



Positive and negative selective assortment

Segismundo S. Izquierdo ^{a,*}, Luis R. Izquierdo ^b, Christoph Hauert ^c

^a Department of Industrial Organization and BioEcoUva, Universidad de Valladolid, Spain

^b Department of Management Engineering, Universidad de Burgos, Spain

^c Department of Mathematics and Department of Zoology, University of British Columbia, Vancouver, Canada

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ABSTRACT

In populations subject to evolutionary processes, the assortment of players with different genes or strategies can have a large impact on players' payoffs and on the expected evolution of each strategy in the population. Here we consider assortment generated by a process of partner choice known as *selective assortment*. Under selective assortment, players looking for a mate can observe the strategies of a sample of potential mates or co-players, and select one of them to interact with. This selection mechanism can generate positive assortment (preference for players using the same strategy), or negative assortment (preference for players using a different strategy). We study the impact of selective assortment in the evolution and in the equilibria of a population, providing results for different games under different evolutionary dynamics (including the replicator dynamics).

1. Introduction

The standard model for random encounters of agents in Evolutionary Game Theory (Weibull, 1995; Sandholm, 2010) assumes uniform random matching in large (technically, infinite) well-mixed populations, meaning that any agent is equally likely to meet any other agent. Thus, under uniform random matching, the probability of interacting with an agent who uses strategy i (an i -player) equals the fraction of i -players in the population. Many scholars have pointed out that such situations are probably rare in nature, and argued in favor of studying deviations from the well-mixed model.

A first natural extension of the well-mixed model is to let the probability that two players interact depend on their individual strategies. For instance, if there is *positive assortment*, individuals preferentially interact with individuals of the same type; on the other hand, if there is *negative assortment*, individuals preferentially interact with individuals of a different type. We consider processes in which the average assortment of a type determines its expected payoff, and expected payoffs determine the population dynamics. An alternative and more detailed way of departing from the framework of well-mixed populations is to assume that players are embedded on an underlying network. In networks, the probability that a certain individual interacts with other individuals depends on the network configuration (the distribution of strategies over the locations of the network). Thus, in networks, local assortment can present considerable fluctuations with respect to average assortment.

Eshel and Cavalli-Sforza (1982) discuss two potential sources of assortment, focusing on positive assortment. The first source is called *struc-*

tural assortment, and is associated with situations in which players with different strategies happen to find themselves in different mating environments. This could be due, for instance, to spatial effects: descendants, who share common traits, are usually in close spatial proximity. Local reproduction or local imitation, combined with local interactions, also tend to generate positive assortment.

The second source of assortment is called *selective assortment*, and it assumes that, when looking for a mate or partner, players can meet a (small) number k of potential mates, observe their strategy (or some reliable and highly correlated proxy), and select one of those potential mates.

Naturally, both sources of assortment (structural and selective) can take place simultaneously: players may actively select mates in different potential-mate environments.

A reference model for positive assortment is the so-called *two-pool assortative matching process with constant assortativity* α (Eshel and Cavalli-Sforza, 1982; Bergstrom, 2003, 2013), which Eshel and Cavalli-Sforza (1982) interpret as a model of structural assortment. This model assumes that a player interacts:

- with probability $\alpha > 0$, with a player who uses the same strategy, and
- with probability $(1 - \alpha)$, with a random player from the population.

Equivalently (in terms of expected payoffs), one can suppose that all players in the population are matched in pairs, in a way such that a fraction α of the population is matched assortatively to individuals of their same strategy, and a fraction $(1 - \alpha)$ is matched uniformly at random.

* Corresponding author.

E-mail address: segismundo.izquierdo@uva.es (S.S. Izquierdo).

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Most models of assortment in the literature on population games (Alger and Weibull, 2013; Grafen, 1979; Alger and Weibull, 2010; Nax and Rigos, 2016; Newton, 2017; Iyer and Killingback, 2020; Holdahl and Wu, 2023) have focused on variations of the two-pool positive assortment process with constant assortativity. As indicated before, this process can be understood as a result of matching or allocating all players into pairs. Jensen and Rigos (2018) provide a general framework for matching rules that allocate all individuals into groups with different compositions (i.e., different frequencies for each strategy), and van Veelen (2011) studies the replicator dynamics in two-strategy games with allocation into groups. Wu (2016) studies two-strategy coordination games in which the index of assortativity is chosen by majority voting.

In this paper we focus on *selective* assortment, which does not assume that all players are matched in pairs. Specifically, we extend Eshel and Cavalli-Sforza's (1982) model of selective assortment to allow for more than two strategies and also for negative assortment. Interestingly, to the best of our knowledge, there are no reference models for negative assortment. Probably, one of the reasons lies in the difficulties of extending matching processes with constant assortativity to negative assortment (Jensen and Rigos, 2018). In particular, in Appendix A we show that extending the *two-pool assortative matching process with constant assortativity* to model negative assortment can give rise to several undesirable issues, both from a mathematical point of view (discontinuous payoff functions) and in terms of obtaining realistic models.

For the two-strategy case, following a different but related approach, Taylor and Nowak (2006) discuss replicator dynamics with non-uniform interaction rates. These non-uniform interaction rates can also be interpreted in terms of assortment. Interestingly, the phase portraits obtained under selective assortment that we present for the specific case of two strategies and replicator dynamics show some parallels with the phase portraits in Taylor and Nowak (2006). Friedman and Sinervo (2016) present a general framework for assortative interactions based on matching (or encounter) matrices, whose terms can also be interpreted as measures of the frequency with which i -players receive payoffs from interactions with j -players. Hauert and Miękisz (2018) consider a model in which players who interact together are also more likely to be competitors for reproduction, which leads to deviations from well-mixed populations and mimics structural assortment.

Most of the proofs are presented in Appendix B. All phase portraits shown in this paper can be easily replicated with freely available open-source software, which performs exact computations of rest points and exact linearization analyses (Izquierdo et al., 2024).

2. Setting and notation

We consider a population of individuals who may interact with each other in pairs, to play a symmetric two-player game. Each individual has a type or strategy $i \in S = \{1, 2, \dots, n\}$. An i -player who interacts with a j -player obtains a payoff U_{ij} .

Let x_i be the proportion, or fraction, of type i in the population and let $\mathbf{x} = (x_i)_{i \in S}$ be the *population state*: the vector describing the distribution of types in the population. Since $x_i \geq 0$ and $\sum_{i \in S} x_i = 1$, the population state $\mathbf{x}$ lives in the simplex $\Delta^{n-1} = \{\mathbf{x} \in \mathbb{R}_+^n : \sum_{i \in S} x_i = 1\}$. The monomorphic states in which all players use the same strategy i are represented by the unit vectors $\mathbf{e}_i$.

At a population state $\mathbf{x}$, each type, or strategy, i is assumed to have an average or expected payoff

$$\pi_i(\mathbf{x}) = \sum_{j=1}^n p_{ji}(\mathbf{x}) U_{ij} \quad (1)$$

where $p_{ji}(\mathbf{x})$ is the conditional probability that an i -player interacts with a j -player, receiving payoff U_{ij} .¹ Thus, in this setting, we allow the prob-

abilities of interaction $p_{ji}(\mathbf{x})$ to depend on individual's type. In this way, we generalize the standard framework of well-mixed populations, where $p_{ji}(\mathbf{x}) = x_j$ holds for all $i, j \in S$, which we refer to as *neutral assortment*.

The conditional probabilities $p_{ji}(\mathbf{x})$, with $\sum_{j=1}^n p_{ji}(\mathbf{x}) = 1$, define the assortment of interactions and determine the expected payoff for each strategy type, at any population state $\mathbf{x}$. We assume that these probabilities are defined at every population state $\mathbf{x} \in \Delta^{n-1}$.

Note that the expected payoff $\pi_i(\mathbf{x})$ for strategy i (1) is a convex combination of the game payoffs $\{U_{ij}\}$. Hence, $\pi_i(\mathbf{x})$ lies between the minimum and the maximum payoffs for i , i.e., $\pi_i(\mathbf{x}) \in [\min_j U_{ij}, \max_j U_{ij}]$.

2.1. Positive and negative assortments

In the literature, the terms *assortment* or *assortative mating* are often used to indicate that individuals interact with their own type with more probability than under random matching. Some authors (see e.g. Iyer and Killingback, 2020) distinguish between positive and negative assortment, depending on whether the conditional interaction probabilities p_{ji} are bigger or smaller than x_i (random matching). In the following, we use the term assortment to refer to the set of functions $p_{ji} : \Delta^{n-1} \rightarrow [0, 1]$ that characterize the conditional interaction probabilities (for each strategy pair) at every state. Comparing the probabilities with uniform random matching, we say that an assortment is:

- Positive if $p_{ii}(\mathbf{x}) \geq x_i$ for every $i \in S$ and every state, with strict inequality at least at one state.
- Negative if $p_{ii}(\mathbf{x}) \leq x_i$ for every $i \in S$ and every state, with strict inequality at least at one state.
- Neutral if $p_{ij}(\mathbf{x}) = x_i$ for every $i, j \in S$ and every state.

When referring to an assortment at a specific state $\mathbf{x}$, we say that an assortment is

- positive at $\mathbf{x}$ if $p_{ii}(\mathbf{x}) > x_i$ for every $i \in S$,
- negative at $\mathbf{x}$ if $p_{ii}(\mathbf{x}) < x_i$ for every $i \in S$, and
- neutral at $\mathbf{x}$ if $p_{ij}(\mathbf{x}) = x_i$ for every $i, j \in S$.

Note that the reference interaction probabilities (those corresponding to random or neutral matching) depend on the state $\mathbf{x}$. In particular, having large values for every same-type interaction probability $p_{ii}(\mathbf{x})$ at a population state does not guarantee that there is positive assortment at that state. For instance, if at some state $\mathbf{x}$, $p_{ii}(\mathbf{x}) = 0.9$ for every $i \in S$, but there is a strategy j such that $x_j > 0.9$, then there is no positive assortment. Similarly, having low values for every same-type interaction probability at a state does not guarantee negative assortment at that state. For instance, if at some state $\mathbf{x}$, $p_{ii}(\mathbf{x}) = 0.1$ for every $i \in S$, but there is a strategy j such that $x_j < 0.1$, then there is no negative assortment.

2.2. Balanced and boundary-compatible assortments

In many cases, it seems natural to assume that if at some population state $\mathbf{x}$ there are no j -players (i.e., if $x_j = 0$), then the conditional probability of meeting a j -player at such a state $p_{ji}(\mathbf{x})$ must be 0. An assortment that satisfies this condition is said to be *boundary-compatible*. Specifically, an assortment is *boundary-compatible* if

$$x_j = 0 \implies p_{ji}(\mathbf{x}) = 0$$

for every state $\mathbf{x} \in \Delta^{n-1}$, and every $i, j \in S$.

At a monomorphic state $\mathbf{e}_i$ (where all players use strategy i), the conditional probabilities of a boundary-compatible assortment satisfy $p_{ij}(\mathbf{e}_i) = 1$ for every j (i.e., if there are only i -players, any j -player entering the population will meet an i -player), leading to payoffs $\pi_j(\mathbf{e}_i) = U_{ji}$. Note that $\pi_j(\mathbf{e}_i)$ can be interpreted as the payoff obtained by a single j -player who enters a monomorphic population of i -players.

Another interesting property to take into account is *balance*. We say that an assortment is *balanced* if:

$$x_j p_{ji}(\mathbf{x}) = x_j p_{ij}(\mathbf{x}) \text{ for every } i, j \in S \text{ and } \mathbf{x} \in \Delta^{n-1}.$$

¹ Consequently, $p_{ii}(\mathbf{x})$ is the conditional probability that an i -player interacts with an i -player, receiving payoff U_{ii} , and $p_{ij}(\mathbf{x})$ is the conditional probability that a j -player interacts with an i -player, receiving payoff U_{ji} .

To interpret this condition, note in Eq. (1) that $p_{j|i}(\mathbf{x})$ is the relative weight of $(i - j)$ interactions in the payoff to i -players. If the frequency of interaction is the same for every player, and all interactions are considered equally relevant, then the balancing condition is met: in payoff-relevant $(i - j)$ interactions, the number of i -players equals the number of j -players. However, if players with different strategies interact at different rates, or if there is some asymmetry in the relevance of some interactions, the assortment will typically be non-balanced.

For instance, uniform random matching generates the neutral assortment $p_{i|j}(\mathbf{x}) = x_i$, which is boundary-compatible and balanced: $x_i p_{j|i}(\mathbf{x}) = x_i x_j = x_j p_{i|j}(\mathbf{x})$. Complete matching (which assumes that every player plays once with every other player) generates the same assortment. The two-pool process with constant assortativity generates a balanced assortment, but it is not boundary-compatible: the minimum value for the probability $p_{i|i}(\mathbf{x})$, i.e. for an i -player to meet another i -player, is $\alpha > 0$ —even at states where there are no i -players.

Selective assortment, on the other hand, is boundary-compatible but it is non-balanced. Specifically, under selective assortment, the relevant payoffs for a player are those obtained when actively selecting a co-player for an interaction. The number of payoff-relevant $i \rightarrow j$ interactions (those initiated by i -players who select a j -player) can differ from the number of payoff-relevant $j \rightarrow i$ interactions (those initiated by j -players who select an i -player). For example, consider two types (1 and 2), and a state $x_1 = 0.9$ ($x_2 = 1 - x_1 = 0.1$): under (strong) negative selective assortment we can obtain $p_{1|2} \approx 1$ and $p_{2|1} \approx 1$, i.e., almost all players are able to find a partner who is using the other strategy (see Section 3 for the formulas). This implies that the ratio between the frequency of payoff-relevant $1 \rightarrow 2$ interactions and the frequency of payoff-relevant $2 \rightarrow 1$ interactions is $\frac{x_1 p_{2|1}}{x_2 p_{1|2}} \approx 9$.

Similarly, for processes where players have different expected number of interactions per period (e.g., heterogeneous structured populations (Maciejewski et al., 2014)), and assuming that the relevant payoff is the average payoff per interaction, the associated assortment is typically non-balanced.

2.3. Two-type games. Positive assortment vs positive index of assortativity

For a two-type random matching process where each individual is matched with exactly one partner, Bergstrom (2003, 2013) defines the *index of assortativity* at state $\mathbf{x}$ as the difference between the conditional probability of interacting with a type (e.g., type 1) if the player is of that same type (type 1) minus that conditional probability if the player is of the other type (type 2):

$$\alpha(\mathbf{x}) \equiv p_{1|1}(\mathbf{x}) - p_{1|2}(\mathbf{x}). \quad (2)$$

It follows from $\sum_j p_{j|i} = 1$ that $\alpha(\mathbf{x}) = p_{2|2}(\mathbf{x}) - p_{2|1}(\mathbf{x})$.

The assortment generated by the *two-pool assortative matching process with constant assortativity* α presents a constant assortativity index $\alpha(\mathbf{x}) = \alpha$ at every state $\mathbf{x}$ (note that $\alpha(\mathbf{x})$ is the assortativity index at state $\mathbf{x}$ and α is a non-negative constant).

In the same context of random matching processes where each player is matched to another player once, a popular measure of assortative mating is Wright's *F-statistic* or correlation coefficient (Wright, 1965). Bergstrom (2003, 2013) shows that, in this setting, the index of assortativity is equal to Wright's *F-statistic*.

The index of assortativity in Eq. (2), however, can be used for any assortment, not only for cases in which each player is matched to another player once. From $p_{1|2} + p_{2|2} = 1$ and (2) we have

$$\alpha(\mathbf{x}) = p_{1|1}(\mathbf{x}) + p_{2|2}(\mathbf{x}) - 1$$

This identity leads to the following observation for two-type games:

Observation 1. *Positive assortment at $\mathbf{x}$ implies positive index of assortativity $\alpha(\mathbf{x}) > 0$. Negative assortment at $\mathbf{x}$ implies negative index of assortativity $\alpha(\mathbf{x}) < 0$.*

For a two-type balanced assortment, it can be shown² that $p_{i|i}(\mathbf{x}) = x_i + \alpha(\mathbf{x})(1 - x_i)$. This implies that, in the special case of balanced assortments, at any interior³ state $\mathbf{x}$ there is an equivalence between positive (negative) index of assortativity and positive (negative) assortment.

However, in general, the converse of Observation 1 is not true: a positive (negative) index of assortativity at a state does not guarantee positive (negative) assortment at that state. Suppose, for instance, that at state $(x_1, x_2) = (0.5, 0.5)$ we have $p_{1|1} = 0.7$, $p_{2|1} = 0.3$, $p_{1|2} = 0.6$, $p_{2|2} = 0.4$. At this state we have a positive index of assortativity $\alpha(0.5, 0.5) = 0.7 - 0.6 = 0.4 - 0.3 = 0.1$, but there is no positive assortment, because $p_{2|2} = 0.4 < x_2 = 0.5$. In general, positive assortment at a state $\mathbf{x}$ is a stronger condition than a positive index of assortativity $\alpha(\mathbf{x}) > 0$ at that state.

3. Selective assortment

In this section we extend the two-type model of positive selective assortment defined by Eshel and Cavalli-Sforza (1982) to any number of types as well as to negative assortment.

Under selective assortment, a player looking for a mate obtains a sample of $k \geq 1$ random players, or potential mates, to interact with. The selection of a mate depends on the assortment:

Positive assortment: select a mate using the same strategy as the player, provided there are any in the sample.

Negative assortment: select a mate at random that is using a different strategy, provided there are any in the sample.

If the sample lacks the desired strategy, then a random mate is chosen.

The special case $k = 1$, which corresponds to neutral assortment, is included as a reference for both positive and negative assortment.

3.1. Positive selective assortment

The probability $p_{j|i}^{k+}(\mathbf{x})$ denotes the conditional probability that an i -player selects a j -player under positive assortment (marked by $+$) given a sample size $k \geq 1$.

The probability that a sample of size k has no i -players is $(1 - x_i)^k$. Similarly, the probability that the sample includes at least one i -player is $1 - (1 - x_i)^k$. Now, assuming that every player with a strategy $j \neq i$ is treated equally by an i -player, we show (see Appendix B):

$$p_{j|i}^{k+}(\mathbf{x}) = \begin{cases} 1 - (1 - x_i)^k & \text{if } j = i \\ (1 - x_i)^{k-1} x_j & \text{if } j \neq i \end{cases} \quad (3)$$

At monomorphic states we have $\pi_j^{k+}(\mathbf{e}_i) = U_{ji}$. This holds for every sample size k , i.e., payoffs at monomorphic states are not affected by the sample size.

At interior states, we have $\lim_{k \rightarrow \infty} \pi_i^{k+}(\mathbf{x}) = U_{ii}$ due to positive assortment. This means that the payoff for every strategy at interior states converges, as $k \rightarrow \infty$, to the strategy's same-type-interaction payoff U_{ii} .

3.2. Negative selective assortment

We now derive the assortment probabilities $p_{j|i}^{k-}(\mathbf{x})$ for negative selective assortment (marked by $-$) with sample size $k \geq 1$.

The probability that all players in a sample are i -players is x_i^k . Again, assuming that every player with strategy $j \neq i$ is treated equally by an

² The proof is a straightforward adaptation of the proof of Eq. (A.1) in Appendix A.

³ A state $\mathbf{x}$ is interior if $x_i > 0$ for every strategy i .

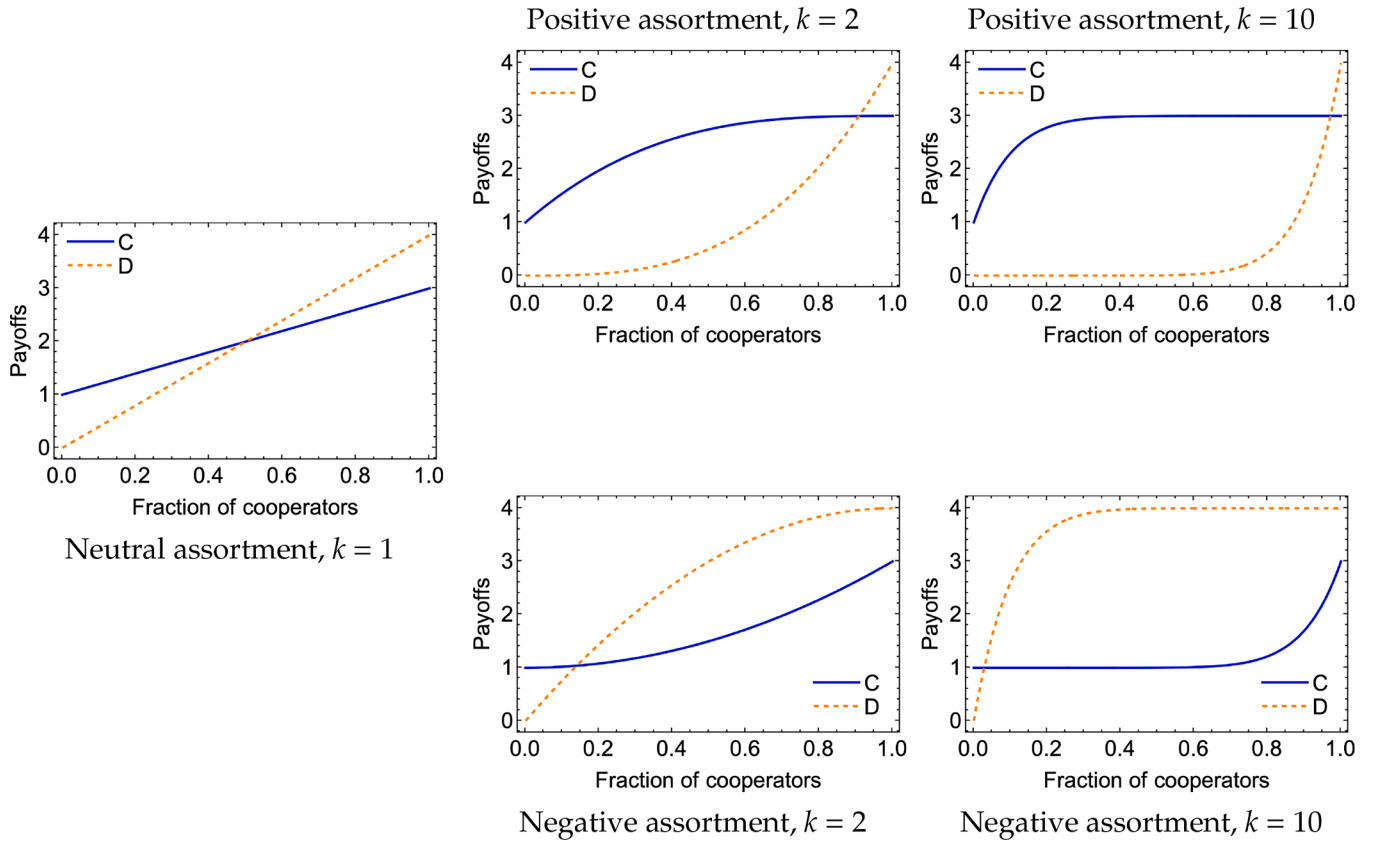


Fig. 1. Payoffs for each strategy as a function of the fraction of cooperators in the Snowdrift game with payoffs $\{P = 0, S = 1, R = 3, T = 4\}$ under neutral assortment ($k = 1$), and under positive and negative selective assortment with sample sizes $k = 2$ and $k = 10$. The unique interior ESS moves with k in opposite directions depending on whether the assortment is positive or negative.

i -player, we show (see [Appendix B](#)):

$$p_{j|i}^{k-}(\mathbf{x}) = \begin{cases} x_i^k & \text{if } j = i \\ (1 - x_i^k) \frac{x_j}{1 - x_i} & \text{if } j \neq i \text{ and } x_i < 1 \\ 0 & \text{if } j \neq i \text{ and } x_i = 1 \end{cases} \quad (4)$$

As before, at monomorphic states $\pi_j^{k-}(\mathbf{e}_i) = U_{ji}$ holds, regardless of the sample size k .

At interior states, as $k \rightarrow \infty$, the probability of same type interactions $p_{i|i}^{k-}(\mathbf{x})$ tends to 0 due to negative assortment. Moreover, $p_{j|i}^{k-}(\mathbf{x}) \rightarrow \frac{x_j}{1 - x_i}$ for $j \neq i$, i.e., the probability that an i -player selects a j -player approaches the relative frequency of j -players among non- i -players.

4. Selective assortment in games with two strategies

In this section we analyze the impact of positive and negative selective assortment in 2-player 2-strategy symmetric games (henceforth, 2×2 games). We name the strategies C for cooperate and D for defect, and characterize the population state by the fraction of C -players x_c . Payoffs are $U_{CC} = R, U_{DD} = P, U_{DC} = T$, and $U_{CD} = S$:

$$\begin{matrix} & C & D \\ C & \begin{pmatrix} R & S \end{pmatrix} \\ D & \begin{pmatrix} T & P \end{pmatrix} \end{matrix} \quad (5)$$

Let us study the different cases, focusing on generic games (i.e., assuming that the four payoffs are different). Without loss of generality, let us assume that mutual cooperation (CC) is more efficient than mutual defection (DD), i.e., $R > P$.

4.1. Payoff functions

From (1) and (3), the payoff functions for positive selective assortment are

$$\begin{aligned} \pi_C^{k+}(x_c) &= R - (1 - x_c)^k (R - S) \\ \pi_D^{k+}(x_c) &= P + x_c^k (T - P) \end{aligned} \quad (6)$$

From (1) and (4), the payoff functions for negative selective assortment are

$$\begin{aligned} \pi_C^{k-}(x_c) &= S + x_c^k (R - S) \\ \pi_D^{k-}(x_c) &= T - (1 - x_c)^k (T - P) \end{aligned} \quad (7)$$

Fig. 1 shows some illustrative examples of these functions for a Snowdrift game ($T > R > S > P$).

At every interior state, the formulas for the payoffs show:

Positive assortment: as k grows, the payoff of each strategy i converges to its same-type-payoff U_{ii} along the diagonal:

$$\pi_C^{k+}(x_c) \xrightarrow{k \rightarrow \infty} R \text{ and } \pi_D^{k+}(x_c) \xrightarrow{k \rightarrow \infty} P.$$

Negative assortment: as k grows, the payoff of each strategy i converges to its different-type-payoff U_{ij} along the anti-diagonal (for 2×2 games):

$$\pi_C^{k-}(x_c) \xrightarrow{k \rightarrow \infty} S \text{ and } \pi_D^{k-}(x_c) \xrightarrow{k \rightarrow \infty} T.$$

The payoff functions in (6) and (7) are monotonic in x_c . The differences $(R - S)$ and $(T - P)$ determine whether the payoff functions are increasing or decreasing as well as their concavity. For instance, for the Snowdrift game ($R - S > 0, T - P > 0$), $\pi_C^{k+}(x_c)$ is increasing and concave, while $\pi_D^{k+}(x_c)$ is increasing and convex (see [Fig. 1](#)).

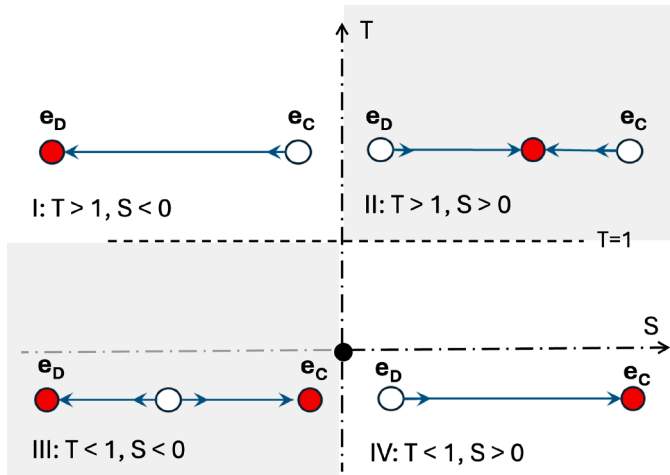


Fig. 2. Phase portraits for 2×2 symmetric games, in the replicator dynamics under neutral assortment. The phase portraits are different in each of the four regions (I–IV). Red dots are attractors and white dots are repellers.

4.2. The replicator dynamics

Let us study the replicator dynamics (Taylor and Jonker, 1978; Weibull, 1995)

$$\dot{x}_i = x_i \left(\pi_i(\mathbf{x}) - \sum_j x_j \pi_j(\mathbf{x}) \right) \quad (\text{RD})$$

in every 2×2 generic game under selective assortment based on monotonicity and concavity of the payoff functions. The phase portrait for the replicator dynamics (RD) does not change by adding a constant to every payoff, or by multiplying all payoffs by the same positive constant (see (6) and (7)). Hence, we can assume normalized payoffs $R = 1$ and $P = 0$ without loss of generality (normalized by subtracting P and dividing by $R - P$). We then define the following four regions:

Region I: $T > 1$ and $S < 0$. Example: Prisoner's Dilemma.

Region II: $T > 1$ and $S > 0$. Example: Snowdrift.

Region III: $T < 1$ and $S < 0$. Example: Stag Hunt.

Region IV: $T < 1$ and $S > 0$. Example: Harmony.

A rest point or equilibrium point is a state $\mathbf{x}$ such that $\dot{x}_i = 0$ for every strategy i . If a dynamic process reaches (or starts at) a rest point, it remains there. Under neutral assortment, the phase portraits for the replicator dynamics in each region are shown in Fig. 2. In region I, D is strictly dominant, so $\mathbf{e}_D$ is a rest point that attracts all interior trajectories. In region II, a unique interior, evolutionarily stable rest point exists and attracts all interior trajectories. In region III, there is bi-stability of $\mathbf{e}_C$ and $\mathbf{e}_D$, with an internal unstable rest point separating their basins of attraction. Finally, in region IV, C is strictly dominant, so $\mathbf{e}_C$ attracts all interior trajectories.

Under positive assortment, a new phase portrait appears in region I (Prisoner's Dilemma) for large enough sample size (see Fig. 3). This new phase portrait includes an additional attractor (close to $\mathbf{e}_C$, for large k) as well as an additional repeller (close to $\mathbf{e}_D$, for large k). Furthermore, both the level of cooperation and the size of the basin of attraction of this new attractor (where cooperators and defectors coexist) increase as the sample size k grows. Note, however, that positive selective assortment cannot stabilize full cooperation in the Prisoner's Dilemma, not even for very large k , because a single D -player entering a population of C -players obtains payoff $\pi_D^{k+}(\mathbf{e}_C) = T > R = \pi_C^{k+}(\mathbf{e}_C)$. In region II, for large k , the interior attractor is close to $\mathbf{e}_C$. In region III, for large k ,

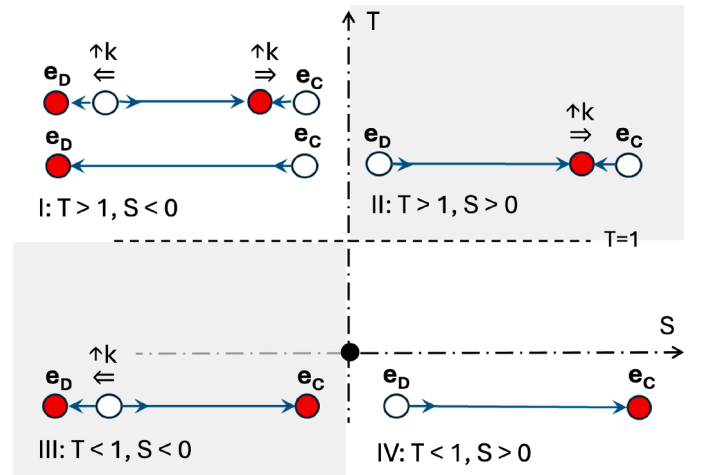


Fig. 3. Phase portraits for 2×2 symmetric games in the RD under positive selective assortment. The phase portraits are different in each of the four regions (I–IV). The symbol ($\uparrow k$) over an interior rest point, with an associated double arrow below, indicates that, for large values of k , the interior rest point is close to the corresponding edge. Red dots are attractors and white dots are repellers.

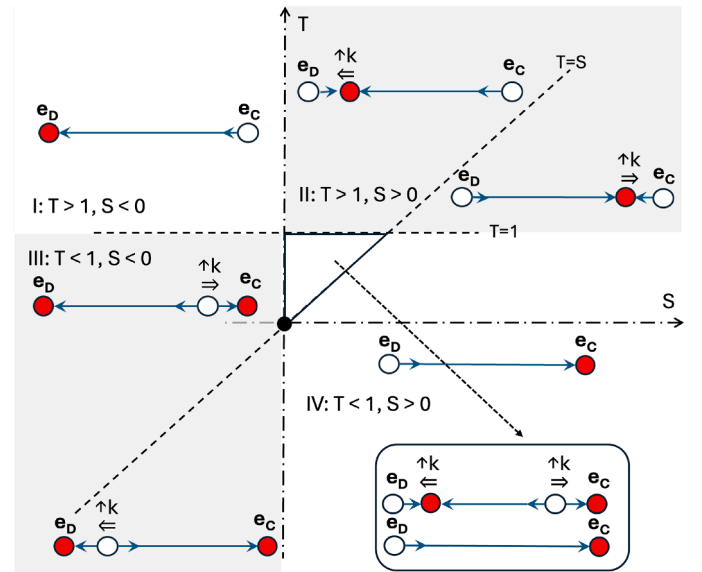


Fig. 4. Phase portraits for 2×2 symmetric games in the RD under negative selective assortment. The phase portraits are different in each of the four regions (I–IV). The box in the bottom right corresponds to the triangle of region IV where $T > S$. The symbol ($\uparrow k$) over an interior rest point, with an associated double arrow below, indicates that, for large values of k , the interior rest point is close to the corresponding edge. Red dots are attractors and white dots are repellers.

the interior repeller is close to $\mathbf{e}_D$. In every case, for large k , the flow in most of the state space points towards $\mathbf{e}_C$, which is the most efficient monomorphic state, leading to a stable state (either at $\mathbf{e}_C$ or close to it). For small k , the phase portraits approach those of neutral assortment (which corresponds to $k = 1$, Fig. 2).

The corresponding results for negative assortment are shown in Fig. 4. In region I (Prisoner's Dilemma) there is no significant change as compared to neutral assortment (c.f. Fig. 2). In region II, for large k , the interior attractor is either close to $\mathbf{e}_C$ (if $S > T$) or close to $\mathbf{e}_D$ (if $T > S$). Similarly, in region III, for large k , the interior repeller is either close to $\mathbf{e}_D$ (if $S > T$) or close to $\mathbf{e}_C$ (if $T > S$). Finally, in region IV, a new phase portrait appears for large enough sample size and $T > S$ (white triangle in the center of Fig. 4). This phase portrait includes an

additional attractor (close to $\mathbf{e}_D$, for large k) and an additional repeller (close to $\mathbf{e}_C$, for large k). In every case, for large k :

- If $T > S$, the flow in most of the state space points towards $\mathbf{e}_D$, leading to a stable state (either at or close to $\mathbf{e}_D$) in which all or most of the population is using strategy D .
- If $T < S$, the flow in most of the state space points towards $\mathbf{e}_C$, leading to a stable state (either at or close to $\mathbf{e}_C$) in which all or most of the population is using strategy C .

5. Selective assortment in games with any number of strategies

In this section we consider symmetric two-player games with any number of strategies for different evolutionary dynamics with positive or negative selective assortment.

We use the concept of a symmetric Nash strategy profile from classical game theory (a profile being a pair of strategies), and the concept of a Nash population state from evolutionary game theory (which refers to a distribution of strategies in the population). For completeness, we formally define these concepts below, before discussing their relationship under selective assortment.

Definition 1. A strategy profile (i, i) is a Nash profile if $U_{ji} \leq U_{ii}$ for every $j \neq i$. It is a strict Nash profile if the condition holds with strict inequality.

For instance, the payoff matrix for the 1-2-3 coordination game, with strategies 1, 2 and 3 is given by:

$$\begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \end{matrix} \quad (8)$$

It is easy to check that the three strategy profiles (1, 1), (2, 2) and (3, 3) are strict Nash profiles. Simply note that the maximum payoff in column i is U_{ii} .

In population games, the payoff function $\pi_i(\mathbf{x})$ denotes the payoff to strategy i in a population with state $\mathbf{x}$. This yields the following standard definitions of Nash and strict Nash monomorphic states.

Definition 2. A monomorphic state $\mathbf{e}_i$ is a Nash state if $\pi_j(\mathbf{e}_i) \leq \pi_i(\mathbf{e}_i)$ for every $j \neq i$. It is a strict Nash state if the condition holds with strict inequality.

At any monomorphic state $\mathbf{e}_i$ we have:

- for positive selective assortment with sample size k , $\pi_j^{k+}(\mathbf{e}_i) = U_{ji}$, and
- for negative selective assortment with sample size k , $\pi_j^{k-}(\mathbf{e}_i) = U_{ji}$,

which yields the following observation:

Observation 2. A monomorphic state $\mathbf{e}_i$ is a (strict) Nash state of a game with selective assortment if and only if strategy profile (i, i) is a (strict) Nash profile of the game.

For instance, for the 1-2-3 coordination game (8), the three monomorphic states $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are strict Nash states under selective assortment (either positive or negative, and for every sample size k).

Strict Nash states are asymptotically stable in the replicator dynamics, under the weak condition of having Lipschitz continuous payoff functions (Hofbauer and Sigmund, 2003). In our setting, this continuity condition corresponds to the basic requirement that if two population states are close, then, for every strategy i , the two payoffs obtained by strategy i at both states are also close.⁴ From Eqs. (1), (3) and (4), it

is easy to check that the payoff functions for selective assortment are Lipschitz continuous. We consequently have the following result:

Observation 3. If (i, i) is a strict Nash profile, then the monomorphic state $\mathbf{e}_i$ is asymptotically stable in the replicator dynamics under selective assortment (either positive or negative, and for every sample size k).

Indeed, Observation 3 extends to every dynamics for which strict Nash states are asymptotically stable. This includes best response dynamics, payoff monotonic imitation dynamics and, more generally, any myopic adjustment dynamics (Hofbauer and Sigmund, 2003).

Observation 4. If (i, i) is not a Nash profile, then the monomorphic state $\mathbf{e}_i$ is an unstable rest point of the replicator dynamics under selective assortment (either positive or negative, and for every sample size k).

Similarly, Observation 4 extends to every dynamics for which non-Nash states are unstable, such as every payoff monotonic imitation dynamics. Furthermore, for many dynamics, such as best response dynamics, only Nash states can be rest points (Sandholm, 2010). Thus, if (i, i) is not a Nash profile then $\mathbf{e}_i$ is not even a rest point under such dynamics.

For positive assortment and large k , an attractor of the replicator dynamics exists at or close to the most efficient monomorphic state. Moreover, most trajectories converge to this attractor. For example, in the 1-2-3 coordination game (see Example 2 below) the attractor is located at the most efficient monomorphic state, whereas it lies close to it in the Prisoner's Dilemma (Example 1) and in the Traveler's Dilemma (Example 3).

In order to show this, we use the concept of a relative neighborhood O . For any given state $\mathbf{x}$, a relative neighborhood O of $\mathbf{x}$ is any set of states that contains $\mathbf{x}$ and the states closest to $\mathbf{x}$. More precisely, any set of states that contains $\{\mathbf{x} \in \Delta^{n-1} : \|\mathbf{x} - \mathbf{x}\| < r\}$ for some positive $r > 0$. Note that r is the radius of a ball centered at $\mathbf{x}$.

Proposition 1. Assume positive selective assortment. Suppose that there is a unique most-efficient monomorphic state $\mathbf{e}_i$, i.e., $U_{ii} > \max_{j \neq i} U_{ji}$, which always holds for generic games. Then:

- Given any interior point $\mathbf{x}$, strategy $\hat{i}$ becomes the unique best strategy at $\mathbf{x}$ for a sufficiently large sample size k , i.e.,

$$\pi_i^{k+}(\mathbf{x}) > \pi_j^{k+}(\mathbf{x}) \quad \text{for every } j \neq \hat{i}.$$

- Any relative neighborhood O of $\mathbf{e}_i$ eventually (i.e., for large enough k) contains an attractor of the replicator dynamics. Moreover, given any interior state $\mathbf{x}$, the trajectory that starts at $\mathbf{x}$ eventually converges to O .

Note that $\mathbf{e}_i$ may not be Nash, and hence the most efficient monomorphic state is potentially unstable. Convergence to a (small) neighborhood of $\mathbf{e}_i$ implies that the dynamic process approaches $\mathbf{e}_i$ (if it is not already close to it) and remains close to it forever. Thus, most of the population ends up adopting strategy $\hat{i}$.

Proposition 1 extends to every dynamics satisfying

$$\pi_i(\mathbf{x}) > \sum_j x_j \pi_j(\mathbf{x}) \implies \dot{x}_i > 0,$$

such as aggregate monotonic imitation dynamics (Hofbauer and Sigmund, 2003).

Example 1. In the Prisoner's Dilemma (see Fig. 5), $\mathbf{e}_D$ is a strict Nash state: it is asymptotically stable in the replicator dynamics under every selective assortment. The cooperative state $\mathbf{e}_C$ is the most efficient monomorphic state but not a Nash state: it is unstable in the replicator dynamics under every selective assortment. However, for positive assortment and large enough k , an attractor exists close to $\mathbf{e}_C$, which eventually attracts most trajectories (see Fig. 5(c) for $k = 10$).

Example 2. In the 1-2-3 coordination game (payoff matrix (8)) with positive selective assortment (Fig. 6) the three monomorphic states are strict Nash states. Consequently, they are asymptotically stable in the

⁴ Technically, the condition requires the existence of some constant M such that, for any two states $\mathbf{x}$ and $\mathbf{y}$, and for every strategy i , $|\pi_i(\mathbf{x}) - \pi_i(\mathbf{y})| < M \|\mathbf{x} - \mathbf{y}\|$.

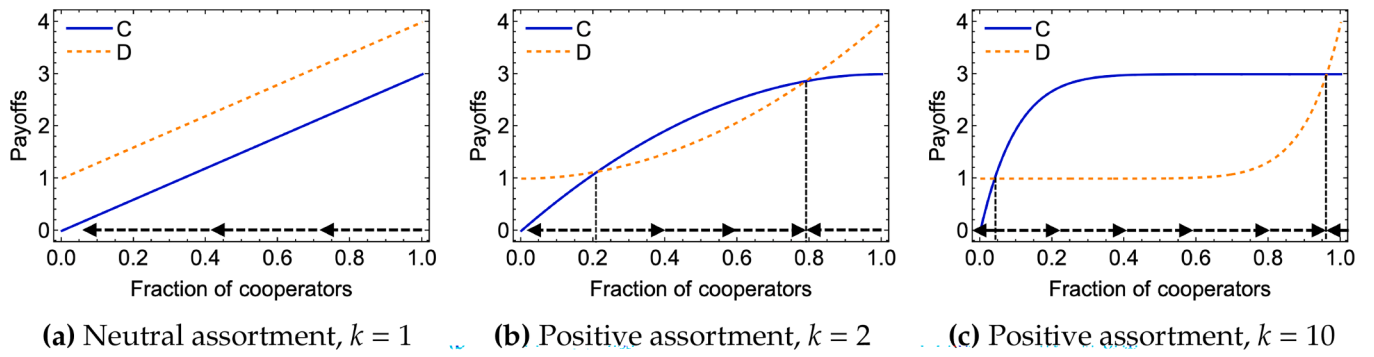


Fig. 5. Payoffs for each strategy as a function of the fraction of cooperators in the Prisoner's Dilemma with payoffs $\{S = 0, P = 1, R = 3, T = 4\}$ under neutral assortment ($k = 1$) and under positive selective assortment with sample sizes $k = 2$ and $k = 10$. The arrows show the phase portrait for the replicator dynamics.

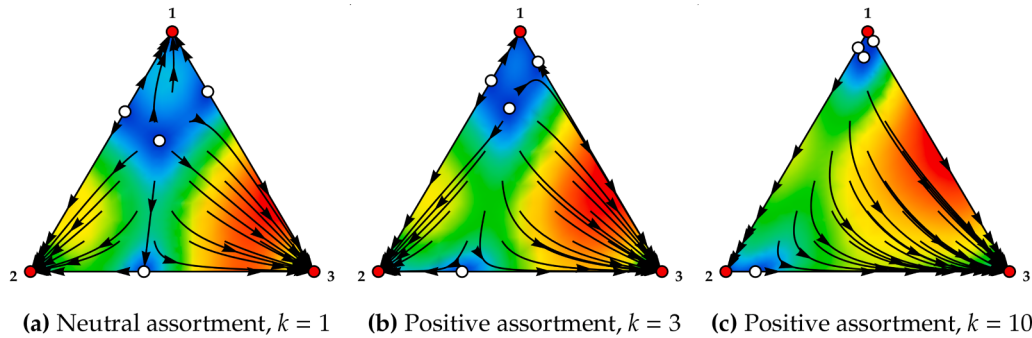


Fig. 6. Replicator dynamics for positive selective assortments, for the 1-2-3-coordination game with payoff matrix (8). Red dots are attractors, while white dots are unstable rest points. The background colors represent speed (blue for slow, yellow for medium, red for fast).

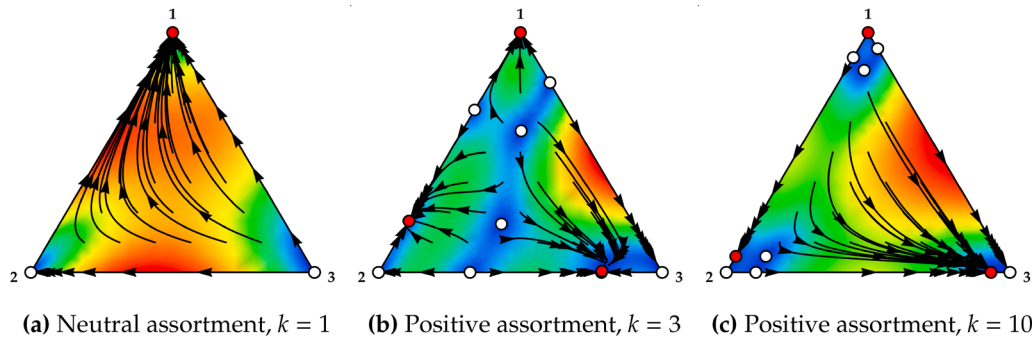


Fig. 7. Replicator dynamics for positive selective assortments, for the Traveler's Dilemma game with payoff matrix (9). Red dots are attractors, while white dots are unstable rest points. The background colors represent speed (blue for slow, yellow for medium, red for fast).

replicator dynamics under (every) selective assortment. The most efficient monomorphic state is e_3 . For positive assortment and increasing k , most trajectories eventually converge to e_3 . More specifically, all those with an initial x_3 that exceeds a threshold converge, and the threshold decreases with k .

Example 3. The Traveler's dilemma game is exemplified by the following payoff matrix for three strategies:

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 2 & 4 & 4 \\ 2 & 0 & 3 & 5 \\ 3 & 0 & 1 & 4 \end{array} \quad (9)$$

With positive selective assortment (Fig. 7) the inefficient e_1 is a strict Nash state and hence asymptotically stable. In contrast, e_2 and e_3 are not Nash states and hence unstable for every k . The least and most efficient monomorphic states are e_1 and e_3 , respectively. Under neutral

assortment, the least efficient state e_1 attracts all interior trajectories. For positive assortment and increasing k , most trajectories converge to an attractor close to the most efficient monomorphic state e_3 . More precisely, all those where the initial x_3 exceeds a threshold converge. For increasing k not only the threshold decreases but also the attractor gets closer to e_3 .

For negative selective assortment and large k , our next proposition provides conditions that guarantee the existence of an attractor close to one of the monomorphic states of a game. It may seem surprising that negative assortment (a preference to interact with other types) leads most players to use the same strategy. However, note that, under selective assortment with large sample size k , most interactions in the population take place between players using different strategies even if most players are using the same strategy: when looking for a partner, players using the majority strategy are very likely to find and select a partner using some minority strategy.

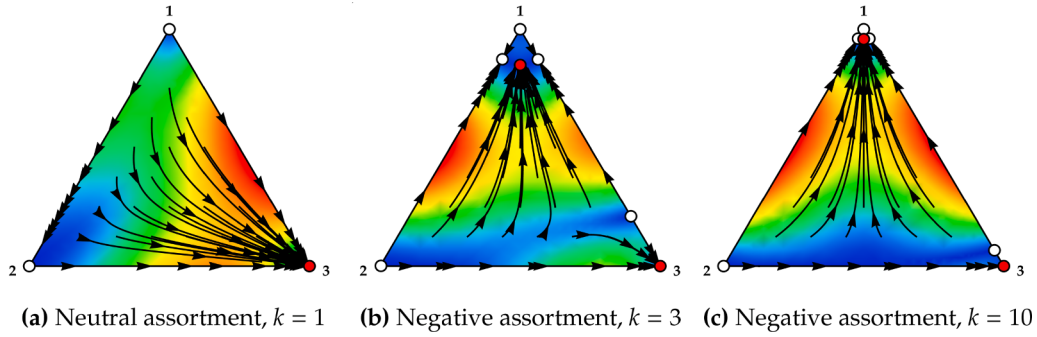


Fig. 8. Replicator dynamics for negative selective assortments, for the game in Example 4. Red dots are attractors, while white dots are unstable rest points. The background colors represent speed (blue for slow, yellow for medium, red for fast). Note how, as k increases from 1 (i) to 10 (iii), the attractor that appears in (ii) close to the top of the simplex (close to e_1) gets closer to e_1 and attracts most trajectories.

Proposition 2. Assume negative selective assortment. If a monomorphic state e_i satisfies $\min_{j \neq i} U_{ij} > \max_{j \neq i} U_{ji}$, then any relative neighborhood O of e_i contains, for sufficiently large k , an attractor of the replicator dynamics.

Additionally, if $\min_{j \neq i} U_{ij} \geq \max_{j \neq i, m \neq j} U_{jm}$, then most trajectories converge to O for large enough k . Specifically, given any interior state x , the trajectory starting at x eventually (i.e., for large enough k) converges to O .

Note that e_i may not be Nash, and hence potentially unstable. Considering the payoff matrix $\{U_{ij}\}$ of the game (see e.g. payoff matrix (10)), the first condition in Proposition 2 compares the minimum non-diagonal payoff in the row for strategy i , $\min_{j \neq i} U_{ij}$, with the maximum non-diagonal payoff in its column. The second condition compares the same value with the maximum non-diagonal payoff in all the other rows. The proof is detailed in Appendix B.

The convergence to a (small) neighborhood of e_i implies that the dynamic process approaches e_i (if it is not already close to it) and remains close to it forever. Note the attractor (red dot) close to e_1 in Fig. 8(b) & (c) for $k = 3$ and for $k = 10$. For $k = 10$ most trajectories converge to that attractor.

Proposition 2 extends to any dynamics under which a single optimal strategy grows unless everybody is using it. More specifically, for any dynamics satisfying

$$\left(\pi_i(x) > \max_{j \neq i} \pi_j(x) \quad \text{and} \quad 0 < x_i < 1 \right) \implies \dot{x}_i > 0,$$

such as best response dynamics.

Example 4. Consider an interaction with the following payoff matrix:

$$\begin{array}{c|cc} & 1 & 2 & 3 \\ \hline 1 & 0 & 3 & 3 \\ 2 & 1 & 3 & 3 \\ 3 & 1 & 3 & 5 \end{array} \quad (10)$$

Under neutral assortment, strategy 1 is dominated by strategies 2 and 3. Strategy 3 is dominant and attracts all interior trajectories of the replicator dynamics (see Fig. 8).

For strategy 1, after excluding U_{11} , the minimum row-payoff is greater than the maximum column-payoff ($3 > 1$). According to Proposition 2 an attractor exists close to e_1 , for negative selective assortment and large enough k . Furthermore, given that $3 \geq \max_{j \neq 1, k \neq j} U_{jk} = 3$, we have that, for any $\epsilon > 0$ and large enough k , all trajectories with an initial value of $x_1 \in [\epsilon, 1 - \epsilon]$ approach this attractor. However, given that e_1 is not a Nash state, e_1 itself is not an attractor.

6. Conclusions

In order to study the effects of positive assortment, the *two-pool assortative matching process* (Eshel and Cavalli-Sforza, 1982; Bergstrom, 2003,

2013) can be considered a standard reference model. In sharp contrast, no standard reference model to study the effects of negative assortment seems to exist. In fact, providing reference models for negative assortment is not immediate: many direct extensions of positive assortment processes (such as extending the two-pool process) can be problematic or unrealistic (see Appendix A for details).

In this paper we have analyzed a model of selective assortment for two-player interactions that extends Eshel and Cavalli-Sforza (1982)'s two-strategy positive assortment model to several strategies and, more importantly, to negative assortment. Through this extension, our contribution now proposes a reference model for negative assortment.

In the replicator dynamics with two strategies the two monomorphic states are rest points under neutral assortment. In generic games, there can be at most one additional, interior rest point, which is either an attractor or a repeller. Selective assortment does not modify the stability of the two monomorphic states, but it can significantly alter the dynamics in the interior of the state space. For instance, for sufficiently strong positive selective assortment, a new interior attractor appears in the Prisoner's Dilemma, where cooperators and defectors coexist. Furthermore, both the level of cooperation and the size of the basin of attraction of this interior attractor increase with the strength of positive assortment. In other games, negative selective assortment can generate a similar effect.

In 2-player games with any number of strategies, and under many evolutionary dynamics, we have shown that positive selective assortment leads to most of the population playing the most efficient strategy, i.e. the strategy with the greatest same-type payoff U_{ii} , for sufficiently large sample size k and most initial conditions. Similarly, for negative assortment, we have identified strategies that are adopted by the majority of the population under many evolutionary dynamics.

Declaration of competing interest

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CRediT authorship contribution statement

Segismundo S. Izquierdo: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization; **Luis R. Izquierdo:** Writing – review & editing, Writing – orig-

inal draft, Visualization, Validation, Supervision, Software, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization; **Christoph Hauert**: Writing – original draft, Visualization, Validation, Supervision, Software, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization.

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Appendix A. Alternative models for negative assortment

Somewhat surprisingly, defining the negative-assortment complement to a process that generates positive assortment is generally not straightforward or even possible. In this appendix, we present three potential extensions of the *two-pool positive assortment model* (Eshel and Cavalli-Sforza, 1982) to include negative assortment, and discuss the issues that each of them presents.

In the two-strategy case, the *two-pool positive assortment model* generates an assortment whose index of assortativity $\alpha(\mathbf{x})$, as defined in (2), is a constant α , i.e., $\alpha(\mathbf{x}) = \alpha$ at every state $\mathbf{x}$. We start by presenting three possible characterizations of the *two-pool positive assortment model* for any number of strategies. Each characterization simply represents one possible way to arrive at the same model of positive assortment. In contrast, each of the three different characterizations of the (same) model of positive assortment leads to a different model for negative assortment.

A.1. Three characterizations of the two-pool positive assortment model

The following characterizations can be used to define the *two-pool positive assortment model* with constant assortativity factors α :

[C-I] Players compute their payoffs by interacting:

- with probability $\alpha > 0$, with a player using the same strategy, and
- with probability $(1 - \alpha)$, with a random player.

This is how the model was initially introduced in Section 1.

[C-II] In the two-type case, the model is characterized by balanced assortment with constant index of assortativity (Section 2.3). For more strategies, the index of assortativity can be extended to *assortativity factors*, $\alpha_{ij}(\mathbf{x}) = p_{i|i}(\mathbf{x}) - p_{i|j}(\mathbf{x})$, defined for every pair of different strategies and every state $\mathbf{x}$. Thus, a constant index of assortativity translates to $p_{i|i}(\mathbf{x}) - p_{i|j}(\mathbf{x}) = \alpha$ for every i and $j \neq i$.

[C-III] A representative fraction α of the population is matched in pairs in a way such that the number of same-strategy pairs is maximized, and the remaining fraction $(1 - \alpha)$ of the population is randomly matched. Note that maximizing the number of same-strategy pairs is equivalent to matching players using the same strategy; thus, [C-III] is equivalent to [C-I].

We now summarize the main properties of this *two-pool positive assortment model* (regardless of how it is characterized).

Conditional probabilities: the probabilities $p_{j|i}(\mathbf{x})$ in this assortment are:

$$p_{j|i}(\mathbf{x}) = \alpha \delta_{ij} + (1 - \alpha) x_j,$$

where δ_{ij} is the Kronecker delta ($\delta_{ii} = 1$, and $\delta_{ij} = 0$ if $i \neq j$).

Proof.

[C-I] The proof is straightforward: simply consider that x_j is the probability that a random player uses strategy j .

[C-II] the condition of being balanced, $x_i p_{j|i}(\mathbf{x}) = x_j p_{i|j}(\mathbf{x})$, leads to $\sum_j x_i p_{j|i}(\mathbf{x}) = \sum_j x_j p_{i|j}(\mathbf{x})$. Considering that $\sum_j p_{j|i}(\mathbf{x}) = 1$ and $p_{i|j}(\mathbf{x}) = p_{i|i}(\mathbf{x}) - \alpha$ for $j \neq i$, we obtain

$$\begin{aligned} \sum_j x_i p_{j|i}(\mathbf{x}) &= \sum_j x_j p_{i|j}(\mathbf{x}) \implies x_i \left(\sum_j p_{j|i}(\mathbf{x}) \right) = x_i p_{i|i}(\mathbf{x}) \\ &\quad + \sum_{j \neq i} x_j (p_{i|j}(\mathbf{x}) - \alpha) \\ &\implies x_i = x_i p_{i|i}(\mathbf{x}) + (1 - x_i)(p_{i|i}(\mathbf{x}) - \alpha) \\ &\implies x_i = p_{i|i}(\mathbf{x}) - \alpha(1 - x_i) \end{aligned} \quad (\text{A.1})$$

From the last equation follows that $p_{i|i}(\mathbf{x}) = \alpha + (1 - \alpha) x_i$ and $p_{j|i}(\mathbf{x}) = p_{j|j}(\mathbf{x}) - \alpha = (1 - \alpha) x_j$ for $i \neq j$.

[C-III] note that a matching of a representative (sub)population that maximizes the number of same-strategy pairs in the subpopulation is a matching in which every player in the subpopulation is paired with another player in the subpopulation who uses the same strategy. The relative prevalence of j -players in the rest of the population does not change, it is still x_j (for every j). Consequently, [C-I] and [C-III] correspond to the same probabilities that each different $i - j$ pairing takes place. $\square$

Payoffs: for the payoffs $\pi_i(\mathbf{x})$, from (1) we have

$$\pi_i(\mathbf{x}) = \sum_{j=1}^n p_{j|i}(\mathbf{x}) U_{ij} = \alpha U_{ii} + \sum_{j=1}^n (1 - \alpha) x_j U_{ij} = \sum_{j=1}^n x_j [\alpha U_{ii} + (1 - \alpha) U_{ij}].$$

Defining the modified game payoffs $\tilde{U}_{ij}$ as

$$\tilde{U}_{ij} = \alpha U_{ii} + (1 - \alpha) U_{ij},$$

we have $\pi_i(\mathbf{x}) = \sum_{j=1}^n x_j \tilde{U}_{ij}$. Thus, the payoffs from this assortment are linear in $\mathbf{x}$ and are equivalent to the payoffs obtained under uniform random matching using the modified game payoffs $\tilde{U}_{ij}$.

Balance: this assortment is balanced and positive, but not boundary compatible since $p_{i|i}(\mathbf{x}) \geq \alpha$ at every state. This implies that even if there are no i -players in a population ($x_i = 0$), a potential invader using strategy i is assumed to be able to interact with another i -player with probability at least α (see Bergstrom, 2013 for an alternative model). This is probably the main drawback of this model of positive assortment.

The following subsections present and discuss potential ways to model negative assortment, taking each of the three characterizations of the *two-pool positive assortment model* as a starting point.

A.2. Extension from [C-I]. Proportional negative assortment

A natural way to model negative assortment in the spirit of characterization [C-I] is: Players compute their payoffs by interacting:

- with probability $\alpha > 0$, with a player using a different strategy, and
- with probability $(1 - \alpha)$, with a random player.

In contrast to positive assortment, here we must also specify the probability of selecting each of the different strategies. A natural way of doing this is proportional to their frequencies. In that case, the probabilities $p_{j|i}(\mathbf{x})$ are:

$$p_{j|i}(\mathbf{x}) = \alpha \frac{x_j}{1 - x_i} (1 - \delta_{ij}) + (1 - \alpha) x_j$$

However, the previous formula is not defined at monomorphic states where $x_i = 1$. Moreover, in generic games with more than two strategies, payoffs at monomorphic states cannot be defined in a continuous way. For example,

$$\lim_{\epsilon \rightarrow 0} \pi_1(1 - \epsilon, \epsilon, 0, \dots, 0) = \alpha U_{12} + (1 - \alpha) U_{11}, \text{ but}$$

$$\lim_{\epsilon \rightarrow 0} \pi_1(1 - \epsilon, 0, \epsilon, 0, \dots, 0) = \alpha U_{13} + (1 - \alpha) U_{11}.$$

A.3. Extensions from [C-II]. Assortment with constant negative assortativity

A natural way to model negative assortment in the spirit of characterization [C-II] is to ensure that the assortment is balanced and all assortativity factors $\alpha_{ij}(\mathbf{x})$ are equal to a negative constant $-\alpha$, i.e. $p_{i|i}(\mathbf{x}) - p_{i|j}(\mathbf{x}) = -\alpha$ for every i and $j \neq i$.

It turns out that there are no balanced assortments with negative constant index of assortativity or, more generally, with negative constant assortativity factors. Jensen and Rigos (2018) prove this fact for the two-strategy case, but the statement also applies to more than two strategies. To see this, consider any state $\hat{\mathbf{x}} \in \Delta^{n-1}$ such that $\hat{x}_i = 0$ and $\hat{x}_j > 0$. The balancing condition implies $p_{i|i}(\hat{\mathbf{x}}) = 0$, so the corresponding assortativity factor is $\alpha_{ij}(\hat{\mathbf{x}}) = p_{i|i}(\hat{\mathbf{x}}) - p_{i|j}(\hat{\mathbf{x}}) = p_{i|j}(\hat{\mathbf{x}}) \geq 0$. Thus, the assortativity factors of a balanced assortment cannot be a negative constant.

Nevertheless, a non-balanced assortment with constant negative assortativity may still exist.⁵ The only condition we impose is that all assortativity factors $\alpha_{ij}(\mathbf{x})$ are equal to a negative constant $-\alpha$, i.e. $p_{i|i}(\mathbf{x}) - p_{i|j}(\mathbf{x}) = -\alpha$ for every i and $j \neq i$.

Note that this extension implies $p_{i|i}(\mathbf{x}) = \alpha + p_{i|j}(\mathbf{x}) > 0$ for $j \neq i$. As a consequence, such assortments may not be realistic because they imply that any non- i -player can interact with an i -player with strictly positive probability, even in a population without i -players.

A.4. Extension from [C-III]. Matching with maximum number of different-strategy pairs

A natural way to model negative assortment in the spirit of characterization [C-III] is to match a representative fraction α of the population in pairs such that the number of pairs with different strategies is maximized. The remaining fraction $(1 - \alpha)$ of the population is randomly matched.

The two-strategy case is not problematic. Letting $r = x_2/x_1$, the payoffs obtained when maximizing the number of pairs with different strategies are:

$$(\pi_1(\mathbf{x}), \pi_2(\mathbf{x})) = \begin{cases} ((1-r)U_{11} + rU_{12}, U_{21}) & \text{if } x_1 \geq 0.5 \\ (U_{12}, (1-1/r)U_{22} + U_{21}/r) & \text{if } x_1 < 0.5 \end{cases}.$$

For more than three strategies, problems start to appear: at some population states, there are different matchings that maximize the number of pairs with different strategies. Moreover, those matching mechanisms can lead to different payoffs. For instance, any matching in pairs of four players who use four different strategies maximizes the number of pairs with different strategies. In any case, for more than two strategies, and considering generic games, this assortment necessarily leads to discontinuous payoff functions, as our next proposition shows.

Proposition 3. *In populations with more than two strategies any matching mechanism that maximizes the number of pairs with different strategies leads to discontinuous payoff functions for generic games.*

Proof. Any matching that maximizes the number of pairs with different strategies satisfies:

- $p_{3|1}(\mathbf{x}^\epsilon) = 1$ at states $\mathbf{x}^\epsilon \equiv (\epsilon, \frac{1}{2} - \epsilon, \frac{1}{2}, 0, \dots, 0)$, for $\epsilon \leq \frac{1}{2}$. This implies $\pi_1(\mathbf{x}^\epsilon) = U_{13}$.
- $p_{2|1}(\mathbf{x}^\delta) = 1$ at states $\mathbf{x}^\delta \equiv (\delta, \frac{1}{2}, \frac{1}{2} - \delta, 0, \dots, 0)$, for $\delta \leq \frac{1}{2}$. This implies $\pi_1(\mathbf{x}^\delta) = U_{12}$.

Let $\mathbf{x}^0 \equiv (0, \frac{1}{2}, \frac{1}{2}, 0, \dots, 0)$. We have $\lim_{\epsilon \rightarrow 0} \mathbf{x}^\epsilon = \mathbf{x}^0$ and $\lim_{\delta \rightarrow 0} \mathbf{x}^\delta = \mathbf{x}^0$, but $\lim_{\epsilon \rightarrow 0} \pi_1(\mathbf{x}^\epsilon) = U_{13}$ and $\lim_{\delta \rightarrow 0} \pi_1(\mathbf{x}^\delta) = U_{12} \neq U_{13}$ for generic games. Thus, $\pi_1(\mathbf{x})$ cannot be continuous at $\mathbf{x}^0$. $\square$

⁵ For the two-strategy case, Friedman and Sinervo (2016) discuss some balanced assortments with (non-constant) negative assortativity.

Appendix B. Proofs

Proof of Eq. (3). We know $p_{i|i}^{k+}(\mathbf{x}) = 1 - (1 - x_i)^k$ and that every player with a strategy that is not i has the same probability of being chosen by an i -player. Consequently, $p_{j|i}^{k+}(\mathbf{x}) = c(\mathbf{x}) x_j$ for some value $c(\mathbf{x})$, so

$$\sum_{j \neq i} p_{j|i}^{k+}(\mathbf{x}) = \sum_{j \neq i} c(\mathbf{x}) x_j = c(\mathbf{x}) \sum_{j \neq i} x_j = c(\mathbf{x}) (1 - x_i). \quad (\text{B.1})$$

From $\sum_j p_{j|i}^{k+}(\mathbf{x}) = 1$ we also have

$$\sum_{j \neq i} p_{j|i}^{k+}(\mathbf{x}) = 1 - p_{i|i}^{k+}(\mathbf{x}) = (1 - x_i)^k. \quad (\text{B.2})$$

Combining (B.1) and (B.2) we find $c(\mathbf{x}) = (1 - x_i)^{k-1}$. Consequently, $p_{j|i}^{k+}(\mathbf{x}) = (1 - x_i)^{k-1} x_j$ for $j \neq i$. $\square$

Proof of Eq. (4). Again, we know $p_{i|i}^k(\mathbf{x}) = x_i^k$ and that every player with a strategy that is not i has the same probability of being chosen by an i -player. Consequently, $p_{j|i}^k(\mathbf{x}) = c(\mathbf{x}) x_j$ for some value $c(\mathbf{x})$, and

$$\sum_{j \neq i} p_{j|i}^k(\mathbf{x}) = \sum_{j \neq i} c(\mathbf{x}) x_j = c(\mathbf{x}) \sum_{j \neq i} x_j = c(\mathbf{x}) (1 - x_i) \quad (\text{B.3})$$

From $\sum_j p_{j|i}^k(\mathbf{x}) = 1$, we also have

$$\sum_{j \neq i} p_{j|i}^k(\mathbf{x}) = 1 - p_{i|i}^k(\mathbf{x}) = 1 - x_i^k \quad (\text{B.4})$$

Combining (B.3) and (B.4) we find $c(\mathbf{x}) = \frac{1-x_i^k}{1-x_i}$ for $x_i \neq 1$. Therefore, $p_{j|i}^{k-}(\mathbf{x}) = \frac{1-x_i^k}{1-x_i} x_j$ for $j \neq i$ and $x_i < 1$. For $x_i = 1$ we have $p_{i|i}^{k-}(\mathbf{e}_i) = 1$ and, consequently, $p_{j|i}^{k-}(\mathbf{e}_i) = 0$ for $j \neq i$. $\square$

Proof of Observation 2. $\pi_j^{k+}(\mathbf{e}_i) = \pi_j^{k-}(\mathbf{e}_i) = U_{ji}$ for every $i, j \in S$ and for every k . $\square$

Proof of Proposition 1. From (1) and (3) we have

$$\pi_i(\mathbf{x}) = [1 - (1 - x_i)^k] U_{ii} + \sum_{j \neq i} (1 - x_i)^{k-1} x_j U_{ij}.$$

It follows that $\lim_{k \rightarrow \infty} \pi_i(\mathbf{x}) = U_{ii}$ at any interior $\mathbf{x}$. Let $\mathbf{e}_i$ be the most efficient monomorphic state. Then, $\lim_{k \rightarrow \infty} \pi_j(\mathbf{x}) = U_{jj} < U_{ii}$ for $j \neq i$, which proves the first part of Proposition 1.

For the second part, let $\bar{\pi}(\mathbf{x}) = \sum_j x_j \pi_j(\mathbf{x})$ be the average payoff at state $\mathbf{x}$, let $D_1 \equiv U_{ii} = \max_j U_{jj}$ and let $D_2 \equiv \max_{j \neq i} U_{jj} < D_1$. We prove that, for every $\mathbf{x}$ with $x_i > 0$,

$$\lim_{k \rightarrow \infty} \bar{\pi}(\mathbf{x}) \leq x_i D_1 + (1 - x_i) D_2. \quad (\text{B.5})$$

Then, for $0 < x_i < 1$,

$$\lim_{k \rightarrow \infty} \bar{\pi}(\mathbf{x}) \leq x_i D_1 + (1 - x_i) D_2 < D_1 = \lim_{k \rightarrow \infty} \pi_i(\mathbf{x}),$$

from which the result follows. Note, for $\epsilon > 0$, large enough k and $x_i \in [\epsilon, 1 - \epsilon]$, we have $\pi_i(\mathbf{x}) > \bar{\pi}(\mathbf{x})$, which implies $\dot{x}_i > 0$ for $x_i \in [\epsilon, 1 - \epsilon]$.

It only remains to show (B.5). In the following bound, we use the value $M \equiv \max_{j \neq i, m \neq j} U_{jm}$.

$$\begin{aligned} \bar{\pi}(\mathbf{x}) &= x_i \pi_i(\mathbf{x}) + \sum_{j \neq i} x_j \pi_j(\mathbf{x}) \leq x_i \pi_i(\mathbf{x}) + \sum_{j \neq i} x_j [1 - (1 - x_j)^k] D_2 \\ &\quad + \sum_{j \neq i} x_j (1 - x_j)^k M \\ &= x_i \pi_i(\mathbf{x}) + (1 - x_i) D_2 + \sum_{j \neq i} x_j (1 - x_j)^k (M - D_2) \\ &\xrightarrow{k \rightarrow \infty} x_i D_1 + (1 - x_i) D_2. \end{aligned}$$

$\square$

Proof of Proposition 2. From (1) and (4) we have that, for $x_i < 1$,

$$\pi_i(\mathbf{x}) = x_i^k U_{ii} + (1 - x_i^k) \sum_{j \neq i} \frac{x_j}{1 - x_i} U_{ij}.$$

Let i be the strategy satisfying the condition $\min_{j \neq i} U_{ij} > \max_{j \neq i} U_{ji}$. We use the auxiliary variables $B_1 \equiv \min_{j \neq i} U_{ij}$, $B_2 \equiv \max_{j \neq i} U_{ji}$ (so, by hypothesis, $B_1 > B_2$) and $M \equiv \max_{j \neq i, m \neq j} U_{jm}$. At any point $\mathbf{x}$ with $x_i < 1$ we have

$$\lim_{k \rightarrow \infty} \pi_i(\mathbf{x}) \geq B_1.$$

At any point $\mathbf{x}$ with $x_i > 0$, we have

$$\lim_{k \rightarrow \infty} \pi_j(\mathbf{x}) \leq \frac{x_i U_{ji} + (1 - x_i - x_j) M}{1 - x_j} \leq \frac{x_i B_2 + (1 - x_i - x_j) M}{1 - x_j} \quad (\text{B.6})$$

for $j \neq i$. The upper bound in (B.6) is a convex combination of B_2 and M . If $M \leq B_1$ (and considering that $B_2 < B_1$), then for every interior $\mathbf{x}$ with $0 < x_i < 1$ we have $\lim_{k \rightarrow \infty} \pi_i(\mathbf{x}) > \lim_{k \rightarrow \infty} \pi_j(\mathbf{x})$. For large enough k this implies that strategy i is strictly dominant for $x_i \in [\epsilon, 1 - \epsilon]$ (fixing first $\epsilon > 0$, and then taking a large enough k), which proves the result for $M \leq B_1$.

If $M > B_1$, let $\gamma > 0$ be a positive constant. We have from (B.6) that, for $x_i > \gamma$,

$$\lim_{k \rightarrow \infty} \pi_j(\mathbf{x}) \leq B_2 + \left(1 - \frac{x_i}{1 - x_j}\right)(M - B_2) \leq B_2 + (1 - \gamma)(M - B_2)$$

Solving for γ in

$$B_2 + (1 - \gamma)(M - B_2) = B_1$$

we find $\gamma = \frac{M - B_1}{M - B_2} < 1$, so for $x_i \in \left(\frac{M - B_1}{M - B_2}, 1\right)$, we have $\lim_{k \rightarrow \infty} \pi_i(\mathbf{x}) > \lim_{k \rightarrow \infty} \pi_j(\mathbf{x})$. We consequently have compact regions $\left\{x : x_i \in \left[\frac{M - B_1}{M - B_2} + \epsilon, 1 - \epsilon\right]\right\}$ in which strategy i is strictly dominant for small enough $\epsilon > 0$ and large enough k . This completes the proof. $\square$

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