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IDEALISTIC FLOWERS IN THE REDUCTION OF SINGULARITIES

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Abstract. We present here a proof of the classical reduction of singularities based on the idea of "idealistic flowers". We follow the general ideas of Maximal Contact Theory, presented in a recent book of Aroca, Hironaka and Vicente, that recovers three old publications of Jorge Juan Institute. The concept of idealistic flowers deals with the globalization problems arising from the local nature of the maximal contact.

1. Introduction

This paper gives a new proof of the classical theorem of the reduction of singularities of complex analytic spaces, following Maximal Contact Theory due to Hironaka, Aroca and Vicente [5]. This proof uses the same underlying objects and tools as Hironaka-Aroca-Vicente, but it is novel in its approach to the globalization problem. In fact, we have adapted the original language to make it closer to the one used in Differential Geometry, in terms of atlases. A natural equivalence relation is defined in that atlases to get the so-called Hironaka's Flowers. In this setting, the need of globalization of the classical maximal contact is skipped in a direct way.

Let us recall the main statement in [5, Th 6, p. xxiv], that can be viewed as a consequence of the statement Theorem 3.2 in this paper, as we explain below:

Theorem 1.1 (Desingularization of Complex Analytic Spaces)**.** *For a complex analytic subvariety $X \subset M$, where M is compact, there is a finite sequence of permissible blowing-ups such that the strict transform of X is non-singular.*

We have named Hironaka's Flowers, or Idealistic Flowers, the main object that is compatible with the inductive nature of the desingularization proof; of course, it is also implicit in [5] as well as in most of the modern versions of the proof of existence of reduction of singularities for complex analytic spaces. We hope that this work could help the reader to understand the original proof,

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without needing to enter in the technicalities of Category Theory or into the powerful language of Algebraic Geometry.

One of the major difficulties in the reduction of singularities of complex analytic spaces, following Maximal Contact Theory, is the fact that the varieties of maximal contact are only defined in a local way.

The original proof using Maximal Contact Theory was made in the years 1973-1977 in the books [3,4,21] from Jorge Juan Institute in Madrid, where the idea of infinitely near singular points is very important, see also [23]. In view of the natural interest of the subject, these three books have been rewritten without essential changes and the proof has been re-published in [5]. Teissier's introduction of [5] is very clarifying on the essentials of this proof.

In the literature, we find several ways for dealing with the local nature of the maximal contact:

The original idea is based on Hironaka's gardening. The proof uses gardens, forests, groves and polygroves in order to take in account all the infinitely near singular points. These objects are compatible under restriction to open sets and other important operations; these features give indirectly the gluing properties needed to reach a global behavior inside the inductive proof of existence of reduction of singularities.

In more recent proofs, the globalization is a consequence of requirements concerning functoriality, constructiveness or canonicity, added to the wanted reduction of singularities. See the proofs of Villamayor [36] in 1989 and Bierstone and Milman [6, 7], announced in 1991 and with a complete publication in 1997.

There are other methods, always having in mind a functorial procedure, inspired in previous works of Giraud [17, 18] on the differential nature of the maximal contact hypersurfaces. Coefficient and homogenized ideals in [25, 37] are good examples of these situations.

Anyway, there are several introductory texts giving complete proofs of the reduction of singularities in a modern viewpoint. The reader may look at [14, 16, 19, 25, 27]. A good bibliographical reference may be found in Spivakovsky's paper [35].

Idealistic Flowers are global objects that are constructed in the usual style of Differential Geometry, by means of compatible charts and atlases. The chart-patching definition allows us to skip the inconveniences of the local nature of the maximal contact. In this way, we are able to make induction on the dimension as usual, in a clean way, without the consideration of added requirements on functoriality or guiding invariants for the center selection.

We present a complete proof of the existence of reduction of singularities, following essentially the same scheme of proof as in [5]. We have written this paper hoping that this version of the proof may give a guideline in the possible reduction of singularities for foliations, vector fields and other differential objects in characteristic zero.

Let us give now a description of the structure of this paper.

Our starting objects are the idealistic spaces. They are composed of an ambient space and a finite list of principal ideals with assigned multiplicity; the ambient space is a non-singular complex analytic space endowed with a normal crossings divisor. The singular locus is the set of points where Hironaka's order is greater than or equal to one. The reduction of singularities of idealistic spaces implies the reduction of singularities of complex analytic spaces thanks to the idealistic presentation shown by Hironaka (see [22] and [5, Th. 3.3.3]). Thus, the goal is to show the existence of reduction of singularities for idealistic spaces.

We say that two idealistic spaces, over the same ambient space, are equivalent when we obtain the same singular locus after any finite sequence of permissible blowing-ups and pull-back by open projections. In other words, they have the same "permissible test systems". In particular, a reduction of singularities of one of them gives a reduction of singularities for the other one. Thus, the problem of reduction of singularities makes sense for the classes under this equivalence relation.

Once we fix an ambient space (M, E), we can consider idealistic spaces over open subsets $(U, U \cap E)$ of the ambient space. We call them idealistic charts. Two idealistic charts are compatible when their restrictions to the intersection of the domains give equivalent idealistic spaces. In this way, we can consider idealistic atlases for the ambient space (M, E). The notion of singular locus and permissible centers for idealistic atlases makes sense and hence the problem of existence of reduction of singularities has also sense for idealistic atlases. Two idealistic atlases are compatible, or equivalent, when their union is also an idealistic atlas. We call "idealistic exponents" the classes of compatibility of idealistic atlases. This concept is very close to the one introduced in [5, Def. 3.3.5]; we borrow it just for use in this paper and we obviously ask the reader to take the original definition in other contexts.

Note that there is only one non-singular idealistic exponent over the ambient space (M, E). The problem of existence of reduction of singularities for idealistic exponents is the same one as the problem of existence of reduction of singularities for idealistic atlases. A particular case of idealistic atlases is the one given by a single idealistic space; hence if the idealistic atlases admit reduction of singularities, then we also have reduction of singularities for idealistic spaces.

The Maximal Contact Theory intends to reduce the dimension of the ambient space, in the case of idealistic exponents, by projecting the problem of reduction of singularities over a special type of hypersurfaces, called maximal contact hypersurfaces. Such hypersurfaces will appear in the so-called adjusted and reduced case, but unfortunately they don't have a global nature. The objects we naturally produce when we perform the projection are the idealistic flowers of dimension e, or idealistic e-flowers. In this way, we detect the class of objects where the classical induction on the dimension can be implemented, as well as other intermediate, but necessary, reductions of the problem. These objects are the idealistic e-flowers.

The e-flowers will be defined over an ambient space (M, E) of dimension bigger than or equal to e, that remains fixed in the process of projections on the maximal contact hypersurfaces. The blow-ups will always be performed in (M, E) and they induce blowing-ups in the smaller spaces supporting the idealistic exponents. This idea appears also in the definition of $\mathbb{C}$-element in [5, Def. 1.2.1] and it is also considered in the concept of marked ideal by Bierstone and Milman in [8, p. 612].

Let us give the definition of idealistic e-flowers. First of all, we define transverse ambient e-subspaces (M, E, N) of the ambient space (M, E). They are given by an e-dimensional non-singular closed analytic subset $N \subset M$ having normal crossings with E and being transverse to the components of the divisor E. In this way, we get an induced (smaller) e-dimensional ambient space $(N, E|_N)$. An immersed idealistic e-space over (M, E) is the data

$$\mathcal{V} = ((M, E, N), \mathcal{N})$$

where (M, E, N) is a transverse ambient e-subspace of (M, E) and $\mathcal{N}$ is an idealistic space over $(N, E|_N)$. By definition, the singular locus of $\mathcal{V}$ is the one of $\mathcal{N}$, that is $\operatorname{Sing}\mathcal{V} = \operatorname{Sing}\mathcal{N}$. The permissible centers are non-singular closed analytic subsets of $\operatorname{Sing}\mathcal{V}$ having normal crossings with $E|_N$, "a fortiori" they are non-singular closed analytic subsets of M having normal crossings with E.

We can state also the problem of existence of reduction of singularities for immersed idealistic e-spaces over (M, E); note that when $e = \dim M$, we get the original situation of idealistic spaces.

We extend the definition of equivalence of idealistic spaces to immersed idealistic e-spaces, by saying that two immersed idealistic e-spaces are equivalent when they have the same permissible test systems. Let us note that two equivalent immersed idealistic e-spaces over (M, E) need not to be supported by the same transverse ambient e-subspace. Only the compatibility of the singular locus will be assured.

An immersed idealistic e-chart for (M, E) is an immersed idealistic e-space over an open subset $(U, E \cap U)$ of (M, E). Two such charts are compatible when their restrictions to the intersection of the domains are equivalent. In this way, we can define immersed idealistic e-atlases. An idealistic e-flower over the ambient space (M, E) is a class of compatibility of immersed idealistic e-atlases. Of course, the notion of singular locus is well defined, as well as the ideas of permissible centers and transforms under permissible blowing-ups. Thus, the problem of reduction of singularities for e-flowers makes sense; moreover, it can be seen as a natural extension of the problem of reduction of singularities for idealistic exponents and idealistic spaces. Now, the basic induction statement is the following one:

$$\begin{matrix}\text{Reduction of singularities} \\ \text{for } (e-1)\text{-flowers}\end{matrix} \Rightarrow \begin{matrix}\text{Reduction of singularities} \\ \text{for } e\text{-flowers}\end{matrix}$$

Let us see how we organize the proof of this statement. First we reduce the problem to the adjusted and reduced case. "Adjusted" means that Hironaka's order at the singular points is exactly equal to one. Hironaka's order is well defined for e-flowers, but it is not stable for equivalent idealistic flowers of different dimensions. "Reduced" means that the singular locus does not contain hypersurfaces of the small ambient subspaces in the corresponding immersed idealistic e-charts. This reduction of the problem passes naturally through a combinatorial process of desingularization expressed in terms of polyhedra.

Once we are dealing with an adjusted and reduced idealistic e-flower $\mathcal{F}$, we look for an idealistic $(e-1)$-flower $\mathcal{H}$ over the same ambient space having maximal contact with $\mathcal{F}$. That is, we ask $\mathcal{F}$ and $\mathcal{H}$ to be equivalent in the sense that they have the same permissible test systems. For this purpose, we develop a procedure of projection of adjusted and reduced idealistic exponents over hypersurfaces, by using a tool, called projecting axis, inspired in some works of Panazzolo [32]. Finally, we get locally and with an empty divisor the maximal contact hypersurface, just by the classical Tschirnhaus transformations. Once we remove from the singular locus the old components of the divisor E, we get the desired $(e-1)$-flower having maximal contact, whose reduction of singularities induces a reduction of singularities of $\mathcal{F}$.

Let us end this introduction with a few related references. The paper of Lipman [26] is a good general reference. The papers [12, 13, 24] for the positive characteristic case, the book [14] on monomialization of morphisms, the papers [9, 11, 29, 32] in the case of differential objects, the papers [1, 2, 28] for the use of weighted blowing-ups. The approaches through Zariski Local Uniformization [38] and Hironaka's Voute Etoilee [15, 20] are also very important; the open questions in positive characteristic and the differential cases could maybe approached through these ideas, see for instance [10, 31, 33].

Part 1. Objects and statements

We present here the main concepts and statements in this work.

2. Ambient spaces

Our *ambient spaces* are pairs $((M,K),(E,E\cap K))$, where the space (M,K) is the germ of a non-singular complex analytic space M over a compact subset $K\subset M$ and $(E,E\cap K)$ is the germ over $E\cap K$ of a normal crossings divisor $E\subset M$. If there is no possible confusion, we simply denote (M,E) the ambient space.

An *open subset* of (M,K) is an open immersion of germs

$$(U,L)\subset(M,K),$$

where $L\subset K$ is a compact. We simply denote $U\subset M$. Let us remark that we always have that $(U,L)=(M,L)$, viewed as germified spaces.

An *open covering* of (M, E) is a family $\{(M_\alpha, E_\alpha)\}_{\alpha\in\Lambda}$ of open ambient spaces, where the M_α are open subsets of M such that $E_\alpha = E \cap M_\alpha$ and the compacts K_α of germification for M_α satisfy that $K = \cup_{\alpha\in\Lambda} K_\alpha$.

A *closed analytic subset of* (M, K) is a closed immersion of germs

$$(N, K \cap N) \subset (M, K),$$

where $N \subset M$ is a closed analytic subset.

An *e-dimensional closed ambient subspace* (M, E, N) of an ambient space (M, E) is given by a purely e-dimensional non-singular closed analytic subset N of M having normal crossings with E. In this way, we obtain an e-dimensional ambient space $(N, E|_N)$, where $E|_N$ denotes the union of the intersections with N of the irreducible components of E not containing N (locally). When N is not locally contained in any of the irreducible components of E, we say that (M, E, N) is a *transverse closed ambient subspace*. In this case we have that $E|_N = E \cap N$.

A *hypersurface* (M, E, H) *of the ambient space* (M, E) is just a closed ambient subspace of (M, E) such that $\dim H = \dim M - 1$. In this case, we have $E|_H = E^* \cap H$, where E^* is the union of the irreducible components of E not contained in H; the hypersurface is transverse if and only if $E^* = E$.

Remark 2.1. Let $(N, K \cap N)$ be a closed analytic subset of (M, K). Sometimes we refer to the complement $M \setminus N$. The complement does not have the nature of a germ of analytic space over a compact subset; its interpretation must be done in terms of appropriate representatives. Sometimes we consider the restriction to such complements of sheaves and other objects; the interpretation must be done in terms of points "close enough" to the compacts of germification. We hope that these notations will not produce confusion and, on the other hand, they will contribute to simplify the exposition.

3. Idealistic spaces

A *marked principal ideal* $\mathcal{I}$ over an ambient space (M, E) is a pair $\mathcal{I} = (I, d)$, where $I \subset \mathcal{O}_M$ is a sheaf of principal ideals and d is a positive integer number that we call the *assigned multiplicity of* $\mathcal{I}$. A point $P \in M$ is a *singular point for* $\mathcal{I}$ if $\nu_P I \geq d$, where $\nu_P I$ stands for the multiplicity of I at P. The *singular locus* $\operatorname{Sing}\mathcal{I}$ is the closed analytic subset of M given by the singular points. Since $I_P \neq 0$ for any point P, the dimension of the singular locus is strictly lower than the dimension of M.

An *idealistic space* $\mathcal{M}$ *over* (M, E) is a triple $\mathcal{M} = (M, E, \mathcal{L})$, where $\mathcal{L}$ is a finite list of marked principal ideals

$$\mathcal{L} = \{\mathcal{I}_j = (I_j, d_j)\}_{j=1}^k.$$

The *singular locus* $\operatorname{Sing}\mathcal{L}$ is defined by $\operatorname{Sing}\mathcal{L} = \bigcap_{j=1}^k \operatorname{Sing}\mathcal{I}_j$. We also say that $\operatorname{Sing}\mathcal{L}$ is the *singular locus of* $\mathcal{M}$ and we write $\operatorname{Sing}\mathcal{M} = \operatorname{Sing}\mathcal{L}$.

Remark 3.1. For practical reasons, we can admit zero ideals in the list $\mathcal{L}$, but we always ask that at least one of the marked ideals is non-null. More precisely, a list with some zero ideals represents by definition the same idealistic space as the list obtained by skipping all the zero ideals.

3.1. Transformations of idealistic spaces

We consider two types of transformations $\sigma : (M', E') \to (M, E)$ of the ambient space (M, E): open projections and blowing-ups with non-singular centers having normal crossings with E (the transformation by isomorphisms is evident and we will not insist on that). When we apply a blowing-up to an idealistic space $\mathcal{M}$, we ask the center Y to be *permissible* in the sense that it is contained in the singular locus of $\mathcal{M}$.

Let $\mathcal{M} = (M, E, \mathcal{L} = \{(I_j, d_j)\}_{j=1}^{k})$ be an idealistic space and let us denote $\sigma : (M', E') \to (M, E)$, one of that morphisms. *The transform*

$$\mathcal{M}' = (M', E', \mathcal{L}' = \{(I'_j, d_j)\}_{j=1}^{k})$$

of $\mathcal{M}$ *by* σ is given as follows, depending on the nature of σ:

If σ is an open inclusion the *restriction* $\mathcal{M}' = \mathcal{M}|_{M'}$ *of* $\mathcal{M}$ *to* M' is defined in an evident way. Assume that σ is a projection on the first factor

$$\sigma : (M', E') = (M \times (\mathbb{C}^m, 0), E \times (\mathbb{C}^m, 0)) \to (M, E).$$

We take $I'_j = \sigma^{-1} I_j$, for $j = 1, 2, \ldots, k$. The singular locus satisfies that

$$\operatorname{Sing} \mathcal{M}' = (\operatorname{Sing} \mathcal{M}) \times (\mathbb{C}^m, 0).$$

Note that the projection contains, as a datum, the functions

$$\omega_i : M' \to (\mathbb{C}, 0), \quad i = 1, 2, \ldots, m,$$

obtained from the natural projections $(\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$. We say that σ is an *open projection* if it is composition of an open inclusion and a projection on the first factor. We define the transform $\mathcal{M}'$ by making first the transform by the open inclusion and secondly the transform by the projection.

Assume that σ is the blowing-up of (M, E) with a permissible center $Y \subset M$. That is, the morphism σ is given by the blowing-up $M' \to M$ with center Y, where we take $E' = \sigma^{-1}(E \cup Y)$. Each I'_j is *the controlled transform of* I_j *with assigned multiplicity* d_j, for $j = 1, 2, \ldots, k$. This means that

$$I'_j = \mathcal{J}_{D'}^{-d_j} \pi^{-1}(I_j), \quad D' = \pi^{-1}(Y),$$

where $\mathcal{J}_{D'} \subset \mathcal{O}_{M'}$ is the ideal sheaf of the exceptional divisor D'.

Note that in all cases $\mathcal{L}'$ has the same assigned multiplicities as $\mathcal{L}$.

3.2. Reduction of singularities of idealistic spaces

In the context of idealistic spaces, the classical Hironaka's reduction of singularities may be stated as follows:

Theorem 3.2. *Let $\mathcal{M}$ be an idealistic space over the ambient (M, E). There is a morphism $\sigma : (M', E') \to (M, E)$ that is composition of a finite sequence of blowing-ups with permissible centers, such that*

$$\operatorname{Sing} \mathcal{M}' = \emptyset,$$

where $\mathcal{M}'$ is the transform of $\mathcal{M}$ by σ.

In other words, there is a *reduction of singularities* for any idealistic space $\mathcal{M}$. This result will be a consequence of the existence of reduction of singularities for more general objects that we call *idealistic flowers.*

4. Test systems and equivalence of idealistic spaces

Consider an ambient space (M, E). A *test system $\mathcal{S}$ over (M, E) of length* $k \geq 1$ is a family $\mathcal{S} = \{(Y_{j-1}, \sigma_j)\}_{j=1}^{k}$ with

$$\sigma_j : (M_j, E_j) \to (M_{j-1}, E_{j-1}),$$

where $(M_0, E_0) = (M, E)$, each Y_{j-1} is the empty set or a non-singular closed analytic subset $Y_{j-1} \subset M_{j-1}$ having normal crossings with E_{j-1}. If $Y_{j-1} = \emptyset$, then σ_j is an open projection. If $Y_{j-1} \neq \emptyset$, then σ_j is the blowing-up with center Y_{j-1}. A *test system of length* 0 is just the identity $(M, E) \to (M, E)$, understood as an open projection.

For each $0 \leq \ell \leq k$, we define the *truncation* $\mathcal{S}^\ell$ to be obtained from $\mathcal{S}$, by taking just the indices $j \leq \ell$.

4.1. Restriction of a test system to an open subset

Let $\mathcal{S}$ be a test system over (M, E) of length k. Consider a non-empty open set $U \subset M$. Let us define the *restriction $\mathcal{S}_U$ of $\mathcal{S}$ to U*. It is a test system over $(U, U \cap E)$ of length $k' \leq k$, that we write

$$\mathcal{S}_U = \{(Y'_{j-1}, \sigma'_j)\}_{j=1}^{k'}, \quad \sigma'_j : (M'_j, E'_j) \to (M'_{j-1}, E'_{j-1}),$$

such that the following properties hold:

(1) $(M'_0, E'_0) = (U, U \cap E)$.
(2) If $j < k'$, then $M'_{j+1} = \sigma_j^{-1}(M'_j) \neq \emptyset$.
(3) If $k' < k$, then $\sigma_{k'+1}^{-1}(M'_{k'}) = \emptyset$.
(4) We have that $Y'_{j-1} = M'_{j-1} \cap Y_{j-1}$, for any $1 \leq j \leq k'$.
 - If $Y'_{j-1} \neq \emptyset$, then σ'_j is the blowing-up of (M'_{j-1}, E'_{j-1}) with center Y'_{j-1}.
 - If $Y'_{j-1} = \emptyset$, then σ'_j is the restriction $(M'_j, E'_j) \to (M'_{j-1}, E'_{j-1})$ of σ_j, understood as an open projection.

Let us remark that $M'_j \subset M_j$ is an open set and σ'_j is a restriction of σ_j.

Remark 4.1. The length k' of $\mathcal{S}_U$ is smaller or equal than k. For practical reasons, we introduce the truncation $\mathcal{S}^\ell$ with respect to $\ell \geq k$ to be given by $\mathcal{S}^\ell = \mathcal{S}$. In this situation, we have that

$$(\mathcal{S}_U)^{k-1} = (\mathcal{S}^{k-1})_U. \tag{1}$$

We shall write S_U^{k-1} to denote both $(\mathcal{S}_U)^{k-1}$ and $(\mathcal{S}^{k-1})_U$.

4.2. Permissible test systems

Let us consider an idealistic space $\mathcal{M}$ and a test system $\mathcal{S}$ of length $k \geq 0$ over (M, E). We define, by induction on the length k, the concept of test systems that are *permissible for* $\mathcal{M}$, or $\mathcal{M}$*-permissible test systems*, and the concept of *transform of* $\mathcal{M}$ *by* $\mathcal{M}$*-permissible test systems.*

If $k = 0$, the test system is permissible and the transform of $\mathcal{M}$ is $\mathcal{M}$ itself. Assume that $k \geq 1$. We say that $\mathcal{S}$ is permissible for $\mathcal{M}$ if $\mathcal{S}^{k-1}$ is permissible for $\mathcal{M}$ and the following hold: either $Y_{k-1} = \emptyset$, or $Y_{k-1} \neq \emptyset$ and it is permissible for the transform $\mathcal{M}_{k-1}$ of $\mathcal{M}$ by $\mathcal{S}^{k-1}$. In both cases, we define the transform $\mathcal{M}_k$ to be the transform of $\mathcal{M}_{k-1}$ by σ_k, where we consider σ_k as an open projection if $Y_{k-1} = \emptyset$ or as a blowing-up, otherwise.

The concept of permissible center is of a local nature. In the same way, the concept of permissible test system is also of a local nature, as it is stated in the next proposition, that is a direct consequence of the local character of permissible blowing-up centers.

Proposition 4.2. *Consider an idealistic space* $\mathcal{M}$ *and a test system* $\mathcal{S}$ *over* (M, E)*. Let* $\{(M_\alpha, E_\alpha)\}_{\alpha \in \Lambda}$ *be an open covering of* (M, E)*. The test system* $\mathcal{S}$ *is permissible for* $\mathcal{M}$ *if and only if the restriction* $\mathcal{S}_{M_\alpha}$ *is permissible for* $\mathcal{M}|_{M_\alpha}$*, for each* $\alpha \in \Lambda$.

4.3. Equivalent idealistic spaces

Two idealistic spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ over the same ambient space (M, E) are called to be *equivalent* if they have exactly the same permissible test systems.

A direct consequence of this definition is that two equivalent idealistic spaces have the same singular locus and that their transforms by a permissible test system are equivalent idealistic spaces. In particular, if $\mathcal{M}_1$ and $\mathcal{M}_2$ are equivalent, a reduction of singularities of $\mathcal{M}_1$ is also a reduction of singularities of $\mathcal{M}_2$, and conversely.

Next statement concerns to the local character of the equivalence between idealistic spaces:

Proposition 4.3. *Consider two idealistic spaces* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *over* (M, E) *and an open covering* $\{(M_\alpha, E_\alpha)\}_{\alpha \in \Lambda}$ *of* (M, E)*. We have that* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are equivalent if and only if the restrictions* $\mathcal{M}_1|_{M_\alpha}$ *and* $\mathcal{M}_2|_{M_\alpha}$ *are equivalent, for all* $\alpha \in \Lambda$.

Proof. Assume that $M_1|_{M_\alpha}$ and $\mathcal{M}_2|_{M_\alpha}$ are equivalent, for all $\alpha \in \Lambda$. Take an $\mathcal{M}_1$-permissible test system $\mathcal{S}$. By Proposition 4.2 we have that $\mathcal{S}_{M_\alpha}$ is permissible for $\mathcal{M}_1|_{M_\alpha}$ and hence for $\mathcal{M}_2|_{M_\alpha}$, thus $\mathcal{S}$ is permissible for $\mathcal{M}_2$.

Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are equivalent. Take $\alpha \in \Lambda$ and a test system $\mathcal{S}$ that is permissible for $\mathcal{M}_1|_{M_\alpha}$. Let $\mathcal{S}^*$ be the test system over (M, E) obtained by adding to $\mathcal{S}$ the inclusion $M_\alpha \subset M$ as the first element. Since the transform of $\mathcal{M}_1$ by the inclusion $M_\alpha \subset M$ is exactly $\mathcal{M}_1|_{M_\alpha}$, we get that $\mathcal{S}^*$ is permissible for $\mathcal{M}_1$ and hence for $\mathcal{M}_2$. The fact that $\mathcal{S}^*$ is permissible for $\mathcal{M}_2$ implies that $\mathcal{S}$ is permissible for $\mathcal{M}_2|_{M_\alpha}$. □

4.4. Examples of equivalent idealistic spaces

Here we consider some useful examples of equivalent idealistic spaces.

Normalization of an idealistic space. We say that an idealistic space is *normalized* when all the assigned multiplicities are the same ones. Let $\mathcal{M} = (M, E, \mathcal{L})$ be an idealistic space, where $\mathcal{L} = \{(I_j, d_j)\}_{j=1}^k$ and take a common multiple d of the d_j. The idealistic space

$$\mathcal{M}' = (M, E, \{(I'_j, d)\}_{j=1}^k), \quad I'_j = (I_j)^{d/d_j}$$

is normalized and it is equivalent to $\mathcal{M}$.

Redundant marked ideals. Let $\mathcal{M} = (M, E, \mathcal{L})$ and $\mathcal{M}' = (M, E, \mathcal{L}')$ be two idealistic spaces. Assume that $\mathcal{L} = \{(I_j, d_j)\}_{j=1}^k$ and $\mathcal{L}' = \cup_{j=1}^k \mathcal{L}_j$, where

$$\mathcal{L}_j = \{(I_j, d_j)\} \cup \{(I_{js}, d_j)\}_{s=1}^{k_j},$$

with $I_{js} \subset I_j$, for all $s = 1, 2, \ldots, k_j$. Then $\mathcal{M}$ and $\mathcal{M}'$ are equivalent idealistic spaces.

4.5. Infinite test systems

An *infinite test system* $\mathcal{S}^\infty$ *over* (M, E) is an infinite sequence

$$\mathcal{S}^\infty = \{(Y_{j-1}, \sigma_j)\}_{j=1}^\infty$$

such that the truncation $\mathcal{S}^k = \{(Y_{j-1}, \sigma_j)\}_{j=1}^k$ of $\mathcal{S}^\infty$ is a test system over (M, E) of length k, for any $k \geq 0$. We say that $\mathcal{S}^\infty$ is *permissible for an idealistic space* $\mathcal{M}$ if $\mathcal{S}^k$ is permissible for $\mathcal{M}$, for any $k \geq 0$.

5. Idealistic atlases and idealistic exponents

In this section we introduce the concept of idealistic exponent. It is defined in terms of equivalence classes of idealistic atlases, in a parallel way to the classical language of Differential Geometry. Anyway, the idealistic atlases have their own interest and sometimes we will work with specific types of idealistic atlases belonging to a given idealistic exponent.

5.1. Idealistic atlases

Let us consider an ambient space (M, E). An *idealistic atlas $\mathcal{A}$ over* (M, E) is a finite family $\mathcal{A} = \{\mathcal{M}_\alpha\}_{\alpha\in\Lambda}$ such that:

(1) The $\mathcal{M}_\alpha$ are idealistic spaces over open sets $(M_\alpha, E_\alpha) \subset (M, E)$, where $\{(M_\alpha, E_\alpha)\}_{\alpha\in\Lambda}$ is an open covering of (M, E).

(2) (*Compatibility property*) For any pair of indices $\alpha, \beta \in \Lambda$, the idealistic spaces
$$\mathcal{M}_{\alpha\beta} = \mathcal{M}_\alpha|_{M_{\alpha\beta}}, \quad \mathcal{M}_{\beta\alpha} = \mathcal{M}_\beta|_{M_{\beta\alpha}}$$
are equivalent, where $M_{\alpha\beta} = M_{\beta\alpha} = M_\alpha \cap M_\beta$.

Each idealistic space $\mathcal{M}_\alpha$ will be called an *idealistic chart of* $\mathcal{A}$. More generally, an *idealistic chart over* (M, E) is any idealistic space of the form $(U, U \cap E, \mathcal{L})$, where U is an open subset of M.

Let us assume that $\mathcal{A}$ is an idealistic atlas over (M, E) and let us denote by S_α the singular locus of $\mathcal{M}_\alpha$. Denote also $S_{\alpha\beta} = S_\alpha \cap M_{\alpha\beta}$, we know that $S_{\alpha\beta} = \mathrm{Sing}(\mathcal{M}_{\alpha\beta})$. Since $\mathcal{M}_{\alpha\beta}$ is equivalent to $\mathcal{M}_{\beta\alpha}$, we have that $S_{\beta\alpha} = S_{\alpha\beta}$. This allows us to gluing together the singular loci S_α in a closed analytic subset $S \subset M$ such that $S \cap M_\alpha = S_\alpha$, for all $\alpha \in \Lambda$. We say that S is the *singular locus of* $\mathcal{A}$ and we denote it as $S = \mathrm{Sing}\,\mathcal{A}$.

The transformations of $\mathcal{A}$ by restriction to an open set of M and by a projection on the first factor are directly defined from the case of idealistic spaces.

A *permissible center for* $\mathcal{A}$ is a non-singular closed analytic subset of $\mathrm{Sing}\,\mathcal{A}$ having normal crossings with E. The local character of the permissible centers, expressed in next Proposition 5.1, is a consequence of the local nature of the equivalence between idealistic spaces given in Proposition 4.3.

Proposition 5.1. *A closed analytic subset Y of M is a permissible center for $\mathcal{A}$ if and only if the following equivalent properties hold*:

(1) *For any point $P \in M$, there is an open subset $U \subset M$, with $P \in U$, such that $Y \cap U$ is a permissible center for $\mathcal{A}|_U$.*

(2) *For any open subset $U \subset M$, the intersection $Y \cap U$ is a permissible center for $\mathcal{A}|_U$.*

(3) *Given $\alpha \in \Lambda$, the intersection $Y \cap M_\alpha$ is permissible for $\mathcal{M}_\alpha$.*

Let us consider a blowing-up $\sigma : (M', E') \to (M, E)$ with permissible center $Y \subset M$. For each $\alpha \in \Lambda$, the restriction of σ is the blowing-up
$$\sigma_\alpha : (\sigma^{-1}(M_\alpha), E'_\alpha = E' \cap \sigma^{-1}(M_\alpha)) \to (M_\alpha, E \cap M_\alpha)$$
of $(M_\alpha, E \cap M_\alpha)$ with center $Y_\alpha = Y \cap M_\alpha$ (the identity when $Y_\alpha = \emptyset$). Note that Y_α is a permissible center for $\mathcal{M}_\alpha$. Let $\mathcal{M}'_\alpha$ be the transform of $\mathcal{M}_\alpha$ by σ_α. Given two indices $\alpha, \beta \in \Lambda$, we have an induced blowing-up of $M_{\alpha\beta}$ with center $Y_{\alpha\beta}$, this implies that $\mathcal{M}'_{\alpha\beta}$ and $\mathcal{M}'_{\beta\alpha}$ are equivalent. In this way, we define the *transform $\mathcal{A}'$ of $\mathcal{A}$ by σ* to be the idealistic atlas over (M', E') given by the family of the $\mathcal{M}'_\alpha$.

5.2. Idealistic exponents

Consider an idealistic atlas $\mathcal{A}$ over the ambient space (M, E). In the same way as for the case of idealistic spaces in Subsection 4.2, we can define the concept of test system that is *permissible for* $\mathcal{A}$, or $\mathcal{A}$*-permissible*, and the *transform of* $\mathcal{A}$ by an $\mathcal{A}$-permissible test system.

As in Proposition 4.2, the property of being permissible a test system for an idealistic atlas has local nature as follows:

Proposition 5.2. *Consider an idealistic atlas $\mathcal{A}$ over the ambient space (M, E). A test system $\mathcal{S}$ over (M, E) is permissible for $\mathcal{A}$ if and only if the following equivalent statements hold*:

(1) *For any point $P \in M$, there is an open subset $U \subset M$, with $P \in U$, such that $\mathcal{S}_U$ is a permissible test system for $\mathcal{A}|_U$.*
(2) *For any open subset $U \subset M$, the restriction $\mathcal{S}_U$ of $\mathcal{S}$ to U is permissible for $\mathcal{A}|_U$.*
(3) *For any idealistic chart $\mathcal{M}_\alpha = (M_\alpha, E_\alpha, \mathcal{L}_\alpha)$ of $\mathcal{A}$, the restricted test system $\mathcal{S}_{M_\alpha}$ is permissible for $\mathcal{M}_\alpha$.*

Proof. Follows from Proposition 5.1. □

An idealistic chart $\mathcal{C}$ over (M, E) is *compatible* with the idealistic atlas $\mathcal{A}$ if $\mathcal{A} \cup \{\mathcal{C}\}$ is again an idealistic atlas.

Proposition 5.3. *Given two idealistic atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ over (M, E), the following properties are equivalent*:

(1) *Any idealistic chart of $\mathcal{A}_2$ is compatible with $\mathcal{A}_1$.*
(2) *Any idealistic chart of $\mathcal{A}_1$ is compatible with $\mathcal{A}_2$.*
(3) *The union $\mathcal{A}_1 \cup \mathcal{A}_2$ is an idealistic atlas.*
(4) *$\mathcal{A}_1$ and $\mathcal{A}_2$ have the same permissible test systems.*

Proof. The statement comes from Proposition 5.2. □

Definition. Two idealistic atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ over (M, E) are called to be *equivalent* if $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same permissible test systems (and hence the equivalent properties in Proposition 5.3 hold).

Definition. The equivalence classes of idealistic atlases over (M, E) are called *idealistic exponents over* (M, E).

The properties and concepts invariant by the equivalence relation of the idealistic atlases define properties and concepts concerning idealistic exponents $\mathcal{E}$ over (M, E). Thus, we have well defined:

- Singular locus $\operatorname{Sing} \mathcal{E}$.
- Concept of permissible centers.
- Transformations by open projections.
- Transformations by blowing-ups of permissible centers.
- Concept of $\mathcal{E}$-permissible test systems.

- Transformations by $\mathcal{E}$-permissible test systems.
- Existence of reduction of singularities.

Remark 5.4. In next parts we will introduce other concepts concerning idealistic exponents. The most relevant among them will be the “order” and the property of “being monomial”.

Let us note that there is only one non-singular idealistic exponent over a given ambient space. So, the problem of the existence of reduction of singularities for idealistic exponents consists on finding a finite sequence of permissible blowing-ups to obtain the non-singular idealistic exponent from a given one. Note also, that the existence of reduction of singularities for idealistic exponents implies Theorem 3.2.

6. Immersed idealistic spaces

Let (M, E) be an ambient space. An *immersed idealistic e-space* $\mathcal{V}$ *over* (M, E) is a datum

$$\mathcal{V} = (M, E, N, \mathcal{L}),$$

where (M, E, N) is a transverse e-dimensional closed ambient subspace of (M, E) and $\mathcal{N} = (N, E|_N, \mathcal{L})$ is an idealistic space over the ambient space $(N, E|_N)$ as defined in Section 2. We also say that $\mathcal{V}$ is an *immersed idealistic space of dimension e*.

The *singular locus* $\operatorname{Sing}\mathcal{V}$ is, by definition, the singular locus of $\mathcal{N}$. It is a closed analytic subset of N and hence it is also a closed analytic subset of M. The permissible centers of $\mathcal{V}$ are, by definition, the permissible centers of $\mathcal{N}$; they have normal crossings with E in M.

We need to include the possibility that $N = \emptyset$. In this case, we postulate the existence of a unique immersed idealistic e-space

$$\mathcal{V}_\emptyset = (M, E, \emptyset, \mathcal{L}_\emptyset),$$

where $\mathcal{L}_\emptyset$ is empty. The singular locus of $\mathcal{V}_\emptyset$ is also the empty set.

The transformations of an immersed idealistic e-space $\mathcal{V}$ are defined from morphisms of the ambient space (M, E). Let us precise them.

The *restriction* $\mathcal{V}|_U$ to an open subset $U \subset M$ is given by

$$\mathcal{V}|_U = (U, E \cap U, N \cap U, \mathcal{L}|_{N\cap U}).$$

A projection on the first factor $\sigma : M \times (\mathbb{C}^m, 0) \to M$ defines a projection on the first factor $\bar\sigma : N \times (\mathbb{C}^m, 0) \to N$. In this way, we define the *transform* $\mathcal{V}' = (M', E', N', \mathcal{L}')$ *of* $\mathcal{V}$ *by* σ, by putting

$$M' = M \times (\mathbb{C}^m, 0), \quad E' = E \times (\mathbb{C}^m, 0), \quad N' = N \times (\mathbb{C}^m, 0)$$

and we take $\mathcal{L}'$ to be the transform of $\mathcal{L}$ by $\bar\sigma$.

Let Y be a permissible center for $\mathcal{V}$. Recall that we have $Y \subset N \subset M$. Consider the blowing-up $\pi : (M', E') \to (M, E)$ with center Y and denote by

N' the strict transform of N. Note that (M', E', N') is a transverse closed ambient subspace of (M', E'). The restriction of π induces the blowing-up

$$\bar{\pi} : (N', E'|_{N'}) \to (N, E|_N)$$

of $(N, E|_N)$ with center Y. We define the *transform* $\mathcal{V}'$ *of* $\mathcal{V}$ by π, by putting $\mathcal{V}' = (M', E', N', \mathcal{L}')$, where $\mathcal{L}'$ is the transform of $\mathcal{L}$ by $\bar{\pi}$.

Proceeding as in the non-immersed case, we define when a test system $\mathcal{S}$ over (M, E) is *$\mathcal{V}$-permissible* and what is the *transform of $\mathcal{V}$ by a $\mathcal{V}$-permissible test system.*

We extend the definition of equivalence for idealistic spaces to the immersed case as follows:

Definition. Consider two immersed idealistic spaces $\mathcal{V}_\alpha$ and $\mathcal{V}_\beta$ over (M, E) of respective dimensions e_α and e_β. We say that $\mathcal{V}_\alpha$ and $\mathcal{V}_\beta$ are *equivalent* if they have the same permissible test systems.

Remark 6.1. Two equivalent immersed idealistic spaces

$$\mathcal{V}_\alpha = (M, E, N_\alpha, \mathcal{L}_\alpha), \quad \mathcal{V}_\beta = (M, E, N_\beta, \mathcal{L}_\beta)$$

have the same singular locus $S = \operatorname{Sing} \mathcal{V}_\alpha = \operatorname{Sing} \mathcal{V}_\beta$. The subspaces N_α and N_β are not necessarily equal, they can even have different dimensions, but $S \subset N_\alpha \cap N_\beta$. In the case when $N = N_\alpha = N_\beta$, we have that $\mathcal{V}_\alpha$ is equivalent to $\mathcal{V}_\beta$ if and only if the (non-immersed) idealistic spaces $(N, E|_N, \mathcal{L}_\alpha)$ and $(N, E|_N, \mathcal{L}_\beta)$ are equivalent.

7. Idealistic flowers

Let us consider an ambient space (M, E). An *immersed idealistic atlas* $\mathcal{P}$ *over* (M, E) is a finite family $\mathcal{P} = \{\mathcal{V}_\alpha\}_{\alpha \in \Lambda}$ such that:

1. The $\mathcal{V}_\alpha$ are immersed idealistic spaces over open sets (M_α, E_α) of (M, E), where $\{(M_\alpha, E_\alpha)\}_{\alpha\in\Lambda}$ is an open covering of (M, E).
2. (*Compatibility property*) For any pair of indices $\alpha, \beta \in \Lambda$, the immersed idealistic spaces
$$\mathcal{V}_{\alpha\beta} = \mathcal{V}_\alpha|_{M_{\alpha\beta}}, \quad \mathcal{V}_{\beta\alpha} = \mathcal{V}_\beta|_{M_{\beta\alpha}}$$
are equivalent, where $M_{\alpha\beta} = M_{\beta\alpha} = M_\alpha \cap M_\beta$.

Each immersed idealistic space $\mathcal{V}_\alpha$ will be called an *immersed idealistic chart of* $\mathcal{P}$. When all the immersed idealistic charts $\mathcal{V}_\alpha$ of $\mathcal{P}$ have the same dimension e, we say that $\mathcal{P}$ is an *immersed idealistic e-atlas.*

An *immersed idealistic e-chart over* (M, E) is any e-dimensional immersed idealistic space $\mathcal{C}$ of the form $\mathcal{C} = (U, U \cap E, N, \mathcal{L})$, where U is an open subset of $\mathcal{M}$. We say that an immersed idealistic chart $\mathcal{C}$ is *compatible* with an immersed idealistic atlas $\mathcal{P}$ if $\mathcal{P} \cup \{\mathcal{C}\}$ is again an immersed idealistic atlas.

Let $\mathcal{P} = \{\mathcal{V}_\alpha\}_{\alpha\in\Lambda}$ be an immersed idealistic atlas, where

$$\mathcal{V}_\alpha = (M_\alpha, E_\alpha, N_\alpha, \mathcal{L}_\alpha).$$

Proceeding as in the non-immersed case (see Section 5), we can define in a coherent way the following notions concerning $\mathcal{P}$:

- The singular locus $\operatorname{Sing}\mathcal{P} = \bigcup_{\alpha\in\Lambda}\operatorname{Sing}\mathcal{V}_\alpha$. It is a closed analytic subset of M, such that $M_\alpha \cap \operatorname{Sing}\mathcal{P} = \operatorname{Sing}\mathcal{V}_\alpha$.
- The permissible centers Y for $\mathcal{P}$. They are locally defined by the property that $Y \cap M_\alpha$ is permissible for $\mathcal{V}_\alpha$.
- The $\mathcal{P}$-permissible test systems over (M, E) and the transforms of $\mathcal{P}$ by $\mathcal{P}$-permissible test systems.

In the same way as for the case of (non-immersed) idealistic atlases, we have the following result:

Proposition 7.1. *Given two immersed idealistic atlases $\mathcal{P}_1$ and $\mathcal{P}_2$ over (M, E), the following properties are equivalent*:

(1) *Any immersed idealistic chart of $\mathcal{P}_2$ is compatible with $\mathcal{P}_1$.*
(2) *Any immersed idealistic chart of $\mathcal{P}_1$ is compatible with $\mathcal{P}_2$.*
(3) *The union $\mathcal{P}_1 \cup \mathcal{P}_2$ is an immersed idealistic atlas.*
(4) *$\mathcal{P}_1$ and $\mathcal{P}_2$ have the same permissible test systems.*

Definition. Two immersed idealistic atlases $\mathcal{P}_1$ and $\mathcal{P}_2$ over (M, E) are called to be *equivalent* if they have the same permissible test systems (and hence the equivalent properties in Proposition 7.1 hold).

Definition. The equivalence classes of immersed idealistic atlases over (M, E) are called *idealistic flowers over* (M, E).

The properties and concepts that are invariant by the equivalence relation of immersed idealistic atlases define properties and concepts concerning idealistic flowers $\mathcal{F}$ over (M, E). Thus, we have well-defined:

- Singular locus $\operatorname{Sing}\mathcal{F}$.
- Concept of permissible centers.
- Transformations by open projections.
- Transformations by blowing-ups of permissible centers.
- Concept of $\mathcal{F}$-permissible test systems.
- Transformations by $\mathcal{F}$-permissible test systems.
- Existence of reduction of singularities.

7.1. Idealistic e-flowers

The immersed idealistic atlases belonging to a given idealistic flower do not have necessarily a fixed dimension. Anyway, in order to prove the reduction of singularities by induction on the dimension, we need also to consider idealistic flowers of a fixed dimension. Thus, we take the following definition:

Definition. Let (M, E) be an ambient space of dimension n. Consider an integer number $0 \le e \le n$. An *idealistic e-flower over* (M, E) is an equivalence class of immersed idealistic e-atlases over (M, E).

Let us remark that the equivalence relation of immersed idealistic atlases is the same one in both cases, but it concerns two different sets:

$$\left\{ \begin{array}{c} \text{immersed idealistic } e\text{-atlases} \\ \text{over } (M,E) \end{array} \right\} \subset \left\{ \begin{array}{c} \text{immersed idealistic atlases} \\ \text{over } (M,E) \end{array} \right\}.$$

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be idealistic flowers of respective dimensions e_1 and e_2 over the same ambient space (M,E). We say that $\mathcal{F}_1$ and $\mathcal{F}_2$ are *equivalent* if they are contained in a common idealistic flower $\mathcal{F}$. This is the same to say that any atlas of $\mathcal{F}_1$ is equivalent to any other of $\mathcal{F}_2$. It is also the same that asking $\mathcal{F}_1$ and $\mathcal{F}_2$ to have the same permissible test systems. Note that, when $e_1 = e_2$, then $\mathcal{F}_1$ and $\mathcal{F}_2$ are equal or disjoint. In the particular case that $e_2 = e_1 - 1$, we say that $\mathcal{F}_2$ has *maximal contact* with $\mathcal{F}_1$.

7.2. Description in terms of idealistic exponents

Let (M,E) be an ambient space. An *immersed idealistic exponent* over (M,E) is a data

$$\mathcal{W} = (M,E,N,\mathcal{E}),$$

where (M,E,N) is a transverse closed ambient subspace of (M,E) and $\mathcal{E}$ is an idealistic exponent over $(N,E|_N)$. The *singular locus* $\operatorname{Sing}\mathcal{W}$ is given by $\operatorname{Sing}\mathcal{W} = \operatorname{Sing}\mathcal{E}$. The *dimension* of $\mathcal{W}$ is the dimension of N.

The immersed idealistic exponents are well-transformed by restriction to open sets, projections over the first factor and permissible blowing-ups, where the permissible centers are, by definition, the permissible centers for $\mathcal{E}$. Thus, the permissible test systems and the corresponding transforms are also defined. Two immersed idealistic exponents are *equivalent* if they have the same permissible test systems.

Remark 7.2. Let us consider two immersed idealistic exponents

$$\mathcal{W}_\alpha = (M,E,N_\alpha,\mathcal{E}_\alpha), \quad \mathcal{W}_\beta = (M,E,N_\beta,\mathcal{E}_\beta).$$

If $\mathcal{W}_\alpha$ and $\mathcal{W}_\beta$ are equivalent and $N_\alpha = N_\beta$, then $\mathcal{E}_\alpha = \mathcal{E}_\beta$. Nevertheless, we can have that $\mathcal{W}_\alpha$ and $\mathcal{W}_\beta$ are equivalent with $N_\alpha \neq N_\beta$. In this case, we can assure that $\operatorname{Sing}(\mathcal{W}_\alpha) = \operatorname{Sing}(\mathcal{W}_\beta) \subset N_\alpha \cap N_\beta$.

An *immersed exp-idealistic atlas* is a finite family

$$\mathcal{Q} = \{\mathcal{W}_\alpha\}_{\alpha\in\Lambda}, \quad \mathcal{W}_\alpha = (M_\alpha,E_\alpha,N_\alpha,\mathcal{E}_\alpha),$$

where the $\mathcal{W}_\alpha$ are *immersed idealistic exponents* over (M_α,E_α), the family $\{(M_\alpha,E_\alpha)\}_{\alpha\in\Lambda}$ is an open covering of (M,E) and we have the usual compatibility condition among the $\mathcal{W}_\alpha$. When all the charts $\mathcal{W}_\alpha$ have dimension equal to e, we say that $\mathcal{Q}$ *has dimension* e; we also say that $\mathcal{Q}$ is an *immersed exp-idealistic* e*-atlas*.

The singular locus, permissible centers, permissible test systems and transforms by a permissible test systems are defined as usual. Two immersed exp-idealistic atlases over (M,E) are *equivalent* if they have the same permissible test systems.

Proposition 7.3. *Let (M,E) be an ambient space. There is a natural bijection between equivalence classes of immersed exp-idealistic atlases over (M,E) and idealistic flowers over (M,E). In the same way, there is a natural bijection between equivalence classes of immersed exp-idealistic e-atlases over (M,E) and idealistic e-flowers over (M,E).*

Proof. Let us consider the map

$$\Psi : \left\{ \begin{array}{c} \text{immersed idealistic} \\ \text{atlases over } (M,E) \end{array} \right\} \to \left\{ \begin{array}{c} \text{immersed exp-idealistic} \\ \text{atlases over } (M,E) \end{array} \right\},$$

defined as follows. Take an immersed idealistic atlas $\mathcal{P} = \{\mathcal{V}_\alpha\}_{\alpha\in\Lambda}$ over (M,E), where

$$\mathcal{V}_\alpha = (M_\alpha, E_\alpha, N_\alpha, \mathcal{L}_\alpha).$$

The idealistic space $(N_\alpha, E_\alpha|_{N_\alpha}, \mathcal{L}_\alpha)$ defines an idealistic exponent $\mathcal{E}_\alpha$ over $(N_\alpha, E_\alpha|_{N_\alpha})$. We take $\Psi(\mathcal{P}) = \{\mathcal{W}_\alpha\}_{\alpha\in\Lambda}$, where

$$\mathcal{W}_\alpha = (M_\alpha, E_\alpha, N_\alpha, \mathcal{E}_\alpha).$$

The map Ψ is well-defined and it induces a bijection up to equivalence. The case when the dimension is fixed in done in the same way. □

7.3. Change of codimension

Let (M,E) be an ambient space and (M,E,N) a closed transverse subspace. Consider two immersed idealistic charts

$$\mathcal{V}_A = (N, E|_N, A, \mathcal{L}_A), \quad \mathcal{V}_B = (N, E|_N, B, \mathcal{L}_B)$$

over the ambient space $(N, E|_N)$. We can consider two new immersed idealistic charts

$$\mathcal{V}^*_A = (M, E, A, \mathcal{L}_A), \quad \mathcal{V}^*_B = (M, E, B, \mathcal{L}_B)$$

over the ambient space (M,E). We have the following property:

Proposition 7.4. *The immersed idealistic charts $\mathcal{V}_A$ and $\mathcal{V}_B$ are equivalent if and only if $\mathcal{V}^*_A$ and $\mathcal{V}^*_B$ are also equivalent.*

We have the same result if we consider immersed exp-idealistic charts

$$\mathcal{W}_A = (N, E|_N, A, \mathcal{E}_A), \quad \mathcal{W}_B = (N, E|_N, B, \mathcal{E}_B)$$

and $\mathcal{W}^*_A = (M, E, A, \mathcal{E}_A)$, $\mathcal{W}^*_B = (M, E, B, \mathcal{E}_B)$.

8. Reduction of singularities

Consider an ambient space (M,E) and let $\mathcal{O}$ be an object over (M,E) belonging to one of the following classes:

- Idealistic Spaces over (M,E).
- Idealistic Atlases over (M,E).
- Idealistic Exponents over (M,E).
- Immersed Idealistic e-Spaces over (M,E).
- Immersed Idealistic e-Atlases over (M,E).

- Idealistic e-Flowers over (M, E).

We say that there is a *reduction of singularities for* $\mathcal{O}$ if and only if there is a morphism $\pi : (M', E') \to (M, E)$ that is a finite composition of permissible blowing-ups such that the transform $\mathcal{O}'$ of $\mathcal{O}$ has empty singular locus. In view of the description of the objects and the transformations, the object $\mathcal{O}$ defines in a proper way an idealistic e-flower $\mathcal{F}_{\mathcal{O}}$ and any reduction of singularities of $\mathcal{F}_{\mathcal{O}}$ induces a reduction of singularities of $\mathcal{O}$ in its own class.

Then, the main result of this notes is the following one:

Theorem 8.1. *Let $\mathcal{F}$ be an idealistic e-flower over an ambient space (M, E). Then, there is a reduction of singularities for $\mathcal{F}$.*

We will provide a proof of this result, working essentially by induction on the dimension e of the idealistic flower $\mathcal{F}$.

Part 2. Hironaka's order

In this part we develop the definition and properties of the *order* of an idealistic e-flower. We do that for idealistic spaces, idealistic atlases, immersed idealistic e-atlases and finally for idealistic e-flowers.

The concept of "order" is important to define the so-called "adjusted" idealistic e-flowers. The most important step in the proof of Theorem 8.1 consists in proving the existence of reduction of singularities for adjusted (and reduced) e-flowers under the induction hypothesis that there is reduction of singularities for any $(e-1)$-flower. This process is founded in the Maximal Contact Theory [5], as we shall see in next parts.

9. Order of an idealistic space

Let $\mathcal{M} = (M, E, \mathcal{L})$ be an idealistic space, where $\mathcal{L} = \{(I_j, d_j)\}_{i=1}^{k}$. Consider a point $P \in M$. Following Hironaka [5], the *order* $\delta_P\mathcal{L}$ is defined by

$$\delta_P\mathcal{L} = \min\left\{\frac{\nu_P I_j}{d_j}; \quad j = 1, 2, \ldots, k\right\} \in \mathbb{Q}_{\geq 0}.$$

Note that P is a singular point if and only if $\delta_P\mathcal{L} \geq 1$. We also use the notation $\delta_P\mathcal{M} = \delta_P\mathcal{L}$, although the divisor E has no relevance in the definition of the order. In a similar way, we define the *generic order* $\delta_Y\mathcal{L}$ along an irreducible closed subspace Y of M.

Proposition 9.1. *Consider an idealistic space $\mathcal{M} = (M, E, \mathcal{L})$ and a permissible irreducible center $Y \subset M$. Let $\pi : (M', E') \to (M, E)$ be the blowing-up with center Y and let $\mathcal{M}'$ be the transform of $\mathcal{M}$ by π. We have that*

$$\delta_{D'}\mathcal{M}' = \delta_Y\mathcal{M} - 1, \tag{2}$$

where $D' = \pi^{-1}(Y)$ is the exceptional divisor of π.

Proof. Write $\mathcal{L} = \{(I_j, d_j)\}_{j=1}^k$ and let $\mathcal{J}_{D'}$ the ideal sheaf defining the exceptional divisor D'. Recall that the transformed list $\mathcal{L}' = \{(I'_j, d_j)\}_{j=1}^k$ is given by

$$I'_j = \mathcal{J}_{D'}^{-d_j} \pi^{-1}(I_j), \quad j = 1, 2, \ldots, k.$$

Put $m_j = \nu_Y(I_j)$. The strict transform $I_j^* \subset \mathcal{O}_{M'}$ satisfies that

$$\pi^{-1}(I_j) = \mathcal{J}_{D'}^{m_j} I_j^*, \quad \nu_{D'}(I_j^*) = 0.$$

We have $\nu_{D'}(\pi^{-1}(I_j)) = m_j$, for all $j = 1, 2, \ldots, k$. We conclude the equality $\nu_{D'}(I'_j) = \nu_Y(I_j) - d_j$. Hence, we have that

$$\nu_{D'}(I'_j)/d_j = (\nu_Y(I_j)/d_j) - 1, \quad j = 1, 2, \ldots, k.$$

Taking the minimal values, we get the equality $\delta_{D'}\mathcal{M}' = \delta_Y \mathcal{M} - 1$. □

9.1. Curve-divisor situation

In this subsection, we introduce a construction, usually called "Hironaka's trick", that is useful for proving that the order is an invariant under the equivalence of idealistic spaces.

A *curve-divisor situation* $(\mathcal{M}, X, D, P)$ is given by:

(1) An idealistic space $\mathcal{M} = (M, E, \mathcal{L})$.
(2) A non-singular closed irreducible curve $X \subset \operatorname{Sing} \mathcal{M}$.
(3) A non-singular closed irreducible hypersurface $D \subset M$ having normal crossings with E and with X, such that $P = X \cap D$ (note that P is a singular point).

Consider a curve-divisor situation $(\mathcal{M}, X, D, P)$. We are going to define an $\mathcal{M}$-permissible infinite test system $\mathcal{S}_{\mathcal{M},X,D,P}$ associated to the curve-divisor situation.

Write $\mathcal{S}_{\mathcal{M},X,D,P} = \{(Y_{j-1}, \sigma_j)\}_{j=1}^{\infty}$. The morphisms σ_j are only of blowing-up type and we give inductively the centers Y_{j-1} as follows. If $j = 1$, we put

(1) $Y_0 = D$, if D is a permissible center for $\mathcal{M}$.
(2) $Y_0 = \{P\}$, otherwise.

We respectively denote by $\mathcal{M}'$, X', D' the transform of $\mathcal{M}$, the strict transform of X and the exceptional divisor with respect to σ_1. Since $\delta_{X'}\mathcal{M}' = \delta_X \mathcal{M}$, the curve X' is in the singular locus of $\mathcal{M}'$, hence we get a new curve-divisor situation $(\mathcal{M}', X', D', P')$, where $P' = X' \cap D'$. We proceed indefinitely in this way.

Lemma 9.2. *We have the following properties*:

(1) $\delta_P \mathcal{M} \geq \delta_X \mathcal{M} + \delta_D \mathcal{M}$.
(2) *If* $\delta_P \mathcal{M} = \delta_X \mathcal{M} + \delta_D \mathcal{M}$, *then* $\delta_{P'}\mathcal{M}' = \delta_{X'}\mathcal{M}' + \delta_{D'}\mathcal{M}'$.

Proof. Let (I,d) be a marked ideal of $\mathcal{L}$ with $\delta_P\mathcal{L}=\nu_P I/d$. Let us denote $m=\nu_D I$ and write $I=\mathcal{J}_D^m J$, where $\mathcal{J}_D$ defines D; hence $\nu_D J=0$. Note that $m/d\geq\delta_D\mathcal{L}$. On the other hand, we have

$$\delta_P\mathcal{L}=\nu_P I/d=\nu_P J/d+m/d\geq\nu_P J/d+\delta_D\mathcal{L}.$$

Since $X\not\subset D$ we have that $\nu_X(\mathcal{J}_D)=0$ and $\nu_X I=\nu_X J$. Moreover, since $P\in X$, we see that $\nu_P J\geq\nu_X J=\nu_X I$. Then, we conclude

$$\delta_P\mathcal{L}=\nu_P I/d=\nu_P J/d+m/d\geq\nu_X I/d+\delta_D\mathcal{L}\geq\delta_X\mathcal{L}+\delta_D\mathcal{L}.$$

This shows (1). Let us see (2). If the center of σ_1 is D, the morphism σ_1 is the identity and we have

$$\delta_{P'}\mathcal{L}'=\delta_P\mathcal{L}-1,\quad \delta_{X'}\mathcal{L}'=\delta_X\mathcal{L},\quad \delta_{D'}\mathcal{L}'=\delta_D\mathcal{L}-1.$$

Now, Statement (2) is straightforward. Assume now that σ_1 is the blowing-up with center P. Let us consider the list $\widetilde{\mathcal{L}}=\{(I,d)\}\subset\mathcal{L}$ given by the only marked ideal (I,d). We know that

$$\delta_P\widetilde{\mathcal{L}}=\delta_P\mathcal{L},\quad \delta_X\widetilde{\mathcal{L}}\geq\delta_X\mathcal{L},\quad \delta_D\widetilde{\mathcal{L}}\geq\delta_D\mathcal{L}.$$

By hypothesis, the equality $\delta_P\mathcal{L}=\delta_X\mathcal{L}+\delta_D\mathcal{L}$ holds. Then, we have that $\delta_P\widetilde{\mathcal{L}}\leq\delta_X\widetilde{\mathcal{L}}+\delta_D\widetilde{\mathcal{L}}$. Applying Statement (1), it follows that

$$\delta_P\widetilde{\mathcal{L}}=\delta_X\widetilde{\mathcal{L}}+\delta_D\widetilde{\mathcal{L}},\quad \delta_X\widetilde{\mathcal{L}}=\delta_X\mathcal{L},\quad \delta_D\widetilde{\mathcal{L}}=\delta_D\mathcal{L}.$$

Now, assume that the following equality holds

$$\delta_{P'}\widetilde{\mathcal{L}}'=\delta_{X'}\widetilde{\mathcal{L}}'+\delta_{D'}\widetilde{\mathcal{L}}'. \tag{3}$$

Let us see how to end the proof of Statement (2). Recall that

$$\delta_{D'}\widetilde{\mathcal{L}}'=\delta_P\widetilde{\mathcal{L}}-1,\ \delta_{D'}\mathcal{L}'=\delta_P\mathcal{L}-1,\ \delta_{X'}\mathcal{L}'=\delta_X\mathcal{L},\ \delta_{X'}\widetilde{\mathcal{L}}'=\delta_X\widetilde{\mathcal{L}}.$$

Moreover, since $\widetilde{\mathcal{L}}'\subset\mathcal{L}'$, we have that $\delta_{P'}\mathcal{L}'\leq\delta_{P'}\widetilde{\mathcal{L}}'$. We conclude that

$$\begin{aligned}\delta_{P'}\mathcal{L}'&\leq\delta_{P'}\widetilde{\mathcal{L}}'\\&=\delta_{X'}\widetilde{\mathcal{L}}'+\delta_{D'}\widetilde{\mathcal{L}}'\\&=\delta_X\widetilde{\mathcal{L}}+\delta_P\widetilde{\mathcal{L}}-1\\&=\delta_X\mathcal{L}+\delta_P\mathcal{L}-1\\&=\delta_{X'}\mathcal{L}'+\delta_{D'}\mathcal{L}'.\end{aligned}$$

By Statement (1), we have $\delta_{P'}\mathcal{L}'=\delta_{X'}\mathcal{L}'+\delta_{D'}\mathcal{L}'$.

Now, it is enough to show the equality in Equation (3). Following the proof of Statement (1), the fact $\delta_P\widetilde{\mathcal{L}}=\delta_X\widetilde{\mathcal{L}}+\delta_D\widetilde{\mathcal{L}}$ implies that $\nu_X J=\nu_P J$. That is, the ideal J is equimultiple along X around P. Denote by J' the strict transform of J by σ_1. We have that $\nu_{P'}J'\leq\nu_P J$. Since $P'\in X'$, we also have that $\nu_{P'}J'\geq\nu_{X'}J'$. Combining these properties with the facts that $\nu_{X'}J'=\nu_X J$ and $\nu_P J=\nu_X J$, we conclude that

$$\nu_{P'}J'=\nu_P J=\nu_X J=\nu_{X'}J'.$$

Recall that the strict transform of I coincides with the strict transform of J, at the point P', since the strict transform of D does not go through P'. Then, we can write the controlled transform I' of I at P' as $I' = \mathcal{J}_{D'}^{m'} J'$, where $m' = \nu_{D'} I'$. This implies that

$$d \cdot \delta_{X'} \widetilde{\mathcal{L}}' = \nu_{X'} I' = \nu_{X'} J' + m' \cdot \nu_{X'} \mathcal{J}_{D'} = \nu_{X'} J' = \nu_{P'} J',$$

since $\nu_{X'} \mathcal{J}_{D'} = 0$, in view of the fact that X' is not contained in D'. Finally, we have that

$$\delta_{P'} \widetilde{\mathcal{L}}' = \nu_{P'} I'/d = \nu_{P'} J'/d + \nu_{D'} I'/d = \delta_{X'} \widetilde{\mathcal{L}}' + \delta_{D'} \widetilde{\mathcal{L}}'.$$

This ends the proof. □

Remark 9.3. Assume that $\mathcal{M}_\alpha$ y $\mathcal{M}_\beta$ are two equivalent idealistic spaces over (M, E). Assume that $(\mathcal{M}_\alpha, X, D, P)$ is a curve-divisor situation. Then $(\mathcal{M}_\beta, X, D, P)$ is also a curve-divisor situation and we have that the two associated infinite test systems coincide, that is

$$\mathcal{S}_{\mathcal{M}_\alpha, X, D, P} = \mathcal{S}_{\mathcal{M}_\beta, X, D, P}.$$

9.2. Invariance under equivalence

In this subsection we give a proof of the following statement:

Proposition 9.4. *Let $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ be two equivalent idealistic spaces over the ambient (M, E). Let S be their common singular locus. For any $P \in S$, we have that $\delta_P \mathcal{M}_\alpha = \delta_P \mathcal{M}_\beta$.*

The proof of Proposition 9.4 is based in the construction of a particular curve-divisor situation, obtained after performing a projection on the first factor, as follows.

Take an idealistic space $\mathcal{M}$ over (M, E) and a point $P \in \operatorname{Sing} \mathcal{M}$. Let σ_1 be the projection on the first factor

$$\sigma_1 : (M_1, E_1) = (M \times (\mathbb{C}, 0), E \times (\mathbb{C}, 0)) \to (M, E).$$

Denote by $\mathcal{M}_1$ the transform of $\mathcal{M}$ by σ_1 and let us put

$$X_1 = \{P\} \times (\mathbb{C}, 0),\ D_1 = M \times \{0\},\ P_1 = (P, 0).$$

We have a curve-divisor situation $(\mathcal{M}_1, X_1, D_1, P_1)$. Note that

$$\delta_{X_1} \mathcal{M}_1 = \delta_{P_1} \mathcal{M}_1 = \delta_P \mathcal{M} \geq 1, \quad \delta_{D_1} \mathcal{M}_1 = 0.$$

In particular $\delta_{P_1} \mathcal{M}_1 = \delta_{X_1} \mathcal{M}_1 + \delta_{D_1} \mathcal{M}_1$.

Consider the test system $\mathcal{S}_{\mathcal{M}_1, X_1, D_1, P_1} = \{(Y_{j-1}, \sigma_j)\}_{j=2}^{\infty}$ and denote by $(\mathcal{M}_j, X_j, D_j, P_j)$ the transformed curve-divisor situations, for $j \geq 2$.

Let us make now a computation of orders by an extensive application of Proposition 9.1 and Lemma 9.2. In order to simplify notations, we will write $e = \delta_P \mathcal{M} = \delta_{X_1} \mathcal{M}_1$ and

$$a_j = \delta_{D_j} \mathcal{M}_j, \quad e_j = \delta_{P_j} \mathcal{M}_j, \quad j \geq 1.$$

The first remark is that $e = e_1 = \delta_{X_1}\mathcal{M}_1 = \delta_{X_j}\mathcal{M}_j$, for all $j \geq 2$. We also know that $a_1 = 0$. By Lemma 9.2, we get the relations

$$e_j = e + a_j, \quad j \geq 1. \tag{4}$$

We are going to describe the sequence $\{a_j\}_{j=2}^{\infty}$ recurrently from the rational number $e \geq 1$. If $j = 2$, by Proposition 9.1, we have that

$$a_2 = e - 1,$$

Assume $j > 2$. If $a_j \geq 1$, then D_j is contained in the singular locus of $\mathcal{M}_j$ and we have $Y_j = D_j$. In this case, we see that

$$a_{j+1} = a_j - 1. \tag{5}$$

If $a_j < 1$, the divisor D_j is not in the singular locus and $Y_j = \{P_j\}$. By Proposition 9.1, we have $a_{j+1} = e_j - 1$; in view of Equation (4), we get

$$a_{j+1} = a_j + e - 1. \tag{6}$$

Thus, the sequence $\{a_j\}_{j=2}^{\infty}$ is inductively obtained from e as follows:

a) $a_2 = e - 1$.
b) If $a_j \geq 1$, then $a_{j+1} = a_j - 1$.
c) If $a_j < 1$, then $a_{j+1} = a_j + e - 1$.

Since the values $a_j \geq 0$ are rational numbers having a common denominator, a diophantine computation shows that there is a first index $j_0 \geq 2$ such that $a_{j_0} = 0$.

Now, we can conclude the proof of Proposition 9.4. Since $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ are equivalent, we have that $\mathcal{M}_{\alpha,1}$ is equivalent to $\mathcal{M}_{\beta,1}$ and hence the infinite test systems $\mathcal{S}_{\mathcal{M}_{\alpha,1},X_1,D_1,P_1}$ and $\mathcal{S}_{\mathcal{M}_{\beta,1},X_1,D_1,P_1}$ coincide, in view of Remark 9.3. In particular, we have that

$$a_j^\alpha \geq 1 \Leftrightarrow a_j^\beta \geq 1, \quad j \geq 2.$$

In other words, the construction of the sequences $\{a_j^\alpha\}$ and $\{a_j^\beta\}$ follows the same steps, starting with $e^\alpha = \delta_P\mathcal{M}_\alpha$ and $e^\beta = \delta_P\mathcal{M}_\beta$, respectively. We have to prove that $e^\alpha = e^\beta$. Assume by contradiction that $e^\alpha > e^\beta$. We have that $a_j^\alpha > a_j^\beta$, for all $j \geq 2$. Now, choosing $j_0 \geq 2$ such that $a_{j_0}^\alpha = 0$, we conclude that $a_{j_0}^\beta < 0$, which is impossible.

10. Order for idealistic exponents and e-flowers

Let $\mathcal{A} = \{\mathcal{M}_\alpha\}_{\alpha\in\Lambda}$ be an idealistic atlas over (M, E). Given a singular point $P \in \operatorname{Sing}\mathcal{A}$, we define the *order* $\delta_P\mathcal{A}$ by

$$\delta_P\mathcal{A} = \delta_P\mathcal{M}_\alpha,$$

where (M_α, E_α) is the ambient space of an idealistic chart $\mathcal{M}_\alpha$ satisfying that $P \in M_\alpha$. In view of Proposition 9.4, the definition does not depend on the particular idealistic chart $\mathcal{M}_\alpha$ such that $P \in M_\alpha$. On the other hand, since $P \in \operatorname{Sing}(\mathcal{M}_\alpha)$, we have that $\delta_P\mathcal{A} \geq 1$.

Given an idealistic exponent $\mathcal{E}$ over (M,E) and a point $P \in \operatorname{Sing}\mathcal{E}$, the *order* $\delta_P\mathcal{E}$ is $\delta_P\mathcal{E} = \delta_P\mathcal{A}$, where $\mathcal{A}$ is any atlas defining $\mathcal{E}$.

Consider now an immersed idealistic e-space $\mathcal{V} = (M,E,N,\mathcal{L})$ and a point $P \in \operatorname{Sing}\mathcal{V}$. We define the *order* $\delta_P\mathcal{V}$ to be

$$\delta_P\mathcal{V} = \delta_P\mathcal{N}, \quad \mathcal{N} = (N, E|_N, \mathcal{L}).$$

The reader can verify that the arguments in Subsections 9.1 and 9.2 work in a similar way for immersed idealistic spaces, with fixed dimension e. More precisely, we have:

Proposition 10.1. *Let $\mathcal{V}_\alpha$ and $\mathcal{V}_\beta$ be two equivalent immersed idealistic spaces over (M,E) of the same dimension e. For any point P in the common singular locus $\operatorname{Sing}\mathcal{V}_\alpha = \operatorname{Sing}\mathcal{V}_\beta$, we have that $\delta_P\mathcal{V}_\alpha = \delta_P\mathcal{V}_\beta$.*

Remark 10.2. An example that the equality of the dimension is necessary in the statement of Proposition 10.1 is the following one. Take $\mathcal{V}_\alpha$ and $\mathcal{V}_\beta$ over $(M,E) = ((\mathbb{C}^2,0),\emptyset)$, given by $N_\alpha = M$, $N_\beta = (y=0)$, with $\mathcal{L}_\alpha = \{\mathcal{I}_\alpha\}$, $\mathcal{L}_\beta = \{\mathcal{I}_\beta\}$, where:

$$\mathcal{I}_\alpha = \left((y^2 - x^3)\mathcal{O}_{\mathbb{C}^2,0}, 2\right), \quad \mathcal{I}_\beta = \left((x^3\mathcal{O}_{\mathbb{C},0}, 2)\right).$$

By Maximal Contact Theory [5], we know that $\mathcal{V}_\alpha$ and $\mathcal{V}_\beta$ are equivalent. Nevertheless, we have that $\delta_0\mathcal{V}_\alpha = 1$ and $\delta_0\mathcal{V}_\beta = 3/2$.

Thus, we can define the *order for immersed idealistic e-atlases and for idealistic e-flowers*, but this is not possible for general immersed idealistic atlases and idealistic flowers.

Part 3. Guide for the reduction of singularities

Here we give a quick guide for the proof of the existence of reduction of singularities for idealistic e-flowers. The original result (Theorem 3.2) we want to prove is the following one:

Theorem 10.3. *Given an ambient space (M,E), there is a reduction of singularities for any idealistic space $\mathcal{M}$ over (M,E).*

This result is a consequence of the following one:

Theorem 10.4. *Given an ambient space (M,E), there is a reduction of singularities for any idealistic atlas $\mathcal{A}$ over (M,E).*

Now, Theorem 10.4 is equivalent to the next one:

Theorem 10.5. *Given an ambient space (M,E), there is a reduction of singularities for any idealistic exponent $\mathcal{E}$ over (M,E).*

We also have the immersed statements:

Theorem 10.6. *Given an ambient space (M,E) and a natural number e with $1 \le e \le \dim M$, there is a reduction of singularities for any immersed idealistic e-space $\mathcal{V}$ over (M,E).*

Theorem 10.7. *Given an ambient space (M, E) and a natural number e with $1 \leq e \leq \dim M$, there is a reduction of singularities for any immersed idealistic e-atlas $\mathcal{P}$ over (M, E).*

Theorem 10.8 (see Theorem 8.1)**.** *Given an ambient space (M, E) and a natural number e with $1 \leq e \leq \dim M$, there is a reduction of singularities for any idealistic e-flower $\mathcal{F}$ over (M, E).*

In view of the developments of the concepts in Part 1, we have the following implications:

$$\begin{array}{ccccc} \text{Th10.8} & \Leftrightarrow & \text{Th10.7} & \Rightarrow & \text{Th10.6} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{Th10.5} & \Leftrightarrow & \text{Th10.4} & \Rightarrow & \text{Th10.3} \end{array}$$

We have to prove Theorem 10.8 or, equivalently, Theorem 10.7.

Thanks to the behavior of the order presented in Part 2, we can define *adjusted idealistic e-flowers* to be the ones such that the order in all the singular points is exactly equal to one. We also consider *reduced idealistic e-flowers*, given by the property that the singular locus has dimension less than or equal to $e - 2$. In our general procedure, we need as well the concept of *monomial idealistic e-flower* to be developed further, that implies the existence of a very "combinatorial" reduction of singularities.

Let us see how to organize the proof of Theorem 10.8. Consider the following statements:

- RedSing(e): *Idealistic e-flowers have reduction of singularities.*
- RedSing(monomial): *Monomial idealistic t-flowers have reduction of singularities, for any $t \geq 1$.*
- RedSing(e, adjusted): *Adjusted idealistic e-flowers have reduction of singularities.*
- RedSing(e, adjusted-reduced): *Adjusted and reduced idealistic e-flowers have reduction of singularities.*

The verification that RedSing(1) holds is straightforward. Our objective is to prove the induction step

$$\text{RedSing}(e-1) \Rightarrow \text{RedSing}(e). \tag{7}$$

We do it in several steps:

Monomial case: RedSing(monomial) holds.

Reduction to the Adjusted Case: This step corresponds to the following implication

$$\left.\begin{array}{l} \text{RedSing}(e, \text{adjusted}) \\ \text{RedSing}(\text{monomial}) \end{array}\right\} \Rightarrow \text{RedSing}(e). \tag{8}$$

Reduction to the Adjusted-Reduced Case: This step corresponds to the following implication

$$\left.\begin{array}{l} \text{RedSing}(e-1) \\ \text{RedSing}(e, \text{adjusted-reduced}) \end{array}\right\} \Rightarrow \text{RedSing}(e, \text{adjusted}). \tag{9}$$

The Adjusted-Reduced Case: This step corresponds to the following implication

$$\text{RedSing}(e-1) \Rightarrow \text{RedSing}(e, \text{adjusted-reduced}). \tag{10}$$

The statement (10) is proven in terms of Maximal Contact Theory [5]. The classical difficulty for the globalization of the maximal contact is the reason for introducing idealistic flowers, instead of working simply with idealistic exponents.

Part 4. First reductions

In this part, we prove the statements "Monomial Case", "Reduction to the Adjusted Case" and "Reduction to the Adjusted-Reduced Case" presented in Part 3.

11. Monomial idealistic *e*-flowers

Let us consider an ambient space (M, E). We say that a sheaf Z of principal ideals on M is *logarithmic for* (M, E) if it is of the form

$$Z = \prod_D \mathcal{J}_D^{a_D}, \quad a_D \in \mathbb{Z}_{\geq 0}, \tag{11}$$

where D runs over the irreducible components of E and $\mathcal{J}_D$ is the ideal sheaf of D. A *logarithmic idealistic space with assigned multiplicity* d *over* (M, E) is any idealistic space $\mathcal{Z}$ of the form

$$\mathcal{Z} = (M, E, \mathcal{L}_{\mathcal{Z}} = \{(Z, d)\}),$$

where Z is a logarithmic sheaf for (M, E).

Let (M, E, N) be a transverse ambient subspace of (M, E) and let Z be a logarithmic sheaf. Note that we can define the restriction of $\mathcal{L}_{\mathcal{Z}}$ to N just by taking $\mathcal{L}_{\mathcal{Z}}|_N = \{(Z|_N, d)\}$, where $Z|_N$ is the restriction of the principal ideal sheaf Z to N. In this way we obtain an immersed idealistic space $\mathcal{Z}|_N$ given by

$$\mathcal{Z}|_N = (M, E, N, \mathcal{L}_{\mathcal{Z}}|_N).$$

Definition. Let $\mathcal{P} = \{\mathcal{V}_\alpha\}_{\alpha \in \Lambda}$ be an immersed idealistic e-atlas over (M, E), where $\mathcal{V}_\alpha = (M_\alpha, E_\alpha, N_\alpha, \mathcal{L}_\alpha)$. We say that $\mathcal{P}$ is *monomial*, or *quasi-ordinary*, if there is a logarithmic idealistic space $\mathcal{Z}$ over (M, E) such that, for any $\alpha \in \Lambda$, the immersed idealistic e-spaces $\mathcal{V}_\alpha$ and $\mathcal{Z}|_{N_\alpha}$ are equivalent. An idealistic e-flower $\mathcal{F}$ is *monomial* or *quasi-ordinary* if it contains a monomial immersed idealistic e-atlas.

Remark 11.1. If $\mathcal{P}$ is monomial, then the family $\mathcal{P}_{\mathcal{Z}} = \{\mathcal{Z}|_{N_\alpha}\}_{\alpha \in \Lambda}$ is an immersed idealistic e-atlas equivalent to $\mathcal{P}$, and conversely.

The reduction of singularities of logarithmic idealistic spaces is of combinatorial nature and well-known. It is a direct consequence of Hironaka's Game, solved by Spivakovsky [34]. The precise statement we need is the following one:

Proposition 11.2. *Let $\mathcal{Z} = (M, E, \{(Z, d)\})$ be a logarithmic idealistic space over (M, E). There is a reduction of singularities for $\mathcal{Z}$ obtained by the composition of a finite sequence of permissible blowing-ups, where the centers are connected components of intersections of the irreducible components of the divisor E in the support of Z.*

Proof. See, for example [30, Proposition 3]. □

Corollary 11.3. *There is a reduction of singularities for any monomial idealistic e-flower $\mathcal{F}$.*

Proof. Let $\mathcal{Z}$ be a logarithmic idealistic space associated to $\mathcal{F}$. The blowing-ups of the reduction of singularities of $\mathcal{Z}$ induce a reduction of singularities for $\mathcal{F}$ as follows. Assume that Y is a permissible center for $\mathcal{Z}$ and let

$$\pi : (M', E') \to (M, E)$$

be the blowing-up with center Y. Up to taking an open subset of M, we can assume that there is a particular immersed idealistic e-space

$$\mathcal{Z}|_N = (M, E, N, \mathcal{L}_{\mathcal{Z}}|_N)$$

belonging to $\mathcal{F}$. Since N has normal crossings with E and Y is intersection of components of E not containing N, the restriction of π to the strict transform N' of N is the blowing-up

$$\bar{\pi} : (N', E'|_{N'}) \to (N, E|_N)$$

with center $Y \cap N$, that is permissible for $(M, E, N, \mathcal{L}_{\mathcal{Z}}|_N)$. The needed commutativity properties hold and we obtain a reduction of singularities for $\mathcal{F}$. □

12. Adjusted idealistic e-flowers

Let us consider an ambient space (M, E) and an idealistic e-flower $\mathcal{F}$ over (M, E). We recall that $\mathcal{F}$ is adjusted if $\delta_P \mathcal{F} = 1$, for any $P \in \operatorname{Sing} \mathcal{F}$.

We have the following stability result:

Proposition 12.1. *Let $\mathcal{F}$ be an adjusted idealistic e-flower over (M, E). Consider a test system $\mathcal{S}$ that is permissible for $\mathcal{F}$ and let $\mathcal{F}'$ the transform of $\mathcal{F}$ by $\mathcal{S}$. Then, the idealistic e-flower $\mathcal{F}'$ is adjusted.*

Proof. The statement for open projections is straightforward. In the case of permissible blowing-ups, the result is a direct consequence of the stability of the multiplicity of an hypersurface under blowing-up with equimultiple centers. □

12.1. Logarithmic factors

Let (M, E) be an ambient space and consider an immersed idealistic e-atlas $\mathcal{P} = \{\mathcal{V}_\alpha\}_{\alpha \in \Lambda}$ over (M, E), where

$$\mathcal{V}_\alpha = (M_\alpha, E_\alpha, N_\alpha, \mathcal{L}_\alpha), \quad \mathcal{L}_\alpha = \{(I_{\alpha,j}, d_{\alpha j})\}_{j=1}^{k_\alpha}. \tag{12}$$

Let d be a positive integer. We say that $\mathcal{P}$ is *d-normalized* if $d = d_{\alpha j}$, for any $\alpha \in \Lambda$, $j = 1, 2, \ldots, k_\alpha$.

Remark 12.2. Any immersed idealistic e-atlas over (M, E) is equivalent to a normalized one, see Subsection 4.4.

Assume now that $\mathcal{P}$ is d-normalized. A *logarithmic factor Z for $\mathcal{P}$* is a logarithmic sheaf Z for (M, E) such that there is a factorization

$$I_{\alpha,j} = Z|_{N_\alpha} \cdot J_{\alpha,j}$$

of principal ideal sheaves on N_α, for any $\alpha \in \Lambda$ and $j = 1, 2, \ldots, k_\alpha$.

Let us note that there is at least one logarithmic factor given by the ideal sheaf $Z = \mathcal{O}_M$, which has empty support.

Definition. Let $\mathcal{P}$ be a d-normalized immersed idealistic e-atlas over (M, E) and consider a logarithmic factor Z for $\mathcal{P}$. The *co-factorial order $\mu_Z \mathcal{P}$* is given by

$$\mu_Z \mathcal{P} = \max\{d\delta_P \mathcal{P} - \nu_P Z;\ P \in \operatorname{Sing} \mathcal{P}\}.$$

When $\operatorname{Sing} \mathcal{P} = \emptyset$, we put $\mu_Z \mathcal{P} = -\infty$.

For any $P \in N_\alpha$, we have that $\nu_P(Z|_{N_\alpha}) = \nu_P Z$, since N_α has normal crossings with E and it is not locally contained in the support of Z. In particular, we have that

$$d\delta_P \mathcal{P} - \nu_P Z = d\delta_P \mathcal{V}_\alpha - \nu_P(Z|_{N_\alpha}) \in \mathbb{Z}_{\geq 0}, \tag{13}$$

when $P \in \operatorname{Sing} \mathcal{P}$. The condition $\mu_Z \mathcal{P} = -\infty$ means exactly that $\operatorname{Sing} \mathcal{P} = \emptyset$.

Proposition 12.3. *Let $\mathcal{P}$ be a d-normalized immersed idealistic e-atlas over (M, E). Assume that Z is a logarithmic factor for $\mathcal{P}$ such that $\mu_Z \mathcal{P} = 0$. Then $\mathcal{P}$ is monomial.*

Proof. Denote $\mathcal{Z} = (M, E, \{(Z, d)\})$ and for any $\alpha \in \Lambda$, denote

$$\mathcal{Z}|_{N_\alpha} = (M_\alpha, E_\alpha, N_\alpha, \{(Z|_{N_\alpha}, d)\}).$$

By Remark 11.1, it is enough to prove that the family $\mathcal{P}_{\mathcal{Z}} = \{\mathcal{Z}|_{N_\alpha}\}_{\alpha \in \Lambda}$ is an immersed idealistic e-atlas that is equivalent to $\mathcal{P}$.

Consider an immersed idealistic e-chart

$$\mathcal{V}_\alpha = (M_\alpha, E_\alpha, N_\alpha, \mathcal{L}_\alpha = \{(I_{\alpha,j}, d)\}_{j=1}^{k_\alpha})$$

belonging to $\mathcal{P}$. Let us prove that $\mathcal{V}_\alpha$ is equivalent to $\mathcal{Z}|_{N_\alpha}$. By the local character of the equivalence, it is enough to do it locally at any point $P \in N_\alpha$ (see Proposition 4.3).

Assume that $P \notin \operatorname{Sing} \mathcal{V}_\alpha$. There is a j_0 such that

$$d > \nu_P(I_{\alpha,j_0}) = \nu_P Z + \nu_P J_{\alpha,j_0} \geq \nu_P Z.$$

Then $P \notin \operatorname{Sing} \mathcal{Z}|_{N_\alpha}$ and we are done.

Assume that $P \in \operatorname{Sing} \mathcal{V}_\alpha$. Since $\mu_Z \mathcal{P} = 0$, we have the equality of germs $(I_{\alpha,j_0})_P = (Z|_{N_\alpha})_P$, for an index j_0. This implies that, locally at P, the immersed idealistic spaces $\mathcal{V}_\alpha$ and $\mathcal{Z}|_{N_\alpha}$ are equivalent, see the examples with redundant ideals in Subsection 4.4. □

Corollary 12.4. *If $\mu_Z \mathcal{P} = 0$, then $\mathcal{P}$ has a reduction of singularities.*

12.2. Adjustment by logarithmic factors

Consider a d-normalized immersed idealistic e-atlas $\mathcal{P} = \{\mathcal{V}_\alpha\}_{\alpha\in\Lambda}$ over (M, E) as in Equation (12) and take a logarithmic factor Z for $\mathcal{P}$. Take an integer number $m \geq 1$ with $m \geq \mu_Z\mathcal{P}$ and denote by $\mathcal{P}^{Z,m} = \{\mathcal{V}_\alpha^{Z,m}\}_{\alpha\in\Lambda}$ the family of immersed idealistic e-charts

$$\mathcal{V}_\alpha^{Z,m} = (M_\alpha, E_\alpha, N_\alpha, \mathcal{L}_\alpha^{Z,m}), \quad \mathcal{L}_\alpha^{Z,m} = \mathcal{L}_\alpha \cup \{(J_{\alpha,j}, m)\}_{j=1}^{k_\alpha}.$$

Recall that $I_{\alpha,j} = Z|_{N_\alpha} J_{\alpha,j}$.

Remark 12.5. We have that $\mathrm{Sing}(\mathcal{V}_\alpha^{Z,m}) \subset \mathrm{Sing}(\mathcal{V}_\alpha)$. The property

$$m > \mu_Z\mathcal{P}$$

holds if and only if $\mathrm{Sing}(\mathcal{V}_\alpha^{Z,m}) = \emptyset$, for any $\alpha \in \Lambda$. In particular, if $m = \mu_Z\mathcal{P}$, there is an index $\alpha \in \Lambda$ such that $\mathrm{Sing}(\mathcal{V}_\alpha^{Z,m}) \neq \emptyset$.

The objective of this subsection is to prove the following result:

Proposition 12.6. *The family $\mathcal{P}^{Z,m} = \{\mathcal{V}_\alpha^{Z,m}\}_{\alpha\in\Lambda}$ is an adjusted immersed idealistic e-atlas over (M, E).*

The fact that $\mathcal{P}^{Z,m}$ is adjusted comes from the definition of $\mu_Z\mathcal{P}$. It remains to show that $\mathcal{P}^{Z,m}$ is an immersed idealistic e-atlas over (M, E).

Recall the notation $M_{\alpha\beta} = M_\alpha \cap M_\beta$ and denote $S_\alpha^{Z,m} = \mathrm{Sing}(\mathcal{V}_\alpha^{Z,m})$, for any pair of indices $\alpha, \beta \in \Lambda$.

Lemma 12.7. $M_{\alpha\beta} \cap S_\alpha^{Z,m} = M_{\alpha\beta} \cap S_\beta^{Z,m}$.

Proof. Consider a point $P \in M_{\alpha\beta} \cap S_\alpha^{Z,m}$. We have to show that

$$\delta_P(\mathcal{V}_\beta) \geq 1, \quad \nu_P(J_{\beta,j}) \geq m,\ j = 1, 2, \ldots, k_\beta.$$

Since $\delta_P(\mathcal{V}_\alpha) \geq 1$ and $\delta_P(\mathcal{V}_\beta) = \delta_P(\mathcal{V}_\alpha)$, then $\delta_P(\mathcal{V}_\beta) \geq 1$.

On the other hand, we have that

$$d\delta_P\mathcal{V}_\alpha - \nu_P Z = m.$$

Indeed, if $d\delta_P\mathcal{V}_\alpha - \nu_P Z < m$ we get $P \notin S_\alpha^{Z,m}$, which is not possible. Now, for any $j = 1, 2, \ldots, k_\beta$, we have

$$\nu_P(J_{\beta,j}) = \nu_P(I_{\beta,j}) - \nu_P Z \geq d\delta_P\mathcal{V}_\beta - \nu_P Z = d\delta_P\mathcal{V}_\alpha - \nu_P Z = m.$$

We conclude that $P \in S_\beta^{Z,m}$, as desired. □

In order to prove Proposition 12.6, up to restrict ourselves to $M_{\alpha\beta}$, it is enough to consider the particular case when

$$\Lambda = \{\alpha, \beta\}, \quad (M, E) = (M_\alpha, E_\alpha) = (M_\beta, E_\beta).$$

Let us prove that $\mathcal{V}_\alpha^{Z,m}$ and $\mathcal{V}_\beta^{Z,m}$ are equivalent. In view of Lemma 12.7, we have that

$$S^{Z,m} = \mathrm{Sing}(\mathcal{V}_\alpha^{Z,m}) = \mathrm{Sing}(\mathcal{V}_\beta^{Z,m}).$$

Now, it is enough to show that this situation repeats under open projections and permissible blowing-ups. More precisely, let

$$\sigma : (M', E') \to (M, E)$$

be either an open projection or a blowing-up with a center $Y \subset S^{Z,m}$ that is a non-singular closed analytic subspace of M having normal crossings with E. We have to find a logarithmic factor Z' for the transform $\mathcal{P}'$ such that $\mu_{Z'}\mathcal{P}' \leq m$ and the following commutativity property holds:

$$(\mathcal{V}_\alpha^{Z,m})' = (\mathcal{V}'_\alpha)^{Z',m}, \quad (\mathcal{V}_\beta^{Z,m})' = (\mathcal{V}'_\beta)^{Z',m}. \tag{14}$$

Since $\mathcal{V}'_\alpha$ and $\mathcal{V}'_\beta$ are equivalent, the situation repeats, as desired.

The construction of Z' in the case of an open projection is straightforward. In the case of a blowing-up, we take

$$Z' = \mathcal{J}_{D'}^{m-d}\pi^{-1}Z, \quad D' = \pi^{-1}(Y).$$

This ends the proof of Proposition 12.6. □

Remark 12.8. The commutativity expressed in Equation (14) extends to d-normalized immersed idealistic e-atlas, in the sense that we have

$$(\mathcal{P}^{Z,m})' = (\mathcal{P}')^{Z',m}, \tag{15}$$

for the transforms under an open projection or a blowing-up with center that is permissible for $\mathcal{P}^{Z,m}$.

12.3. Reduction to the adjusted case

In this subsection, we give a proof of the statement corresponding to Equation (8). We state the result as follows:

Proposition 12.9. *Let $\mathcal{F}$ be an idealistic e-flower over the ambient space (M, E). Assume that the following statements are true:*

a) *The monomial idealistic e-flowers have reduction of singularities.*
b) *The adjusted idealistic e-flowers have reduction of singularities.*

Then $\mathcal{F}$ has reduction of singularities.

Proof. Take a d-normalized immersed idealistic e-atlas $\mathcal{P}$ belonging to $\mathcal{F}$. Let us see that $\mathcal{P}$ has reduction of singularities. Fix a logarithmic factor Z for $\mathcal{P}$. We do induction on the co-factorial order $\mu_Z\mathcal{P}$. If $\mu_Z\mathcal{P} = 0$, we are done, since $\mathcal{P}$ is monomial by Proposition 12.3.

Assume that $\mu_Z\mathcal{P} = m \geq 1$. We know that $\mathcal{P}^{Z,m}$ is an adjusted immersed idealistic e-atlas over (M, E) such that

$$\emptyset \neq \mathrm{Sing}(\mathcal{P}^{Z,m}) \subset \mathrm{Sing}(\mathcal{P}).$$

The permissible centers for $\mathcal{P}^{Z,m}$ are also permissible for $\mathcal{P}$. In view of our hypothesis, there is a sequence $\{\pi_i\}_{i=1}^p$ of permissible blowing-ups for $\mathcal{P}^{Z,m}$, such that

$$\mathrm{Sing}\left((\mathcal{P}^{Z,m})^{(p)}\right) = \emptyset.$$

By the commutativity in Remark 12.8, the centers of π_i are permissible for the successive transforms of $\mathcal{P}$ and we have that

$$(\mathcal{P}^{Z,m})^{(p)} = (\mathcal{P}^{(p)})^{Z^{(p)},m}.$$

Hence $\operatorname{Sing}(\mathcal{P}^{(p)})^{Z^{(p)},m} = \emptyset$ and thus $\mu_{Z^{(p)}}\mathcal{P}^{(p)} < m$. We end by the induction hypothesis. □

13. Reduction to the adjusted-reduced case

Here we provide a proof of the reduction to the adjusted-reduced case, corresponding to Equation (9). We state the result as follows:

Proposition 13.1. *Let $\mathcal{F}$ be an adjusted idealistic e-flower over the ambient space (M, E). Assume that the following statements are true*:

a) *The idealistic $(e-1)$-flowers have reduction of singularities.*

b) *The adjusted-reduced idealistic e-flowers have reduction of singularities.*

Then $\mathcal{F}$ has reduction of singularities.

Let us recall that an idealistic e-flower $\mathcal{F}$ is called to be *reduced* if we have that $\dim \operatorname{Sing} \mathcal{F} \leq e-2$. The following stability result holds:

Proposition 13.2. *Let $\mathcal{F}$ be an idealistic e-flower over the ambient space (M, E). Consider a permissible test system $\mathcal{S}$ for $\mathcal{F}$ and denote by $\mathcal{F}'$ the transform of $\mathcal{F}$ by $\mathcal{S}$. If $\mathcal{F}$ is adjusted and reduced, then $\mathcal{F}'$ is also adjusted and reduced.*

Proof. The result is consequence of the fact that when we blow-up a center that is equimultiple for a hypersurface, then the exceptional divisor is not in the locus of maximal multiplicity. We leave the details to the reader. □

Let us start the proof of Proposition 13.1. Take an adjusted idealistic e-flower $\mathcal{F}$ over (M, E). If $\dim \operatorname{Sing} \mathcal{F} \leq e-2$, we are done. On the other hand we know that $\dim \operatorname{Sing} \mathcal{F} \leq e-1$, hence we can assume that $\dim \operatorname{Sing} \mathcal{F} = e-1$. Let $L_1, L_2, \ldots, L_s$ be the irreducible components of dimension $\dim L_i = e-1$ of $\operatorname{Sing} \mathcal{F}$. We reason by induction on the number s. Consider the following result:

Proposition 13.3. *Assume that $s \geq 1$. There is an open set $U \subset M$ such that $U \cap \operatorname{Sing} \mathcal{F} = L_1$ and L_1 is a non-singular closed analytic subset $L_1 \subset M$.*

Proof. Take a point $P \in L_1$. It is enough to show that $\operatorname{Sing} \mathcal{F}$ is non-singular at P. Take an immersed idealistic e-chart

$$\mathcal{V} = (U, E \cap U, N, \mathcal{L}), \quad \mathcal{L} = \{(I_j, d_j)\}_{j=1}^{k},$$

belonging to $\mathcal{F}$, with $P \in U$. We have that $\nu_{L_1 \cap U}(I_j) \geq d_j$, for any index $j = 1, 2, \ldots, k$. Since $\mathcal{F}$ is adjusted, there is an index j_0 such that

$$\nu_P I_{j_0} = \nu_{L_1 \cap U} I_{j_0} = d_{j_0}.$$

This means that $L_1 \cap U$ is d_{j_0}-equimultiple for I_{j_0}. Noting that $L_1 \cap U$ is a hypersurface in N, we have that

$$I_{j_0} = (\mathcal{J}_{L_1 \cap U})^{d_{j_0}},$$

(locally at P, that is, up to make U smaller) where $\mathcal{J}_{L_1 \cap U}$ is the ideal sheaf of $\mathcal{O}_N$ defining $L_1 \cap U$. We conclude that the singular locus $\operatorname{Sing} \mathcal{F}$ is equal to $L_1 \cap U$, near P, and it is non-singular. □

As a consequence of Proposition 13.3 and in view of the induction hypothesis on s, we can restrict ourselves to the case when

$$\operatorname{Sing} \mathcal{F} = L,$$

where L is a non-singular irreducible $(e-1)$-dimensional closed analytic subset of M. If L has normal crossings with E, a single blowing-up with center L makes L to disappear from the singular locus and we are done. Now, it suffices to obtain the property that L has normal crossings with E by means of blowing-ups with centers contained in L. This is solved in next subsection.

13.1. Normal crossings for a non-singular closed subspace

The result we present here is what we need for the end of the proof of Proposition 13.1:

Proposition 13.4. *Consider an n-dimensional ambient space (M, E) and a closed non-singular subset $L \subset M$ of dimension $e - 1 < n$. Let us assume that any idealistic $(e-1)$-flower has reduction of singularities. Then there is a finite sequence of blowing-ups*

$$(M, E) \leftarrow (M_1, E_1) \leftarrow \cdots \leftarrow (M_k, E_k) \qquad (16)$$

of ambient spaces with centers contained in the successive strict transforms of L, such that the last strict transform L_k has normal crossings with E_k.

Proof. Let us consider L as the support of a new ambient space. Write

$$E = E^* \cup D \cup F,$$

where F is the union of components of E containing L and the divisor D is union of other components of E, in such a way that $F \cup D$ has normal crossings with L (such a D exists, but it is not necessarily unique, even if we chose a maximal one). Let us write $E^* = \cup_{i \in A} E_i^*$ the decomposition of E^* into its irreducible components. For any $B \subset A$, let us denote $E_B^* = \bigcap_{i \in B} E_i^*$. Consider the set

$$\Sigma = \{B \subset A;\ E_B^* \cap L \neq \emptyset\}.$$

Let $s = \max\{\#B;\ B \in \Sigma\}$, $\Sigma_s = \{B \in \Sigma;\ \#B = s\}$ and $t = \#\Sigma_s$. If $s = 0$, we are done, since then $L \cap E^* = \emptyset$. We reason by induction on the lexicographical invariant (s, t). Take $B \in \Sigma_s$ and consider the list $\mathcal{L}_{M,E^*,L,B}$ of marked ideals in L given by

$$\mathcal{L}_{M,E^*,L,B} = \{(I_{E_i^*}|_L, 1)\}_{i \in B}.$$

Take the idealistic space $\mathcal{N}$ over $(L, L\cap D)$ defined as

$$\mathcal{N} = \mathcal{N}_{M,E,E^*,L,B} = (L, L\cap D, \mathcal{L}_{M,E^*,L,B}).$$

Note that $\operatorname{Sing}\mathcal{N} = E^*_B \cap L$.

Let Y be a permissible center for $\mathcal{N}$. We have that:

(1) $Y \subset L$ is non-singular.
(2) $Y \subset F_i$, for each irreducible component of F.
(3) Y has normal crossings with $L\cap D$.
(4) Y is equimultiple for I_{E^*}, where I_{E^*} is the ideal sheaf of $\mathcal{O}_M$ defining E^*. More precisely, we have that $Y \subset E^*_B$ and for any index $i \in A\setminus B$, we have that $E^*_i \cap Y = \emptyset$.

We conclude that Y is equimultiple for $I_{F\cup E^*} = I_{E^*}I_F$ and it has normal crossings with D. Then, we have that Y has normal crossings with E. Thus, we can perform the blowing-up of ambient spaces

$$\pi_Y : (M', E') \to (M, E)$$

centered at Y, with $E' = (E^*)' \cup \tilde{D} \cup F'$, where $\tilde{D} = \pi_Y^{-1}(D\cup Y)$ and $(E^*)'$, F' are the respective strict transforms of E^*, F. If L' is the strict transform of L, the restriction

$$\bar{\pi}_Y : (L', L'\cap \tilde{D}) \to (L, L\cap D),$$

is the blowing-up of $(L, L\cap D)$ with center Y.

Let $\mathcal{N}'$ be the transform of $\mathcal{N}$ by $\bar{\pi}_Y$. If $\operatorname{Sing}(\mathcal{N}') = \emptyset$, the new invariant (s', t') is strictly smaller than (s, t) and we are done by induction. If $\operatorname{Sing}(\mathcal{N}') \neq \emptyset$, then $(s', t') = (s, t)$, we have that $A' = A$, $\Sigma' = \Sigma$ and the following commutativity property holds:

$$\mathcal{N}' = \mathcal{N}_{M',E',(E^*)',L',B}.$$

Now, we end by performing a reduction of singularities of $\mathcal{N}$. □

Part 5. Projections of idealistic exponents

Let $\mathcal{E}$ be an adjusted and reduced idealistic exponent over an n-dimensional ambient space (M, E) and consider a hypersurface (M, E, H), recall that H is non-singular and it has normal crossings with E. In this part, we introduce a procedure for obtaining a new idealistic exponent $\operatorname{pr}_H \mathcal{E}$ over $(H, E|_H)$ that we call *the projection of* $\mathcal{E}$ *on* (M, E, H) whose main properties are the following ones:

- $H \cap \operatorname{Sing}\mathcal{E} = \operatorname{Sing}(\operatorname{pr}_H \mathcal{E})$.
- A closed analytic subset $Y \subset H$ is a permissible center for $\operatorname{pr}_H \mathcal{E}$ if and only if it is a permissible center for $\mathcal{E}$.
- $(\operatorname{pr}_H \mathcal{E})' = \operatorname{pr}_{H'}(\mathcal{E}')$, for the transforms under a morphism that is either an open projection or a blowing-up with permissible center Y for $\mathcal{E}$, with $Y \subset H$.

Let us note that the assignment $((M, E, H), \mathcal{E}) \mapsto \operatorname{pr}_H \mathcal{E}$ is necessarily unique if it exists. Indeed, we deduce from the above properties that if we have two assignments pr and $\widetilde{\operatorname{pr}}$, then $\operatorname{pr}_H \mathcal{E} = \widetilde{\operatorname{pr}}_H \mathcal{E}$, since they have the same permissible test systems.

Let us also note that (M, E, H) may be transverse or not, see the definitions in Section 2. Thus, there is a union E^* of irreducible components of E such that $E|_H = E^* \cap H$, where E^* is the union of the irreducible components of E not coinciding (locally) with H.

We end this part with the construction of the projection of an idealistic e-flower on (M, E, H) in the particular case that H is a disjoint union of irreducible components of E.

Once the projections are constructed, the existence of a reduction of singularities of $\operatorname{pr}_H \mathcal{E}$ implies the existence of a morphism

$$\sigma : (M', E') \to (M, E)$$

that is a composition of a finite sequence of permissible blowing-ups for $\mathcal{E}$ in such a way that $H' \cap \operatorname{Sing} \mathcal{E}' = \emptyset$, where H' is the strict transform of H by σ. This is a key observation for the Maximal Contact Theory.

The projections are done with the help of *projecting axes*. We have built these structures inspired in a part of the work of Panazzolo in [32].

14. Projecting axes

The construction of projecting axes is done "around H" instead of considering "the whole ambient space". More precisely, we will consider the open set (M_H, E_H) of (M, E) defined as follows. Recalling that M is a germ over the compact set $K \subset M$, we define M_H to be the germ of M on the compact set $K \cap H$ and E_H to be the germ of E over the compact set $E \cap K \cap H$.

Denote by $\Theta_M[\log E^*]$ the sheaf of germs of vector fields over M that are tangent to E^*. A *projecting chart for* (M, E, H) is a pair

$$\mathfrak{c} = (U, \xi),$$

where U is an open set of M_H and $\xi \in \Theta_M[\log E^*](U)$ is a non-singular vector field transverse to $H \cap U$ at every point. Two projecting charts $\mathfrak{c}_1 = (U_1, \xi_1)$ and $\mathfrak{c}_2 = (U_2, \xi_2)$ for (M, E, H) are *compatible* if there is a unit u_{12}, defined in $U_{12} = U_1 \cap U_2$, such that

$$\xi_2|_{U_{12}} = u_{12}\xi_1|_{U_{12}}, \quad \xi_1|_{U_{12}}(u_{12}) = 0.$$

Note that we ask u_{12} to be a first integral of $\xi_1|_{U_{12}}$. Automatically, we have that u_{12} is also a first integral of $\xi_2|_{U_{12}}$.

Given $P \in H$, there is always a projecting chart $\mathfrak{c} = (U, \xi)$, with $P \in U$. It is enough to take local coordinates $\mathbf{x}, z$ around P defined on U and adapted to E, such that $H \cap U = (z = 0)$ and

$$\xi = \partial/\partial z.$$

Conversely, given a projecting chart $\mathfrak{c} = (U, \xi)$ and a point $P \in H \cap U$, there are local coordinates $\mathbf{x}, z$ around P defined on an open set $V \subset U$ that are adapted to E, such that $H \cap U = (z = 0)$ and $\xi = \partial/\partial z$.

This suggests the following definition:

Definition. A *rectified projecting chart for* (M, E, H) is a data $(\mathfrak{c}, \mathbf{x}, z)$, where $\mathfrak{c} = (U, \xi)$ is a projecting chart for (M, E, H) and $\mathbf{x}, z$ are coordinate functions defined on U, adapted to E, such that $H \cap U = (z = 0)$ and that $\xi = \partial/\partial z$. A given projecting chart $\mathfrak{c}$ for (M, E, H) is *rectifiable* if there are coordinates $\mathbf{x}, z$ such that $(\mathfrak{c}, \mathbf{x}, z)$ is a rectified projecting chart for (M, E, H).

We define a *projecting atlas* $\mathfrak{a}$ *for* (M, E, H) to be a finite family $\mathfrak{a} = \{\mathfrak{c}_\alpha\}_{\alpha \in \Lambda}$, such that the $\mathfrak{c}_\alpha$ are pairwise compatible projecting charts for (M, E, H), whose definition domains cover H; equivalently, the definition domains cover M_H.

Definition. Two projecting atlases $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of (M, E) over the hypersurface (M, E, H) are *compatible* if their union $\mathfrak{a}_1 \cup \mathfrak{a}_2$ is also a projecting atlas for (M, E, H). The compatibility classes of projecting atlases are called *projecting axes for* (M, E, H). Given a projecting axis $\mathfrak{E}$ and a projecting chart $\mathfrak{c} = (U, \xi)$ for (M, E, H), we say that $\mathfrak{c}$ *belongs to* $\mathfrak{E}$ if $\mathfrak{c}$ belongs to some of the projecting atlases defining $\mathfrak{E}$.

Given a projecting atlas $\mathfrak{a}$, there is another compatible projecting atlas $\widetilde{\mathfrak{a}}$ such that all the charts in $\widetilde{\mathfrak{a}}$ are rectifiable, by a convenient “refinement” of the charts. In particular, there is always an atlas composed of rectifiable charts among the atlases defining a given projecting axis.

Remark 14.1. A projecting axis for (M, E, H) is exactly the same object as a projecting axis for (M_H, E_H, H).

14.1. First integrals of projecting axes

Let $\mathfrak{E}$ be a projecting axis for (M, E, H). We denote by $\mathcal{O}_{M_H}$ the sheaf of germs of holomorphic functions of M_H, that is $\mathcal{O}_{M_H} = \mathcal{O}_M|_{M_H}$. Let us build the *sheaf of first integrals* $\mathrm{Int}\mathfrak{E}$ *of* $\mathfrak{E}$. For each open subset $V \subset M_H$, we define $\mathrm{Int}\mathfrak{E}(V)$ to be the subring

$$\mathrm{Int}\mathfrak{E}(V) \subset \mathcal{O}_{M_H}(V) = \mathcal{O}_M(V)$$

whose elements are the holomorphic functions h defined in V satisfying the following equivalent properties:

a) Given a point $P \in V$, there is a projecting chart $\mathfrak{c} = (U, \xi)$ belonging to $\mathfrak{E}$ such that $P \in U \subset V$ and $\xi(h|_U) = 0$.
b) Given a projecting chart $\mathfrak{c} = (U, \xi)$ belonging to $\mathfrak{E}$ such that $U \subset V$, we have that $\xi(h|_U) = 0$.

The sheaf $\mathrm{Int}\mathfrak{E}$ is a subsheaf of rings of $\mathcal{O}_{M_H}$. In particular, we have that $\mathcal{O}_{M_H}$ is a $\mathrm{Int}\mathfrak{E}$-module.

Remark 14.2. Let $(\mathfrak{c}, \mathbf{x}, z)$ be a rectified projecting chart for (M, E, H) such that $\mathfrak{c} = (U, \xi)$ belongs to a given projecting axis $\mathfrak{E}$ for (M, E, H). A function $h \in \mathcal{O}_M(U)$ is a first integral for $\mathfrak{E}$ if and only if $\partial h/\partial z = 0$.

14.2. Local nature of projecting axes

Let $\mathfrak{c} = (U, \xi)$ be a projecting chart for (M, E, H). The *restriction* $\mathfrak{c}|_V$ *of* $\mathfrak{c}$ *to an open set* V *of* M_H is given in a natural way by

$$\mathfrak{c}|_V = (U \cap V, \xi|_{U\cap V}).$$

It is a projecting chart for $(V, E \cap V, H \cap V)$.

Let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be two projecting charts for (M, E, H). The following properties are equivalent:

(1) The charts $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are compatible.
(2) For any open set $V \subset M_H$, the restrictions $\mathfrak{c}_1|_V$ and $\mathfrak{c}_2|_V$ are compatible.
(3) The restrictions $\mathfrak{c}_1|_V$ and $\mathfrak{c}_2|_V$ are compatible for the open sets $V \subset M_H$ belonging to an open cover of M_H.

Consider a projecting atlas $\mathfrak{a} = \{\mathfrak{c}_\alpha\}_{\alpha\in\Lambda}$ for (M, E, H). The *restriction* $\mathfrak{a}|_V$ *of* $\mathfrak{a}$ *to an open set* V *of* M_H is given by

$$\mathfrak{a}|_V = \{\mathfrak{c}_\alpha|_V\}_{\alpha\in\Lambda}.$$

It is a projecting atlas for $(V, E \cap V, H \cap V)$. Moreover, if $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are two compatible projecting atlases for (M, E, H), their restrictions $\mathfrak{a}_1|_V$ and $\mathfrak{a}_2|_V$ are also compatible. This allows us to define without ambiguity the *restriction* $\mathfrak{E}|_V$ to V of a projecting axis $\mathfrak{E}$ for (M, E, H). It is, of course, a projecting axis for $(V, E \cap V, H \cap V)$.

The following "gluing" result, concerning the local nature of projecting axes, follows directly from the local nature of the compatibility of charts:

Lemma 14.3. *Consider an open covering* $\{U_\beta\}_{\beta\in B}$ *of* M_H. *Assume that for each* $\beta \in B$, *there is a projecting axis* $\mathfrak{E}_\beta$ *for* $(U_\beta, E \cap U_\beta, H \cap U_\beta)$. *If the equality*

$$\mathfrak{E}_\beta|_{U_{\beta\gamma}} = \mathfrak{E}_\gamma|_{U_{\beta\gamma}}, \quad U_{\beta\gamma} = U_\beta \cap U_\gamma,$$

holds for any $\beta, \gamma \in B$, *there is a unique projecting axis* $\mathfrak{E}$ *for* (M, E, H) *such that* $\mathfrak{E}|_{U_\beta} = \mathfrak{E}_\beta$, *for all* $\beta \in B$.

14.3. Projections on the first factor and projecting axes

Consider a projection on the first factor

$$\sigma : (M', E') = (M \times (\mathbb{C}^m, 0), E \times (\mathbb{C}^m, 0)) \to (M, E)$$

and put $H' = \sigma^{-1}(H)$. Note that $M'_{H'} = \sigma^{-1}(M_H)$ and that (M', E', H') is a hypersurface of (M', E'). Recall that we have the functions

$$\omega_i : M' \to (\mathbb{C}, 0), \quad i = 1, 2, \dots, m,$$

obtained by composition of $M \times (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$ with the i-th coordinate function of $(\mathbb{C}^m, 0)$.

Let $\mathfrak{c} = (U, \xi)$ be a projecting chart for (M, E, H). The *transform*

$$\sigma^{-1}\mathfrak{c} = (U', \xi')$$

of $\mathfrak{c}$ *by* σ is obtained by putting $U' = \sigma^{-1}(U)$ and by taking ξ' to be the unique vector field over U' satisfying

(1) $(d\sigma) \circ \xi' = \xi \circ \sigma$. (The vector fields ξ and ξ' are σ-related).
(2) $\xi'(\omega_i) = 0$, for all $i = 1, 2, \ldots, m$.

We have that $\sigma^{-1}\mathfrak{c}$ is a projecting chart for (M', E', H'). If $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are two compatible projecting charts, then, the transformed charts $\sigma^{-1}\mathfrak{c}_1$ and $\sigma^{-1}\mathfrak{c}_2$ are also compatible. In this way, we define the *transform* $\sigma^{-1}\mathfrak{a}$ *of a projecting atlas* $\mathfrak{a}$ as well as the *transform* $\sigma^{-1}\mathfrak{E}$ *of a projecting axis* $\mathfrak{E}$ *for* (M, E, H). We obtain, respectively, a projecting atlas and a projecting axis for (M', E', H').

Remark 14.4. Let $\mathfrak{E}' = \sigma^{-1}\mathfrak{E}$. The sheaf of first integrals $\mathrm{Int}\mathfrak{E}'$ is related with $\mathrm{Int}\mathfrak{E}$ as follows. Consider an open subset $V' \subset M'_{H'}$. Note that V' is of the form $V' = U \times (\mathbb{C}^m, 0)$. A function h' defined over V' is a first integral for $\mathfrak{E}'$ if and only if it factorizes through a first integral h of $\mathfrak{E}$ defined in U, that is $h' = h \circ \sigma$.

14.4. Blowing-up projecting axes

Before studying the transformation of a projecting axis by a blowing-up, let us consider the following lemma:

Lemma 14.5. *Let* $\mathfrak{E}_1$ *and* $\mathfrak{E}_2$ *be two projecting axis for* (M, E, H). *Assume that there is a closed analytic subset* $Z \subset M_H$ *of codimension greater than or equal to one, such that* $\mathfrak{E}_1|_{M_H \setminus Z} = \mathfrak{E}_2|_{M_H \setminus Z}$. *Then* $\mathfrak{E}_1 = \mathfrak{E}_2$.

Proof. It is enough to show that any two projecting charts $\mathfrak{c}_1 = (U_1, \xi_1)$ and $\mathfrak{c}_2 = (U_2, \xi_2)$ of $\mathfrak{E}_1$ and $\mathfrak{E}_2$, respectively, are compatible.

Write $U = U_1 \cap U_2$. If $U = \emptyset$, there is nothing to prove. Assume that $U \neq \emptyset$ and denote $\bar{\mathfrak{c}}_i = \mathfrak{c}_i|_U = (U, \bar{\xi}_i)$, for $i = 1, 2$. We need to show the existence of a unit $u \in \mathcal{O}_M(U)$ satisfying that $\bar{\xi}_2 = u\bar{\xi}_1$, with $\bar{\xi}_1(u) = 0$. Consider the subset

$$A = \{P \in U;\ \text{ there is } f_P \in \mathcal{O}_{M,P} \text{ such that } \xi_{2,P} = f_P \xi_{1,P}\}.$$

Let us see that A is a closed analytic subset of U. Recalling that $\bar{\xi}_1$ and $\bar{\xi}_2$ are non-singular vector fields, the set $A \subset U$ is defined by the equation $\bar{\xi}_1 \wedge \bar{\xi}_2 = 0$. Hence $A \subset U$ is a closed analytic subset. Let us write $W = U \setminus Z$. Since $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are compatible charts in W, we have that $W \subset A \subset U$, and hence $A = U$. Then, given $P \in U$, we have the relation

$$\xi_{2,P} = f_P \xi_{1,P}.$$

Once again, since they are non-singular vector fields, we conclude that the germs f_P are unique and they are units. These germs are "glued" in a unit $u \in \mathcal{O}_M(U)$ such that $u_P = f_P$, for all $P \in U$. Hence $\bar{\xi}_2 = u\bar{\xi}_1$.

On the other hand, the compatibility of $\mathfrak{c}_1$ and $\mathfrak{c}_2$ in W implies that

$$\bar{\xi}_1(u) = 0, \text{ over } W.$$

Again, since $\bar{\xi}_1(u) = 0$ defines a closed analytic subset of U, we have that $\bar{\xi}_1(u) = 0$, in the whole U. We conclude that $\mathfrak{c}_1$ are $\mathfrak{c}_2$ compatible charts in U as desired. □

Let Y be a non-singular irreducible closed analytic subset of H having normal crossings with E. Assume that the codimension of Y in H is grater than or equal to one and thus Y does not coincide with any connected component of H. Let us perform the blowing-up

$$\pi : (M', E') \to (M, E)$$

centered at Y and denote by H' the strict transform of H by π. We have that (M', E', H') is a hypersurface of (M', E') and π induces a blowing-up

$$\bar{\pi} : (H', E'|_{H'}) \to (H, E|_H).$$

Let us note that $M'_{H'} \subset \pi^{-1}(M_H)$ and that π induces an identification between $M'_{H'} \setminus \pi^{-1}(Y)$ and $\pi(M'_{H'}) \setminus Y$, as a consequence of the identification between $M' \setminus \pi^{-1}(Y)$ and $M \setminus Y$.

Proposition 14.6. *Let $\mathfrak{E}$ be a projecting axis for (M, E, H). There is a unique projecting axis $\mathfrak{E}'$ for (M', E', H') such that*

$$\mathfrak{E}'|_{M'_{H'} \setminus \pi^{-1}(Y)} = \mathfrak{E}|_{\pi(M'_{H'}) \setminus Y},$$

where we have taken the identification $M' \setminus \pi^{-1}(Y) \to M \setminus Y$ induced by the blowing-up π.

Proof. Uniqueness is a direct consequence of Lemma 14.5. Let us see the existence. In order to do it, we are going to prove the existence of a covering of M_H by open subsets $U \subset M_H$ with the following property:

There is a projecting axis $\mathfrak{E}'$ for $(U^\star, E^\star, H^\star)$ such that

$$\mathfrak{E}'|_{U^\star_{H^\star} \setminus \pi^{-1}(Y)} = \mathfrak{E}|_{\pi(U^\star_{H^\star}) \setminus Y}, \tag{17}$$

where $U^\star = \pi^{-1}(U)$ and $E^\star$, $H^\star$ are the corresponding intersections of E', H' with $U^\star$.

Assume this result is proved and take two of these open subsets U_β and U_γ. By invoking the uniqueness, we have the equality

$$\mathfrak{E}'_\beta|_{U^\star_{\beta\gamma}} = \mathfrak{E}'_\gamma|_{U^\star_{\beta\gamma}}, \quad U^\star_{\beta\gamma} = U^\star_\beta \cap U^\star_\gamma.$$

Now, we obtain $\mathfrak{E}'$ by application of the gluing Lemma 14.3.

Around each point of H, we can choose a rectified projecting chart

$$(\mathfrak{c} = (U, \xi), \mathbf{x}, z)$$

of $\mathfrak{E}$ with the additional condition that if $Y \cap U \neq \emptyset$, then

$$Y \cap U = (z = 0) \cap (x_i = 0;\ i = 1, 2, \ldots, t),$$

where $t \geq 1$ is the codimension of Y in H. In this way, we cover M_H by the domains of that charts. It remains to prove that given one of that charts $(\mathfrak{c} = (U, \xi), \mathbf{x}, z)$, there is a projecting axis $\mathfrak{E}'$ for $(U^\star, E^\star, H^\star)$ satisfying to the property stated in Equation (17).

Let us consider $U_i^\star = U^\star \setminus L_i^\star$, where $L_i^\star$ is the strict transform of $(x_i = 0)$ by π, for any $i = 1, 2, \ldots, t$. Note that

$$H^\star \subset U_{H^\star}^\star \subset \widetilde{U}, \text{ where } \widetilde{U} = \bigcup_{i=1}^t U_i^\star.$$

Let us write $V_i = U \setminus (x_i = 0)$ and $W = \cup_{i=1}^t V_i$. We have the equality

$$\pi(U_i^\star \setminus \pi^{-1}(Y)) = V_i, \quad i = 1, 2, \ldots, t.$$

Then, we have $\pi(\widetilde{U} \setminus \pi^{-1}(Y)) = W$. Since $U_{H^\star}^\star \subset \widetilde{U}$, it is enough to prove the existence of a projecting axis $\mathfrak{E}'$ defined in $\widetilde{U}$ satisfying that

$$\mathfrak{E}'|_{\widetilde{U} \setminus \pi^{-1}(Y)} = \mathfrak{E}|_W,$$

where we have taken the identification $\widetilde{U} \setminus \pi^{-1}(Y) \to W$ induced by the blowing-up π outside of $\pi^{-1}(Y)$.

In order to obtain $\mathfrak{E}'$, we are going to define $\mathfrak{E}'_i$ on $U_i^\star$ by

$$\mathfrak{E}'_i|_{U_i^\star \setminus \pi^{-1}(Y)} = \mathfrak{E}|_{V_i}, \quad i = 1, 2, \ldots, t. \tag{18}$$

Indeed, in view of Lemma 14.5, the property in Equation (18) implies that

$$\mathfrak{E}'_i|_{U_i^\star \cap U_j^\star} = \mathfrak{E}'_j|_{U_i^\star \cap U_j^\star}, \quad i, j \in \{1, 2, \ldots, t\}$$

and we obtain the desired $\mathfrak{E}'$ by gluing the $\mathfrak{E}'_i$, for $i = 1, 2, \ldots, t$.

Let us define the projecting axis $\mathfrak{E}'_i$. Consider the vector field

$$\xi_i = x_i \xi,$$

defined in U, and the projecting chart $\mathfrak{c}_i = (V_i, \xi_i|_{V_i})$. Since x_i is a first integral and a unit in V_i, the chart $\mathfrak{c}_i$ belongs to the axis $\mathfrak{E}$. Thus, the chart $\mathfrak{c}_i$ defines exactly $\mathfrak{E}|_{V_i}$. The vector field ξ_i lifts by π to a unique vector field ξ'_i over $U_i^\star$, that is written in appropriate coordinates as

$$\xi'_i = \partial/\partial z',$$

where $H^\star \cap U_i^\star = (z' = 0)$. This allows us to define $\mathfrak{E}'_i$ from the projecting chart $(U_i^\star, \xi'_i)$. □

Definition. Let $\mathfrak{E}$ be a projecting axis. The projecting axis $\mathfrak{E}'$ given in Proposition 14.6 is called *the transform of $\mathfrak{E}$ by the blowing-up π*.

15. Projections of idealistic spaces

Consider a hypersurface (M, E, H) of an ambient space (M, E). Let us take an adjusted and reduced idealistic space $\mathcal{M}$ of (M, E).

In this section, we construct projections of $\mathcal{M}$ over (M, E, H) associated to projecting axes and projectable generators of the marked ideals of $\mathcal{M}$. These constructions are compatible with the equivalence of idealistic spaces, with open projections and with permissible blowing-ups whose centers are contained in H. Thanks to these properties, in the next section, we will be able to build the projection on (M, E, H) for adjusted-reduced idealistic exponents of (M, E).

15.1. Projecting systems. Local nature

Before giving the definition of projecting systems, we need some concepts concerning only a projecting axis $\mathfrak{E}$ for (M, E, H).

We say that a sheaf $\mathbf{J} \subset \mathcal{O}_{M_H}$ is an *$\mathfrak{E}$-projectable module over M_H* if it is a locally principal Int$\mathfrak{E}$-submodule of $\mathcal{O}_{M_H}$. This means that there is an open covering $\{U_\beta\}_{\beta\in B}$ of M_H together with holomorphic functions $F_\beta \in \mathcal{O}_M(U_\beta)$, satisfying the following properties:

(1) The germ of F_β is non-zero at each point.
(2) For each open subset V of M_H and any index $\beta \in B$, we have

$$\mathbf{J}(V \cap U_\beta) = \mathrm{Int}\mathfrak{E}(V \cap U_\beta) \cdot F_\beta|_{V\cap U_\beta} \subset \mathcal{O}_M(V \cap U_\beta).$$

Note that, for each pair of indices $\beta, \gamma \in B$, we have that

$$F_\gamma|_{U_\beta\cap U_\gamma} = u_{\beta\gamma} F_\gamma|_{U_\beta\cap U_\gamma},$$

where $u_{\beta\gamma} \in \mathrm{Int}\mathfrak{E}(U_\beta \cap U_\gamma)$ is a first integral that is also a unit; in particular, it is also a unit in $\mathcal{O}_M(U_\beta \cap U_\gamma)$.

Definition. Let us consider a locally principal ideal sheaf $I \subset \mathcal{O}_{M_H}$. An *$\mathfrak{E}$-projectable generator* $\mathbf{J}$ *of* I is any $\mathfrak{E}$-projectable module $\mathbf{J}$, with $\mathbf{J} \subset I$, that generates I as $\mathcal{O}_{M_H}$-ideal sheaf.

In local terms, if $\mathbf{J}$ is given by a family $\{(U_\beta, F_\beta)\}_{\beta\in B}$ of open sets U_β and functions $F_\beta \in \mathcal{O}_M(U_\beta)$ as before, the ideal $I(U_\beta) \subset \mathcal{O}_M(U_\beta)$ is generated by F_β, for each $\beta \in B$.

Now, we can define the projecting systems:

Definition. Let $\mathcal{M} = (M, E, \mathcal{L})$ be an adjusted-reduced idealistic space. A *projecting system* $\mathfrak{S}$ *for* $(\mathcal{M}, H)$ is a pair $\mathfrak{S} = (\mathfrak{E}, \mathfrak{J})$ satisfying the following properties:

(1) $\mathfrak{E}$ is a projecting axis for (M, E, H).
(2) If $\mathcal{L} = \{(I_j, d_j)\}_{j=1}^k$, then $\mathfrak{J}$ is a list $\mathfrak{J} = \{\mathbf{J}_j\}_{j=1}^k$, where each $\mathbf{J}_j$ is an $\mathfrak{E}$-projectable generator of $I_j|_{M_H}$.

Given an open set V of M_H, the restriction $\mathfrak{S}|_V$ of a projecting system $\mathfrak{S} = (\mathfrak{E}, \mathfrak{J})$ is naturally given by

$$\mathfrak{S}|_V = (\mathfrak{E}|_V, \mathfrak{J}|_V), \quad \mathfrak{J}|_V = \{\mathbf{J}_j|_V\}_{j=1}^k.$$

It is a projecting system for $(\mathcal{M}|_V, H \cap V)$.

We have local determination and local gluing procedures for projecting systems, since the corresponding properties hold for $\mathcal{M}$, $\mathfrak{E}$ and $\mathfrak{J}$.

15.2. Projections on the first factor and projecting systems

Consider a projection on the first factor

$$\sigma : (M', E') = (M \times (\mathbb{C}^m, 0), E \times (\mathbb{C}^m, 0)) \to (M, E)$$

and an idealistic space $\mathcal{M} = (M, E, \mathcal{L})$, with $\mathcal{L} = \{(I_j, d_j)\}_{j=1}^k$. Take a projecting system $\mathfrak{S} = (\mathfrak{E}, \mathfrak{J})$ for $(\mathcal{M}, H)$. The *transform* $\mathfrak{S}'$ *of* $\mathfrak{S}$ *by* σ will be denoted by

$$\mathfrak{S}' = \left(\mathfrak{E}',\ \mathfrak{J}' = \{\mathbf{J}'_j\}_{j=1}^k\right),$$

where $\mathfrak{E}'$ is the transform of $\mathfrak{E}$ by σ.

Let us detail which are the $\mathfrak{E}'$-projectable generators $\mathbf{J}'_j$ of I'_j. Take an open subset U' of $M'_{H'}$. Since we are working with the germified space $(\mathbb{C}^m, 0)$, we know that $U' = U \times (\mathbb{C}^m, 0)$, where $U \subset M_H$ is an open subset. Recall that $I'_j(U')$ is given by

$$I'_j(U') = \{f \circ \sigma;\ f \in I_j(U)\} \cdot \mathcal{O}_{M'}(U'),$$

which is an ideal of the ring $O_{M'}(U')$. On the other hand, we define

$$\mathbf{J}'_j(U') = \{f \circ \sigma;\ f \in \mathbf{J}_j(U)\} \cdot \mathrm{Int}\mathfrak{E}'(U').$$

Since $\mathrm{Int}\mathfrak{E}'(U') = \{h \circ \sigma;\ h \in \mathrm{Int}\mathfrak{E}(U)\}$, we see directly that

$$\mathbf{J}'_j(U') = \{f \circ \sigma;\ f \in \mathbf{J}_j(U)\}.$$

In this way, we obtain the projecting system $\mathfrak{S}'$ for $(\mathcal{M}', H')$, where $\mathcal{M}'$ is the transform of $\mathcal{M}$ by σ and $H' = H \times (\mathbb{C}^m, 0)$.

15.3. Blowing-up projecting systems

Let $Y \subset H$ be a permissible center for $\mathcal{M}$. Since $\mathcal{M}$ is reduced, we know that Y has codimension greater than or equal to two in M, hence it has codimension greater than or equal to one in H. Let us consider the blowing-up

$$\pi : (M', E') \to (M, E)$$

centered at Y. As we know, the morphism π induces the blowing-up

$$\bar{\pi} : (H', E'|_{H'}) \to (H, E|_H)$$

centered at Y, where H' is the strict transform of H by π and the divisor $E'|_{H'}$ coincides with $\bar{\pi}^{-1}(E|_H \cup Y)$.

Let us define the *transform* $\mathfrak{S}'$ *of* $\mathfrak{S}$ *by* π. We put $\mathfrak{S}' = (\mathfrak{E}', \mathfrak{J}')$, where $\mathfrak{E}'$ is the transform of $\mathfrak{E}$ by π. It remains to describe $\mathfrak{J}' = \{\mathbf{J}'_j\}_{j=1}^k$. We consider the morphism

$$\sigma : (M'_{H'}, E'_{H'}) \to (M, E)$$

given as composition of the open inclusion $M'_{H'} \subset M'$ with the blowing-up π. Let $\mathcal{J}_{\pi^{-1}(Y)}$ be the ideal sheaf in $M'_{H'}$ defining the exceptional divisor $\pi^{-1}(Y) \cap M'_{H'}$. We denote

$$\boldsymbol{\mathcal{J}}_{\pi^{-1}(Y)} = \mathcal{J}_{\pi^{-1}(Y)} \cap \mathrm{Int}\mathfrak{E}'.$$

Noting that $\mathcal{J}_{\pi^{-1}(Y),P'}$ is generated by a first integral of $\mathfrak{E}'$ at the points P' in $M'_{H'}$, we see that $\boldsymbol{\mathcal{J}}_{\pi^{-1}(Y)}$ is an $\mathfrak{E}'$-projectable generator of $\mathcal{J}_{\pi^{-1}(Y)}$. Consider now the $\mathfrak{E}$-projectable generator $\mathbf{J}_j$ of I_j in $\mathfrak{J}$. Recall that

$$\sigma^{-1}I_j = I'_j\mathcal{J}^{d_j}_{\pi^{-1}(Y)}.$$

Denote by $\sigma^*\mathbf{J}_j$ the sheaf of $\mathrm{Int}(\mathfrak{E}')$-modules generated by $f\circ\sigma$, where f varies over the sections of $\mathbf{J}_j$. In terms of germs, given a point $P'\in M'_{H'}$ and $P=\sigma(P')$, we have that

$$(\sigma^*\mathbf{J}_j)_{P'} = \{f\circ\sigma;\ f\in \mathbf{J}_{j,P}\}\cdot \mathrm{Int}\mathfrak{E}'_{P'}.$$

Then, we have that $\sigma^*\mathbf{J}_j$ is an $\mathfrak{E}'$-projectable generator of $\sigma^{-1}I_j$. Since $\sigma^{-1}I_j$ is divisible by $\mathcal{J}^{d_j}_{\pi^{-1}(Y)}$, then $\sigma^*\mathbf{J}_j$ is divisible by $\boldsymbol{\mathcal{J}}^{d_j}_{\pi^{-1}(Y)}$. In this way, we obtain an $\mathfrak{E}'$-projectable generator $\mathbf{J}'_j$ of I'_j given by the relation

$$\sigma^*\mathbf{J}_j = \mathbf{J}'_j\cdot\boldsymbol{\mathcal{J}}^{d_j}_{\pi^{-1}(Y)}.$$

Thus, we obtain a projecting system $\mathfrak{S}'$ for $(\mathcal{M}',H')$, where $\mathcal{M}'$ is the transform of $\mathcal{M}$ by σ and H' is the strict transform of H.

Remark 15.1. Let us show how these objects are described in terms of coordinates. Fix two points $P\in Y$ and $P'\in H'\cap\pi^{-1}(P)$. We can choose a rectified projecting chart $(\mathfrak{c}=(U,\xi),\mathbf{x},z)$ around P such that

$$Y\cap U = (z=0)\cap(x_1=x_2=\cdots=x_t=0).$$

Moreover, we can assume that the blowing-up π is given at P' in coordinates $\mathbf{x}',z'$ by the relations $x_1=x'_1$, $z=x'_1z'$,

$$x_s = x'_1(x'_s+\lambda_s),\quad \lambda_s\in\mathbb{C},\quad s=2,3,\ldots,t$$

and $x_s=x'_s$, for $s=t+1,t+2,\ldots,n-1$, where $n=\dim M$. We know that $\mathbf{J}_j=\{hF_j;\partial h/\partial z=0\}$, where F_j generates I_j at P. The ideal I'_j is generated at P' by

$$F'_j = (x'_1)^{-d_j}(F_j\circ\sigma).$$

The transformed projecting axis is given by $\partial/\partial z'$ in the coordinates $\mathbf{x}',z'$ and the $\mathfrak{E}'$-projectable generator of I'_j is $\mathbf{J}'_j=\{h'F'_j;\ \partial h'/\partial z'=0\}$.

15.4. Projected space of a projecting system

Let us consider a projecting system $\mathfrak{S}=(\mathfrak{E},\mathfrak{J})$ for $(\mathcal{M},H)$. In this subsection we construct an idealistic space

$$\mathfrak{S}_{\mathcal{M},H} = (H,E|_H,\mathcal{N}),\quad \mathcal{N}=\{(N_{js},d_j-s)\}_{1\le j\le k,\,0\le s\le d_j-1}$$

of $(H,E|_H)$ that we call the *projected space of* $\mathcal{M}$ *by* $\mathfrak{S}$ *over* (M,E,H).

Let us define the ideal sheaves $N_{js}\subset\mathcal{O}_H$. Take a projecting chart $\mathfrak{c}=(U,\xi)$ belonging to the projecting axis $\mathfrak{E}$. For each pair j,s, let

$$\xi^s(\mathbf{J}_j)$$

be the iterated s-fold application of ξ to the $\mathfrak{E}$-projectable generator $\mathbf{J}_j$ of I_j. Making a local computation for a section hF_j of $\mathbf{J}_j$, we have that

$$\xi(hF_j) = h\xi F_j, \quad h \in \mathrm{Int}\mathfrak{E}.$$

We conclude that $\xi^s(\mathbf{J}_j)$ is an $\mathfrak{E}|_U$-projectable module. We define $N_{js}|_{U\cap H}$ to be the ideal sheaf of $\mathcal{O}_H|_{H\cap U}$ generated by the restriction

$$\xi^s(\mathbf{J}_j)|_{H\cap U}$$

of $\xi^s(\mathbf{J}_j)$ to $H \cap U$. This definition is compatible with the other projecting charts in the intersection of the domains. We define the ideal sheaves N_{js} by a gluing procedure from the $N_{js}|_{U\cap H}$.

Before continuing with the properties of these objects, let us see how are the ideals N_{js} in appropriated local coordinates. Consider a rectified projecting chart $(\mathfrak{c} = (U, \xi), \mathbf{x}, z)$ belonging to the axis $\mathfrak{E}$. Assume also that the ideals $I_j|_U$ are generated by functions $F_j \in \mathcal{O}_M(U)$. We can write

$$F_j = \textstyle\sum_{s=0}^{\infty} G_{js}(\mathbf{x}) z^s. \tag{19}$$

Then, each ideal sheaf $N_{js}|_{H\cap U}$ is generated by $G_{js}(\mathbf{x})$.

Remark 15.2. The assumption that the idealistic space $\mathcal{M}$ is reduced guaranties that not all the ideals N_{js} are zero. Indeed, in terms of equations, if all that ideals are zero, we get that $z = 0$ must be in the singular locus.

A basic property of the projected space $\mathfrak{S}_{\mathcal{M},H}$ is the following one:

Proposition 15.3. *The singular locus of $\mathfrak{S}_{\mathcal{M},H}$ is the restriction of the singular locus of $\mathcal{M}$ to H, that is*

$$\mathrm{Sing}(\mathfrak{S}_{\mathcal{M},H}) = H \cap \mathrm{Sing}\,\mathcal{M}.$$

Proof. Follows directly by taking local equations and coordinates. □

Remark 15.4. In the above situation, consider a non-singular closed analytic subset $Y \subset H$. Then we have that Y has normal crossings with E if and only if Y has normal crossings with $E|_H$, inside H. In the case that (M, E, H) is a transverse hypersurface, the observation is straightforward. If (M, E, H) is not transverse, then H coincides locally with an irreducible component of E, that is, we locally have that $E = E^* \cup H$. The normal crossings property between Y and $E|_H = E^* \cap H$ and the fact that $Y \subset H$ assure the normal crossings property with E.

As a consequence, the permissible centers for $\mathfrak{S}_{\mathcal{M},H}$ coincide with the permissible centers for $\mathcal{M}$ that are contained in H.

16. Commutativity and equivalence

Assume that we have a projecting system $\mathfrak{S} = (\mathfrak{E}, \mathfrak{J})$ for $(\mathcal{M}, H)$. Consider a morphism

$$\sigma : (M', E') \to (M, E)$$

that is either an open projection or the blowing-up of (M,E) with center Y permissible for $\mathcal{M}$ and such that $Y \subset H$. Since $\mathcal{M}$ is adjusted-reduced, the codimension of Y in M is greater than or equal to two, hence its codimension in H is greater than or equal to one. In both cases, we have induced a morphism

$$\bar\sigma : (H', E'|_{H'}) \to (H, E|_H),$$

that is an open projection or a blowing-up centered at Y, respectively. Note that in the case of a blowing-up, the center Y is also permissible for the projected space $\mathfrak{S}_{\mathcal{M},H}$, in view of Proposition 15.3.

Let $\mathfrak{S}'$ be the transform of $\mathfrak{S}$ by σ and let $(\mathfrak{S}_{\mathcal{M},H})'$ be the transform by $\bar\sigma$ of the projected space $\mathfrak{S}_{\mathcal{M},H}$. Denote by $\mathfrak{G}'$, $\mathcal{M}'$ and H' the transform of $\mathfrak{G}$, $\mathcal{M}$ and the strict transform of H by σ, respectively.

Proposition 16.1. *We have that* $(\mathfrak{S}_{\mathcal{M},H})' = \mathfrak{S}'_{\mathcal{M}',H'}$.

Proof. We use the notations and computations done in Subsection 15.1.

Let V be a non-empty subset of M_H. The commutativity property for the restriction to V is written as

$$\mathfrak{S}_{\mathcal{M},H}|_V = (\mathfrak{S}|_V)_{\mathcal{M}|_V, H\cap V}. \tag{20}$$

It follows directly from the definitions. In the case that σ is a projection on the first factor, the commutativity property is also deduced automatically from the definitions.

Assume that σ is a permissible blowing-up for $\mathcal{M}$, centered at $Y \subset H$. The commutativity property can be checked in local coordinates. Let us fix two points $P \in Y$ and $P' \in H' \cap \pi^{-1}(P)$, a rectified projecting chart $(\mathfrak{c} = (U,\xi), \mathbf{x}, z)$ belonging to $\mathfrak{E}$, centered at P, and such that

$$Y \cap U = (z = 0) \cap (x_1 = x_2 = \cdots = x_t = 0).$$

Moreover, let us assume without loss of generality that σ is given at P' in coordinates $\mathbf{x}', z'$ as in Remark 15.1.

We know that $\mathbf{J}_j = \{hF_j;\ \partial h/\partial z = 0\}$, where F_j generates I_j. Write $F_j = \sum_{s=0}^{\infty} G_{sj}(\mathbf{x}) z^s$. Since Y belongs to the singular locus of $\mathcal{M}$, we can write

$$G_{sj}(\mathbf{x}) = (x'_1)^{d_j - s} G'_{sj}(\mathbf{x}'),$$

for the indices $s \le d_j$. The ideal I'_j is generated by

$$F'_j = (x'_1)^{-d_j} F_j(\mathbf{x}, z) = \textstyle\sum_{s=0}^{d_j-1} G'_{sj}(\mathbf{x}')(z')^s + (z')^{d_j}\tilde F_j.$$

The transform $\mathfrak{c}'$ of the chart $\mathfrak{c}$ belongs to the transformed axis $\mathfrak{E}'$ and is given by $\partial/\partial z'$ in the coordinates $\mathbf{x}', z'$. The $\mathfrak{E}'$-projectable generator of I'_j is $\mathbf{J}'_j = \{h'F'_j;\ \partial h'/\partial z' = 0\}$, locally at P'. The projected space $\mathfrak{S}_{\mathcal{M},H}$ is given (locally) by the list

$$\mathcal{N} = \{(G_{sj}\mathcal{O}_{H,P}, d_j - s)\}_{0 \le j \le k, 0 \le s \le d_j - 1},$$

and the projected space of $\mathfrak{S}'_{\mathcal{M}',H'}$ coincides with the transform of $\mathcal{N}$ and it is given by the list

$$\mathcal{N}' = \{(G'_{sj}\mathcal{O}_{H',P'}, d_j - s)\}_{0\leq j\leq k, 0\leq s\leq d_j-1}.$$

Thus, we obtain the desired commutativity property. □

16.1. Basic properties of the projections

Let us summarize here the results in Proposition 15.3, Proposition 16.1 and Remark 15.4, stated for any adjusted and reduced idealistic space $\mathcal{M}$ of (M, E) and any hypersurface (M, E, H):

- $\mathrm{Sing}(\mathfrak{S}_{\mathcal{M},H}) = H \cap \mathrm{Sing}\mathcal{M}$.
- The permissible centers for $\mathcal{M}$ contained in H coincide with the permissible centers of $\mathfrak{S}_{\mathcal{M},H}$.
- $(\mathfrak{S}_{\mathcal{M},H})' = \mathfrak{S}'_{\mathcal{M}',H'}$, for the transforms under open projections and blowing-up with permissible centers.

As a first consequence of these properties we obtain the following equivalence result:

Proposition 16.2. *Assume that $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ are two equivalent idealistic spaces over (M, E) and let $\mathfrak{S}^\alpha$ and $\mathfrak{S}^\beta$ be two projecting systems for $(\mathcal{M}_\alpha, H)$ and $(\mathcal{M}_\beta, H)$, respectively. Then $\mathfrak{S}^\alpha_{\mathcal{M}_\alpha,H}$ and $\mathfrak{S}^\beta_{\mathcal{M}_\beta,H}$ are equivalent idealistic spaces over $(H, E|_H)$.*

16.2. Projection of idealistic exponents

Consider an adjusted and reduced idealistic exponent $\mathcal{E}$ over (M, E). In this subsection we construct an idealistic exponent $\mathrm{pr}_H\,\mathcal{E}$ over $(H, E|_H)$, that we call the *projected idealistic exponent of $\mathcal{E}$ over (M, E, H)*, satisfying the three properties stated in the beginning of this Part 5.

Take a point $P \in H$. By using suitable equations we can construct a rectifiable projecting chart

$$\mathfrak{c}_P = (U_P, \xi_P)$$

for (M, E, H), where $P \in U_P \subset M_H$. Reducing the size of U_P, we also find an idealistic chart $\mathcal{U}_P = (U_P, E \cap U_P, \mathcal{L}_P)$ of $\mathcal{E}$ in such a way that the marked ideals of $\mathcal{L}_P$ are generated by global functions that have a decomposition as in Equation (19). Finally, using the equations, we obtain a projecting system $\mathfrak{S}^P = (\mathfrak{C}_P, \mathfrak{J}_P)$ for $(\mathcal{U}_P, H \cap U_P)$. Choosing a finite open covering of M_H by open subsets $\{U_\alpha\}_{\alpha\in\Lambda}$, among the U_P, we get a finite family of projecting systems $\mathfrak{S}^\alpha$ for $(\mathcal{U}_\alpha, H_\alpha)$, where $H_\alpha = H \cap U_\alpha$ and each $\mathcal{U}_\alpha$ is an idealistic chart of $\mathcal{E}$, defined in U_α.

Given two indices $\alpha, \beta \in \Lambda$, we have that $\mathcal{U}_{\alpha\beta}$ and $\mathcal{U}_{\beta\alpha}$ are equivalent, since $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ belong to $\mathcal{E}$ (we take the usual notations). Consider now the collection of idealistic charts

$$\mathfrak{S}^\alpha_{\mathcal{U}_\alpha,H_\alpha} = (H_\alpha, E_\alpha|_{H_\alpha}, \mathcal{N}_\alpha)\,, \quad \alpha \in \Lambda$$

In view of Proposition 16.2, we conclude that the family $\{\mathfrak{S}^{\alpha}_{\mathcal{U}_\alpha, H_\alpha}\}_{\alpha\in\Lambda}$ is an idealistic atlas over $(H, E|_H)$. Even more, any other idealistic atlas obtained by this procedure is equivalent to it. In this way we define without ambiguity the projected idealistic exponent $\mathrm{pr}_H\,\mathcal{E}$ of $\mathcal{E}$ over (M, E, H).

We obtain the three required properties from the case of a single idealistic space presented in Subsection 16.1.

17. Projections of e-flowers on the divisor

Let (M, E) be an ambient space and let F be an irreducible component of the divisor E, thus, we have a hypersurface (M, E, F). Consider an adjusted and reduced e-flower $\mathcal{F}$ over (M, E). In this section we build the *projection* $\mathrm{pr}_F\,\mathcal{F}$. It will be the only idealistic $(e-1)$-flower over $(F, E|_F)$, satisfying the three basic properties:

- $\mathrm{Sing}(\mathrm{pr}_F\,\mathcal{F}) = F \cap \mathrm{Sing}\,\mathcal{F}$.
- The permissible centers of $\mathcal{F}$ contained in F are exactly the permissible centers of $\mathrm{pr}_F\,\mathcal{F}$.
- We have the commutativity property $\mathrm{pr}_{F'}(\mathcal{F}') = (\mathrm{pr}_F\,\mathcal{F})'$ for the transforms under an open projection or a blowing-up with a permissible center contained in F.

The uniqueness is assured. The existence comes from the case of idealistic exponents as we detail now. Let us consider an immersed exp-idealistic e-atlas $\mathcal{Q}$ belonging to $\mathcal{F}$ given by

$$\mathcal{Q} = \{\mathcal{W}_\alpha\}_{\alpha\in\Lambda}, \quad \mathcal{W}_\alpha = (M_\alpha, E_\alpha, N_\alpha, \mathcal{E}_\alpha).$$

Take an index $\alpha \in \Lambda$, put $F_\alpha = F \cap M_\alpha$. Recall that N_α is transverse to F and hence

$$(N_\alpha, E_\alpha|_{N_\alpha}, F_\alpha \cap N_\alpha), \quad E_\alpha|_{N_\alpha} = E_\alpha \cap N_\alpha,$$

is a hypersurface of $(N_\alpha, E_\alpha|_{N_\alpha})$. Note that $F_\alpha \cap N_\alpha$ is a disjoint union of irreducible components of $E_\alpha|_{N_\alpha}$. Recalling that $\mathcal{Q}$ is adjusted and reduced, we can project $\mathcal{E}_\alpha$ over $(N_\alpha, E_\alpha|_{N_\alpha}, F_\alpha \cap N_\alpha)$ to obtain an $(e-1)$-dimensional idealistic exponent $\mathrm{pr}_{F_\alpha\cap N_\alpha}\,\mathcal{E}_\alpha$ over $(F_\alpha \cap N_\alpha, E_\alpha|_{F_\alpha\cap N_\alpha})$ and hence an immersed $(e-1)$-dimensional idealistic exponent

$$\widetilde{\mathcal{W}}_\alpha = (F_\alpha, E_\alpha|_{F_\alpha}, F_\alpha \cap N_\alpha, \mathrm{pr}_{F_\alpha\cap N_\alpha}\,\mathcal{E}_\alpha).$$

Consider the following proposition, that can be proved by a systematic use of the three basic properties of the projections of idealistic exponents:

Proposition 17.1. *The family* $\widetilde{\mathcal{Q}} = \{\widetilde{\mathcal{W}}_\alpha\}_{\alpha\in\Lambda}$ *is an* $(e-1)$*-dimensional immersed exp-idealistic atlas over the ambient space* $(F, E|_F)$*. Moreover, the following properties hold*:

(1) $\mathrm{Sing}(\widetilde{\mathcal{Q}}) = F \cap \mathrm{Sing}(\mathcal{Q})$.

(2) *The permissible centers for* $\widetilde{\mathcal{Q}}$ *are exactly the permissible centers for* $\mathcal{Q}$ *contained in* F.

(3) *We have the commutativity* $\widetilde{\mathcal{Q}'} = \widetilde{\mathcal{Q}}'$ *under open projections and blowing-ups with permissible centers contained in* F.

Now, it suffices to define $\mathrm{pr}_F\,\mathcal{F}$ to be the idealistic $(e-1)$-flower over $(F, E|_F)$ given by the atlas $\widetilde{\mathcal{Q}}$.

Part 6. Maximal contact

In this part, we end the proof of Theorem 10.8. In view of the results in the previous parts, it is enough to prove the statement corresponding to the adjusted-reduced case, presented in Equation (10) of Part 3. More precisely, we have to prove the following statement:

Proposition 17.2. *Assume that all the idealistic* $(e-1)$*-flowers have reduction of singularities. Consider an adjusted and reduced idealistic* e*-flower* $\mathcal{F}$ *over* (M, E). *Then* $\mathcal{F}$ *has reduction of singularities.*

We start by recalling the following definition of *maximal contact* in terms of idealistic flowers:

Definition. Let $\mathcal{F}$ be an idealistic e-flower over an ambient space (M, E). We say that an idealistic $(e-1)$-flower $\mathcal{H}$ over (M, E) has *maximal contact with* $\mathcal{F}$ if $\mathcal{F}$ and $\mathcal{H}$ are equivalent as idealistic flowers.

Let us also recall that being equivalent means that the two idealistic flowers $\mathcal{F}$ and $\mathcal{H}$ have the same permissible test systems.

Remark 17.3. The following ones are direct consequences of the definition:

1. If $\mathcal{H}_1$ and $\mathcal{H}_2$ have maximal contact with $\mathcal{F}$, then $\mathcal{H}_1 = \mathcal{H}_2$.
2. If there is $\mathcal{H}$ having maximal contact with $\mathcal{F}$, then $\mathcal{F}$ is reduced, since the codimension of the singular locus is greater than or equal to $e-2$.
3. If there is $\mathcal{H}$ having maximal contact with $\mathcal{F}$, then $\mathcal{F}$ is adjusted. In fact, if the order at a point is greater than one, by a curve-divisor procedure, as shown in Subsection 9.2, there is a permissible test system $\mathcal{S}$ for $\mathcal{F}$ that gives a transform $\mathcal{F}'$ whose singular locus has codimension $n-e+1$, where n is the dimension of the ambient space (M, E). By the maximal contact property, the sequence $\mathcal{S}$ is also permissible for $\mathcal{H}$ and the singular locus $\mathrm{Sing}\,\mathcal{H}'$ of the transform $\mathcal{H}'$ coincides with the singular locus of $\mathcal{F}'$. Then $\mathrm{Sing}\,\mathcal{H}'$ has codimension $n-e+1$; this is not possible since $n-e+1$ is actually the codimension of $\mathcal{H}'$.
4. Any reduction of singularities of $\mathcal{H}$ induces a reduction of singularities of $\mathcal{F}$.

Thus, in order to obtain a proof of Proposition 17.2, and hence a proof of Theorem 10.8, it is enough to prove the following statement:

Proposition 17.4. *Assume that all the idealistic* $(e-1)$*-flowers have reduction of singularities. Consider an adjusted and reduced idealistic e-flower* $\mathcal{F}$ *over* (M, E)*. We can find a morphism*

$$\sigma : (M', E') \to (M, E),$$

composition of a finite sequence of permissible blowing-ups such that there is an idealistic $(e-1)$*-flower* $\mathcal{H}'$ *over* (M', E') *having maximal contact with the transform* $\mathcal{F}'$ *of* $\mathcal{F}$ *by* σ*.*

We are going to prove Proposition 17.4 in several steps. The first step is to show how to separate the "old components of the divisor" from the singular locus. The second step is to find "maximal contact hypersurfaces", obtained from Tschirnhaus' coordinate changes, when we have an empty divisor. In this way, we are done in the special case when $E = \emptyset$. Thanks to the first step, after finitely many permissible blowing-ups, we eliminate the old components and using the hypersurfaces obtained for the case of an empty divisor, we get the idealistic $(e-1)$-flower $\mathcal{H}'$ that gives maximal contact with $\mathcal{F}'$.

18. Separating old components

Here we separate "old components" of the divisor from the singular locus. To manage the idea of "old component", we consider *splittings*

$$E = E^* \cup D,$$

where both E^* and D are unions of disjoint sets of irreducible components of E. The divisor E^* will stand for the "old components".

Assume that $\pi : (M', E') \to (M, E)$ is the blowing-up of (M, E) with a center Y having normal crossings with E. We know that

$$E' = \pi^{-1}(E \cup Y).$$

The *transformed splitting* $E' = E'^* \cup D'$ of $E = E^* \cup D$ is given by taking E'^* to be the strict transform of E^* by π and $D' = \pi^{-1}(D \cup Y)$.

If we have an open projection $\sigma : (M', E') \to (M, E)$, the *transformed splitting* is $E' = E'^* \cup D'$, where $E'^* = \sigma^{-1}(E^*)$ and $D' = \sigma^{-1}(D)$.

Proposition 18.1. *Consider an adjusted and reduced idealistic e-flower* $\mathcal{F}$ *over* (M, E)*. Take a splitting* $E = E^* \cup D$*. Assume that any idealistic* $(e-1)$*-flower has reduction of singularities. There is a composition* $\sigma : (M', E') \to (M, E)$ *of a finite sequence of permissible blowing-ups such that*

$$E'^* \cap \operatorname{Sing} \mathcal{F}' = \emptyset,$$

where $\mathcal{F}'$ *is the transform of* $\mathcal{F}$ *by* σ *and* $E' = E'^* \cup D'$ *is the transformed splitting of* $E = E^* \cup D$ *by* σ*.*

Proof. It is enough to deal with the case when $E^* = F$ is a single component of the exceptional divisor. Let $\operatorname{pr}_F \mathcal{F}$ be the projection of $\mathcal{F}$ on F as constructed in Section 17. Thanks to our hypothesis on the existence of reduction of singularities for idealistic $(e-1)$-flowers, we can take a reduction of singularities of $\mathrm{pr}_F \mathcal{F}$. In view of the three properties stated in Section 17, this reduction of singularities allows us to obtain the situation $F' \cap \operatorname{Sing} \mathcal{F}' = \emptyset$, as desired. □

19. Maximal contact hypersurfaces

Maximal contact hypersurfaces are given by the following definition:

Definition. Let $\mathcal{E}$ be an idealistic exponent over (M, E) and let (M, E, H) be a transverse hypersurface of (M, E). We say that (M, E, H) has *maximal contact with* $\mathcal{E}$ if for any $\mathcal{E}$-permissible test system $\mathcal{S}$, we have the following properties:

(1) The singular locus $\operatorname{Sing} \mathcal{E}'$ of the transform $\mathcal{E}'$ of $\mathcal{E}$ by $\mathcal{S}$ has codimension greater than or equal to two.
(2) $\operatorname{Sing} \mathcal{E}' \subset H'$, where H' is the strict transform of H by $\mathcal{S}$.

Note that the strict transform (M', E', H') of the hypersurface (M, E, H) under the $\mathcal{E}$-permissible test system $\mathcal{S}$ has also maximal contact with $\mathcal{E}'$.

Remark 19.1. By a similar argument to the one in Remark 17.3, we have that the idealistic exponent $\mathcal{E}$ is necessarily adjusted and reduced, otherwise there is no maximal contact hypersurface.

Remark 19.2. Let $\mathcal{E}$ be an idealistic exponent over an n-dimensional ambient space (M, E). Denote by $\mathcal{F}$ the idealistic n-flower over (M, E) defined by the immersed exp-idealistic n-chart

$$\mathcal{W} = (M, E, M, \mathcal{E}).$$

Assume that (M, E, H) is a hypersurface of (M, E) having maximal contact with $\mathcal{E}$. We can project $\mathcal{E}$ onto (M, E, H) to obtain an idealistic exponent

$$\widetilde{\mathcal{E}} = \mathrm{pr}_H(\mathcal{E})$$

over $(H, E|_H)$. This gives to us an immersed exp-idealistic $(n-1)$-chart

$$\widetilde{\mathcal{W}} = (M, E, H, \widetilde{\mathcal{E}})$$

that defines an idealistic $(n-1)$-flower $\widetilde{\mathcal{F}}$ over (M, E). Then $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are equivalent idealistic flowers. In other words, we have that $\widetilde{\mathcal{F}}$ has maximal contact with $\mathcal{F}$.

Remark 19.3. The following one is the basic example in the Theory of Maximal Contact. Take $M = (\mathbb{C}^n, 0)$, with coordinates $\mathbf{x}, z$, such that

$$E \subset \left(\prod_{i=1}^{n-1} x_i = 0 \right).$$

Assume that the adjusted and reduced idealistic exponent $\mathcal{E}$ contains an idealistic chart with a list having the marked ideal $(f\mathcal{O}_{\mathbb{C}^n,0}, d)$, where $\nu_0 f = d$ and f is written as

$$f = z^d + \sum_{i=2}^{d} g_i(\mathbf{x}) z^{d-i}. \tag{21}$$

The hypersurface of maximal contact is $H = (z = 0)$. The proof of this statement is founded in the classical behaviour of the Tschirnhaus form of f given in Equation (21). More precisely, one sees immediately that the singular locus is contained in $z = 0$; moreover, the Tschirnhaus form of f is stable under open projections and permissible blowing-ups at the points of the singular locus.

19.1. Maximal contact without divisor

We recall here the basic result in the Theory of Maximal Contact:

Proposition 19.4. *Let $\mathcal{E}$ be an adjusted and reduced idealistic exponent over the ambient $(M, \emptyset)$. For any $P \in \operatorname{Sing}\mathcal{E}$, there are an open set $U \subset M$ with $P \in U$ and a closed hypersurface $(U, \emptyset, H)$ having maximal contact with $\mathcal{E}|_U$.*

Proof. Up to take a smaller open subset if necessary, we can assume that there is an idealistic chart belonging to $\mathcal{E}$ of the form

$$(M, \emptyset, \mathcal{L}), \quad \mathcal{L} = \{(I, d)\} \cup \{(I_j, d_j)\}_{j=2}^{k}.$$

such that $\nu_P I = d$. In view of Weierstrass Preparation Theorem, we can choose local coordinates $\mathbf{x}, y$ around P and a generator f of $I_P \subset \mathcal{O}_{M,P}$ having the form

$$f = y^d + \tilde{g}_1(\mathbf{x}) y^{d-1} + \tilde{g}_2(\mathbf{x}) y^{d-2} + \cdots + \tilde{g}_d(\mathbf{x}).$$

Note that $\nu_P(\tilde{g}_i) \geq i$, for $i = 1, 2, \ldots, d$. Let us perform the coordinate change

$$z = y + (1/d)\tilde{g}_1(\mathbf{x}) \quad \text{(Tschirnhaus)}.$$

Then, we write f as $f = z^d + g_2(\mathbf{x}) z^{d-2} + \cdots + g_d(\mathbf{x})$. Taking $H = (z = 0)$ we obtain the maximal contact property, in view of Remark 19.3. □

19.2. Systems of maximal contact hypersurfaces

Let (M, E) be an ambient space and let us consider a splitting $E = E^* \cup D$ of E into two normal crossings divisors without common irreducible components. Let us consider an adjusted and reduced idealistic e-flower $\mathcal{F}$ over (M, E).

A *system $\mathfrak{H}$ of maximal contact hypersurfaces for $\mathcal{F}$ associated to the splitting* $E = E^* \cup D$ is a finite family

$$\mathfrak{H} = \{(\mathcal{V}_\alpha, H_\alpha, D_\alpha)\}_{\alpha \in \Lambda}, \quad \mathcal{V}_\alpha = (M_\alpha, E_\alpha, N_\alpha, \mathcal{L}_\alpha),$$

satisfying the following properties:

1. The family $\mathcal{P} = \{\mathcal{V}_\alpha\}_{\alpha \in \Lambda}$ is an immersed idealistic e-atlas over (M, E) belonging to $\mathcal{F}$.

(2) For any $\alpha \in \Lambda$, we have that $D_\alpha = D \cap M_\alpha$ and thus there is a splitting $E_\alpha = E^*_\alpha \cup D_\alpha$, induced by $E = E^* \cup D$.
(3) Each $(N_\alpha, D_\alpha|_{N_\alpha}, H_\alpha)$ is a maximal contact hypersurface for the idealistic space $(N_\alpha, D_\alpha|_{N_\alpha}, \mathcal{L}_\alpha)$. (Recall that $D_\alpha|_{N_\alpha} = D_\alpha \cap N_\alpha$, since $(M_\alpha, D_\alpha, N_\alpha)$ is a transverse ambient subspace of (M_α, D_α)).

The systems of maximal contact hypersurfaces for $\mathcal{F}$ associated to a splitting may be transformed by $\mathcal{F}$-permissible test systems in a natural way, to obtain a new system of maximal contact hypersurfaces for the transform of $\mathcal{F}$, associated to the transformed splitting. The maximal contact hypersurfaces are transformed by taking the strict transform and the immersed idealistic e-atlases are transformed as we have already seen in Section 6.

Remark 19.5. Note that we take immersed idealistic spaces $\mathcal{V}_\alpha$ instead of immersed idealistic exponents. The reason is that an idealistic exponent over (M, E) does not define an idealistic exponent over (M, D), since the permissible test systems are not the same ones. Nevertheless, an immersed idealistic space over (M, E) does define an immersed idealistic space over (M, D).

Proposition 19.6. *Let $\mathfrak{H}$ be a system of maximal contact hypersurfaces for an idealistic e-flower $\mathcal{F}$ over (M, E) associated to the splitting*

$$E = E^* \cup D, \quad E^* = \emptyset, \quad D = E.$$

Let $\mathcal{E}_\alpha$ be the idealistic exponent over $(N_\alpha, E_\alpha|_{N_\alpha})$ defined by $\mathcal{L}_\alpha$ and let $\widetilde{\mathcal{E}}_\alpha$ be the projection of $\mathcal{E}_\alpha$ over the hypersurface $(N_\alpha, E_\alpha|_{N_\alpha}, H_\alpha)$. Then, the family

$$\mathcal{Q}_{\mathfrak{H}} = \{\widetilde{\mathcal{W}}_\alpha = (M_\alpha, E_\alpha, H_\alpha, \widetilde{\mathcal{E}}_\alpha)\}_{\alpha \in \Lambda}$$

is an immersed exp-idealistic $(e-1)$-atlas over (M, E) that defines an idealistic $(e-1)$-flower $\mathcal{H}$ over (M, E) having maximal contact with $\mathcal{F}$.

Proof. We know that the e-flower $\mathcal{F}$ is described by the immersed exp-idealistic e-atlas $\mathcal{Q} = \{\mathcal{W}_\alpha\}_{\alpha \in \Lambda}$, where $\mathcal{W}_\alpha = (M_\alpha, E_\alpha, N_\alpha, \mathcal{E}_\alpha)$. In particular, we have the equivalence

$$\mathcal{W}_\alpha|_{M_{\alpha\beta}} \sim \mathcal{W}_\beta|_{M_{\alpha\beta}}.$$

In view of Remark 19.2 and Subsection 7.3, we know that

$$\mathcal{W}_\alpha \sim \widetilde{\mathcal{W}}_\alpha, \quad \mathcal{W}_\beta \sim \widetilde{\mathcal{W}}_\beta.$$

Making the restriction to $M_{\alpha\beta}$ we conclude that $\widetilde{\mathcal{W}}_\alpha|_{M_{\alpha\beta}} \sim \widetilde{\mathcal{W}}_\beta|_{M_{\alpha\beta}}$. Hence $\mathcal{Q}_{\mathfrak{H}}$ is an immersed exp-idealistic $(e-1)$-atlas over (M, E). Moreover, the equivalences $\mathcal{W}_\alpha \sim \widetilde{\mathcal{W}}_\alpha$ imply that $\mathcal{F}$ is equivalent to $\mathcal{H}$. □

20. Conclusion

Here we prove Proposition 17.4. This ends the proof of Theorem 10.8.

Let us recall that we work under the induction assumption that the idealistic $(e-1)$-flowers have reduction of singularities. Consider an adjusted and reduced idealistic e-flower $\mathcal{F}$ over an ambient space (M, E).

By Proposition 19.4, there is a system $\mathfrak{H}$ of maximal contact hypersurfaces for $\mathcal{F}$ associated to the splitting $E = E^* \cup D$, with $D = \emptyset$.

By Proposition 18.1, there is a composition $\sigma : (M', E') \to (M, E)$ of a finite sequence of $\mathcal{F}$-permissible blowing-ups such that

$$E'^* \cap \operatorname{Sing} \mathcal{F}' = \emptyset.$$

Take an open subset $U \subset M'$ containing the singular locus of $\mathcal{F}'$ such that $U \cap E'^* = \emptyset$. Now, finding a maximal contact $(e-1)$-flower $\mathcal{H}'$ for $\mathcal{F}'$ is the same problem as finding such a maximal contact $(e-1)$-flower for $\mathcal{F}'|_U$. Let $\mathfrak{H}'$ be transformed of $\mathfrak{H}$ by σ and consider the restriction $\mathfrak{H}'|_U$. Since $U \cap E'^* = \emptyset$, we have that $E' \cap U = D' \cap U$. We are in the situation of Proposition 19.6, that provides the desired idealistic $(e-1)$-flower of maximal contact with $\mathcal{F}'|_U$. This ends the proof. □

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