbol{m}}(u)}{u}du+C\leq \ln(Ce^{\pi})\nu_{\boldsymbol{m}}(Ce^{\pi}y)+C,\qquad y\geq 0.\end{aligned}$$

Finally, by Lemma 1.1.32, we deduce that

$$\kappa_{\nu_{\boldsymbol{m}}}(y) \le D\nu_{\boldsymbol{m}}(y) + D, \qquad y \ge 0,$$

for suitable $D > 0$. This is condition $(\omega_{\mathrm{snq}})$ for $\nu_{\boldsymbol{m}}$ and, by Lemma 1.1.32, we may conclude that $\gamma(\boldsymbol{M}) > 1$.

(ii)⇒(i) Condition $\gamma(\boldsymbol{M}) > 1$ implies that $\gamma(\boldsymbol{M}) > 0$, and amounts to condition $(\gamma_1)$ for $\boldsymbol{m}$ (see (1.4)), so that $\breve{\boldsymbol{M}}$ clearly satisfies (nq). By Lemma 1.1.32, the condition $\gamma(\boldsymbol{M}) > 1$ is equivalent to the existence of a constant $C > 0$ such that

$$\kappa_{\nu_{\boldsymbol{m}}}(y) \le C\nu_{\boldsymbol{m}}(y) + C, \qquad y \ge 0. \tag{3.17}$$

Then, from (3.11), (3.10) and the above inequality we deduce that

$$P_{\omega_{\boldsymbol{M}}}(iy) \le \kappa_{\omega_{\boldsymbol{M}}}(y) = \omega_{\boldsymbol{M}}(y) + \kappa_{\nu_{\boldsymbol{m}}}(y) \le \omega_{\boldsymbol{M}}(y) + C\nu_{\boldsymbol{m}}(y) + C, \qquad y \ge 0.$$

By (3.16), we have from above that

$$P_{\omega_{\boldsymbol{M}}}(iy) \le \omega_{\boldsymbol{M}}(e^C y) + C, \qquad y \ge 0,$$

which completes the proof. □

**Remark 3.2.14.** The condition $\gamma(\boldsymbol{M}) > 1$ is the same as $\gamma(\breve{\boldsymbol{M}}) > 0$, or equivalently, (snq) for $\breve{\boldsymbol{M}}$ (even if $\breve{\boldsymbol{M}}$ might not satisfy (lc), we can apply [34, Corollary 3.13] to obtain this equivalence). So, for a weight sequence $\boldsymbol{M}$ satisfying (snq), Langenbruch's condition allows to pass from (nq) to (snq) for $\breve{\boldsymbol{M}}$.

Observe also that, by [34, Lemma 3.20], the condition (nq) for $\breve{\boldsymbol{M}}$ implies that the index $\omega(\breve{\boldsymbol{M}})$, introduced in [68] and studied in detail in [34], is nonnegative, and so $\omega(\boldsymbol{M}) = \omega(\breve{\boldsymbol{M}})+1 \ge 1$. As one only knows that $\gamma(\boldsymbol{M}) \le \omega(\boldsymbol{M})$ in general, and these indices can perfectly be different, one may better understand the effect of Langenbruch's condition.

**Remark 3.2.15.** On the one hand, as said before, for a weight sequence $\boldsymbol{M}$ the condition $\gamma(\boldsymbol{M}) > 1$ amounts to the condition $(\gamma_1)$ for $\boldsymbol{m}$, and it is well-known (see [38, Prop. 4.4]) that then $\omega_{\boldsymbol{M}}$ satisfies $(\omega_{\mathrm{snq}})$. As it can be deduced from [48, Prop. 1.7], this last fact is, in its turn, equivalent to the existence of a constant $C > 0$ such that

$$P_{\omega_{\boldsymbol{M}}}(iy) \le C\omega_{\boldsymbol{M}}(y) + C, \quad y \ge 0.$$

On the other hand, in [38, Prop. 3.6] the condition (mg) for a weight sequence $\boldsymbol{M}$ is shown to be equivalent to the fact that $2\omega_{\boldsymbol{M}}(y) \le \omega_{\boldsymbol{M}}(Dy) + D$ for all $y \ge 0$ and suitable $D > 0$. Gathering these estimates, we conclude that if $\boldsymbol{M}$ is strongly regular then $\gamma(\boldsymbol{M}) > 1$ if, and only if, $\boldsymbol{M}$ satisfies Langenbruch's condition. This was basically the reasoning that allowed V. Thilliez to obtain optimal $\{\boldsymbol{M}\}$-flat functions, in the very same way as we are doing in the next result, but dropping now the moderate growth condition by means of Proposition 3.2.13.

Thanks to the previous result, we will construct optimal $\{\boldsymbol{M}\}$-flat functions in the right half plane as long as $\gamma(\boldsymbol{M}) > 1$.

**Proposition 3.2.16.** *Let $\boldsymbol{M}$ be a weight sequence. If $\gamma(\boldsymbol{M}) > 1$, then the function*

$$G(z) = \exp(-P_{\omega_{\boldsymbol{M}}}(i/z) - iQ_{\omega_{\boldsymbol{M}}}(i/z))$$

*is an optimal $\{\boldsymbol{M}\}$-flat function in the halfplane $S_1$, where $Q_{\omega_{\boldsymbol{M}}}$ is the harmonic conjugate of $P_{\omega_{\boldsymbol{M}}}$ in the upper half plane.*

*Proof.* It is clear that the function $G$ is holomorphic in $S_1$. On the one hand, by taking into account (3.8), for $z \in S_1$ we have that

$$|G(z)| = \exp(-P_{\omega_{\boldsymbol{M}}}(i/z)) \leq \exp(-\omega_{\boldsymbol{M}}(1/|z|)) = h_{\boldsymbol{M}}(|z|).$$

On the other hand, the condition $\gamma(\boldsymbol{M}) > 1$ implies, by Proposition 3.2.13, that there exists $C > 0$ such that $P_{\omega_{\boldsymbol{M}}}(ix) \leq \omega_{\boldsymbol{M}}(Cx) + C$ for every $x > 0$. Since one can easily check that $Q_{\omega_{\boldsymbol{M}}}(i/x) = 0$, we have that

$$G(x) = \exp(-P_{\omega_{\boldsymbol{M}}}(i/x)) \geq \exp(-\omega_{\boldsymbol{M}}(C/x) - C) = \exp(-C)h_{\boldsymbol{M}}(x/C),$$

as desired. □

By a ramification of the variable we can extend this method to an arbitrary weight sequence with $\gamma(\boldsymbol{M}) > 0$ and any sector whose opening is less than $\pi\gamma(\boldsymbol{M})$.

**Proposition 3.2.17.** *Let $\boldsymbol{M}$ be a weight sequence with $\gamma(\boldsymbol{M}) > 0$. Then, for any $0 < \gamma < \gamma(\boldsymbol{M})$ there exists an optimal $\{\boldsymbol{M}\}$-flat function in $S_\gamma$.*

*Proof.* Let $s > 0$ be such that $\gamma < 1/s < \gamma(\boldsymbol{M})$. Then, by [34, Th. 3.10, Prop. 3.6] we have that $\gamma(\boldsymbol{M}^s) = s\gamma(\boldsymbol{M}) > 1$, where $\boldsymbol{M}^s := (M_p^s)_{p\in\mathbb{N}_0}$ is again a weight sequence. We apply the last result to the sequence $\boldsymbol{M}^s$, so there exist an optimal $\{\boldsymbol{M}^s\}$-flat function $G$ in $S_1$. It is important to note that the bounds for $G$ appearing in Definition 3.2.3 will be in terms of $h_{\boldsymbol{M}^s}$, instead of $h_{\boldsymbol{M}}$. Moreover, the following relation between the functions $\omega_{\boldsymbol{M}^s}$ and $\omega_{\boldsymbol{M}}$ is straightforward:

$$\omega_{\boldsymbol{M}}(t^{1/s}) = \frac{1}{s}\omega_{\boldsymbol{M}^s}(t), \qquad t \geq 0. \tag{3.18}$$

Now, let us prove that the function $F(z)=(G(z^s))^{1/s}$, $z\in S_\gamma$, is an optimal $\{\boldsymbol{M}\}$-flat function in $S_\gamma$. From the fact that $G$ is an optimal $\{\boldsymbol{M}^s\}$-flat function, (1.13) and (3.18), we get

$$\begin{aligned}F(x)=(G(x^s))^{1/s}&\geq K_1^{1/s}\exp(-s^{-1}\omega_{\boldsymbol{M}^s}(1/(K_2x^s)))\\&\geq K_1^{1/s}\exp(-\omega_{\boldsymbol{M}}(1/(K_2^{1/s}x)))=K_1^{1/s}h_{\boldsymbol{M}}(K_2^{1/s}x),\qquad x>0,\end{aligned}$$

for suitable constants $K_1,K_2>0$. Moreover, we have that

$$\begin{aligned}|F(z)|&\leq K_3^{1/s}\exp(-s^{-1}\omega_{\boldsymbol{M}^s}(1/(K_4|z|^s)))\\&\leq K_3^{1/s}\exp(-\omega_{\boldsymbol{M}}(1/(K_4^{1/s}|z|)))=K_3^{1/s}h_{\boldsymbol{M}}(K_4^{1/s}|z|),\qquad z\in S_\gamma,\end{aligned}$$

for suitable constants $K_3,K_4>0$, and we are done. □

### 3.2.2 Surjectivity of the Borel map for regular sequences

We will describe next how, by means of an optimal flat function, one can obtain extension operators, right inverses for the Borel map, for ultraholomorphic classes defined by regular sequences.

If $G$ is an optimal $\{\boldsymbol{M}\}$-flat function in $\widetilde{\mathcal{A}}^u_{\{\boldsymbol{M}\}}(S)$, we define the kernel function $e\colon S\to\mathbb{C}$ given by

$$e(z):=G\left(\frac{1}{z}\right),\quad z\in S.$$

It is obvious that $e(x)>0$ for all $x>0$, and there exist $K_1,K_2,K_3,K_4>0$ such that

$$K_1h_{\boldsymbol{M}}\left(\frac{K_2}{x}\right)\leq e(x),\quad x>0,\qquad\text{and}\qquad |e(z)|\leq K_3h_{\boldsymbol{M}}\left(\frac{K_4}{|z|}\right),\quad z\in S. \tag{3.19}$$

For every $p\in\mathbb{N}_0$ we define the *p-th moment* of the function $e(z)$, given by

$$\mu(p):=\int_0^\infty t^pe(t)\,dt.$$

Note that the positive value $\mu(0)$ need not be equal to 1.

The following result is crucial for our aim.

**Proposition 3.2.18.** *Suppose $\boldsymbol{M}$ is a weight sequence with $\gamma(\boldsymbol{M})>0$, and $G$ is an optimal $\{\boldsymbol{M}\}$-flat function in $\widetilde{\mathcal{A}}^u_{\{\boldsymbol{M}\}}(S)$ for some unbounded sector $S$. Consider the sequence of moments $\mu:=(\mu(p))_{p\in\mathbb{N}_0}$ associated with the kernel function $e(z)=G(1/z)$. Then, $\boldsymbol{M}$ satisfies* (dc) *if, and only if, there exist $B_1,B_2>0$ such that*

$$\mu(0)B_1^pM_p\leq\mu(p)\leq\mu(0)B_2^pM_p,\quad p\in\mathbb{N}_0. \tag{3.20}$$

*In other words, $\boldsymbol{M}$ and $\mu$ are equivalent.*

**Proof.** First, we suppose that $\boldsymbol{M}$ has (dc), and therefore $\widehat{\boldsymbol{M}}$ is a regular sequence. Observe that we only need to reason for $p \in \mathbb{N}$. On the one hand, because of the right-hand inequalities in (3.19) and Lemma 1.1.28.(ii), for every $p \in \mathbb{N}$ and $s > 0$ we may write

$$\begin{aligned}\mu(p) &= \int_0^s t^p e(t)\,dt + \int_s^\infty \frac{1}{t^2} t^{p+2} e(t)\,dt \\ &\leq K_3 \int_0^s t^p\,dt + K_3 \sup_{t>0} t^{p+2} h_{\boldsymbol{M}}\left(\frac{K_4}{t}\right) \int_s^\infty \frac{1}{t^2}\,dt \\ &= K_3 \frac{s^{p+1}}{p+1} + K_3 \frac{1}{s} K_4^{p+2} M_{p+2} \leq K_3 \left( \frac{s^{p+1}}{p+1} + \frac{(K_4 D)^{p+2} M_p}{s} \right).\end{aligned}$$

Note that in the last equality we have used (1.9), and then we have applied (dc) with a suitable constant $D > 0$. Since $s > 0$ was arbitrary, we finally get

$$\begin{aligned}\mu(p) \leq \inf_{s>0} K_3 \left( \frac{s^{p+1}}{p+1} + \frac{(K_4 D)^{p+2} M_p}{s} \right) &= K_3 \frac{p+2}{p+1} (K_4 D)^{p+1} (M_p)^{(p+1)/(p+2)} \\ &\leq \mu(0) B_2^p M_p,\end{aligned}$$

for a suitably enlarged constant $B_2 > 0$ (observe that $p \geq 1$ and that, eventually, $M_p \geq 1$).

On the other hand, by the left-hand inequalities in (3.19) and Lemma 1.1.28.(i), for every $p \in \mathbb{N}$ and $s > 0$ we may estimate

$$\mu(p) \geq \int_0^s t^p e(t)\,dt \geq K_1 \int_0^s t^p h_{\boldsymbol{M}}\left(\frac{K_2}{t}\right) dt \geq K_1 h_{\boldsymbol{M}}\left(\frac{K_2}{s}\right) \frac{s^{p+1}}{p+1}.$$

Then, again by (1.9), we deduce that

$$\mu(p) \geq \frac{K_1}{p+1} \sup_{s>0} h_{\boldsymbol{M}}\left(\frac{K_2}{s}\right) s^{p+1} = \frac{K_1}{p+1} K_2^{p+1} M_{p+1} \geq \mu(0) B_1^p M_p$$

for a suitable constant $B_1 > 0$ (note that $\boldsymbol{M}$ is eventually nondecreasing).

Now, suppose that $\boldsymbol{M}$ and $\mu$ are equivalent and therefore (3.20) holds for suitable $B_1, B_2 > 0$. The above estimate (first inequality) shows for every $p \in \mathbb{N}$ that

$$M_{p+1} \leq \frac{p+1}{K_1} \left(\frac{1}{K_2}\right)^{p+1} \mu(p) \underbrace{\leq}_{(3.20)} \frac{\mu(0)}{K_1} \left(\frac{2B_2}{K_2}\right)^{p+1} M_p,$$

and so $\boldsymbol{M}$ has (dc). □

We can already state the following main result. The forthcoming implication $(ii) \Rightarrow (v)$ for strongly regular sequences $\boldsymbol{M}$ was first obtained by V. Thilliez [80, Th. 3.2.1], and the proof heavily rested on the moderate growth condition, both for the construction [80, Th. 2.3.1] of optimal $\{\boldsymbol{M}\}$-flat functions in sectors $S_\gamma$ for every $\gamma > 0$ such that $\gamma < \gamma(\boldsymbol{M})$, and for the subsequent use of Whitney extension results in the ultradifferentiable setting. In [42] the implication $(ii) \Rightarrow (iii)$ was proved again for strongly regular sequences, but with a completely different technique, and it is this approach which allows here for the weakening of condition (mg) into (dc).

**Theorem 3.2.19.** *Let $\widehat{\boldsymbol{M}}$ be a regular sequence (that is, $\boldsymbol{M}$ is a weight sequence and satisfies (dc)) with $\gamma(\boldsymbol{M}) > 0$, and let $\gamma > 0$ be given. Each of the following statements implies the next one:*

*(i)* $\gamma < \gamma(\boldsymbol{M})$.

*(ii) There exists an optimal $\{\boldsymbol{M}\}$-flat function in $\widetilde{\mathcal{A}}^u_{\{\boldsymbol{M}\}}(S_\gamma)$.*

*(iii) There exists $c > 0$ such that for every $h > 0$ there exists a linear continuous map $T_{\boldsymbol{M},h}\colon \mathbb{C}[[z]]_{\boldsymbol{M},h} \to \widetilde{\mathcal{A}}^u_{\boldsymbol{M},ch}(S_\gamma)$ such that $\widetilde{\mathcal{B}} \circ T_{\boldsymbol{M},h}$ is the identity map in $\mathbb{C}[[z]]_{\boldsymbol{M},h}$ (i.e., $T_{\boldsymbol{M},h}$ is an extension operator, right inverse for $\widetilde{\mathcal{B}}$).*

*(iv) The Borel map $\widetilde{\mathcal{B}}\colon \widetilde{\mathcal{A}}^u_{\{\boldsymbol{M}\}}(S_\gamma) \to \mathbb{C}[[z]]_{\{\boldsymbol{M}\}}$ is surjective. In other words, $(0,\gamma] \subset \widetilde{S}^u_{\{\boldsymbol{M}\}}$.*

*(v)* $(0,\gamma) \subset S_{\{\widehat{\boldsymbol{M}}\}}$.

*(vi)* $\gamma \le \gamma(\boldsymbol{M})$.

**Proof.** $(i) \Rightarrow (ii)$ See Proposition 3.2.17, valid for any weight sequence $\boldsymbol{M}$.

$(ii) \Rightarrow (iii)$ Let $h > 0$ and $\widehat{f} = \sum_{p=0}^\infty a_p z^p \in \mathbb{C}[[z]]_{\boldsymbol{M},h}$ be given. Let $(\mu(p))_{p\in\mathbb{N}_0}$ be the sequence of moments associated to the function $e(z) = G(1/z)$, where $G$ is an optimal $\{\boldsymbol{M}\}$-flat function in $\widetilde{\mathcal{A}}^u_{\{\boldsymbol{M}\}}(S_\gamma)$. By the definition of the norm in $\mathbb{C}[[z]]_{\boldsymbol{M},h}$ (see (3.3)), we have

$$|a_p| \le |\widehat{f}|_{\boldsymbol{M},h} h^p M_p, \quad p \in \mathbb{N}_0.$$

From the left-hand inequalities in (3.20), we deduce that

$$\left|\frac{a_p}{\mu(p)}\right| \le \frac{|\widehat{f}|_{\boldsymbol{M},h}}{\mu(0)} \left(\frac{h}{B_1}\right)^p, \quad p \in \mathbb{N}_0. \tag{3.21}$$

Hence, the formal Borel-like transform of $\widehat{f}$,

$$\widehat{g} = \sum_{p=0}^\infty \frac{a_p}{\mu(p)} z^p,$$

is convergent in the disc $D(0,R)$ for $R = B_1/h > 0$, and it defines a holomorphic function $g$ there. Choose $R_0 := B_1/(2h) < R$, and define

$$T_{\boldsymbol{M},h}(\widehat{f})(z) := \frac{1}{z}\int_0^{R_0} e\left(\frac{u}{z}\right) g(u)\,du, \qquad z \in S_\gamma,$$

which is a truncated Laplace-like transform of $g$ with kernel $e$. By virtue of Leibniz's theorem for parametric integrals and the properties of $e$, we deduce that this function, denoted by $f$ for the sake of brevity, is holomorphic in $S_\gamma$. We will prove that $f \sim^u_{\{\boldsymbol{M}\}} \hat{f}$ uniformly in $S_\gamma$, and that the map $\widehat{f} \mapsto f$, which is obviously linear, is also continuous from $\mathbb{C}[[z]]_{\boldsymbol{M},h}$ into $\widetilde{\mathcal{A}}^u_{\boldsymbol{M},ch}(S_\gamma)$ for suitable $c > 0$ independent from $h$.

Let $p \in \mathbb{N}_0$ and $z \in S_\gamma$. We have

$$\begin{aligned} f(z) - \sum_{n=0}^{p-1} a_n z^n &= f(z) - \sum_{n=0}^{p-1} \frac{a_n}{\mu(n)}\mu(n) z^n \\ &= \frac{1}{z}\int_0^{R_0} e\left(\frac{u}{z}\right)\sum_{n=0}^{\infty}\frac{a_n}{\mu(n)}u^n\,du - \sum_{n=0}^{p-1}\frac{a_n}{\mu(n)}\int_0^\infty v^n e(v)\,dv\,z^n. \end{aligned}$$

After a change of variable $u = zv$ in the last integral, one may use Cauchy's residue theorem and the right-hand estimates in (3.19) in order to rotate the path of integration and obtain

$$z^n\int_0^\infty v^n e(v)dv = \frac{1}{z}\int_0^\infty u^n e\left(\frac{u}{z}\right)\,du.$$

So, we can write the preceding difference as

$$\frac{1}{z}\left(\int_0^{R_0} e\left(\frac{u}{z}\right)\sum_{n=p}^{\infty}\frac{a_n}{\mu(n)}u^n\,du - \int_{R_0}^{\infty} e\left(\frac{u}{z}\right)\sum_{n=0}^{p-1}\frac{a_n}{\mu(n)}u^n\,du\right).$$

Then, we have

$$\left|f(z) - \sum_{n=0}^{p-1} a_n z^n\right| \le \frac{1}{|z|}(f_1(z) + f_2(z)), \tag{3.22}$$

where

$$f_1(z) = \left|\int_0^{R_0} e\left(\frac{u}{z}\right)\sum_{n=p}^{\infty}\frac{a_n}{\mu(n)}u^n\,du\right|, \quad f_2(z) = \left|\int_{R_0}^{\infty} e\left(\frac{u}{z}\right)\sum_{n=0}^{p-1}\frac{a_n}{\mu(n)}u^n\,du\right|.$$

We now estimate $f_1(z)$ and $f_2(z)$. Observe that for every $u \in (0, R_0]$ we have $0 < hu/B_1 \leq 1/2$. So, from (3.21) we get

$$\sum_{n=p}^{\infty} \frac{|a_n|}{\mu(n)} u^n \leq \frac{|\widehat{f}|_{\boldsymbol{M},h}}{\mu(0)} \sum_{n=p}^{\infty} \left(\frac{hu}{B_1}\right)^n \leq \frac{2|\widehat{f}|_{\boldsymbol{M},h}}{\mu(0)} \left(\frac{h}{B_1}\right)^p u^p.$$

Hence,

$$f_1(z) \leq \frac{2|\widehat{f}|_{\boldsymbol{M},h}}{\mu(0)} \left(\frac{h}{B_1}\right)^p \int_0^{R_0} \left| e\left(\frac{u}{z}\right) \right| u^p \, du. \tag{3.23}$$

Regarding $f_2(z)$, for $u \geq R_0$ and $0 \leq n \leq p-1$ we have $(u/R_0)^n \leq (u/R_0)^p$, so $u^n \leq R_0^n u^p / R_0^p$. Again by (3.21), and taking into account the value of $R_0$, we may write

$$\sum_{n=0}^{p-1} \frac{|a_n|}{\mu(n)} u^n \leq \frac{|\widehat{f}|_{\boldsymbol{M},h}}{\mu(0)} \frac{u^p}{R_0^p} \sum_{n=0}^{p-1} \left(\frac{hR_0}{B_1}\right)^n \leq \frac{|\widehat{f}|_{\boldsymbol{M},h}}{\mu(0)} \left(\frac{2h}{B_1}\right)^p u^p.$$

Then, we get

$$f_2(z) \leq \frac{|\widehat{f}|_{\boldsymbol{M},h}}{\mu(0)} \left(\frac{2h}{B_1}\right)^p \int_{R_0}^{\infty} \left| e\left(\frac{u}{z}\right) \right| u^p \, du. \tag{3.24}$$

In order to conclude, note that the second inequality in (3.19), followed by the first one, and the fact that $e(x) > 0$ for $x > 0$, together imply that for every $z \in S_\gamma$ and every $u > 0$ we have

$$|e(u/z)| \leq K_3 h_{\boldsymbol{M}} \left(K_4 \frac{|z|}{u}\right) \leq \frac{K_3}{K_1} e\left(\frac{K_2 u}{K_4 |z|}\right).$$

We use this fact, a simple change of variable and the right-hand estimates in (3.20), and obtain that

$$\begin{aligned}\int_0^{\infty} \left| e\left(\frac{u}{z}\right) \right| u^p \, du &\leq \int_0^{\infty} \frac{K_3}{K_1} e\left(\frac{K_2 u}{K_4|z|}\right) u^p \, du \\ &= \frac{K_3}{K_1} \left(\frac{K_4 |z|}{K_2}\right)^{p+1} \mu(p) \leq \frac{\mu(0) K_3 K_4}{K_1 K_2} \left(\frac{K_4 B_2}{K_2}\right)^p M_p |z|^{p+1}.\end{aligned}$$

This estimate can be taken into both (3.23) and (3.24), and from (3.22) we easily get that for every $p \in \mathbb{N}_0$,

$$\left| f(z) - \sum_{n=0}^{p-1} a_n z^n \right| \leq \frac{3K_3K_4}{K_1K_2} |\widehat{f}|_{\boldsymbol{M},h} \left(\frac{2K_4B_2h}{K_2B_1}\right)^p M_p |z|^p, \quad z \in S_\gamma,$$

and so $f$ admits $\widehat{f}$ as its uniform $\{\boldsymbol{M}\}$-asymptotic expansion in $S_\gamma$. Moreover, recalling the definition (3.2) of the norm in these spaces with uniform asymptotics and fixed type, if we put $c := 2K_4B_2/(K_2B_1) > 0$, we see that $f \in \widetilde{\mathcal{A}}^u_{\boldsymbol{M},ch}(S_\gamma)$ and

$$\|f\|_{\boldsymbol{M},ch,\widetilde{u}} \leq \frac{3K_3K_4}{K_1K_2}|\widehat{f}|_{\boldsymbol{M},h},$$

what proves the continuity of the linear map $T_{\boldsymbol{M},h}$.

*(iii)* ⇒ *(iv)* Immediate for any weight sequence $\boldsymbol{M}$.

*(iv)* ⇒ *(v)* It follows from (3.4), again valid for any weight sequence.

*(v)* ⇒ *(vi)* This statement is a consequence of Theorem 3.2.2. □

We note that the condition (dc) is only used in the implications *(ii)* ⇒ *(iii)* and *(v)* ⇒ *(vi)*.

**Remark 3.2.20.** The facts in Theorem 3.2.19.*(iii)* and Proposition 3.1.4.*(ii)* together guarantee that for every $\delta \in (0,\gamma)$ there exists $c' > 0$ such that for every $h > 0$ there exists a linear and continuous extension operator from $\mathbb{C}[[z]]_{\boldsymbol{M},h}$ into $\mathcal{A}_{\widehat{\boldsymbol{M}},c'h}(S_\delta)$. In fact, V. Thilliez stated his main result in this regard [80, Th. 3.2.1] in terms of the existence of such extension operators for every $\delta < \gamma(\boldsymbol{M})$ and $\boldsymbol{M}$ a strongly regular sequence.

The following three corollaries become now clear.

**Corollary 3.2.21.** *Let $\widehat{\boldsymbol{M}}$ be a regular sequence, and $\gamma > 0$. The following are equivalent:*

- *(i)* $\gamma(\boldsymbol{M}) > \gamma$,
- *(ii) There exists $\gamma_1 > \gamma$ such that the space $\widetilde{\mathcal{A}}^u_{\{\boldsymbol{M}\}}(S_{\gamma_1})$ contains optimal $\{\boldsymbol{M}\}$-flat functions.*
- *(iii) There exists $\gamma_1 > \gamma$ such that the Borel map $\widetilde{\mathcal{B}} : \widetilde{\mathcal{A}}^u_{\{\boldsymbol{M}\}}(S_{\gamma_1}) \to \mathbb{C}[[z]]_{\{\boldsymbol{M}\}}$ is surjective., i.e., $\gamma_1 \in \widetilde{S}^u_{\{\boldsymbol{M}\}}$.*

**Proof.** *(ii)* ⇒ *(iii)* and *(iii)* ⇒ *(i)* are respectively contained in Theorem 3.2.19 and Theorem 3.2.2, under weaker hypotheses. *(i)* ⇒ *(ii)* is immediately deduced from Proposition 3.2.17. □

As a consequence of Proposition 1.1.23 and Theorem 3.2.19 we get the following result.

**Corollary 3.2.22.** *Let $\widehat{\boldsymbol{M}}$ be a regular sequence. The following are equivalent:*

- *(i) $\boldsymbol{M}$ satisfies (snq).*
- *(ii) There exists $\gamma > 0$ such that the space $\widetilde{\mathcal{A}}^u_{\{\boldsymbol{M}\}}(S_\gamma)$ contains optimal $\{\boldsymbol{M}\}$-flat functions.*

(*iii*) *There exists* $\gamma > 0$ *such that the Borel map* $\widetilde{\mathcal{B}} : \widetilde{\mathcal{A}}^u_{\{\boldsymbol{M}\}}(S_\gamma) \to \mathbb{C}[[z]]_{\{\boldsymbol{M}\}}$ *is surjective. In other words,* $\widetilde{S}^u_{\{\boldsymbol{M}\}} \neq \emptyset$*.*

Note that, according to Proposition 3.1.4, in the previous items (*ii*) and (*iii*) one could change $\widetilde{\mathcal{A}}^u_{\{\boldsymbol{M}\}}(S_\gamma)$ and $\widetilde{S}^u_{\{\boldsymbol{M}\}}$ into $\mathcal{A}_{\{\widehat{\boldsymbol{M}}\}}(S_\gamma)$ and $S_{\{\widehat{\boldsymbol{M}}\}}$, respectively.

**Corollary 3.2.23.** *Let* $\widehat{\boldsymbol{M}}$ *be a regular sequence, and* $\gamma > 0$*. The following are equivalent:*

(*i*) $\gamma(\boldsymbol{M}) > \gamma$*,*

(*ii*) *There exists* $\gamma_1 > \gamma$ *such that the space* $\mathcal{A}_{\{\widehat{\boldsymbol{M}}\}}(S_{\gamma_1})$ *contains optimal* $\{\boldsymbol{M}\}$*-flat functions,*

(*iii*) *There exists* $\gamma_1 > \gamma$ *such that* $\widetilde{\mathcal{B}} : \mathcal{A}_{\{\widehat{\boldsymbol{M}}\}}(S_{\gamma_1}) \to \mathbb{C}[[z]]_{\{\boldsymbol{M}\}}$ *is surjective, i.e.,* $\gamma_1 \in S_{\{\widehat{\boldsymbol{M}}\}}$*.*

### 3.2.3 Optimal flat functions and strongly regular sequences

Under the moderate growth condition, the implication (*ii*) $\Rightarrow$ (*i*) in the version of Corollary 3.2.22 for the space $\mathcal{A}_{\{\widehat{\boldsymbol{M}}\}}(S_\gamma)$ can be shown independently by using a result from J. Bruna [9], where a precise formula for nontrivial flat functions in Carleman-Roumieu ultradifferentiable classes, appearing in a work of T. Bang [3], is exploited. For the sake of completeness, we will present this proof below.

**Theorem 3.2.24.** *Let* $\boldsymbol{M}$ *be a weight sequence satisfying (mg). If there exists* $\gamma > 0$ *such that* $\mathcal{A}_{\{\widehat{\boldsymbol{M}}\}}(S_\gamma)$ *contains optimal* $\{\boldsymbol{M}\}$*-flat functions, then* $\boldsymbol{M}$ *is strongly regular.*

The proof requires two auxiliary results which we state and prove now.

First, given a weight sequence $\boldsymbol{M}$, the sequence of quotients $\boldsymbol{m} = (m_p)_{p\in\mathbb{N}_0}$ is nondecreasing and tends to infinity, but it can happen that it remains constant on large intervals $[p_0, p_1]$ of indices, so that the counting function $\nu_{\boldsymbol{m}}$ defined in (1.10) yields $\nu_{\boldsymbol{m}}(m_{p_0}) = \nu_{\boldsymbol{m}}(m_{p_1}) = p_1 + 1$. However, in some applications or proofs it would be convenient to have $\nu_{\boldsymbol{m}}(m_p) = p + 1$ for all $p \geq 0$. This can be assumed without loss of generality by the following result.

**Lemma 3.2.25.** *Let* $a = (a_p)_{p\geq 1}$ *be a nondecreasing sequence of positive real numbers satisfying* $\lim_{p\to+\infty} a_p = +\infty$ *(it suffices that* $a_{p-1} < a_p$ *holds true for infinitely many indices* $p$*). Then there exists a sequence* $b = (b_p)_{p\geq 1}$ *of positive real numbers such that* $p \mapsto b_p$ *is strictly increasing and satisfies*

$$0 < \inf_{p\geq 1} \frac{b_p}{a_p} \leq \sup_{p\geq 1} \frac{b_p}{a_p} < +\infty.$$

So, in the language of weight sequences, we prove that for any weight sequence $\boldsymbol{M}$ there exists a strongly equivalent weight sequence $\boldsymbol{L}$ (and so $\boldsymbol{M} \approx \boldsymbol{L}$) such that $\nu_{\boldsymbol{\ell}}(\ell_p) = p+1$ for all $p \in \mathbb{N}_0$. Note that equivalent weight sequences define the same Carleman-Roumieu ultraholomorphic classes and associated weighted classes of formal power series.

**Proof.** Since $a$ is nondecreasing and $\lim_{p\to+\infty} a_p = +\infty$ there exists a sequence $(p_j)_{j\geq 1}$ of indices such that $a_{p_j-1} < a_{p_j} = \cdots = a_{p_{j+1}-1} < a_{p_{j+1}}$ for all $j \geq 1$ (and so $p_1 \geq 2$). For all $j \geq 1$ we have now $a_{p_j}/(a_{p_j-1}) > 1 + \varepsilon_j$ for a sequence $(\varepsilon_j)_{j\geq 1}$ with possibly small strictly positive numbers $\varepsilon_j$. Finally we put $p_0 := 1$.

We take some arbitrary $A > 1$ and choose $\delta_j > 0$ small enough so as to have $(1+\delta_j)^{p_{j+1}-p_j-1} \leq \min\{A, 1+\varepsilon_{j+1}\}$. Then the sequence $(\delta_j)_{j\geq 0}$ satisfies

$$(1+\delta_j)^{p_{j+1}-p_j-1} \leq 1+\varepsilon_{j+1} < \frac{a_{p_{j+1}}}{a_{p_{j+1}-1}}, \quad (1+\delta_j)^{p_{j+1}-p_j-1} \leq A, \quad j \geq 0. \tag{3.25}$$

We define now $b$ as follows:

$$b_q := a_q \text{ if } q = p_j,\ j \geq 0, \qquad b_q := (1+\delta_j) b_{q-1} \text{ if } 1+p_j \leq q \leq p_{j+1}-1,\ j \geq 0. \tag{3.26}$$

So we have by iteration $b_q = (1+\delta_j)^{q-p_j} b_{p_j} = (1+\delta_j)^{q-p_j} a_{p_j} = (1+\delta_j)^{q-p_j} a_q > a_q$ for all $q$ with $1+p_j \leq q \leq p_{j+1}-1$, $j \geq 0$. On each such interval of indices the mapping $q \mapsto b_q$ is now clearly strictly increasing since $1+\delta_j > 1$ for all $j$. Moreover, by the first half in (3.25), we have $b_{p_{j+1}-1} = (1+\delta_j)^{p_{j+1}-p_j-1} a_{p_j} < b_{p_{j+1}}$. Hence the sequence $q \mapsto b_q$ is strictly increasing.

By definition (3.26) we have $b_q = a_q$ for all $q = p_j$, $j \geq 0$, and $b_q > a_q$ otherwise. We conclude if we show that $b_q \leq A a_q$ for all $q$ with $1+p_j \leq q \leq p_{j+1}-1$, $j \geq 0$. For this, since $q \mapsto b_q$ is strictly increasing, it suffices to observe that, thanks to the second half in (3.25), we have $b_{p_{j+1}-1} = (1+\delta_j)^{p_{j+1}-p_j-1} a_{p_j} \leq A a_{p_j} = A a_{p_{j+1}-1}$. □

The second result is the following.

**Lemma 3.2.26.** *Let $\boldsymbol{M}$ be a weight sequence. Then $\boldsymbol{M}$ satisfies (mg) if and only if $\omega_{\boldsymbol{M}}(t) = O(\nu_{\boldsymbol{m}}(t))$ as $t \to +\infty$.*

**Proof.** The condition (mg) for $\boldsymbol{M}$ is equivalent to $m_n \leq A(M_n)^{1/n}$ for some $A \geq 1$ and all $n \in \mathbb{N}$ (e.g., see [61, Lemma 2.2]). It is also known that $\omega_{\boldsymbol{M}}(m_n) = \log(m_n^n/M_n)$ for $n \in \mathbb{N}$ (see [46, Chapitre I]). So, if $m_{n-1} \leq t < m_n$ for some $n \geq 1$, we get

$$\omega_{\boldsymbol{M}}(t) \leq \omega_{\boldsymbol{M}}(m_n) = n \log\left(\frac{m_n}{M_n^{1/n}}\right) \leq n\log(A) = \log(A)\nu_{\boldsymbol{m}}(t),$$

that is, $\omega_{\boldsymbol{M}}(t) = O(\nu_{\boldsymbol{m}}(t))$ as $t \to +\infty$.

Conversely, suppose that there exists $A \geq 1$ such that $\omega_{\boldsymbol{M}}(t) \leq A\nu_{\boldsymbol{m}}(t)$ for all $t \geq m_0$. By [61, Lemma 2.2], (mg) for $\boldsymbol{M}$ holds true if and only if there exists $H \geq 1$ such that for all $t$ large enough one has $2\nu_{\boldsymbol{m}}(t) \leq \nu_{\boldsymbol{m}}(Ht) + H$, and this we will prove. Take $H \geq \exp(2A)$ and $t \geq m_0$. Using (1.15), and since $\nu_{\boldsymbol{m}}$ is nondecreasing, we estimate

$$\begin{aligned}\nu_{\boldsymbol{m}}(Ht) \geq A^{-1}\omega_{\boldsymbol{M}}(Ht) &= A^{-1}\int_{m_0}^{Ht} \frac{\nu_{\boldsymbol{m}}(\lambda)}{\lambda} d\lambda \geq A^{-1}\int_{t}^{Ht} \frac{\nu_{\boldsymbol{m}}(\lambda)}{\lambda} d\lambda \\ &\geq A^{-1}\nu_{\boldsymbol{m}}(t)\int_{t}^{Ht} \frac{1}{\lambda} d\lambda = A^{-1}\log(H)\nu_{\boldsymbol{m}}(t) \geq 2\nu_{\boldsymbol{m}}(t),\end{aligned}$$

as desired.

We mention that an alternative, more abstract proof can be based in the theory of O-regular variation and Matuszewska indices for functions. By [34, Th. 4.4] we have that the lower Matuszewska indices of $\nu_{\boldsymbol{m}}$ and $\omega_{\boldsymbol{M}}$ agree, that is, $\beta(\nu_{\boldsymbol{m}}) = \beta(\omega_{\boldsymbol{M}})$, and by [34, Cor.2.17 and Cor. 4.2] we know $\boldsymbol{M}$ has (mg) if and only if $\beta(\nu_{\boldsymbol{m}}) > 0$. So, if $\beta(\nu_{\boldsymbol{m}}) > 0$, by [34, Th. 4.3] we have that $\liminf_{t\to\infty} \frac{\nu_{\boldsymbol{m}}(t)}{\omega_{\boldsymbol{M}}(t)} > 0$, and we deduce that $\omega_{\boldsymbol{M}}(t) = O(\nu_{\boldsymbol{m}}(t))$ as $t \to +\infty$. Conversely, if $\omega_{\boldsymbol{M}}(t) = O(\nu_{\boldsymbol{m}}(t))$ as $t \to +\infty$, then $\liminf_{t\to\infty} \frac{\nu_{\boldsymbol{m}}(t)}{\omega_{\boldsymbol{M}}(t)} > 0$, so by [34, Th. 4.3] we have that $\beta(\omega_{\boldsymbol{M}}) > 0$, and we are done. □

**Proof of Theorem 3.2.24.** We follow the proof of necessity for [9, Th. 2.2]. By Lemma 3.2.25 and the remark following it, we can assume without loss of generality that $\boldsymbol{m}$ is strictly increasing.

Let $G$ be an optimal $\{\boldsymbol{M}\}$-flat function in $\mathcal{A}_{\{\widehat{\boldsymbol{M}}\}}(S_\gamma)$ for some $\gamma > 0$. So, there exists some $h > 0$ such that

$$p_{\boldsymbol{M},h}(G) := \sup_{n\in\mathbb{N}_0, x\in(0,+\infty)} \frac{|G^{(n)}(x)|}{h^n n! M_n} < +\infty.$$

This shows that the Carleman-Roumieu ultradifferentiable class $\mathcal{E}_{\{\widehat{\boldsymbol{M}}\}}((-\varepsilon, +\infty))$, consisting of all smooth complex-valued functions $g$ defined on the interval $(-\varepsilon, \infty)$ for some $\varepsilon > 0$, and such that

$$\sup_{n\in\mathbb{N}_0, x\in(-\varepsilon,+\infty)} \frac{|g^{(n)}(x)|}{H^n n! M_n} < +\infty$$

for suitable $H > 0$, contains nontrivial flat functions (it suffices to extend $G$ by 0 for $x \in (-\varepsilon, 0])$. Then, the well-known Denjoy-Carleman theorem (e.g., see [25, Th. 1.3.8]) yields that $\boldsymbol{M}$ satisfies (nq).

Let now

$$R_n := \sum_{k\geq n} \frac{1}{(k+1)m_k} < +\infty, \quad n \in \mathbb{N}_0,$$

and let the function $F$ be defined by $F(t) := n$ if $R_{n+1} < t \leq R_n$, $n \in \mathbb{N}_0$.

By [3, (14), p. 142] we obtain that

$$G(x) = |G(x)| \leq p_{\boldsymbol{M},h}(G) \exp\left(-F(hex)\right), \quad x \in (0, +\infty).$$

Combining this with (3.5), with (1.13) and setting $C := p_{\boldsymbol{M},h}(G)$, we get

$$\exp\left(F(hex)\right) \leq \frac{C}{G(x)} \leq CK_1^{-1} \exp(\omega_{\boldsymbol{M}}(1/(K_2 x))), \quad x > 0.$$

If we put $t = hex$ and $B := he/K_2$, we obtain that for every $t > 0$,

$$F(t) \leq \log(CK_1^{-1}) + \omega_{\boldsymbol{M}}(B/t). \tag{3.27}$$

By Lemma 3.2.26, there exists $C_1 \geq 1$ such that $\omega_{\boldsymbol{M}}(s) \leq C_1\nu_{\boldsymbol{m}}(s) + C_1$ for $s > 0$. Choosing $t = B/m_n$ in (3.27), we see that

$$\begin{aligned} F(B/m_n) &\leq \log(CK_1^{-1}) + \omega_{\boldsymbol{M}}(m_n) \leq \log(CK_1^{-1}) + C_1\nu_{\boldsymbol{m}}(m_n) + C_1 \\ &= \log(CK_1^{-1}) + C_1(n+1) + C_1, \end{aligned}$$

since $\boldsymbol{m}$ is strictly increasing. Hence, $F(B/m_n) \leq C_2(n+1)$ for some $C_2 \in \mathbb{N}$ and all $n \in \mathbb{N}_0$. By definition of $F$, we get $R_{C_2(n+1)+1} \leq B/m_n$, i.e.,

$$m_n \sum_{k\geq C_2(n+1)+1} \frac{1}{(k+1)m_k} \leq B, \quad n \in \mathbb{N}_0.$$

Finally,

$$\begin{aligned} m_n \sum_{k\geq n} \frac{1}{(k+1)m_k} &= m_n \sum_{k\geq C_2(n+1)+1} \frac{1}{(k+1)m_k} + m_n \sum_{k=n}^{C_2(n+1)} \frac{1}{(k+1)m_k} \\ &\leq B + m_n \frac{n(C_2-1)+C_2+1}{(n+1)m_n} \leq B + 2C_2, \end{aligned}$$

which is (snq) for $\boldsymbol{M}$. □

## 3.3 Construction of optimal flat functions for a family of non strongly regular sequences

As deduced in Theorem 3.2.19, the construction of optimal $\{\boldsymbol{M}\}$-flat functions in sectors within an ultraholomorphic class, given by a regular sequence $\widehat{\boldsymbol{M}}$, provides extension operators and surjectivity results. Although such general construction has been shown in Proposition 3.2.17, we wish to present here a family of (non strongly) regular sequences for which an alternative, more explicit technique works.

We recall that, for logarithmically convex sequences $(M_p)_{p\in\mathbb{N}_0}$, the condition (dc) is equivalent to the condition $\log(M_p)=O(p^2)$, $p\to\infty$ (see [46, Ch. 6]). On the other hand, the condition (mg) implies that the sequence is below some Gevrey order (there exists $\alpha>0$ such that $M_p=O(p!^\alpha)$ as $p\to\infty$; see e.g. [47, 80]).

We will work, for $q>1$ and $1<\sigma\le 2$, with the sequences $\boldsymbol{M}_{q,\sigma}:=(q^{p^\sigma})_{p\in\mathbb{N}_0}$. They are clearly weight sequences and, by (1.5), it is immediate that $\gamma(\boldsymbol{M}_{q,\sigma})=\infty$, so they satisfy (snq) (see (1.1.23)). According to the previous comments, they satisfy (dc) but not (mg). So, $\widehat{\boldsymbol{M}}_{q,\sigma}$ is regular, but $\boldsymbol{M}_{q,\sigma}$ is not strongly regular.

The case $\sigma=2$ is well-known, as it corresponds to the so-called *$q$-Gevrey sequences*, appearing in the study of formal and analytic solutions for $q$-difference equations, see for example [4, 17] and the references therein.

First, we will construct a holomorphic function on $\mathbb{C}\setminus(-\infty,0]$ which will provide, by restriction, an optimal $\{\boldsymbol{M}_{q,\sigma}\}$-flat function in any unbounded sector $S_\gamma$ with $0<\gamma<2$. Subsequently, we will obtain such functions on general sectors of the Riemann surface $\mathcal{R}$ of the logarithm by ramification. This, according to Theorems 3.2.2 and 3.2.19, agrees with the fact that $\gamma(\boldsymbol{M}_{q,\sigma})=\infty$.

### 3.3.1 Flatness in the class given by $\boldsymbol{M}_{q,\sigma}$

It will be convenient to note that for a fixed $\sigma\in(1,2]$, there exists a unique $s\ge 2$ such that $\sigma=s/(s-1)$.

We start by suitably estimating the function

$$\omega_{\boldsymbol{M}_{q,\sigma}}(t)=\sup_{p\in\mathbb{N}_0}\ln\left(\frac{t^p}{q^{p^\sigma}}\right)=\sup_{p\in\mathbb{N}_0}(p\ln(t)-p^{s/(s-1)}\ln(q)),\quad t>0.$$

Due to the fact that $\omega_{\boldsymbol{M}_{q,\sigma}}(t)=0$ for $t\le 1$ (since $m_0=M_1/M_0=M_1=q>1$ and by (1.15)), we will restrict our attention to the case $t>1$. Obviously, $\omega_{\boldsymbol{M}_{q,\sigma}}(t)$ is bounded above by the supremum of $x\ln(t)-x^{s/(s-1)}\ln(q)$ when $x$ runs over $(0,\infty)$, which is easily obtained by elementary calculus and occurs at the point

$$x_0=\left(\frac{(s-1)\ln(t)}{s\ln(q)}\right)^{s-1}.$$

If we put

$$b_{q,s} := \frac{1}{s}\left(\frac{s-1}{s\ln(q)}\right)^{s-1}, \tag{3.28}$$

then

$$\omega_{\boldsymbol{M}_{q,\sigma}}(t) \le \left(\frac{(s-1)\ln(t)}{s\ln(q)}\right)^{s-1}\ln(t) - \left(\frac{(s-1)\ln(t)}{s\ln(q)}\right)^{s}\ln(q) = b_{q,s}\ln^s(t), \quad t>1. \tag{3.29}$$

On the other hand, for $t > q^{s/(s-1)}$ (what amounts to $x_0 > 1$) we also have that $\omega_{\boldsymbol{M}_{q,\sigma}}(t)$ is at least the value of $x\ln(t) - x^{s/(s-1)}\ln(q)$ at $x = \lfloor x_0 \rfloor$, that is,

$$\begin{aligned}
\omega_{\boldsymbol{M}_{q,\sigma}}(t) &\ge \left\lfloor\left(\frac{(s-1)\ln(t)}{s\ln(q)}\right)^{s-1}\right\rfloor\ln(t) - \left\lfloor\left(\frac{(s-1)\ln(t)}{s\ln(q)}\right)^{s-1}\right\rfloor^{s/(s-1)}\ln(q)\\
&\ge \left(\left(\frac{(s-1)\ln(t)}{s\ln(q)}\right)^{s-1} - 1\right)\ln(t) - \left(\frac{(s-1)\ln(t)}{s\ln(q)}\right)^{s}\ln(q)\\
&= b_{q,s}\ln^s(t) - \ln(t).
\end{aligned} \tag{3.30}$$

**Lemma 3.3.1.** *For every $t \ge q^{2s/(s-1)}$ it holds*

$$b_{q,s}\ln^s(t) - \ln(t) \ge b_{q,s}\ln^s\left(\frac{t}{q^{s/(s-1)}}\right) - \ln\left(q^{s/(s-1)}\right). \tag{3.31}$$

**Proof.** Observe that every $t \ge q^{2s/(s-1)}$ may be written as $t = q^{ys/(s-1)}$ for some $y \ge 2$. Then, we have that

$$b_{q,s}\ln^s(t) - b_{q,s}\ln^s\left(\frac{t}{q^{s/(s-1)}}\right) = b_{q,s}\left(\frac{s\ln(q)}{s-1}\right)^s\left(y^s - (y-1)^s\right) = \frac{\ln(q)}{s-1}\left(y^s - (y-1)^s\right).$$

By the mean value theorem, $y^s - (y-1)^s > s(y-1)^{s-1}$, and since $s \ge 2$ and $y \ge 2$, we have $(y-1)^{s-1} \ge y-1$. So we deduce that

$$\frac{\ln(q)}{s-1}\left(y^s - (y-1)^s\right) > \frac{s\ln(q)}{s-1}(y-1) = \ln(t) - \ln\left(q^{s/(s-1)}\right),$$

as desired. □

Combining (3.29) with (3.30) and (3.31), and using (1.13), we get

$$\exp\left(-b_{q,s}\ln^s\left(\frac{1}{t}\right)\right) \le h_{\boldsymbol{M}_{q,\sigma}}(t) \le q^{s/(s-1)}\exp\left(-b_{q,s}\ln^s\left(\frac{1}{q^{s/(s-1)}t}\right)\right), \tag{3.32}$$

for all $0 < t \le q^{-2s/(s-1)}$, and therefore we can say that these estimates express optimal $\{\boldsymbol{M}_{q,\sigma}\}$-flatness.

### 3.3.2 Optimal $\{\boldsymbol{M}_{q,\sigma}\}$-flat function in $S_2$

The estimates in (3.32) suggest considering the function $\exp\big(-b_{q,s}\log^s(1/z)\big)$, with, say, principal branches, as a candidate for providing optimal flat functions. However, its analyticity in wide sectors is not guaranteed. Moreover, even in small sectors around the direction $d=0$, its behaviour at $\infty$ might not be as desired: For example, when $s=2$ it tends to 0 as $0<x\to\infty$, what excludes the possibility of proving the inequality in (3.5).

Because of these reasons, we will first define a suitably modified function in the sector $S_2=\mathbb{C}\setminus(-\infty,0]$, prove its flatness there, and then turn to general sectors by composing it with an appropriate ramification.

We define

$$G_2^{q,s}(z):=\exp\left(-b_{q,s}\log^s\left(1+\frac{1}{z}\right)\right),\quad z\in S_2, \tag{3.33}$$

where the principal branch of the logarithm is chosen for both log and the power $w\mapsto w^s=\exp(s\log(w))$ involved. Observe that if $z\in S_2$, then $1+1/z\in\mathbb{C}\setminus(-\infty,1]$, and so $\log(1+1/z)=\ln(|1+1/z|)+i\arg(1+1/z)\notin(-\infty,0]$. This ensures that the map

$$z\mapsto\log^s\left(1+\frac{1}{z}\right)=\exp\left(s\log\left(\log(1+\frac{1}{z})\right)\right)$$

is also holomorphic in $S_2$, and so is $G_2^{q,s}$.

In order that $G_2^{q,s}$ is an optimal $\{\boldsymbol{M}_{q,\sigma}\}$-flat function in $S_2$, we are only left with proving the estimates (3.5) and (3.6). It turns out to be more convenient to work with the associated kernel

$$e_2(z):=G_2^{q,s}(1/z)=\exp(-b_{q,s}\log^s(1+z)),\quad z\in S_2,$$

and verify the following result.

**Lemma 3.3.2.** *There exist positive constants $C_1,C_2$ such that*

$$|e_2(z)|\le C_1e_2(C_2|z|),\quad z\in S_2.$$

**Proof.** In the first place, we observe that for every $z\in S_2$,

$$\begin{aligned}\Re(\log^s(z+1))&=|\log^s(z+1)|\cos(\arg(\log^s(z+1)))\\&=|\log(z+1)|^s\cos(s\arg(\log(z+1))).\end{aligned}\tag{3.34}$$

Now,

$$s|\arg(\log(z+1))|=s\left|\arctan\left(\frac{\arg(z+1)}{\ln|z+1|}\right)\right|\le s\left|\arctan\left(\frac{\pi}{\ln|z+1|}\right)\right|.\tag{3.35}$$

Hence, setting

$$R_0 := 1 + \exp\left(\frac{\pi}{\tan(\pi/(2s))}\right) \geq 2,$$

we get that $|z| > R_0$ implies that $|z+1| > R_0 - 1 \geq 1$, and therefore $\ln|z+1| > 0$ and

$$\frac{\pi}{\ln|z+1|} < \tan\left(\frac{\pi}{2s}\right).$$

From this and (3.35) we deduce that $\cos(s\arg(\log(z+1))) > 0$. Then, continuing with (3.34),

$$\begin{aligned}\Re(\log^s(z+1)) &\geq |\Re(\log(z+1))|^s \cos(s\arg(\log(z+1)))\\ &= \ln^s|z+1| - \ln^s|z+1| \frac{\sin^2(s\arg(\log(z+1)))}{1+\cos(s\arg(\log(z+1)))}. \end{aligned} \tag{3.36}$$

Now, from the equality in (3.35) we see that $s\arg(\log(z+1)) \to 0$ as $z \to \infty$ in $S_2$, and moreover

$$\lim_{\substack{z\to\infty \\ z\in S_2}} \left[\left(\frac{\sin^2(s\arg(\log(z+1)))}{1+\cos(s\arg(\log(z+1)))}\right) \Big/ \left(\frac{s^2\arg^2(z+1)}{2\ln^2|z+1|}\right)\right] = 1.$$

Therefore, there exist $R_1 \geq R_0$ and $C > 0$ such that

$$\frac{\sin^2(s\arg(\log(z+1)))}{1+\cos(s\arg(\log(z+1)))} \leq C\frac{1}{\ln^2|z+1|}, \qquad |z| > R_1.$$

We deduce from (3.36) that for $z \in S_2$ with $|z| > R_1$,

$$\Re(\log^s(z+1)) \geq \ln^s|z+1| - C\ln^{s-2}|z+1| \geq \ln^s(|z|-1) - C\ln^{s-2}(|z|+1). \tag{3.37}$$

We would be almost done if we obtain, for the right-hand side in (3.37), a lower bound in terms of, say, $\ln^s(1+|z|/2)$ for $|z|$ sufficiently large.

This is easy in case $s = 2$, for it suffices to take $|z| > 4$ in order to have $3 < 1 + |z|/2 < |z| - 1$, and so if $|z| \geq R_2 := \max\{R_1, 4\}$ we have

$$\Re(\log^s(z+1)) \geq \ln^s(|z|-1) - C \geq \ln^s\left(1 + \frac{|z|}{2}\right) - C.$$

In case $s > 2$, it is not difficult to check that

$$\lim_{x\to+\infty}\left(\ln^s(x-1) - C\ln^{s-2}(x+1) - \ln^s\left(1+\frac{x}{2}\right)\right) = +\infty,$$

so that, according to (3.37), there exists $R_2 \geq R_1$ such that for $z \in S_2$ with $|z| \geq R_2$ one has

$$\Re(\log^s(z+1)) \geq \ln^s\left(1+\frac{|z|}{2}\right).$$

In any case, we can deduce an upper estimate of the form

$$\begin{aligned}|e_2(z)| &= \exp\big(-b_{q,s}\Re(\log^s(z+1))\big)\\ &\le e^C\exp\left(-b_{q,s}\ln^s\left(1+\frac{|z|}{2}\right)\right) = e^C e_2\left(\frac{|z|}{2}\right), \quad z\in S_2,\ |z|>R_2.\end{aligned}$$

Finally, since the function $|e_2(z)|$ stays bounded and bounded away from 0 for bounded $|z|$ (in particular, it tends to 1 when $z$ tends to 0 in $S_2$), the previous estimate can be extended to the whole of $S_2$ by suitably enlarging the constant $C$. □

We are ready for the main objective of this section.

**Theorem 3.3.3.** *The function $G_2^{q,s}$ defined in* (3.33) *is an optimal $\{\boldsymbol{M}_{q,\sigma}\}$-flat function in $S_2$.*

**Proof.** The previous lemma ensures that there exist positive constants $C_1$, $C_2$ such that

$$|G_2^{q,s}(z)| \le C_1\exp\left(-b_{q,s}\ln^s\left(1+\frac{C_2}{|z|}\right)\right), \quad z\in S_2. \tag{3.38}$$

Observe that this inequality guarantees that $|G_2^{q,s}(z)|$ is bounded. As the same is true for $h_{\boldsymbol{M}_{q,\sigma}}(t)$ for every $t\ge t_0$ and any fixed $t_0>0$ (see Lemma 1.1.28), we only need to check the estimate (3.6) for small enough $|z|$.

For $|z|\le C_2$ it is clear that $\ln(1+C_2/|z|)>\ln(C_2/|z|)\ge 0$. Then, from (3.32) we have that

$$\begin{aligned}|G_2^{q,s}(z)| &\le C_1\exp\left(-b_{q,s}\ln^s\left(1+\frac{C_2}{|z|}\right)\right)\\ &\le C_1\exp\left(-b_{q,s}\ln^s\left(\frac{C_2}{|z|}\right)\right)\le C_1 h_{\boldsymbol{M}_{q,\sigma}}\left(\frac{|z|}{C_2}\right), \quad |z|\le C_2 q^{-2s/(s-1)},\end{aligned}$$

and we have proved (3.6).

Now, let us note that $G_2^{q,s}(x)$ is bounded away from 0 as soon as $x\ge r$ for any fixed $r>0$, since then

$$\exp\left(-b_{q,s}\ln^s(1+1/r)\right)\le G_2^{q,s}(x).$$

Again, we only need to check the estimate (3.5) for small enough $x$. Indeed, we have for $x>0$ that

$$G_2^{q,s}(x)=\exp\left(-b_{q,s}\ln^s\left(\frac{1}{x}\right)\right)\exp\left(-b_{q,s}\left[\ln^s\left(1+\frac{1}{x}\right)-\ln^s\left(\frac{1}{x}\right)\right]\right).$$

The mean value theorem gives that $\ln^s(1+1/x)-\ln^s(1/x)$ tends to zero if $x\searrow 0$, and we deduce that there exists $L$ such that

$$G_2^{q,s}(x)\geq L\exp\left(-b_{q,s}\ln^s\left(\frac{1}{x}\right)\right),\quad x\leq q^{-s/(s-1)}.$$

The second inequality in (3.32) implies now that, as long as $x\leq q^{-s/(s-1)}$, we have

$$G_2^{q,s}(x)\geq Lq^{-s/(s-1)}h_{\boldsymbol{M}_{q,\sigma}}\left(\frac{x}{q^{s/(s-1)}}\right),$$

and so (3.5) holds. □

### 3.3.3 Optimal $\{\boldsymbol{M}_{q,\sigma}\}$-flat function in arbitrary sectors

Let us consider a sector $S_\gamma\subset\mathcal{R}$ with $\gamma>2$, and define the function

$$G_\gamma^{q,s}(z):=\exp\left(-b_{q,s}\left(\frac{\gamma}{2}\right)^s\log^s\left(1+z^{-2/\gamma}\right)\right)=\left(G_2^{q,s}(z^{2/\gamma})\right)^{(\gamma/2)^s},\quad z\in S_\gamma. \tag{3.39}$$

The map $z\mapsto z^{2/\gamma}$ is holomorphic from $S_\gamma$ into $S_2$, and so $G_\gamma^{q,s}$ is holomorphic in $S_\gamma$. We will prove that this function is an optimal $\{\boldsymbol{M}_{q,\sigma}\}$-flat function in this sector.

As before, we consider the kernel

$$e_\gamma(z):=G_\gamma^{q,s}(1/z)=\exp\left(-b_{q,s}\left(\frac{\gamma}{2}\right)^s\log^s\left(1+z^{2/\gamma}\right)\right)=\left(e_2(z^{2/\gamma})\right)^{(\gamma/2)^s},\quad z\in S_\gamma.$$

**Lemma 3.3.4.** *There exist constants $B_1,B_2>0$ such that*

$$|e_\gamma(z)|\leq B_1e_2(B_2|z|),\quad z\in S_\gamma. \tag{3.40}$$

**Proof.** According to the definition of $e_\gamma$ and by applying Lemma 3.3.2, there exist constants $C_1,C_2>0$ such that for every $z\in S_\gamma$ one has

$$|e_\gamma(z)|=\left|e_2(z^{2/\gamma})\right|^{(\gamma/2)^s}\leq\left(C_1e_2(C_2|z|^{2/\gamma})\right)^{(\gamma/2)^s}.$$

We recall that the function $|e_2(z)|$ stays bounded for bounded $|z|$; from the previous estimates, the same can be said about $|e_\gamma(z)|$, and so we can prove (3.40) by restricting our considerations to large enough values of $|z|$ and well chosen $B_2>0$, and then suitably enlarging the constant $B_1>0$ involved. Let us observe that

$$\left(e_2(C_2|z|^{2/\gamma})\right)^{(\gamma/2)^s}=\exp\left(-b_{q,s}\ln^s\left[(1+C_2|z|^{2/\gamma})^{\gamma/2}\right]\right),$$
$$e_2(B_2|z|)=\exp\left(-b_{q,s}\ln^s(1+B_2|z|)\right).$$

So, we will be done if we see that

$$\ln^s(1+B_2|z|)-\ln^s[(1+C_2|z|^{2/\gamma})^{\gamma/2}],$$

admits an upper bound for large enough $|z|$ and suitably chosen $B_2>0$. But this follows from the clear fact that

$$\ln^s(1+B_2|z|)-\ln^s\left[\left(1+C_2|z|^{2/\gamma}\right)^{\gamma/2}\right]\sim -s\ln\left(\frac{C_2^{\gamma/2}}{B_2}\right)\ln^{s-1}(1+B_2|z|),\quad |z|\to\infty,$$

where $\sim$ means that the quotient of both expressions tends to 1. Indeed, in view of this equivalence it suffices to choose any $B_2<C_2^{\gamma/2}$ in order to have the desired estimation for suitably large $B_1$ and $|z|$. $\square$

**Corollary 3.3.5.** *The function $G_\gamma^{q,s}$ defined in* (3.39) *is an optimal $\{\boldsymbol{M}_{q,\sigma}\}$-flat function in $S_\gamma$.*

**Proof.** By the previous lemma, there exist $B_1,B_2>0$ such that

$$|G_\gamma^{q,s}(z)|\le B_1\exp\left(-b_{q,s}\ln^s\left(1+\frac{B_2}{|z|}\right)\right),\quad z\in S_\gamma.$$

Note that this estimate is essentially that in (3.38), and so the conclusion follows in exactly the same way as in the proof of Theorem 3.3.3. $\square$

**Remark 3.3.6.** We mention that a similar approach has been followed in the preprint [27], by A. Lastra and J. Jiménez-Garrido and J. Sanz, in order to construct extension operators for the ultraholomorphic classes associated with the sequences $\boldsymbol{M}^{\tau,\sigma}=(p^{\tau p^\sigma})_{p\in\mathbb{N}_0}$, for $\tau>0$ and $\sigma\in(1,2)$. These sequences have appeared in a series of papers by S. Pilipović, N. Teofanov and F. Tomić [53, 54, 55, 56], inducing ultradifferentiable spaces of so-called extended Gevrey regularity. However, in that case the construction of suitable kernels for our technique involves the Lambert function, whose handling is not so convenient. This fact has caused our results to be available only in sectors strictly contained in $S_2$, in spite of the fact that $\gamma(\boldsymbol{M}_{\tau,\sigma})=\infty$, what would in principle allow for such extension operators to exist in sectors of arbitrary opening.

## 3.4 Convolved sequences, flat functions and extension results

We show in this section that whenever two weight sequences are given and there exist optimal flat functions in the respectively associated classes, then optimal flat functions exist in the class defined by the so-called convolved sequence as well (given by the point-wise product). Moreover, the extension technique works if one of the convolved sequences satisfies (dc).

On the one hand the abstract statement is a straight-forward consequence of a result by H. Komatsu, see Remark 3.4.1 for more details. On the other hand this approach can be useful for constructing (counter-)examples. In general even for nice sequences the convolved sequence can behave in a complicated way, see Sect. 3.4.3, and so a direct explicit construction of optimal flat functions in the class defined by the convolved sequence will be challenging.

### 3.4.1 Convolved sequences

Let $\boldsymbol{M}^1 = (M^1_p)_{p\in\mathbb{N}_0}$, $\boldsymbol{M}^2 = (M^2_p)_{p\in\mathbb{N}_0}$ be two sequences of positive real numbers, then the *convolved sequence* $\boldsymbol{L} := \boldsymbol{M}^1 \star \boldsymbol{M}^2$ is $(L_p)_{p\in\mathbb{N}_0}$ given by

$$L_p := \min_{0\le q\le p} M^1_q M^2_{p-q}, \quad p \in \mathbb{N}_0,$$

see [38, (3.15)]. Hence, obviously $\boldsymbol{M}^1 \star \boldsymbol{M}^2 = \boldsymbol{M}^2 \star \boldsymbol{M}^1$.

For all $p \in \mathbb{N}_0$ we have $L_p \le \min\{M^1_0 M^2_p, M^2_0 M^1_p\}$. So, if in addition $M^1_0 = M^2_0 = 1$, then we get $L_0 = 1$ and

$$L_p \le \min\{M^1_p, M^2_p\}, \quad p \in \mathbb{N}_0. \tag{3.41}$$

Given $\boldsymbol{M} = (M_p)_{p\in\mathbb{N}_0}$ with $M_0 = 1$, put $\boldsymbol{L} = (L_p)_{p\in\mathbb{N}_0} = \boldsymbol{M} \star \boldsymbol{M}$. The condition (mg) states precisely that there exists $A > 0$ such that $M_p \le A^p L_p$ for every $p \in \mathbb{N}_0$; according to (3.41), $\boldsymbol{M}$ satisfies (mg) if and only if $\boldsymbol{M}$ and $\boldsymbol{M} \star \boldsymbol{M}$ are equivalent.

**Remark 3.4.1.** Let $\boldsymbol{M}, \boldsymbol{M}^1, \boldsymbol{M}^2$ be weight sequences.

(i) In [38, Lemma 3.5] the following facts are shown: $\boldsymbol{M}^1 \star \boldsymbol{M}^2$ is again a weight sequence. The corresponding quotient sequence $\boldsymbol{m}^1 \star \boldsymbol{m}^2$ is obtained when rearranging resp. ordering the sequences $\boldsymbol{m}^1$ and $\boldsymbol{m}^2$ in the order of growth. This yields, by definition of the counting function (see (6)), that for all $t \ge 0$ one has

$$\nu_{\boldsymbol{m}^1 \star \boldsymbol{m}^2}(t) = \nu_{\boldsymbol{m}^1}(t) + \nu_{\boldsymbol{m}^2}(t);$$

so, by (7) we get

$$\omega_{\boldsymbol{M}^1 \star \boldsymbol{M}^2}(t) = \omega_{\boldsymbol{M}^1}(t) + \omega_{\boldsymbol{M}^2}(t), \quad t \ge 0,$$

and by (4) we obtain

$$h_{\boldsymbol{M}^1 \star \boldsymbol{M}^2}(t) = h_{\boldsymbol{M}^1}(t) h_{\boldsymbol{M}^2}(t), \quad t > 0. \tag{3.42}$$

(ii) If either $\boldsymbol{M}^1$ or $\boldsymbol{M}^2$ has (dc), then $\boldsymbol{M}^1 \star \boldsymbol{M}^2$ as well: As said before, for sequences $(M_p)_{p\in\mathbb{N}_0}$ satisfying (lc), the condition (dc) amounts to the condition $\log(M_p) = O(p^2)$, $p \to \infty$. Then, it suffices to apply (3.41).

(iii) As seen in item (i), for every $t \geq 0$ we have

$$2\omega_{\boldsymbol{M}}(t) = \omega_{\boldsymbol{M}\star\boldsymbol{M}}(t).$$

Since $\boldsymbol{M}$ satisfies (mg) if and only if there exists $H \geq 1$ such that

$$2\omega_{\boldsymbol{M}}(t) \leq \omega_{\boldsymbol{M}}(Ht) + H, \quad t \geq 0$$

(see [38, Prop. 3.6]), it turns out that (mg) amounts to the fact that

$$\omega_{\boldsymbol{M}\star\boldsymbol{M}}(t) \leq \omega_{\boldsymbol{M}}(Ht) + H, \quad t \geq 0,$$

for some $H \geq 1$, or in other words,

$$h_{\boldsymbol{M}}(t) \leq e^H h_{\boldsymbol{M}\star\boldsymbol{M}}(Ht), \quad t > 0.$$

### 3.4.2 Optimal flat functions and extension procedure

Let $\boldsymbol{M}^1$ and $\boldsymbol{M}^2$ be weight sequences such that optimal flat functions $G_{\boldsymbol{M}^1}$ and $G_{\boldsymbol{M}^2}$ exist in the corresponding classes with uniform asymptotic expansion in a given sector $S$. Then, we claim that $G_{\boldsymbol{M}^1\star\boldsymbol{M}^2} := G_{\boldsymbol{M}^1} \cdot G_{\boldsymbol{M}^2}$ is an optimal flat function (on the same sector $S$) in the class associated with the sequence $\boldsymbol{M}^1\star\boldsymbol{M}^2$. Suppose $K_m$ and $J_m$, $m = 1, 2, 3, 4$, are the constants appearing in (3.5) and (3.6) for $G_{\boldsymbol{M}^1}$ and $G_{\boldsymbol{M}^2}$, respectively. By (3.42) we get that, for all $z \in S$,

$$\begin{aligned}|G_{\boldsymbol{M}^1}(z) \cdot G_{\boldsymbol{M}^2}(z)| &\leq K_3 h_{\boldsymbol{M}^1}(K_4|z|) J_3 h_{\boldsymbol{M}^2}(J_4|z|)\\ &\leq K_3J_3 h_{\boldsymbol{M}^1}(D|z|) h_{\boldsymbol{M}^2}(D|z|) = C h_{\boldsymbol{M}^1\star\boldsymbol{M}^2}(D|z|),\end{aligned}$$

with $C := K_3J_3$ and $D := \max\{K_4, J_4\}$, since each function $h_{\boldsymbol{M}}$ is nondecreasing. Similarly, for $x > 0$ we can estimate

$$\begin{aligned}G_{\boldsymbol{M}^1}(x) \cdot G_{\boldsymbol{M}^2}(x) &\geq K_1 h_{\boldsymbol{M}^1}(K_2x) J_1 h_{\boldsymbol{M}^2}(J_2x)\\ &\geq K_1J_1 h_{\boldsymbol{M}^1}(D_1x) h_{\boldsymbol{M}^2}(D_1x) = C_1 h_{\boldsymbol{M}^1\star\boldsymbol{M}^2}(D_1x),\end{aligned}$$

with $C_1 := K_1J_1$ and $D_1 := \min\{K_2, J_2\}$, and the conclusion follows.

In case at least one of the sequences $\boldsymbol{M}^1$ and $\boldsymbol{M}^2$ satisfies (dc), $\boldsymbol{M}^1 \star \boldsymbol{M}^2$ does so, and the extension operators from Theorem 3.2.19 will be available for the convolved sequence.

### 3.4.3 Some examples

Fix $q > 1$ and $\sigma \in (1, 2]$. Let us put $\boldsymbol{L}_{q,\sigma} := \boldsymbol{M}_{q,\sigma} \star \boldsymbol{M}_{q,\sigma}$, $\boldsymbol{L}_{q,\sigma} = (L_p)_{p\in\mathbb{N}_0}$. It is not difficult to check that

$$L_{2p} = q^{2p^\sigma}, \quad L_{2p+1} = q^{p^\sigma+(p+1)^\sigma}, \qquad p \in \mathbb{N}_0.$$

Observe that $2p^\sigma = 2^{1-\sigma}(2p)^\sigma$, so that $L_{2p}$ equals the $2p$-th term of the sequence $\boldsymbol{M}_{q^{2^{1-\sigma}},\sigma}$. Regarding the odd terms, it is a consequence of Taylor's formula at $x = 0$ for the functions of the form $x \mapsto (1 + x)^\alpha$, $\alpha > 0$, that

$$p^\sigma + (p + 1)^\sigma - 2^{1-\sigma}(2p + 1)^\sigma = O(p^{\sigma-2}), \quad p \to \infty.$$

Since $\sigma \in (1, 2]$, we deduce that $\boldsymbol{L}_{q,\sigma}$ is equivalent to $\boldsymbol{M}_{q^{2^{1-\sigma}},\sigma}$.

According to Subsection 3.4.2, an optimal flat function in the class associated with $\boldsymbol{L}_{q,\sigma}$ in, say, the sector $S_2$ is the function

$$G(z) := G_2^{q,s}(z)G_2^{q,s}(z) = \exp\left(-2b_{q,s}\log^s\left(1 + \frac{1}{z}\right)\right), \quad z \in S_2.$$

It is not a surprise that, from the definition (3.28) of $b_{q,s}$ and the relation between $\sigma$ and $s$, one obtains $b_{q^{2^{1-\sigma}},s} = 2b_{q,s}$, and so $G$ is precisely $G_2^{q^{2^{1-\sigma}},s}$, what agrees with the aforementioned equivalence of sequences.

If we consider instead $1 < \sigma < 2$ and $\boldsymbol{J} := \boldsymbol{M}_{q,\sigma} \star \boldsymbol{M}_{q,2}$, $\boldsymbol{J} = (J_p)_{p\in\mathbb{N}_0}$, the computation of the terms $J_p$ is no longer possible in closed form, since their values depend for general $p$ on the position of $\sigma$ within the interval $(1, 2)$. However, the previous subsection shows that, for $s$ associated with $\sigma$ as usual, the function

$$G(z) := G_2^{q,s}(z)G_2^{q,2}(z) = \exp\left(-b_{q,s}\log^s\left(1 + \frac{1}{z}\right) - b_{q,2}\log^2\left(1 + \frac{1}{z}\right)\right), \quad z \in S_2,$$

is an optimal flat function in the class associated with $\boldsymbol{J}$ in $S_2$. Note that $s$ is not equal to 2, hence the very aspect of the exponent in this function, and the fact that the restriction $G|_{(0,\infty)}$ is closely related to the function $h_{\boldsymbol{J}}$ (see Definition 3.2.3), shows that $\boldsymbol{J}$ is not equivalent to any of the sequences $\boldsymbol{M}_{q,\sigma}$. Since the sequence $\boldsymbol{J}$ does satisfy (dc), the extension procedure described in this paper is available for the classes associated with $\boldsymbol{J}$.

Observe that these examples of optimal flat functions can also be provided in general sectors $S_\gamma$, $\gamma > 2$, by using the functions $G_\gamma^{q,s}$ introduced in (3.39).

## 3.5 An improved Borel-Ritt theorem in the Roumieu case

We describe next a new condition on a weight sequence $\boldsymbol{M}$, which, as long as $\gamma(\boldsymbol{M}) > 0$, will amount to the equivalence of the sequence $\boldsymbol{M}$ shifted one position to the left with the sequence of Stieltjes moments of a suitable kernel function defined from an optimal flat function. This fact motivates the terminology, and will be extremely important for deducing an improved Borel-Ritt theorem, as this condition is much weaker than (dc).

### 3.5.1 The condition of shifted moments

We start with the definition.

**Definition 3.5.1.** Let $\boldsymbol{M}$ be a sequence. We say that $\boldsymbol{M}$ has *shifted moments*, denoted by (sm), if there exist some constants $C_0>0$ and $H>1$ such that

$$\log(m_{p+1}/m_p)\le C_0H^{p+1},\qquad p\in\mathbb{N}_0.$$

**Remark 3.5.2.** Property (sm) is generally kept when going from $\boldsymbol{M}$ to $\widehat{\boldsymbol{M}}$ or from $\boldsymbol{M}$ to $\widecheck{\boldsymbol{M}}$; while this statement is well-known for (dc) and (mg), the one for the, up to our knowledge, new condition (sm) stems from the inequalities

$$\log\left(\frac{m_{p+1}}{m_p}\right)\le\log\left(\frac{m_{p+1}}{m_p}\right)+\log\left(\frac{p+2}{p+1}\right)=\log\left(\frac{\widehat{m}_{p+1}}{\widehat{m}_p}\right)\le\log\left(\frac{m_{p+1}}{m_p}\right)+\log(2),$$

(when applied to $\boldsymbol{M}$ or to $\widecheck{\boldsymbol{M}}$).

The next lemma shows that this condition is weaker than (dc).

**Lemma 3.5.3.** *Let $\boldsymbol{M}$ be a sequence such that $a_0:=\inf_{p\in\mathbb{N}_0}m_p>0$ (in particular, this holds if $\boldsymbol{M}$ is* (lc)*). Then,* (dc) *implies* (sm)*.*

*Proof.* Since $m_p\le C_0H^{p+1}$ and $p+1\le 2^p$ for every $p\in\mathbb{N}_0$ and some $C_0>0$ and $H>1$, one has

$$\log\left(\frac{m_{p+1}}{m_p}\right)\le\log(m_{p+1})-\log(a_0)\le\log(C_0/a_0)+(p+2)\log(H)\le C_1H_1^{p+1}$$

for every $p\in\mathbb{N}_0$, with the choices $C_1=\log(HC_0^{1/2}/a_0^{1/2})>0$ (note that $a_0\le C_0H<C_0H^2$) and $H_1=2$. □

It is straightforward that (dc) and (mg) are stable under equivalence for general sequences, and the same can be deduced for (nq) and (snq) for weight sequences by indirect methods, as these two last conditions characterize the non injectivity and the surjectivity, respectively, of the Borel map in Carleman ultradifferentiable classes, by the classical Denjoy-Carleman theorem (see, for example, [67]) and the results of H.-J. Petzsche [52, Cor. 3.2] (see [34, Cor. 3.14] for a direct proof of a more general statement about the stability of (snq)). We prove that (sm) is stable under equivalence for general sequences.

**Lemma 3.5.4.** *The property* (sm) *is preserved under equivalence of sequences.*

*Proof.* Suppose $\boldsymbol{M}$ satisfies (sm), and consider $\boldsymbol{L} = (L_p)_p$ such that $\boldsymbol{M} \approx \boldsymbol{L}$. There exists $C > 1$ such that $C^{-p-1}L_p \leq M_p \leq C^{p+1}L_p$, $p \in \mathbb{N}_0$. Consequently, for all $p \in \mathbb{N}_0$ one has

$$l_p = \frac{L_{p+1}}{L_p} \leq \frac{C^{p+2}C^{p+1}M_{p+1}}{M_p} = C^{2p+3}m_p, \quad l_p \geq \frac{M_{p+1}}{C^{p+2}C^{p+1}M_p} = C^{-2p-3}m_p.$$

So, taking into account that $p+1 \leq 2^p$ for every $p$, we have

$$\begin{aligned}\log\left(\frac{l_{p+1}}{l_p}\right) &\leq \log\left(C^{4p+8}\frac{m_{p+1}}{m_p}\right) \leq (4p+8)\log(C) + C_0H^{p+1} \\ &\leq 4\log(C) + 2\log(C)2^{p+1} + C_0H^{p+1} \leq C_1H_1^{p+1}\end{aligned}$$

for the choices $C_1 = 4\log(C) + C_0 > 0$ and $H_1 = \max\{2, H\} > 1$, so that $\boldsymbol{L}$ also satisfies (sm). $\square$

Moreover, it is clear that $\boldsymbol{M}$ satisfies (sm) if, and only if, $\boldsymbol{M}^r := (M_p^r)_p$ does for some/every $r > 0$. Now, we present some examples, compare with the previous ones given in 1.1.7.

**Examples 3.5.5.** (i) The sequences $\boldsymbol{M}_{q,\sigma} = (q^{p^\sigma})_p$ ($q > 1$, $0 < \sigma \leq 2$) and $\boldsymbol{M} = (p^{\tau p^\sigma})_p$ ($\tau > 0$, $1 < \sigma < 2$) are such that $\widehat{\boldsymbol{M}}$ is regular, and therefore they have (sm). In particular, the $q$-Gevrey sequences $\boldsymbol{M}_{q,2} = (q^{p^2})_p$ ($q > 1$) appear in the study of $q$-difference equations.

(ii) The weight sequences $\boldsymbol{M}_{q,\sigma} = (q^{p^\sigma})_p$ ($q > 1$, $\sigma > 2$) and $\boldsymbol{M} = (p^{\tau p^\sigma})_p$ ($\tau > 0$, $\sigma \geq 2$) do not satisfy (dc), so that $\widehat{\boldsymbol{M}}$ is not regular, but they still satisfy (sm). The sequences of the family $\{(p^{\tau p^\sigma})_p\}_{\sigma>1}$ have appeared as the defining sequences for some generalized ultradifferentiable classes "beyond Gevrey regularity", deeply studied in a series of papers by S. Pilipović, N. Teofanov and F. Tomić [53, 54, 55, 56, 78], J. Jiménez-Garrido, A. Lastra and J. Sanz [27] and J. Jiménez-Garrido, D. N. Nenning and G. Schindl [31].

(iii) The rapidly growing weight sequences $\boldsymbol{M} = (q^{p^p})_p$ ($q > 1$; $M_0 := 1$) are weight sequences which do not satisfy (sm). As it will be seen, they are the only ones in the list to which the results in the following sections cannot be applied.

### 3.5.2 A new Borel-Ritt theorem in the Roumieu case

In this section we present an improvement in the Borel-Ritt-Gevrey theorem (see Theorem 3.2.19), where the condition (dc) is changed into the much weaker condition (sm). First, we characterize this new condition in terms of the equivalence already announced.

**Proposition 3.5.6.** *Suppose $\boldsymbol{M}$ is a weight sequence with $\gamma(\boldsymbol{M}) > 0$, and $G$ is an optimal $\{\boldsymbol{M}\}$-flat function in $\widetilde{\mathcal{A}}^u_{\{\boldsymbol{M}\}}(S)$, where $S$ is an unbounded sector bisected by the positive real line. Consider the sequence of moments $\boldsymbol{\mu} := (\mu(p))_{p\in\mathbb{N}_0}$ associated with the kernel function $e(z) = G(1/z)$. Then, $\boldsymbol{M}$ satisfies* (sm) *if, and only if, $\boldsymbol{M}_{+1} := (M_{p+1})_{p\in\mathbb{N}_0}$ and $\boldsymbol{\mu}$ are equivalent.*

*Proof.* Suppose $\boldsymbol{M}$ satisfies (sm). On the one hand, because of the right-hand inequalities in (3.19) and the definition of $h_{\boldsymbol{M}}$, we have

$$t^p e(t) \leq K_3 K_4^p M_p, \quad p \in \mathbb{N}_0,\ t > 0.$$

So, we may write

$$\begin{aligned}
\mu(p) &= \int_0^{K_4 m_p} t^p e(t)\,dt + \int_{K_4 m_p}^{K_4 m_{p+1}} t^p e(t)\,dt + \int_{K_4 m_{p+1}}^{\infty} \frac{1}{t^2} t^{p+2} e(t)\,dt \\
&\leq K_3 K_4 m_p K_4^p M_p + K_3 \int_{K_4 m_p}^{K_4 m_{p+1}} t^p h_{\boldsymbol{M}}\left(\frac{K_4}{t}\right) dt + K_3 K_4^{p+2} M_{p+2} \frac{1}{K_4 m_{p+1}} \\
&= 2K_3 K_4^{p+1} M_{p+1} + K_3 K_4^{p+1} M_{p+1} \log\left(\frac{m_{p+1}}{m_p}\right),
\end{aligned}$$

where in the last equality we have used (1.8). Since there exists $C_0 > 0$ and $H > 1$ such that $\log(m_{p+1}/m_p) \leq C_0 H^{p+1}$ for every $p$, we get

$$\mu(p) \leq K_3 K_4^{p+1} M_{p+1}(2 + C_0 H^{p+1}) \leq K_3 K_4 (2 + C_0 H)(K_4 H)^p M_{p+1},$$

and so $\boldsymbol{\mu} \precsim \boldsymbol{M}_{+1}$.

On the other hand, by the left-hand inequalities in (3.19), for every $p \in \mathbb{N}_0$ we may estimate

$$\mu(p) \geq \int_0^s t^p e(t)\,dt \geq K_1 \int_0^s t^p h_{\boldsymbol{M}}\left(\frac{K_2}{t}\right) dt \geq K_1 h_{\boldsymbol{M}}\left(\frac{K_2}{s}\right) \frac{s^{p+1}}{p+1}.$$

Then, by (1.9) we deduce that

$$\mu(p) \geq \frac{K_1}{p+1} \sup_{s>0} h_{\boldsymbol{M}}\left(\frac{K_2}{s}\right) s^{p+1} = \frac{K_1}{p+1} K_2^{p+1} M_{p+1} \geq K_1 K_2 \left(\frac{K_2}{2}\right)^p M_{p+1},$$

and so $\boldsymbol{M}_{+1} \precsim \boldsymbol{\mu}$, as desired.

Conversely, suppose $\boldsymbol{M}_{+1} \approx \boldsymbol{\mu}$. In particular, there exist $C, h > 0$ such that $\mu(p) \leq Ch^p M_{p+1}$ for $p \in \mathbb{N}_0$. By the left-hand inequalities in (3.19), we may estimate

$$\mu(p) \geq \int_{K_2 m_p}^{K_2 m_{p+1}} t^p e(t)\, dt \geq K_1 \int_{K_2 m_p}^{K_2 m_{p+1}} t^p h_{\boldsymbol{M}}\left(\frac{K_2}{t}\right) dt$$
$$= K_1 K_2^{p+1} M_{p+1} \log\left(\frac{m_{p+1}}{m_p}\right), \quad p \in \mathbb{N}_0.$$

Therefore,

$$K_1 K_2^{p+1} M_{p+1} \log\left(\frac{m_{p+1}}{m_p}\right) \leq Ch^p M_{p+1}, \quad p \in \mathbb{N}_0,$$

and $\boldsymbol{M}$ satisfies (sm). $\square$

We can already state the following main result, whose proof is an adaptation of the one for Theorem 3.2.19. Regrettably, we are not able to deduce $\gamma \leq \gamma(\boldsymbol{M})$ from the surjectivity of the Borel map in classes on sectors $S_\gamma$ under this weaker condition (sm).

**Theorem 3.5.7.** *Let $\boldsymbol{M}$ be a weight sequence satisfying* (sm) *and with $\gamma(\boldsymbol{M}) > 0$, and let $\gamma > 0$ be given. Then, each of the following statements implies the next one:*

*(i)* $\gamma < \gamma(\boldsymbol{M})$.

*(ii) There exists $c > 0$ such that for every $h > 0$ there exists an extension operator from $\mathbb{C}[[z]]_{\boldsymbol{M},h}$ into $\widetilde{\mathcal{A}}^u_{\boldsymbol{M},ch}(S_\gamma)$.*

*(iii) The Borel map $\widetilde{\mathcal{B}}\colon \widetilde{\mathcal{A}}^u_{\{\boldsymbol{M}\}}(S_\gamma) \to \mathbb{C}[[z]]_{\{\boldsymbol{M}\}}$ is surjective. In other words, $(0, \gamma] \subset \widetilde{S}^u_{\{\boldsymbol{M}\}}$.*

*(iv)* $(0, \gamma) \subset S_{\{\widehat{\boldsymbol{M}}\}}$.

*In particular, one has $(0, \gamma(\boldsymbol{M})) \subset S_{\{\widehat{\boldsymbol{M}}\}} \subset \widetilde{S}^u_{\{\boldsymbol{M}\}}$.*

*Proof.* (i) $\Rightarrow$ (ii) By Proposition 3.2.17, valid for any weight sequence $\boldsymbol{M}$ with $\gamma(\boldsymbol{M}) > 0$, we can consider an optimal $\{\boldsymbol{M}\}$-flat function $G$ in $\widetilde{\mathcal{A}}^u_{\{\boldsymbol{M}\}}(S_\gamma)$. Let $(\mu(p))_{p\in\mathbb{N}_0}$ be the sequence of moments associated with the function $e(z) = G(1/z)$. Given $h > 0$ and $\widehat{f} = \sum_{p=0}^{\infty} a_p z^p \in \mathbb{C}[[z]]_{\boldsymbol{M},h}$, by the definition of the norm in $\mathbb{C}[[z]]_{\boldsymbol{M},h}$ (see (3.3)), we have

$$|a_p| \leq |\widehat{f}|_{\boldsymbol{M},h} h^p M_p, \quad p \in \mathbb{N}_0.$$

Because of Proposition 3.5.6, there exist $h_1, h_2 > 0$ such that

$$h_1^{p+1} M_{p+1} \le \mu(p) \le h_2^{p+1} M_{p+1}, \quad p \in \mathbb{N}_0. \tag{3.43}$$

So, we deduce that

$$\left| \frac{a_{p+1}}{\mu(p)} \right| \le |\widehat{f}|_{\boldsymbol{M},h} \left( \frac{h}{h_1} \right)^{p+1}, \quad p \in \mathbb{N}_0. \tag{3.44}$$

Hence, the formal Borel-like transform of $\widehat{f} - a_0$, defined as

$$\widehat{g} = \sum_{p=0}^{\infty} \frac{a_{p+1}}{\mu(p)} z^p,$$

is convergent in the disc $D(0,R)$ for $R = h_1/h > 0$, and it defines a holomorphic function $g$ there. Choose $R_0 := h_1/(2h) < R$, and define

$$I_{\boldsymbol{M},h}(\widehat{f}\,)(z) := \int_0^{R_0} e\left(\frac{u}{z}\right) g(u)\, du, \qquad z \in S_\gamma,$$

which is a truncated Laplace-like transform of $g$ with kernel $e$. By Leibniz's theorem for parametric integrals and the properties of $e$, this function is holomorphic in $S_\gamma$. We will prove that $I_{\boldsymbol{M},h}(\widehat{f}\,) \sim^u_{\{\boldsymbol{M}\}} \hat{f} - a_0$ uniformly in $S_\gamma$.

Let $p \in \mathbb{N}$ and $z \in S_\gamma$. We have

$$\begin{aligned} I_{\boldsymbol{M},h}(\widehat{f}\,)(z) - \sum_{n=1}^{p-1} a_n z^n &= I_{\boldsymbol{M},h}(\widehat{f}\,)(z) - \sum_{n=1}^{p-1} \frac{a_n}{\mu(n-1)} \mu(n-1) z^n \\ &= \int_0^{R_0} e\left(\frac{u}{z}\right) \sum_{n=1}^{\infty} \frac{a_n}{\mu(n-1)} u^{n-1}\, du \\ &\quad - \sum_{n=1}^{p-1} \frac{a_n}{\mu(n-1)} \int_0^{\infty} v^{n-1} e(v)\, dv\, z^n. \end{aligned}$$

A change of variables $u = zv$ in the last integral, Cauchy's residue theorem and the right-hand estimates in (3.19) allow us to rotate the path of integration and obtain

$$z^n \int_0^{\infty} v^{n-1} e(v) dv = \int_0^{\infty} u^{n-1} e\left(\frac{u}{z}\right)\, du.$$

So, the preceding difference can be written as

$$\int_0^{R_0} e\left(\frac{u}{z}\right) \sum_{n=p}^{\infty} \frac{a_n}{\mu(n-1)} u^{n-1}\, du - \int_{R_0}^{\infty} e\left(\frac{u}{z}\right) \sum_{n=1}^{p-1} \frac{a_n}{\mu(n-1)} u^{n-1}\, du.$$

Then, we have

$$\left| I_{\boldsymbol{M},h}(\widehat{f})(z) - \sum_{n=1}^{p-1} a_n z^n \right| \leq I_{1,p}(z) + I_{2,p}(z), \tag{3.45}$$

where

$$I_{1,p}(z) = \left| \int_0^{R_0} e\left(\frac{u}{z}\right) \sum_{n=p}^{\infty} \frac{a_n}{\mu(n-1)} u^{n-1}\, du \right|,$$

$$I_{2,p}(z) = \left| \int_{R_0}^{\infty} e\left(\frac{u}{z}\right) \sum_{n=1}^{p-1} \frac{a_n}{\mu(n-1)} u^{n-1}\, du \right|.$$

We first estimate $I_{1,p}(z)$. Since for every $u \in (0, R_0]$ we have $0 < hu/h_1 \leq 1/2$, from (3.44) we get

$$\sum_{n=p}^{\infty} \frac{|a_n|}{\mu(n-1)} u^{n-1} \leq |\widehat{f}|_{\boldsymbol{M},h} \frac{h}{h_1} \sum_{n=p}^{\infty} \left(\frac{hu}{h_1}\right)^{n-1} \leq 2|\widehat{f}|_{\boldsymbol{M},h} \left(\frac{h}{h_1}\right)^p u^{p-1}.$$

Hence,

$$I_{1,p}(z) \leq 2|\widehat{f}|_{\boldsymbol{M},h} \left(\frac{h}{h_1}\right)^p \int_0^{R_0} \left| e\left(\frac{u}{z}\right) \right| u^{p-1}\, du. \tag{3.46}$$

Regarding $I_{2,p}(z)$, for $u \geq R_0$ and $1 \leq n \leq p-1$ we have $(u/R_0)^{n-1} \leq (u/R_0)^{p-1}$, so $u^{n-1} \leq R_0^{n-1} u^{p-1}/R_0^{p-1}$. Again by (3.44), and taking into account that $R_0 = h_1/(2h)$, we may write

$$\begin{aligned}\sum_{n=1}^{p-1} \frac{|a_n|}{\mu(n-1)} u^{n-1} &\leq |\widehat{f}|_{\boldsymbol{M},h} \frac{h u^{p-1}}{h_1 R_0^{p-1}} \sum_{n=1}^{p-1} \left(\frac{hR_0}{h_1}\right)^{n-1} \\ &\leq 2|\widehat{f}|_{\boldsymbol{M},h} \frac{h}{h_1 R_0^{p-1}} u^{p-1} = |\widehat{f}|_{\boldsymbol{M},h} \left(\frac{2h}{h_1}\right)^p u^{p-1}.\end{aligned}$$

Then,

$$I_{2,p}(z) \leq |\widehat{f}|_{\boldsymbol{M},h} \left(\frac{2h}{h_1}\right)^p \int_{R_0}^{\infty} \left| e\left(\frac{u}{z}\right) \right| u^{p-1}\, du,$$

and together with (3.45) and (3.46) we deduce

$$\left| I_{\boldsymbol{M},h}(\widehat{f})(z) - \sum_{n=1}^{p-1} a_n z^n \right| \leq |\widehat{f}|_{\boldsymbol{M},h} \left(\frac{2h}{h_1}\right)^p \int_0^{\infty} \left| e\left(\frac{u}{z}\right) \right| u^{p-1}\, du.$$

We estimate the last integral using first the second inequality in (3.19), then the first one, and the fact that $e(x) > 0$ for $x > 0$, so that for every $z \in S_\gamma$ and every $u > 0$ we have

$$\left| e\left(\frac{u}{z}\right) \right| \leq K_3 h_{\boldsymbol{M}} \left( K_4 \frac{|z|}{u} \right) \leq \frac{K_3}{K_1} e\left( \frac{K_2 u}{K_4 |z|} \right).$$

A change of variables and the right-hand estimates in (3.43) lead to

$$\begin{aligned}\int_0^\infty \left|e\left(\frac{u}{z}\right)\right| u^{p-1}\,du &\leq \int_0^\infty \frac{K_3}{K_1}\, e\left(\frac{K_2 u}{K_4|z|}\right) u^{p-1}\,du\\ &= \frac{K_3}{K_1}\left(\frac{K_4|z|}{K_2}\right)^p \mu(p-1) \leq \frac{K_3}{K_1}\left(\frac{K_4 h_2}{K_2}\right)^p M_p|z|^p.\end{aligned}$$

So, for every $p\in\mathbb{N}_0$ we have

$$\left|I_{\boldsymbol{M},h}(\widehat{f}\,)(z) - \sum_{n=1}^{p-1} a_n z^n\right| \leq \frac{K_3|\widehat{f}|_{\boldsymbol{M},h}}{K_1}\left(\frac{2K_4h_2h}{K_2h_1}\right)^p M_p|z|^p, \quad z\in S_\gamma,$$

and $I_{\boldsymbol{M},h}(\widehat{f}\,)$ admits $\widehat{f}-a_0$ as its uniform $\{\boldsymbol{M}\}$-asymptotic expansion in $S_\gamma$. We consider the map $T_{\boldsymbol{M},h}$ defined in $\mathbb{C}[[z]]_{\boldsymbol{M},h}$ as

$$T_{\boldsymbol{M},h}(\widehat{f}\,)(z) := I_{\boldsymbol{M},h}(\widehat{f}\,)(z) + a_0, \qquad z\in S_\gamma,$$

which is obviously linear. Moreover, if we set $c := 2K_4h_2/(K_2h_1) > 0$,

$$\begin{aligned}\left|T_{\boldsymbol{M},h}(\widehat{f}\,)(z) - \sum_{n=0}^{p-1} a_n z^n\right| &= \left|I_{\boldsymbol{M},h}(\widehat{f}\,)(z) - \sum_{n=1}^{p-1} a_n z^n\right|\\ &\leq \frac{K_3|\widehat{f}|_{\boldsymbol{M},h}}{K_1}(ch)^p M_p|z|^p, \quad z\in S_\gamma,\end{aligned}$$

which proves that $T_{\boldsymbol{M},h}(\widehat{f}\,)\in\widetilde{\mathcal{A}}^u_{\boldsymbol{M},ch}(S_\gamma)$ and, according to (3.2),

$$\|T_{\boldsymbol{M},h}(\widehat{f}\,)(z)\|_{\boldsymbol{M},ch,\widetilde{u}} \leq \frac{K_3}{K_1}|\widehat{f}|_{\boldsymbol{M},h}, \quad \widehat{f}\in\mathbb{C}[[z]]_{\boldsymbol{M},h},$$

so that the continuity of $T_{\boldsymbol{M},h}\colon \mathbb{C}[[z]]_{\boldsymbol{M},h}\to\widetilde{\mathcal{A}}^u_{\boldsymbol{M},ch}(S_\gamma)$ is obtained.

*(ii)* ⇒ *(iii)* Immediate for any weight sequence $\boldsymbol{M}$.

*(iii)* ⇒ *(iv)* It follows from (3.4), again valid for any weight sequence.

□

**Remark 3.5.8.** Theorem 3.5.7 and Proposition 3.1.4.*(ii)* together guarantee that for every $\gamma\in(0,\gamma(\boldsymbol{M}))$ there exists $a>0$ such that for every $h>0$ there exists a local extension operator from $\mathbb{C}[[z]]_{\boldsymbol{M},h}$ into $\mathcal{A}_{\widehat{\boldsymbol{M}},ah}(S_\gamma)$.

## 3.6 New surjectivity results in the Beurling Case

In this last section we collect some results on surjectivity and existence of right inverses for the asymptotic Borel maps for Beurling ultraholomorphic classes. We split the results into two subsections according to the condition imposed on the sequence, (dc) or (sm).

### 3.6.1 Continuous right inverses under derivation closedness

A first result on the length of the interval $S_{(\widehat{\boldsymbol{M}})}$ was already provided by J. Schmets and M. Valdivia [74, Theorems 4.4 and 4.6], and it can be rephrased as follows. Here $\lfloor x\rfloor$ stands for the greatest integer which is less than or equal to the real number $x$.

**Theorem 3.6.1.** *Let $\widehat{\boldsymbol{M}}$ be a regular sequence and $r>0$. If there is a global extension operator $U_{\boldsymbol{M}}:\mathbb{C}[[z]]_{(\boldsymbol{M})}\to\mathcal{A}_{(\widehat{\boldsymbol{M}})}(S_r)$, then $\gamma(\boldsymbol{M})>\lfloor r\rfloor$.*

Later on, and regarding strongly regular sequences, the aforementioned result of V. Thilliez [80, Corollary 3.4.1] showed that $(0,\gamma(\boldsymbol{M}))\subset S_{(\widehat{\boldsymbol{M}})}$, and A. Debrouwere [15, Corollary 1.3] has recently proved that surjectivity comes with global extension operators. It is worth noting that [15, Theorem 1.2] gives a complete solution to the Borel-Ritt problem in non-uniform Beurling classes (which are not treated in this paper) defined by strongly regular sequences.

Going back to results without assuming the condition (mg), we first mention that the hypotheses in Theorem 3.6.1 can be improved by using some techniques included in [74], and Proposition 4.3 therein, in the same line of ideas that inspired the proof of a similar statement [33, Theorem 4.14(i)] in the Roumieu case. Note that we will exchange the existence of the extension operator into just the surjectivity of the Borel map, and that $\mathcal{A}_{(\widehat{\boldsymbol{M}})}(S_r)\subset\widetilde{\mathcal{A}}^{u}_{(\boldsymbol{M})}(S_r)$, what again weakens the forthcoming assumption.

**Theorem 3.6.2.** *Let $\widehat{\boldsymbol{M}}$ be a regular sequence. If $r>0$ is such that $\widetilde{\mathcal{B}}:\widetilde{\mathcal{A}}^{u}_{(\boldsymbol{M})}(S_r)\to\mathbb{C}[[z]]_{(\boldsymbol{M})}$ is surjective, then $\gamma(\boldsymbol{M})>\lfloor r\rfloor$.*

We include the proof for the reader's convenience. First, for $r\in\mathbb{N}$ we need to introduce the space $\mathcal{N}_{r,(\boldsymbol{M})}([0,\infty))$ consisting of the functions $f\in\mathcal{C}^{\infty}([0,\infty))$ such that:

(a) $f^{(pr+j)}(0)=0$ for every $p\in\mathbb{N}_0$ and $j\in\{1,\dots,r-1\}$ (this condition is empty when $r=1$),

(b) for every $h>0$ one has

$$\sup_{p\in\mathbb{N}_0,\,x\in[0,\infty)}\frac{|f^{(pr)}(x)|}{h^pM_p}<\infty.$$

We recall the following crucial result.

**Proposition 3.6.3** ([74], Prop. 4.3)**.** *Let $r \in \mathbb{N}$ and $\boldsymbol{M}$ be a weight sequence. If the map $\mathcal{B}_r : \mathcal{N}_{r,(\boldsymbol{M})}([0,\infty)) \longrightarrow \mathbb{C}[[z]]_{(\widetilde{\boldsymbol{M}})}$ sending $f$ to the formal power series $\sum_{p=0}^{\infty}(f^{(pr)}(0)/p!)z^p$ is surjective, then the sequence $\boldsymbol{m}$ satisfies the condition $(\gamma_r)$ or, in other words, $\gamma(\boldsymbol{M}) > r$.*

*Proof of Theorem 3.6.2.* If $r \in (0,1)$, then it suffices to observe that $\widetilde{S}^u_{(\boldsymbol{M})}$ is not empty in order to conclude, by Lemma 3.2.1, that $\gamma(\boldsymbol{M}) > 0 = \lfloor r \rfloor$ (note that (dc) has not been used in this case).

Suppose now that $r \geq 1$ and put $r_0 = \lfloor r \rfloor \in \mathbb{N}$. We will show that $\mathcal{B}_{r_0} : \mathcal{N}_{r_0,(\boldsymbol{M})}([0,\infty)) \longrightarrow \mathbb{C}[[z]]_{(\widetilde{\boldsymbol{M}})}$ is surjective, and then $\gamma(\boldsymbol{M}) > r_0 = \lfloor r \rfloor$ by Proposition 3.6.3, as desired.

Given $\widehat{g} = \sum_{p=0}^{\infty} a_p z^p \in \mathbb{C}[[z]]_{(\widetilde{\boldsymbol{M}})}$, we write $b_p := a_p p!$ for all $p \in \mathbb{N}_0$, and for every $h > 0$ there exists $C_1 > 0$ such that

$$|b_p| \leq C_1 h^p p! \widetilde{M}_p = C_1 h^p M_p, \quad p \in \mathbb{N}_0. \tag{3.47}$$

Consider the formal power series $\widehat{f} = \sum_{p=0}^{\infty}(-1)^{pr_0} b_p z^p \in \mathbb{C}[[z]]_{(\boldsymbol{M})}$. By hypothesis, there exists $\psi \in \widetilde{\mathcal{A}}^u_{(\widehat{\boldsymbol{M}})}(S_r)$ such that $\widetilde{\mathcal{B}}(\psi) = \widehat{f}$, and so there exists $C_2 > 0$ such that for every $p \in \mathbb{N}_0$ one has

$$\left|\psi(z) - \sum_{k=0}^{p-1}(-1)^{kr_0} b_k z^k\right| \leq C_2 h^p M_p |z|^p, \qquad z \in S_r. \tag{3.48}$$

The function $\varphi : S_{r/r_0} \to \mathbb{C}$ given by $\varphi(w) = \psi(w^{-r_0}) - b_0$, is well defined and holomorphic in $S_{r/r_0} \supseteq S_1$. Moreover, according to (3.48) for $p = 1$, for every $w \in S_1$ one has

$$\left|\frac{\varphi(w)}{w}\right| = \frac{1}{|w|}|\psi(w^{-r_0}) - b_0| \leq \frac{C_2 h M_1}{|w|^{r_0+1}}. \tag{3.49}$$

So, the function $f : \mathbb{R} \to \mathbb{C}$ given by

$$f(t) = \frac{1}{2\pi i}\int_{1-\infty i}^{1+\infty i} e^{tu}\frac{\varphi(u)}{u}\,du$$

is well defined and continuous on $\mathbb{R}$. By the classical Hankel formula for the reciprocal Gamma function, for every natural number $p \geq 2$ and every $t \in \mathbb{R}$ we may write

$$f(t) - \sum_{k=1}^{p-1}(-1)^{kr_0} b_k \frac{t^{kr_0}}{(kr_0)!} = \frac{1}{2\pi i}\int_{1-\infty i}^{1+\infty i} e^{tu}\left(\frac{\varphi(u)}{u} - \sum_{k=1}^{p-1}\frac{(-1)^{kr_0} b_k}{u^{kr_0+1}}\right) du. \tag{3.50}$$

Since, again by (3.48), we have

$$\left|\frac{\varphi(u)}{u}-\sum_{k=1}^{p-1}(-1)^{kr_0}b_k\frac{1}{u^{kr_0+1}}\right|=\frac{1}{|u|}\left|\psi(u^{-r_0})-\sum_{k=0}^{p-1}(-1)^{kr_0}b_k(u^{-r_0})^k\right|\leq\frac{C_2h^pM_p}{|u|^{pr_0+1}}, \tag{3.51}$$

for every $u\in S_1$, we can apply Leibniz's theorem for parametric integrals and deduce that the function

$$f(t)-\sum_{k=1}^{p-1}(-1)^{kr_0}b_k\frac{t^{kr_0}}{(kr_0)!},$$

belongs to $\mathcal{C}^{pr_0-1}(\mathbb{R})$. Moreover, all of its derivatives of order $m\leq pr_0-1$ at $t=0$ vanish, see the proof of [33, Theorem 4.14(i)].

As $p$ is arbitrary, we have that $f\in\mathcal{C}^\infty(\mathbb{R})$ and, moreover,

$$f^{(m)}(0)=\begin{cases}(-1)^{pr_0}b_p & \text{if } m=pr_0 \text{ for some } p\geq 1,\\ 0 & \text{otherwise.}\end{cases}$$

Finally, we define the function

$$F(t)=b_0+f(-t),\quad t\geq 0.$$

Obviously, $F\in\mathcal{C}^\infty([0,\infty))$ and $F^{(pr_0)}(0)=b_p$, $p\in\mathbb{N}_0$; $F^{(m)}(0)=0$ otherwise. In order to conclude, we estimate the derivatives of $F$ of order $pr_0$ for some $p\in\mathbb{N}_0$. For $p=0$ and $t\geq 0$, we take into account (3.47) and (3.49) in order to obtain that

$$|F^{(0)}(t)|\leq|b_0|+\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-t}\frac{C_2hM_1}{|1+yi|^{r_0+1}}\,dy\leq C_1+\frac{C_2hM_1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(1+y^2)^{(r_0+1)/2}}\,dy, \tag{3.52}$$

and so $F$ is bounded. For $p\geq 1$ we may write formula (3.50) evaluated at $-t$ as

$$f(-t)-\sum_{k=1}^{p}b_k\frac{t^{kr_0}}{(kr_0)!}=\frac{1}{2\pi i}\int_{1-\infty i}^{1+\infty i}e^{-tz}\left(\frac{\varphi(z)}{z}-\sum_{k=1}^{p}\frac{(-1)^{kr_0}b_k}{z^{kr_0+1}}\right)dz.$$

Then,

$$\begin{aligned}F^{(pr_0)}(t)&=b_p+\left(f(-t)-\sum_{k=1}^{p}b_k\frac{t^{kr_0}}{(kr_0)!}\right)^{(pr_0)}(t)\\&=b_p+\frac{1}{2\pi i}\int_{1-\infty i}^{1+\infty i}e^{-tz}(-z)^{pr_0}\left(\frac{\varphi(z)}{z}-\sum_{k=1}^{p}\frac{(-1)^{kr_0}b_k}{z^{kr_0+1}}\right)dz,\end{aligned}$$

and we may apply (3.47), and (3.51) in order to obtain

$$|F^{(pr_0)}(t)| \le C_1 h^p M_p + \frac{C_2 h^{p+1} M_{p+1}}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(1+y^2)^{(r_0+1)/2}}\,dy. \tag{3.53}$$

From (3.52) and (3.53), and since $\boldsymbol{M}$ satisfies (dc), we deduce that there exist $C_3, C_4 > 0$ and $H > 1$ such that for every $p \in \mathbb{N}_0$ one has

$$|F^{(pr_0)}(t)| \le C_1 h^p M_p + C_3 (Hh)^p M_p \le C_4 (Hh)^p M_p, \quad t \ge 0.$$

Since $h$ is arbitrary and $H$ does not depend on it, we see $F \in \mathcal{N}_{r_0,(\boldsymbol{M})}([0,\infty))$ and $\mathcal{B}_{r_0}(F) = \widehat{g}$, and so $\mathcal{B}_{r_0}$ is surjective. $\square$

In a recent work, A. Debrouwere [14] characterized the surjectivity of the asymptotic Borel map in the right half-plane for regular sequences in the terms of the gamma index associated with this sequence.

**Theorem 3.6.4** ([14], Theorem. 7.4)**.** *Suppose $\widehat{\boldsymbol{M}}$ is a regular sequence. The following are equivalent:*

*(i) The Borel map $\widetilde{\mathcal{B}}\colon \mathcal{A}_{(\widehat{\boldsymbol{M}})}(S_1) \to \mathbb{C}[[z]]_{(\boldsymbol{M})}$ is surjective.*

*(ii) There exists a global extension operator $U_{\boldsymbol{M}} : \mathbb{C}[[z]]_{(\boldsymbol{M})} \to \mathcal{A}_{(\widehat{\boldsymbol{M}})}(S_1)$.*

*(iii) $\gamma(\boldsymbol{M}) > 1$.*

Note that the implication $(ii) \Rightarrow (iii)$ corresponds to Theorem 3.6.1 for $r = 1$, and that $(iii) \Rightarrow (i)$ was obtained by V. Thilliez, as already mentioned. Also, the implication $(i) \Rightarrow (iii)$ is slightly weaker than our previous result applied for $r = 1$. However, the full equivalence is a powerful result, as $(ii)$ is deduced from any of the other two conditions.

As it occurs in the Roumieu case, see [37], this information can be taken into the case of Beurling classes in a general sector by applying general Laplace and Borel integral transforms of order $\alpha > 0$, which basically arise from the classical transforms (inverse of each other) combined with ramifications of exponent $\alpha$. We sketch the information needed, as the details can be found in Sections 5.5 and 5.6 of [1].

For $0 < \alpha < 2$, to the Laplace kernel function

$$e_\alpha(z) := \frac{1}{\alpha} z^{1/\alpha} \exp(-z^{1/\alpha}), \qquad z \in S_\alpha,$$

there corresponds the moment function

$$m_\alpha(\lambda) := \int_0^\infty t^{\lambda-1} e_\alpha(t)\,dt = \Gamma(1+\alpha\lambda), \qquad \Re(\lambda) \ge 0,$$

and the Borel kernel function

$$E_\alpha(z) := \sum_{p=0}^{\infty} \frac{z^p}{m_\alpha(p)} = \sum_{p=0}^{\infty} \frac{z^p}{\Gamma(1+\alpha p)}, \qquad z \in \mathbb{C},$$

which is the classical Mittag-Leffler function of order $\alpha$.

Given a function $f$ holomorphic in a sector $S = S(d,\beta) := \{z \in \mathcal{R} : |\arg(z) - d| < \beta\pi/2\}$ (for some $\beta > 0$ and $d \in \mathbb{R}$) with suitable growth, the *$\alpha$-Laplace transform of $f$ in a direction $\tau$ in $S$* (i. e. $\tau \in \mathbb{R}$ and $|\tau - d| < \beta\pi/2$) is defined as

$$(\mathcal{L}_{\alpha,\tau} f)(z) := \int_0^{\infty(\tau)} e_\alpha(u/z) f(u) \frac{du}{u}, \quad |\arg(z) - \tau| < \alpha\pi/2,\ |z| \text{ small enough},$$

where integration is along the half-line parameterized by $t \in (0,\infty) \mapsto te^{i\tau}$. The family $\{\mathcal{L}_{\alpha,\tau} f\}_{\tau \,\mathrm{in}\, S}$ defines, by analytic continuation, a function $\mathcal{L}_\alpha f$ named the *$\alpha$-Laplace transform* of $f$, which is holomorphic in a sectorial region (see [1] for details) bisected by $d$ of opening $\pi(\beta+\alpha)$.

Now, let $S = S(d,\beta,r) := \{z \in S(d,\beta) \colon |z| < r\}$ be a sector with $\beta > \alpha$, and $f : S \to \mathbb{C}$ be holomorphic in $S$ and continuous at 0 (that is, the limit of $f$ at 0 exists when $z$ tends to 0 in every proper subsector of $S$). For $\tau \in \mathbb{R}$ such that $|\tau - d| < (\beta - \alpha)\pi/2$ we consider a path $\delta_\alpha(\tau)$ in $S$ consisting of a segment from the origin to a point $z_0$ with $\arg(z_0) = \tau + \alpha(\pi+\varepsilon)/2$ (for some suitably small $\varepsilon \in (0,\pi)$), then the circular arc $|z| = |z_0|$ from $z_0$ to the point $z_1$ on the ray $\arg(z) = \tau - \alpha(\pi+\varepsilon)/2$ (traversed clockwise), and finally the segment from $z_1$ to the origin. The *$\alpha$-Borel transform of $f$ in direction $\tau$* is defined as

$$(\mathcal{B}_{\alpha,\tau} f)(u) := \frac{-1}{2\pi i} \int_{\delta_\alpha(\tau)} E_\alpha(u/z) f(z) \frac{dz}{z}, \quad u \in S(\tau,\varepsilon_0), \quad \varepsilon_0 \text{ small enough}.$$

The family $\{\mathcal{B}_{\alpha,\tau} f\}_\tau$ defines the *$\alpha$-Borel transform* of $f$, holomorphic in the sector $S(d,\beta-\alpha)$ and denoted by $\mathcal{B}_\alpha f$.

In case $\alpha \geq 2$, $\mathcal{L}_\alpha f$ and $\mathcal{B}_\alpha f$ are defined by combining suitable ramification operators with the previous ones, see again [1].

The formal $\alpha$-Laplace and $\alpha$-Borel transforms, defined from $\mathbb{C}[[z]]$ into $\mathbb{C}[[z]]$, are respectively given by

$$\widehat{\mathcal{L}}_\alpha\Big(\sum_{p=0}^{\infty} a_p z^p\Big) := \sum_{p=0}^{\infty} \Gamma(1+\alpha p) a_p z^p, \qquad \widehat{\mathcal{B}}_\alpha\Big(\sum_{p=0}^{\infty} a_p z^p\Big) := \sum_{p=0}^{\infty} \frac{a_p}{\Gamma(1+\alpha p)} z^p.$$

The following result for the Roumieu case appeared in [37, Theorem 3.5]. The fact that the constants $C$ and $c$, appearing in the next items, do not depend on the value of $h$, makes the result valid also for the Beurling case in a straightforward way, and its proof is therefore omitted. Recall that we use the notation $\boldsymbol{M} \cdot \boldsymbol{\Gamma}_\alpha$, respectively $\boldsymbol{M}/\boldsymbol{\Gamma}_\alpha$, for the sequences which are termwise product, resp. quotient, of $\boldsymbol{M}$ and $\boldsymbol{\Gamma}_\alpha = (\Gamma(1+\alpha p))_p$.

**Theorem 3.6.5.** *Suppose $\boldsymbol{M}$ is an arbitrary sequence, and $\alpha, \gamma > 0$. Let $f \in \widetilde{\mathcal{A}}^u_{[\boldsymbol{M}]}(S_\gamma)$ and $f \sim^u_{[\boldsymbol{M}]} \widehat{f}$. Then, the following hold:*

*(i) For every $\beta$ with $0 < \beta < \gamma$ one has*

$$\mathcal{L}_\alpha f \in \widetilde{\mathcal{A}}^u_{[\boldsymbol{M}\cdot\boldsymbol{\Gamma}_\alpha]}(S_{\beta+\alpha}) \quad \text{and} \quad \mathcal{L}_\alpha f \sim^u_{[\boldsymbol{M}\cdot\boldsymbol{\Gamma}_\alpha]} \widehat{\mathcal{L}}_\alpha \widehat{f}.$$

*Moreover, there exist $C, c > 0$, depending only on $\alpha$, $\beta$ and $\gamma$, such that for every $h > 0$ and every $f \in \widetilde{\mathcal{A}}^u_{\boldsymbol{M},h}(S_\gamma)$ one has $\|\mathcal{L}_\alpha f\|_{\boldsymbol{M}\cdot\boldsymbol{\Gamma}_\alpha,ch,\widetilde{u}} \le C\|f\|_{\boldsymbol{M},h,\widetilde{u}}$, and therefore the maps $\mathcal{L}_\alpha \colon \widetilde{\mathcal{A}}^u_{\boldsymbol{M},h}(S_\gamma) \to \widetilde{\mathcal{A}}^u_{\boldsymbol{M}\cdot\boldsymbol{\Gamma}_\alpha,ch}(S_{\beta+\alpha})$ and $\mathcal{L}_\alpha \colon \widetilde{\mathcal{A}}^u_{[\boldsymbol{M}]}(S_\gamma) \to \widetilde{\mathcal{A}}^u_{[\boldsymbol{M}\cdot\boldsymbol{\Gamma}_\alpha]}(S_{\beta+\alpha})$ are continuous.*

*(ii) Suppose $\gamma > \alpha$. For every $\beta$ with $\alpha < \beta < \gamma$ one has*

$$\mathcal{B}_\alpha f \in \widetilde{\mathcal{A}}^u_{[\boldsymbol{M}/\boldsymbol{\Gamma}_\alpha]}(S_{\beta-\alpha}) \quad \text{and} \quad \mathcal{B}_\alpha f \sim^u_{[\boldsymbol{M}/\boldsymbol{\Gamma}_\alpha]} \widehat{\mathcal{B}}_\alpha \widehat{f}.$$

*Moreover, there exist $C, c > 0$, depending only on $\alpha$, $\beta$ and $\gamma$, such that for every $h > 0$ and every $f \in \widetilde{\mathcal{A}}^u_{\boldsymbol{M},h}(S_\gamma)$ one has $\|\mathcal{B}_\alpha f\|_{\boldsymbol{M}/\boldsymbol{\Gamma}_\alpha,ch,\widetilde{u}} \le C\|f\|_{\boldsymbol{M},h,\widetilde{u}}$, and therefore the maps $\mathcal{B}_\alpha \colon \widetilde{\mathcal{A}}^u_{\boldsymbol{M},h}(S_\gamma) \to \widetilde{\mathcal{A}}^u_{\boldsymbol{M}/\boldsymbol{\Gamma}_\alpha,ch}(S_{\beta-\alpha})$ and $\mathcal{B}_\alpha \colon \widetilde{\mathcal{A}}^u_{[\boldsymbol{M}]}(S_\gamma) \to \widetilde{\mathcal{A}}^u_{[\boldsymbol{M}/\boldsymbol{\Gamma}_\alpha]}(S_{\beta-\alpha})$ are continuous.*

Note that the formal Laplace and Borel transforms, $\widehat{\mathcal{L}}_\alpha$ and $\widehat{\mathcal{B}}_\alpha$, are topological isomorphisms between the space $\mathbb{C}[[z]]_{(\boldsymbol{M})}$ and $\mathbb{C}[[z]]_{(\boldsymbol{M}\cdot\boldsymbol{\Gamma}_\alpha)}$, respectively $\mathbb{C}[[z]]_{(\boldsymbol{M}/\boldsymbol{\Gamma}_\alpha)}$, for an arbitrary sequence $\boldsymbol{M}$.

The use of Laplace and Borel transforms of arbitrary positive order allows us to generalize Theorem 3.6.4 for arbitrary sectors. The idea of proof for the Roumieu case [37, Th. 4.2] applies to the Beurling case, we include it for the sake of completeness. We note that this result may also be deduced from the results in [15] about classes with non-uniform asymptotics, but we think it interesting to provide an argument contained in our framework. This procedure makes Theorem 3.6.2 necessary.

**Theorem 3.6.6.** *Suppose $\widehat{\boldsymbol{M}}$ is a regular sequence, and let $r > 0$. Each of the following statements implies the next one:*

*(i) $r < \gamma(\boldsymbol{M})$.*

*(ii) There exists a global extension operator $U_{\boldsymbol{M},r} : \mathbb{C}[[z]]_{(\boldsymbol{M})} \to \widetilde{\mathcal{A}}^u_{(\boldsymbol{M})}(S_r)$.*

*(iii) The Borel map* $\widetilde{\mathcal{B}}:\widetilde{\mathcal{A}}^u_{(\boldsymbol{M})}(S_r)\to\mathbb{C}[[z]]_{(\boldsymbol{M})}$ *is surjective.*

*(iv)* $r\leq\gamma(\boldsymbol{M})$.

*Proof.* (i)⇒(ii) We consider two cases:

(a.1) Suppose $r>1$, and take a real number $r'$ with $r<r'<\gamma(\boldsymbol{M})$. The sequence $\boldsymbol{P}_1:=\widehat{\boldsymbol{M}}/\boldsymbol{\Gamma}_{r'}$ satisfies (dc) and thanks to (1.7), $\gamma(\boldsymbol{P}_1)=\gamma(\boldsymbol{M})-r'+1>1$. By Lemma 1.1.25, there exists a weight sequence $\boldsymbol{P}$ such that $\boldsymbol{p}\simeq\boldsymbol{m}/\overline{\boldsymbol{g}}^{r'-1}$, satisfies (dc) and $\gamma(\boldsymbol{P})=\gamma(\boldsymbol{M})+1-r'>1$. Since the classes associated with $\boldsymbol{P}$ and $\boldsymbol{M}/\boldsymbol{\Gamma}_{r'-1}$ agree, Theorem 3.6.4 provides an extension operator

$$U\colon\mathbb{C}[[z]]_{\left(\boldsymbol{M}/\boldsymbol{\Gamma}_{r'-1}\right)}\to\mathcal{A}_{\left(\widehat{\boldsymbol{M}}/\boldsymbol{\Gamma}_{r'-1}\right)}(S_1).$$

By Proposition 3.1.4.(i), we have that $\mathcal{A}_{\left(\widehat{\boldsymbol{M}}/\boldsymbol{\Gamma}_{r'-1}\right)}(S_1)\hookrightarrow\widetilde{\mathcal{A}}^u_{\left(\boldsymbol{M}/\boldsymbol{\Gamma}_{r'-1}\right)}(S_1)$, and therefore this induces an extension operator

$$\widetilde{U}\colon\mathbb{C}[[z]]_{\left(\boldsymbol{M}/\boldsymbol{\Gamma}_{r'-1}\right)}\to\widetilde{\mathcal{A}}^u_{\left(\boldsymbol{M}/\boldsymbol{\Gamma}_{r'-1}\right)}(S_1).$$

Theorem 3.6.5.(i) implies that the composition

$$\mathcal{L}_{r'-1}\circ\widetilde{U}\circ\widehat{\mathcal{B}}_{r'-1}:\mathbb{C}[[z]]_{(\boldsymbol{M})}\to\widetilde{\mathcal{A}}^u_{(\boldsymbol{M})}(S_\rho),\qquad 0<\rho<r'=1+(r'-1),$$

will be an extension operator. Thanks to the fact that $r<r'$, the restriction of the elements of this last space to $S_r$ provides the extension operator

$$U_{\boldsymbol{M},r}:\mathbb{C}[[z]]_{(\boldsymbol{M})}\to\widetilde{\mathcal{A}}^u_{(\boldsymbol{M})}(S_r)$$

that we were looking for.

(a.2) If $r\leq 1$, consider $\alpha$ such that $\alpha+r>1$, and take $r'$ with $r<r'<\gamma(\boldsymbol{M})$. The weight sequence $\boldsymbol{M}\cdot\boldsymbol{\Gamma}_\alpha$ satisfies (dc) and $\gamma(\boldsymbol{M}\cdot\boldsymbol{\Gamma}_\alpha)=\gamma(\boldsymbol{M})+\alpha>r'+\alpha>1$. By item (a.1), there exists an extension operator

$$U\colon\mathbb{C}[[z]]_{(\boldsymbol{M}\cdot\boldsymbol{\Gamma}_\alpha)}\to\widetilde{\mathcal{A}}^u_{(\boldsymbol{M}\cdot\boldsymbol{\Gamma}_\alpha)}(S_{r'+\alpha}).$$

Again Theorem 3.6.5.(ii) implies that

$$\mathcal{B}_\alpha\circ\widetilde{U}\circ\widehat{\mathcal{L}}_\alpha:\mathbb{C}[[z]]_{(\boldsymbol{M})}\to\widetilde{\mathcal{A}}^u_{(\boldsymbol{M})}(S_\rho),\qquad 0<\rho<r',$$

will be an extension operator, and the restriction of the elements of this space to $S_r$ provides the desired extension operator as before.

(ii)⇒(iii) The existence of $U_{\boldsymbol{M},r}$ implies that the corresponding Borel map $\widetilde{\mathcal{B}}$ : $\widetilde{\mathcal{A}}^u_{(\boldsymbol{M})}(S_r) \to \mathbb{C}[[z]]_{(\boldsymbol{M})}$ is surjective in $S_r$.
(iii)⇒(iv) Let us see that $r \leq \gamma(\boldsymbol{M})$. We again have different cases:

(b.1) If $0 < r < 1$, consider positive real numbers $\alpha, r'$ with $1-\alpha < r' < r$. By applying the Laplace transform $\mathcal{L}_\alpha\colon \widetilde{\mathcal{A}}^u_{(\boldsymbol{M})}(S_r) \to \widetilde{\mathcal{A}}^u_{(\boldsymbol{M}\cdot\boldsymbol{\Gamma}_\alpha)}(S_{r'+\alpha})$, Theorem 3.6.5.(i) shows that the map

$$\widetilde{\mathcal{A}}^u_{(\boldsymbol{M}\cdot\boldsymbol{\Gamma}_\alpha)}(S_{r'+\alpha}) \to \mathbb{C}[[z]]_{(\boldsymbol{M}\cdot\boldsymbol{\Gamma}_\alpha)},$$

is surjective. Observe that $r'+\alpha > 1$, so we deduce by restriction to the half-plane $S_1$ that, according to Proposition 3.1.4.(ii), also the map

$$\mathcal{A}_{(\widehat{\boldsymbol{M}}\cdot\boldsymbol{\Gamma}_\alpha)}(S_1) \to \mathbb{C}[[z]]_{(\boldsymbol{M}\cdot\boldsymbol{\Gamma}_\alpha)},$$

is surjective. Theorem 3.6.4 implies then that $\gamma(\boldsymbol{M}\cdot\boldsymbol{\Gamma}_\alpha) > 1$ or, equivalently by (1.6), $\gamma(\boldsymbol{M}) > 1-\alpha$. Since $\alpha$ can be chosen arbitrarily while keeping $1-\alpha < r$, we deduce $\gamma(\boldsymbol{M}) \geq r$.

(b.2) If $r \in \mathbb{N}$, we know that $\gamma(\boldsymbol{M}) > r$ by Theorem 3.6.2.

(b.3) If $r \in (1,\infty)\backslash\mathbb{N}$, again by Theorem 3.6.2 we deduce that $\gamma(\boldsymbol{M}) > \lfloor r \rfloor$, so that the sequence $\boldsymbol{P}_1 := \widehat{\boldsymbol{M}/\boldsymbol{\Gamma}}_{\lfloor\gamma\rfloor}$ is such that $\gamma(\boldsymbol{P}_1) > 1$ by using the properties of gamma index. Hence, by Lemma 1.1.25 there exists a weight sequence $\boldsymbol{P}$ such that $\boldsymbol{P} \approx \boldsymbol{M}/\boldsymbol{\Gamma}_{\lfloor\gamma\rfloor}$, $\gamma(\boldsymbol{P}) = \gamma(\boldsymbol{M}) - \lfloor\gamma\rfloor$ and $\boldsymbol{P}$ will also satisfy (dc). Consider a value $r'$ with $\lfloor r \rfloor < r' < r$. By applying the Borel transform $\mathcal{B}_{\lfloor r\rfloor}\colon \widetilde{\mathcal{A}}^u_{(\boldsymbol{M})}(S_r) \to \widetilde{\mathcal{A}}^u_{(\boldsymbol{M}/\boldsymbol{\Gamma}_{\lfloor r\rfloor})}(S_{r'-\lfloor r\rfloor})$, Theorem 3.6.5.(ii) shows that the map

$$\widetilde{\mathcal{A}}^u_{(\boldsymbol{M}/\boldsymbol{\Gamma}_{\lfloor r\rfloor})}(S_{r'-\lfloor r\rfloor}) \to \mathbb{C}[[z]]_{(\boldsymbol{M}/\boldsymbol{\Gamma}_{\lfloor r\rfloor})},$$

is surjective, or equivalently, thanks to the equivalence $\boldsymbol{P} \approx \boldsymbol{M}/\boldsymbol{\Gamma}_{\lfloor\gamma\rfloor}$, the map

$$\widetilde{\mathcal{A}}^u_{(\boldsymbol{P})}(S_{r'-\lfloor r\rfloor}) \to \mathbb{C}[[z]]_{(\boldsymbol{P})},$$

is also surjective. Since $r' - \lfloor r \rfloor \in (0,1)$, we may invoke item (b.1) and deduce that $\gamma(\boldsymbol{P}) \geq r' - \lfloor r \rfloor$, what amounts to $\gamma(\boldsymbol{M}) \geq r'$. We conclude by making $r'$ tend to $r$.

□

### 3.6.2 Surjectivity for Beurling classes under condition (sm)

We end by proving a surjectivity result for Beurling classes when their defining weight sequence satisfies our new condition (sm). The technique used by V. Thilliez in [80, Th. 3.4.1] will be followed, and we first need to recall two auxiliary results from the work of J. Chaumat and A.-M. Chollet [11].

**Lemma 3.6.7** ([11], Lemma 14)**.** *Let $\boldsymbol{L} = (L_p)_p$ be a sequence of nonnegative real numbers and $\boldsymbol{M} = (M_p)_p$ be a sequence of positive real numbers. The following conditions are equivalent:*

*(i) For all $h > 0$, there exists a constant $C(h) > 0$ such that $L_p \le C(h)h^p M_p$ for every $p \in \mathbb{N}_0$.*

*(ii)* $\displaystyle\lim_{p\to\infty} \left(\frac{L_p}{M_p}\right)^{1/p} = 0.$

*(iii) There exists a sequence $\mathbb{E} = (\varepsilon_p)_{p\in\mathbb{N}_0}$ of positive real numbers tending to zero such that $L_p \le \varepsilon_0\varepsilon_1 \cdots \varepsilon_{p-1} M_p$, $p \in \mathbb{N}$.*

*Moreover, if (i) or (ii) are satisfied, then (iii) holds true for a nonincreasing sequence $\mathbb{E}$.*

**Lemma 3.6.8** ([11], Lemma 16)**.** *Let $\boldsymbol{A} = (A_p)_p$ be a sequence of nonnegative real numbers such that $\sum_{p=0}^{\infty} A_p$ is convergent, and let $\boldsymbol{B} = (B_p)_p$ and $\boldsymbol{D} = (D_p)_p$ be sequences of positive real numbers such that* $\lim_{p\to\infty} B_p = 0$*, and $\boldsymbol{D}$ is nonincreasing and* $\lim_{p\to\infty} D_p = 0$*. Then, there exists a nondecreasing sequence $\boldsymbol{E} = (E_p)_p$ of positive real numbers such that:*

*(i)* $\lim_{p\to\infty} E_p = \infty$*.*

*(ii)* $\sum_{p=q}^{\infty} E_p A_p \le 8E_q \sum_{p=q}^{\infty} A_p$*, $q \in \mathbb{N}_0$.*

*(iii) The sequence $\boldsymbol{E} \cdot \boldsymbol{D} = (E_p D_p)_p$ is nonincreasing.*

*(iv)* $\lim_{p\to\infty} E_p B_p = 0$*.*

The next result is an adaptation of a similar result, [11, Prop. 17], in which the condition (mg) has now been substituted by (sm).

**Theorem 3.6.9.** *Let $\boldsymbol{L} = (L_p)_p$ be a weight sequence satisfying* (snq) *and* (sm)*. If $\boldsymbol{A} = (A_p)_p$ is a sequence of nonnegative real numbers such that for all $h > 0$ there exists $C(h) > 0$ such that $A_p \le C(h)h^p L_p$ for every $p \in \mathbb{N}_0$, then there exists a weight sequence $\boldsymbol{K} = (K_p)_p$ which satisfies* (snq), (sm) *and such that:*

*i) There exists a constant $D > 0$ such that $A_p \le DK_p$, for all $p \in \mathbb{N}_0$.*

*ii) For all $h > 0$, there exists $C'(h) > 0$ such that $K_p \leq C'(h) h^p L_p$, $p \in \mathbb{N}_0$.*

*Proof.* By Lemma 3.6.7, there exist a nonincreasing sequence $\mathfrak{E} = (\varepsilon_p)_{p \in \mathbb{N}_0}$ which tends to zero, and such that

$$A_p \leq \varepsilon_0 \varepsilon_1 \cdots \varepsilon_{p-1} L_p, \qquad p \in \mathbb{N}. \tag{3.54}$$

Consider the sequence $\{u_p\}_{p \in \mathbb{N}_0}$ defined as

$$u_p = \frac{1}{(p+1)\ell_p},$$

where $(\ell_p)_{\in \mathbb{N}_0}$ is the sequence of quotients of $\boldsymbol{L}$. $\{u_p\}_{p \in \mathbb{N}_0}$ is nonincreasing and tends to zero, and since $\boldsymbol{L}$ satisfies (snq), there exists some constant $A > 0$ with

$$\sum_{p=q}^{\infty} u_p \leq A(q+1) u_q, \qquad q \in \mathbb{N}_0. \tag{3.55}$$

As $((p+1)u_p\}_{p \in \mathbb{N}_0}$ is nonincreasing and tends zero, we can apply Lemma 3.6.8 with

$$A_p = u_p, \qquad B_p = \max\{\varepsilon_p, (p+1)u_p\}, \qquad D_p = (p+1)u_p, \qquad p \in \mathbb{N}_0.$$

So, there exists a nondecreasing sequence $\boldsymbol{E}$ which tends to $\infty$, and such that:

$$\sum_{p=q}^{\infty} u_p E_p \leq 8 E_q \sum_{p=q}^{\infty} u_p, \qquad q \in \mathbb{N}_0. \tag{3.56}$$

$$\text{The sequence } ((p+1)u_p E_p)_{p \in \mathbb{N}_0} \text{ is nonincreasing.} \tag{3.57}$$

$$\lim_{p \to \infty} \varepsilon_p E_p = 0, \qquad \lim_{p \to \infty} (p+1) u_p E_p = 0. \tag{3.58}$$

Let us consider the sequence $(k_p)_{p \in \mathbb{N}_0}$ defined as

$$k_p = \frac{\ell_p}{E_p} = \frac{1}{(p+1)u_p E_p}, \qquad p \in \mathbb{N}_0.$$

Then, from (3.55) and (3.56) we deduce that

$$\sum_{p=q}^{\infty} \frac{1}{(p+1)k_p} = \sum_{p=q}^{\infty} u_p E_p \leq 8 E_q \sum_{p=q}^{\infty} u_p \leq 8A(q+1) u_q E_q = 8A \frac{1}{k_q}, \qquad q \in \mathbb{N}_0.$$

Therefore, the sequence $\mathbb{K}$ defined as

$$K_0 = 1, \qquad K_p = k_0 k_1 \cdots k_{p-1}, \qquad p \in \mathbb{N},$$

satisfies (snq). Moreover, the sequence $\boldsymbol{K}$ is a weight sequence due to (3.57) and (3.58). Now, since $\boldsymbol{E}$ is nondecreasing we deduce that

$$\frac{k_{p+1}}{k_p} = \frac{\ell_{p+1}E_p}{\ell_p E_{p+1}} \leq \frac{\ell_{p+1}}{\ell_p}, \qquad p \in \mathbb{N}_0,$$

and so $\boldsymbol{K}$ satisfies (sm) too (with the same constants as $\boldsymbol{L}$). Taking into account that

$$K_p = \frac{1}{E_0 \dots E_{p-1}} L_p, \qquad p \in \mathbb{N}, \tag{3.59}$$

we deduce from (3.54) that

$$A_p \leq \varepsilon_0\varepsilon_1 \cdots \varepsilon_{p-1} L_p = \varepsilon_0 E_0 \varepsilon_1 E_1 \cdots \varepsilon_{p-1} E_{p-1} K_p, \qquad p \in \mathbb{N}.$$

We observe in (3.58) that the sequence $(\varepsilon_p E_p)_{p\in\mathbb{N}_0}$ tends to zero and, therefore, Lemma 3.6.7 provides for all $t > 0$ a constant $D(t) > 0$ such that $A_p \leq D(t)t^p K_p$ for every $p \in \mathbb{N}_0$ and, in particular, for $t = 1$ we obtain that $A_p \leq D(1)K_p$. Finally, from (3.59) and the fact that $(1/E_p)_{p\in\mathbb{N}_0}$ tends to zero, we can apply again Lemma 3.6.7 and we deduce that for all $h > 0$ there exists some constant $C'(h) > 0$ such that $K_p \leq C'(h)h^p L_p$ for every $p \in \mathbb{N}_0$. □

We are ready for the proof of our last result.

**Theorem 3.6.10.** *Let $\boldsymbol{M}$ be a weight sequence with $\gamma(\boldsymbol{M}) > 0$ and that satisfies* (sm)*, and let $0 < r < \gamma(\boldsymbol{M})$ be given. Then, the Borel map $\widetilde{\mathcal{B}}\colon \widetilde{\mathcal{A}}^u_{(\boldsymbol{M})}(S_r) \to \mathbb{C}[[z]]_{(\boldsymbol{M})}$ is surjective. So, $(0, \gamma(\boldsymbol{M})) \subset \widetilde{S}^u_{(\boldsymbol{M})}$.*

*Proof.* Let $\widehat{f} = \sum_{p=0}^{\infty} a_p z^p \in \mathbb{C}[[z]]_{(\boldsymbol{M})}$ be given. By the definition of $\mathbb{C}[[z]]_{(\boldsymbol{M})}$ (see also (3.3)), for every $h > 0$ there exists $C(h) > 0$ such that

$$|a_p| \leq C(h)h^p M_p, \quad p \in \mathbb{N}_0. \tag{3.60}$$

On the one hand, by the properties of the gamma index we have that $\gamma(\boldsymbol{M}^{1/r}) = \gamma(\boldsymbol{M})/r > 1$, and Lemma 1.1.25 provides a weight sequence $\boldsymbol{L} = (L_p)_p$ such that $\gamma(\widehat{\boldsymbol{L}}) > 1$ and there exists $a > 0$ such that $a^{-1}(\ell_p(p+1)) \leq m_p^{1/r} \leq a(\ell_p(p+1))$ for all $p \in \mathbb{N}_0$. In particular, we have that $\gamma(\boldsymbol{L}) > 0$, and so $\boldsymbol{L}$ satisfies (snq). Moreover, it is clear that $a^{-p}p!L_p \leq M_p^{1/r} \leq a^p p! L_p$ for every $p$, and so $\boldsymbol{M} \approx (\widehat{\boldsymbol{L}})^r$, what implies that the classes defined by both sequences coincide. Because of the stability properties of (sm) described in Subsection 3.5.1, $\boldsymbol{L}$ inherits (sm) from $\boldsymbol{M}$.

On the other hand, from (3.60) we obtain

$$\frac{|a_p|^{1/r}}{p!} \leq C(h)^{1/r}(ah^{1/r})^p L_p, \quad p \in \mathbb{N}_0,$$

and so we are in a position to apply Theorem 3.6.9 to the sequences $\boldsymbol{A} = (|a_p|^{1/r}/p!)_p$ and $\boldsymbol{L}$. Hence, there exists a weight sequence $\boldsymbol{K} = (K_p)_p$ which satisfies (snq) (i. e., $\gamma(\boldsymbol{K}) > 0$) and (sm), such that there exists $D > 0$ with $|a_p|^{1/r}/p! \le DK_p$ for all $p \in \mathbb{N}_0$, and such that for all $h > 0$, there exists $C'(h) > 0$ such that

$$K_p \le C'(h)h^p L_p, \quad p \in \mathbb{N}_0. \tag{3.61}$$

The first estimates state that $|a_p| \le D^r(p!K_p)^r$, and so $\widehat{f} \in \mathbb{C}[[z]]_{\{\boldsymbol{N}\}}$ for the weight sequence $\boldsymbol{N} := (\widehat{\boldsymbol{K}})^r$. Again $\boldsymbol{N}$ inherits (sm) from $\boldsymbol{K}$, and moreover $\gamma(\boldsymbol{N}) = r(\gamma(\boldsymbol{K}) + 1) > r$. So, we can apply Theorem 3.5.7 to deduce that $\widetilde{\mathcal{B}}\colon \widetilde{\mathcal{A}}^u_{\{\boldsymbol{N}\}}(S_r) \to \mathbb{C}[[z]]_{\{\boldsymbol{N}\}}$ is surjective. Hence, there exists $f \in \widetilde{\mathcal{A}}^u_{\{\boldsymbol{N}\}}(S_r)$ such that $\widetilde{\mathcal{B}}(f) = \widehat{f}$. Finally, observe that from (3.61) we get

$$N_p = (p!K_p)^r \le C'(h)^r (h^r)^p (\widehat{L}_p)^r, \quad p \in \mathbb{N}_0,$$

and so $\widetilde{\mathcal{A}}^u_{\{\boldsymbol{N}\}}(S_r) \subset \widetilde{\mathcal{A}}^u_{((\widehat{\boldsymbol{L}})^r)}(S_r) = \widetilde{\mathcal{A}}^u_{(\boldsymbol{M})}(S_r)$, from where the conclusion follows. $\square$

# Chapter 4

# A new Stieltjes moment problem in Gelfand-Shilov spaces defined by weight sequences with shifted moments

This final chapter of the dissertation deals with a modified Stieltjes moment problem whose consideration is motivated by the condition (sm) of shifted moments. It turns out that, in the framework of Gelfand-Shilov spaces of Roumieu type defined by weight sequences, one can extend the classical target space of the Stieltjes moment mapping as long as (dc) is substituted by (sm), and then study the injectivity and surjectivity of this mapping in this new context.

## 4.1 Preliminaries

In this section we introduce the classes that we are going to use in this chapter, specially Gelfand-Shilov spaces. Moreover, we define the Fourier and Laplace transform, and we analyze the effect of these transformation over Gelfand-Shilov spaces.

For a given open set $\Omega$ in the complex plane $\mathbb{C}$, we denote by $\mathcal{H}(\Omega)$ the space of holomorphic functions in $\Omega$. In particular, we write $\mathbb{H}$ for the open upper half-plane of $\mathbb{C}$, and consider, as in Chapter 2, the ultraholomorphic class

$$\mathcal{A}_{\{\boldsymbol{M}\}}(\mathbb{H}) = \bigcup_{h>0} \mathcal{A}_{\boldsymbol{M},h}(\mathbb{H}),$$

where $\mathcal{A}_{\boldsymbol{M},h}(\mathbb{H})$ is the space consisting of all $f \in \mathcal{H}(\mathbb{H})$ such that

$$\|f\|_{\boldsymbol{M},h} := \sup_{p\in\mathbb{N}_0} \sup_{z\in\mathbb{H}} \frac{|f^{(p)}(z)|}{h^p M_p} < \infty.$$

The following result follows from the fact that the elements of $\mathcal{A}_{\boldsymbol{M},h}(\mathbb{H})$ together with all their derivatives are Lipschitz on $\mathbb{H}$.

**Lemma 4.1.1.** *Let $\boldsymbol{M}$ be a sequence and let $f \in \mathcal{A}_{\boldsymbol{M},h}(\mathbb{H})$ for some $h > 0$. Then,*

$$f_p(x) = \lim_{z\in\mathbb{H}, z\to x} f^{(p)}(z) \in \mathbb{C}$$

*exists for all $x \in \mathbb{R}$ and $p \in \mathbb{N}_0$. Moreover, $f_0 \in C^\infty(\mathbb{R})$, $f_0^{(p)} = f_p$ for all $p \in \mathbb{N}$, and*

$$\sup_{p\in\mathbb{N}_0} \sup_{x\in\mathbb{R}} \frac{|f_0^{(p)}(x)|}{h^p M_p} < \infty.$$

In the sequel, we shall frequently write $f(x) = \lim_{z\in\mathbb{H}, z\to x} f(z)$ for $x \in \mathbb{R}$, and so $f$ and all its derivatives can be considered to be continuous on $\overline{\mathbb{H}}$ and satisfy the same global estimates there. Now, we present the space of sequences that admit a control in terms of a given sequence $\boldsymbol{M}$.

**Definition 4.1.2.** Let $\boldsymbol{M}$ be a sequence and $h > 0$. We define $\Lambda_{\boldsymbol{M},h}$ as the space consisting of all sequences $(c_p)_p \in \mathbb{C}^{\mathbb{N}_0}$ such that

$$|(c_p)_p|_{\boldsymbol{M},h} := \sup_{p\in\mathbb{N}_0} \frac{|c_p|}{h^p M_p} < \infty.$$

$(\Lambda_{\boldsymbol{M},h}, |\cdot|_{\boldsymbol{M},h})$ is a Banach space, and

$$\Lambda_{\{\boldsymbol{M}\}} := \bigcup_{h>0} \Lambda_{\boldsymbol{M},h}$$

is the corresponding $(LB)$ space.

The *asymptotic Borel mapping* may be defined as

$$\mathcal{B} : \mathcal{A}_{\{\boldsymbol{M}\}}(\mathbb{H}) \to \Lambda_{\{\boldsymbol{M}\}}, \quad f \mapsto (f^{(p)}(0))_p,$$

by Lemma 4.1.1. Since $|\mathcal{B}(f)|_{\boldsymbol{M},h} \leq \|f\|_{\boldsymbol{M},h}$ for every $f \in \mathcal{A}_{\boldsymbol{M},h}(\mathbb{H})$, it is linear and continuous both between $\mathcal{A}_{\boldsymbol{M},h}(\mathbb{H})$ and $\Lambda_{\boldsymbol{M},h}$ for every $h > 0$, and between $\mathcal{A}_{\{\boldsymbol{M}\}}(\mathbb{H})$ and $\Lambda_{\{\boldsymbol{M}\}}$. As already indicated, an updated account on the injectivity and surjectivity of the asymptotic Borel mapping on various ultraholomorphic classes defined on arbitrary sectors may be found in the works of J. Jiménez-Garrido, J. Sanz and G. Schindl [33, 37] and in the previous chapter, a part of which is already published [28] or in preparation [30]. There, two indices $\gamma(\boldsymbol{M})$ (see section 1.1.3) and $\omega(\boldsymbol{M})$, associated to the sequence $\boldsymbol{M}$, play a prominent role. In [26, Ch. 2] and [34, Sect. 3], the connections between these indices, the growth properties usually imposed on sequences, and the theory of O-regular variation, have been thoroughly studied.

Thanks to the gamma index, the surjectivity of the asymptotic Borel mapping in a half-plane can be characterized as follows. Note that the condition $\gamma(\boldsymbol{M}) > 1$ amounts, in view of (1.4) and the easy equality $\gamma(\widehat{\boldsymbol{M}}) = \gamma(\boldsymbol{M}) + 1$ (see (1.6)), to the fact that $\widehat{\boldsymbol{M}}$ satisfies $(\gamma_2)$, which is the condition appearing in [14, Thm. 7.4.(b)].

**Theorem 4.1.3** ([14], Theorem 7.4.(b))**.** *Let* $\boldsymbol{M}$ *be a weight sequence and satisfy* (dc)*. The following are equivalent:*

*(i)* $\mathcal{B}\colon \mathcal{A}_{\{\boldsymbol{M}\}}(\mathbb{H}) \to \Lambda_{\{\boldsymbol{M}\}}$ *is surjective.*

*(ii)* $\gamma(\boldsymbol{M}) > 2$*.*

For the study of the injectivity of the asymptotic Borel map, J. Sanz [68] introduced the growth index $\omega(\boldsymbol{M})$.

**Definition 4.1.4.** Let $\boldsymbol{M}$ be a sequence. We define the $\omega(\boldsymbol{M})$ index by

$$\omega(\boldsymbol{M}) := \liminf_{p\to\infty} \frac{\log(m_p)}{\log(p)} \in [0,\infty].$$

Moreover, it turns out that

$$\begin{aligned}\omega(\boldsymbol{M}) &= \sup\{\mu > 0 \,|\, \sum_{p=0}^{\infty} \frac{1}{(m_p)^{1/\mu}} < \infty\} \\ &= \sup\{\mu > 0 \,|\, \sum_{p=0}^{\infty} \frac{1}{((p+1)m_p)^{1/(\mu+1)}} < \infty\}.\end{aligned}$$

Concerning the injectivity of the asymptotic Borel mapping, we have the next result.

**Theorem 4.1.5.** ([64, Thm. 12], [33, Thm. 3.4]) *Let* $\boldsymbol{M}$ *be a weight sequence. Then,* $\mathcal{B} : \mathcal{A}_{\{\boldsymbol{M}\}}(\mathbb{H}) \to \Lambda_{\{\boldsymbol{M}\}}$ *is injective if and only if*

$$\sum_{p=0}^{\infty} \frac{1}{m_p^{1/2}} = \infty,$$

*which in turn implies that* $\omega(\boldsymbol{M}) \le 2$*.*

Finally, we mention that if $\boldsymbol{M}$ is a weight sequence, the asymptotic Borel mapping $\mathcal{B}:\mathcal{A}_{\{\boldsymbol{M}\}}(\mathbb{H})\to\Lambda_{\{\boldsymbol{M}\}}$ is not bijective [33, Thm. 3.17].

### 4.1.1 Gelfand-Shilov spaces and the Fourier transform

In this section, we introduce briefly the Gelfand-Shilov spaces, as a subclass of functions that are infinitely differentiable. After that, we discuss the behavior of the Fourier transform over these spaces.

**Definition 4.1.6.** Let $\boldsymbol{M}$ and $\boldsymbol{A}$ be sequences of positive real numbers. For $h>0$ we define $\mathcal{S}^{\boldsymbol{A},h}_{\boldsymbol{M},h}(\mathbb{R})$ as the space consisting of all $\varphi\in C^\infty(\mathbb{R})$ such that

$$s^{\boldsymbol{A},h}_{\boldsymbol{M},h}(\varphi):=\sup_{p,q\in\mathbb{N}_0}\sup_{x\in\mathbb{R}}\frac{|x^p\varphi^{(q)}(x)|}{h^{p+q}M_pA_q}<\infty.$$

$(\mathcal{S}^{\boldsymbol{A},h}_{\boldsymbol{M},h}(\mathbb{R}),s^{\boldsymbol{A},h}_{\boldsymbol{M},h})$ is a Banach space. In addition, we set

$$\mathcal{S}^{\{\boldsymbol{A}\}}_{\{\boldsymbol{M}\}}(\mathbb{R})=\bigcup_{h>0}\mathcal{S}^{\boldsymbol{A},h}_{\boldsymbol{M},h}(\mathbb{R}),$$

which is an $(LB)$ space.

If $M_p^{1/p}\to\infty$ as $p\to\infty$, and in particular if $\boldsymbol{M}$ is a weight sequence, notice that $\varphi\in C^\infty(\mathbb{R})$ belongs to $\mathcal{S}^{\boldsymbol{A},h}_{\boldsymbol{M},h}(\mathbb{R})$ if and only if

$$\sup_{q\in\mathbb{N}_0}\sup_{x\in\mathbb{R}}\frac{|\varphi^{(q)}(x)|e^{\omega_{\boldsymbol{M}}(|x|/h)}}{h^qA_q}<\infty$$

or, in other words, for every $q\in\mathbb{N}_0$ one has

$$|\varphi^{(q)}(x)|\le s^{\boldsymbol{A},h}_{\boldsymbol{M},h}(\varphi)h^qA_qe^{-\omega_{\boldsymbol{M}}(|x|/h)}=s^{\boldsymbol{A},h}_{\boldsymbol{M},h}(\varphi)h^qA_qh_{\boldsymbol{M}}(h/|x|),\quad x\in\mathbb{R}.\tag{4.1}$$

Analogously, we define the spaces $\mathcal{S}_{\boldsymbol{M},h}(\mathbb{R})$, for $h>0$ and $\mathcal{S}_{\{\boldsymbol{M}\}}(\mathbb{R})$.

**Definition 4.1.7.** Let $\boldsymbol{M}$ be a sequence and $h>0$. We define $\mathcal{S}_{\boldsymbol{M},h}(\mathbb{R})$, as the space consisting of all $\varphi\in C^\infty(\mathbb{R})$ such that, for all $q\in\mathbb{N}_0$,

$$s^q_{\boldsymbol{M},h}(\varphi):=\sup_{p\in\mathbb{N}_0}\sup_{x\in\mathbb{R}}\frac{|x^p\varphi^{(q)}(x)|}{h^pM_p}<\infty;$$

$s^q_{\boldsymbol{M},h}$ is a seminorm and therefore $(\mathcal{S}_{\boldsymbol{M},h}(\mathbb{R}),(s^q_{\boldsymbol{M},h})_{q\in\mathbb{N}_0})$ is a Fréchet space. Moreover, we set

$$\mathcal{S}_{\{\boldsymbol{M}\}}(\mathbb{R})=\bigcup_{h>0}\mathcal{S}_{\boldsymbol{M},h}(\mathbb{R}),$$

endowed with its natural $(LF)$ space structure.

We also define

$$\mathcal{S}^{\{\boldsymbol{A}\}}_{\{\boldsymbol{M}\}}(0,\infty) := \{\varphi \in \mathcal{S}^{\{\boldsymbol{A}\}}_{\{\boldsymbol{M}\}}(\mathbb{R}) \mid \operatorname{supp}\varphi \subseteq [0,\infty)\}$$

and

$$\mathcal{S}_{\{\boldsymbol{M}\}}(0,\infty) := \{\varphi \in \mathcal{S}_{\{\boldsymbol{M}\}}(\mathbb{R}) \mid \operatorname{supp}\varphi \subseteq [0,\infty)\},$$

whose relative topologies from their ambient spaces coincide, as long as $\boldsymbol{M}$ is a weight sequence, with the corresponding $(LB)$ and $(LF)$ structures obtained from the similarly defined Banach subspaces $\mathcal{S}^{\boldsymbol{A},h}_{\boldsymbol{M},h}(0,\infty)$ or Fréchet subspaces $\mathcal{S}_{\boldsymbol{M},h}(0,\infty)$, see [14, Lemma 3.3 and page 24].

**Remark 4.1.8.** Observe that $\mathcal{S}^{\{\boldsymbol{A}\}}_{\{\boldsymbol{M}\}}(\mathbb{R}) \subset \mathcal{S}_{\{\boldsymbol{M}\}}(\mathbb{R})$ and $\mathcal{S}^{\{\boldsymbol{A}\}}_{\{\boldsymbol{M}\}}(0,\infty) \subset \mathcal{S}_{\{\boldsymbol{M}\}}(0,\infty)$. If $\boldsymbol{A}$ satisfies (lc), then $\mathcal{S}^{\{\boldsymbol{A}\}}_{\{\boldsymbol{M}\}}(0,\infty)$ is non-trivial (i. e., it contains non identically zero functions) if and only if $\sum_{p=0}^{\infty} 1/a_p < \infty$, as follows from the Denjoy-Carleman theorem.

In the remainder of this subsection we investigate the image of the spaces $\mathcal{S}^{\{\boldsymbol{A}\}}_{\{\boldsymbol{M}\}}(\mathbb{R})$ and $\mathcal{S}^{\{\boldsymbol{A}\}}_{\{\boldsymbol{M}\}}(0,\infty)$ under the Fourier transform (cf. [24, Sect. IV.6]), which we define as follows:

**Definition 4.1.9.** We define the Fourier transform of an integrable function, $\varphi \in L^1(\mathbb{R})$, as

$$\mathcal{F}(\varphi)(\xi) = \widehat{\varphi}(\xi) = \int_{-\infty}^{\infty} \varphi(x) e^{ix\xi} \mathrm{d}x.$$

With this definition it is well-known that the inverse Fourier transform is defined as

$$\mathcal{F}^{-1}(\varphi)(\xi) = \frac{1}{2\pi}\mathcal{F}(\varphi)(-\xi), \quad \varphi \in \mathcal{S}(\mathbb{R}),\ \xi \in \mathbb{R}, \tag{4.2}$$

where $\mathcal{S}(\mathbb{R})$ is the Schwartz space of rapid decreasing functions.

In our next statements, the sequence

$$\boldsymbol{M}_{+1} := (M_{p+1})_{p\in\mathbb{N}_0}$$

will play a prominent role. Note that its first term $M_1$ will not be generally equal to 1.

**Proposition 4.1.10.** *Let $\boldsymbol{M}$ be a weight sequence satisfying* (sm)*, and $\boldsymbol{A}$ be either an almost increasing sequence, or a sequence such that $\liminf_{p\to\infty} A_p^{1/p} > 0$ and $\widehat{\boldsymbol{A}}$ satisfies* (lc)*. Then:*

*(i) There exists $a > 0$ such that for every $h \geq 1$ one has $\mathcal{F}(\mathcal{S}^{\widehat{\boldsymbol{A}},h}_{\boldsymbol{M},h}(\mathbb{R})) \subset \mathcal{S}^{\boldsymbol{M}_{+1},ah}_{\widehat{\boldsymbol{A}},ah}(\mathbb{R})$, and $\mathcal{F}\colon \mathcal{S}^{\widehat{\boldsymbol{A}},h}_{\boldsymbol{M},h}(\mathbb{R}) \to \mathcal{S}^{\boldsymbol{M}_{+1},ah}_{\widehat{\boldsymbol{A}},ah}(\mathbb{R})$ is continuous.*

*(ii)* $\mathcal{F}(\mathcal{S}^{\{\widehat{\boldsymbol{A}}\}}_{\{\boldsymbol{M}\}}(\mathbb{R})) \subset \mathcal{S}^{\{\boldsymbol{M}_{+1}\}}_{\{\widehat{\boldsymbol{A}}\}}(\mathbb{R})$ *and* $\mathcal{F}\colon \mathcal{S}^{\{\widehat{\boldsymbol{A}}\}}_{\{\boldsymbol{M}\}}(\mathbb{R}) \to \mathcal{S}^{\{\boldsymbol{M}_{+1}\}}_{\{\widehat{\boldsymbol{A}}\}}(\mathbb{R})$ *is continuous.*

*(iii) The two previous statements are valid when replacing* $\mathcal{F}$ *by* $\mathcal{F}^{-1}$*.*

*Proof.* The statements about $\mathcal{F}^{-1}$ will be valid if the ones about $\mathcal{F}$ are, due to (4.2). (ii) is immediate from (i), which we prove now. Let $h \geq 1$ and $\varphi \in \mathcal{S}^{\widehat{\boldsymbol{A}},h}_{\boldsymbol{M},h}(\mathbb{R})$ be arbitrary. Then,

$$\sup_{x\in\mathbb{R}} |x^p \varphi^{(q)}(x)| \leq s^{\widehat{\boldsymbol{A}},h}_{\boldsymbol{M},h}(\varphi) h^{p+q} M_p q! A_q, \qquad p, q \in \mathbb{N}_0. \tag{4.3}$$

After applying Leibniz's theorem for parametric integrals and integration by parts, we get

$$\sup_{\xi\in\mathbb{R}} |\xi^q \widehat{\varphi}^{(p)}(\xi)| \leq \sum_{j=0}^{\min\{p,q\}} \binom{q}{j} \frac{p!}{(p-j)!} \int_{-\infty}^{\infty} |x^{p-j} \varphi^{(q-j)}(x)| \mathrm{d}x. \tag{4.4}$$

We split each of the integrals into five intervals, and use (4.3) in order to estimate the first of them:

$$\begin{aligned}\int_{-\infty}^{-hm_{p+1-j}} |x^{p+2-j} \varphi^{(q-j)}(x)| \frac{\mathrm{d}x}{x^2} &\leq s^{\widehat{\boldsymbol{A}},h}_{\boldsymbol{M},h}(\varphi) h^{p+q+2-2j} M_{p+2-j} (q-j)! A_{q-j} \frac{1}{hm_{p+1-j}} \\ &= s^{\widehat{\boldsymbol{A}},h}_{\boldsymbol{M},h}(\varphi) h^{p+q+1-2j} M_{p+1-j} (q-j)! A_{q-j}. \end{aligned} \tag{4.5}$$

One can proceed similarly for the integral over $(hm_{p+1-j}, \infty)$. On the interval $(hm_{p-j}, hm_{p+1-j})$ one uses (4.1), then (1.8) and finally (sm) to obtain

$$\begin{aligned}\int_{hm_{p-j}}^{hm_{p+1-j}} x^{p-j} |\varphi^{(q-j)}(x)| \mathrm{d}x &\leq s^{\widehat{\boldsymbol{A}},h}_{\boldsymbol{M},h}(\varphi) h^{q-j} (q-j)! A_{q-j} \int_{hm_{p-j}}^{hm_{p+1-j}} x^{p-j} h_{\boldsymbol{M}}(h/x) \mathrm{d}x \\ &= s^{\widehat{\boldsymbol{A}},h}_{\boldsymbol{M},h}(\varphi) h^{q-j} (q-j)! A_{q-j} \int_{hm_{p-j}}^{hm_{p+1-j}} x^{p-j} \frac{h^{p+1-j} M_{p+1-j}}{x^{p+1-j}} \mathrm{d}x \\ &= s^{\widehat{\boldsymbol{A}},h}_{\boldsymbol{M},h}(\varphi) h^{p+q+1-2j} M_{p+1-j} (q-j)! A_{q-j} \log(\frac{m_{p+1-j}}{m_{p-j}}) \\ &\leq C_0 s^{\widehat{\boldsymbol{A}},h}_{\boldsymbol{M},h}(\varphi) H^{p+1-j} h^{p+q+1-2j} M_{p+1-j} (q-j)! A_{q-j}. \end{aligned} \tag{4.6}$$

The integral on the interval $(-hm_{p+1-j}, -hm_{p-j})$ is treated analogously. Finally, again (4.3) provides

$$\begin{aligned}\int_{-hm_{p-j}}^{hm_{p-j}} |x^{p-j} \varphi^{(q-j)}(x)| \mathrm{d}x &\leq 2hm_{p-j} s^{\widehat{\boldsymbol{A}},h}_{\boldsymbol{M},h}(\varphi) h^{p+q-2j} M_{p-j} (q-j)! A_{q-j} \\ &= 2 s^{\widehat{\boldsymbol{A}},h}_{\boldsymbol{M},h}(\varphi) h^{p+q+1-2j} M_{p+1-j} (q-j)! A_{q-j}. \end{aligned} \tag{4.7}$$

Taking (4.5), (4.6) and (4.7) into (4.4), we obtain

$$\sup_{\xi\in\mathbb{R}} |\xi^q \widehat{\varphi}^{(p)}(\xi)| \leq \sum_{j=0}^{\min\{p,q\}} \binom{q}{j}\binom{p}{j} j! 2 s^{\widehat{\boldsymbol{A}},h}_{\boldsymbol{M},h}(\varphi) h^{p+q+1-2j} M_{p+1-j}(q-j)! A_{q-j}(2+C_0 H^{p+1-j}). \tag{4.8}$$

Every weight sequence is almost increasing. In case $\boldsymbol{A}$ also is, there exists $D \geq 1$ such that $A_j \leq DA_p$ and $M_j \leq DM_p$ for all $j \leq p$. This fact, together with the elementary inequalities $j!(q-j)! \leq q!$ and $\binom{q}{j} \leq 2^q$ for every $0 \leq j \leq q$, allow us to write

$$\begin{aligned}
\sup_{\xi\in\mathbb{R}} |\xi^q \widehat{\varphi}^{(p)}(\xi)| &\leq 2 s^{\widehat{\boldsymbol{A}},h}_{\boldsymbol{M},h}(\varphi) D^2 h^{p+q+1} 2^q M_{p+1} q! A_q (2 + C_0 H) \sum_{j=0}^{p} \binom{p}{j} h^{-2j} H^{p-j} \\
&= 2(2 + C_0 H) D^2 h s^{\widehat{\boldsymbol{A}},h}_{\boldsymbol{M},h}(\varphi) \left(\frac{1}{h^2} + H\right)^p 2^q h^{p+q} M_{p+1} q! A_q \\
&\leq 2(2 + C_0 H) D^2 h s^{\widehat{\boldsymbol{A}},h}_{\boldsymbol{M},h}(\varphi) \left(2(1 + H)h\right)^{p+q} M_{p+1} q! A_q
\end{aligned}$$

for all $p, q \in \mathbb{N}_0$.

In case $\widehat{\boldsymbol{A}}$ satisfies (lc), we use Lemma 1.1.6 (vi) for $\widehat{\boldsymbol{A}}$, so $(q-j)!A_{q-j} \leq q!A_q/(j!A_j)$ for $0 \leq j \leq q$. If moreover $\liminf_{p\to\infty} A_p^{1/p} > 0$, there exists $c > 0$ such that $A_p \geq c^p$ for every $p$, and going back to (4.8), we get

$$\begin{aligned}
\sup_{\xi\in\mathbb{R}} |\xi^q \widehat{\varphi}^{(p)}(\xi)| &\leq 2 s^{\widehat{\boldsymbol{A}},h}_{\boldsymbol{M},h}(\varphi) D h^{p+q+1} 2^q M_{p+1} q! A_q (2 + C_0 H) \sum_{j=0}^{p} \binom{p}{j} h^{-2j} c^{-j} H^{p-j} \\
&= 2(2 + C_0 H) D h s^{\widehat{\boldsymbol{A}},h}_{\boldsymbol{M},h}(\varphi) \left(\frac{1}{ch^2} + H\right)^p 2^q h^{p+q} M_{p+1} q! A_q \\
&\leq 2(2 + C_0 H) D h s^{\widehat{\boldsymbol{A}},h}_{\boldsymbol{M},h}(\varphi) \left(2\Big(\frac{1}{c} + H\Big)h\right)^{p+q} M_{p+1} q! A_q
\end{aligned}$$

for all $p, q \in \mathbb{N}_0$. Hence, we have proved the first statement with $a = 2(H+1)$ in the first case, and $a = 2(H + 1/c)$ in the second one. □

Resting on Proposition 4.1.10, the next result can be shown in a similar way as the corresponding implication in [12, Prop. 2.1].

**Proposition 4.1.11.** *Let $\boldsymbol{M}$ be a weight sequence satisfying* (sm), *and $\boldsymbol{A}$ be either an almost increasing sequence, or a sequence such that $\liminf_{p\to\infty} A_p^{1/p} > 0$ and $\widehat{\boldsymbol{A}}$ satisfies* (lc). *If $\psi \in \mathcal{S}^{\{\boldsymbol{M}_{+1}\}}_{\{\widehat{\boldsymbol{A}}\}}(\mathbb{R})$ and there is $\Psi : \overline{\mathbb{H}} \to \mathbb{C}$ satisfying the following conditions:*

*(i)* $\Psi_{|\mathbb{R}} = \psi$.

(*ii*) $\Psi$ *is continuous on* $\overline{\mathbb{H}}$ *and analytic on* $\mathbb{H}$.

(*iii*) $\lim_{\zeta\in\overline{\mathbb{H}},\zeta\to\infty}\Psi(\zeta)=0$,

*then* $\psi\in\mathcal{F}(\mathcal{S}^{\{\widehat{\boldsymbol{A}}\}}_{\{\boldsymbol{M}\}}(0,\infty))$.

### 4.1.2 The Laplace transform

In order to introduce the second integral transformation, the Laplace transform, we need an auxiliary space. More precisely:

**Definition 4.1.12.** Let $\boldsymbol{M}$ be a sequence. We define $C_{\boldsymbol{M},h}(0,\infty)$ as the space consisting of all $\varphi\in C((0,\infty))$ such that

$$s^0_{\boldsymbol{M},h}(\varphi)=\sup_{p\in\mathbb{N}_0}\sup_{x\in(0,\infty)}\frac{x^p|\varphi(x)|}{h^pM_p}<\infty.$$

Note that, $(C_{\boldsymbol{M},h}(0,\infty),s^0_{\boldsymbol{M},h})$ is a Banach space. Moreover, we set

$$C_{\{\boldsymbol{M}\}}(0,\infty)=\bigcup_{h>0}C_{\boldsymbol{M},h}(0,\infty),$$

and endow it with its natural $(LB)$ space structure.

If $\boldsymbol{M}$ is a weight sequence and $\varphi\in C_{\boldsymbol{M},h}(0,\infty)$, this amounts to having

$$|\varphi(x)|\le s^0_{\boldsymbol{M},h}(\varphi)e^{-\omega_{\boldsymbol{M}}(|x|/h)}=s^0_{\boldsymbol{M},h}(\varphi)h_{\boldsymbol{M}}(h/|x|),\quad x>0, \tag{4.9}$$

as in (4.1).

**Definition 4.1.13.** We define the *Laplace transform* of $\varphi\in C_{\{\boldsymbol{M}\}}(0,\infty)$ as

$$\mathcal{L}(\varphi)(\zeta)=\int_0^\infty\varphi(x)e^{ix\zeta}\mathrm{d}x,\qquad\zeta\in\overline{\mathbb{H}}.$$

**Remark 4.1.14.** Let $\boldsymbol{M}$ and $\boldsymbol{A}$ be sequences. We have that $\mathcal{S}^{\{\boldsymbol{A}\}}_{\{\boldsymbol{M}\}}(0,\infty)\subset\mathcal{S}_{\{\boldsymbol{M}\}}(0,\infty)\subset C_{\{\boldsymbol{M}\}}(0,\infty)$ with continuous inclusions, since clearly $\mathcal{S}^{\boldsymbol{A},h}_{\boldsymbol{M},h}(0,\infty)\subset\mathcal{S}_{\boldsymbol{M},h}(0,\infty)\subset C_{\boldsymbol{M},h}(0,\infty)$ for every $h>0$, the norm in $C_{\boldsymbol{M},h}(0,\infty)$ enters the family of seminorms defining the topology of $\mathcal{S}_{\boldsymbol{M},h}(0,\infty)$, and

$$s^q_{\boldsymbol{M},h}(\varphi)\le h^qA_qs^{\boldsymbol{A},h}_{\boldsymbol{M},h}(\varphi),\quad\varphi\in\mathcal{S}^{\boldsymbol{A},h}_{\boldsymbol{M},h}(0,\infty),\ q\in\mathbb{N}_0.$$

Note that $\mathcal{L}(\varphi)_{|\mathbb{R}}=\widehat{\varphi}$ for all $\varphi\in\mathcal{S}^{\{\boldsymbol{A}\}}_{\{\boldsymbol{M}\}}(0,\infty)$.

**Lemma 4.1.15.** *Let $\boldsymbol{M}$ be a weight sequence satisfying* (sm)*. Then, for every $h > 0$ one has $\mathcal{L}(C_{\boldsymbol{M},h}(0,\infty)) \subset \mathcal{A}_{\boldsymbol{M}_{+1},Hh}(\mathbb{H})$, where $H > 1$ is the constant appearing in* (sm)*, and $\mathcal{L}\colon C_{\boldsymbol{M},h}(0,\infty) \to \mathcal{A}_{\boldsymbol{M}_{+1},Hh}(\mathbb{H})$ is continuous. So, the mapping $\mathcal{L} : C_{\{\boldsymbol{M}\}}(0,\infty) \to \mathcal{A}_{\{\boldsymbol{M}_{+1}\}}(\mathbb{H})$ is well-defined and continuous, and it is moreover injective.*

*Proof.* Suppose $\varphi \in C_{\{\boldsymbol{M}\}}(0,\infty)$, and choose $h > 0$ such that (4.9) holds. Given $\zeta \in \mathbb{H}$ and $p \in \mathbb{N}_0$, since $\Re(ix\zeta) = -x\Im(\zeta) < 0$ for every $x > 0$ (where $\Im$ denotes the imaginary part), we have

$$
\begin{aligned}
|(\mathcal{L}(\varphi))^{(p)}(\zeta)| &\leq \int_0^\infty x^p|\varphi(x)|\mathrm{d}x \qquad (4.10)\\
&\leq \int_0^{hm_p} x^p|\varphi(x)|\mathrm{d}x + s^0_{\boldsymbol{M},h}(\varphi)\int_{hm_p}^{hm_{p+1}} x^p\frac{h^{p+1}M_{p+1}}{x^{p+1}}\mathrm{d}x\\
&+ \int_{hm_{p+1}}^\infty \frac{1}{x^2}x^{p+2}|\varphi(x)|\mathrm{d}x\\
&\leq s^0_{\boldsymbol{M},h}(\varphi)\left(hm_ph^pM_p + h^{p+1}M_{p+1}\log\left(\frac{m_{p+1}}{m_p}\right) + h^{p+2}M_{p+2}\frac{1}{hm_{p+1}}\right)\\
&\leq s^0_{\boldsymbol{M},h}(\varphi)h^{p+1}M_{p+1}\left(2 + C_0H^{p+1}\right) \leq (2 + C_0H)hs^0_{\boldsymbol{M},h}(\varphi)(Hh)^pM_{p+1},
\end{aligned}
$$

where in the next-to-last inequality (sm) has been applied. Hence, $\mathcal{L}$ is well-defined and continuous from $C_{\boldsymbol{M},h}(0,\infty)$ into $\mathcal{A}_{\boldsymbol{M}_{+1},hH}(\mathbb{H})$.

The proof of injectivity can be found in [16, Lemma 2.10]. □

**Remark 4.1.16.** If we suppose that $\boldsymbol{M}$ satisfies the stronger condition (dc) instead of (sm), then the Laplace transform $\mathcal{L}$ sends $C_{\{\boldsymbol{M}\}}(0,\infty)$ into $\mathcal{A}_{\{\boldsymbol{M}\}}(\mathbb{H})$. This is easily seen by splitting the integral in (4.10) into only two subintervals, $(0, hm_p)$ and $(hm_p,\infty)$ and estimating similarly.

## 4.2 A new Stieltjes moment problem in Gelfand-Shilov spaces

In this section we present the main results. Firstly, the use of optimal flat functions in ultraholomorphic classes (see Proposition 3.2.17), allows to determine the appropriate target space in the moment problem according to whether (dc) or (sm) is satisfied. After some auxiliary results, Theorem 4.2.6 characterizes the injectivity of the Stieltjes moment mapping under condition (sm) for $\boldsymbol{M}$. Finally, Theorem 4.2.7 studies the surjectivity problem and its connection to the existence of local right inverses for $\mathcal{M}$ with a uniform scaling of the parameter defining the Banach spaces under consideration.

We recall the following definition of the $p$-th moment associated with a function in $C_{\{\boldsymbol{M}\}}(0,\infty)$.

**Definition 4.2.1.** Let $\boldsymbol{M}$ be a weight sequence. The $p$-th moment, $p \in \mathbb{N}_0$, of an element $\varphi \in C_{\{\boldsymbol{M}\}}(0,\infty)$ is defined as

$$\mu_p(\varphi) := \int_0^\infty x^p \varphi(x)\mathrm{d}x.$$

The formula

$$\mathcal{L}(\varphi)^{(p)}(0) = i^p \mu_p(\varphi), \qquad \varphi \in C_{\{\boldsymbol{M}\}}(0,\infty), p \in \mathbb{N}_0,$$

guarantees, according to Lemma 4.1.15, that whenever $\boldsymbol{M}$ satisfies (sm) the *Stieltjes moment mapping*

$$\mathcal{M} : C_{\{\boldsymbol{M}\}}(0,\infty) \to \Lambda_{\{\boldsymbol{M}_{+1}\}};\ \varphi \mapsto (\mu_p(\varphi))_p$$

is well-defined and continuous. Indeed, for every $h > 0$ and $\varphi \in C_{\boldsymbol{M},h}(0,\infty)$ one has

$$|\mathcal{M}(\varphi)|_{\boldsymbol{M}_{+1},Hh} \leq (2 + C_0 H) h s^0_{\boldsymbol{M},h}(\varphi).$$

However, if $\boldsymbol{M}$ satisfies (dc), Remark 4.1.16 shows that $\mathcal{M}$ sends $C_{\{\boldsymbol{M}\}}(0,\infty)$ into $\Lambda_{\{\boldsymbol{M}\}}$. This latter situation was studied in [16], while the former one is our objective now. In order to stress the relevance of conditions (dc) and (sm) for the Stieltjes moment problem, we need to consider optimal flat functions (see section 3.2).

We recall (see section 3.2.2) that if $G$ is an optimal $\{\boldsymbol{M}\}$-flat function in $S_\gamma$, we define the kernel function $e\colon S_\gamma \to \mathbb{C}$ given by

$$e(z) := G\left(\frac{1}{z}\right), \quad z \in S_\gamma.$$

Because of (3.5) and (3.6), we have that

$$K_1 h_{\boldsymbol{M}}\left(\frac{K_2}{x}\right) \leq e(x) \leq K_3 h_{\boldsymbol{M}}\left(\frac{K_4}{x}\right), \quad x > 0, \tag{4.11}$$

and according to (4.9), we see that (the restriction to $(0,\infty)$ of) $e$ belongs to $C_{\{\boldsymbol{M}\}}(0,\infty)$.

The following result, partially obtained in Proposition 3.2.18, shows the key role of such kernel functions. We include the whole proof for the reader's convenience.

**Proposition 4.2.2.** *Suppose $\boldsymbol{M}$ is a weight sequence with $\gamma(\boldsymbol{M}) > 0$. Then, $\mathcal{M}(C_{\{\boldsymbol{M}\}}(0,\infty)) \subset \Lambda_{\{\boldsymbol{M}\}}$ if, and only if, $\boldsymbol{M}$ satisfies* (dc).

*Proof.* As indicated in Remark 4.1.16, the condition is sufficient. Conversely, suppose now that $\mathcal{M}(C_{\{\boldsymbol{M}\}}(0,\infty)) \subset \Lambda_{\{\boldsymbol{M}\}}$. Consider an optimal $\{\boldsymbol{M}\}$-flat function $G$ in a suitably narrow sector $S$ bisected by the positive real axis, and let $e$ be the corresponding kernel function. Since (the restriction to $(0,\infty)$ of) $e \in C_{\{\boldsymbol{M}\}}(0,\infty)$, there exists $C, h > 0$ such that $\mu_p(e) \leq Ch^p M_p$ for every $p \in \mathbb{N}_0$. On the other hand, by the left-hand inequalities in (4.11) and the monotonicity of $h_{\boldsymbol{M}}$, for every $p \in \mathbb{N}_0$ and $s > 0$ we may estimate

$$\mu_p(e) \geq \int_0^s t^p e(t)\,dt \geq K_1 \int_0^s t^p h_{\boldsymbol{M}}\left(\frac{K_2}{t}\right) dt \geq K_1 h_{\boldsymbol{M}}\left(\frac{K_2}{s}\right)\frac{s^{p+1}}{p+1}.$$

Then, by (1.9) we deduce that

$$\mu_p(e) \geq \frac{K_1}{p+1}\sup_{s>0} h_{\boldsymbol{M}}\left(\frac{K_2}{s}\right)s^{p+1} = \frac{K_1}{p+1}K_2^{p+1}M_{p+1} \geq K_1K_2\left(\frac{K_2}{2}\right)^p M_{p+1}.$$

From the estimates for $\mu_p(e)$ from above and below we deduce that (dc) is satisfied. □

Similarly, we have the following characterization, which was again partially included in Proposition 3.5.6.

**Proposition 4.2.3.** *Suppose* $\boldsymbol{M}$ *is a weight sequence with* $\gamma(\boldsymbol{M}) > 0$. *Then,* $\mathcal{M}(C_{\{\boldsymbol{M}\}}(0,\infty)) \subset \Lambda_{\{\boldsymbol{M}_{+1}\}}$ *if, and only if,* $\boldsymbol{M}$ *satisfies* (sm).

*Proof.* As previously said, Lemma 4.1.15 implies the condition is sufficient. Suppose now that $\mathcal{M}(C_{\{\boldsymbol{M}\}}(0,\infty)) \subset \Lambda_{\{\boldsymbol{M}_{+1}\}}$, and consider $e \in C_{\{\boldsymbol{M}\}}(0,\infty)$ as before, so that there exist $C, h > 0$ such that $\mu_p(e) \leq Ch^p M_{p+1}$ for every $p \in \mathbb{N}_0$. On the other hand, by the left-hand inequalities in (4.11), for every $p \in \mathbb{N}_0$ we have that

$$\begin{aligned}\mu_p(e) \geq \int_{K_2m_p}^{K_2m_{p+1}} t^p e(t)\,dt \geq K_1 \int_{K_2m_p}^{K_2m_{p+1}} t^p h_{\boldsymbol{M}}\left(\frac{K_2}{t}\right) dt \\ = K_1K_2^{p+1}M_{p+1}\log\left(\frac{m_{p+1}}{m_p}\right),\end{aligned}$$

where the last equality is a consequence of (1.8). Again the estimates for $\mu_p(e)$ from above and below imply that (sm) is satisfied. □

We will reduce, via the Laplace transform, the study of the injectivity and surjectivity of the Stieltjes moment mapping in this new setting to their counterparts for the asymptotic Borel mapping (Theorems 4.1.5 and 4.1.3), as it was already done by A. L. Durán and R. Estrada in [19], and later on by several authors [12, 43, 44, 16].

The next lemma provides an auxiliary function, already appearing in the work [19] and later adapted to our needs, see [16]. We set $\mathbb{H}_{-1} = \{z \in \mathbb{C} \,|\, \Im z > -1\}$.

**Lemma 4.2.4.** *([16, Lemma 3.1]) Let $\boldsymbol{A}$ be a sequence satisfying* (nq)*, and such that $\widehat{\boldsymbol{A}}$ is a weight sequence. Then, there is $G \in \mathcal{H}(\mathbb{H}_{-1})$ satisfying the following conditions:*

(*i*) *$G$ does not vanish on $\mathbb{H}_{-1}$.*

(*ii*) $\sup_{z\in\mathbb{H}_{-1}} |G(z)|e^{\omega_{\widehat{A}}(|z|)} < \infty.$

(*iii*) $\displaystyle\sup_{p\in\mathbb{N}}\sup_{x\in\mathbb{R}} \frac{|G^{(p)}(x)|e^{\omega_{\widehat{A}}(|x|/2)}}{2^p p!} < \infty.$

Proposition 4.1.11 and Lemma 4.2.4 imply the following general result. A similar proof for classical Gelfand-Shilov spaces (i. e., when $\boldsymbol{M}$ is a Gevrey sequence $(p!^{\alpha})_p$ with $\alpha > 1$) can be found in [43, Prop. 4.13], and a similar statement for strongly regular sequences $\boldsymbol{M}$ with $\gamma(\boldsymbol{M}) > 1$ was included in [44, Prop. 6.6] without proof. Another version, disregarding continuity and under stronger conditions than the ones imposed here, appeared in [16, Lemma 3.3].

**Lemma 4.2.5.** *Let $\boldsymbol{M}$ be a sequence satisfying* (lc) *and such that $(p!)_p \precsim \boldsymbol{M}$ (equivalently, $\liminf_{p\to\infty} \breve{M}_p^{1/p} > 0$), and let $\boldsymbol{A}$ be a sequence satisfying* (nq) *and such that $\widehat{\boldsymbol{A}}$ is a weight sequence. Consider the function $G$ from Lemma 4.2.4. Then, there exists $a > 0$ such that for every $h \geq 1$ and for every $f \in \mathcal{A}_{\boldsymbol{M}_{+1},h}(\mathbb{H})$ one has $(fG)|_{\mathbb{R}} \in \mathcal{S}^{\boldsymbol{M}_{+1},ah}_{\widehat{\boldsymbol{A}},ah}(\mathbb{R})$, and the map so defined is continuous from $\mathcal{A}_{\boldsymbol{M}_{+1},h}(\mathbb{H})$ into $\mathcal{S}^{\boldsymbol{M}_{+1},ah}_{\widehat{\boldsymbol{A}},ah}(\mathbb{R})$.*
*Moreover, if $\boldsymbol{M}$ satisfies also* (sm)*, then $(fG)|_{\mathbb{R}} \in \mathcal{F}(\mathcal{S}^{\{\widehat{\boldsymbol{A}}\}}_{\{\boldsymbol{M}\}}(0,\infty))$ for every $f \in \mathcal{A}_{\{\boldsymbol{M}_{+1}\}}(\mathbb{H})$.*

*Proof.* According to Lemma 4.2.4.(*iii*), (1.13) and the definition of $h_{\widehat{\boldsymbol{A}}}$, there exists $C > 0$ such that for every $p, j \in \mathbb{N}_0$ and $x \in \mathbb{R}$ one has

$$|G^{(j)}(x)| \leq C2^j j! h_{\widehat{\boldsymbol{A}}}(2/|x|) \leq C2^j j! \widehat{A}_p \left(\frac{2}{|x|}\right)^p.$$

Hence, for $h \geq 1$, $f \in \mathcal{A}_{\boldsymbol{M}_{+1},h}(\mathbb{H})$, $x \in \mathbb{R}$ and $p, q \in \mathbb{N}_0$ we get

$$\begin{aligned} |x^p (fG)^{(q)}(x)| &= \left| x^p \sum_{k=0}^{q} \binom{q}{k} f^{(k)}(x) G^{(q-k)}(x) \right| \\ &\leq C\|f\|_{\boldsymbol{M}_{+1},h} 2^p \widehat{A}_p \sum_{k=0}^{q} \binom{q}{k} h^k M_{k+1} 2^{q-k}(q-k)!. \end{aligned}$$

We apply now Lemma (1.1.6)(vi) for $\boldsymbol{M}$. Also, by hypothesis there exists $B>0$ such that $M_p \geq B^p p!$ for every $p \in \mathbb{N}_0$, so

$$
\begin{aligned}
|x^p (fG)^{(q)}(x)| &\leq C\|f\|_{\boldsymbol{M}_{+1},h} 2^p \widehat{A}_p M_{q+1} \sum_{k=0}^{q} \binom{q}{k} h^k 2^{q-k} \frac{(q-k)!}{M_{q-k}} \\
&\leq C\|f\|_{\boldsymbol{M}_{+1},h} 2^p \widehat{A}_p M_{q+1} \sum_{k=0}^{q} \binom{q}{k} h^k \left(\frac{2}{B}\right)^{q-k} \\
&\leq C\|f\|_{\boldsymbol{M}_{+1},h} \left(2(1+2/B)h\right)^{p+q} \widehat{A}_p M_{q+1}.
\end{aligned}
$$

Hence, the first statement is proved with $a = 2(1+2/B)$. The second assertion stems directly from Proposition 4.1.11. □

We are ready to study the injectivity and surjectivity of the Stieltjes moment mapping.

**Theorem 4.2.6.** *Let $\boldsymbol{M}$ be a weight sequence satisfying* (sm) *and* $(p!)_p \precsim \boldsymbol{M}$*, and let $\boldsymbol{A}$ be a sequence satisfying* (nq) *and such that $\widehat{\boldsymbol{A}}$ is a weight sequence. Then, the following statements are equivalent:*

*(i)* $\displaystyle\sum_{p=0}^{\infty} \frac{1}{m_p^{1/2}} = \infty.$

*(ii)* $\mathcal{B} : \mathcal{A}_{\{\boldsymbol{M}_{+1}\}}(\mathbb{H}) \to \Lambda_{\{\boldsymbol{M}_{+1}\}}$ *is injective.*

*(iii)* $\mathcal{M} : C_{\{\boldsymbol{M}\}}(0,\infty) \to \Lambda_{\{\boldsymbol{M}_{+1}\}}$ *is injective.*

*(iv)* $\mathcal{M} : \mathcal{S}_{\{\boldsymbol{M}\}}(0,\infty) \to \Lambda_{\{\boldsymbol{M}_{+1}\}}$ *is injective.*

*(v)* $\mathcal{M} : \mathcal{S}_{\{\boldsymbol{M}\}}^{\{\widehat{\boldsymbol{A}}\}}(0,\infty) \to \Lambda_{\{\boldsymbol{M}_{+1}\}}$ *is injective.*

*Proof.* $(i) \Rightarrow (ii)$: By Theorem 4.1.5.

$(ii) \Rightarrow (iii)$: Let $\varphi \in C_{\{\boldsymbol{M}\}}(0,\infty)$ be such that $\mu_p(\varphi) = 0$ for all $p \in \mathbb{N}_0$. By Lemma 4.1.15 we have that $\mathcal{L}(\varphi) \in \mathcal{A}_{\{\boldsymbol{M}_{+1}\}}(\mathbb{H})$. Moreover, $\mathcal{L}(\varphi)^{(p)}(0) = i^p \mu_p(\varphi) = 0$ for all $p \in \mathbb{N}_0$ and, thus, $\mathcal{L}(\varphi) \equiv 0$. Since $\mathcal{L}$ is injective (Lemma 4.1.15), we obtain that $\varphi \equiv 0$.

$(iii) \Rightarrow (iv) \Rightarrow (v)$: Obvious.

$(v) \Rightarrow (i)$: In view of Theorem 4.1.5 it suffices to show that $\mathcal{B} : \mathcal{A}_{\{\boldsymbol{M}_{+1}\}}(\mathbb{H}) \to \Lambda_{\{\boldsymbol{M}_{+1}\}}$ is injective. Let $f \in \mathcal{A}_{\{\boldsymbol{M}_{+1}\}}(\mathbb{H})$ be such that $f^{(p)}(0) = 0$ for all $p \in \mathbb{N}_0$. Consider the function $G$ from Lemma 4.2.4. By Lemma 4.2.5 we have that

$(fG)|_{\mathbb{R}} = \widehat{\varphi}$ for some $\varphi \in \mathcal{S}^{\{\widehat{\boldsymbol{A}}\}}_{\{\boldsymbol{M}\}}(0,\infty)$. Observe that

$$\mu_p(\varphi) = (-i)^p \widehat{\varphi}^{(p)}(0) = (-i)^p (fG)^{(p)}(0) = (-i)^p \sum_{j=0}^{p} \binom{p}{j} f^{(j)}(0) G^{(p-j)}(0) = 0, \quad p \in \mathbb{N}_0.$$

Hence, $\varphi \equiv 0$ and, thus, $fG \equiv 0$. Since $G$ does not vanish (Lemma 4.2.4($i$)), we obtain that $f \equiv 0$. □

In the case of surjectivity, it will frequently come with local extension operators, right inverses for the Borel, respectively, the moment mapping, with a uniform scaling of the constant $h$ determining the Banach spaces under consideration.

**Theorem 4.2.7.** *Let $\boldsymbol{M}$ be a weight sequence satisfying* (sm)*, and let $\boldsymbol{A}$ be a sequence satisfying* (nq) *and such that $\widehat{\boldsymbol{A}}$ is a weight sequence. Then:*
*(I) Each of the following statements implies the next one:*

($i$) *There exists $a > 0$ such that for every $h \geq 1$ there exists a linear and continuous operator $R_h\colon \Lambda_{\boldsymbol{M}_{+1},h} \to \mathcal{S}^{\widehat{\boldsymbol{A}},ah}_{\boldsymbol{M},ah}(0,\infty)$ such that $\mathcal{M} \circ R_h$ is the identity map in $\Lambda_{\boldsymbol{M}_{+1},h}$.*

($ii$) *There exists $a > 0$ such that for every $h \geq 1$ there exists a linear and continuous operator $T_h\colon \Lambda_{\boldsymbol{M}_{+1},h} \to \mathcal{S}_{\boldsymbol{M},ah}(0,\infty)$ such that $\mathcal{M} \circ T_h$ is the identity map in $\Lambda_{\boldsymbol{M}_{+1},h}$.*

($iii$) *There exists $a > 0$ such that for every $h \geq 1$ there exists a linear and continuous operator $U_h\colon \Lambda_{\boldsymbol{M}_{+1},h} \to C_{\boldsymbol{M},ah}(0,\infty)$ such that $\mathcal{M} \circ U_h$ is the identity map in $\Lambda_{\boldsymbol{M}_{+1},h}$.*

($iv$) *There exists $a > 0$ such that for every $h \geq 1$ there exists a linear and continuous operator $V_h\colon \Lambda_{\boldsymbol{M}_{+1},h} \to \mathcal{A}_{\boldsymbol{M}_{+1},ah}(\mathbb{H})$ such that $\mathcal{B} \circ V_h$ is the identity map in $\Lambda_{\boldsymbol{M}_{+1},h}$.*

*The following statements are equivalent:*

($i'$) $\mathcal{M} : \mathcal{S}^{\{\widehat{\boldsymbol{A}}\}}_{\{\boldsymbol{M}\}}(0,\infty) \to \Lambda_{\{\boldsymbol{M}_{+1}\}}$ *is surjective.*

($ii'$) $\mathcal{M} : \mathcal{S}_{\{\boldsymbol{M}\}}(0,\infty) \to \Lambda_{\{\boldsymbol{M}_{+1}\}}$ *is surjective.*

($iii'$) $\mathcal{M} : C_{\{\boldsymbol{M}\}}(0,\infty) \to \Lambda_{\{\boldsymbol{M}_{+1}\}}$ *is surjective.*

($iv'$) $\mathcal{B} : \mathcal{A}_{\{\boldsymbol{M}_{+1}\}}(\mathbb{H}) \to \Lambda_{\{\boldsymbol{M}_{+1}\}}$ *is surjective.*

*Any of the statements from (i) to (iv) implies all the conditions from (i') to (iv'). Moreover, the condition*

($v'$) $\gamma(\boldsymbol{M}) > 2$.

*implies (iv).*
*(II) If* $\boldsymbol{A}$ *satisfies in addition the condition* (sm), *then (iv) implies:*

($v$) *There exists* $a > 0$ *such that for every* $h \geq 1$ *there exists a linear and continuous operator* $W_h \colon \Lambda_{\boldsymbol{M}_{+1},h} \to \mathcal{S}^{(\widehat{\boldsymbol{A}})_{+1},ah}_{\boldsymbol{M}_{+1},ah}(0,\infty)$ *such that* $\mathcal{M} \circ W_h$ *is the identity map in* $\Lambda_{\boldsymbol{M}_{+1},h}$.

*(III) If* $\boldsymbol{M}$ *satisfies in addition the condition* (dc), *then we can substitute* $\boldsymbol{M}_{+1}$ *by* $\boldsymbol{M}$ *in all its appearances, and (iv') implies (v'). So, the six conditions (i')-(v') and (iv) are equivalent.*
*(IV) If* $\boldsymbol{M}$ *and* $\boldsymbol{A}$ *satisfy in addition* (dc), *then we can substitute* $\boldsymbol{M}_{+1}$ *by* $\boldsymbol{M}$ *and* $(\widehat{\boldsymbol{A}})_{+1}$ *by* $\widehat{\boldsymbol{A}}$ *in all their appearances, and all the ten previous statements (i)-(v), (i')-(v') are equivalent.*

In the proof of Theorem 4.2.7 we shall use the following lemma, inspired by [19].

**Lemma 4.2.8.** *([16, Lemma 3.6]) Let* $(c_p)_p \in \mathbb{C}^{\mathbb{N}}$ *and let* $G \in C^\infty((-\delta,\delta))$, *for some* $\delta > 0$, *such that* $G(0) \neq 0$. *Set*

$$b_p = \sum_{j=0}^{p} \binom{p}{j} c_j \left(\frac{1}{G}\right)^{(p-j)}(0), \qquad p \in \mathbb{N}_0.$$

*Then,*

$$\sum_{j=0}^{p} \binom{p}{j} b_j G^{(p-j)}(0) = c_p, \qquad p \in \mathbb{N}_0.$$

*Proof of Theorem 4.2.7.* (*I*) (*i*) $\Rightarrow$ (*ii*) $\Rightarrow$ (*iii*): Remark 4.1.14 makes these implications obvious.

(*iii*) $\Rightarrow$ (*iv*): The map $J_0$ sending every $(c_p)_p \in \Lambda_{\boldsymbol{M}_{+1},h}$ into $((-i)^p c_p)_p$ is a topological isomorphism on $\Lambda_{\boldsymbol{M}_{+1},h}$, $h > 0$. Then it suffices to consider $V_h := \mathcal{L} \circ U_h \circ J_0$, which, according to Lemma 4.1.15, is linear and continuous from $\Lambda_{\boldsymbol{M}_{+1},h}$ into $\mathcal{A}_{\boldsymbol{M}_{+1},bh}(\mathbb{H})$ for some $b > 0$ independent from $h \geq 1$. Moreover, $\mu_p(U_h(J_0((c_p)_p))) = (-i)^p c_p$, and so $(V_h((c_p)_p))^{(p)}(0) = i^p \mu_p(U_h(J_0((c_p)_p))) = c_p$ for all $p \in \mathbb{N}_0$, as desired.

($i'$) $\Rightarrow$ ($ii'$) $\Rightarrow$ ($iii'$) Obvious by the corresponding contentions.

$(iii') \Rightarrow (iv')$: Given $(c_p)_p \in \Lambda_{\{\boldsymbol{M}_{+1}\}}$, pick $\varphi \in C_{\{\boldsymbol{M}\}}(0,\infty)$ such that $\mu_p(\varphi) = (-i)^p c_p$ for every $p$. Then, $f := \mathcal{L}(\varphi) \in \mathcal{A}_{\{\boldsymbol{M}_{+1}\}}(\mathbb{H})$ by Lemma 4.1.15, and $f^{(p)}(0) = i^p \mu_p(\varphi) = c_p$ for all $p \in \mathbb{N}_0$.

$(iv') \Rightarrow (i')$: We first note that $(iv')$ implies that $\gamma(\boldsymbol{M}) > 1$ and, by Lemma 1.1.24, we deduce $(p!)_p \precsim \boldsymbol{M}$. To see this, observe that, as indicated in Lemma 4.1.1, a function $f \in \mathcal{A}_{\{\boldsymbol{M}_{+1}\}}(\mathbb{H})$ can be extended to $\overline{\mathbb{H}}$ and its restriction to $[-1,1]$ provides a function $f_0$ such that:

(a) $f_0^{(p)}(0) = f^{(p)}(0)$ for every $p \in \mathbb{N}_0$, and

(b) $f_0 \in \mathcal{E}^{\{\boldsymbol{M}_{+1}\}}([-1,1])$, the space of functions $\varphi \in C^\infty([-1,1])$ such that

$$\sup_{x\in[-1,1],\, p\in\mathbb{N}_0} \frac{|\varphi^{(p)}(x)|}{h^p M_{p+1}} < \infty$$

for suitable $h > 0$.

Hence, by $(iv')$ the Borel map $\mathcal{B}\colon \mathcal{E}^{\{\boldsymbol{M}_{+1}\}}([-1,1]) \to \Lambda_{\{\boldsymbol{M}_{+1}\}}$ will also be surjective, and a classical result of H.-J. Petzsche [52, Th. 3.5] (see also [33, Th. 4.4]) proves that $\boldsymbol{M}_{+1}$ satisfies $(\gamma_1)$. By (1.4), we have $\gamma(\boldsymbol{M}) = \gamma(\boldsymbol{M}_{+1}) > 1$.
Consider now the function $G$ from Lemma 4.2.4, and define the linear map $J$ sending every $(c_p)_p \in \Lambda_{\boldsymbol{M}_{+1},h}$, for some $h \geq 1$, into the sequence $(b_p)_p$ given by

$$b_p = \sum_{j=0}^{p} \binom{p}{j} i^j c_j \left(\frac{1}{G}\right)^{(p-j)}(0), \qquad p \in \mathbb{N}_0.$$

Since this fact will be useful later, we now prove that $J$ is continuous from $\Lambda_{\boldsymbol{M}_{+1},h}$ into $\Lambda_{\boldsymbol{M}_{+1},bh}$ for some $b > 0$ independent from $h$. The function $1/G$ is holomorphic on a disk with center at 0 and radius larger than $1/2$, so there is $C' > 0$ such that $|(1/G)^{(p)}(0)| \leq C' 2^p p!$ for all $p \in \mathbb{N}_0$. Hence,

$$|b_p| \leq C' |(c_p)_p|_{\boldsymbol{M}_{+1},h} \sum_{j=0}^{p} \binom{p}{j} h^j M_{j+1} 2^{p-j} (p-j)!.$$

Since $\boldsymbol{M}$ is (lc), we can use Lemma 1.1.6 (vi). Also, by the previous argument there exists $B > 0$ such that $M_p \geq B^p p!$ for every $p \in \mathbb{N}_0$. So,

$$\begin{aligned} |b_p| &\leq C' |(c_p)_p|_{\boldsymbol{M}_{+1},h} \sum_{j=0}^{p} \binom{p}{j} h^j 2^{p-j} \frac{(p-j)! M_{p+1}}{M_{p-j}} \\ &\leq C' |(c_p)_p|_{\boldsymbol{M}_{+1},h} \left( \left(1 + \frac{2}{B}\right) h \right)^p M_{p+1}, \quad p \in \mathbb{N}_0, \end{aligned}$$

and we are done with $b = 1 + 2/B$. By assumption, there exists $f \in \mathcal{A}_{\{\boldsymbol{M}_{+1}\}}(\mathbb{H})$ such that $f^{(p)}(0) = b_p$ for all $p \in \mathbb{N}_0$. Lemma 4.2.5 guarantees that $(fG)|_{\mathbb{R}} = \widehat{\varphi}$ for some $\varphi \in \mathcal{S}^{\{\widehat{\boldsymbol{A}}\}}_{\{\boldsymbol{M}\}}(0,\infty)$, and Lemma 4.2.8 implies that for each $p \in \mathbb{N}_0$, we have that

$$\mu_p(\varphi) = (-i)^p \widehat{\varphi}^{(p)}(0) = (-i)^p (fG)^{(p)}(0) = (-i)^p \sum_{j=0}^{p} \binom{p}{j} b_j G^{(p-j)}(0) = c_p,$$

so we are done.

It is evident that any of the first four statements $(*)$ implies the corresponding $(*')$, and so any of the statements from (i) to (iv) implies all the equivalent conditions from (i') to (iv').

$(v') \Rightarrow (iv)$ Note that $(v')$ amounts to $\gamma(\boldsymbol{M}_{+1}) > 2$, and this fact implies in particular (see Lemma 1.1.25) that $\boldsymbol{M}_{+1} \approx \widehat{\boldsymbol{N}}$ for a weight sequence $\boldsymbol{N}$ with $\gamma(\boldsymbol{N}) = \gamma(\boldsymbol{M}_{+1}) - 1 > 1$. Since the condition (sm) is stable under equivalence and also under passing from $\widehat{\boldsymbol{N}}$ to $\boldsymbol{N}$, it turns out that $\boldsymbol{N}$ satisfies (sm) as well, and we can apply Remark 3.5.8 and deduce (iv) for the spaces defined in terms of $\widehat{\boldsymbol{N}}$. Since the equivalence of sequences preserves the spaces of functions or sequences defined by them, this means that $(iv)$ holds as stated.

(II) Observe first that $(iv)$ implies $(iv')$, and as shown in $(iv') \Rightarrow (i')$, we then have $(p!)_p \precsim \boldsymbol{M}$. Since $\boldsymbol{A}$ satisfies (sm), we consider the function $G$, and the operator $J$, sending $(c_p)_p \in \Lambda_{\boldsymbol{M}_{+1},h}$, for some $h \geq 1$, into the sequence $(b_p)_p$ as before. By hypothesis, $f := V_h \circ J((c_p)_p) \in \mathcal{A}_{\{\boldsymbol{M}_{+1}\}}(\mathbb{H})$ is such that $f^{(p)}(0) = b_p$ for all $p \in \mathbb{N}_0$. Now we set $Tf := (fG)|_{\mathbb{R}}$, and define $W_h := \mathcal{F}^{-1} \circ T \circ V_h \circ J$. According to the behavior described in Lemma 4.2.5 and Proposition 4.1.10 (the hypotheses of the later are easily checked, as $\boldsymbol{M}_{+1}$ is almost increasing), the map $W_h$ is linear and continuous from $\Lambda_{\boldsymbol{M}_{+1},h}$ into $\mathcal{S}^{(\widehat{\boldsymbol{A}})_{+1},ah}_{\boldsymbol{M}_{+1},ah}(0,\infty)$ for some $a > 0$ independent from $h$, and $\mathcal{M} \circ W_h$ is the identity map in $\Lambda_{\boldsymbol{M}_{+1},h}$ by arguing as in $(iv') \Rightarrow (i')$.

(III) Since (dc) for the weight sequence $\boldsymbol{M}$ amounts to $\boldsymbol{M}_{+1} \approx \boldsymbol{M}$, the substitution keeps the considered spaces unchanged. Then, it suffices to apply Theorem 4.1.3 to see that $(iv') \Rightarrow (v')$.

(IV) If both $\boldsymbol{M}$ and $\boldsymbol{A}$ satisfy (dc), we have $\boldsymbol{M}_{+1} \approx \boldsymbol{M}$ and $(\widehat{\boldsymbol{A}})_{+1} \approx \widehat{\boldsymbol{A}}$, and so the statements (i) and (v) are equivalent. This fact and the previous implications allow for the conclusion. □

**Remark 4.2.9.** The equivalence of the five conditions $(i') - (v')$ when $\boldsymbol{M}$ is strongly regular (and so also $\boldsymbol{M}_{+1} \approx \boldsymbol{M}$) was already shown in [16, Th. 3.5], while the case when $\boldsymbol{M}$ is (dc) is deduced in [14, Th. 6.1.(b) and Th. 7.2.(b)]. The novelty in this situation consists in the equivalence with $(iv)$. One should also note that in [14] the Stieltjes moment problem is also solved for Beurling-like classes, and the existence of global right inverses in both the Roumieu and Beurling classes is characterized. However, the techniques used there seem to heavily depend on the condition (dc) (see, for example, Lemmas 3.6.(b) and 3.7.(b) and Proposition 5.1 in [14]), so they are not available under the weaker condition (sm). Nevertheless, (sm) allows for the construction of the local extension operators $V_h$ in $(iv)$, and this has been the motivation for this new insight. Note that, when $\boldsymbol{M}$ and $\boldsymbol{A}$ are (dc), the construction of local right inverses for $\mathcal{M}$ as the ones in $(i)-(iii)$, with a uniform scaling of the parameter $h$ entering the definition of the corresponding Banach spaces, is new, although (ii) was previously obtained in [44] when $\boldsymbol{M}$ is strongly regular.

# Conclusiones y trabajo futuro

La tesis ha tratado varios problemas relevantes relacionados con clases de funciones complejas, ya sea holomorfas en sectores no acotados de la superficie de Riemann del logaritmo, o indefinidamente derivables en la recta real y con soporte en $[0, \infty)$, cuyas derivadas están sujetas a cierta restricción en su crecimiento dada en términos de algún tipo de peso, bien una sucesión, una función o una matriz.

El primer objetivo, que se logró satisfactoriamente, fue caracterizar varias propiedades de estabilidad, como el cierre por inversas o por composición, para clases ultraholomorfas de funciones en sectores no acotados de la superficie de Riemann del logaritmo, de tipo Roumieu y definidas en términos de una matriz peso. Los resultados previamente conocidos a este respecto se debieron a J. Siddiqi y M. Ider [76] en 1987, y solo consideraron clases definidas por sucesiones peso y en sectores no más amplios que un semiplano. Nuestros resultados amplían y completan los suyos, ya que el trabajo con matrices peso abarca el caso de las sucesiones peso, y hemos resuelto el problema para sectores con apertura arbitraria. Una herramienta clave, que puede resultar útil en otros contextos, ha sido la construcción, bajo hipótesis suficientemente generales, de funciones características, que tienen una naturaleza maximal muy concreta dentro de estas clases. Como subproducto, también obtenemos nuevos resultados de estabilidad cuando el control del crecimiento en estas clases se expresa en términos de una función peso en el sentido de Braun-Meise-Taylor. Por supuesto, se pueden formular y estudiar algunas otras propiedades de estabilidad, pero todavía no hemos considerado otros problemas similares.

Hasta donde sabemos, los resultados de estabilidad para las clases de Beurling no se han estudiado previamente en la literatura. Hemos podido hacerlo aquí, aunque sólo para sectores que no sean más amplios que un semiplano. Esta limitación se debe a la falta de funciones características en este contexto para sectores generales, y nos ha obligado a aplicar una técnica completamente diferente, basada en la teoría de las álgebras de Fréchet multiplicativamente convexas. Una posible tarea futura es la extensión de estos resultados a sectores de apertura arbitraria. Esto podría necesitar un nuevo enfoque, ya que parece difícil obtener una familia, que juegue el papel de las exponenciales, que tenga un comportamiento similar en sectores arbitrariamente amplios.

Se puede realizar una aproximación diferente a las clases ultraholomorfas, cambiando las estimaciones de las derivadas por las correspondientes para los restos que aparecen en el desarrollo asintótico en el vértice, como se hizo en el tercer capítulo. Se pueden considerar las propiedades de estabilidad en este nuevo marco, y en la literatura se conocen algunos resultados para las clases de Gevrey. Ya hemos obtenido información parcial sobre la estabilidad bajo composición en clases definidas por una sucesión peso general, pero aún queda trabajo por hacer para presentar un resultado satisfactorio para el caso de matrices peso, ya que surgen algunas complejidades debidas al cambio en la estructura de peso.

El segundo logro importante de la disertación es la construcción de funciones planas óptimas en sectores de apertura adecuada para clases ultraholomorfas definidas en términos de una sucesión peso general. Consideramos que estas funciones podrían desempeñar un papel en muchos otros contextos donde aparecen estructuras ponderadas. Han sido extremadamente útiles para nuestro objetivo de mejorar los resultados conocidos de tipo Borel-Ritt, que tratan de la sobreyectividad de la aplicación de Borel asintótica en clases ultraholomorfas de Carleman asociadas a sucesiones peso fuertemente no casianalíticas generales. Mediante ddichas funciones, se pueden definir transformadas formales tipo Borel y transformadas tipo Laplace truncadas adecuadas, que permiten diseñar un procedimiento constructivo general para obtener operadores de extensión lineal continua, inversas por la derecha de la aplicación de Borel, para el caso de sucesiones peso regulares en el sentido de Dyn'kin, es decir, aquellas que satisfacen la condición de cierre por derivación. Más aún, la longitud del intervalo de sobreyectividad ha sido determinada para sucesiones peso que satisfacen (dc).

Además, se ha demostrado que una condición mucho más débil para la sucesión peso, la de tener momentos desplazados, es suficiente para obtener estos resultados de extensión. De esta manera, para todas las sucesiones que aparecen en las aplicaciones tenemos resultados satisfactorios de sobreyectividad y extensión en sectores cuya apertura es menor que un valor bien determinado, que depende de un índice de O-variación regular asociado con la sucesión.

Sin embargo, aún quedan algunos problemas pendientes por resolver en este sentido:

- Aunque sólo algunas sucesiones de crecimiento muy rápido quedan fuera de nuestras consideraciones, como por ejemplo $(q^{p^p})_p$, nos gustaría obtener resultados generales sobre la sobreyectividad para clases ultraholomorfas definidas por sucesiones peso completamente arbitrarias.

- La longitud del intervalo de sobreyectividad, es decir, el conjunto de valores positivos $\gamma$ tales que la aplicación de Borel es sobreyectiva para la clase definida en el sector de apertura $\pi\gamma$, no está determinada para sucesiones peso que no satisfagan (dc). En otras palabras, no tenemos pruebas de que la sobreyectividad implique $\gamma \leq \gamma(\boldsymbol{M})$, como sabemos bajo la condición (dc). La única información para sucesiones peso generales se puede encontrar en el artículo [33], donde se muestra que dicha longitud será como máximo la parte entera de $\gamma(\boldsymbol{M})$ más 1. Este es un problema interesante que estamos estudiando actualmente.

- Incluso cuando se sabe que la longitud del intervalo de sobreyectividad es $\gamma(\boldsymbol{M})$, o sea, para sucesiones regulares, la situación para la apertura del sector igual a $\pi\gamma(\boldsymbol{M})$ no está resuelta en muchos casos. Por ejemplo, queda pendiente el caso en que $\boldsymbol{M}$ es fuertemente regular y $\gamma(\boldsymbol{M})$ es un número irracional, a menos que la sucesión peso admita un orden aproximado no nulo, cuando sabemos que la aplicación de Borel no es sobreyectiva para esa apertura. Está resuelta (también en sentido negativo) la situación en que $\boldsymbol{M}$ es fuertemente regular y $\gamma(\boldsymbol{M})$ es un número racional, o si $\boldsymbol{M}$ satisface (dc) y $\gamma(\boldsymbol{M})$ es un número natural, pero no se sabe que se cumpla ninguna otra afirmación general.

- Parecen ser necesarias nuevas técnicas para demostrar la sobreyectividad de la aplicación de Borel en sectores estrechos si no se cumple la condición (sm), ya que la herramienta de las transformadas de Borel y Laplace ya no es aplicable.

- La existencia de operadores de extensión globales (inversos por la derecha para la aplicación de Borel) en el caso de las clases de Roumieu en un semiplano ha sido completamente resuelta por A. Debrouwere bajo la condición (dc), ver [14], y J. Jiménez-Garrido, J. Sanz y G. Schindl dieron una extensión satisfactoria del resultado para sectores arbitrarios en [37]. Sin embargo, esta pregunta está abierta en ausencia de (dc), y nos gustaría obtener alguna respuesta al menos cuando se cumpla la condición (sm). Por supuesto, se puede deducir fácilmente alguna información a partir de la necesidad de la condición ($\beta_2$) de H.-J. Petzsche [52] para la existencia de tales operadores en el marco ultradiferenciable, pero el argumento inverso no está disponible actualmente.

En cuanto a las clases de Beurling, hemos podido mejorar ligeramente un resultado clásico de J. Schmets y M. Valdivia bajo cierre por derivación. Bajo esta condición, observamos que los resultados de A. Debrouwere [15] resuelven completamente el problema para clases de funciones con desarrollo asintótico no uniforme, tanto en el sentido de la sobreyectividad como en lo que respecta a los operadores de extensión. Sus resultados, sin embargo, dependen en gran medida del uso de (dc). Aunque la nueva condición (sm) nos ha permitido demostrar la sobreyectividad de la aplicación de Borel en sectores adecuadamente estrechos, nuestra técnica (adaptada del trabajo de V. Thilliez y basada en las ideas de J. Chaumat y A.-M. Chollet) no proporciona una pista ni para determinar la longitud del intervalo de sobreyectividad ni para la existencia de operadores de extensión. Éste es un interesante problema abierto en el contexto Beurling, y consideramos que una prueba más constructiva de la sobreyectividad podría ayudar en su solución.

Otro problema abierto que no ha sido tratado en esta disertación, pero que entró en nuestros planes iniciales, es el estudio de la inyectividad y sobreyectividad de la aplicación de Borel para clases ultraholomorfas definidas mediante el control de las derivadas en regiones más generales que sectores. S. Mandelbrojt [46] ha dado una solución muy elegante al problema de inyectividad para el caso de desarrollos asintóticos uniformes, pero sólo se han obtenido algunos resultados parciales en el marco antes mencionado por parte de autores de la escuela rusa, ver los trabajos de R. S. Yulmukhametov [84], K. V. Trunov y R. S. Yulmukhametov [82] y R. A. Gaisin [23]. La dificultad de esta tarea merece un esfuerzo mayor en un futuro próximo.

Con respecto a los temas tratados en los capítulos segundo y tercero, nos gustaría mencionar que parece haber una estrecha conexión entre la existencia de funciones planas óptimas en una clase y para un sector dado $S_\gamma$, y la existencia de funciones características en la misma clase (es decir, la definida por la misma estructura de peso) pero en el sector $S_{\gamma+2}$. Este punto merece una aclaración y también será estudiado.

Finalmente, la condición de momentos desplazados ha permitido dar un nuevo enfoque al considerar el problema del momento de Stieltjes dentro de los espacios generales de Gelfand-Shilov definidos mediante sucesiones peso. La novedad consiste en la posibilidad de recubrir un espacio de llegada naturalmente mayor mediante la aplicación de momentos, que envía una función a su sucesión de momentos de Stieltjes. Se ha estudiado y, en algunos casos, se ha caracterizado la inyectividad y sobreyectividad de la aplicación de momentos en este nuevo escenario.

Es natural preguntarse cuál es el espacio de llegada natural para dicha aplicación cuando el espacio de Gelfand-Shilov del que partimos está definido por una sucesión peso totalmente general, y estudiar la inyectividad y la sobreyectividad en este marco. Otro tema a considerar en un futuro muy cercano es el tratamiento general de los mismos problemas para el caso de espacios tipo Beurling, especialmente cuando se supone que se cumple la condición (sm). Es probable que este caso pueda incluirse en la versión de este trabajo que enviaremos para su publicación.

# Conclusions and future work

The dissertation has treated several relevant problems dealing with classes of complex functions, either holomorphic in unbounded sectors of the Riemann surface of the logarithm or smooth in the real line and with support in $[0, \infty)$, whose derivatives are subject to certain restriction on their growth given in terms of some kind of weight, either a sequence, or a function, or a matrix.

The first aim, which has been satisfactorily accomplished, was to characterize several stability properties, such as inverse or composition closedness, for ultraholomorphic function classes in unbounded sectors of the Riemann surface of the logarithm of Roumieu type defined in terms of a weight matrix. The previously known results in this respect were due to J. Siddiqi and M. Ider [76] in 1987, and they only considered classes defined by weight sequences and in sectors not wider than a half-plane. Our results extend and complete theirs, as the work with weight matrices encompasses the weight sequence framework, and we have solved the problem for sectors with arbitrary opening. A key tool, which can be useful in other respects, has been the construction, under fairly general hypotheses, of characteristic functions, which have a precise maximal nature within these classes. As a by-product, we obtain also new stability results when the growth control in these classes is expressed in terms of a weight function in the sense of Braun-Meise-Taylor. Of course, some other stability properties can be formulated and studied, but we have not considered yet any other such problem.

Up to our knowledge, the stability results for Beurling classes have not been studied previously in the literature. We have been able to do so, although only for sectors not wider than a half-plane. This limitation is due to the lack of characteristic functions in this context for general sectors, and has forced us to apply a completely different technique, resting on the theory of multiplicatively convex Fréchet algebras. One possible future task is the extension of these results to sectors of arbitrary opening. This might need a new technique, since it seems difficult to obtain a family, playing the role of the exponentials, that has a similar behavior in arbitrarily wide sectors.

A different approach to ultraholomorphic classes can be made by changing the estimates for the derivatives into the corresponding ones for the remainders appearing in the asymptotic expansion at the vertex, as it has been done in the third chapter. One can consider stability properties in this new framework, and some results are known in the literature for Gevrey classes. We have already obtained some partial information regarding stability under composition in classes defined by a general weight sequence, but still some work has to be done in order to present a satisfactory result for the case of weight matrices, as some intricacies arise as the weight structure is changed.

The second important achievement of the dissertation is the construction of optimal flat functions in sectors of suitable opening for ultraholomorphic classes defined in terms of a general weight sequence. We consider that these functions could play a role in many other contexts where weighted structures appear. They have been extremely useful for our objective of improving known results of Borel-Ritt-type, which deal with the surjectivity of the asymptotic Borel mapping in Carleman ultraholomorphic classes associated to general strongly nonquasianalytic weight sequences. By means of them, one can define suitable formal Borel-like transforms, and truncated Laplace-like transforms, which allow for the design of a general constructive procedure in order to obtain linear continuous extension operators, right inverses of the Borel mapping, for the case of regular weight sequences in the sense of Dyn'kin, i. e., those satisfying derivation closedness. Moreover, the length of the surjectivity interval has been determined for weight sequences satisfying (dc).

Furthermore, a much weaker condition for the weight sequence, that of having shifted moments, is shown to be sufficient to obtain these extension results. In this way, for every sequence appearing in applications we have now satisfactory surjectivity and extension results on sectors whose opening is smaller than a well-determined value depending on an index of O-regular variation associated with the sequence.

However, there are still some pending problems to be solved in this regard:

- Although only some very fast growing sequences fall apart from our considerations, like e. g. $(q^{p^p})_p$, we would like to obtain a general results about surjectivity for ultraholomorphic classes defined by completely unrestricted weight sequences.

- The length of the surjectivity interval, i. e., the set of positive values $\gamma$ such that the Borel mapping is surjective for the class defined on the sector of opening $\pi\gamma$, is not determined for weight sequences not satisfying (dc). In other words, we have no proof that surjectivity implies $\gamma \leq \gamma(\boldsymbol{M})$, as we know under condition (dc). The only information for general weight sequences can be found in the paper [33], where it is shown that such length will be at most the integer part of $\gamma(\boldsymbol{M})$ plus 1. This is a very interesting problem we are currently studying.

- Even when the length of the surjectivity interval is known to be $\gamma(\boldsymbol{M})$, i. e., for regular sequences, the situation for the opening of the sector equal to $\pi\gamma(\boldsymbol{M})$ is not solved in many instances. For example, the case when $\boldsymbol{M}$ is strongly regular and $\gamma(\boldsymbol{M})$ is an irrational number is pending, unless the weight sequence admits a nonzero proximate order, when we know the Borel mapping is not surjective for that opening. The answer is known (again in the negative) if $\boldsymbol{M}$ is strongly regular and $\gamma(\boldsymbol{M})$ is a rational number, or if $\boldsymbol{M}$ satisfies (dc) and $\gamma(\boldsymbol{M})$ is a natural number, but no other general statement is known to hold.

- New techniques seem to be necessary in order to prove surjectivity of the Borel mapping in narrow sectors if condition (sm) is not satisfied, since the Borel and Laplace transforms' technique is no longer applicable.

- The existence of global extension operators (right inverses for the Borel mapping) in the case of Roumieu classes in a half-plane has been completely solved by A. Debrouwere under condition (dc), see [14], and a satisfactory extension of the result for arbitrary sectors was given by J. Jiménez-Garrido, J. Sanz and G. Schindl in [37]. However, this question is open in the absence of (dc), and we would like to obtain some answer at least when condition (sm) is satisfied. Of course, some implication can be easily deduced from the necessity of the condition $(\beta_2)$ of H.-J. Petzsche [52] for the existence of such operators in the ultradifferentiable framework, but the converse argument is not currently available.

Regarding Beurling classes, we have been able to slightly improve a classical result of J. Schmets and M. Valdivia under derivation closedness. Under this condition, we note that the results of A. Debrouwere [15] completely solve the problem for non-uniform asymptotics, both in the sense of surjectivity and as far as extension operators are concerned. His results, however, depend heavily on the use of (dc). Although the new condition (sm) has allowed us to prove surjectivity of the Borel mapping in suitably narrow sectors, our technique (adapted from the work of V. Thilliez and resting on the ideas of J. Chaumat and A.-M. Chollet) does not provide a clue either for determining the length of the surjectivity interval, or for the existence of extension operators. This is an interesting open problem in the Beurling setting, and we consider that a more constructive proof of surjectivity could help in its solution.

Another open problem which has not been treated in this dissertation, but entered our initial plans, is the study of injectivity and surjectivity of the Borel mapping for ultraholomorphic classes defined by the control of the derivatives in regions more general than sectors. S. Mandelbrojt [46] has given a very elegant solution to the injectivity problem for the case of uniform asymptotic expansions, but only some partial results have been obtained in the aforementioned framework by authors form the Russian school, see the works of R. S. Yulmukhametov [84], K. V. Trunov and R. S. Yulmukhametov [82] and R. A. Gaisin [23]. The difficulty of this task deserves a bigger effort in the near future.

With respect to the topics treated in the second and third chapters, we would like to mention that there seems to be a close connection between the existence of optimal flat functions in a class and for a given sector $S_\gamma$, and the existence of characteristic functions in the same class (i. e., the one defined by the same weight structure) but in the sector $S_{\gamma+2}$. This point deserves clarification and it will also be studied.

Finally, the condition of shifted moments allows for a new framework when considering the Stieltjes moment problem within the general Gelfand-Shilov spaces defined via weight sequences. The novelty consists of the possibility of covering a naturally larger target space for the moment mapping, which sends a function to its sequence of Stieltjes moments. The injectivity and surjectivity of the moment mapping in this new setting is studied and, in some cases, characterized.

It is natural to ask which is the natural target space for the moment mapping when the Gelfand-Shilov space we depart from is defined by a general, unrestricted weight sequence, and study the injectivity and surjectivity in this framework in full generality. Another topic to consider in the very near future is the general treatment of the same problems for the Beurling setting, specially when the condition (sm) is assumed to hold. It is likely that this case can be included in the version of this work that we will submit for publication.

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