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Journal of Differential Equations 435 (2025) 113315

Journal of Differential Equations

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Global bifurcation diagrams for coercive third-degree polynomial ordinary differential equations with recurrent nonautonomous coefficients ☆

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Received 6 January 2025; accepted 6 April 2025
Available online 22 April 2025

Abstract

Nonautonomous bifurcation theory is a growing branch of mathematics, for the insight it provides into radical changes in the global dynamics of realistic models for many real-world phenomena, i.e., into the occurrence of critical transitions. This paper describes several global bifurcation diagrams for nonautonomous first order scalar ordinary differential equations generated by coercive third degree polynomials in the state variable. The conclusions are applied to a population dynamics model subject to an Allee effect that is weak in the absence of migration and becomes strong under a migratory phenomenon whose sense and intensity depend on a threshold in the number of individuals in the population.

MSC: 37B55; 37G35; 37N25

☆ C. Elia partially supported by the GNCS-Indam group and the Ministero dell'Università e della Ricerca under research grant PRIN2022PNR P2022M7JZW SAFER MESH - "Sustainable mAnagement oF watEr Resources modEls and numerical MetHods". C. Núñez was supported by Ministerio de Ciencia, Innovación y Universidades (Spain) under project PID2021-125446NB-I00 and by Universidad de Valladolid under project PIP-TCESC-2020, and she is a member of IMUVA.

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https://doi.org/10.1016/j.jde.2025.113315

Keywords: Nonautonomous dynamical systems; Nonautonomous bifurcation theory; Critical transitions; Population models

1. Introduction

Bifurcation theory is a branch of the study of dynamical systems that dates back to the early works of Poincaré [41] at the end of the 19th century. Much more recent is the extension of this theory to non-autonomous dynamical systems. The analysis of these systems arises from the need of the applied branches of science to describe models whose own laws of evolution change with respect to time, which generally allows a more realistic description of the phenomenon. All these models depend on parameters, and very often a small variation in one of these parameters causes a strong variation in the resulting global dynamics. Understanding the mechanisms of occurrence of these changes and, closely related, describing the dynamics for close values of the parameter are, broadly speaking, the objectives of nonautonomous bifurcation theory.

The most common approach to autonomous bifurcation theory for one-parametric families of *scalar* ordinary differential equations (ODEs in what follows) analyzes the evolution, as the parameter varies, of the number and type of critical points, which correspond to the constant solutions of the ODEs. They are classified into hyperbolic attractive, hyperbolic repulsive, and nonhyperbolic, and often determine the global phase line. The object of study is not so clear in the nonautonomous extension of the theory, since a scalar time-dependent ODE $x' = f(t, x)$ does not admit, in general, constant solutions. So, although the overall objective is basically always the same, there is not total agreement on where to place the focus for the analysis. Different approaches are presented in the works of Braaksma et al. [6], Johnson and Mantellini [26], Fabbri et al. [16], Kloeden [29], Langa et al. [32], Rasmussen [44,45], Núñez and Obaya [38,39], Jäger [23], Pötzsche [42,43], Kloeden and Rasmussen [30], Anagnostopoulou and Jäger [1], Anagnostopoulu et al. [2,3], Fuhrmann [17], Longo et al. [34], Remo et al. [46], and Dueñas et al. [11,12,14], as well as in the references therein.

In this work, following in the wake of [11,12,14], we analyze the bifurcation problem given by the variation in ε of an ε-parametric family of third degree coercive polynomial nonautonomous ODEs,

$$x' = -x^3 + \bar{c}(t)\, x^2 + \varepsilon \left(\bar{b}(t)\, x + \bar{a}(t)\right), \tag{1.1}$$

determined by three bounded and uniformly continuous maps $\bar{c}$, $\bar{b}$, $\bar{a}\colon \mathbb{R} \to \mathbb{R}$. With the approach previously established in [38,39,34], we use the skew-product formalism, defining from (1.1) a (possibly local) real continuous flow τ_ε on the trivial bundle $\Omega \times \mathbb{R}$, where Ω is the hull of $(\bar{c}, \bar{b}, \bar{a})$. That is, Ω is the (compact) closure in the compact-open topology of $C(\mathbb{R}, \mathbb{R}^3)$ of the set of time-shifts $\{(\bar{c}, \bar{b}, \bar{a})\cdot t \mid t \in \mathbb{R}\}$, where $\bar{d}\cdot t(s) = \bar{d}(t+s)$. Defining $c(\omega) = \omega_1(0)$, $b(\omega) = \omega_2(0)$ and $a(\omega) = \omega_3(0)$ for $\omega = (\omega_1, \omega_2, \omega_3) \in \Omega$, we obtain, for each $\varepsilon \in \mathbb{R}$, the family of equations

$$x' = -x^3 + c(\omega{\cdot}t)\, x^2 + \varepsilon \left(b(\omega{\cdot}t)\, x + a(\omega{\cdot}t)\right), \qquad \omega \in \Omega, \tag{1.2}$$

whose solutions $v_\varepsilon(t, \omega, x)$ satisfying $v_\varepsilon(0, \omega, x) = x$ yield the fiber-component of the flow τ_ε, which is of skew-product type: $\tau_\varepsilon(t, \omega, x) = (\omega{\cdot}t, v_\varepsilon(t, \omega, x))$. Observe that (1.1) is (1.2) for $\omega = (\bar{c}, \bar{b}, \bar{a})$. We assume the time-shift flow on the hull Ω to be minimal and uniquely ergodic (which is the situation in many nonautonomous mathematical models, as those determined by an almost periodic function $(\bar{c}, \bar{b}, \bar{a})\colon \mathbb{R} \to \mathbb{R}^3$), and choose the minimal subsets of $\Omega \times \mathbb{R}$ as the objects whose variation in number and type (hyperbolic attractive, hyperbolic repulsive of nonhyperbolic) determine the occurrence of bifurcation values of ε. The most basic minimal set is the so-called *copy of the base*, which is the (invariant) graph of a continuous map $\mathfrak{b}_\varepsilon\colon \Omega \to \mathbb{R}$ such that $\mathfrak{b}_\varepsilon(\omega{\cdot}t) = v_\varepsilon(t, \omega, \mathfrak{b}_\varepsilon(\omega))$ for all $(t, \omega) \in \mathbb{R} \times \Omega$: this is the natural extension of a critical value in the autonomous case. So, our approach is quite natural, although unlike in the autonomous framework there may be minimal subsets with a highly complicated dynamics. Some of the first samples of these extremely complex minimal sets, that include strange non-chaotic attractors, can be found in Millionščikov [36,37], Vinograd [51] (see also Lipnitskii [33] for some technical improvements), Johnson [24], and Koltyzhenkov [31] (and in Grebogi et al. [19], Bezhaeva and Oseledets [5], and Keller [28] for discrete instead of continuous flows). In the bifurcation diagrams described in [38,39,34,11,12,14] we observe a phenomenon that appears frequently in the literature: these complex sets can appear only at the bifurcation values of the parameter. This will also be the situation of the problem studied here. So, we get one more sample that the degree of rarity of these sets depends not only on their intern dynamics, but also on their extreme lack of persistence under quite standard perturbations.

Returning to the particular case of (1.2), it is enough to work with autonomous examples to see that the possibilities of the bifurcation diagram are very numerous. In order for the result of our analysis to be of reasonable length, we need to make certain choices at the beginning. Modifying these assumptions will substantially change the results, but the study of many of the cases that we do not consider in this paper can be carried out using the same techniques: classical general methods of topological dynamics and ergodic theory combined with new results and techniques, in the line of those developed in [38,40,11,14]. The main results of this paper are obtained under the conditions $\inf_{\omega\in\Omega} d(\omega) > 0$ for $d = c,\ b,\ -a$; and, like in the autonomous case (with $c > 0$ $b > 0$ and $a < 0$), the relative sizes of c and $-a/b$ determine very different bifurcation situations.

In all these autonomous bifurcation diagrams, only two types of bifurcations appear: local saddle-node bifurcations, when two branches of hyperbolic critical points exist to the left (or right) of ε_0 and collide at this point, giving rise to a unique nonhyperbolic critical point at ε_0 and to the local absence of critical points to its right (or left); and local transcritical bifurcation, when two branches of hyperbolic critical points exist both at the left and right of ε_0, and they collide at a unique nonhyperbolic critical point at ε_0. In the nonautonomous setting, we say that ε_0 is a local saddle-node bifurcation point when two hyperbolic copies of the base which exist for close values of $\varepsilon < \varepsilon_0$ (or $\varepsilon > \varepsilon_0$) approach each other as $\varepsilon \to (\varepsilon_0)^-$ (or as $\varepsilon \to (\varepsilon_0)^+$) until they collide at least at a point (and simultaneously at all the points of a residual subset of Ω), giving rise to a locally unique τ_{ε_0}-minimal set, which is nonhyperbolic, and to the absence of minimal sets “nearby” for close $\varepsilon > \varepsilon_0$ (or $\varepsilon < \varepsilon_0$). And we say that ε_0 is a local transcritical bifurcation point when two hyperbolic copies of the base exist for close values of ε and approach each other as $\varepsilon \to \varepsilon_0$ until they collide at ε_0, giving rise to a locally unique τ_{ε_0}-minimal set which is nonhyperbolic. For our problem, roughly speaking, we prove that

- 0 is always a local saddle-node bifurcation point, which appears as the result of the global collision of two hyperbolic copies of the base as $\varepsilon \to (0)^+$.
- When $\sup_{\omega\in\Omega} c(\omega) < \inf_{\omega\in\Omega}(-a(\omega)/b(\omega))$, there are at least two more values of (possibly partial) collision of hyperbolic copies of the base: $\varepsilon^* > \varepsilon_* > 0$; the three values are local saddle-node bifurcation points if, in addition, a is a real multiple of b; and they are the unique ones if the oscillation of c is not too strong.

- When $\inf_{\omega\in\Omega} c(\omega) > \sup_{\omega\in\Omega}(-a(\omega)/b(\omega))$, there are no strictly positive bifurcation values. Additional conditions determine either the absence of negative bifurcation values or the existence of exactly two of them, also of saddle-node type.
- When $c(\omega) = -a(\omega)/b(\omega) = s$, with s constant, there is a strictly positive bifurcation value, of local transcritical type, and none negative.

The results outlined above are better understood by having a look to the depictions in Figs. 1, 2 and 3, in Section 3. In all the situations, the analysis also involves a description of the evolution of the global attractor $\mathcal{A}_\varepsilon$ of the flow τ_ε and, in most of the cases, the bifurcation values are points of discontinuity of the map $\varepsilon \mapsto \mathcal{A}_\varepsilon$. The hypotheses and results, assumed and proved for the skew-product, are easily rewritten for the initial family (1.1). In this reformulation, instead of considering the evolution in the type and number of minimal sets, we focus on the number and type of hyperbolic solutions.

The results are applied to describe the evolution of a single population in a given habitat, subject to an Allee effect (see Courchamp at al. [9]) which is weak in the absence of migration, and to a particular type of migration whose intensity depends on a threshold in the number of individuals in the habitat. The bifurcation points can be read in terms of critical transitions (see Scheffer [48]): significant changes in the state of a complex system that occur as consequences of small variations in its inputs.

We complete the Introduction with a brief sketch of the structure of the paper. Section 2 contains the basic concepts and properties required to understand the rest of the paper: we introduce the skew-product framework we work in; we recall the concepts of equilibria, hyperbolic and nonhyperbolic minimal set, and global attractor; we summarize some properties of the Lyapunov exponents; and we describe with more detail the hull construction outlined above. The core of the paper is Section 3, where we obtain the global bifurcation diagrams mentioned above (and additional results under less restrictive hypotheses, not described in these first paragraphs), and where we indicate how to particularize each of them to a parametric family of processes instead of flows. Finally, in Section 4, we apply our previous results to analyze the occurrence of critical transitions in a particular population dynamics model.

2. Some preliminary results

In this section we recall the main concepts and tools required to prove the main results in Section 3. A *(real and continuous) flow* on a topological space $\mathbb{Y}$ is a continuous map $\sigma\colon \mathcal{V} \subseteq \mathbb{R}\times\mathbb{Y} \to \mathbb{Y}$ defined in an open subset $\mathcal{V} \supseteq \{0\}\times\mathbb{Y}$ such that, for all $y \in \mathbb{Y}$, $\sigma(0, y) = y$ and $\sigma(t + s, y) = \sigma(t, \sigma(s, y))$ if the right-hand term is defined. It is *global* if $\mathcal{V} = \mathbb{R}\times\mathbb{Y}$. The definitions of orbit, forward and backward semiorbit, invariant set, α-limit set and ω-limit set, which we omit, can be found in the basic texts of topological dynamics, as [15]. We also omit the definitions of (regular) invariant and ergodic measures for the flow: see, e.g., [35].

Let (Ω, σ) be a global flow on a compact metric space Ω, and let us consider the family of equations

$$x' = f(\omega{\cdot}t, x)\,, \qquad \omega\in\Omega\,, \tag{2.1}$$

where $f \in C^{0,1}(\Omega\times\mathbb{R}, \mathbb{R})$; i.e., we assume that the partial derivative f_x globally exists, and that f and f_x are jointly continuous. (In Section 2.3 we will briefly explain how such a family arises from a suitable single ODE.) We represent by τ the (possibly local) skew-product flow induced by (2.1) on $\Omega \times \mathbb{R}$, namely

$$\tau\colon \mathcal{V} \subseteq \mathbb{R} \times \Omega \times \mathbb{R} \to \Omega \times \mathbb{R}\,, \quad (t, \omega, x) \mapsto (\omega{\cdot}t, v(t, \omega, x)) \tag{2.2}$$

where $\mathcal{V} \supseteq \{0\} \times \Omega \times \mathbb{R}$ is an open subset and $v(t, \omega, x)$ is the maximal solution of the equation (2.1) corresponding to ω with initial condition $v(0, \omega, x) = x$. We will write $v'(t, \omega, x) = f(\omega{\cdot}t, v(t, \omega, x))$. So, v' represents $(d/dt)\, v$.

2.1. Compact invariant sets, upper and lower solutions, and global attractor

The next concepts will play a fundamental role in some of the proofs. A map $\mathfrak{b}\colon \Omega \to \mathbb{R}$ is a *τ-equilibrium* if $\mathfrak{b}(\omega{\cdot}t) = v(t, \omega, \mathfrak{b}(\omega))$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$. A *τ-copy of the base* is the graph of a continuous τ-equilibrium. (The prefix τ will be sometimes omitted if there is no risk of confusion.)

Let $\mathcal{K} \subset \Omega \times \mathbb{R}$ be a compact τ-invariant set projecting onto the whole set Ω. The set $\mathcal{K}$ is *pinched* if the section $(\mathcal{K})_\omega := \{x \in \mathbb{R} \mid (\omega, x) \in \mathcal{K}\}$ reduces to a point at least at a point $\omega \in \Omega$. It is easy to check that its *lower* (resp. *upper*) *equilibrium*, given by $\mathfrak{l}_{\mathcal{K}}(\omega) := \sup\{x \in \mathbb{R} \mid (\omega, x) \in \mathcal{K}\}$ (resp. $\mathfrak{u}_{\mathcal{K}}(\omega) := \inf\{x \in \mathbb{R} \mid (\omega, x) \in \mathcal{K}\}$) is a lower (resp. upper) semicontinuous equilibrium, and hence it is continuous at the points of a residual subset of Ω.

The flow (Ω, σ) (or the set Ω) is *minimal* if every σ-orbit is dense in Ω. A τ-invariant compact subset $\mathcal{M} \subset \Omega \times \mathbb{R}$ is τ-minimal if $(\mathcal{M}, \tau|_{\mathcal{M}})$ is minimal; or, equivalently, if the τ-orbit of any element of $\mathcal{M}$ is dense in $\mathcal{M}$. Any τ-invariant compact set contains a τ-minimal set. If Ω is minimal, then any τ-invariant compact set projects on the whole set Ω, and any copy of the base is a τ-minimal set. As already mentioned, the copies of the base are the simplest minimal sets, playing in many cases the equivalent role of the equilibrium points for autonomous ODEs. We represent by $\mathcal{M} = \{\mathfrak{b}\}$ the τ-minimal set defined by a continuous copy of the base $\mathfrak{b}\colon \Omega \to \mathbb{R}$.

The next result, for a minimal base flow, is basically proved in [7, Section 2]. A more detailed proof is given in [10, Proposition 1.32 and Corollary 1.33].

Proposition 2.1. *Let the flow (Ω, σ) be minimal.*

(i) *Let $\mathfrak{b}\colon \Omega \to \mathbb{R}$ be a semicontinuous equilibrium and let ω_0 be any continuity point of $\mathfrak{b}$. Then,*

$$\mathcal{M}^b = \mathrm{cl}_{\Omega\times\mathbb{R}}\{(\omega_0{\cdot}t, \mathfrak{b}(\omega_0{\cdot}t)) \mid t \in \mathbb{R}\} \tag{2.3}$$

is a minimal set, it is independent of the choice of ω_0, and its section $(\mathcal{M}^b)_\omega$ reduces to $\mathfrak{b}(\omega)$ for all the points of the residual σ-invariant subset of Ω given by the continuity points of $\mathfrak{b}$. In addition, the sections $(\mathcal{M})_\omega$ of any τ-minimal set $\mathcal{M} \subset \Omega \times \mathbb{R}$ are singletons for all the points ω in a residual σ-invariant subset of Ω.

(ii) *Two different τ-minimal sets $\mathcal{M}_1$ and $\mathcal{M}_2$ are fiber-ordered: if $x_1^0 < x_2^0$ for two points $(\omega^0, x_1^0) \in \mathcal{M}_1$ and $(\omega^0, x_2^0) \in \mathcal{M}_2$, then $x_1 < x_2$ whenever $(\omega, x_1) \in \mathcal{M}_1$ and $(\omega, x_2) \in \mathcal{M}_2$.*

A *bounded global lower solution* for $x' = f(\omega{\cdot}t, x)$ is a bounded map $\mathfrak{b}\colon \Omega \to \mathbb{R}$ such that $t \mapsto \mathfrak{b}(\omega{\cdot}t)$ is C^1 and $\mathfrak{b}'(\omega) \le f(\omega, \mathfrak{b}(\omega))$ for all $\omega \in \Omega$, where $\mathfrak{b}'(\omega) = (d/dt)\, \mathfrak{b}(\omega{\cdot}t)|_{t=0}$, and it is *strict* if the inequality is strict for all $\omega \in \Omega$. By changing the sign of the inequalities we obtain the definition of (*strict*) *bounded global upper solution*.

The constant lower and upper solutions r (respectively characterized by the conditions $f(\omega, r) > 0$ and $f(\omega, r) < 0$ for all $\omega \in \Omega$) will be a useful tool for many points in the proofs of the main results. The next property will be used often:

Proposition 2.2. *Let $m_1 < m_2$ be real constants, and assume that one of them is a global upper solution and the other one a lower global solution. Then, there exists a minimal set contained in $\Omega \times [m_1, m_2]$. If, in addition, m_1 (resp. m_2) is strict, then the minimal set is contained in $\Omega \times (m_1, m_2]$ (resp. $\Omega \times [m_1, m_2)$).*

Proof. We choose any $\omega \in \Omega$. It is easy to check that $\Omega \times [m_1, m_2]$ contains the forward (resp. backward) semiorbit of $(\omega, (m_1 + m_2)/2)$ if m_2 (resp. m_1) is the global upper solution. Hence, $\Omega \times [m_1, m_2]$ also contains a minimal subset of the ω-limit set (resp. α-limit set) of this orbit. Let us prove the last assertion in the case that m_2 is a strict global upper solution, assuming for contradiction the existence of a point $(\bar{\omega}, m_2)$ in the ω-limit set of $(\omega, (m_1 + m_2)/2)$. Then, since $v'(0, \bar{\omega}, m_2) = f(\bar{\omega}, v(0, \bar{\omega}, m_2)) = f(\bar{\omega}, m_2) < (m_2)' = 0$, we get $v(t, \bar{\omega}, m_2) > m_2$ for small $t < 0$. But this is impossible, since the ω-limit set is τ-invariant and contained in $\Omega \times [m_1, m_2]$. The proofs of the three remaining cases are analogous. □

A set $\mathcal{A} \subset \Omega \times \mathbb{R}$ is the *global attractor for the flow* τ if it is a compact τ-invariant set that attracts every bounded set $\mathcal{C} \subset \Omega \times \mathbb{R}$. This attraction property means that all the forward τ-semiorbits of points of $\mathcal{C}$ are globally defined (i.e., $[0, \infty) \times \mathcal{C} \subset \mathcal{V}$) and that $\lim_{t \to \infty} \operatorname{dist}(\tau_t(\mathcal{C}), \mathcal{A}) = 0$, where $\tau_t(\mathcal{C}) := \{\tau(t, \omega, x) \mid (\omega, x) \in \mathcal{C}\}$ and

$$\operatorname{dist}(\mathcal{C}_1, \mathcal{C}_2) = \sup_{(\omega_1, x_1) \in \mathcal{C}_1} \left(\inf_{(\omega_2, x_2) \in \mathbb{C}_2} \left(\operatorname{dist}_{\Omega \times \mathbb{R}}((\omega_1, x_1), (\omega_2, x_2)) \right) \right).$$

The next properties are proved in [11, Theorem 5.1].

Theorem 2.3. *Assume the coercivity condition $\lim_{x \to \pm\infty} f(\omega, x) = \mp\infty$ uniformly on Ω. Then all the forward semiorbits are global (i.e., $[0, \infty) \times \Omega \times \mathbb{R} \subset \mathcal{V}$), and there exists the global attractor $\mathcal{A}$ for τ, which is given by the union of the graphs of all the bounded solutions of the family of equations (2.1) and takes the form*

$$\mathcal{A} = \bigcup_{\omega \in \Omega} \left(\{\omega\} \times [\mathfrak{l}(\omega), \mathfrak{u}(\omega)] \right).$$

Remarks 2.4. In what follows we give some properties of global attractors, bounded global upper and lower solutions and copies of the base that will be needed in Section 3. The coercivity condition of Theorem 2.3 is assumed.

1. If there exists $r_1 \in \mathbb{R}$ (resp. $r_2 \in \mathbb{R}$) such that $f(\omega, x) > 0$ if $x \le r_1$ (resp. $f(\omega, x) < 0$ if $x \ge r_2$) for all $\omega \in \Omega$, then $\mathcal{A} \subset \Omega \times (r_1, \infty)$ (resp. $\mathcal{A} \subset \Omega \times (-\infty, r_2)$). Let us prove the first assertion, assuming for contradiction that $l := \inf_{\omega \in \Omega} \mathfrak{l}(\omega) = \mathfrak{l}(\bar{\omega}) \le r_1$. Then, $\mathfrak{l}(\bar{\omega}{\cdot}t) = v(t, \bar{\omega}, \mathfrak{l}(\bar{\omega})) = v(t, \bar{\omega}, l) < l$ for $t < 0$, and this is impossible. Similarly, if $f(\omega, x) > 0$ for $x < r_1$ and for all $\omega \in \Omega$, then $\mathcal{A} \subset \Omega \times [r_1, \infty)$. The remaining proofs are analogous.

2. If $\mathfrak{b} \colon \Omega \to \mathbb{R}$ is C^1 along the base orbits and $\mathfrak{b}'(\omega) \le f(\omega, \mathfrak{b}(\omega))$ (resp. $\mathfrak{b}'(\omega) \ge f(\omega, \mathfrak{b}(\omega))$) for all $\omega \in \Omega$, then $\mathfrak{b} \le \mathfrak{u}$ (resp. $\mathfrak{b} \ge \mathfrak{l}$). If $\mathfrak{b}'(\omega) < f(\omega, \mathfrak{b}(\omega))$ (resp. $\mathfrak{b}'(\omega) >$

$f(\omega, \mathfrak{b}(\omega)))$ for all $\omega \in \Omega$, then $\mathfrak{b} < \mathfrak{u}$ (resp. $\mathfrak{b} > \mathfrak{l}$). These properties, based on classical comparison arguments, are proved in [11, Theorem 5.1(iii)].

3. If $\mathfrak{b}\colon \Omega \to \mathbb{R}$ is upper semicontinuous and a bounded global (strict) upper solution, then its graph is (strictly) above the ω-limit set $\mathcal{O}$ of any point $(\omega_0, \mathfrak{b}(\omega_0))$ (i.e., of the corresponding orbit); that is, $x \le \mathfrak{b}(\omega)$ ($x < \mathfrak{b}(\omega)$) for any point $(\omega, x) \in \mathcal{O}$. And if, in addition, there exists the α-limit set of a point $(\omega_0, \mathfrak{b}(\omega_0))$, then this set is (strictly) above the graph of $\mathfrak{b}$. Consequently, in the strict case, no point $(\omega_0, \mathfrak{b}(\omega_0))$ belongs to any minimal set. Analogous properties with reverse orders hold in the case of a bounded global (strict) lower solution given by a lower semicontinuous maps. The proofs of these properties are based on comparison results: we first prove the non-strict inequalities, and then deduce the strict ones in the strict cases by easy contradiction arguments, solving the equation in the reverse sense of the time.

2.2. *Lyapunov exponents and hyperbolic minimal sets*

Let $\mathcal{K} \subset \Omega \times \mathbb{R}$ be a τ-invariant compact set projecting onto the whole set Ω. A value $\gamma \in \mathbb{R}$ is a *Lyapunov exponent of* $\mathcal{K}$ if there exists $(\omega, x) \in \mathcal{K}$ such that

$$\gamma = \lim_{t\to\pm\infty} (1/t) \int_0^t f_x(\tau(r, \omega, x))\, dr\,. \tag{2.4}$$

Let us assume that (Ω, σ) is uniquely ergodic, and let us call m the unique σ-invariant (and ergodic) measure. Using Riesz' Representation Theorem, Kryloff-Bogoliuboff's Theorem, Birkhoff's Ergodic Theorem, [18, Theorem 4.1] and [4, Theorem 1.8.4], it is possible to check that γ is a Lyapunov exponent of $\mathcal{K}$ if and only if there exists an m-measurable equilibrium $\mathfrak{b}\colon \Omega \to \mathbb{R}$ with graph contained in $\mathcal{K}$ such that

$$\gamma = \int_\Omega f_x(\omega, \mathfrak{b}(\omega))\, dm\,.$$

A detailed proof of this assertion, in a much more general case, can be found in [10, Sections 1.1.3 and 1.2.4].

A τ-copy of the base $\{\mathfrak{b}\}$ is *hyperbolic attractive* if it is uniformly exponentially stable (on the fiber) as time increases; i.e., if there exists $\rho > 0$, $k \ge 1$ and $\gamma > 0$ such that: if, for any $\omega \in \Omega$, $|\mathfrak{b}(\omega) - x| < \rho$, then $v(t, \omega, x)$ is defined for all $t \ge 0$, and in addition $|\mathfrak{b}(\omega{\cdot}t) - v(t, \omega, x)| \le k\, e^{-\gamma t}\, |\mathfrak{b}(\omega) - x|$ for $t \ge 0$. Changing $t \ge 0$ by $t \le 0$ and γ by $-\gamma$ provides the definition of *repulsive hyperbolic* τ-copy of the base. We will also say that a τ-minimal set is *hyperbolic attractive* (resp. *repulsive*) if it is a hyperbolic attractive (resp. repulsive) τ-copy of the base; and, otherwise, it is *nonhyperbolic*.

Remark 2.5. An attractive (resp. repulsive) hyperbolic copy of the base $\{\mathfrak{b}\}$ does not intersect the α-limit set (resp. ω-limit set) of any (ω, x) with $x \ne \mathfrak{b}(\omega)$. This intuitive property is proved in [13, Proposition 2.6(ii)].

The next result, which will be repeatedly used, is basically proved in [7, Corollary 2.10 and Theorem 3.4], and a more detailed proof of (i) and (ii) is included in the proof of [10, Theorem 1.40].

Theorem 2.6. *Let the flow (Ω, σ) be minimal. Then,*

(i) *a minimal set is hyperbolic attractive if and only if its upper Lyapunov exponent is negative.*
(ii) *A minimal set is hyperbolic repulsive if and only if its lower Lyapunov exponent is positive.*
(iii) *If the coercivity condition of Theorem 2.3 holds, then the global attractor $\mathcal{A}$ is an attractive hyperbolic τ-copy of the base if and only if all the τ-minimal sets are hyperbolic attractive.*

In particular, in the uniquely ergodic case, with ergodic measure m, a τ-copy of the base $\{\mathfrak{b}\}$ is: hyperbolic attractive if and only if $\int_\Omega f_x(\omega, \mathfrak{b}(\omega))\, dm < 0$, hyperbolic repulsive if and only if $\int_\Omega f_x(\omega, \mathfrak{b}(\omega))\, dm > 0$, and (hence) nonhyperbolic if and only if $\int_\Omega f_x(\omega, \mathfrak{b}(\omega))\, dm = 0$. And, in the conditions of (iii)*, $\mathcal{A}$ is an attractive hyperbolic copy of the base if and only if $\int_\Omega f_x(\omega, \mathfrak{b}(\omega))\, dm < 0$ for any m-measurable bounded τ-equilibrium.*

2.3. The hull construction

Let us now consider a single ODE

$$x' = \bar{f}(t, x) \tag{2.5}$$

where $\bar{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ belongs to $C^{0,1}(\mathbb{R}, \mathbb{R})$; i.e., the derivative $\bar{f}_x$ with respect to x globally exists, and the restrictions of the maps f and f_x to $\mathbb{R} \times \mathcal{J}$ are bounded and uniformly continuous for any compact set $\mathcal{J} \subset \mathbb{R}$. Let us define $\bar{f}_t(s, x) := \bar{f}(t+s, x)$. The *hull* Ω of $\bar{f}$ is the closure of the set $\{\bar{f}_t \mid t \in \mathbb{R}\}$ on the set $C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ provided with the compact-open topology. Then: the set Ω is a compact metric space contained in $C^{0,1}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, the time-shift map $\sigma : \mathbb{R} \times \Omega \to \Omega$, $(t, \omega) \mapsto \omega{\cdot}t := \omega{\cdot}t$ defines a global continuous flow, and the map f given by $f(\omega, x) = \omega(0, x)$ belongs to $C^{0,1}(\Omega \times \mathbb{R}, \mathbb{R})$. The proof of these properties can be found in [49, Theorem IV.3] and [50, Theorem I.3.1]. Note that (2.5) is one of the equations of the corresponding family (2.1): it is given by the element $\omega = \bar{f} \in \Omega$. Note also that (Ω, σ) is a *transitive flow*, i.e., there exists a dense σ-orbit: that of the point $\bar{f}$. The map $\bar{f}$ is *recurrent* if (Ω, σ) is a *minimal flow*.

The flow τ given by (2.2) from the family (2.1) constructed from (2.5) is the *skew-product flow induced by $\bar{f}$ on its hull*. A standard procedure in nonautonomous dynamics is: to construct this skew-product flow, use techniques from topological dynamics and ergodic theory to describe the behavior of its orbits, and derive consequences for the dynamics induced by (2.5). This is basically the approach of this paper: the results are formulated for minimal and uniquely ergodic flows; but then we show how to extract conclusions for a single recurrent equation giving rise to a uniquely ergodic hull, and apply them to the analysis of a particular model.

3. Some global bifurcation diagrams

Let (Ω, σ) be a global real continuous flow on a compact metric space, minimal and uniquely ergodic, and let m be the unique σ-invariant measure on Ω. Let $a, b, c : \Omega \to \mathbb{R}$ be continuous maps, and let us consider the one-parameter family of families of scalar ODEs

$$x' = p_\varepsilon(\omega{\cdot}t, x)\,, \qquad \omega \in \Omega\,, \tag{3.1}$$

where ε varies in $\mathbb{R}$ and

$$p_\varepsilon(\omega,x) := -x^3 + c(\omega)\,x^2 + \varepsilon\left(b(\omega)\,x + a(\omega)\right). \tag{3.2}$$

Recall that a family of this type appears by the hull procedure from a single ODE: see Sections 2.3 and 3.2. We will write $(3.1)_\varepsilon$ and $(3.1)_\varepsilon^\omega$ to refer to the ω-family for a fixed ε and to a particular equation, respectively. We also represent by τ_ε the (possibly local) skew-product flow induced by $(3.1)_\varepsilon$ on $\Omega\times\mathbb{R}$, so that

$$\tau_\varepsilon\colon \mathcal{V}_\varepsilon \subseteq \mathbb{R}\times\Omega\times\mathbb{R} \to \Omega\times\mathbb{R}\,,\quad (t,\omega,x)\mapsto (\omega{\cdot}t, v_\varepsilon(t,\omega,x))$$

where $\mathcal{V}_\varepsilon \supset \{0\}\times\Omega\times\mathbb{R}$ is open. Note that p_ε satisfies the most restrictive conditions of [11], which are the coercivity property $\lim_{x\to\pm\infty} p_\varepsilon(\omega,x)/x = -\infty$ uniformly on Ω, and the strict concavity of the derivative of $x\mapsto p_\varepsilon(\omega,x)$ for all $\omega\in\Omega$. Some of the results of that paper, as well as some of [13] (in turn, strongly based on [40] and [34]), will be used in the description of the possibilities for the global τ_ε-dynamics. As stated in Theorem 2.3, the coercivity condition ensures that $v_\varepsilon(t,\omega,x)$ is defined and bounded for all $t\ge 0$ (i.e., $[0,\infty)\times\Omega\times\mathbb{R}\subset\mathcal{V}_\varepsilon$), and there exists the global attractor $\mathcal{A}_\varepsilon$ for τ_ε, which is given by the union of the graphs of all the bounded solutions of the family of equations $(3.1)_\varepsilon$, and takes the form

$$\mathcal{A}_\varepsilon = \bigcup_{\omega\in\Omega}\left(\{\omega\}\times[\mathfrak{l}_\varepsilon(\omega),\mathfrak{u}_\varepsilon(\omega)]\right).$$

Recall also (see Section 2.1) that the maps $\mathfrak{l}_\varepsilon,\,\mathfrak{u}_\varepsilon\colon\Omega\to\mathbb{R}$ are lower and upper semicontinuous equilibria, respectively, and that each of them is continuous at the points of a residual subset of Ω.

Theorem 3.1. *There are three possibilities for the number of τ_ε-minimal sets:*

(1) *There are exactly three τ_ε-minimal sets. In this case, they are copies of the base: $\{\mathfrak{l}_\varepsilon\}$, $\{\mathfrak{m}_\varepsilon\}$ and $\{\mathfrak{u}_\varepsilon\}$, with $\mathfrak{l}_\varepsilon<\mathfrak{m}_\varepsilon<\mathfrak{u}_\varepsilon$. In addition, $\{\mathfrak{l}_\varepsilon\}$ and $\{\mathfrak{u}_\varepsilon\}$ are hyperbolic attractive and $\{\mathfrak{m}_\varepsilon\}$ is hyperbolic repulsive.*
(2) *There are exactly two τ_ε-minimal sets. In this case, there are two possibilities: either $\{\mathfrak{l}_\varepsilon\}$ is an attractive hyperbolic τ_ε-copy of the base and the other one, nonhyperbolic, is constructed as the closure of $\{(\omega_0{\cdot}t,\mathfrak{u}_\varepsilon(\omega_0{\cdot}t))\mid t\in\mathbb{R}\}$ for a continuity point ω_0 of $\mathfrak{u}_\varepsilon$, and it is a pinched set; or $\{\mathfrak{u}_\varepsilon\}$ is an attractive hyperbolic τ_ε-copy of the base and the other one, nonhyperbolic, is constructed as the closure of $\{(\omega_0{\cdot}t,\mathfrak{l}_\varepsilon(\omega_0{\cdot}t))\mid t\in\mathbb{R}\}$ for a continuity point ω_0 of $\mathfrak{l}_\varepsilon$, and it is a pinched set.*
(3) *There is only one τ_ε-minimal set, in which case $\mathfrak{l}_\varepsilon$ and $\mathfrak{u}_\varepsilon$ coincide on the residual set of common continuity points, and hence the global attractor is a pinched set. If this minimal set is hyperbolic, then it is an attractive hyperbolic τ_ε-copy of the base, given by $\{\mathfrak{l}_\varepsilon\}=\{\mathfrak{u}_\varepsilon\}$, and it coincides with $\mathcal{A}_\varepsilon$.*

Proof. The existence of the global attractor (which is compact and τ_ε-invariant) ensures the existence of at least one τ_ε-minimal set: see Section 2.1. According to [11, Theorem 4.2], there are at most three of them, and the situation is that of (1) if there are three. Let us define $\mathcal{M}_\varepsilon^l$ and $\mathcal{M}_\varepsilon^u$ from $\mathfrak{l}_\varepsilon$ and $\mathfrak{u}_\varepsilon$ as in Proposition 2.1(i). Since any minimal set projects on the whole set Ω (see Section 2.1), the existence of more than one of them ensures that $\mathfrak{l}_\varepsilon(\omega)<\mathfrak{u}_\varepsilon(\omega)$ for all $\omega\in\Omega$. So, if there are exactly two, then they are $\mathcal{M}_\varepsilon^l$ and $\mathcal{M}_\varepsilon^u$: see again Proposition 2.1(i).

In addition, [11, Theorem 5.13(iii)] ensures that one of them is an attractive hyperbolic copy of the base, and it follows from [11, Proposition 5.3(ii)] that the other one is nonhyperbolic. So, we are in the situation (2). Finally, if there exists exactly one minimal set, then $\mathcal{M}^l_\varepsilon = \mathcal{M}^u_\varepsilon$, and Proposition 2.1(i) guarantees that $\mathfrak{l}_\varepsilon(\omega) = \mathfrak{u}_\varepsilon(\omega)$ at all the common continuity points of both maps; so, the section $(\mathcal{A}_\varepsilon)_\omega$ reduces to one element at these points, and hence $\mathcal{A}_\varepsilon$ is pinched. If, in addition, $\mathcal{M}^l_\varepsilon = \mathcal{M}^u_\varepsilon$ is hyperbolic, then [11, Proposition 5.3(i)] precludes the possibility that it is repulsive, so it is attractive. Hence, "all" the minimal sets are hyperbolic attractive, which, according to Theorem 2.6(iii), ensures that $\mathcal{A}_\varepsilon$ is an attractive hyperbolic copy of the base. That is, the situation is that described in (3). □

Remarks 3.2. 1. Whenever the dynamics of $(3.1)_\varepsilon$ fits in situation (1) of Theorem 3.1, we represent by $\{\mathfrak{m}_\varepsilon\}$ the repulsive hyperbolic τ_ε-copy of the base.

2. A detailed description of the global dynamics (i.e., of the asymptotic behavior of the solutions) can be done in each case of Theorem 3.1. We omit this, which is basically done in [11,13], and we refer to the numerical simulations of Section 4 for some clues in this regard.

As said in the Introduction, we will perform our analysis in the case that $a(\omega) < 0$ and $c(\omega) > 0$ for all $\omega \in \Omega$. These are the unique conditions required in Theorem 3.4 (and in the auxiliary Proposition 3.3) to establish the first basic bifurcation properties.

Proposition 3.3. *Assume that $a(\omega) < 0$ and $c(\omega) > 0$ for each $\omega \in \Omega$. Then, there exists $\varepsilon_0 > 0$ such that*

(i) *the map $x \mapsto p_\varepsilon(\omega, x)$ has three real roots if $\varepsilon \in (0, \varepsilon_0]$ for all $\omega \in \Omega$: $x^1_\varepsilon(\omega) > x^2_\varepsilon(\omega) > 0 > x^3_\varepsilon(\omega)$. In addition, $\lim_{\varepsilon\to 0^+}(x^1_\varepsilon(\omega) - c(\omega)) = 0$ and $\lim_{\varepsilon\to 0^+} x^2_\varepsilon(\omega) = \lim_{\varepsilon\to 0^+} x^3_\varepsilon(\omega) = 0$, and the three limits are uniform on Ω.*

(ii) *The map $x \mapsto p_\varepsilon(\omega, x)$ has only one real root if $\varepsilon \in [-\varepsilon_0, 0)$, $x^1_\varepsilon(\omega)$, with $\lim_{\varepsilon\to 0^-}(x^1_\varepsilon(\omega) - c(\omega)) = 0$ uniformly on Ω. In addition, $x^1_\varepsilon(\omega) > 0$ for all $\omega \in \Omega$.*

Proof. A classical algebraic result (see, e.g., [20, Exercises 10.14 and 10.17]) establishes that the existence of one or three real roots of the third degree polynomial $p_\varepsilon(\omega, x)$ depends on the sign of its *discriminant* $\Delta_\varepsilon(\omega)$, given by

$$\Delta_\varepsilon := \varepsilon\left(-4\,a\,c^3 + \varepsilon\,b^2\,c^2 - 18\,\varepsilon\,a\,b\,c - 27\,\varepsilon\,a^2 + 4\,\varepsilon^2\,b^3\right): \tag{3.3}$$

there is only one real root if $\Delta_\varepsilon(\omega) < 0$ and three of them if $\Delta_\varepsilon(\omega) > 0$. Hence, since $\Delta_\varepsilon(\omega)$ is jointly continuous in (ε, ω), and since $\lim_{\varepsilon\to 0}\Delta_\varepsilon(\omega)/\varepsilon = -4\,a(\omega)\,c^3(\omega) > 0$, there are three real roots $x^1_\varepsilon(\omega) > x^2_\varepsilon(\omega) > x^3_\varepsilon(\omega)$ if $\varepsilon > 0$ is small enough, and a real root $x^1_\varepsilon(\omega)$ (plus two complex ones) if $-\varepsilon > 0$ is small enough. In addition, the roots (considered as complex numbers) can be written as continuous maps of the coefficients $\varepsilon\,a$, $\varepsilon\,b$ and c of p_ε. Hence, the limits of the three solutions as $\varepsilon \to 0$ are $c(\omega)$, 0 and 0 (the roots of $x^2\,(c(\omega) - x) = p_0(\omega, x) = \lim_{\varepsilon\to 0} p_\varepsilon(\omega, x)$), and they are uniform on Ω (since the limits $\lim_{\varepsilon\to 0}\varepsilon\,a(\omega) = \lim_{\varepsilon\to 0}\varepsilon\,b(\omega) = 0$ are uniform on Ω). In both cases, the upper (unique for small $\varepsilon < 0$) real solution converges to $c(\omega)$ as $\varepsilon \to 0$, so $x^1_\varepsilon > 0$ for $|\varepsilon| > 0$ small enough. Finally, for $\varepsilon > 0$, $p_\varepsilon(\omega, 0) = \varepsilon\,a(\omega) < 0$ and $\lim_{x\to -\infty} p_\varepsilon(\omega, x) = \infty$. So, if there were more than one negative root, there would be no positive ones, which as just seen is precluded for $\varepsilon > 0$ small enough. The conclusion is $x^1_\varepsilon(\omega) > x^2_\varepsilon(\omega) > 0 > x^3_\varepsilon(\omega)$ for such an $\varepsilon > 0$. □

We fix some notation which will be used in the rest of the paper:

$$
\begin{aligned}
a_- &:= \inf_{\omega\in\Omega} a(\omega) \quad \text{and} \quad a_+ := \sup_{\omega\in\Omega} a(\omega)\,,\\
b_- &:= \inf_{\omega\in\Omega} b(\omega) \quad \text{and} \quad b_+ := \sup_{\omega\in\Omega} b(\omega)\,,\\
c_- &:= \inf_{\omega\in\Omega} c(\omega) \quad \text{and} \quad c_+ := \sup_{\omega\in\Omega} c(\omega)\,.
\end{aligned}
$$

Recall that we say that there exists a (nonautonomous) local saddle-node bifurcation point at ε_0 when two hyperbolic copies of the base exist for $\varepsilon < \varepsilon_0$ (or $\varepsilon > \varepsilon_0$) close to ε_0 and they approach each other as $\varepsilon \to (\varepsilon_0)^-$ (or as $\varepsilon \to (\varepsilon_0)^+$), giving rise to a locally unique nonhyperbolic τ_{ε_0}-minimal set $\mathcal{M}_{\varepsilon_0}$, and to the absence of minimal sets "nearby $\mathcal{M}_{\varepsilon_0}$" for close $\varepsilon > \varepsilon_0$ (or $\varepsilon < \varepsilon_0$).

Theorem 3.4. *Assume that $a(\omega) < 0$ and $c(\omega) > 0$ for each $\omega \in \Omega$. Then,*

- (i) *the unique τ_0-minimal sets are $\{\mathfrak{l}_0\} = \{0\}$, which is nonhyperbolic, and $\{\mathfrak{u}_0\}$, which is hyperbolic attractive and satisfies $c_- \le \mathfrak{u}_0 \le c_+$. In addition, either $\mathfrak{u}_0 \equiv c$ and these maps are constant or, for all $\omega \in \Omega$, there exists a strictly increasing two-sided sequence $(t_n)_{n\in\mathbb{Z}}$ with $\lim_{n\to\pm\infty} t_n = \pm\infty$ such that $c(\omega{\cdot}t_{2n}) - \mathfrak{u}_0(\omega{\cdot}t_{2n}) > 0$ and $c(\omega{\cdot}t_{2n+1}) - \mathfrak{u}_0(\omega{\cdot}t_{2n+1}) < 0$.*
- (ii) *For all $\varepsilon > 0$, $\mathfrak{l}_\varepsilon < 0$, $\{\mathfrak{l}_\varepsilon\}$ is an attractive hyperbolic τ_ε-copy of the base, and $(0,\infty) \to C(\Omega,\mathbb{R})$, $\varepsilon \mapsto \mathfrak{l}_\varepsilon$ is a continuous map in the uniform topology of $C(\Omega,\mathbb{R})$.*
- (iii) *The set $\mathcal{I} := \{\varepsilon_+ > 0 \,|\,$ there are three hyperbolic τ_ε-copies of the base for all $\varepsilon \in (0,\varepsilon_+)\}$ is nonempty and open; $\mathfrak{l}_\varepsilon < 0 < \mathfrak{m}_\varepsilon < \mathfrak{u}_\varepsilon$ for all $\varepsilon \in \mathcal{I}$; $\lim_{\varepsilon\to 0^+}(\mathfrak{u}_\varepsilon(\omega) - \mathfrak{u}_0(\omega)) = \lim_{\varepsilon\to 0^+}\mathfrak{l}_\varepsilon(\omega) = \lim_{\varepsilon\to 0^+}\mathfrak{m}_\varepsilon(\omega) = 0$, all of them uniformly on Ω; the maps $\mathcal{I}\cup\{0\} \to C(\Omega,\mathbb{R})$, $\varepsilon \mapsto \mathfrak{l}_\varepsilon, \mathfrak{m}_\varepsilon, \mathfrak{u}_\varepsilon$ are continuous in the uniform topology of $C(\Omega,\mathbb{R})$, where $\mathfrak{m}_0 := 0$; and there exists $\varepsilon_0 \in (0, \sup\mathcal{I}]$ such that the maps $(0,\varepsilon_0) \to C(\Omega,\mathbb{R})$, $\varepsilon \mapsto -\mathfrak{l}_\varepsilon, \mathfrak{m}_\varepsilon$ are strictly increasing.*
- (iv) *For all $\varepsilon < 0$, $\mathfrak{u}_\varepsilon > 0$.*
- (v) *If, in addition, $c_+ < 3\,c_-$, then there exists $\varepsilon_- < 0$ such that, if $\varepsilon \in (\varepsilon_-, 0)$, then $\mathcal{A}_\varepsilon = \{\mathfrak{u}_\varepsilon\}$ is the unique τ_ε-minimal set; it is hyperbolic attractive; and $\lim_{\varepsilon\to 0^-}(\mathfrak{u}_\varepsilon(\omega) - \mathfrak{u}_0(\omega)) = 0$ uniformly on Ω. In particular, there is a local saddle-node bifurcation at $\varepsilon = 0$.*

Proof. (i) Since $p_0(\omega,x) = -x^3 + c(\omega)\,x^2$, the (unique) Lyapunov exponent of the τ_0-copy of the base $\{0\}$ is $\int_\Omega (p_0)_x(\omega,0)\,dm = 0$, and hence $\{0\}$ is a nonhyperbolic τ_0-minimal set: see Theorem 2.6. In particular, $\mathfrak{l}_0 \le 0$. Since $p_0(\omega,r) = -r^2(r - c(\omega)) > 0$ for all $r < 0$, Remark 2.4.1 ensures that $\mathfrak{l}_0 \ge 0$, and hence $\mathfrak{l}_0 \equiv 0$. In addition, $p_0(\omega,c_-) = (c_-)^2(c(\omega) - c_-) \ge 0$ and $p_0(\omega,c_+) = (c_+)^2(c(\omega) - c_+) \le 0$, and hence Proposition 2.2 ensures the existence of a minimal set contained in $\Omega \times [c_-,c_+] \subset \Omega \times (0,\infty)$. So, we are necessarily in case (2) of Theorem 3.1, and hence the second minimal set is $\{\mathfrak{u}_0\}$ and it is hyperbolic attractive. The first assertions in (i) are proved.

Now, observe that $\mathfrak{u}_0'(\omega{\cdot}t)/\mathfrak{u}_0(\omega{\cdot}t) = \mathfrak{u}_0(\omega{\cdot}t)(-\mathfrak{u}_0(\omega{\cdot}t) + c(\omega{\cdot}t))$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$. So, if $\mathfrak{u}_0 \equiv c$, then $t \mapsto \mathfrak{u}_0(\omega{\cdot}t)$ is constant for all ω and hence the continuous map $\omega \mapsto \mathfrak{u}_0(\omega) = c(\omega)$ is constant by the minimality of the base. Otherwise, Birkhoff's Ergodic Theorem yields $0 = \int_\Omega (\mathfrak{u}_0'(\omega)/\mathfrak{u}_0(\omega))\,dm = \int_\Omega \mathfrak{u}_0(\omega)(-\mathfrak{u}_0(\omega) + c(\omega))\,dm = 0$, which precludes the global inequalities $c > \mathfrak{u}_0$ and $c < \mathfrak{u}_0$. Hence, the sets $\mathcal{U}_\pm := \{\omega \in \Omega \,|\, \pm(\mathfrak{u}_0(\omega) - c(\omega)) > 0\}$ are nonempty and open. The minimality of Ω yields, for a fixed $\omega \in \Omega$, two increasing sequences

$(s_n^{\pm})_{n\in\mathbb{N}}$ with limit ∞ and two decreasing ones $(\bar{s}_n^{\pm})_{n\in\mathbb{N}}$ with limit $-\infty$ such that $\omega{\cdot}s_n^{\pm}\in\mathcal{U}_{\pm}$ and $\omega{\cdot}\bar{s}_n^{\pm}\in\mathcal{U}_{\pm}$, which implies the existence of the sequence $(t_n)_{n\in\mathbb{Z}}$ of the last assertion in (i).

(ii)&(iv) Since $p_\varepsilon(\omega,0)=\varepsilon\,a(\omega)$, we have $(0)'=0>p_\varepsilon(\omega,0)$ if $\varepsilon>0$ and $(0)'=0<p_\varepsilon(\omega,0)$ if $\varepsilon<0$ for all $\omega\in\Omega$. So, Remark 2.4.2 guarantees $\mathfrak{l}_\varepsilon<0$ for $\varepsilon>0$ (in (ii)) and property (iv). To prove the second assertion in (ii), we fix $\varepsilon>0$, define

$$q_\varepsilon(\omega,x):=\begin{cases}-x^3+c(\omega)\,x^2+\varepsilon\,\big(b(\omega)\,x+a(\omega)\big) & \text{if } x\le 0\,,\\ c(\omega)\,x^2+\varepsilon\,\big(b(\omega)\,x+a(\omega)\big) & \text{if } x>0\,,\end{cases}$$

and consider the induced skew-product $\bar{\tau}_\varepsilon(t,\omega,x)=(\omega{\cdot}t,\bar{v}_\varepsilon(t,\omega,x))$. It is easy to check that q_ε is globally continuous and C^2 with respect to x, and that its second derivative $(q_\varepsilon)_{xx}$ is strictly positive. Hence, $x\mapsto q_\varepsilon(\omega,x)$ is strictly convex for all ω, which ensures that $y\mapsto -q_\varepsilon(\omega,y)$ is strictly concave for all ω. Let us consider the time-reversed flow $\sigma^-\colon\mathbb{R}\times\Omega\to\Omega$, $(t,\omega)\mapsto\omega{\cdot}(-t)$. It is easy to check that (Ω,σ^-) is minimal and uniquely ergodic. Note that $-q_\varepsilon(\omega,0)=-\varepsilon\,a(\omega)>0$ for all $\omega\in\Omega$, and that there exists $r>0$ such that $-q_\varepsilon(\omega,\pm r)<0$ for all $\omega\in\Omega$. Hence, Proposition 2.2 applied to the skew-product flow $(\Omega\times\mathbb{R},\bar{\tau}_\varepsilon^-)$ defined from $y'=-q_\varepsilon(\omega{\cdot}(-t),y)$ (over (Ω,σ^-)) ensures the existence of at least two minimal sets, strictly below and above $\Omega\times\{0\}$. Consequently, the dynamics for $\bar{\tau}_\varepsilon^-$ is determined by an attractor-repeller pair of copies of the base (see, e.g., [13, Theorem 3.6]), and this ensures that the lower minimal set is a repulsive hyperbolic $\bar{\tau}_\varepsilon^-$-copy of the base, and that there are no bounded $\bar{\tau}_\varepsilon^-$-orbits below it. The change $x(t)=y(-t)$ takes $y'=-q_\varepsilon(\omega{\cdot}(-t),y)$ to $x'=q_\varepsilon(\omega{\cdot}t,x)$, and it is easy to deduce from $\bar{\tau}_\varepsilon^-(t,\omega,x)=(\omega{\cdot}(-t),\bar{v}_\varepsilon(-t,\omega,x))$ that the lower minimal set is an attractive hyperbolic $\bar{\tau}_\varepsilon$-copy of the base, say $\{\bar{\mathfrak{l}}_\varepsilon\}$, and that there are no bounded $\bar{\tau}_\varepsilon$-orbits below it. Since each one of the maps $t\mapsto\bar{\mathfrak{l}}_\varepsilon(\omega{\cdot}t)$ and $t\mapsto\mathfrak{l}_\varepsilon(\omega{\cdot}t)$ (bounded and negative) solve $x'=p_\varepsilon(\omega{\cdot}t,x)$ and $x'=q_\varepsilon(\omega{\cdot}t,x)$, $\bar{\mathfrak{l}}_\varepsilon\ge\mathfrak{l}_\varepsilon$ (since $t\mapsto\mathfrak{l}_\varepsilon(\omega{\cdot}t)$ is the lower bounded solution for $x'=p_\varepsilon(\omega{\cdot}t,x)$), and $\mathfrak{l}_\varepsilon\ge\bar{\mathfrak{l}}_\varepsilon$ (since $t\to\bar{\mathfrak{l}}_\varepsilon(\omega{\cdot}t)$ is the lower bounded solution for $x'=q_\varepsilon(\omega{\cdot}t,x)$). Therefore, they are equal. In particular, $\int_\Omega p_x(\omega{\cdot}t,\mathfrak{l}_\varepsilon(\omega))\,dm=\int_\Omega q_x(\omega{\cdot}t,\bar{\mathfrak{l}}_\varepsilon(\omega))\,dm$ (where m is the unique σ-invariant measure on Ω). That is, the unique Lyapunov exponent for $\{\mathfrak{l}_\varepsilon\}$ for the flow τ_ε coincides with that of $\{\bar{\mathfrak{l}}_\varepsilon\}$ for $\bar{\tau}_\varepsilon$, and hence Theorem 2.6 ensures that it is negative (since it is the unique Lyapunov exponent of the attractive hyperbolic $\bar{\tau}_\varepsilon$-copy of the base $\{\bar{\mathfrak{l}}_\varepsilon\}$) and that $\{\mathfrak{l}_\varepsilon\}$ is an attractive hyperbolic τ_ε-copy of the base (since its unique Lyapunov exponent is negative).

Finally, the classical result of robustness of the existence of hyperbolic copies of the base and their continuous variation in the uniform topology (see, e.g., [14, Theorem 2.3]) proves the continuity of $\varepsilon\mapsto\mathfrak{l}_\varepsilon$ stated in (ii).

(iii) Let us take $\rho\in(0,c_-)$. Proposition 3.3(i) allows us to take $\varepsilon_\rho>0$ small enough to ensure that $x_\varepsilon^3(\omega)<0<x_\varepsilon^2(\omega)<\rho<x_\varepsilon^1(\omega)$ for all $\omega\in\Omega$ and $\varepsilon\in(0,\varepsilon_\rho)$. Since $p_\varepsilon(\omega,x)=-(x-x_\varepsilon^1(\omega))(x-x_\varepsilon^2(\omega))(x-x_\varepsilon^3(\omega))$, we have $p_\varepsilon(\omega,\rho)>0$ for all $\omega\in\Omega$. We also look for $k_-<0$ and $k_+>\rho$ such that $p_\varepsilon(\omega,k_-)>0$ and $p_\varepsilon(\omega,k_+)<0$ for all $\varepsilon\in[0,\varepsilon_\rho]$ and all $\omega\in\Omega$. Then: k_- and ρ are strict global upper solutions, and 0 and k_+ are strict global lower solutions. Since $k_-<0<\rho<k_+$, Proposition 2.2 ensures the existence of three minimal sets $\mathcal{M}_\varepsilon^1$, $\mathcal{M}_\varepsilon^2$ and $\mathcal{M}_\varepsilon^3$, with $\mathcal{M}_\varepsilon^1\subset\Omega\times(k_-,0)$, $\mathcal{M}_\varepsilon^2\subset\Omega\times(0,\rho)$ and $\mathcal{M}_\varepsilon^3\subset\Omega\times(\rho,k_+)$. Theorem 3.1 ensures that $\mathcal{M}_\varepsilon^1=\{\mathfrak{l}_\varepsilon\}$ (attractive), $\mathcal{M}_\varepsilon^2=\{\mathfrak{m}_\varepsilon\}$ (repulsive), and $\mathcal{M}_\varepsilon^3=\{\mathfrak{u}_\varepsilon\}$ (attractive). So, $(0,\varepsilon_\rho)\subseteq\mathcal{I}$, with $\mathcal{I}$ defined in (iii). The robustness of the existence of hyperbolic copies of the base and their continuous variation in the uniform topology (see [14, Theorem 2.3]) prove that $\mathcal{I}$ is open and that the maps $\mathcal{I}\to C(\Omega,\mathbb{R})$, $\varepsilon\mapsto\mathfrak{l}_\varepsilon$, $\mathfrak{m}_\varepsilon$, $\mathfrak{u}_\varepsilon$ are continuous. Note also that, if $\varepsilon>0$ the inequality $p_\varepsilon(\omega,0)<0$ for all ω precludes the existence of a point $(\omega,0)$ in any τ_ε-minimal set $\mathcal{M}_\varepsilon$: see Remark 2.4.3. So, since $\mathfrak{m}_\varepsilon > 0$ for small $\varepsilon > 0$ and $\varepsilon \to \mathfrak{m}_\varepsilon$ is continuous on $\mathcal{I}$, we conclude that $\mathfrak{m}_\varepsilon > 0$ and hence $\mathfrak{l}_\varepsilon < 0 < \mathfrak{m}_\varepsilon < \mathfrak{u}_\varepsilon$ for all $\varepsilon \in \mathcal{I}$.

Keeping the notation of Proposition 3.3, we define $(x^3_\varepsilon)_- := \inf_{\omega\in\Omega} x^3_\varepsilon(\omega)$ for $\varepsilon > 0$ small enough. Then, $p_\varepsilon(\omega, r) > 0$ for all $r < (x^3_\varepsilon)_-$, which ensures that $\mathfrak{l}_\varepsilon > (x^3_\varepsilon)_-$: see Remark 2.4.1. Proposition 3.3 ensures that $\lim_{\varepsilon\to 0^+}(x^3_\varepsilon)_- = 0$, which combined with $l_\varepsilon < 0$ for $\varepsilon > 0$ yields $\lim_{\varepsilon\to 0^+} \mathfrak{l}_\varepsilon(\omega) = 0$ uniformly on Ω. Now, we choose the initial $\rho > 0$ as close to 0 as desired, and observe that $0 < \mathfrak{m}_\varepsilon < \rho$ if ε is small enough. This proves the assertion concerning $\lim_{\varepsilon\to 0^+} \mathfrak{m}_\varepsilon$. Finally, [14, Theorem 2.3] ensures the existence of an attractive hyperbolic copy of the base as uniformly close as desired to $\{\mathfrak{u}_0\}$ for small $\varepsilon > 0$, which must be $\{\mathfrak{u}_\varepsilon\}$, and this proves the assertion about $\lim_{\varepsilon\to 0^+}(\mathfrak{u}_\varepsilon - \mathfrak{u}_0)$. These properties and (i) complete the proof of the continuity on $\mathcal{I} \cup \{0\}$.

The proofs of the monotonicity properties of $\varepsilon \mapsto \mathfrak{l}_\varepsilon, \mathfrak{m}_\varepsilon$ take arguments from the proof of [11, Theorem 5.10], which we repeat here for the reader's convenience. The previous uniform limiting properties allow us to ensure that, if $\varepsilon > 0$ is small enough, then $b(\omega)\, x + a(\omega) < 0$ if $x \in [\mathfrak{l}_\varepsilon(\omega), \mathfrak{m}_\varepsilon(\omega)]$ for all $\omega \in \Omega$. We choose ε_0 so that this holds for $\varepsilon \in (0, \varepsilon_0)$. If $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$, then $\mathfrak{l}'_{\varepsilon_1}(\omega) = p_{\varepsilon_1}(\omega, \mathfrak{l}_{\varepsilon_1}(\omega)) > p_{\varepsilon_2}(\omega, \mathfrak{l}_{\varepsilon_1}(\omega))$, and hence $\mathfrak{l}_{\varepsilon_1} > \mathfrak{l}_{\varepsilon_2}$ (see Remark 2.4.2). Let us complete the proof of (iii) checking that $\varepsilon \mapsto \mathfrak{m}_\varepsilon$ is strictly increasing on $(0, \varepsilon_0)$. We fix $\varepsilon_1 \in (0, \varepsilon_0)$. The previously checked continuous variation of the copies of the base allows us to take $\varepsilon_2 < \varepsilon_1$ in $(0, \varepsilon_0)$ close enough to ensure $\mathfrak{l}_{\varepsilon_1} < \mathfrak{m}_{\varepsilon_2} < \mathfrak{u}_{\varepsilon_1}$. So, for a fixed $\bar\omega \in \Omega$, $v_{\varepsilon_1}(t, \bar\omega, \mathfrak{m}_{\varepsilon_2}(\bar\omega)) > v_{\varepsilon_1}(t, \bar\omega, \mathfrak{l}_{\varepsilon_1}(\bar\omega)) = \mathfrak{l}_{\varepsilon_1}(\bar\omega{\cdot}t)$ for all $t \in \mathbb{R}$: the τ_{ε_1}-orbit of $(\bar\omega, \mathfrak{m}_{\varepsilon_2}(\bar\omega))$ is above $\{\mathfrak{l}_{\varepsilon_1}\}$, and hence globally bounded. This ensures the existence of the corresponding α-limit set. In addition, $\mathfrak{m}'_{\varepsilon_2}(\omega) < p_{\varepsilon_1}(\omega, \mathfrak{m}_{\varepsilon_2}(\omega))$ for all $\omega \in \Omega$, since $\varepsilon_2\,(b(\omega)\,\mathfrak{m}_{\varepsilon_2}(\omega) + a(\omega)) < \varepsilon_1\,(b(\omega)\,\mathfrak{m}_{\varepsilon_2}(\omega) + a(\omega))$. According to Remark 2.4.3, the previous α-limit set is strictly below the graph of $\mathfrak{m}_{\varepsilon_2}$. Let $\mathcal{N}$ be a τ_{ε_1}-minimal contained in this α-limit set. This α-limit set cannot intersect $\{\mathfrak{l}_{\varepsilon_1}\}$ or $\{\mathfrak{u}_{\varepsilon_1}\}$ (see Remark 2.5), and hence $\mathcal{N} = \{\mathfrak{m}_{\varepsilon_1}\}$. This ensures that $\mathfrak{m}_{\varepsilon_1} < \mathfrak{m}_{\varepsilon_2}$, as asserted.

(v) We assume that $c_+ < 3\,c_-$ and work with $\varepsilon < 0$. Initially, we take values of $\varepsilon < 0$ close enough to 0 to ensure that, for all $\omega \in \Omega$, $x^1_\varepsilon(\omega) > 0$ is the unique real root of $x \mapsto p_\varepsilon(\omega, x)$ (see Proposition 3.3), and define $(x^1_\varepsilon)_- := \inf_{\omega\in\Omega} x^1_\varepsilon(\omega)$. Then $\mathfrak{l}_\varepsilon > (x^1_\varepsilon)_-$, since $p_\varepsilon(\omega, r) > 0$ for all $r < (x^1_\varepsilon)_-$ (see again Remark 2.4.1). Since $\lim_{\varepsilon\to 0^-}(x^1_\varepsilon(\omega) - c(\omega)) = 0$ uniformly on Ω (see again Proposition 3.3), we have $\lim_{\varepsilon\to 0^-}(x^1_\varepsilon)_- = c_-$. We fix $r \in (c_+/3, c_-)$ and look for $\varepsilon_- < 0$ close enough to 0 to ensure that $(x^1_\varepsilon)_- > r$ for all $\varepsilon \in (\varepsilon_-, 0)$. Now, we fix $\varepsilon \in (\varepsilon_-, 0)$ and define $q_\varepsilon(\omega, x)$ as the $C^{0,2}(\Omega \times \mathbb{R}, \mathbb{R})$ function which coincides with $p_\varepsilon(\omega, x)$ for $x \ge r$ and is given by a second degree polynomial for $x \le r$. In particular, $q_\varepsilon(\omega, r) > 0$. In addition, for all $x \le r$, $(\partial^2/\partial x^2)\, q_\varepsilon(\omega, x) = (\partial^2/\partial x^2)\, q_\varepsilon(\omega, r) = (\partial^2/\partial x^2)\, p_\varepsilon(\omega, r) = -6r + 2\,c(\omega) \le -6r + 2\,c_+ < 0$. And, if $x > r$, then $(\partial^2/\partial x^2)\, q_\varepsilon(\omega, x) = (\partial^2/\partial x^2)\, p_\varepsilon(\omega, x) = -6x + 2\,c(\omega) < -6r + 2\,c_+ < 0$. That is, the map $x \mapsto q_\varepsilon(\omega, x)$ is strictly concave for all $\omega \in \Omega$. Moreover, $\lim_{x\to-\infty} q_\varepsilon(\omega, x) = -\infty$ (as in the case of a concave second-degree polynomial) and $\lim_{x\to\infty} q_\varepsilon(\omega, x) = \lim_{x\to\infty} p_\varepsilon(\omega, x) = -\infty$, uniformly on Ω in both cases. These properties mean that q_ε satisfies all the conditions **c1-c4** of [13, Section 3]. Since $\mathfrak{l}_\varepsilon \ge (x^1_\varepsilon)_- > r$, any τ_ε-minimal set is contained in $\Omega \times (r, \infty)$, and hence is also a $\tilde\tau_\varepsilon$-minimal set for the skew-product flow $\tilde\tau_\varepsilon$ defined on $\Omega \times \mathbb{R}$ by $x' = q_\varepsilon(\omega{\cdot}t, x)$. Conversely, any $\tilde\tau_\varepsilon$-minimal set contained in $\Omega \times (r, \infty)$ is also a τ_ε-minimal set. Since $q_\varepsilon(\omega, r) > 0$ and $q_\varepsilon(\omega, \pm s) < 0$ for all $\omega \in \Omega$ if s is large enough, Proposition 2.2 ensures the existence of a $\tilde\tau_\varepsilon$-minimal set $\mathcal{M}^u_\varepsilon$ in $\Omega \times (r, \infty)$ and of another in $\Omega \times (-\infty, r)$. According to [13, Theorem 3.3], $\mathcal{M}^u_\varepsilon$ is an attractive hyperbolic $\tilde\tau_\varepsilon$-copy of the base, and the unique $\tilde\tau_\varepsilon$-minimal set above r. The conclusion is that $\mathcal{M}^u_\varepsilon$ is also the unique τ_ε-minimal set. Since its Lyapunov exponents are the same for p_ε as for q_ε (due to the equality $(p_\varepsilon)_x = (q_\varepsilon)_x$ on $\Omega \times (r, \infty)$), they are negative, and Theorem 2.6 ensures that $\mathcal{A}_\varepsilon = \mathcal{M}_\varepsilon$ is an attractive hyperbolic τ_ε-copy of the base, as asserted. Finally, the last assertion in (v) follows from the previous ones and (iii). $\square$

Remarks 3.5. 1. Observe that the hypothesis $c_+ < 3\,c_-$ of Theorem 3.4(v) always holds if the map c is a positive constant. Otherwise, the range of "allowed" values of c increases as c_- increases. Note also that, if all the conditions assumed in Theorem 3.4(v) hold, then the local saddle-node bifurcation at $\varepsilon = 0$ has an extra property: the collision as $\varepsilon \to 0^+$ of the two approaching hyperbolic τ_ε-copies of the base is total, giving rise to the nonhyperbolic τ_0-copy of the base $\{0\}$. So, in contrast to the possibly very complex dynamics at the bifurcation values $\varepsilon \neq 0$ which we will find later (due to the possibly very complex dynamics of the nonhyperbolic minimal set), here the dynamics for $\varepsilon = 0$ is simple: a nonautonomous reproduction of the autonomous dynamics around $\varepsilon = 0$ of, for instance, $x' = -x^3 + x^2 - \varepsilon$. In fact, since $p_0(\omega, r) > 0$ for all $r \in (0, c_-)$, it is easy to check that $\{0\}$ is the α-limit set of the τ_0-orbit of all $(\omega, r) \in \Omega \times [0, c_-)$ and that $\{\mathfrak{u}_0\}$ is the unique τ_0-minimal set contained in the ω-limit set.

2. If, under the hypotheses of Theorem 3.4, we also assume $b \equiv 0$, then an easy extension of the results of [11] (see also [14]) provides a complete description of the global bifurcation diagram of (3.1):

- there are exactly two bifurcation points, 0 and a certain $\varepsilon_* > 0$;
- the maps $(-\infty, \varepsilon_*) \to C(\Omega, \mathbb{R})$, $\varepsilon \mapsto \mathfrak{u}_\varepsilon$ and $(0, \infty) \to C(\Omega, \mathbb{R})$, $\varepsilon \mapsto \mathfrak{l}_\varepsilon$ are continuous and strictly decreasing, and $\{\mathfrak{u}_\varepsilon\}$ (resp. $\{\mathfrak{l}_\varepsilon\}$) is an attractive hyperbolic τ_ε-copy of the base for all $\varepsilon < \varepsilon_*$ (resp. $\varepsilon > 0$);
- $\mathcal{A}_\varepsilon = \{\mathfrak{l}_\varepsilon\} = \{\mathfrak{u}_\varepsilon\}$ for $\varepsilon \notin [0, \varepsilon_*]$, with $\lim_{\varepsilon \to \pm\infty} \mathfrak{u}_\varepsilon = \mp\infty$ uniformly on Ω;
- there are three hyperbolic τ_ε-copies of the base for $\varepsilon \in (0, \varepsilon_*)$ given by $\mathfrak{l}_\varepsilon < \mathfrak{m}_\varepsilon < \mathfrak{u}_\varepsilon$, and $\varepsilon \mapsto \mathfrak{m}_\varepsilon$ is strictly increasing on $(0, \varepsilon_*)$;
- and there are two τ_ε minimal sets for $\varepsilon = 0$ (resp. $\varepsilon = \varepsilon_*$): $\{\mathfrak{u}_0\}$ (resp. $\{\mathfrak{l}_{\varepsilon_*}\}$), which is hyperbolic attractive, and a nonhyperbolic one given by the collision of the two lower (resp. upper) copies of the base as $\varepsilon \to 0^+$ (resp. $\varepsilon \to (\varepsilon_*)^-$).

So, the two bifurcations points are of local saddle-node type. The interested reader can find in [11, Figure 1] a similar bifurcation diagram, which must be horizontally inverted to get ours. In this situation, Theorem 3.4 adds just a little piece of information to these facts: as explained in the previous remark, the lower nonhyperbolic minimal set for $\varepsilon = 0$ reduces to the copy of the base $\{0\}$. Note also that this description shows that the situation that Theorem 3.6(iii) describes cannot hold if $b \equiv 0$.

The hypotheses $b \geq 0$ or $b > 0$, added in the next result, allow us to delve deeper into the dynamical changes as ε varies.

Theorem 3.6. *Assume that $a(\omega) < 0$, $b(\omega) \geq 0$ and $c(\omega) > 0$ for all $\omega \in \Omega$. Then, in addition to the information provided by Theorem 3.4,*

(i) *$\mathfrak{l}_\varepsilon < 0$ for all $\varepsilon > 0$, $\mathfrak{l}_0 \equiv 0$, and $\mathfrak{l}_\varepsilon > s_0 > 0$ for all $\varepsilon < 0$, where $s_0 := \min(\inf_{\omega\in\Omega} c(\omega), \inf_{\omega\in\Omega,\, b(\omega)\neq 0}(-a(\omega)/b(\omega)))$ if $b \not\equiv 0$, and $s_0 := \inf_{\omega\in\Omega} c(\omega)$ if $b \equiv 0$. In particular, there is a local saddle-node bifurcation at $\varepsilon = 0$.*

(ii) *The continuous map* $(0,\infty)\to C(\Omega,\mathbb{R})$, $\varepsilon\mapsto \mathfrak{l}_\varepsilon$ *is strictly decreasing, with* $\lim_{\varepsilon\to 0^+}\mathfrak{l}_\varepsilon(\omega)=0$ *and* $\lim_{\varepsilon\to\infty}\mathfrak{l}_\varepsilon(\omega)=-\infty$*, both of them uniformly on* Ω.

If, in addition, $b(\omega)>0$ *for all* $\omega\in\Omega$*, then*

(iii) *the set* $\mathcal{J}:=\{\varepsilon^+>0\,|$ *there are three hyperbolic* τ_ε*-copies of the base for all* $\varepsilon>\varepsilon^+\}$ *is nonempty and open, the maps* $\mathcal{J}\to C(\Omega,\mathbb{R})$, $\varepsilon\mapsto\mathfrak{l}_\varepsilon,\,\mathfrak{m}_\varepsilon,\,\mathfrak{u}_\varepsilon$ *are continuous in the uniform topology of* $C(\Omega,\mathbb{R})$*, and* $\lim_{\varepsilon\to\infty}\mathfrak{u}_\varepsilon(\omega)=\infty$ *uniformly on* Ω*. In addition, there exists* $\varepsilon^0\ge\inf\mathcal{J}$ *such that the continuous map* $(\varepsilon^0,\infty)\to C(\Omega,\mathbb{R})$, $\varepsilon\mapsto\mathfrak{u}_\varepsilon$ *is strictly increasing.*

Proof. (i) The two first assertions in (i) (as well as the continuity stated in (ii)) are proved in Theorem 3.4(i)-(ii). Let us now define s_0 as in the statement of (i), and take $r<s_0$. Then, if $\varepsilon<0$, $p_\varepsilon(\omega,r)>p_0(\omega,r)>0$: the first inequality follows from $\varepsilon\,(b(\omega)\,r+a(\omega))>0$, and the second one from $c(\omega)-r>0$. This ensures that $\mathcal{A}_\varepsilon\subset\Omega\times(s_0,\infty)$ (see Remark 2.4.1) and hence proves that $\mathfrak{l}_\varepsilon>s_0$ for all $\varepsilon<0$. This fact precludes the existence of τ_ε-minimal sets "close" to $\{0\}$ if $\varepsilon<0$. Consequently, there is a local saddle-node bifurcation at $\varepsilon=0$, due to the collision of $\mathfrak{l}_\varepsilon$ and $\mathfrak{m}_\varepsilon$ as $\varepsilon\to 0^+$: see Theorem 3.4(iii).

(ii) Since $\mathfrak{l}_\varepsilon<0$ for $\varepsilon>0$, we have $b(\omega)\,\mathfrak{l}_\varepsilon(\omega)+a(\omega)<0$ for all $\omega\in\Omega$. So, if $0<\varepsilon_1<\varepsilon_2$, then $\mathfrak{l}'_{\varepsilon_1}(\omega)=p_{\varepsilon_1}(\omega,\mathfrak{l}_{\varepsilon_1}(\omega))>p_{\varepsilon_2}(\omega,\mathfrak{l}_{\varepsilon_1}(\omega))$, and hence $\mathfrak{l}_{\varepsilon_1}>\mathfrak{l}_{\varepsilon_2}$: see Remark 2.4.2. Theorem 3.4(v) proves the assertion about $\lim_{\varepsilon\to 0^+}\mathfrak{l}_\varepsilon$. Now, we fix $r<0$ and note that $\lim_{\varepsilon\to\infty}p_\varepsilon(\omega,r)=-\infty$ uniformly on Ω, since $a(\omega)+b(\omega)\,r\le a_+<0$. Hence, there exists $\varepsilon_r>0$ such that $p_\varepsilon(\omega,r)<0$ for all $\varepsilon\ge\varepsilon_r$. According to Remark 2.4.2, $\mathfrak{l}_\varepsilon<r$ for all $\varepsilon>\varepsilon_r$, which proves the last assertion in (ii).

(iii) The first goal is to find a strictly positive constant providing a global strict lower solution if $\varepsilon>0$ is large enough: as we will see later, the conclusions follow from this property. Note that, if $\varepsilon>0$ and $x>0$, then $p_\varepsilon(\omega,x)\ge\bar p_\varepsilon(x):=-x^3+c_-\,x^2+\varepsilon\,b_-\,x+\varepsilon\,a_-$ for all $\omega\in\Omega$. The points x_ε^+ and x_ε^- given by

$$x_\varepsilon^\pm=\frac{c_-\pm\sqrt{c_-^2+3\,\varepsilon\,b_-}}{3}$$

are the local minimum and maximum of $\bar p_\varepsilon(x)$, respectively. We observe that $x_\varepsilon^+>0$ and write it as $x_\varepsilon^+=(c_-+r_\varepsilon)/3$, with $r_\varepsilon:=\sqrt{c_-^2+3\,\varepsilon\,b_-}>\sqrt{3\,\varepsilon\,b_-}$. A straightforward computation shows that

$$\begin{aligned}27\,\bar p_\varepsilon(x_\varepsilon^+)&=6\,\varepsilon\,b_-\,r_\varepsilon+9\,\varepsilon\,b_-\,c_-+27\,\varepsilon\,a_-+2\,c_-^2\,r_\varepsilon+2\,c_-^3\\&>6\sqrt{3}\,\varepsilon^{3/2}\,b_-^{3/2}+\varepsilon\,(27\,a_-+9\,b_-\,c_-)\,,\end{aligned}$$

and hence $p_\varepsilon(\omega,x_\varepsilon^+)\ge\bar p_\varepsilon(x_\varepsilon^+)>0$ for all $\omega\in\Omega$ if $\varepsilon>0$ is large enough.

So, for $\varepsilon>0$ large enough, the constant x_ε^+ is a global strict lower solution. The expression of $p_\varepsilon(\omega,x)$ shows that a sufficiently large constant $s_\varepsilon>0$ is a global strict upper solution, and 0 is a global strict upper solution (due to $p_\varepsilon(\omega,0)=\varepsilon\,a(\omega)<0$). So, Proposition 2.2 ensures the existence of two minimal sets: $\mathcal{M}_\varepsilon^u\subset\Omega\times(x_\varepsilon^+,\infty)$ and $\mathcal{M}_\varepsilon^m\subset\Omega\times(0,x_\varepsilon^+)$. Since Theorem 3.4(ii) shows the existence of a third minimal set $\mathcal{M}_\varepsilon^l=\{\mathfrak{l}_\varepsilon\}\subset\Omega\times(-\infty,0)$, Theorem 3.1 ensures that

$\mathcal{M}^u_\varepsilon = \{\mathfrak{u}_\varepsilon\}$ and it is hyperbolic attractive, and that $\mathcal{M}^m_\varepsilon$ is a repulsive hyperbolic τ_ε-copy of the base, say $\mathcal{M}^m_\varepsilon = \{\mathfrak{m}_\varepsilon\}$.

Therefore, $\mathcal{J}$ is nonempty. It is open, as a consequence of the persistence of hyperbolic copies of the base under small variations of ε (see again [14, Theorem 2.3]), which also shows the continuity asserted in (iii). In addition, $\lim_{\varepsilon\to\infty} \mathfrak{u}_\varepsilon(\omega) = +\infty$ uniformly on Ω, since $\mathfrak{u}_\varepsilon > x^+_\varepsilon$ and $\lim_{\varepsilon\to\infty} x^+_\varepsilon = \infty$. This last property ensures the existence of $\varepsilon^0 \ge \inf \mathcal{J}$ such that $\inf_{\omega\in\Omega} \mathfrak{u}_\varepsilon(\omega) \ge \sup_{\omega\in\Omega}(-a(\omega)/b(\omega))$ for $\varepsilon > \varepsilon^0$. So, if $\varepsilon_2 > \varepsilon_1 > \varepsilon^0$, then $\mathfrak{u}'_{\varepsilon_1}(\omega) = p_{\varepsilon_1}(\omega, \mathfrak{u}_{\varepsilon_1}(\omega)) < p_{\varepsilon_2}(\omega, \mathfrak{u}_{\varepsilon_1}(\omega))$ for all $\omega \in \Omega$, and hence $\mathfrak{u}_{\varepsilon_1} < \mathfrak{u}_{\varepsilon_2}$ (see Remark 2.4.2). The proof is complete. □

3.1. Four different nonautonomous bifurcation diagrams

The next three results add extra conditions to $a < 0$, $b \ge 0$ with $b \not\equiv 0$ and $c > 0$, related to the relative sizes of $b\,c \ge 0$ and $a < 0$. These conditions can be considered nonautonomous extensions of the three possibilities arising in the autonomous case, namely $b\,c + a < 0$, $b\,c + a > 0$ and $b\,c + a = 0$. The three cases which we consider are far away from covering the infinitely many possibilities that arise in the nonautonomous case, but there are mathematical models that justify their interest, such as the case of the population dynamics that we analyze in Section 4. Recall that we assume the existence of a unique σ-invariant measure m on Ω.

Theorem 3.7. *Assume that $a(\omega) < 0$, $b(\omega) \ge 0$, $c(\omega) > 0$, and $b(\omega)\,c_+ + a(\omega) < 0$ for all $\omega \in \Omega$. Let $\mathcal{I} = (0, \varepsilon_*)$ be the open interval of Theorem 3.4*(iii)*. Then, in addition to the information provided by Theorems 3.4 and 3.6,*

(i) *$\varepsilon_* \in \mathbb{R}$, the continuous maps $\mathcal{I} \to C(\Omega, \mathbb{R})$, $\varepsilon \mapsto \mathfrak{l}_\varepsilon$, $-\mathfrak{m}_\varepsilon$, $\mathfrak{u}_\varepsilon$ are strictly decreasing, and $\mathfrak{m}_{\varepsilon_*}(\omega) := \lim_{\varepsilon\to(\varepsilon_*)^-} \mathfrak{m}_\varepsilon(\omega)$ and $\bar{\mathfrak{u}}_{\varepsilon_*}(\omega) := \lim_{\varepsilon\to(\varepsilon_*)^-} \mathfrak{u}_\varepsilon(\omega)$ define two semicontinuous τ_{ε_*}-equilibria which coincide with $\mathfrak{u}_{\varepsilon_*}(\omega)$ at all ω in a σ-invariant residual subset $\mathcal{R}_{\varepsilon_*} \subseteq \Omega$. In particular, there exist exactly two τ_{ε_*}-minimal sets: $\{l_{\varepsilon_*}\} \subset \Omega \times (-\infty, 0)$, which is hyperbolic attractive, and $\mathcal{M}_{\varepsilon_*} \subset \Omega \times (0, c_+)$, which is nonhyperbolic.*
(ii) *For all $\varepsilon < 0$, $\mathcal{A}_\varepsilon \subset \Omega \times (c_-, \infty)$.*
(iii) *If, in addition, $c_+ < 3\,c_-$, then $\mathcal{A}_\varepsilon = \{\mathfrak{u}_\varepsilon\}$ is the unique τ_ε-minimal set for all $\varepsilon < 0$, it is hyperbolic attractive, and the map $(-\infty, \varepsilon_*) \to C(\Omega, \mathbb{R})$, $\varepsilon \mapsto \mathfrak{u}_\varepsilon$ is continuous in the uniform topology of $C(\Omega, \mathbb{R})$.*

Assume also that $b(\omega) > 0$ for all $\omega \in \Omega$, and define $s_- := \inf_{\omega\in\Omega}(-a(\omega)/b(\omega))$ and $s_+ := \sup_{\omega\in\Omega}(-a(\omega)/b(\omega))$. Then,

(iv) *for all $\varepsilon \le 0$, $\mathfrak{u}_\varepsilon < s_+$; there exists $\bar{\varepsilon} \le 0$ such that $\mathcal{A}_\varepsilon = \{\mathfrak{u}_\varepsilon\}$ is an attractive hyperbolic copy of the base for all $\varepsilon < \bar{\varepsilon}$; and if, in addition, $c_+ < 3\,c_-$, then there exists $\underline{\varepsilon}$ with $-\infty \le \underline{\varepsilon} < 0$ such that $\varepsilon \mapsto \mathfrak{u}_\varepsilon$ is strictly decreasing on $(\underline{\varepsilon}, 0)$.*
(v) *Let $\mathcal{J} = (\varepsilon^*, \infty)$ be the open set defined in Theorem 3.6*(iii)*. Then, $\varepsilon_* < \varepsilon^*$, $\mathfrak{m}_\varepsilon > s_-$ for all $\varepsilon \in \mathcal{J}$, and there are exactly two τ_{ε^*}-minimal sets: $\{\mathfrak{l}_{\varepsilon^*}\}$ and $\mathcal{M}_{\varepsilon^*} \subset \Omega \times (s_-, \infty) \subset \Omega \times (c_+, \infty)$.*
(vi) *If, in addition, $a/b = -s \in \mathbb{R}$, then the maps $(\varepsilon^*, \infty) \to C(\Omega, \mathbb{R})$, $\varepsilon \mapsto \mathfrak{u}_\varepsilon$, $-\mathfrak{m}_\varepsilon$ are strictly increasing; the maps $\mathfrak{m}_{\varepsilon^*}(\omega) := \lim_{\varepsilon\to(\varepsilon^*)^+} \mathfrak{m}_\varepsilon(\omega)$ and $\bar{\mathfrak{u}}_{\varepsilon^*}(\omega) := \lim_{\varepsilon\to(\varepsilon^*)^+} \mathfrak{u}_\varepsilon(\omega)$ define two semicontinuous τ_{ε^*}-equilibria which coincide with $\mathfrak{u}_{\varepsilon^*}(\omega)$ at all ω in a σ-invariant residual subset $\mathcal{R}_{\varepsilon^*} \subseteq \Omega$; the continuous map $(-\infty, \varepsilon_*) \to C(\Omega, \mathbb{R})$, $\varepsilon \mapsto \mathfrak{u}_\varepsilon$ is strictly*

decreasing; $\lim_{\varepsilon\to-\infty} \mathfrak{u}_\varepsilon(\omega) = \lim_{\varepsilon\to\infty} \mathfrak{m}_\varepsilon(\omega) = s$ *uniformly on* Ω*; and* $\{\mathfrak{l}_\varepsilon\}$ *is the unique* τ_ε *minimal set for* $\varepsilon \in (\varepsilon_*, \varepsilon^*)$*. So,* 0*,* ε_* *and* ε^* *are three bifurcation points, all of them of local saddle-node type, and they are the unique ones if* $c_+ < 3\,c_-$*.*

Proof. (i) In what follows, we use the notation and information of Proposition 3.3. Let us take $\varepsilon_0 \in \mathcal{I}$ small enough to ensure that, if $\varepsilon \in (0, \varepsilon_0]$, then there exist three real roots of $p_\varepsilon(\omega, x)$ which satisfy $x^3_\varepsilon(\omega) < 0 < x^2_\varepsilon(\omega) < x^1_\varepsilon(\omega)$ and $x^2_\varepsilon(\omega) < c_+$ for all $\omega \in \Omega$, and take $\varepsilon \in (0, \varepsilon_0]$. Since $p_\varepsilon(\omega, c_+) = (c_+)^2(c(\omega) - c_+) + \varepsilon\,(b(\omega)\,c_+ + a(\omega)) < 0$ if $\omega \in \Omega$, either $c_+ < x^2_\varepsilon(\omega)$ or $c_+ > x^1_\varepsilon(\omega)$ for each $\omega \in \Omega$, so that $c_+ > x^1_\varepsilon(\omega)$ for all $\omega \in \Omega$. So, $p_\varepsilon(\omega, r) < 0$ for any $r > c_+$, and hence $\mathfrak{u}_\varepsilon < c_+$: see Remark 2.4.1. Now assume that $\mathfrak{u}_\varepsilon \le c_+$ for $\varepsilon = \varepsilon_1, \varepsilon_2$ with $0 \le \varepsilon_1 < \varepsilon_2$. Then, $\mathfrak{u}'_{\varepsilon_2}(\omega) = p_{\varepsilon_2}(\omega, \mathfrak{u}_{\varepsilon_2}(\omega)) > p_{\varepsilon_1}(\omega, \mathfrak{u}_{\varepsilon_2}(\omega))$, since $b(\omega)\,\mathfrak{u}_\varepsilon(\omega) + a(\omega) \le b(\omega)\,c_+ + a(\omega) < 0$. Hence, $\mathfrak{u}_{\varepsilon_2} < \mathfrak{u}_{\varepsilon_1} \le c_+$ (see Remark 2.4.2). In particular, $\varepsilon \mapsto \mathfrak{u}_\varepsilon$ strictly decreases on $[0, \varepsilon_0]$. Let $\mathcal{I}_u \subset \mathcal{I}$ the interval of persistence of this property. There exists $\delta > 0$ such that, for any $\varepsilon \in \mathcal{I}_u$, $\mathfrak{u}_\varepsilon < \mathfrak{u}_0 - \delta \le c_+ - \delta$. This, the continuity of $\varepsilon \mapsto \mathfrak{u}_\varepsilon$ on $\mathcal{I}$, and the previously proved property, preclude the possibility that $\sup\mathcal{I}_u < \sup\mathcal{I}$, and hence $\varepsilon \mapsto \mathfrak{u}_\varepsilon$ strictly decreases on $\mathcal{I}$.

The strictly decreasing character of the continuous map $\varepsilon \mapsto \mathfrak{l}_\varepsilon$ on $\mathcal{I} \subseteq (0, \infty)$ is proved in Theorem 3.6(ii). To check that $\varepsilon \mapsto \mathfrak{m}_\varepsilon$ is strictly increasing on $\mathcal{I}$, we adapt the argument of the proof of Theorem 3.4(iii). First note that, for all $\varepsilon \in \mathcal{I}$, $\mathfrak{m}_\varepsilon < \mathfrak{u}_\varepsilon < c_+$ and hence $b(\omega)\,\mathfrak{m}_\varepsilon(\omega) + a(\omega) < 0$ for all $\omega \in \Omega$. So: we fix $\varepsilon_1 \in \mathcal{I}$ and take $\varepsilon_2 > \varepsilon_1$ in $\mathcal{I}$ close enough to ensure $\mathfrak{u}_{\varepsilon_1} > \mathfrak{m}_{\varepsilon_2} > \mathfrak{l}_{\varepsilon_1}$, so that the τ_{ε_1}-orbit of $(\omega, \mathfrak{m}_{\varepsilon_2}(\omega))$ is above $\{\mathfrak{l}_{\varepsilon_1}\}$; we check that $\mathfrak{m}_{\varepsilon_2}$ is a global strict lower solution for τ_{ε_1}; we use these properties and Remark 2.4.3 to ensure that the α-limit set for τ_{ε_1} of $(\omega, \mathfrak{m}_{\varepsilon_2}(\omega))$ exists and is strictly below the graph of $\mathfrak{m}_{\varepsilon_2}$; and we deduce that the unique τ_{ε_1}-minimal set contained in this α-limit set is $\{\mathfrak{m}_{\varepsilon_1}\}$, so that $\mathfrak{m}_{\varepsilon_1} < \mathfrak{m}_{\varepsilon_2}$.

Now, we assume for contradiction that $\sup\mathcal{I} =: \varepsilon_* = \infty$. Note that $\mathfrak{u}_\varepsilon > \mathfrak{m}_\varepsilon > 0$ for all $\varepsilon \in \mathcal{I}$. Given any $\rho > 0$, we take $\varepsilon_\rho > \max_{\omega\in\Omega,\, x\in[-\rho, c_+]}(x^3 - c(\omega)\,x^2)/(b(\omega)\,x + a(\omega)) \ge 0$ (note that $b(\omega)\,x + a(\omega) < 0$ if $x \le c_+$). Then, $p_\varepsilon(\omega, r) < 0$ for all $r \in [-\rho, c_+]$ if $\varepsilon > \varepsilon_\rho$. Let us deduce that $\mathfrak{u}_\varepsilon \le -\rho < 0$ if $\varepsilon \ge \varepsilon_\rho$, which provides the sought-for contradiction. Recall that $\mathfrak{u}_\varepsilon$ is continuous and $\mathfrak{u}_\varepsilon < c_+$ for all $\varepsilon \in \mathcal{I} = (0, \infty)$. Again for contradiction, we assume that $\max_{\omega\in\Omega} \mathfrak{u}_\varepsilon(\omega) =: s \in (-\rho, c_+)$, and take $\bar\omega \in \Omega$ with $\mathfrak{u}_\varepsilon(\bar\omega) = s$. Then, $p_\varepsilon(\bar\omega, \mathfrak{u}_\varepsilon(\bar\omega)) < 0$, and hence $\mathfrak{u}_\varepsilon(\bar\omega{\cdot}t) = v_\varepsilon(t, \bar\omega, \mathfrak{u}_\varepsilon(\bar\omega)) > s$ for small values of $t < 0$, which contradicts the definition of s. The conclusion is that ε_* is finite, as asserted.

Recall that $\varepsilon_* \notin \mathcal{I}$, since $\mathcal{I}$ is open. Therefore, there exists at most a τ_{ε_*}-minimal set $\mathcal{M}_{\varepsilon_*}$ different from $\{\mathfrak{l}_{\varepsilon_*}\}$ (see Theorems 3.1 and 3.4(ii)), which is nonhyperbolic. The remaining part of this proof is very similar to part of that of [11, Theorem 5.10(i)], but we detail it for the reader's convenience. The monotonicity properties of $0 < \mathfrak{m}_\varepsilon < \mathfrak{u}_\varepsilon$ ensure the global existence of the limits $\mathfrak{m}_{\varepsilon_*}$ and $\bar{\mathfrak{u}}_{\varepsilon_*}$, with $0 < \mathfrak{m}_\varepsilon < \mathfrak{m}_{\varepsilon_*} \le \bar{\mathfrak{u}}_{\varepsilon_*} < \mathfrak{u}_\varepsilon$ for $\varepsilon \in \mathcal{I}$. It is easy to check that they are τ_{ε_*}-equilibria. Since they are monotone limits of continuous functions, they are semicontinuous on Ω: $\mathfrak{m}_{\varepsilon_*}$ is lower semicontinuous and $\bar{\mathfrak{u}}_{\varepsilon_*}$ is upper semicontinuous. Let $\mathcal{M}^m_{\varepsilon_*}$ be the minimal set associated to $\mathfrak{m}_{\varepsilon_*}$ by (2.3). The lower semicontinuity of $\mathfrak{m}_{\varepsilon_*}$ yields $x \ge \mathfrak{m}_{\varepsilon_*}(\omega) \ge 0$ for any $(\omega, x) \in \mathcal{M}$. In particular, $\mathcal{M}^m_{\varepsilon_*} \ne \{\mathfrak{l}_{\varepsilon_*}\}$ (see Theorem 3.4(ii)), which yields $\mathcal{M}^m_{\varepsilon_*} = \mathcal{M}_{\varepsilon_*}$. The nonexistence of a third minimal set, the inequalities $\mathfrak{m}_{\varepsilon_*} \le \bar{\mathfrak{u}}_{\varepsilon_*} \le \mathfrak{u}_{\varepsilon_*}$, and Proposition 2.1 ensure that $\mathcal{M}_{\varepsilon_*}$ is also associated to $\bar{\mathfrak{u}}_{\varepsilon_*}$ and to $\mathfrak{u}_{\varepsilon_*}$ by (2.3). Of course, $x \le \mathfrak{u}_{\varepsilon_*}(\omega)$ for any $(\omega, x) \in \mathcal{M}_{\varepsilon_*}$. In particular, $\mathfrak{m}_{\varepsilon_*}(\omega) = \bar{\mathfrak{u}}_{\varepsilon_*}(\omega) = \mathfrak{u}_{\varepsilon_*}(\omega)$ for all ω in the residual subset of Ω formed by their common continuity points, and hence $\mathcal{M}_{\varepsilon_*}$ is contained in $\bigcup_{\omega\in\Omega}\big(\{\omega\} \times [\mathfrak{m}_{\varepsilon_*}(\omega), \mathfrak{u}_{\varepsilon_*}(\omega)]\big)$, which is a compact τ_{ε_*}-invariant pinched subset of $\Omega \times (0, c_+)$.

(ii) Since $c_- \le c_+ \le -a(\omega)/b(\omega)$ for all $\omega \in \Omega$ with $b(\omega) \ne 0$, we have $c_- \le s_0$, where s_0 is defined in Theorem 3.6(i). This result proves (ii).

(iii) We fix $\varepsilon < 0$, assume $c_+ < 3\,c_-$, and proceed in a similar way to the proof of Theorem 3.4(v): we fix $r \in (c_+/3, c_-)$; we define $q_\varepsilon(\omega, x)$ as the $C^{0,2}(\Omega \times \mathbb{R}, \mathbb{R})$ function which coincide with $p_\varepsilon(\omega, x)$ for $x \ge r$ and is given by a second degree polynomial for $x \le r$; we check that $(\partial^2/\partial x^2)\, q_\varepsilon(\omega, x) = -6r + 2\,c(\omega) \le -6r + 2\,c_+ < 0$ for all $x \le r$ and $(\partial^2/\partial x^2)\, q_\varepsilon(\omega, x) = -6x + 2\,c(\omega) < -6r + 2\,c_+ < 0$ for all $x > r$; we deduce that q_ε satisfies all the conditions **c1**-**c4** of [13, Section 3]; we use $q_\varepsilon(\omega, c_-) = p_\varepsilon(\omega, c_-) > 0$ and $q_\varepsilon(\omega, \pm s) < 0$ for all $\omega \in \Omega$ if $s > c_-$ is large enough to deduce from Proposition 2.2 and [13, Theorem 3.3] the existence of exactly one minimal set $\mathcal{M}_\varepsilon$ strictly above $\Omega \times \{c_-\}$ for the skew-product flow $\tilde{\tau}_\varepsilon$ defined by $x' = q_\varepsilon(\omega{\cdot}t, x)$, which is an attractive hyperbolic copy of the base; and we conclude from $\mathcal{A}_\varepsilon \subset \Omega \times (c_-, \infty)$ (proved in (ii)) that $\mathcal{M}_\varepsilon$ is also the unique τ_ε-minimal set, and from the coincidence of the Lyapunov exponents of the minimal sets for both flows which are above $\Omega \times \{c_-\}$ and from Theorem 2.6 that $\mathcal{A}_\varepsilon = \mathcal{M}_\varepsilon$ is an attractive hyperbolic copy of the base, i.e., $\mathcal{A}_\varepsilon = \{\mathfrak{u}_\varepsilon\}$. The continuity of $(-\infty, \varepsilon_*) \to C(\Omega, \mathbb{R})$, $\varepsilon \mapsto \mathfrak{u}_\varepsilon$ follows, again, from the robustness of the existence of hyperbolic copies of the base and their continuous variation in the uniform topology (see, e.g., [14, Theorem 2.3]).

(iv) Let us take $\varepsilon < 0$. If $r \ge s_+$, then $p_\varepsilon(\omega, r) = r^2(c(\omega) - r) + \varepsilon\,(b(\omega)\,r + a(\omega)) < 0$, since $b(\omega)\,r + a(\omega) \ge b(\omega)\,s_+ + a(\omega) \ge 0$ and $c(\omega) - r \le c(\omega) - s_+ \le c_+ + a(\omega)/b(\omega) < 0$. According Remark 2.4.2, $\mathfrak{u}_\varepsilon < s_+$, as asserted. Our next goal is to check that, if $-\varepsilon$ is large enough, then all the Lyapunov exponents of $\mathcal{A}_\varepsilon$ are strictly negative: according to Theorem 2.6, this property proves that $\mathcal{A}_\varepsilon = \{\mathfrak{u}_\varepsilon\}$ is an attractive hyperbolic copy of the base. As explained in Section 2.2, it suffices to check that

$$\int_\Omega (p_\varepsilon)_x(\omega, \mathfrak{b}_\varepsilon(\omega))\, dm = \int_\Omega \left(-3(\mathfrak{b}_\varepsilon(\omega))^2 + 2\,\mathfrak{b}_\varepsilon(\omega)\, c(\omega) + \varepsilon\, b(\omega)\right) dm < 0$$

for any m-measurable equilibrium $\mathfrak{b}_\varepsilon \colon \Omega \to \mathbb{R}$ with graph contained in $\mathcal{A}_\varepsilon$. Since $c_- \le \mathfrak{b}_\varepsilon \le s_+$ for any such τ_ε-equilibrium $\mathfrak{b}_\varepsilon$, and since $b_- > 0$, the inequality holds if $-\varepsilon > 0$ is large enough, say $\varepsilon < \bar{\varepsilon} < 0$. On the other hand, according to Theorem 3.4, $\mathfrak{u}_0 \le c_+$ (and hence $\mathfrak{u}_0 < s_-$) and if, in addition, $c_+ < 3\,c_-$, then $\varepsilon \mapsto \mathfrak{u}_\varepsilon \in C(\Omega, \mathbb{R})$ is continuous at $\varepsilon = 0$. So, there exists $\underline{\varepsilon} \le 0$ (perhaps $\underline{\varepsilon} = -\infty$) such that, if $\varepsilon \in (\underline{\varepsilon}, 0)$, then $\mathfrak{u}_\varepsilon(\omega) \le s_-$ for all $\omega \in \Omega$. It is easy to deduce that, if $\underline{\varepsilon} < \varepsilon_1 < \varepsilon_2 \le 0$, then $\mathfrak{u}'_{\varepsilon_2}(\omega) = p_{\varepsilon_2}(\omega, \mathfrak{u}_{\varepsilon_2}(\omega)) < p_{\varepsilon_1}(\omega, \mathfrak{u}_{\varepsilon_2}(\omega))$ for all $\omega \in \Omega$, and hence $\mathfrak{u}_{\varepsilon_2} < \mathfrak{u}_{\varepsilon_1}$ (see once more Remark 2.4.2).

(v) It is clear that ε_* does not belong to $\mathcal{J}$, since according to (i) there are exactly two τ_{ε_*}-minimal sets. This ensures that $\varepsilon_* \le \varepsilon^*$. In addition, there exist at most two τ_{ε^*}-minimal sets, since $\varepsilon^* \notin J$. We will check below that there are indeed two of them: $\{\mathfrak{l}_{\varepsilon^*}\}$ (which is hyperbolic attractive and below $\Omega \times \{0\}$, as proved by Theorem 3.4(ii)) and $\mathcal{M}_{\varepsilon^*}$, which is above $\Omega \times \{s_-\}$ for $s_- := \inf_{\omega\in\Omega}(-a(\omega)/b(\omega)) > c_+$. Since, as seen in (i), $\mathcal{M}_{\varepsilon_*}$ is below $\Omega \times \{c_+\}$, we conclude that $\varepsilon_* < \varepsilon^*$. The proof will also show that $\mathfrak{m}_\varepsilon > s_-$ for all $\varepsilon \in \mathcal{J}$, and this completes the list of assertions in (v).

It is easy to check that $p_\varepsilon(\omega, s_-) < 0$ for all $\omega \in \Omega$ if $\varepsilon > 0$. So, s_- is a global strict upper solution for τ_ε. On the other hand, since $\lim_{\varepsilon\to\infty} \mathfrak{u}_\varepsilon(\omega) = \infty$ uniformly on Ω (see Theorem 3.6(iii)), there exists a minimum $\varepsilon_1 \ge \varepsilon^*$ such that $\mathfrak{u}_\varepsilon(\omega) \ge s_-$ for all $\omega \in \Omega$ if $\varepsilon > \varepsilon_1$. These properties, combined with $\mathfrak{l}_\varepsilon < 0 < s_-$ for all $\varepsilon > 0$ (see Theorem 3.4(ii)) mean that, for any fixed $\omega \in \Omega$ and all $\varepsilon \ge \varepsilon_1$, the map $t \mapsto v_\varepsilon(t, \omega, s_-)$ is globally bounded, and that $\Omega \times \{s_-\}$ is always strictly above the ω-limit set of (ω, s_-) and strictly below its α-limit set: see Remark 2.4.3. Remark 2.5

ensures that the α-limit set cannot contain $\{\mathfrak{u}_\varepsilon\}$. Hence it necessarily contains $\{\mathfrak{m}_\varepsilon\}$, and the ω-limit set contains the unique τ_ε-minimal set below $\{\mathfrak{m}_\varepsilon\}$, which is $\{\mathfrak{l}_\varepsilon\}$. In particular, $\mathfrak{m}_\varepsilon > s_-$ for all $\varepsilon > \varepsilon_1$. Let us check that $\varepsilon_1 = \varepsilon^*$, assuming for contradiction that $\varepsilon_1 > \varepsilon^*$. Using the continuity of $\varepsilon \mapsto \mathfrak{u}_\varepsilon$ on $\mathcal{J}$ ensured by Theorem 3.6(iii), we deduce that $\mathfrak{u}_{\varepsilon_1}(\omega) \ge s_-$ for all $\omega \in \Omega$, and that there exists $\omega_0 \in \Omega$ such that $\mathfrak{u}_{\varepsilon_1}(\omega_0) = s_-$. But, as just seen, this yields $\mathfrak{u}_{\varepsilon_1} > \mathfrak{m}_{\varepsilon_1} > s_-$, which provides the contradiction. Altogether, we have $\mathfrak{m}_\varepsilon > s_- > \mathfrak{l}_\varepsilon$ for all $\varepsilon \in \mathcal{J}$.

Note that we have proved that $\Omega \times \{s_-\} \subset \mathcal{A}_\varepsilon$ for all $\varepsilon > \varepsilon^*$. Since there exists $\rho > 0$ such that $s_- < \mathfrak{m}_\varepsilon < \mathfrak{u}_\varepsilon \le \rho$ for all $\varepsilon \in (\varepsilon^*, \varepsilon^* + 1]$, we conclude that $v_{\varepsilon^*}(t, \omega, s_-) = \lim_{\varepsilon \to (\varepsilon^*)^+} v_\varepsilon(t, \omega, s_-) \le \rho$ for all $t < 0$ and $\omega \in \Omega$. So, $\mathfrak{l}_{\varepsilon^*} < s_- \le \mathfrak{u}_{\varepsilon^*}$, and hence, as seen above, there exists the α-limit set of any point (ω, s_-) for τ_{ε^*} and it is strictly above $\Omega \times \{s_-\}$. Hence, there exists a τ_{ε^*}-minimal set contained in $\Omega \times (s_-, \infty)$, as asserted.

(vi) Assume that $-a(\omega)/b(\omega) = s$, constant. Let us first analyze the situation in (ε^*, ∞). Recall that $\mathfrak{u}_\varepsilon > \mathfrak{m}_\varepsilon > s$ if $\varepsilon > \varepsilon^*$: see (v). An argument similar to that of the first (resp. second) paragraph of the proof of (i) proves that $\varepsilon \mapsto \mathfrak{u}_\varepsilon$ (resp. $\varepsilon \mapsto -\mathfrak{m}_\varepsilon$) is strictly increasing on (ε^*, ∞). In particular, there exist the pointwise limits $\bar{\mathfrak{u}}_{\varepsilon^*} := \lim_{\varepsilon \to (\varepsilon^*)^+} \mathfrak{u}_\varepsilon$ and $\mathfrak{m}_{\varepsilon^*} := \lim_{\varepsilon \to (\varepsilon^*)^+} \mathfrak{m}_\varepsilon$, whose additional properties are checked as those of $\bar{\mathfrak{u}}_{\varepsilon_*}$ and $\mathfrak{m}_{\varepsilon_*}$ in (i). Let us prove that $\lim_{\varepsilon \to \infty} \mathfrak{m}_\varepsilon(\omega) = s$ uniformly on Ω. We take $\delta > 0$ such that, for an $\varepsilon_2 > \varepsilon^*$, $\mathfrak{l}_\varepsilon < 0 < s + \delta < \mathfrak{u}_\varepsilon$ if $\varepsilon > \varepsilon_2$, and work for these values of ε. Note that $p_\varepsilon(\omega, s + \delta) = (s + \delta)^2 (c(\omega) - s - \delta) + \varepsilon\, b(\omega)\, \delta$, so that $s + \delta$ is a global strict lower solution if $\varepsilon > 0$ is large enough. According to Remark 2.4.3, $\Omega \times \{s + \delta\}$ is strictly above a τ_ε-minimal set contained in the α-limit set of a point $(\omega, s + \delta)$ (that exists since this point belongs to $\mathcal{A}_\varepsilon$), which is necessarily $\{\mathfrak{m}_\varepsilon\}$, as Remark 2.5 guarantees. That is, $s \le \mathfrak{m}_\varepsilon \le s + \delta$ if ε is large enough, and this proves the assertion.

Let us now analyze the situation for $\varepsilon \in (-\infty, \varepsilon_*)$. First of all, we check that $\lim_{\varepsilon \to -\infty} \mathfrak{u}_\varepsilon(\omega) = s$ uniformly on Ω. For $\delta > 0$, $p_\varepsilon(\omega, s - \delta) = (s - \delta)^2 (c(\omega) - s + \delta) - \varepsilon\, b(\omega)\, \delta$, so $s - \delta$ is a global strict lower solution if $-\varepsilon > 0$ is large enough. According to Remark 2.4.3, $\Omega \times \{s - \delta\}$ is strictly below a τ_ε-minimal set contained in the ω-limit set of a point $(\omega, s - \delta)$; and, according to (iv), this τ_ε-minimal set is necessarily $\{\mathfrak{u}_\varepsilon\}$ if $\varepsilon < \bar{\varepsilon}$. That is, $s - \delta < \mathfrak{u}_\varepsilon < s$ if $-\varepsilon > 0$ is large enough (see (iv)), which proves the assertion. To check that $\varepsilon \mapsto \mathfrak{u}_\varepsilon$ is strictly decreasing on $(-\infty, \varepsilon_*)$, we repeat the argument of (iv), which is possible since, by (iv) and (i), $\mathfrak{u}_\varepsilon < s$ for all these values of ε. (Recall that, in addition, $\mathcal{A}_\varepsilon = \{\mathfrak{u}_\varepsilon\}$ for all $\varepsilon \le 0$ if $c_+ < 3\, c_-$: see (iii).)

Let us see what happens for $\varepsilon \in (\varepsilon_*, \varepsilon^*)$. We fix $\bar{\varepsilon} \in (\varepsilon_*, \varepsilon^*)$. As a first step, we will check that $\mathcal{A}_{\bar{\varepsilon}} \subset \Omega \times (-\infty, s)$; i.e., that $s > \mathfrak{u}_{\bar{\varepsilon}}(\omega)$ for all $\omega \in \Omega$. Since $\mathfrak{l}_{\bar{\varepsilon}} < 0$ (see Theorem 3.4(ii)), it suffices to assume that the $\tau_{\bar{\varepsilon}}$-orbit of a point (ω_0, s) is bounded (i.e., that $s \le \mathfrak{u}_{\bar{\varepsilon}}(\omega)$) and reach a contradiction. Since $p_{\bar{\varepsilon}}(\omega, s) = s^2\, (c(\omega) - s) < 0$, s is a constant strict upper solution for $\tau_{\bar{\varepsilon}}$. Then, as seen in the proof of (v), $\Omega \times \{s\}$ is strictly above the ω-limit set of (ω_0, s), which ensures that $s < \mathfrak{u}_{\bar{\varepsilon}}$. So, we can repeat once again the argument of (iv) to check that $\varepsilon \mapsto \mathfrak{u}_\varepsilon$ is strictly increasing on $(\bar{\varepsilon}, \infty)$. In particular, the τ_ε-orbit of (ω_0, s) is bounded for all $\varepsilon \ge \bar{\varepsilon}$. Let $\bar{\omega}$ be a continuity point of the semicontinuous map $\mathfrak{u}_{\varepsilon^*}$, so that $(\mathcal{M}_{\varepsilon^*})(\bar{\omega}) = \{\mathfrak{u}_{\varepsilon^*}(\bar{\omega})\}$ (see Proposition 2.1). The α-limit set of (ω_0, s) for τ_{ε^*} contains a minimal set which cannot be hyperbolic attractive (see again Remark 2.5), so it is $\mathcal{M}_{\varepsilon^*}$. That is, there exists $(t_n) \downarrow -\infty$ such that $(\bar{\omega}, \mathfrak{u}_{\varepsilon^*}(\bar{\omega})) = \lim_{n \to \infty} (\omega_0 {\cdot} t_n, v_{\varepsilon^*}(t_n, \omega_0, s))$. We can assume without restriction the existence of $\bar{x} := \lim_{n \to \infty} v_{\bar{\varepsilon}}(t_n, \omega_0, s)$, and observe that $\bar{x} \le \mathfrak{u}_{\bar{\varepsilon}}(\bar{\omega})$, since $v_{\bar{\varepsilon}}(t, \bar{\omega}, \bar{x})$ is bounded ($v_{\bar{\varepsilon}}(t, \bar{\omega}, \bar{x}) = \lim_{n \to \infty} v_{\bar{\varepsilon}}(t_n + t, \omega_0, s)$). Then, since $v_{\bar{\varepsilon}}(t, \omega_0, s) > v_{\varepsilon^*}(t, \omega_0, s)$ for all $t < 0$ (as we deduce from $p_{\bar{\varepsilon}}(\omega, r) < p_{\varepsilon^*}(\omega, r)$ for all $\omega \in \Omega$ if $r \ge s$ and from $v_\varepsilon(t, \omega_0, s) > s$ for all $t < 0$ and all $\varepsilon > 0$), we can conclude that $\bar{x} \ge \mathfrak{u}_{\varepsilon^*}(\bar{\omega}) > \mathfrak{u}_{\bar{\varepsilon}}(\bar{\omega})$, which provides the sought-for contradiction.

So, we have $\mathcal{A}_{\bar\varepsilon} \subset \Omega \times (-\infty, s)$. Now we assume for contradiction the existence of a $\tau_{\bar\varepsilon}$-minimal set $\mathcal{N}_{\bar\varepsilon} > \{\mathfrak{l}_{\bar\varepsilon}\}$. We choose a point $\bar\omega \in \Omega$ at which the sections of $\mathcal{M}_{\varepsilon_*}$ and $\mathcal{N}_{\bar\varepsilon}$ are singletons: $(\mathcal{M}_{\varepsilon_*})_{\bar\omega_1} = \{\mathfrak{u}_{\varepsilon_*}(\bar\omega)\} = \{\mathfrak{m}_{\varepsilon_*}(\bar\omega)\}$ and $(\mathcal{N}_{\bar\varepsilon})_{\bar\omega} = \{\bar x\}$, so that $\bar x \in (\mathfrak{l}_\varepsilon(\bar\omega), s)$. The information regarding monotonicity, continuity, limiting behavior as $\varepsilon \to -\infty$, and the shape of $\mathcal{M}_{\varepsilon_*}$ provided so far, allows us to choose an $\varepsilon_0 < \bar\varepsilon$ and a unique (bounded) τ_ε-equilibrium, say $\mathfrak{b}_{\varepsilon_0}$, such that $\mathfrak{b}_{\varepsilon_0}(\bar\omega) = \bar x$: if $\bar x \in (\mathfrak{l}_\varepsilon(\bar\omega), 0]$, then $\varepsilon_0 \in [0, \bar\varepsilon)$ and $\mathfrak{b}_{\varepsilon_0} = \mathfrak{l}_{\varepsilon_0}$; if $\bar x \in (0, \mathfrak{m}_{\varepsilon_*}(\bar\omega)) = (0, \mathfrak{u}_{\varepsilon_*}(\bar\omega))$, then $\varepsilon \in (0, \varepsilon_*)$ and $\mathfrak{b}_{\varepsilon_0} = \mathfrak{m}_{\varepsilon_0}$; and if $\bar x \in [\mathfrak{u}_{\varepsilon_*}(\bar\omega), s)$, then $\varepsilon_0 \in (-\infty, \varepsilon_*]$ and $\mathfrak{b}_{\varepsilon_0} = \mathfrak{u}_{\varepsilon_0}$. In any case, $\mathfrak{b}_{\varepsilon_0}$ is a global strict upper solution for $\tau_{\bar\varepsilon}$, since $\mathfrak{b}'_{\varepsilon_0}(\omega) = p_{\varepsilon_0}(\omega, \mathfrak{b}_{\varepsilon_0}(\omega)) > p_{\bar\varepsilon}(\omega, \mathfrak{b}_{\varepsilon_0}(\omega))$ due to the inequality $\mathfrak{b}_\varepsilon < s$. As explained in Remark 2.4.3, this ensures that the ω-limit of the point $(\bar\omega, \mathfrak{b}_{\varepsilon_0}(\bar\omega)) = (\bar\omega, \bar x)$ (which is, of course, $\mathcal{N}_{\bar\varepsilon}$) is strictly below the graph of $\mathfrak{b}_{\varepsilon_0}$. This fact precludes $(\bar\omega, \bar x) \in \mathcal{N}_{\bar\varepsilon}$ and provides the sought-for contradiction.

The assertions of the last sentence of (vi) follow from the previous description and Theorems 3.6(i) and 3.4(v). □

Fig. 1 provides a depiction of the "three saddle-node bifurcation diagram" of (3.1) under the most restrictive conditions of Theorem 3.7. It is interesting to remark that the dynamics of the nonhyperbolic minimal sets $\mathcal{M}_{\varepsilon_*}$ and $\mathcal{M}_{\varepsilon^*}$ at the bifurcation points ε_* and ε^* can be extremely complicated, even with the occurrence of SNAs described, for instance, in [27], [38], [21,22] (based on the classical examples of [36,37], [51] and [25]). A more detailed description of these dynamical possibilities can be easily adapted to this case from that made in [11, Proposition 5.11]. In particular, as there explained, the measure m of the residual subsets $\mathcal{R}^{\varepsilon_*}$ and $\mathcal{R}^{\varepsilon^*}$ of Ω at whose points the upper and lower equilibria of $\mathcal{M}_{\varepsilon_*}$ and $\mathcal{M}_{\varepsilon^*}$ respectively collide can be 0 or 1.

Theorem 3.8. *Assume that $a(\omega) < 0$, $b(\omega) > 0$, $c(\omega) > 0$ and $b(\omega)\, c_- + a(\omega) > 0$ for all $\omega \in \Omega$, and call $s_+ := \sup_{\omega\in\Omega}(-a(\omega)/b(\omega))$ and $s_- := \inf_{\omega\in\Omega}(-a(\omega)/b(\omega))$. Then, in addition to the information provided by Theorems 3.4 and 3.6,*

- (i) *for all $\varepsilon > 0$, there are three hyperbolic copies of the base, with $\mathfrak{l}_\varepsilon < 0 < \mathfrak{m}_\varepsilon < s_+ < \mathfrak{u}_\varepsilon$. That is, $\mathcal{I} = (0, \infty)$, where $\mathcal{I}$ is defined in Theorem 3.4(iii). In addition, the maps $(0, \infty) \to C(\Omega, \mathbb{R})$, $\varepsilon \mapsto -\mathfrak{l}_\varepsilon, \mathfrak{u}_\varepsilon$ are strictly increasing; and there exists ε^* with $0 < \varepsilon^* \le \infty$ such that the map $(0, \varepsilon^*) \to C(\Omega, \mathbb{R})$, $\varepsilon \mapsto \mathfrak{m}_\varepsilon$ is strictly increasing. In particular, there are no strictly positive bifurcation values.*
- (ii) *For all $\varepsilon < 0$, $\mathcal{A}_\varepsilon \subset \Omega \times (s_-, c_+)$; there exists ε_* with $-\infty \le \varepsilon_* < 0$ such that the map $(\varepsilon_*, \infty) \to C(\Omega, (s_-, c_+))$, $\varepsilon \mapsto \mathfrak{u}_\varepsilon$ is strictly increasing; and there exists $\varepsilon_0 \le 0$ such that $\mathcal{A}_\varepsilon$ is an attractive hyperbolic copy of the base for $\varepsilon \in (-\infty, \varepsilon_0)$.*
- (iii) *If, in addition, $c_+ < 3\, s_-$, then $\varepsilon_0 = 0$. Hence, $\{\mathfrak{u}_\varepsilon\}$ is an attractive hyperbolic copy of the base for all $\varepsilon \in \mathbb{R}$ and the map $\mathbb{R} \to C(\Omega, \mathbb{R})$, $\varepsilon \mapsto \mathfrak{u}_\varepsilon$ is continuous. Consequently, in this case, 0 is the unique bifurcation value, of local saddle-node type.*
- (iv) *If, in addition, $a/b = -s \in \mathbb{R}$, then $\varepsilon^* = \infty$ and $\varepsilon_* = -\infty$; the map $(-\infty, 0) \to C(\Omega, (s, c_+))$, $\varepsilon \mapsto \mathfrak{l}_\varepsilon$ is strictly increasing; and $\lim_{\varepsilon\to-\infty} \mathfrak{u}_\varepsilon(\omega) = \lim_{\varepsilon\to\infty} \mathfrak{m}_\varepsilon(\omega) = s$ uniformly on Ω. There are three possibilities for $\varepsilon \in (-\infty, 0)$:*
  - *$\mathcal{A}_\varepsilon = \{\mathfrak{u}_\varepsilon\}$ is an attractive hyperbolic copy of the base for all $\varepsilon < 0$, in which case the map $\mathbb{R} \to C(\Omega, \mathbb{R})$, $\varepsilon \mapsto \mathfrak{u}_\varepsilon$ is continuous, and 0 is the unique bifurcation value, of local saddle-node type. (This happens if $c_+ < 3\, s$.)*
  - *There exist $\underline{\varepsilon} < \bar\varepsilon < 0$ such that: there are three τ_ε hyperbolic copies of the base for any $\varepsilon \in (\underline{\varepsilon}, \bar\varepsilon)$; $\mathcal{A}_\varepsilon$ is an attractive hyperbolic copy of the base for $\varepsilon \in (0, \infty) - [\underline{\varepsilon}, \bar\varepsilon\,]$; there are two $\tau_{\underline{\varepsilon}}$-minimal sets, $\{\mathfrak{l}_{\underline{\varepsilon}}\}$ (which is hyperbolic attractive) and a nonhyperbolic one*

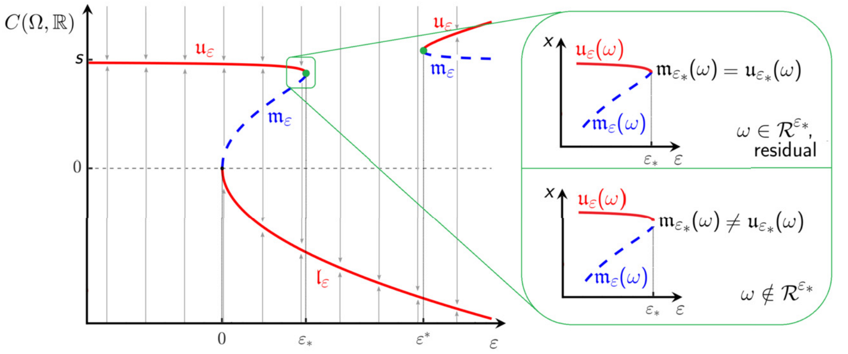

Fig. 1. The figure in the left depicts the bifurcation diagram of the ε-parametric family (3.1) when $c > 0$, $c_+ < 3c_-$, $b > 0$, $a = -s\,b$ for a constant $s > 0$, and $b\,c_+ + a < 0$, which is described in Theorem 3.7 in combination with Theorems 3.4 and 3.6. The three strictly monotone solid red curves represent the families of attractive hyperbolic copies of the base that determine the upper and lower equilibria of the global attractor. The two strictly monotone dashed blue curves represent the families of repulsive hyperbolic copies of the base. The light grey arrows partly depict the dynamics of the rest of the orbits. There are three bifurcation points, all of them of saddle-node type: $\varepsilon = 0$, where $\mathfrak{m}_\varepsilon$ and $\mathfrak{l}_\varepsilon$ globally collide on 0; and ε_* and ε^*, where $\mathfrak{m}_\varepsilon$ and $\mathfrak{u}_\varepsilon$ partly collide, giving rise to semicontinuous but perhaps noncontinuous maps. This fact is depicted by two large green points, and explained for ε_* in the zoom at the right. (Figs. 2 and 3 will also use "large green points" to depict similar situations, as well as red and blue curves, and grey arrows.) (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

given by the collision of $\{\mathfrak{u}_\varepsilon\}$ and $\{\mathfrak{m}_\varepsilon\}$ as $\varepsilon \to (\underline{\varepsilon})^+$; there are two $\tau_{\bar\varepsilon}$-minimal sets, $\{\mathfrak{u}_{\bar\varepsilon}\}$ (which is hyperbolic attractive) and a nonhyperbolic one given by the collision of $\{\mathfrak{l}_\varepsilon\}$ and $\{\mathfrak{m}_\varepsilon\}$ as $\varepsilon \to (\bar\varepsilon)^-$; the maps $(-\infty, \bar\varepsilon) \to C(\Omega,\mathbb{R})$, $\varepsilon \to \mathfrak{l}_\varepsilon$ and $(\underline{\varepsilon}, 0) \to C(\Omega,\mathbb{R})$, $\varepsilon \to \mathfrak{u}_\varepsilon$ are continuous and strictly increasing; and the map $(\underline{\varepsilon}, \bar\varepsilon) \to C(\Omega,\mathbb{R})$, $\varepsilon \to \mathfrak{m}_\varepsilon$ is continuous and strictly decreasing. So, there are exactly three bifurcation values, $\underline{\varepsilon}$, $\bar\varepsilon$ and 0, all of them of local saddle-node type.

- *There is a unique negative value $\varepsilon_1 < 0$ such that $\{\mathfrak{u}_{\varepsilon_1}\}$ is not a hyperbolic copy of the base, in which case $\mathcal{A}_{\varepsilon_1}$ is a pinched set containing a unique τ_{ε_1}-minimal set.*

Proof. (i) If $\varepsilon \geq 0$, then $p_\varepsilon(\omega, s_+) = (s_+)^2(-s_+ + c(\omega)) + \varepsilon\,(b(\omega)\,s_+ + a(\omega)) > 0$, since $b(\omega)\,s_+ + a(\omega) \geq 0$ for all $\omega \in \Omega$ and $c_- > -a(\omega)/b(\omega)$ for all ω (and hence $c(\omega) \geq c_- > s_+$). In addition, $p_\varepsilon(\omega, 0) = \varepsilon\,a(\omega) < 0$ if $\varepsilon > 0$. We fix $\varepsilon > 0$, look for $r_1 < 0 < s_+ < r_2$ with $p_\varepsilon(\omega, r_1) > 0$ and $p_\varepsilon(\omega, r_2) < 0$ for all $\omega \in \mathbb{R}$, and deduce the first assertion in (i) from Proposition 2.2 and Theorem 3.1. If $0 < \varepsilon_1 < \varepsilon_2$, then $\mathfrak{u}'_{\varepsilon_1}(\omega) < p_{\varepsilon_2}(\omega, \mathfrak{u}_{\varepsilon_1}(\omega))$ (since $b(\omega)\,\mathfrak{u}_{\varepsilon_1}(\omega) + a(\omega) > b(\omega)\,s_+ + a(\omega) \geq 0$), and hence, as explained in Remark 2.4.2, $\mathfrak{u}_{\varepsilon_1} < \mathfrak{u}_{\varepsilon_2}$: $\varepsilon \mapsto \mathfrak{u}_\varepsilon$ is strictly increasing on $(0, \infty)$. To prove the monotonicity of $\varepsilon \mapsto \mathfrak{m}_\varepsilon$, we use an argument analogous to that of the last paragraph in the proof of Theorem 3.4(iii), working on the interval $(0, \varepsilon^*)$ (perhaps $(0, \infty)$) on which $b(\omega)\,\mathfrak{m}_\varepsilon(\omega) + a(\omega) < 0$: this happens if $\mathfrak{m}_\varepsilon < s_-$, and hence at least for small values of $\varepsilon > 0$, since $\lim_{\varepsilon\to 0^+} \mathfrak{m}_\varepsilon(\omega) = 0$ uniformly on Ω. The lack of strictly positive bifurcation values of ε is a trivial consequence of the previous properties.

(ii) Let us fix $\varepsilon < 0$. By hypotheses, $s_- = s_0$, with s_0 defined in Theorem 3.6(i), and this result ensures that $\mathcal{A}_\varepsilon \subset \Omega \times (s_-, \infty)$. In addition, if $r \geq c_+$, then $p_\varepsilon(\omega, r) < 0$ for all $\omega \in \Omega$, since $c(\omega) - r \leq 0$ and $b(\omega)\,r + a(\omega) \geq b(\omega)\,c_- + a(\omega) > 0$. According to Remark 2.4.1, $\mathcal{A}_\varepsilon \subset \Omega \times (s_-, c_+)$.

Since $\{\mathfrak{u}_0\}$ is a hyperbolic copy of the base and $\mathfrak{u}_0 \geq c_- > s_+$ (see Theorem 3.4(i)), the persistence ensured by [14, Theorem 2.3] guarantees the existence of a hyperbolic copy of the base strictly greater than s_+ for $\varepsilon < 0$ close enough to 0, and hence the definition of $\mathfrak{u}_\varepsilon$ shows that $\mathfrak{u}_\varepsilon > s_+$ for these values of ε. Let $(\varepsilon_*, 0) \subseteq (-\infty, 0)$ be the interval of persistence of this property (on which we cannot guarantee the continuity of $\mathfrak{u}_\varepsilon$). If $\varepsilon_* < \varepsilon_1 < \varepsilon_2 < 0$, then $\mathfrak{u}'_{\varepsilon_1}(\omega) < p_{\varepsilon_2}(\omega, \mathfrak{u}_{\varepsilon_1}(\omega))$ (since $b(\omega)\,\mathfrak{u}_{\varepsilon_1}(\omega) + a(\omega) > b(\omega)\,s_+ + a(\omega) \geq 0$), and hence Remark 2.4.2 ensures that $\mathfrak{u}_{\varepsilon_1} < \mathfrak{u}_{\varepsilon_2}$, as asserted.

Let us check that $\mathcal{A}_\varepsilon$ is an attractive hyperbolic copy of the base if $-\varepsilon$ is large enough. We deduce from $b_- > 0$ that $\int_\Omega (p_\varepsilon)_x(\omega, \mathfrak{b}_\varepsilon(\omega))\,dm < 0$ for all m-measurable equilibrium $\mathfrak{b}_\varepsilon \colon \Omega \to (s_-, c^+)$ (i.e., with graph contained in $\mathcal{A}_\varepsilon$) if, let's say, $\varepsilon < \varepsilon_0 < 0$. As explained in the proof of Theorem 3.7(iv), this ensures that all the Lyapunov exponents of the global attractor are strictly negative, and hence the assertion follows from Theorem 2.6.

(iii) Let us assume that $c_+ < 3\,s_-$. To check the first assertion in (iii), we use again an argument similar to that used in the proofs of Theorems 3.4(v) and 3.7(iii). We fix $\varepsilon < 0$; we define $q_\varepsilon(\omega, x)$ as the $C^{0,2}(\Omega \times \mathbb{R}, \mathbb{R})$ function which coincides with $p_\varepsilon(\omega, x)$ for $x \geq s_-$ and is given by a second degree polynomial for $x \leq s_-$; we check that $(\partial^2/\partial x^2)\,q_\varepsilon(\omega, x) < 0$ for $x \geq s_-$ and for $x < s_-$; and we deduce from this and its shape that q_ε satisfies all the conditions **c1**-**c4** of [13, Section 3]. In addition, s_- is a strict global lower solution for our $\varepsilon < 0$, since $q_\varepsilon(\omega, s_-) = p_\varepsilon(\omega, s_-) > 0$ (which follows from $s_- < c_+$). Hence, there exists exactly a minimal set $\mathcal{M}^u_\varepsilon$ for the skew-product flow $\tilde{\tau}_\varepsilon$ defined by $x' = q_\varepsilon(\omega{\cdot}t, x)$, and it is hyperbolic attractive and strictly above $\Omega \times \{s_-\}$. Since any τ_ε-minimal set is strictly above $\Omega \times \{s_-\}$, $\mathcal{M}^u_\varepsilon$ is the unique one, and since its Lyapunov exponents are the same for p_ε as for q_ε (i.e., negative), then Theorem 2.6 ensures that $\mathcal{A}_\varepsilon = \mathcal{M}_\varepsilon$ is an attractive hyperbolic copy of the base, $\mathcal{A}_\varepsilon = \{\mathfrak{u}_\varepsilon\}$. The usual persistence argument shows the continuity of $\varepsilon \mapsto \mathfrak{u}_\varepsilon$ on $\mathbb{R}$, and the last assertion in (iii) is a consequence of the previous analysis.

(iv) Let us assume that $a/b = -s \in \mathbb{R}$. By reviewing the proofs of (i) and (ii), we see that $\varepsilon^* = \infty$ (since $s_- = s_+$ and hence $\mathfrak{m}_\varepsilon < s_-$ for all $\varepsilon > 0$) and $\varepsilon_* = -\infty$ (since $s_+ = s_-$ and hence $\mathfrak{u}_\varepsilon > s_+$ for all $\varepsilon < 0$). We fix $\varepsilon_1 < \varepsilon_2 < 0$ and deduce from $\mathfrak{l}_{\varepsilon_2} > s_- = s_+$ (see (ii)) that $\mathfrak{l}'_{\varepsilon_2}(\omega) > p_{\varepsilon_1}(\omega, \mathfrak{l}_{\varepsilon_2}(\omega))$ for all $\omega \in \Omega$, so $\mathfrak{l}_{\varepsilon_2}(\omega) > \mathfrak{l}_{\varepsilon_1}(\omega)$ (see Remark 2.4.2). Let us check that $\lim_{\varepsilon \to \infty} \mathfrak{m}_\varepsilon(\omega) = s$ uniformly on Ω. We take $\delta > 0$. Since $p_\varepsilon(\omega, s-\delta) = (s-\delta)^2(c(\omega) - s + \delta) - \varepsilon\, b(\omega)\,\delta$, we have that $s - \delta$ is a global strict upper solution if $\varepsilon > 0$ is large enough, and $(\omega, s-\delta) \in \mathcal{A}_\varepsilon$ for all $\omega \in \Omega$ and a large enough ε (see (i)), Remark 2.4.3 shows that, for these values of ε, $\Omega \times \{s - \delta\}$ is strictly below a τ_ε-minimal set contained in the α-limit set of a point $(\omega, s-\delta)$, which according to Remark 2.5 is necessarily $\{\mathfrak{m}_\varepsilon\}$. That is, $\mathfrak{m}_\varepsilon > s - \delta$ if ε is large enough, which combined with $\mathfrak{m}_\varepsilon < s$ proves the assertion. A similar argument, working with $s + \delta$, shows that $\lim_{\varepsilon \to -\infty} \mathfrak{u}_\varepsilon(\omega) = s$ uniformly on Ω.

The property $\mathcal{A}_\varepsilon \subset \Omega \times (s, c_+)$ for any $\varepsilon < 0$ (see (iii)) allows us to check that, if $\varepsilon_1 < \varepsilon_2 < 0$, any τ_{ε_1}-equilibrium is a global strict lower solution for τ_{ε_2}, and any τ_{ε_2}-equilibrium is a global strict upper solution for τ_{ε_1}. These properties are required several times in the steps leading to a detailed proof of the remaining assertions, which we only sketch.

Let us assume the existence of an $\varepsilon_1 < 0$ such that $\mathcal{A}_{\varepsilon_1}$ is not a hyperbolic copy of the base (which, according to (iii), ensures that $\varepsilon_0 \leq \varepsilon_1$ and is not possible if $c_+ < 3\,s$). To start with, we also assume that there are three τ_{ε_1}-minimal sets. We call $\mathcal{I}$ the maximal interval containing ε_1 at which this property holds. We know that $\mathcal{I} \subset [\varepsilon_0, 0]$ (see point (ii) and Theorem 3.4(i)), and that it is open. This property and those mentioned in the previous paragraph allow us to repeat the arguments leading to the proof of [11, Theorem 5.10] to conclude the existence of $\underline{\varepsilon}$ and $\bar{\varepsilon}$ with

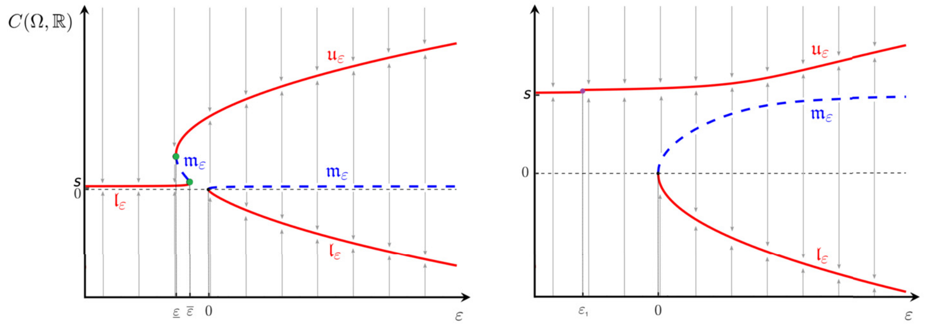

Fig. 2. The two panels represent two of the three possibilities for the bifurcation diagram of the ε-parametric family (3.1) when $c > 0$, $b > 0$, $a = -s\,b$ for $s \in (0, \infty)$, and $s < c$. See Theorem 3.8 (in combination with Theorems 3.4 and 3.6) for the results, and the caption of Fig. 1 for the meaning of the different elements. In the left panel, there are three bifurcation values of the parameter, all of them of local saddle-node type. In the right panel, the purple point over ε_1 depicts a pinched attractor containing a unique nonhyperbolic τ_{ε_1}-minimal set. In the non depicted bifurcation diagram, which holds at least if, in addition, $c < 3s$, a solid red upper continuous curve would represent the evolution of $\mathfrak{u}_\varepsilon$ as ε varies in $\mathbb{R}$, and hence $\varepsilon = 0$ would be the unique bifurcation value, of local saddle-node type.

$\varepsilon_0 \le \underline{\varepsilon} < \varepsilon_1 < \bar{\varepsilon} \le 0$ satisfying the stated properties. To check that $\bar{\varepsilon} < 0$, it is enough to observe that the lower $\tau_{\bar{\varepsilon}}$-minimal set is strictly above $\{0\}$, which is the lower τ_0-minimal set (see again Theorem 3.4(i)).

Now we assume that there are exactly two τ_{ε_1}-minimal sets. According to [11, Theorem 5.13(iii)], one of them is hyperbolic attractive. This allows us to repeat the arguments of [11, Theorem 5.12] to conclude the existence of three τ_{ε_2}-minimal sets for an $\varepsilon_2 < 0$ close to ε_1, and hence we are in the same situation of the previous paragraph (being in this case ε_1 one of the two negative bifurcation values).

The remaining case is that $\mathcal{A}_{\varepsilon_1}$, which is not a hyperbolic copy of the base, contains just one τ_{ε_1}-minimal set, which is necessarily nonhyperbolic: see Theorem 2.6(iii). The previous analysis shows that there exists just one τ_ε-minimal set for any $\varepsilon < 0$, and hence we can reason as in [11, Theorems 5.14 and 5.15] to conclude that the situation is the last one described in the statement of the theorem. □

It is easy to find autonomous examples of the three cases described in the previous point (iv), what makes sense of this case study. Fig. 2 depicts two of these three bifurcation diagrams of (3.1), appearing under the most restrictive conditions of Theorem 3.8.

The hypotheses of the next theorem are considerably more restrictive than those of the previous ones. We include this result by completeness, and point out that it completes the description of all the possibilities for the bifurcation diagrams in the autonomous case with $c > 0$, $b > 0$ and $a < 0$. Recall that ε_0 is a (nonautonomous) local transcritical bifurcation point when two hyperbolic copies of the base exist for close values of ε and approach each other as $\varepsilon \to (\varepsilon_0)$, giving rise to a locally unique τ_{ε_0}-minimal set, which is nonhyperbolic.

Theorem 3.9. *Assume that $b(\omega) > 0$, $c(\omega) \equiv s > 0$ (constant), and $a(\omega) = -s\,b(\omega)$ for all $\omega \in \Omega$. Let $\varepsilon_* := \sup \mathcal{I}$, with $\mathcal{I}$ defined in Theorem 3.4(iii). Then, $\varepsilon_* = s^2 / \int_\Omega b(\omega)\,dm$. In addition, $\{s\}$ is a τ_ε-copy of the base for all $\varepsilon \in \mathbb{R}$, it is hyperbolic attractive with $s = \mathfrak{u}_\varepsilon$ if and only if $\varepsilon < \varepsilon_*$, and it is hyperbolic repulsive with $s < \mathfrak{u}_\varepsilon$ if and only if $\varepsilon > \varepsilon_*$. More precisely, in addition to the information provided by Theorems 3.4 and 3.6,*

(i) *if* $\varepsilon < 0$*, then* $\mathcal{A}_\varepsilon = \{s\}$*, and it is hyperbolic attractive.*
(ii) *There exist exactly two* τ_0*-minimal sets:* $\{\mathfrak{l}_0\} = \{0\}$*, which is nonhyperbolic, and* $\{\mathfrak{u}_0\} = \{s\}$*, hyperbolic attractive.*
(iii) *If* $\varepsilon > 0$ *and* $\varepsilon \neq \varepsilon_*$*, there are three hyperbolic copies of the base, given by* $\mathfrak{l}_\varepsilon < \mathfrak{m}_\varepsilon < s$ *for* $\varepsilon \in (0, \varepsilon_*)$ *and by* $\mathfrak{l}_\varepsilon < s < \mathfrak{u}_\varepsilon$ *for* $\varepsilon > \varepsilon_*$*.*
(iv) *There exist exactly two* τ_{ε_*}*-minimal sets:* $\{\mathfrak{l}_\varepsilon\}$*, which is hyperbolic attractive, and* $\{s\}$*, nonhyperbolic.*
(v) *The maps* $[0, \infty) \to C(\Omega, \mathbb{R})$*,* $\varepsilon \mapsto -\mathfrak{l}_\varepsilon$*,* $[0, \varepsilon_*) \to C(\Omega, \mathbb{R})$*,* $\varepsilon \mapsto \mathfrak{m}_\varepsilon$ *(with* $\mathfrak{m}_0 := 0$*), and* $(\varepsilon_*, \infty) \to C(\Omega, \mathbb{R})$*,* $\varepsilon \mapsto \mathfrak{u}_\varepsilon$ *are continuous and strictly increasing. In addition, the semicontinuous maps* $\mathfrak{u}_{\varepsilon_*}$ *and* $\mathfrak{m}_{\varepsilon_*} := \lim_{\varepsilon \to (\varepsilon^*)^-} \mathfrak{m}_\varepsilon$ *take the value* s *at their continuity points.*

Therefore, there exist exactly two bifurcation points: 0*, of local saddle-node type, and* ε_**, of local transcritical type.*

Proof. Note that the hypotheses guarantee that $p_\varepsilon(\omega, x) = (x - s)(-x^2 + \varepsilon\, b(\omega))$. Since $p_\varepsilon(\omega, s) = 0$ for all $\omega \in \Omega$ and $\varepsilon \in \mathbb{R}$, $\{s\}$ is a τ_ε-copy of the base; and the remaining initial assertions follow from the fact that $\int_\Omega p_x(\omega, s)\, dm = -s^2 + \varepsilon \int_\Omega b(\omega)\, dm$ is its unique Lyapunov exponent: see Section 2.2.

(i) For $\varepsilon < 0$, $p_\varepsilon(\omega, r) < 0$ for all $\omega \in \Omega$ if $r > s$ and $p_\varepsilon(\omega, r) > 0$ for all $\omega \in \Omega$ if $r < s$. So, Remark 2.4.1 shows that $\mathcal{A}_\varepsilon = \{s\}$.

(ii) Since $\{s\}$ is an attractive hyperbolic τ_0-copy of the base, Theorem 3.4(i) proves (ii).

(iii) For $\varepsilon \in (0, \varepsilon_*)$, $l_\varepsilon < 0$ determines an attractive hyperbolic copy of the base (see Theorem 3.4(ii)), and s another one. Hence, there exists a repulsive hyperbolic copy of the base, $\{\mathfrak{m}_\varepsilon\}$, with $\mathfrak{l}_\varepsilon < \mathfrak{m}_\varepsilon < s$: see Theorem 3.1. For $\varepsilon > \varepsilon_*$, $\{\mathfrak{l}_\varepsilon\}$ provides an attractive hyperbolic copy of the base and $\{s\}$ a repulsive one, with $\mathfrak{l}_\varepsilon < s$. Hence, Theorem 3.1 ensures that $\{\mathfrak{u}_\varepsilon\}$ is also an attractive hyperbolic copy of the base above $\{s\}$.

(iv) Theorem 3.4(ii) shows that $\{\mathfrak{l}_{\varepsilon_*}\}$ is an attractive hyperbolic τ_{ε_*}-copy of the base. Since $\{s\}$ is a nonhyperbolic one, there are no more: see Theorem 3.1.

(v) Theorem 3.6(ii) proves the assertions concerning $\mathfrak{l}_\varepsilon$. Note now that the sets $\mathcal{I}$ of Theorem 3.4(iii) and $\mathcal{J}$ of Theorem 3.6(iii) are $(0, \varepsilon_*)$ and (ε_*, ∞). Since $\mathfrak{u}_\varepsilon > s$ for $\varepsilon > \varepsilon^*$, we get $(\mathfrak{u}_{\varepsilon_1})'(\omega) \geq p_{\varepsilon_2}(\omega, \mathfrak{u}_{\varepsilon_1}(\omega))$ for all $\omega \in \Omega$ if $\varepsilon_* < \varepsilon_1 < \varepsilon_2$. According to Remark 3.2, this ensures that $\mathfrak{u}_{\varepsilon_1} < \mathfrak{u}_{\varepsilon_2}$. This fact combined with Theorem 3.6(iii) proves the assertions concerning $\mathfrak{u}_\varepsilon$ on (ε_*, ∞). By reasoning as in the proof of Theorem 3.4(iii), we check that $(0, \varepsilon_*) \to C(\Omega, \mathbb{R})$, $\varepsilon \mapsto \mathfrak{m}_\varepsilon$ is strictly increasing, which combined with Theorem 3.4(iii) proves the assertions concerning $\mathfrak{u}_\varepsilon$ on $[0, \varepsilon_*)$. The monotonicity properties ensure the existence of the limits $\lim_{\varepsilon \to (\varepsilon_*)^+} \mathfrak{u}_\varepsilon \geq s$ and $\lim_{\varepsilon \to (\varepsilon_*)^-} \mathfrak{m}_\varepsilon \leq s$, and that they provide semicontinuous τ_{ε_*}-equilibria, and Proposition 2.1 ensures that they coincide with s at their continuity points. □

The bifurcation diagram described in Theorem 3.9 is depicted in Fig. 3.

Remark 3.10. For further purposes, we point out that the complete analysis can be repeated for if we change (3.1) by $x' = d(\omega)\, p_\varepsilon(\omega, x)$ with $d\colon \Omega \to (0, \infty)$ continuous and p_ε given by (3.2). In fact, d does not have influence on the global shape of the bifurcation diagram in the analysed cases: they still depend on the relation between c and $-b/a$.

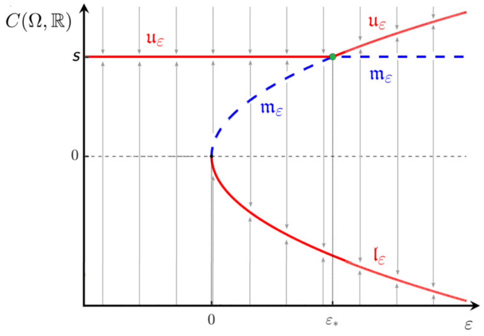

Fig. 3. The bifurcation diagram of the ε-parametric family (3.1) when $c = s$ for $s \in (0, \infty)$, $b > 0$ and $a = -s\,b$. In this case, 0 is a local saddle-node bifurcation point, and $\varepsilon^* := s^2 / \int_\Omega b(\omega)\, dm$ is a transcritical bifurcation point. This is proved in Theorem 3.9, combined with Theorems 3.4 and 3.6. The meaning of the different elements is explained in Fig. 1.

We complete the description of these bifurcation diagrams by pointing out that the bifurcation values ε_* and ε^* of Fig. 1, $\underline{\varepsilon}$, $\bar{\varepsilon}$ and 0 (resp. 0) of the left (resp. right) panel of Fig. 2, and 0 of Fig. 3, are points of discontinuity of the map $\varepsilon \to \mathcal{A}_\varepsilon$, in the sense explained in [8, Chapter 3]. (In fact, the map is upper-semicontinuous but not lower-semicontinuous at those points.)

3.2. The results for an ε-parametric family of ODEs

The results obtained so far in Section 3 provide a wealth of information about the evolution of the dynamics induced by the ε-parametric family of ODEs

$$x' = \bar{p}_\varepsilon(t, x)\,, \tag{3.4}$$

where ε varies in $\mathbb{R}$ and

$$\bar{p}_\varepsilon(t, x) := -x^3 + \bar{c}(t)\, x^2 + \varepsilon \left(\bar{b}(t)\, x + \bar{a}(t)\right)$$

for bounded and uniformly continuous maps $\bar{c}$, $\bar{b}$, $\bar{a}\colon \mathbb{R} \to \mathbb{R}$ such that the corresponding hull is minimal and uniquely ergodic.

More precisely, let Ω be the joint hull of $\bar{\omega} := (\bar{c}, \bar{b}, \bar{a})$, defined as the closure in the compact open topology of $C(\mathbb{R}, \mathbb{R}^3)$ of the time-shifts $\omega_t = (\bar{c}_t, \bar{b}_t, \bar{a}_t)$ (see Section 2.3), and let us define $c(\omega) = \omega_1(0)$, $b(\omega) = \omega_2(0)$ and $a(\omega) = \omega_3(0)$ for $\omega = (\omega_1, \omega_2, \omega_3) \in \Omega$. Note that $c(\bar{\omega}_t) = \bar{c}(t)$, $b(\bar{\omega}_t) = \bar{b}(t)$ and $a(\bar{\omega}_t) = \bar{a}(t)$. That is, (3.4) is one of the elements of the family

$$x' = -x^3 + c(\omega_t)\, x^2 + \varepsilon \left(b(\omega_t)\, x + a(\omega_t)\right), \qquad \omega \in \Omega\,. \tag{3.5}$$

There are well-known conditions ensuring the minimality and unique ergodicity of the time-shift flow on Ω, what we assume from now on. For instance, this is the case if $\bar{c}$, $\bar{b}$, $\bar{a}\colon \mathbb{R} \to \mathbb{R}$ are almost periodic functions.

Let us represent by $(3.4)_\varepsilon$ the equation corresponding to the value ε. The analysis of the global dynamics induced by $(3.4)_\varepsilon$ requires the analysis of the set $\mathcal{B}_\varepsilon$ of bounded solutions of $(3.4)_\varepsilon$, which is closely related to the attractor $\mathcal{A}_\varepsilon$ of $(3.5)_\varepsilon$: if $\mathfrak{l}_\varepsilon$ and $\mathfrak{u}_\varepsilon$ are the lower and upper bounded equilibria, and if $l_\varepsilon(t)=\mathfrak{l}_\varepsilon(\bar\omega{\cdot}t)$ and $u_\varepsilon(t)=\mathfrak{u}_\varepsilon(\bar\omega{\cdot}t)$ (for $\bar\omega=(\bar c,\bar b,\bar a)$), then $\mathcal{B}_\varepsilon:=\{(t,x)\,|\,l_\varepsilon(t)\le x\le u_\varepsilon(t)\}$. This is a consequence of Theorem 2.3. The analysis also relies on the number and type of hyperbolic solutions. Recall that a bounded solution $b(t)$ of $(3.4)_\varepsilon$ is *hyperbolic attractive* (resp. *hyperbolic repulsive*) if there exist $k\ge 1$ and $\gamma>0$ such that $\exp\Big(\int_s^t(\bar p_\varepsilon)_x(r,b(r))\,dr\Big)\le ke^{-\gamma(t-s)}$ whenever $t\ge s$ (resp. $\exp\Big(\int_s^t(\bar p_\varepsilon)_x(r,b(r))\,dr\Big)\le ke^{\gamma(t-s)}$ whenever $t\le s$). According to, e.g., [13, Theorems 5.3 and 5.6], $(3.4)_\varepsilon$ has at most three hyperbolic solutions, in which case l_ε and u_ε are hyperbolic attractive and the "middle one" is repulsive. In this case, the dynamics of $(3.4)_\varepsilon$ is completely determined by its hyperbolic solutions: see, e.g., [13, Theorem 5.6].

It is easy to establish conditions on $\bar c$, $\bar b$ and $\bar a$ which are equivalent to the hypotheses on c, b and a required in Theorems 3.4, 3.6, 3.7, 3.8 and 3.9:

- the conditions $c>0$, $a<0$, $b\ge 0$ and $b>0$ hold if and only if $\inf_{t\in\mathbb{R}}\bar c(t)>0$, $\sup_{t\in\mathbb{R}}\bar a(t)<0$, $\inf_{t\in\mathbb{R}}\bar b(t)\ge 0$ and $\inf_{t\in\mathbb{R}}\bar b(t)>0$, respectively;
- the conditions $b(\omega)\,c_+ + a(\omega)<0$, $b(\omega)\,c_- + a(\omega)>0$, $c_+<3\,c_-$ and $c_+<3\,s_-$ for $s_-=\inf_{\omega\in\Omega}(-a(\omega)/b(\omega))$ are equivalent to $\sup_{t\in\mathbb{R}}\bar c(t)<\inf_{t\in\mathbb{R}}(-\bar a(t)/\bar b(t))$, $\inf_{t\in\mathbb{R}}\bar c(t)>\sup_{t\in\mathbb{R}}(-\bar a(t)/\bar b(t))$, $\sup_{t\in\mathbb{R}}\bar c(t)<3\inf_{t\in\mathbb{R}}\bar c(t)$ and $\sup_{t\in\mathbb{R}}\bar c(t)<3\inf_{t\in\mathbb{R}}(-\bar a(t)/\bar b(t))$, respectively;
- and the equality $a=-s\,b$ for a constant $s\in\mathbb{R}$ holds if and only if $\bar a=-s\,\bar b$.

Let us give an example of how to apply the previous results to the analysis of the parametric variation of (3.4). (Another one, more precise, will be given in Section 4.)

Proposition 3.11. *Let us assume that* $\inf_{t\in\mathbb{R}}\bar c(t)>0$, $\inf_{t\in\mathbb{R}}\bar b(t)>0$, $\bar a(t)=-s\,\bar b(t)$ *for a constant* $s\in(0,\infty)$ *and all* $t\in\mathbb{R}$, *and* $\inf_{t\in\mathbb{R}}\bar c(t)>s$. *Then,*

(i) l_0 *is an attractive hyperbolic solution of* $(3.4)_0$ *and* u_0 *is a nonhyperbolic solution, with* $\inf_{t\in\mathbb{R}}\bar c(t)<u_0(t)<\sup_{t\in\mathbb{R}}\bar c(t)$. *In addition,* $u_0(t)-c(t)$ *changes sign at the points of a strictly increasing two-sided sequence* $(s_n)_{n\in\mathbb{Z}}$.

(ii) *For all* $\varepsilon>0$, $(3.4)_\varepsilon$ *has three hyperbolic solutions* $l_\varepsilon<m_\varepsilon<u_\varepsilon$, *with* $l_\varepsilon<0<m_\varepsilon<s<u_\varepsilon$, *with* l_ε *and* u_ε *attractive and* m_ε *repulsive. In addition,* $\lim_{\varepsilon\to\infty}u_\varepsilon(t)=\infty$, $\lim_{\varepsilon\to\infty}l_\varepsilon(t)=-\infty$, $\lim_{\varepsilon\to\infty}m_\varepsilon(t)=s$ *and* $\lim_{\varepsilon\to 0^+}(u_\varepsilon(t)-u_0(t))=\lim_{\varepsilon\to 0^+}l_\varepsilon(t)=\lim_{\varepsilon\to 0^+}m_\varepsilon(t)=0$, *all of them uniformly on* $\mathbb{R}$, *and the maps* $(0,\infty)\to C(\mathbb{R},\mathbb{R})$, $\varepsilon\mapsto -l_\varepsilon, m_\varepsilon, u_\varepsilon$ *are strictly increasing.*

(iii) *For all* $\varepsilon<0$, $\mathcal{A}_\varepsilon\subset\Omega\times(s,\sup_{t\in\mathbb{R}}\bar c(t))$; *the map* $(\infty,0]\to\mathbb{R}$, $\varepsilon\mapsto u_\varepsilon$ *is strictly increasing; and there exists* $\varepsilon_0\le 0$ *such that, for* $\varepsilon\in(-\infty,\varepsilon_0)$, *there is a unique bounded solution (given by* $l_\varepsilon=u_\varepsilon$*), which is hyperbolic attractive. If, in addition,* $\sup_{t\in\mathbb{R}}\bar c(t)<3s$, *then* $\varepsilon_0=0$,

(iv) *There are three possibilities for* $\varepsilon<0$*:*
   - *the value* ε_0 *of* (iii) *is* 0. *(This is the situation if* $\sup_{t\in\mathbb{R}}\bar c(t)<3s$.*)*
   - *There exist* $\underline{\varepsilon}<\bar\varepsilon<0$ *such that:* $(3.4)_\varepsilon$ *has three hyperbolic solutions for any* $\varepsilon\in(\underline{\varepsilon},\bar\varepsilon)$; $l_\varepsilon=u_\varepsilon$ *and it is hyperbolic attractive for* $\varepsilon\in(0,\infty)-[\underline{\varepsilon},\ \bar\varepsilon\,]$; $l_{\underline{\varepsilon}}$ *is the unique hyperbolic solution of* $(3.4)_{\underline{\varepsilon}}$, *it is attractive, and it is uniformly separated from* $u_{\underline{\varepsilon}}$; $u_{\bar\varepsilon}$ *is the unique hyperbolic solution of* $(3.4)_{\bar\varepsilon}$, *it is attractive, and it is uniformly separated from* $l_{\bar\varepsilon}$; *the maps* $(-\infty,\bar\varepsilon)\to C(\mathbb{R},\mathbb{R})$, $\varepsilon\to l_\varepsilon$ *and* $(\underline{\varepsilon},0)\to C(\mathbb{R},\mathbb{R})$, $\varepsilon\to u_\varepsilon$ *are strictly increasing; and the map* $(\underline{\varepsilon},\bar\varepsilon)\to C(\mathbb{R},\mathbb{R})$, $\varepsilon\to m_\varepsilon$ *is strictly decreasing.*

- *There is a unique point* $\varepsilon_1 < 0$ *such that* $(3.4)_{\varepsilon_1}$ *has bounded solutions but not hyperbolic ones. In this case,* $\inf_{t\in\mathbb{R}}(u_{\varepsilon_1}(t) - l_{\varepsilon_1}(t)) = 0$.

The proof relies on applying Theorem 3.8 to the families $(3.5)_\varepsilon$ constructed from $(3.5)_\varepsilon$. It also uses that: a hyperbolic copy of the base $\{\mathfrak{b}\}$ for $(3.5)_\varepsilon$ determines a hyperbolic solution of each equation $(3.5)_\varepsilon^\omega$, given by $t \mapsto \mathfrak{b}(\omega{\cdot}t)$ (see, e.g., [13, Proposition 2.7]); and that, since Ω is minimal, if $\mathcal{M}$ is a nonhyperbolic τ_ε-minimal set and $(\omega, x) \in \mathcal{M}$, then the solution $v_\varepsilon(t, \omega, x)$ of $(3.5)_\varepsilon^\omega$ is nonhyperbolic (see, e.g., [10, Proposition 1.54]). We leave the (easy) details to the reader.

We complete this section pointing out that, in the conditions of the previous result, the possibly complicated dynamics arising at $\underline{\varepsilon}$ reads as: it is possible that $u_{\underline{\varepsilon}}$ is the pointwise limit of u_ε and of m_ε as $\varepsilon \to (\underline{\varepsilon})^+$, but not sure. This happens if the point $\bar{\omega} = (\bar{c}, \bar{b}, \bar{a})$ belongs to a residual subset of Ω, impossible to determine a priori, which in addition can have measure (m) 0 or 1. In many situations (as when the initial coefficients are constants or periodic maps) this residual set is the whole Ω. If not, we just know that $u_{\underline{\varepsilon}} \geq \bar{u}_{\underline{\varepsilon}} \geq m_{\underline{\varepsilon}}$, with $\bar{u}_{\underline{\varepsilon}}(t) := \lim_{\varepsilon\to(\underline{\varepsilon})^+} u_\varepsilon(t)$ and $m_{\underline{\varepsilon}}(t) := \lim_{\varepsilon\to(\underline{\varepsilon})^+} m_\varepsilon(t)$. The situation is analogous at $\bar{\varepsilon}$ with $l_{\bar{\varepsilon}}$ and limits as $\varepsilon \to (\bar{\varepsilon})^-$; but not necessarily the same, since the residual set can be different.

4. Numerical simulations in a population dynamics model

In this section, we study a single species population model that undergoes quasiperiodic fluctuations (see for example [47] and references therein where experimental evidence of quasi-periodic behavior in population dynamics can be found). We take into account the interplay between the Allee effect (see for example [9]) and migration phenomena, both affected by seasonality. The model is

$$x' = r(t)\, x^2\, (1 - x/k(t)) + \varepsilon\, b(t)\, (x - s)\,, \tag{4.1}$$

with $\varepsilon \geq 0$. The value $\varepsilon = 0$ of the parameter corresponds to the absence of migration. The maps $r(t)$ and $k(t)$ are positively bounded from below: $r(t)$ represents the intrinsic growth rate, i.e., the growth rate in case of unlimited resources, and the function $k(t)$ is closely related to the carrying capacity although not exactly equal (as in the autonomous case): it measures the threshold below which the per capita population growth rate x'/x decreases. Changing the x factor of logistic models to x^2 is a common way to include the weak Allee effect: the per capita population growth rate is reduced at lower density (as the solution $x = 0$ loses hyperbolicity).

The additive term $\varepsilon\, b(t)\, (x - s)$ is related to migration. The map $\varepsilon\, b(t)$, where b is positively bounded from below, represents the (seasonality-dependent) intensity of migration, while s is a positive constant representing the threshold of population attractiveness: there is immigration (population increases) if the population is sufficiently high, and emigration if it is below s. The idea fits well with that of the Allee effect: a sufficient number of individuals increases the chances of survival of the population. In fact, as we will explain later, for small and large values of $\varepsilon > 0$ (or even for all $\varepsilon > 0$) there appear two strictly positive hyperbolic solutions of $(4.1)_\varepsilon$: a critical population $m_\varepsilon(t)$ (repulsive) that provides a threshold below which the population will die out, and a stable (attractive) healthy population $u_\varepsilon(t)$ above this threshold. This is the usual situation under strong Allee effect. Note that $x = 0$ is not a solution of $(4.1)_\varepsilon$ if $\varepsilon > 0$, so that extinction in finite time is possible: at the moment in which x reaches 0, the population disappears and the model becomes meaningless. (Also observe that, for $b > 0$, a negative value of ε changes the role of s: there would be immigration for a lower number of individuals, and this makes sense too. Although we will focus on $\varepsilon \geq 0$, the previously obtained theory also provides conclusions for $\varepsilon < 0$.)

A variation of ε means a variation in the migratory intensity which may give rise to different population dynamics. The coefficient maps r, k and b are chosen to get a quasiperiodic map $(r, k, b)\colon \mathbb{R} \to \mathbb{R}^3$, so the flow on its hull is minimal and uniquely ergodic. Note that (4.1) is $x' = (r(t)/k(t))\,(-x^3 + k(t)\,x^2 + \varepsilon\, k(t)\, b(t)\,(x - s)/r(t))$. Since $\inf_{t\in\mathbb{R}} r(t)/k(t) > 0$, the hull extension provides one of the families considered in Remark 3.10, and hence all the results of Section 3 can be applied to describe the bifurcation diagram of the ε-parametric family of skew-product flows. In line with Section 3.2, in what follows we just focus on the family of processes instead of flows. It is easy to check that the conditions $\sup_{t\in\mathbb{R}} k(t) < s$, $\inf_{t\in\mathbb{R}} k(t) > s$ and $k(t) \equiv s$ lead us respectively to the situations of Theorems 3.7, 3.8 and 3.9, under their most restrictive hypotheses. So, in all these cases, we have already proved the previously mentioned existence of two strictly positive hyperbolic solutions $m_\varepsilon < u_\varepsilon$ for small or large $\varepsilon > 0$, where m_ε is repulsive and u_ε is attractive, and where $\lim_{\varepsilon\to 0^+} m_\varepsilon(t) = 0$ uniformly on $\mathbb{R}$. The solution m_ε acts as a threshold for survival: if, for an $\varepsilon > 0$, $x_\varepsilon(0) < m_\varepsilon(0)$ or, equivalently, if $x_\varepsilon(t_0) < m_\varepsilon(t_0)$ for any $t_0 > 0$, then the population becomes extinct in finite time; whereas, if $x_\varepsilon(0) > m_\varepsilon(0)$, the population eventually "reaches" (i.e., "approaches until being undistinguishable from") the healthy steady population $u_\varepsilon(t)$. In addition, $\lim_{\varepsilon\to\infty} u_\varepsilon(t) = \infty$ uniformly on $\mathbb{R}$, and so the resources that the stable population requires exceed the capacity of the environment if ε is sufficiently large: the increase of u_ε is due to a massive influx of individuals, difficult to imagine for any reasonable population. But the model makes perfect sense for a not too large migratory intensity. In addition, for $\varepsilon = 0$, any initial number of individuals gives rise to a population which eventually reaches the (hyperbolic attractive) stable population $u_0(t)$, where $\inf_{t\in\mathbb{R}} k(t) < u_0(t) < \sup_{t\in\mathbb{R}} k(t)$ for all $t \in \mathbb{R}$, and where $u_0 - k$ changes sign infinitely many times as t increases. This proves the aforementioned close relation between $k(t)$ and the steady population in the absence of migration.

First, let us assume $\inf_{t\in\mathbb{R}} k(t) > s$, which is the situation of Proposition 3.11: roughly speaking, the threshold of population attractiveness is below the carrying capacity in the absence of migration. So, everything works properly: only a population that is initially too low dies out, since, for (ε-relatively) small x, emigration dominates over intrinsic growth. More precisely, Proposition 3.11 ensures the absence of strictly positive bifurcation values of ε: the strictly positive hyperbolic solutions m_ε and u_ε exist for all $\varepsilon > 0$. However, their monotonicity properties with respect to ε (see Fig. 2), and their behavior as $\varepsilon \to 0^+$ and as $\varepsilon \to \infty$, give rise to a critical value of ε_{x_0} if we fix an initial number of individuals $x_\varepsilon(0) = x_0 < s$: this ensures emigration for small $t > 0$, and implies that the population $x_\varepsilon(t)$ survives reaching the steady one if and only if this emigration is not too intense. More precisely, there exists $\varepsilon_{x_0} > 0$ such that $x_\varepsilon(0) < m_\varepsilon(0)$ if $0 < \varepsilon < \varepsilon_{x_0}$ and $x_\varepsilon(0) > m_\varepsilon(0)$ if $\varepsilon > \varepsilon_{x_0}$, and this means survival if $\varepsilon < \varepsilon_{x_0}$ (and in the unstable situation $\varepsilon = \varepsilon_{x_0}$, with $x_{\varepsilon_{x_0}} = m_{\varepsilon_{x_0}}$), and extinction in finite time for $\varepsilon > \varepsilon_{x_0}$. If the initial population is $x_0 \geq s$, then it survives for all $\varepsilon > 0$.

Now, let us analyze the case $k \equiv s$, adapting the information of Theorem 3.9 (see also Fig. 3). In this case, the threshold of attractiveness coincides with the (constant) carrying capacity in the absence of migration. If $\varepsilon_* = s^2/\bar{b}$ for $\bar{b} = \lim_{t\to\infty}(1/t)\int_0^t b(s)\,ds$, then s is a constant solution for all $\varepsilon > 0$, hyperbolic attractive if and only if $\varepsilon < \varepsilon_*$ and hyperbolic repulsive if and only if $\varepsilon > \varepsilon_*$. As in the previous case, an initial population $x_\varepsilon(0) = x_0 < s$ only survives while the migration intensity is low enough to yield $x_0 < m_\varepsilon(0)$. Now, this situation ends for sure at a value $\varepsilon_{x_0} \leq \varepsilon_*$ of the parameter, beyond which our population is doomed to extinction. In addition, again as in the previous case, the population survives for any ε if $x_0 \geq s$. In terms of hyperbolic solutions, there are two strictly positive ones $m_\varepsilon < u_\varepsilon$ for all $\varepsilon > 0$, $\varepsilon \neq \varepsilon_*$, with $u_\varepsilon = s$ for $\varepsilon < \varepsilon_*$ and $m_\varepsilon = s$ for $\varepsilon > \varepsilon^*$. They approach each other as $\varepsilon \to \varepsilon_*$, and their limits at this point are non uniformly separated bounded solutions (which may coincide, as in the autonomous and periodic cases). So, there is a (nonautonomous) transcritical bifurcation point at ε_*.

In the rest of this section, we analyze the case $\sup_{t\in\mathbb{R}} k(t) < s$, which we illustrate below with the help of a specific example. Now, the threshold of population attractiveness is above the (again roughly speaking) carrying capacity of the environment without migration. For instance, a low population may lose attractiveness even with sufficient resources, leading to emigration; and immigration may occur if there are nearby patches occupied by the same species but with fewer resources (and this causes population stress). The precise information about the variation as ε increases is provided by Theorem 3.7 (see also Fig. 1), that shows the existence of two bifurcation values $\varepsilon_* < \varepsilon^*$ on $(0, \infty)$ for the ε-parametric family of skew-products. They can also be read as bifurcation values for our initial process (4.1). Note that a population that reaches a number of individuals less than s stays below s (since s is an upper solution), and its survival is only possible (not sure) if $\varepsilon \leq \varepsilon_*$ is small (see Fig. 1). In other words, a low population is subject to emigration, and even if the first term on the right-hand side of $(4.1)_\varepsilon$ is positive, it is not sufficient to prevent extinction if the migration intensity is relatively large.

Let us now analyze the situation when $\varepsilon \in (0, \varepsilon_*]$. If $0 < x_\varepsilon(0) = x_0 < m_{\varepsilon_*}(0) := \lim_{\varepsilon\to(\varepsilon_*)^-} m_\varepsilon(0)$, then the population gets extinct in finite time when the migratory intensity exceeds a certain value $\varepsilon_{x_0} \in (0, \varepsilon_*)$ (with $x_0 = m_{\varepsilon_{x_0}}(0)$). If $x_0 \geq m_{\varepsilon_*}(0)$, then the population survives for all $\varepsilon \in [0, \varepsilon_*]$. But the steady population u_ε eventually reached, which is below s, decreases as ε increases: even if the initial population is above s and hence there is immigration during a time, the lack of resources causes a decrease, so the population eventually reaches s, there is emigration forever, and a more intense emigration means a lower steady population.

For $\varepsilon \in (\varepsilon_*, \varepsilon^*)$, no matter how large the initial population is, extinction arrives in finite time: the intensity of emigration once the population is below s causes a decrease which is not compensated by the intrinsic growth rate. Finally, for $\varepsilon \geq \varepsilon^*$: if $0 < x_0 < m_\varepsilon(0)$ (as for all $\varepsilon \geq \varepsilon^*$ if $x_0 \leq s$), then the population gets extinct in finite time; and if $x_0 > s$, then population survives for $\varepsilon \geq \varepsilon^{x_0}$, where this second critical value $\varepsilon^{x_0} \geq \varepsilon^*$ satisfies $m_{\varepsilon^{x_0}}(0) \leq x_0$ (which exists since $m_\varepsilon(0)$ decreases to s as $\varepsilon \to (\varepsilon^*)^+$). So, a high enough initial population can compensate the stress caused by immigration. But now, the only factor that allows the population to survive is immigration, and hence the model loses credibility for a high value of ε.

In terms of hyperbolic solutions, ε_* and ε^* are points at which two hyperbolic solutions m_ε and u_ε approach each other (as $\varepsilon \to (\varepsilon_*)^-$ and as $\varepsilon \to (\varepsilon^*)^+$), giving rise to the local absence of hyperbolic solutions: they are replaced by limits of the monotonic families $\{u_\varepsilon\}$ and $\{m_\varepsilon\}$, which may globally coincide or not, but which are never uniformly separated.

Let us illustrate these theoretical results with a particular example. The quasiperiodic fluctuations in the population dynamics are represented by taking $r \equiv 1$, and $k > 0$ and $b > 0$ periodic with incommensurate oscillation frequencies. More precisely, we work with

$$x' = -\frac{1}{2 + 0.5\sin(\sqrt{3}\,t)}\,x^3 + x^2 + \varepsilon\,(2.1 + 0.3\cos(t))\,(x - 2.6)\,, \tag{4.2}$$

so that $b(t) := 2.1 + 0.3\cos(t)$ and $s := 2.6 > 2 + 0.5\sin(\sqrt{3}\,t) =: k(t)$ for all $t \in \mathbb{R}$: we are in the third of the cases described above. In what follows, we detect numerically the bifurcation values

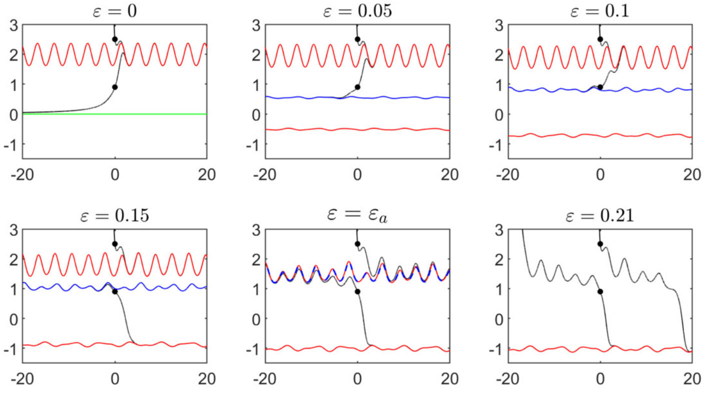

Fig. 4. Equation (4.2). Panels 1 to 5: Depiction of u_ε, m_ε and l_ε as ε increases towards ε_a. Panel 6: behavior of solutions for $\varepsilon > \varepsilon_a$.

ε_* and ε^* for (4.2) and approximate the hyperbolic copies of the base for different parameter values.

In order to approximate ε_*, we reason as follows. We choose ε_a^0 and ε_b^0 such that $(4.2)_\varepsilon$ has three hyperbolic solutions for $\varepsilon = \varepsilon_a^0$ and just one for $\varepsilon = \varepsilon_b^0$. The existence of the hyperbolic solutions has been detected numerically solving (4.2) with a 4-th order Runge-Kutta method and constant discretization stepsize $h = 2^{-10}$. We obtain analogous numerical results also for smaller and larger stepsizes. We solve initial value problems in $[-10^4, 10^4]$ for forward integration and in $[10^4, -10^4]$ for backward integration. We then apply a bisection procedure to the starting interval $[\varepsilon_a^0, \varepsilon_b^0]$ and locate ε_* in $[\varepsilon_a, \varepsilon_b] = [0.201945926862769 \;\; 0.201945926863700]$. Note that $(\varepsilon_b - \varepsilon_a) = O(10^{-12})$. We reason in a similar way for ε^* and locate it inside the interval $[9.129175817935083, \;\; 9.129175817935174]$.

In Fig. 4 we depict solutions of (4.2) for different values of ε in a neighborhood of ε_*. In all six panels we plot in red the attractive hyperbolic solutions (including the lower bounded solution l_ε, which is negative for $\varepsilon > 0$), in blue the repulsive one, and in black other solutions. We use same initial conditions in all panels for the solutions in black, namely $x_\varepsilon(0) = 2.5$ and $x_\varepsilon(0) = 0.9$. These initial data are marked on all panels. The first panel of Fig. 4 corresponds to $\varepsilon = 0$: the model does not contemplate migration. In addition to the already described behavior, it is worth to observe that an initially low number of individuals results in a low population over a long period (as a consequence of the weak Allee effect). The solution $l_0 = m_0 = 0$ is depicted in green, since it is nonhyperbolic. Panels 2, 3, and 4 correspond to three values of $\varepsilon < \varepsilon_*$, so that $(4.2)_\varepsilon$ has three hyperbolic solutions: u_ε and l_ε, depicted in red and m_ε, in blue. The solution m_ε acts as a threshold for survival. We call attention to the variation of the solution depicted in black with initial condition $x_\varepsilon(0) = x_0 = 0.9$: in panels 2 and 3, it converges towards u_ε, while in panel 4 it converges towards l_ε and hence it gets extinct in finite time; so, the critical value ε_{x_0} described above is inside $(0.1, 0.15)$. For $\varepsilon = \varepsilon_a$ we can not distinguish between u_ε and m_ε: the two solutions seem to collide (for sure, they are not uniformly separated at ε_*) and hence they lose hyperbolicity. Populations above u_{ε_a} survive, while populations below it get extinct. Finally for $\varepsilon = 0.21$, which is between ε_* and ε^*, the population is always doomed to extinction.

To complete the analysis, we explore the behavior of u_ε and m_ε as ε increase towards ε_a, to graphically show that they approximate each other and to illustrate their hyperbolicity loss as $\varepsilon \to (\varepsilon_*)^+$. In the first and second panel of Fig. 5 we respectively plot u_ε and m_ε for different

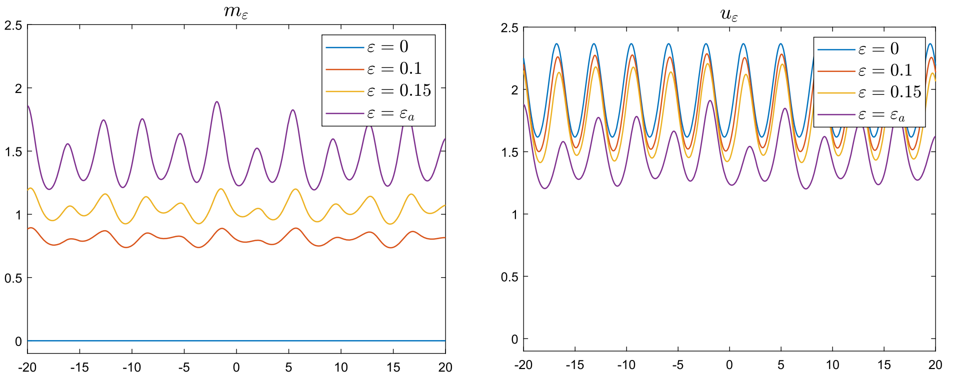

Fig. 5. Equation (4.2). Behavior of m_ε and u_ε as ε increases towards ε_a. The maps m_{ε_a} and u_{ε_a} are indistinguishable in the representation window.

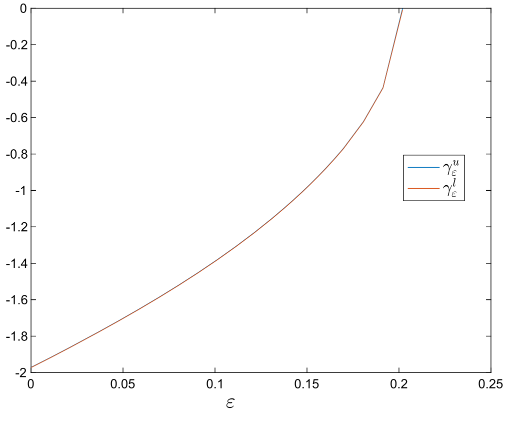

Fig. 6. Truncated Lyapunov exponents of u_ε as ε varies.

values of ε. The plots show that u_ε decreases towards u_{ε_a} and that m_ε increases towards m_{ε_a}: see Theorem 3.7(i). In order to verify the loss of hyperbolicity, we compute the Lyapunov exponent of u_ε as ε increases towards ε_a. The Lyapunov exponent is computed truncating (2.4) at a large enough time T. Equation (4.2) is quasiperiodic, hence the Lyapunov exponent of u_{ε_a} exists as a limit. Nonetheless, we compute upper and lower approximations of γ_{ε_a} in order to locate the Lyapunov exponent in a given interval. To this purpose, together with T, we also use a finite time τ, with $0 \ll \tau < T$. We then take as lower (resp. upper) approximation the minimum (maximum) of all the truncated exponents for $t \in [\tau, T]$. We denote this lower and upper approximations respectively as $\gamma^l_{\varepsilon_a}$ and $\gamma^u_{\varepsilon_a}$. The values that we obtain are showed in Table 1. From this table, the linear convergence of the Lyapunov exponent to 0 is evident, confirming the loss of hyperbolicity of u_ε at the bifurcation value. Finally, in Fig. 6, we plot the Lyapunov exponents γ^l_ε and γ^u_ε of u_ε as ε varies in $[0, \varepsilon_a]$. In the plot, the upper and lower exponent approach zero as ε approaches ε_a, witnessing a loss of hyperbolicity of u_ε as $\varepsilon \to \varepsilon_*$.

Table 1
Lower and upper approximations of the Lyapunov exponent of u_{ε_a} for different values of the truncated time τ and T.

T	τ	$\gamma^l_{\varepsilon_a}$	$\gamma^u_{\varepsilon_a}$
10^3	10^2	-1.8×10^{-2}	3.1×10^{-3}
10^4	10^3	-9.4×10^{-3}	-1.2×10^{-3}
10^5	10^4	-1.3×10^{-3}	-1.5×10^{-4}

Data availability

No data was used for the research described in the article.

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