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ORIGINAL ARTICLE

On some properties of the asymptotic Samuel function

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Abstract

The asymptotic Samuel function generalizes to arbitrary rings the usual order function of a regular local ring. Here, we explore some natural properties in the context of excellent, equidimensional rings containing a field. In addition, we establish some results regarding the Samuel slope of a local ring. This is an invariant related with algorithmic resolution of singularities of algebraic varieties. Among other results, we study its behavior after certain faithfully flat extensions.

KEYWORDS
asymptotic Samuel function, integral closure

1 | INTRODUCTION

Let X be an algebraic variety over a field k. To give a constructive resolution of singularities of X means to describe a procedure to construct the centers of a finite sequence of blow ups at regular centers,

$$X_0 = X \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \tag{1.1}$$

so that X_n is regular. This is usually accomplished (when known to exist) by defining some upper-continuous functions

$$\Gamma_i : X_i \to (\Lambda, \leq),$$

where $(\Lambda, \leq)$ is some well-ordered set, and where the maximum value of Γ_i determines the center of the monoidal transformation $\pi_i : X_i \to X_{i-1}$, for $i = 1, \ldots, n$. These *resolution functions* also provide us with a criterion to determine that the variety X_{i+1} is *less singular* than X_i.

The construction of the functions Γ_i is somehow involved, and yet, it is strongly supported on the usual order function that one defines in a regular local ring. Furthermore, it vastly exploits the nice properties of the order function when defined in a smooth scheme of finite type over a perfect field. See for instance the approach to resolution followed in [17] where this fact becomes quite evident.

Some properties of the order function in regular rings that play a key role in resolution

Let S be a regular ring and let $\mathfrak{q} \subset S$ be a prime ideal. The usual order of an element $s \in S$ at $\mathfrak{q}$ is

$$\nu_{\mathfrak{q}S_{\mathfrak{q}}}(s) := \max\{\ell \,:\, s \in \mathfrak{q}^{\ell} S_{\mathfrak{q}}\}.$$

(A) The function $\nu_{\mathfrak{q}S_{\mathfrak{q}}}$ is a valuation, and therefore for $a, b \in S$, $\nu_{\mathfrak{q}S_{\mathfrak{q}}}(ab) = \nu_{\mathfrak{q}S_{\mathfrak{q}}}(a) + \nu_{\mathfrak{q}S_{\mathfrak{q}}}(b)$.

(B) When S is essentially of finite type over a perfect field k, for a fixed $s \in S$ the function

$$\begin{array}{rccc} \nu(s): & \operatorname{Spec}(S) & \longrightarrow & \mathbb{N} \cup \{\infty\} \\ & \mathfrak{q} & \mapsto & \nu_{\mathfrak{q}S_{\mathfrak{q}}}(s) \end{array}$$

$\nu(s)$ is upper semicontinuous. In particular, for $\mathfrak{q}_1 \subset \mathfrak{q}_2$

$$\nu_{\mathfrak{q}_1 S_{\mathfrak{q}_1}}(s) \leq \nu_{\mathfrak{q}_2 S_{\mathfrak{q}_2}}(s). \tag{1.2}$$

Actually, the inequality (1.2) holds for regular rings in general (see [16]), and it can also be read in terms of the symbolic powers of $\mathfrak{q}_i$, namely, for all $\ell \in \mathbb{N}$,

$$\mathfrak{q}_1^{(\ell)} \subseteq \mathfrak{q}_2^{(\ell)}. \tag{1.3}$$

(C) When S contains a field, and $\mathfrak{q}$ defines a regular subscheme in $\operatorname{Spec}(S)$, that is, if $S/\mathfrak{q}$ is regular, then the ordinary and the symbolic powers of $\mathfrak{q}$ coincide,

$$\mathfrak{q}^{(\ell)} = \mathfrak{q}^{\ell}, \tag{1.4}$$

or in other words, for all $s \in S$,

$$\nu_{\mathfrak{q}}(s) = \nu_{\mathfrak{q}S_{\mathfrak{q}}}(s). \tag{1.5}$$

This last property plays a special role, for instance, to control the transforms of the resolution invariants after each of the blow ups at the regular centers in sequence (1.1).

The order function is used to define the resolution functions

Let us start by considering a special case. Let S be a smooth k-algebra of finite type over a perfect field k, let $f(Z) \in S[Z]$ be a polynomial defining a hypersurface X of maximum multiplicity $m > 1$ in $\operatorname{Spec}(S[Z])$, and suppose we can write,

$$f(Z) = Z^m + a_1 Z^{m-1} + \cdots + a_m. \tag{1.6}$$

Already the order stratifies the singularities of f into locally closed strata. Thus, to approach a resolution of X, one may think that the problem can be reduced to lowering the maximum order of a strict transform of f below m. This is usually referred to as *resolving the pair* $(\langle f \rangle, m)$. However, just this information might not be enough to construct the resolution function $\Gamma : X \to (\Lambda, \geq)$. In particular, resolving the pair $(\langle f \rangle, m)$ requires the definition of new functions, usually referred as *resolution invariants*. And, again, the main source to defining them relies, in one way or another, on some order function of a suitably defined local regular ring. Thus, Γ would look something like this:

$$\begin{array}{rcl} \Gamma: \quad X & \longrightarrow & \Lambda := \mathbb{N} \times \mathbb{Q}_{>0} \times \cdots \times \mathbb{Q}_{>0} \\ \xi & \mapsto & (m, h_1(\xi), h_2(\xi), \ldots, h_\ell(\xi)), \end{array} \tag{1.7}$$

where the set Λ is ordered lexicographically. In particular this means that if $\xi, \eta \in X$ are two points with the same multiplicity and if $\xi \in \overline{\eta}$, then,

$$h_1(\eta) \leq h_1(\xi). \tag{1.8}$$

The value of $h_1(\xi)$ is the *weighted-order* at ξ of some ideal $J \subset S$ that collects information coming from the coefficients of f. There are different strategies to define h_1: the so-called *δ-invariant* coming from *Hironaka's polyhedron of the singularity*, the order of the *coefficient ideal of the pair* $(\langle f \rangle, m)$, or the order of an *elimination algebra of* $(\langle f \rangle, m)$, $\mathrm{ord}_X^{(d)}$, or the function $\mathrm{H{-}ord}_X$, among others (see [4–6, 8, 11, 12, 20, 30, 33]). The rest of the functions h_i, $i > 1$, depend in some sense on the construction of h_1.

The previous example covers the case of a hypersurface, since the defining equation f of a hypersurface X can be assumed to have the form in (1.6) after choosing a suitable local (étale) embedding in a neighborhood of a singular point. In addition, the case of an arbitrary variety can be *reduced to the hypersurface case*, also, after considering a suitable local (étale) embedding (see [21], or [34]). Thus, our initial example already gives us a rough picture of a procedure to construct the function Γ.

Note that the definition of the resolution functions strongly uses local-étale embeddings of X into smooth ambient spaces, where the good properties of the usual order function come in handy. As a counterpart, some (non-trivial) work has to be done in order to show that the resolution functions are independent of the embeddings. This is needed to prove, for instance, that the centers to blow up in sequence (1.1), which are determined locally, patch as to define global centers on X that ultimately lead to a resolution of singularities of X. And sometimes the use of the étale topology is not enough. For instance, the invariants provided by Hironaka's polyhedron are constructed at the completion of a local regular ring where the ideal of the variety is defined. In this line, we should mention the works of Cossart–Piltant in [13] and Cossart–Schober in [14], where it is shown that to construct the Hironaka's polyhedron, the completion can be avoided.

We can go one step further and explore properties of the local rings at the singular points of X that allow us to collect information regarding the resolution functions: can we avoid the use of a local embedding in a regular ring and get information directly from the singular local ring of a variety?

When $\xi \in X$ is a singular point, it is still possible to consider the order function at $\mathcal{O}_{X,\xi}$, but this does not behave very nicely. To start with, it is far from being upper semicontinuous. A function that has a much nicer behavior is the *asymptotic Samuel function*.

In this paper, we study some properties of the asymptotic Samuel function which could be useful to understand resolution functions from an intrinsic point of view. In this sense, we approach two different problems. On one hand, we explore the properties of the asymptotic Samuel function in comparison to the properties **(A), (B),** and **(C)** listed before. In particular, we will see that **(B)** and **(C)** hold for equimultiple primes (compare to the inequality (1.8)). On the other hand, we continue the work started in [2], where we used the asymptotic Samuel function to define an invariant, the *Samuel slope* of a local ring, which can be defined for any local Noetherian ring. In [2], we showed that the Samuel slope is connected to some resolution functions that appear naturally when working with algebraic varieties over perfect fields of prime characteristic. In particular, there seems to be a strong connection between the Samuel slope of a singular point and the value of the function h_1 mentioned above (see [3]). Here, we do not restrict to algebraic varieties and explore further properties of this invariant in the wider context of excellent local equicharacteristic equidimensional rings, and, among other results, we prove inequalities in the line of (1.8) when comparing the Samuel slope at equimultiple primes.

Some definitions and main results:

Definition 1.1. Let A be a Noetherian ring and let $I \subset A$ be a proper ideal. The *asymptotic Samuel function at* I, $\bar{\nu}_I : A \to \mathbb{R} \cup \{\infty\}$, is defined as follows:

$$\bar{\nu}_I(f) = \lim_{n\to\infty} \frac{\nu_I(f^n)}{n}, \qquad f \in A. \tag{1.9}$$

This function was first introduced by Samuel in [29], when studying the behavior of powers of ideals. Afterward, Rees pursued the use of this function in [25, 26] where it is shown that the limit exists, see also [32, Lemma 6.9.2], [27, 28]. If $(A, \mathfrak{m})$ is a local regular ring, then $\overline{\nu}_{\mathfrak{m}}$ is the ordinary order function at the maximal ideal of A (and then we write $\nu_{\mathfrak{m}}$).

The asymptotic Samuel function measures how deep a given element lies into the integral closure of an ideal, that is,

$$\bar{\nu}_I(f) \geq \frac{n}{\ell} \Longleftrightarrow f^{\ell} \in \overline{I^n}, \tag{1.10}$$

see [32, Corollary 6.9.1]. If I is not contained in a minimal prime of A and $\{v_1, \dots, v_s\}$ is a set of Rees valuations of I, then

$$\bar{\nu}_I(f) = \min\left\{\frac{v_i(f)}{v_i(I)} \mid i = 1, \dots, s\right\}, \tag{1.11}$$

(see [32, Lemma 10.1.5, Theorem 10.2.2] and [31, Proposition 2.2]). Therefore, if $f \in A$ is not nilpotent, $\overline{\nu}_I(f) \in \mathbb{Q}$.

Remark 1.2. The asymptotic Samuel function is an order function. It can be checked that the following hold:

(i) for $f, g \in A$, $\overline{\nu}_I(f+g) \geq \min\{\overline{\nu}_I(f), \overline{\nu}_I(g)\}$, with equality if $\overline{\nu}_I(f) \neq \overline{\nu}_I(g)$;
(ii) for $f, g \in A$, $\overline{\nu}_I(fg) \geq \overline{\nu}_I(f) + \overline{\nu}_I(g)$.

In addition, it is worthwhile noticing that for $f \in A$ and $\ell \in \mathbb{N}$, $\overline{\nu}_I(f^{\ell}) = \ell \cdot \overline{\nu}_I(f)$.

Definition 1.1 can be extended to the case in which arbitrary filtrations of ideals are considered. This has been studied by Cutkosky and Praharaj in [15]. On the other hand, we refer to the work of Hickel in [19] for some results on the explicit computation of the asymptotic Samuel function on complete local rings. Some of these results will play a role in our arguments, and they will be precisely stated and properly referred in Section 2.

Properties (A), (B), and (C) 1.3. In the following lines, we will revisit properties **(A)**, **(B)**, and **(C)** in the case of the asymptotic Samuel function.

(A) In general, the asymptotic Samuel function is not a valuation and it is not hard to find examples.

Example 1.4. Let k be a field. Consider the ring $B = k[x, y, z]/\langle xy - z^3\rangle$, with maximal ideal $\mathfrak{m} = \langle \overline{x}, \overline{y}, \overline{z}\rangle$. We have that:

$$\overline{\nu}_{\mathfrak{m}}(\overline{x}) = \overline{\nu}_{\mathfrak{m}}(\overline{y}) = \overline{\nu}_{\mathfrak{m}}(\overline{z}) = 1;$$

$$\overline{\nu}_{\mathfrak{m}}(\overline{xy}) = 3;\;\; \overline{\nu}_{\mathfrak{m}}(\overline{xz}) = 2;\;\; \overline{\nu}_{\mathfrak{m}}(\overline{yz}) = 2.$$

Here, it can be checked that $\langle \overline{x}, \overline{y}\rangle$ is not a reduction of $\mathfrak{m}$. However, there are minimal reductions of $\mathfrak{m}$ which contain the element $\overline{z}$. In fact, under some assumptions, using minimal reductions of $\mathfrak{m}$, one can identify a regular subring of B where the restriction of $\overline{\nu}_{\mathfrak{m}}$ behaves as a valuation. The following result clarifies what is going on in the previous example, in fact, a little bit more can be stated:

Proposition 2.6. *Let $(B, \mathfrak{m})$ be an equidimensional excellent equicharacteristic local ring of dimension $d \geq 1$ containing a field k. Suppose $\mathfrak{m}$ has a reduction generated by d elements, $y_1, \dots, y_d \in \mathfrak{m}$. Set $A := k[y_1, \dots, y_d]_{\langle y_1, \dots, y_d\rangle} \subset B$. Then, for $a \in A$ and $b \in B$,*

$$\overline{\nu}_{\mathfrak{m}}(ab) = \overline{\nu}_{\mathfrak{m}}(a) + \overline{\nu}_{\mathfrak{m}}(b).$$

(B), (C) Let B be a non-necessarily regular ring. For our discussions, the case $\dim(B) = 0$ can be left out, thus through the paper we will be assuming that $\dim(B) \geq 1$. For a prime ideal $\mathfrak{p} \subset B$, using the asymptotic Samuel function we can define the following filterations: for $r \in \mathbb{Q}_{>0}$,

$$\mathfrak{p}^{\geq r} := \{b \in B : \overline{\nu}_{\mathfrak{p}}(b) \geq r\}, \qquad \mathfrak{p}^{(\geq r)} := \{b \in B : \overline{\nu}_{\mathfrak{p}B_{\mathfrak{p}}}(b) \geq r\}.$$

In general, we do not expect that properties **(B)** and **(C)** hold:

Example 1.5. Let k be a field, and let $B = \left(k[x,y,z]/\langle y^2 + zx^3\rangle\right)_{\mathfrak{m}}$, where $\mathfrak{m} = \langle \overline{x}, \overline{y}, \overline{z}\rangle$. Set $\mathfrak{p} = \langle \overline{y}, \overline{z}\rangle$. Notice that

$$\overline{\nu}_{\mathfrak{m}}(\overline{z}) = 1 \leq \overline{\nu}_{\mathfrak{p}B_{\mathfrak{p}}}(\overline{z}) = 2.$$

In addition, $\mathfrak{p}$ defines a regular prime in B, i.e. $B/\mathfrak{p}$ is regular, however,

$$1 = \overline{\nu}_{\mathfrak{p}}(\overline{z}) < \overline{\nu}_{\mathfrak{p}B_{\mathfrak{p}}}(\overline{z}) = 2.$$

As indicated before, we will see that, to expect similar properties as in the regular case, we have to restrict to primes with the same multiplicity. In other words, the asymptotic Samuel function *behaves as expected* when restricted to a (locally closed) stratum of constant multiplicity of Spec(B).

To fix notation, for a prime $\mathfrak{p}$ in Spec(B) and for a $\mathfrak{p}$-primary ideal, $\mathfrak{a} \subset B$, we will use $e_{B_{\mathfrak{p}}}(\mathfrak{a}B_{\mathfrak{p}})$ to denote the multiplicity of the local ring $B_{\mathfrak{p}}$ at $\mathfrak{a}B_{\mathfrak{p}}$. Now, properties **(B)** and **(C)** have the following reformulation in the context of singular rings:

Theorem 3.1. *Let B be an equidimensional excellent ring containing a field. Let $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset B$ be two prime ideals such that $e_{B_{\mathfrak{p}_1}}(\mathfrak{p}_1 B_{\mathfrak{p}_1}) = e_{B_{\mathfrak{p}_2}}(\mathfrak{p}_2 B_{\mathfrak{p}_2})$. Then $\overline{\nu}_{\mathfrak{p}_1 B_{\mathfrak{p}_1}}(b) \leq \overline{\nu}_{\mathfrak{p}_2 B_{\mathfrak{p}_2}}(b)$ for $b \in B$.*

In particular, this says that for $\mathfrak{p}_1 \subset \mathfrak{p}_2$ as in the theorem, and $r \in \mathbb{Q}_{>0}$,

$$\mathfrak{p}_1^{(\geq r)} \subseteq \mathfrak{p}_2^{(\geq r)}.$$

Theorem 3.3. *Let B be an equidimensional excellent ring containing a field. Let $\mathfrak{p} \subset B$ be a prime in the top multiplicity locus of B, and assume that $B/\mathfrak{p}$ is regular. Then, $\overline{\nu}_{\mathfrak{p}}(b) = \overline{\nu}_{\mathfrak{p}B_{\mathfrak{p}}}(b)$ for $b \in B$.*

Hence, in particular, for $\mathfrak{p}$ as in the theorem, and $r \in \mathbb{Q}_{>0}$,

$$\mathfrak{p}^{(\geq r)} = \mathfrak{p}^{\geq r}.$$

To conclude this part, if $(S, \mathfrak{n})$ is a regular local ring, the usual order induces the natural filtration $\{\mathfrak{n}^{\ell}\}_{\ell \in \mathbb{N}}$, where

$$\mathfrak{n}^{\ell} = \{s \in S : \nu_{\mathfrak{n}}(s) \geq \ell\},$$

which in turns leads to the usual graded ring $\mathrm{Gr}_{\mathfrak{n}}(S) = \bigoplus_{\ell \in \mathbb{N}} \mathfrak{n}^{\ell}/\mathfrak{n}^{\ell+1}$ that is graded over $\mathbb{N}$ and finitely generated over $S/\mathfrak{n}$. For an arbitrary local ring $(B, \mathfrak{m}, k)$, we can consider the graded ring associated to the filtration induced by the asymptotic Samuel function: setting $\mathfrak{m}^{>r} := \{b \in B : \overline{\nu}_{\mathfrak{m}}(b) > r\}$, define

$$\overline{\mathrm{Gr}}_{\mathfrak{m}}(B) := \bigoplus_{r \in \mathbb{Q}_{\geq 0}} \mathfrak{m}^{\geq r}/\mathfrak{m}^{>r}.$$

Note that by (1.11), there is some integer $n \in \mathbb{N}$ so that $\overline{\mathrm{Gr}}_{\mathfrak{m}}(B)$ is graded over $\frac{1}{n}\mathbb{N}$. And actually n can be taken as $m!$ if $m = e_{B_{\mathrm{red}}}(\mathfrak{m}_{\mathrm{red}})$, where B_{red} denotes $B/\mathrm{Nil(B)}$ and $\mathfrak{m}_{\mathrm{red}} = \mathfrak{m}/\mathrm{Nil}(B)$ (see Remark 2.12). Imposing some (mild) conditions on B, we can show that $\overline{\mathrm{Gr}}_{\mathfrak{m}}(B)$ is a k-algebra of finite type:

Theorem 3.4. *Let $(B, \mathfrak{m}, k)$ be an excellent local ring. Then, $\overline{\mathrm{Gr}}_{\mathfrak{m}}(B)$ is a k-algebra of finite type.*

The Samuel slope of a local ring 1.6. The notion of *Samuel slope of a local ring* was introduced in [2]. More precisely, for a Noetherian local ring $(B, \mathfrak{m}, k)$ of dimension d we consider the natural map:

$$\lambda_{\mathfrak{m}} : \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}^{\geq 1}/\mathfrak{m}^{>1}.$$

If $(B, \mathfrak{m}, k)$ is not regular, then $\ker(\lambda_{\mathfrak{m}})$ might be non-trivial, and its dimension as a k-vector space is an invariant of the ring. To start with, it can be proven that $\dim_k(\ker(\lambda_{\mathfrak{m}}))$ is bounded above by the *excess of embedding dimension of* B, exc-emb-dim(B), that is,

$$0 \leq \dim_k(\ker(\lambda_{\mathfrak{m}})) \leq \text{exc-emb-dim}(B) := \dim_k\left(\mathfrak{m}/\mathfrak{m}^2\right) - \dim(B).$$

This follows from the fact that elements $a \in \ker(\lambda_{\mathfrak{m}})$ are nilpotent in $\mathrm{Gr}_{\mathfrak{m}}(B)$.

Local non-regular rings where the upper bound is not achieved seem to have milder singularities than the others, and in such case we say that *Samuel slope of* B, $\mathcal{S}\text{-sl}(B)$, is 1.

When $\dim_k(\ker(\lambda_{\mathfrak{m}})) = \text{exc-emb-dim}(B) > 0$ we say that B is in the *extremal case*, and we define the Samuel slope as follows.

Set $t = \text{exc-emb-dim}(B) > 0$. We say that the elements $\gamma_1, \ldots, \gamma_t \in \mathfrak{m}$ form a $\lambda_{\mathfrak{m}}$*-sequence* if their classes in $\mathfrak{m}/\mathfrak{m}^2$ form a basis of $\ker(\lambda_{\mathfrak{m}})$. Then,

$$\mathcal{S}\text{-sl}(B) := \sup_{\lambda_{\mathfrak{m}}\text{-sequence}} \{\min\{\bar{\nu}_{\mathfrak{m}}(\gamma_1), \ldots, \bar{\nu}_{\mathfrak{m}}(\gamma_t)\}\} \in \mathbb{R}_{\geq 0} \cup \{\infty\},$$

where the supremum is taken over all the $\lambda_{\mathfrak{m}}$-sequences of B.

Equivalently, $\mathcal{S}\text{-sl}(B)$ can also defined in the following way. Let $\mathbf{x} = \{x_1, \ldots, x_{d+t}\} \subset \mathfrak{m}$ be a minimal set of generators of $\mathfrak{m}$. We define the *slope with respect to* $\mathbf{x}$ as

$$\mathrm{sl}_{\mathbf{x}}(B) := \min\{\overline{\nu}_{\mathfrak{m}}(x_{d+1}), \ldots, \overline{\nu}_{\mathfrak{m}}(x_{d+t})\}.$$

And then,

$$\mathcal{S}\text{-sl}(B) := \sup_{\mathbf{x}} \mathrm{sl}_{\mathbf{x}}(B) = \sup_{\mathbf{x}}\{\min\{\bar{\nu}_{\mathfrak{m}}(x_{d+1}), \ldots, \bar{\nu}_{\mathfrak{m}}(x_{d+t})\}\},$$

where the supremum is taken over all possible minimal ordered sets of generators $\mathbf{x}$ of $\mathfrak{m}$. To conclude, if $(B, \mathfrak{m})$ is regular we set $\mathcal{S}\text{-sl}(B) := \infty$.

Example 1.7. Let k be a field of characteristic 2. Set $B = \left(k[x, y]/\langle x^2 + y^4 + y^5\rangle\right)_{\mathfrak{m}}$, where $\mathfrak{m} = \langle \overline{x}, \overline{y}\rangle$. Then, $(B, \mathfrak{m})$ is in the extremal case and $\ker(\lambda_{\mathfrak{m}}) = \langle \mathrm{In}_{\mathfrak{m}}(\overline{x})\rangle$. Both $\{\overline{x}\}$ and $\{\overline{x} + \overline{y}^2\}$ are $\lambda_{\mathfrak{m}}$-sequences. However, $\overline{\nu}_{\mathfrak{m}}(\overline{x}) = 2$, while $\overline{\nu}_{\mathfrak{m}}(\overline{x} + \overline{y}^2) = 5/2$. In fact, $\mathcal{S}\text{-sl}(B) = 5/2$, see Corollary 4.3.

Here, we also study the Samuel slope in the wider setting of equicharacteristic equidimensional excellent local rings. First of all, just from the definition, it is not clear that this invariant is finite in the case of non-regular rings. We show:

Theorem 4.5. *Let $(B, \mathfrak{m}, k)$ be a non-regular reduced equicharacteristic equidimensional excellent local ring. Then,* $\mathcal{S}\text{-sl}(B) \in \mathbb{Q}$.

Next, we consider the case of non-reduced rings. Recall that the multiplicity induces the same stratification on both, B and B_{red}. We show that both rings share the same Samuel slope.

Theorem 4.7. *Let $(B, \mathfrak{m}, k)$ be a non-reduced equicharacteristic equidimensional excellent local ring. Then* $\mathcal{S}\text{-sl}(B) = \mathcal{S}\text{-sl}(B_{\mathrm{red}})$.

From here it follows that $\mathcal{S}\text{-sl}(B) = \infty$ if and only if B_{red} is a regular local ring (Corollary 4.8).

Since the Samuel slope is an invariant of the local ring of a singularity, one would expect that it be preserved under étale extensions and completion. In [2], it was shown that $(B', \mathfrak{m}')$ is a local étale extension of $(B, \mathfrak{m})$ with the same residue field, then $\mathcal{S}\text{-sl}(B) = \mathcal{S}\text{-sl}(B')$. The argument given there also shows that $\mathcal{S}\text{-sl}(B) = \mathcal{S}\text{-sl}(\widehat{B})$, where $\widehat{B}$ is the $\mathfrak{m}$-adic completion of B. Here, we treat the case of arbitrary local étale extensions, which requires a different strategy.

Theorem 5.5. *Let $(B, \mathfrak{m}, k)$ be an equicharacteristic equidimensional excellent local ring, and let $(B, \mathfrak{m}) \to (B', \mathfrak{m}')$ be a local-étale extension. Then, $\mathcal{S}\text{-sl}(B) = \mathcal{S}\text{-sl}(B')$.*

To conclude, as we mentioned the Samuel slope of a local ring seems to be connected to the resolution invariant h_1 in (1.7) which has the following property: if $\mathfrak{p} \subset \mathfrak{m}$ is an equimultiple prime then

$$h_1(\mathfrak{p}) \leq h_1(\mathfrak{m}).$$

In fact, when the characteristic is zero, then h_1 is upper semi-continuous when restricted to points with the same multiplicity. For a Noetherian ring B, we can consider the function

$$\begin{array}{rccc} \mathcal{S}\text{-sl}: & \operatorname{Spec}(B) & \longrightarrow & \mathbb{Q} \cup \{\infty\} \\ & \mathfrak{p} & \mapsto & \mathcal{S}\text{-sl}(B_{\mathfrak{p}}). \end{array}$$

In general, $\mathcal{S}\text{-sl}$ is not upper semicontinuous, see Example 6.1, but it has the following nice property on the maximal spectrum, $\operatorname{MaxSpec}(B)$, of B:

Theorem 6.2. *Let B be an equidimensional equicharacteristic excellent ring and let $\mathfrak{p} \in \operatorname{Spec}(B)$. Then, there is a dense open set $U \subset \operatorname{MaxSpec}(B/\mathfrak{p})$ such that*

$$\mathcal{S}\text{-sl}(B_{\mathfrak{p}}) \leq \mathcal{S}\text{-sl}(B_{\mathfrak{m}}) \qquad \textit{for all} \quad \mathfrak{m}/\mathfrak{p} \in U.$$

The paper is organized as follows. One of the main ingredients in our proofs is the use of the so-called *finite-transversal projections* together with Hickel's result on the computation of the asymptotic Samuel function. These are treated in Section 2, where we also address Proposition 2.6. The proofs of Theorems 3.1, 3.3, and 3.4 are addressed in Section 3. The rest of the paper is dedicated to the Samuel slope of a local ring. Theorems 4.5 and 4.7 are proved in Section 4, while Theorem 5.5 is proved in Section 5. Finally, a proof of Theorem 6.2 is given in Section 6.

2 | FINITE-TRANSVERSAL PROJECTIONS

Finite-transversal projections were considered in [34] for the construction of *local presentations of the multiplicity function* of algebraic varieties defined over a perfect field. The existence of such presentations implies that resolution of singularities of algebraic varieties can be achieved via successive simplifications of the multiplicity (in characteristic zero). Finite-transversal projections were further explored in [1], where they were considered between (non-necessarily regular) algebraic varieties defined over perfect fields. Some other properties of such morphisms are discussed in [7]. In this section, we treat this notion in a more general setting, dropping the assumption that the rings involved be k-algebras of the finite type.

Definition 2.1. Let $S \subset B$ be a finite extension of excellent rings with S regular and B equidimensional and reduced. Let K be the fraction field of S and let $L := B \otimes_S K$. Suppose that no non-zero element of S is a zero divisor in B. We say that the projection $\operatorname{Spec}(B) \to \operatorname{Spec}(S)$ (or the extension $S \subset B$) is *finite-transversal with respect to* $\mathfrak{p} \in \operatorname{Spec}(B)$ if:

$$e_{B_{\mathfrak{p}}}(\mathfrak{p}B_{\mathfrak{p}}) = [L : K].$$

If $(S, \mathfrak{n}) \to (B, \mathfrak{m})$ is a finite-transversal extension of local rings with respect to $\mathfrak{m}$ then we simply say that $(S, \mathfrak{n}) \to (B, \mathfrak{m})$ is a *finite-transversal extension*.

Using Zariski's formula for the multiplicity for finite projections, [35, Theorem 24, p. 297 and Corollary 1, p. 299], one can get the following characterization of finite-transversal projections:

Proposition 2.2 [34, Corollary 4.9]. *Let $S \subset B$ be a finite extension of excellent rings with S regular, and B equidimensional and reduced. Suppose that no non-zero element of S is a zero divisor in B. Let $\mathfrak{p} \subset B$ be a prime ideal, and let $\mathfrak{q} = \mathfrak{p} \cap S$. Then, the following are equivalent:*

(1) $e_{B_\mathfrak{p}}(\mathfrak{p}B_\mathfrak{p}) = [L : K]$.
(2) The following three conditions hold:
  (i) $\mathfrak{p}$ *is the only prime of* B *dominating* $\mathfrak{q}$,
  (ii) $k(\mathfrak{p}) = B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} = S_\mathfrak{q}/\mathfrak{q}S_\mathfrak{q} = k(\mathfrak{q})$,
  (iii) $e_{B_\mathfrak{p}}(\mathfrak{q}B_\mathfrak{p}) = e_{B_\mathfrak{p}}(\mathfrak{p}B_\mathfrak{p})$.

Observe that by Rees' theorem, condition (2) (iii) is equivalent to asking that $\mathfrak{q}B_\mathfrak{p}$ be a reduction of the ideal $\mathfrak{p}B_\mathfrak{p}$. To be able to use Rees' theorem we will be assuming that B is an excellent ring.

On finite-transversal morphisms and the asymptotic Samuel function

Finite-transversal projections will play a central role in our arguments, mainly because of the combination of the outputs of Proposition 2.3, due to Villamayor, and a theorem of Hickel for the computation of the asymptotic Samuel function, Theorem 2.4. In addition, in Section 2.5 we briefly describe how to construct finite-transversal morphisms for some faithfully flat extension of a given ring B. This will be frequently used in the rest of the paper.

Proposition 2.3 [34, Lemma 5.2]. *Let $S \subset B$ be a finite extension such that the non-zero elements of S are non-zero divisors in B. Assume that S is a regular ring and let $K = K(S)$ be the quotient field of S. Let $\theta \in B$ and let $f(Z) \in K[Z]$ be the monic polynomial of minimal degree such that $f(\theta) = 0$. If $S[\theta]$ denotes the S-subalgebra of B generated by θ, then:*

(1) the coefficients of f are in S, that is, $f(Z) \in S[Z]$, and
(2) $S[\theta] \cong S[Z]/\langle f(Z)\rangle$.

Theorem 2.4 [19, Theorem 2.1]. *Let $(B, \mathfrak{m})$ be a Noetherian equicharacteristic equidimensional and excellent local ring of Krull dimension d. Assume that there is a faithfully flat extension $(B, \mathfrak{m}) \to (\widetilde{B}, \widetilde{\mathfrak{m}})$ with $\mathfrak{m}\widetilde{B} = \widetilde{\mathfrak{m}}$ together with a finite-transversal morphism with respect to $\widetilde{\mathfrak{m}}$, $S \subset \widetilde{B}$. Let $b \in B$. If*

$$p(Z) = Z^\ell + a_1 Z^{\ell-1} + \cdots + a_\ell$$

is the minimal polynomial of $b \in \widetilde{B}$ over the fraction field of S, $K(S)$, then $p(Z) \in S[Z]$ and

$$\overline{\nu}_\mathfrak{m}(b) = \overline{\nu}_{\widetilde{\mathfrak{m}}}(b) = \min_i \left\{ \frac{\nu_\mathfrak{n}(a_i)}{i} : i = 1, \dots, \ell \right\}, \tag{2.1}$$

where $\mathfrak{n} = \widetilde{\mathfrak{m}} \cap S$.

Proof. Theorem 2.1 in [19] is stated in the case in which $\widetilde{B} = \widehat{B}$, and then a reduction to the domain case is considered. See [7, Theorem 11.6.8], where it is checked that Hickel's theorem holds in this more general setting. □

Constructing finite-transversal projections 2.5. *Given an excellent reduced equidimensional ring B, and a point $\mathfrak{p} \in$ Spec(B), in general, there might not be a regular ring S and a finite-transversal projection with respect to $\mathfrak{p}$, Spec$(B) \to$ Spec(S). To start with, it is a necessary condition that $\mathfrak{p}$ has a reduction generated by dim$(B_\mathfrak{p})$-elements. But even if such condition is satisfied, the existence of the required finite projection is not guaranteed (see [7, Example 11.3.11]). However, finite-transversal projections can be constructed if we are allowed to extend our ring B. For instance, in [34] (see [9, Proposition 31.1]) it is proven that if B is essentially of finite type over a perfect field k, then a finite-transversal projection can be constructed in some étale extension of B.*

Existence of finite transversal projection. Suppose that $(B, \mathfrak{m})$ is a local equicharacteristic equidimensional excellent reduced ring. Assume also that B contains a reduction of $\mathfrak{m}$ generated by $d = \dim(B)$ elements, $x_1, \dots, x_d \in B$. Now, denote by k' a coefficient field of the $\mathfrak{m}$-adic completion $(\widehat{B}, \widehat{\mathfrak{m}})$ and set $S = k'[[x_1, \dots, x_d]] \subset \widehat{B}$. Since $x_1, \dots, x_d$ are analytically independent, S is a ring of power series in d variables. The extension

$$S = k'[[x_1, \dots, x_d]] \subset \widehat{B} \tag{2.2}$$

is finite by [10, Theorem 8, p. 68] and, moreover, finite-transversal with respect to $\widehat{\mathfrak{m}}$. See also [19, Proof of Theorem 1.1]. The extension $B \to \widehat{B}$ is faithfully flat, this means that for any ideal $I \subset B$ and $b \in B$ we have

$$\overline{\nu}_I(b) = \overline{\nu}_{I\widehat{B}}(b). \tag{2.3}$$

Note that if the residue field of B is infinite then by [32, Proposition 8.3.7], B contains a reduction of $\mathfrak{m}$ generated by $d = \dim(B)$ elements.

Extension to the case with reduction with d elements. If B does not contain a reduction of $\mathfrak{m}$ generated by d elements, then we want to produce a faithfully flat extension $(B, \mathfrak{m}) \to (B_1, \mathfrak{m}_1)$ such that $\mathfrak{m}_1$ has such reduction. We consider two possibilities as follows:

(a) If B contains a field k consider a suitable étale extension of k, $k_1 \supset k$, so that, after localizing at a maximal ideal $\mathfrak{m}_1 \subset B \otimes_k k_1$, the local ring

$$B_1 := (B \otimes_k k_1)_{\mathfrak{m}_1} \tag{2.4}$$

contains a reduction generated by d-elements.

(b) Other possibility is to set the ring

$$B_1 = (B[x])_{\mathfrak{m}[x]} \tag{2.5}$$

which has infinite residue field.

Note that in both cases we have that

$$\overline{\nu}_I(b) = \overline{\nu}_{IB_1}(b), \tag{2.6}$$

for any ideal $I \subset B$ and $b \in B$.

Reduction to reduced rings. In general, we will be dealing with a local ring $(B, \mathfrak{m})$ of Krull dimension $d \geq 1$. And we will be interested in proving results concerning the asymptotic Samuel function, $\overline{\nu}_{\mathfrak{m}} : B \to \mathbb{Q}_{\geq 0}$. Now, observe, first of all, that if $b \in B$ and $\widetilde{b} \in B_{\text{red}}$ is the image of b in $B/\text{Nil}(B)$, then

$$\overline{\nu}_{\mathfrak{m}}(b) = \overline{\nu}_{\mathfrak{m}_{\text{red}}}(\widetilde{b}). \tag{2.7}$$

Hence, in many situations we may reduce our proofs to the case in which the ring in consideration is reduced.

Reduction to complete rings. Summing up, for a local ring $(B, \mathfrak{m})$, let be B_1 as in (2.4) or as in (2.5). Then, we have a chain of faithfully flat extensions,

$$(B, \mathfrak{m}) \to (B_1, \mathfrak{m}_1) \to (\widehat{B}_1, \widehat{\mathfrak{m}}_1). \tag{2.8}$$

For any ideal $I \subset B$ and any $b \in B$, the chain of equalities

$$\overline{\nu}_I(b) = \overline{\nu}_{IB_1}(b) = \overline{\nu}_{I\widehat{B}_1}(b), \tag{2.9}$$

is guaranteed (see [32, Proposition 1.6.2]). In particular, if $I = \mathfrak{m}$, then

$$\overline{\nu}_{\mathfrak{m}}(b) = \overline{\nu}_{\mathfrak{m}B_1}(b) = \overline{\nu}_{\mathfrak{m}_1}(b) = \overline{\nu}_{\mathfrak{m}_1\widehat{B}_1}(b) = \overline{\nu}_{\widehat{\mathfrak{m}}_1}(b). \tag{2.10}$$

Thus, given an excellent, equidimensional, equicharacteristic local ring $(B, \mathfrak{m})$, in most situations we will be able to reduce our proofs to the case of a complete reduced local ring containing either an infinite residue field or a field with sufficient scalars. By assuming that B is excellent we will guarantee that B is formally equidimensional and analytically unramified. The former condition allows us to use Rees' theorem in Proposition 2.2, and the second will be implicitly used when reducing to the case of the completion of a reduced ring.

Good behavior for equimultiple prime ideals. Let be $\mathfrak{p} \subset \mathfrak{m}$ a prime ideal in B such that $e_{B_{\mathfrak{p}}}(\mathfrak{p}B_{\mathfrak{p}}) = e_B(\mathfrak{m})$. Then, this condition is preserved if we consider some of the above extensions. Set $(B', \mathfrak{m}')$ equal to either $B' = B_{\mathrm{red}}$, or $B' = B_1$ as in (2.4) or as in (2.5), or $B' = \widehat{B}_1$ then there exists a prime ideal $\mathfrak{p}' \subset B'$ dominating $\mathfrak{p}$, and

$$e_{B'_{\mathfrak{p}'}}(\mathfrak{p}'B'_{\mathfrak{p}'}) = e_{B'}(\mathfrak{m}'). \tag{2.11}$$

Moreover, if $\mathfrak{p}$ defines a regular subscheme in $\mathrm{Spec}(B)$, then $\mathfrak{p}'$ also defines a regular subscheme in B'.

When does $\overline{\nu}$ behave as a valuation?

After the discussion in Section 2.5, we get the following result:

Proposition 2.6. *Let $(B, \mathfrak{m})$ be an equidimensional excellent equicharacteristic local ring of dimension $d \geq 1$. Suppose $\mathfrak{m}$ has a reduction generated by d elements, $y_1, \dots, y_d \in \mathfrak{m}$, and let $k \subset B$ be a field. Set $A := k[y_1, \dots, y_d]_{\langle y_1,\dots,y_d\rangle} \subset B$. Then, for $a \in A$ and $b \in B$,*

$$\overline{\nu}_{\mathfrak{m}}(ab) = \overline{\nu}_{\mathfrak{m}}(a) + \overline{\nu}_{\mathfrak{m}}(b).$$

Proof. Using the arguments in Section 2.5, we can assume that B is reduced and complete, and consider the finite-transversal extension

$$S = k'[[y_1, \dots, y_d]] \to B,$$

where $k' \supset k$ is a coefficient field of B. Now the result follows from [2, Proposition 2.10]. □

The rest of the section is devoted to the study of some more properties of finite-transversal projections.

On finite-transversal projections and the top multiplicity locus of a ring

The following three statements follow as a consequence of Proposition 2.2 when applied to a local ring $(B, \mathfrak{m})$. They are results concerning the primes in $\mathrm{Spec}(B)$ that have the same multiplicity as that of B at $\mathfrak{m}$, that is, the primes in the top multiplicity locus of $\mathrm{Spec}(B)$.

Proposition 2.7. *Suppose that $(S, \mathfrak{n}) \subset (B, \mathfrak{m})$ is finite-transversal with respect to $\mathfrak{m}$. Let $\mathfrak{p} \subset B$ be a prime ideal with $e_{B_{\mathfrak{p}}}(\mathfrak{p}B_{\mathfrak{p}}) = e_B(\mathfrak{m})$. Let $\mathfrak{q} = \mathfrak{p} \cap S$. Then:*

(i) The extension $S_{\mathfrak{q}} \subset B_{\mathfrak{p}}$ is finite transversal with respect to $\mathfrak{p}$;
(ii) The local ring $B/\mathfrak{p}$ is regular if and only if $S/\mathfrak{q}$ is regular;
(iii) If $B/\mathfrak{p}$ is regular then $S/\mathfrak{q} = B/\mathfrak{p}$.

Proof. Similar results were proven in [34, Corollary 5.9, Proposition 6.3] and [1, Corollary 2.8] in the context of algebraic varieties defined over perfect fields. Here, we check that the statement holds for more general rings under the hypotheses of the proposition.

(i) Consider the following commutative diagram with vertical finite morphisms:

$$\begin{array}{ccccc} B & \longrightarrow & B \otimes_S S_{\mathfrak{q}} & \longrightarrow & L \\ \uparrow & & \uparrow & & \uparrow \\ S & \longrightarrow & S_{\mathfrak{q}} & \longrightarrow & K \end{array}$$

Observe that the generic rank of the extension $S_{\mathfrak{q}} \to B \otimes_S S_{\mathfrak{q}}$ is $m = e_{B_{\mathfrak{p}}}(\mathfrak{p}B_{\mathfrak{p}})$. Hence, by definition, $S_{\mathfrak{q}} \to B \otimes_S S_{\mathfrak{q}}$ is finite-transversal with respect to $\mathfrak{p}$. By Proposition 2.2 (2) (i), $B \otimes_S S_{\mathfrak{q}} = B_{\mathfrak{p}}$. In other words, $S_{\mathfrak{q}} \subset B_{\mathfrak{p}}$ is finite transversal with respect to $\mathfrak{p}$.

(ii) Since $S_{\mathfrak{q}} \subset B_{\mathfrak{p}}$ is finite-transversal with respect to $\mathfrak{p}$, by Proposition 2.2 (2) (ii), $k(\mathfrak{p}) = k(\mathfrak{q})$. Now, consider the commutative diagram with vertical finite extensions,

$$\begin{array}{ccccc} B & \longrightarrow & B/\mathfrak{p} & \longrightarrow & k(\mathfrak{p}) \\ \uparrow & & \uparrow & & \| \\ S & \longrightarrow & S/\mathfrak{q} & \longrightarrow & k(\mathfrak{q}). \end{array}$$

Note that $S/\mathfrak{q} \to B/\mathfrak{p}$ is a finite extension of local rings. Since conditions (2) (i)–(iii) of Proposition 2.2 hold for $(S, \mathfrak{n}) \to (B, \mathfrak{m})$, the same conditions hold for $(S/\mathfrak{q}, \mathfrak{n}/\mathfrak{q}) \to (B/\mathfrak{p}, \mathfrak{m}/\mathfrak{p})$. Now, apply Zariski's multiplicity formula for finite projections (Theorem 2.2) to $S/\mathfrak{q} \to B/\mathfrak{p}$ to obtain,

$$1 = e_{S/\mathfrak{q}} \cdot [k(\mathfrak{p}) : k(\mathfrak{q})] = e_{B/\mathfrak{p}} \cdot [k(\mathfrak{m}) : k(\mathfrak{n}))] = e_{B/\mathfrak{p}},$$

from where the claim in (ii) follows.

(iii) By (ii) if $B/\mathfrak{p}$ is regular, then $S/\mathfrak{q} \subset B/\mathfrak{p}$ is a finite extension of regular local rings with the same quotient field. Since $S/\mathfrak{q}$ is regular, it is normal, and hence $S/\mathfrak{q} = B/\mathfrak{p}$. □

Proposition 2.8 (Presentations of finite-transversal extensions). *Suppose that $(S, \mathfrak{n}) \subset (B, \mathfrak{m})$ is finite-transversal with respect to $\mathfrak{m}$. Let $\mathfrak{p} \subset B$ be a prime ideal with $e_{B_{\mathfrak{p}}}(\mathfrak{p}B_{\mathfrak{p}}) = e_B(\mathfrak{m})$, and assume in addition that $B/\mathfrak{p}$ is a regular local ring. Let $\mathfrak{q} = \mathfrak{p} \cap S$. There are $\theta_1, \dots, \theta_e \in \mathfrak{p}$ such that:*

(i) $B = S[\theta_1, \dots, \theta_e]$;
(ii) $\mathfrak{p} = \mathfrak{q}B + \langle \theta_1, \dots, \theta_e \rangle$.

In addition,

(iii) $\mathfrak{q}B$ *is a reduction of* $\mathfrak{p}$ *(in B).*

Proof. We follow ideas from [34, Lemma 6.4] for part (i), [2, Lemma 8.10] for part (ii) and [1, Lemma 3.6] for part (iii), where similar results were proven in the context of algebraic varieties defined over perfect fields. To facilitate the reading of the paper we check here that the proofs can be adapted to cover a wider class of rings under the hypotheses of the proposition.

(i) Write $B = S[\theta'_1, \dots, \theta'_e]$. By Proposition 2.7 (iii), $S/\mathfrak{q} = B/\mathfrak{p}$, therefore, for each $i \in \{1, \dots, e\}$, there is some $s_i \in S$ such that $\theta'_i - s_i \in \mathfrak{p}$. Set $\theta_i := \theta'_i - s_i$ for $i = 1 \dots, e$. Then, $B = S[\theta_1, \dots, \theta_e]$.

(ii) By (i), we can write $B = S[\theta_1, \dots, \theta_e]$ with $\theta_i \in \mathfrak{p}$ for $i = 1, \dots, e$. Since $S \subset B$ is finite-transversal at $\mathfrak{m}$, $S/\mathfrak{n} = B/\mathfrak{m}$, therefore,

$$\mathfrak{m} = \mathfrak{n} + \langle \theta_1, \dots, \theta_e \rangle. \tag{2.12}$$

If $\mathfrak{p} = \mathfrak{m}$ we are done. Otherwise, since $\mathfrak{q} \subset S$ defines a regular subscheme, there is a regular system of parameters in S, $y_1, \dots, y_d$, such that $\mathfrak{q} = \langle y_1, \dots, y_r \rangle$ for some $r \in \{1, \dots, d-1\}$. Then

$$\mathfrak{q}B + \langle \theta_1, \dots, \theta_e \rangle \subset \mathfrak{p}. \tag{2.13}$$

Now,

$$d - r = \dim(S/\mathfrak{q}) = \dim(B/\mathfrak{p}) \le \dim(B/(\mathfrak{q}B + \langle \theta_1, \dots, \theta_e \rangle)) \le d - r,$$

where the last inequality follows because by (2.12),

$$\mathfrak{m}/(\mathfrak{q} + \langle \theta_1 + \cdots + \theta_e \rangle) = (\mathfrak{n} + \langle \theta_1, \dots, \theta_e \rangle)/(\mathfrak{q} + \langle \theta_1 + \cdots + \theta_e \rangle),$$

and therefore, the maximal ideal $\mathfrak{m}/(\mathfrak{q} + \langle \theta_1, \dots, \theta_e \rangle)$ can be generated by $d - r$ elements. Since $B/\mathfrak{p}$ is an integral domain necessarily the containment in (2.13) is an equality.

(iii) By (i) we can assume that $B = S[\theta_1, \dots, \theta_e]$ with $\theta_i \in \mathfrak{p}$ for $i = 1, \dots, e$. By Proposition 2.7 (i) and by Proposition 2.2, we have that $\mathfrak{q}B_{\mathfrak{p}}$ is a reduction of $\mathfrak{p}B_{\mathfrak{p}}$. To see that $\mathfrak{p}$ is the integral closure of $\mathfrak{q}B$ in B it suffices to check this condition at all the maximal ideals containing $\mathfrak{p}$. Since B is local, this amounts to checking this condition at B.

Since $B/\mathfrak{p}$ is a regular local ring, by Proposition 2.7 (iii), $S/\mathfrak{q}$ is also a regular local ring. Observe that the multiplicity of S is 1, and so is the multiplicity of $S_{\mathfrak{q}}$. Hence, by Theorem 2.9, $\mathrm{ht}(\mathfrak{q}) = l(\mathfrak{q})$ in S.

Now, since the extension $S \subset B$ is finite, $\mathrm{ht}(\mathfrak{q}B) = \mathrm{ht}(\mathfrak{q})$, and the blow up of B at $\mathfrak{q}B$ is finite over the blow up of S at $\mathfrak{q}$. Hence, the fibers over the closed points have the same dimensions, and therefore, $l(\mathfrak{q}B) = l(\mathfrak{q})$. Therefore, $l(\mathfrak{q}B) = \mathrm{ht}(\mathfrak{q}B)$ in B. Recall that $e_{B_{\mathfrak{p}}}(\mathfrak{q}B_{\mathfrak{p}}) = e_{B_{\mathfrak{p}}}(\mathfrak{p}B_{\mathfrak{p}})$, by Proposition 2.2 (iii). Finally, since $\mathfrak{p}$ is the only minimal prime of $\mathfrak{q}B$, the statement follows from Theorem 2.10. □

Theorem 2.9 Hironaka–Schickhoff, [24, Corollary 3, p. 121]. *Let (A, M) be a formally equidimensional local ring, and let $\mathfrak{p} \subset A$ be a prime ideal so that $A/\mathfrak{p}$ is regular. Then, $ht(\mathfrak{p}) = l(\mathfrak{p})$ in A if and only if the local rings A and $A_{\mathfrak{p}}$ have the same multiplicity.*

Theorem 2.10 Böger, [24, Theorems 2 and 3, pp. 115–116], [32, Corollary 11.3.2]. *Let (A, M) be a formally equidimensional local ring. Fix an ideal $I \subset A$ so that $ht(I) = l(I)$. Consider an ideal $J \subset A$ so that $I \subset J \subset \sqrt{I}$. Then, I is a reduction of J if and only if $e_{A_{\mathfrak{q}}}(IA_{\mathfrak{q}}) = e_{A_{\mathfrak{q}}}(JA_{\mathfrak{q}})$ for each minimal prime ideal $\mathfrak{q}$ of I.*

Proposition 2.11 (Intermediate extensions of finite-transversal extensions). *Suppose that $(S, \mathfrak{n}) \subset (B, \mathfrak{m})$ is finite-transversal with respect to $\mathfrak{m}$. Let $\mathfrak{p} \subset B$ be a prime ideal with $e_{B_{\mathfrak{p}}}(\mathfrak{p}B_{\mathfrak{p}}) = e_B(\mathfrak{m})$. Let $B' \subset B$ be an intermediate extension, that is, $S \subset B' \subset B$, and consider the diagram:*

$$\begin{array}{cccc} B & & \mathfrak{m} & \mathfrak{p} \\ \uparrow & \nwarrow & & \\ & B' & \mathfrak{m}' := \mathfrak{m} \cap B', & \mathfrak{p}' := \mathfrak{p} \cap B' \\ & \nearrow & & \\ S & & \mathfrak{n} := \mathfrak{m} \cap S & \mathfrak{q} := \mathfrak{p} \cap S. \end{array}$$

Then:

(i) The ring B' is local with maximal ideal $\mathfrak{m}'$;
(ii) The extension $S \subset B'$ is finite-transversal with respect to $\mathfrak{m}'$ of generic rank $m' = e_{B'}(\mathfrak{m}')$;
(iii) The extension $S_{\mathfrak{q}} \subset B'_{\mathfrak{p}'}$ is finite-transversal with respect to $\mathfrak{p}'$ and $e_{B'_{\mathfrak{p}'}}(\mathfrak{p}'B'_{\mathfrak{p}'}) = m' = e_{B'}(\mathfrak{m}')$;
(iv) For $b \in B'$,

$$\overline{\nu}_{\mathfrak{m}}(b) = \overline{\nu}_{\mathfrak{m}'}(b) \tag{2.14}$$

and

$$\overline{\nu}_{\mathfrak{p}B_{\mathfrak{p}}}(b) = \overline{\nu}_{\mathfrak{p}'B_{\mathfrak{p}'}}(b); \tag{2.15}$$

(v) If $B/\mathfrak{p}$ is regular, then $B'/\mathfrak{p}'$ is regular, and in such case, $\mathfrak{q}B'$ is a reduction of $\mathfrak{p}'$ in B'.

Proof.

(i) This follows from the fact that B is local and the extensions $S \subset B' \subset B$ are finite.
(ii) It suffices to check that the extension $S \subset B'$ satisfies conditions (2)(i)–(iii) of Proposition 2.2. Condition (2)(i) has already been proven, and condition 2(ii) follows from the chain of containments,

$$S/\mathfrak{n} \subset B'/\mathfrak{m}' \subset B/\mathfrak{m} = S/\mathfrak{n}.$$

To check that condition (2)(iii) holds, observe first that $\mathfrak{n}B$ is a reduction of $\mathfrak{m}$, hence $\overline{\mathfrak{n}B} = \mathfrak{m}$. On the other hand,

$$\mathfrak{n}B \subset \mathfrak{m}'B \subset \mathfrak{m} = \overline{\mathfrak{n}B}.$$

Since the extension $B' \subset B$ is finite, by [32, Proposition 1.6.1],

$$\overline{\mathfrak{n}B'} = \overline{\mathfrak{n}B} \cap B' = \mathfrak{m} \cap B' = \mathfrak{m}'.$$

(iii) By Proposition 2.7(i), the extension $S_{\mathfrak{q}} \subset B_{\mathfrak{p}}$ is finite-transversal with respect to $\mathfrak{p}$. Repeating the argument in (ii) we find that $S_{\mathfrak{q}} \subset B'_{\mathfrak{p}'}$ is finite-transversal with respect to $\mathfrak{p}'$. See also [34, Lemma 4.12] for (i), (ii), and (iii), in the case of domains.

(iv) Equality (2.14) follows from (1.10), from the fact that $\mathfrak{m}'B$ is a reduction of $\mathfrak{m}$, see [32, Propositions 8.1.5 and 1.6.1]. Equality (2.15) follows similarly applying the previous argument to the finite-transversal extension $S_{\mathfrak{q}} \subset B'_{\mathfrak{p}'} \subset B_{\mathfrak{p}}$.

(v) By Proposition 2.7 (ii) and (iii) $S/\mathfrak{q}$ is regular and, moreover $S/\mathfrak{q} = B/\mathfrak{p}$. Then, the first part of the statement follows. Finally, by [32, Proposition 1.6.1] and Proposition 2.8(iii), we have $\overline{\mathfrak{q}B'} = \mathfrak{p}'$. □

Remark 2.12. With the same hypotheses and notation as in Proposition 2.11, observe that if $B' = S[\theta]$ for some $\theta \in B$, then, by Proposition 2.3, $S[\theta] \simeq S[Z]/\langle f(Z)\rangle$, where $f(Z) \in S[Z]$ is the minimum polynomial of θ over K, the quotient field of S. The degree of $f(Z)$ is the generic rank of $S \subset S[\theta]$, that is, the dimension of the K-vector space $K \otimes_S S[\theta]$, which is bounded above by $[L : K] = e_B(\mathfrak{m}) = m$. Therefore, by Theorem 2.4, $\overline{\nu}_{\mathfrak{m}}(\theta) \in \frac{1}{m!}\mathbb{N}$. See also [19, Theorem 1.1].

3 | SOME NATURAL PROPERTIES OF THE ASYMPTOTIC SAMUEL FUNCTION

In this section, we are going to explore some natural properties of the asymptotic Samuel function, addressing the proofs of the results presented in the introduction.

Theorem 3.1. *Let B be an equidimensional excellent ring containing a field. Let $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset B$ be two prime ideals such that $e_{B_{\mathfrak{p}_1}}(\mathfrak{p}_1 B_{\mathfrak{p}_1}) = e_{B_{\mathfrak{p}_2}}(\mathfrak{p}_2 B_{\mathfrak{p}_2})$. Then, $\overline{\nu}_{\mathfrak{p}_1 B_{\mathfrak{p}_1}}(b) \leq \overline{\nu}_{\mathfrak{p}_2 B_{\mathfrak{p}_2}}(b)$ for $b \in B$.*

Proof. After localizing at $\mathfrak{p}_2$, we can assume that $(B, \mathfrak{m}, k)$ is a local ring. By the arguments in Section 2.5, see (2.11), we can start by assuming that B is reduced. Consider the $\mathfrak{m}$-adic completion of B, $\widehat{B}$. Let $\mathfrak{p} \subset \widehat{B}$ be a prime dominating $\mathfrak{p}_1 B$. Then:

$$\overline{\nu}_{\mathfrak{p}_1 B_{\mathfrak{p}_1}}(b) \leq \overline{\nu}_{\mathfrak{p}_1 \widehat{B}_{\mathfrak{p}}}(b) \leq \overline{\nu}_{\mathfrak{p}\widehat{B}_{\mathfrak{p}}}(b).$$

Hence, by the arguments detailed in Section 2.5, to prove the theorem we can assume that $(B, \mathfrak{m})$ is a reduced complete local ring and that there is a finite-transversal projection $(S, \mathfrak{n}) \to (B, \mathfrak{m})$, with $S = k[[y_1, \ldots, y_d]]$, where k is the residue field of B, and $\langle y_1, \ldots, y_d\rangle$ generate a reduction of the maximal ideal $\mathfrak{m}$ of B. Let $\mathfrak{q} = \mathfrak{p} \cap S$. By Proposition 2.7, the extension $S_{\mathfrak{q}} \subset B_{\mathfrak{p}}$ is finite-transversal with respect to $\mathfrak{p}$. Now, consider the diagram:

$$\begin{array}{ccc}
B & \longrightarrow & B_{\mathfrak{p}} \\
\uparrow & & \uparrow \\
B' = S[b] & \longrightarrow & B'_{\mathfrak{p}'} = S_{\mathfrak{q}}[b] \\
\uparrow & & \uparrow \\
S = k[[y_1, \ldots, y_d]] & \longrightarrow & S_{\mathfrak{q}}
\end{array}$$

Let $\mathfrak{m}' := \mathfrak{m} \cap S[b]$ and $\mathfrak{p}' := \mathfrak{p} \cap S[b]$. By Proposition 2.11, the extension $S \subset S[b]$ is finite-transversal with respect to $\mathfrak{m}'$ of generic rank $\ell = e_{S[b]_{\mathfrak{m}'}}(\mathfrak{m}')$ and $S_{\mathfrak{q}} \subset S_{\mathfrak{q}}[b]$ is finite transversal with respect to $\mathfrak{p}'$ with the same generic rank, $\ell = e_{S[b]_{\mathfrak{m}'}}(\mathfrak{m}') = e_{B'_{\mathfrak{p}'}}(\mathfrak{p}')$.

Using Proposition 2.11 (iv),

$$\overline{\nu}_{\mathfrak{p}B_{\mathfrak{p}}}(b) = \overline{\nu}_{\mathfrak{p}'B'_{\mathfrak{p}'}}(b) \quad \text{and} \quad \overline{\nu}_{\mathfrak{m}}(b) = \overline{\nu}_{\mathfrak{m}'}(b),$$

hence, it suffices to prove the theorem for B'. Now, by Proposition 2.3, $S[b] \cong S[Z]/\langle f(Z)\rangle$, where $f(Z)$ is the minimal polynomial of r over S. The degree of this polynomial equals the multiplicity of $S[b]$ at $\mathfrak{m}'$, ℓ. By Theorem 2.4, if

$$f(Z) = Z^{\ell} + a_1 Z^{\ell-1} + \cdots + a_l,$$

then

$$\overline{\nu}_{\mathfrak{m}'}(b) = \min_i \left\{ \frac{\nu_{\mathfrak{n}}(a_i)}{i} : i = 1, \dots, \ell \right\}.$$

Now, observe that $S[b]_{\mathfrak{p}'} = S_{\mathfrak{q}}[b]$, and therefore, again by Proposition 2.3, $S[b]_{\mathfrak{p}'} = S_{\mathfrak{q}}[Z]\langle f(Z)\rangle$. Hence, again by Theorem 2.4,

$$\overline{\nu}_{\mathfrak{p}'B'_{\mathfrak{p}'}}(b) = \min_i \left\{ \frac{\nu_{\mathfrak{q}}(a_i)}{i} : i = 1, \dots, \ell \right\}.$$

To conclude by [16, Theorem 2.11], for each $i \in \{1, \dots, \ell\}$, $\frac{\nu_{\mathfrak{q}}(a_i)}{i} \leq \frac{\nu_{\mathfrak{n}}(a_i)}{i}$, thus $\overline{\nu}_{\mathfrak{p}'B'_{\mathfrak{p}'}}(b) \leq \overline{\nu}_{\mathfrak{m}'}(b)$. □

Remark 3.2. Observe that for a given $b \in B$ the function

$$\begin{array}{rccc} \overline{\nu}(b): & \operatorname{Spec}(B) & \longrightarrow & \mathbb{Q} \cup \{\infty\} \\ & \mathfrak{p} & \mapsto & \overline{\nu}_{\mathfrak{p}B_{\mathfrak{p}}}(b) \end{array}$$

might not be upper semicontinuous, even after restricting ourselves to the top multiplicity locus of B. See Example 6.1, where $\overline{\nu}_{\mathfrak{p}B_{\mathfrak{p}}}(\overline{x}) = 1$, whereas $\overline{\nu}_{\mathfrak{m}}(\overline{x}) = (p+1)/p$, for every maximal ideal $\mathfrak{m}$ containing $\mathfrak{p}$.

Theorem 3.3. *Let B be an equidimensional excellent ring containing a field. Let $\mathfrak{p} \subset B$ be a prime in the top multiplicity locus of B and assume that $B/\mathfrak{p}$ is regular. Then, $\overline{\nu}_{\mathfrak{p}}(b) = \overline{\nu}_{\mathfrak{p}B_{\mathfrak{p}}}(b)$ for $b \in B$.*

Proof. Recall that if $n, \ell \in \mathbb{N}$, $\ell \neq 0$, by (1.10), $\overline{\nu}_{\mathfrak{p}}(b) \geq n/\ell$ if and only if $b^{\ell} \in \overline{\mathfrak{p}^n}$. On the other hand, by [32, Proposition 1.1.4(4)], $\overline{\mathfrak{p}^n}B_{\mathfrak{m}} = \overline{\mathfrak{p}^n B_{\mathfrak{m}}}$ for all maximal ideals $\mathfrak{m} \subset B$. As a consequence, $b^{\ell} \in \overline{\mathfrak{p}^n}$ if and only if, $b^{\ell} \in \overline{\mathfrak{p}^n B_{\mathfrak{m}}}$ for all maximal ideals $\mathfrak{m} \subset B$. Thus, it suffices to prove that the equality in the statement holds after localizing at each maximal ideal $\mathfrak{m} \supset \mathfrak{p}$. Hence, we can assume that $(B, \mathfrak{m})$ is local and that $e_{B_{\mathfrak{p}}}(\mathfrak{p}B_{\mathfrak{p}}) = e_B(\mathfrak{m})$. By the arguments in Section 2.5, see (2.11), and the discussion at the beginning of the proof of Theorem 3.1, we can assume that $(B, \mathfrak{m})$ is reduced complete local ring and that there is a finite-transversal extension $S \subset B$. Let $p(z) = z^m + a_1 z^{m-1} + \cdots + a_m \in S[z]$ be the minimal polynomial of b over S, and let $\mathfrak{q} = \mathfrak{p} \cap S$. Then, following the arguments in the first part of the proof of [19, Theorem 2.1],

$$\overline{\nu}_{\mathfrak{p}}(b) \geq \min \left\{ \frac{\nu_{\mathfrak{q}}(a_i)}{i} : i = 1, \dots, m \right\}.$$

By Proposition 2.7, the prime $\mathfrak{q}$ defines a regular prime in $\operatorname{Spec}(S)$. Hence, since S is regular and contains a field we have that the ordinary and symbolic powers of $\mathfrak{q}$ coincide. Therefore,

$$\overline{\nu}_{\mathfrak{p}}(b) \geq \min \left\{ \frac{\nu_{\mathfrak{q}}(a_i)}{i} : i = 1, \dots, m \right\} = \min \left\{ \frac{\nu_{\mathfrak{q}S_{\mathfrak{q}}}(a_i)}{i} : i = 1, \dots, m \right\} = \overline{\nu}_{\mathfrak{p}B_{\mathfrak{p}}}(b) \geq \overline{\nu}_{\mathfrak{p}}(b). \qquad \square$$

As indicated in the introduction, for a local ring $(B, \mathfrak{m})$, the filtration $\{\mathfrak{m}^{\geq r}\}_{r \in \mathbb{Q}_{\geq 0}}$ leads us to the consideration of the graded ring $\overline{\operatorname{Gr}}_{\mathfrak{m}}(B)$, see [23]. Since B is Noetherian, $\overline{\operatorname{Gr}}_{\mathfrak{m}}(B)$ is graded over the rationals with bounded denominators, that is, there is some $\ell \in \mathbb{N}_{\geq 1}$ such that $\overline{\mathrm{Gr}}_{\mathfrak{m}}(B) = \bigoplus_{r\in\frac{1}{\ell}\mathbb{N}_{\geq 0}} \mathfrak{m}^{\geq r}/\mathfrak{m}^{>r}$. If in addition we impose that B is excellent, reduced, equidimensional and equicharacteristic, then by Remark 2.12, ℓ can be taken as $m!$, where m is the multiplicity of the local ring B.

Theorem 3.4. *Let $(B, \mathfrak{m}, k)$ be an excellent local ring. Then, $\overline{\mathrm{Gr}}_{\mathfrak{m}}(B)$ is a k-algebra of the finite type.*

Proof. First observe that by (2.7) and from the definition of $\overline{\mathrm{Gr}}_{\mathfrak{m}}(B)$, we have that $\overline{\mathrm{Gr}}_{\mathfrak{m}}(B) = \overline{\mathrm{Gr}}_{\mathfrak{m}}(B_{\mathrm{red}})$, hence we may assume that B is reduced. If $(B, \mathfrak{m})$ is regular there is nothing to prove. Otherwise, let $\ell := m!$, where m is the multiplicity of the local ring B. Let

$$\mathcal{G} := B \oplus 0W^{\frac{1}{\ell}} \oplus \cdots \oplus 0W^{\frac{\ell-1}{\ell}} \oplus \mathfrak{m}W \oplus 0W^{\frac{\ell+1}{\ell}} \oplus \cdots \oplus 0W^{\frac{2\ell-1}{\ell}} \oplus \mathfrak{m}^2W^2 \oplus \cdots,$$

that is, $\mathcal{G} = \bigoplus_{n\in\mathbb{N}} \mathfrak{m}^{\frac{n}{\ell}} W^{\frac{n}{\ell}}$, where $\mathfrak{m}^0 = B$, $\mathfrak{m}^{\frac{n}{\ell}} = (0)$ if $\frac{n}{\ell} \notin \mathbb{N}$, and W is a variable that helps us keep track of the grading. Define also,

$$\mathcal{H} := B \oplus \mathfrak{m}^{\geq\frac{1}{\ell}} W^{\frac{1}{\ell}} \oplus \cdots \oplus \mathfrak{m}^{\geq\frac{\ell-1}{\ell}} W^{\frac{\ell-1}{\ell}} \oplus \mathfrak{m}^{\geq 1}W \oplus \mathfrak{m}^{\geq\frac{\ell+1}{\ell}} W^{\frac{\ell+1}{\ell}} \oplus \cdots \oplus \mathfrak{m}^{\geq\frac{2\ell-1}{\ell}} W^{\frac{2\ell-1}{\ell}} \oplus \mathfrak{m}^{\geq 2}W^2 \oplus \cdots,$$

that is, $\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \mathfrak{m}^{\geq\frac{n}{\ell}} W^{\frac{n}{\ell}}$.

Then, there is a containment of graded algebras $\mathcal{G} \subset \mathcal{H}$. Observe that $\mathcal{G}$ is finitely generated over B and that $\mathcal{H}$ is integral over $\mathcal{G}$, since, for a homogeneous element $fW^{\frac{n}{\ell}} \in \mathcal{H}$, we have that

$$f^{\ell} \in \overline{\mathfrak{m}^n}.$$

Now, B is excellent, and hence so is $\mathcal{G}$. Let L be the total quotient field of B. The extension $L(W) \subset L(W^{\frac{1}{\ell}})$ is finite and therefore the integral closure of $\mathcal{G}$ in $L(W^{\frac{1}{\ell}})$, $\overline{\mathcal{G}}$, is finite over $\mathcal{G}$. Since $\mathcal{G} \subset \mathcal{H} \subset \overline{\mathcal{G}}$, it follows that $\mathcal{H}$ is finite over $\mathcal{G}$, hence finitely generated over B. To conclude notice that $\overline{\mathrm{Gr}}_{\mathfrak{m}}(B)$ is a quotient of $\mathcal{H}$. □

4 | FINITENESS OF THE SAMUEL SLOPE

We devote the last sections of this paper to study properties of the Samuel slope function defined in Section 1.6. Here, we address the proofs of Theorems 4.5 and 4.7.

In this section, and also in Section 5, we will be using two main facts: first, that the Samuel slope can be computed in the completion of the local ring (Proposition 4.1), and second, if we are given a finite-transversal extension $(S, \mathfrak{n}) \to (B, \mathfrak{m})$ then, there is a procedure to approximate the Samuel slope of B, using translations with elements in S (Proposition 4.2).

Proposition 4.1 [2, Proposition 3.10]. *Let $(B, \mathfrak{m}, k)$ be a Noetherian local ring.*

- *$(B, \mathfrak{m}, k) \to (B', \mathfrak{m}', k')$ is an étale homomorphism such that $k = k'$ then $\mathcal{S}\text{−}\mathrm{sl}(B) = \mathcal{S}\text{−}\mathrm{sl}(B')$.*
- *If $(\hat{B}, \hat{\mathfrak{m}}, k)$ denotes the $\mathfrak{m}$-adic completion of B then $\mathcal{S}\text{−}\mathrm{sl}(B) = \mathcal{S}\text{−}\mathrm{sl}(\hat{B})$.*

Proof. The first assertion is Proposition 3.10 in [2]. And the same proof applies to the completion, since it is enough to observe that $\mathrm{Gr}_{\mathfrak{m}}(B) = \mathrm{Gr}_{\hat{\mathfrak{m}}}(\hat{B})$. □

Proposition 4.2 [2, Lemma 8.9]. *Let $(S, \mathfrak{n}) \to (B, \mathfrak{m})$ be a finite-transversal extension. Write $B = S[\theta_1, \dots, \theta_e]$ for some $\theta_i \in B$, $i = 1, \dots, e$. Set $d = \dim(S) = \dim(B)$. Suppose that the embedding dimension of B is $d + t$, with $t > 0$, and that $\mathcal{S}\text{−}\mathrm{sl}(B) > 1$. Write $\mathfrak{n} = \langle y_1, \dots, y_d\rangle$ for some $y_i \in S$, $i = 1, \dots, d$. Then, there are $s_i \in S$, $i = 1, \dots, e$ such that, after reordering the elements θ_i:*

(i) $B = S[\theta'_1, \dots, \theta'_e]$, where $\theta'_i = \theta_i + s_i$, and

(ii) $\{y_1, \dots, y_d, \theta'_1, \dots, \theta'_t\}$ *is a minimal set of generators of* $\mathfrak{m}$ *with* $t \le e$.

Furthermore,

(iii) For a given a $\lambda_{\mathfrak{m}}$*-sequence,* $\{\delta_1, \dots, \delta_t\} \subset B$*, there are* $s'_i \in S$*,* $i = 1, \dots, e$*, such that if* $\theta''_i := \theta_i + s'_i$ *then,*
 (a) $B = S[\theta''_1, \dots, \theta''_e]$,
 (b) $\min\{\overline{\nu}_{\mathfrak{m}}(\theta''_i) \mid i = 1, \dots, t, \dots, e\} = \min\{\overline{\nu}_{\mathfrak{m}}(\theta''_i) : i = 1, \dots, t\} \ge \min\{\overline{\nu}_{\mathfrak{m}}(\delta_i) : i = 1, \dots, t\}$ *and,*
 (c) $\{\theta''_1, \dots, \theta''_t\}$ *is a* $\lambda_{\mathfrak{m}}$*-sequence.*

Corollary 4.3. *Let* $(B, \mathfrak{m})$ *be non-regular reduced equicharacteristic equidimensional excellent local ring. Let* $\{\theta_1, \dots, \theta_t\}$ *be a* $\lambda_{\mathfrak{m}}$*-sequence such that*

$$\rho = \min\{\overline{\nu}_{\mathfrak{m}}(\theta_1), \dots, \overline{\nu}_{\mathfrak{m}}(\theta_t)\} \in \mathbb{Q} \setminus \mathbb{Z},$$

then $\mathcal{S}-\mathrm{sl}(B) = \rho$.

Proof. Without loss of generality we can assume that $\rho = \overline{\nu}_{\mathfrak{m}}(\theta_1)$. If $\mathcal{S}-\mathrm{sl}(B) > \rho$, then by Proposition 4.2(iii), there exists some $s \in S$ such that $\overline{\nu}_{\mathfrak{m}}(\theta_1 + s) > \overline{\nu}_{\mathfrak{m}}(\theta_1)$. Since $\rho \in \mathbb{Q} \setminus \mathbb{Z}$, observe that $\overline{\nu}_{\mathfrak{m}}(\theta_1) \ne \overline{\nu}_{\mathfrak{m}}(s)$ for all $s \in S$ (since then $\overline{\nu}_{\mathfrak{m}}(s) \in \mathbb{Z}$). Hence, if for some $s \in S$, $\overline{\nu}_{\mathfrak{m}}(\theta_1 + s) \ge \overline{\nu}_{\mathfrak{m}}(\theta_1)$, then $\overline{\nu}_{\mathfrak{m}}(\theta_1 + s) = \overline{\nu}_{\mathfrak{m}}(\theta_1)$, see Remark 1.2. □

The following lemma, which was proven in [2, Proposition 8.6] in the context of a local ring of an algebraic variety, is valid for Noetherian local rings with the same proof, which we briefly sketch here. We will use this result in the proof of Theorem 4.5.

Lemma 4.4. *Let* $(B, \mathfrak{m}, k)$ *be a Noetherian local ring of dimension* $d \ge 1$ *which is in the extremal case. Then, B contains a reduction of* $\mathfrak{m}$ *generated by d elements.*

Proof. By [22, Theorem 10.14], it suffices to find d-elements $\kappa_1, \dots, \kappa_d \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that if $\overline{\kappa_1}, \dots, \overline{\kappa_d}$ denote their images in $\mathfrak{m}_\xi / \mathfrak{m}_\xi^2$, then $\mathrm{Gr}_{\mathfrak{m}_\xi}(\mathcal{O}_{X,\xi}) / \langle \overline{\kappa_1}, \dots, \overline{\kappa_d} \rangle$ is a graded ring of dimension 0. Suppose $\dim_k \mathfrak{m}/\mathfrak{m}^2 = d + t$. By hypothesis $\dim_k \ker(\lambda) = t$, and if $\delta \in \mathfrak{m} \setminus \mathfrak{m}^2$ is so that $\overline{\delta} \in \ker(\lambda_{\mathfrak{m}})$, then $\mathrm{In}_{\mathfrak{m}} \delta \in \mathrm{Gr}_{\mathfrak{m}}(B)$ is nilpotent. Thus, any collection of d-elements in $\mathfrak{m} \setminus \mathfrak{m}^2$ that completes a $\lambda_{\mathfrak{m}}$-sequence to a basis of $\mathfrak{m}/\mathfrak{m}^2$ generates a reduction of $\mathfrak{m}$. □

Theorem 4.5. *Let* $(B, \mathfrak{m}, k)$ *be a non-regular reduced equicharacteristic equidimensional excellent local ring of dimension d. Then,* $\mathcal{S}-\mathrm{sl}(B) \in \mathbb{Q}$.

Proof. First of all, if B is not in the extremal case then $\mathcal{S}-\mathrm{sl}(B) = 1$, and there is nothing to prove. Otherwise, B is in the extremal case, and then it contains a reduction of $\mathfrak{m}$ generated by d elements by Lemma 4.4. By Proposition 4.1, we can assume that B is a local complete ring, and by Section 2.5 we can consider a finite-transversal extension $(S, \mathfrak{n}) \to (B, \mathfrak{m})$.

Assume that the embedding dimension of $(B, \mathfrak{m})$ is $d + t$, where t is the excess embedding dimension of B. By Proposition 4.2, we can write $B = S[\theta_1^{(0)}, \dots, \theta_e^{(0)}]$ with $\overline{\nu}_{\mathfrak{m}}(\theta_j^{(0)}) > 1$, $j = 1, \dots, e$, and $\mathfrak{m} = \mathfrak{n}B + \langle \theta_1^{(0)}, \dots, \theta_t^{(0)} \rangle$. If $\min\{\overline{\nu}_{\mathfrak{m}}(\theta_1^{(0)}), \dots, \overline{\nu}_{\mathfrak{m}}(\theta_t^{(0)})\} \in \mathbb{Q} \setminus \mathbb{Z}$ then by Corollary 4.3 we are done. In fact, if there are some $s_i \in S$, $i = 1, \dots, t$, such that $\min\{\overline{\nu}_{\mathfrak{m}}(\theta_1^{(0)} + s_1), \dots, \overline{\nu}_{\mathfrak{m}}(\theta_t^{(0)} + s_t)\} \in \mathbb{Q} \setminus \mathbb{Z}$ the result follows as well. Therefore, we can assume that $\min\{\overline{\nu}_{\mathfrak{m}}(\theta_1^{(0)} + s_1), \dots, \overline{\nu}_{\mathfrak{m}}(\theta_t^{(0)} + s_t)\} \in \mathbb{Z}$ for every $s_i \in S, i = 1, \dots, t$. Hence, the only way that the $\mathcal{S}-\mathrm{sl}(B) \notin \mathbb{Q}$ is that $\mathcal{S}-\mathrm{sl}(B) = \infty$. Suppose that we can find a sequence of $\lambda_{\mathfrak{m}}$-sequences $\{\{\gamma_1^{(i)}, \dots, \gamma_t^{(i)}\}\}_{i \ge 0}$ such that $\min\{\overline{\nu}_{\mathfrak{m}}(\gamma_1^{(i)}), \dots, \overline{\nu}_{\mathfrak{m}}(\gamma_t^{(i)})\}$ tends to infinity as i grows. Using Proposition 4.2, we can find a sequence $\{\{\theta_1^{(i)}, \dots, \theta_t^{(i)}\}\}_{i \ge 0}$ such that:

(i) There exists some $s_j^{(i)} \in S$ such that $\theta_j^{(i)} = \theta_j^{(0)} + s_j^{(i)}$, for every $j = 1, \dots, t$ and $i \ge 1$,
(ii) $\mathfrak{m} = \mathfrak{n}B + \langle \theta_1^{(i)}, \dots, \theta_t^{(i)} \rangle$,
(iii) $\min\{\overline{\nu}_{\mathfrak{m}}(\theta_j^{(i)}) : j = 1, \dots, t\} \ge \min\{\overline{\nu}_{\mathfrak{m}}(\gamma_j^{(i)}) : j = 1, \dots, t\}$,
(iv) the set $\{\theta_1^{(i)}, \dots, \theta_t^{(i)}\}$ forms a $\lambda_{\mathfrak{m}}$-sequence.

Note that by the condition in (i), $\overline{\nu}_{\mathfrak{m}}(s_j^{(i)}) > 1$ and combining this with condition (iii) it follows that

$$0 \neq \mathrm{In}(\theta_j^{(i)}) = \mathrm{In}(\theta_j^{(0)}) \in \mathfrak{m}/\mathfrak{m}^2. \tag{4.1}$$

Taking a subsequence if necessary, we may assume that the sequence $\{\overline{\nu}_{\mathfrak{m}}(\theta_1^{(i)})\}_{i\geq 1}$ is strictly monotonically increasing (to those purposes note that for every $i \geq 0$, $\overline{\nu}_{\mathfrak{m}}(\theta_1^{(i)}) \neq \infty$ since B is reduced by hypothesis). It follows then that the sequence $\{s_1^{(i)}\}_{i\geq 1}$ is a Cauchy sequence, since

$$\overline{\nu}_{\mathfrak{m}}(\theta_1^{(i)}) = \overline{\nu}_{\mathfrak{m}}(s_1^{(i+1)} - s_1^{(i)}) = \nu_{\mathfrak{n}}(s_1^{(i+1)} - s_1^{(i)}) \in \mathbb{Z}_{\geq 1},$$

where the last equality is a consequence of [2, Proposition 2.10]. Since S is complete, the sequence $\{s_1^{(i)}\}_{i\geq 0}$ converges in S to an element s_1. Hence, the element $\theta = \theta_1^{(0)} + s_1$ is nonzero (see (4.1)) and it is the limit of the sequence $\{\theta_1^{(i)}\}_{i\geq 1} = \{\theta_1^{(0)} + s_1^{(i)}\}_{i\geq 1}$ satisfying that $\overline{\nu}_{\mathfrak{m}}(\theta) = \infty$, contradicting the fact that B is reduced. □

Corollary 4.6. *Let $(B, \mathfrak{m})$ be non-regular reduced equicharacteristic equidimensional excellent local ring. Assume that there is a finite-transversal projection $(S, \mathfrak{n}) \to (B, \mathfrak{m})$, and that $B = S[\theta_1, \ldots, \theta_e]$ for some $\theta_1, \ldots, \theta_e \in B$. Then, there exist $s_1, \ldots, s_e \in S$ such that*

$$\mathcal{S}\text{−sl}(B) = \min\{\overline{\nu}_{\mathfrak{m}}(\theta_i - s_i) \mid i = 1, \ldots, e\}.$$

Proof. This is a direct consequence of Proposition 4.2 and Theorem 4.5. □

Theorem 4.7. *Let $(B, \mathfrak{m}, k)$ be an equicharacteristic equidimensional excellent local ring. Then $\mathcal{S}\text{−sl}(B) = \mathcal{S}\text{−sl}(B_{red})$.*

Proof. The natural surjective morphism of local rings, $(B, \mathfrak{m}, k) \to (B_{\mathrm{red}}, \mathfrak{m}_{\mathrm{red}}, k)$ induces a surjective linear map of k-vector spaces, $h : \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}_{\mathrm{red}}/\mathfrak{m}_{\mathrm{red}}^2$, with $\ker(h) = (\mathrm{Nil}(B) + \mathfrak{m}^2)/\mathfrak{m}^2$, and a commutative diagram of k-vector spaces,

$$\begin{array}{ccc} \mathfrak{m}/\mathfrak{m}^2 & \xrightarrow{\;h\;} & \mathfrak{m}_{\mathrm{red}}/\mathfrak{m}_{\mathrm{red}}^2 \\ \Big\downarrow{\scriptstyle \lambda_{\mathfrak{m}}} & & \Big\downarrow{\scriptstyle \lambda_{\mathfrak{m}_{\mathrm{red}}}} \\ \mathfrak{m}^{(\geq 1)}/\mathfrak{m}^{(>1)} & \xrightarrow{\;h'\;} & \mathfrak{m}_{\mathrm{red}}^{(\geq 1)}/\mathfrak{m}_{\mathrm{red}}^{(>1)}. \end{array}$$

Note that h' is an isomorphism.

There are linear subspaces $L \subset \mathfrak{m}/\mathfrak{m}^2$ and $L_{\mathrm{red}} \subset \mathfrak{m}_{\mathrm{red}}/\mathfrak{m}_{\mathrm{red}}^2$ such that

$$\mathfrak{m}/\mathfrak{m}^2 = L \oplus \ker(\lambda_{\mathfrak{m}}), \qquad \mathfrak{m}_{\mathrm{red}}/\mathfrak{m}_{\mathrm{red}}^2 = L_{\mathrm{red}} \oplus \ker(\lambda_{\mathfrak{m}_{\mathrm{red}}}).$$

Since $\ker(h) \subset \ker(\lambda_{\mathfrak{m}})$, then $L \cong L_{\mathrm{red}}$ and there exists a linear subspace $H \subset \ker(\lambda_{\mathfrak{m}})$ such that $\ker(\lambda_{\mathfrak{m}}) = H \oplus \ker(h)$. Hence $H \cong \ker(\lambda_{\mathfrak{m}_{\mathrm{red}}})$ via h.

If B_{red} is a regular local ring then $t_{\mathrm{red}} = 0$ and $\mathcal{S}\text{−sl}(B_{\mathrm{red}}) = \infty$ by definition. In this case $\ker(\lambda_{\mathfrak{m}}) = \ker(h)$, and hence for any $\lambda_{\mathfrak{m}}$-sequence $\theta_1, \ldots, \theta_t$, we have that $\theta_i \in \mathrm{Nil}(B)$ for all $i = 1, \ldots, t$. The result follows since $\overline{\nu}_{\mathfrak{m}}(\theta_i) = \infty$.

Assume now that B_{red} is non-regular, then $t_{\mathrm{red}} > 0$ and by Theorem 4.5 $\mathcal{S}\text{−sl}(B_{\mathrm{red}}) < \infty$. If $\mathcal{S}\text{−sl}(B_{\mathrm{red}}) = 1$ then $\dim_k(L) = \dim_k(L_{\mathrm{red}}) > d = \dim(B)$, and we have that $\mathcal{S}\text{−sl}(B) = 1$.

If $\mathcal{S}\text{−sl}(B_{\mathrm{red}}) > 1$, then the result follows from (2.7) and because every $\lambda_{\mathfrak{m}_{\mathrm{red}}}$-sequence can be extended to a $\lambda_{\mathfrak{m}}$-sequence with elements in $\ker(h)$, and, reciprocally, every $\lambda_{\mathfrak{m}}$-sequence contains a $\lambda_{\mathfrak{m}_{\mathrm{red}}}$-sequence. □

Corollary 4.8. *Let $(B, \mathfrak{m}, k)$ be an equicharacteristic equidimensional excellent local ring. Then, $\mathcal{S}\text{−sl}(B) = \infty$ if and only if B_{red} is regular.*

5 | THE SAMUEL SLOPE AFTER SOME FAITHFULLY FLAT EXTENSIONS

In this section, we will show that the Samuel slope of a local ring remains the same after the faithfully flat extensions considered in Section 2.5.

Proposition 5.1. *Let $(B, \mathfrak{m})$ be a Noetherian local ring. Set $B' = B[x]_{\mathfrak{m}[x]}$ and let $\mathfrak{m}' = \mathfrak{m}B'$ be the maximal ideal of B'. Then, $\mathcal{S}\text{-sl}(B) = \mathcal{S}\text{-sl}(B')$.*

Proof. If B is regular, there is nothing to prove. Otherwise, observe that the excess of embedding dimension of B, t, is the same as that of B'. If B is not in the extremal case, then B' is not in the extremal case either, and then $\mathcal{S}\text{-sl}(B) = 1 = \mathcal{S}\text{-sl}(B')$. Therefore, it remains to prove the statement if B is in the extremal case, and hence so is B'. The inequality $\mathcal{S}\text{-sl}(B) \leq \mathcal{S}\text{-sl}(B')$ is straightforward. Let us prove that $\mathcal{S}\text{-sl}(B) \geq \mathcal{S}\text{-sl}(B')$.

Every element $\theta' \in B'$ can be expressed, up to a unit, as a polynomial

$$\theta' = \theta_0 + \theta_1 x + \cdots + \theta_r x^r,$$

for some $r \in \mathbb{N}$ and where $\theta_i \in B$. Note that

$$\overline{\nu}_{\mathfrak{m}'}(\theta') = \min\{\overline{\nu}_{\mathfrak{m}}(\theta_0), \ldots, \overline{\nu}_{\mathfrak{m}}(\theta_r)\}.$$

This follows from the fact that

$$\overline{\nu}_{\mathfrak{m}}(\theta_i) \geq \frac{a}{b} \iff \theta_i^b \in \overline{\mathfrak{m}^a}, \qquad \overline{\nu}_{\mathfrak{m}'}(\theta') \geq \frac{a}{b} \iff \theta'^b \in \overline{\mathfrak{m}'^a},$$

and $\overline{\mathfrak{m}'^a} = \overline{\mathfrak{m}^a}B'$, see [32, Lemma 8.4.2(9)].

Assume that $\theta'_1, \ldots, \theta'_t$ is a $\lambda_{\mathfrak{m}'}$-sequence. Up to some units in B', every θ'_i can be expressed as a polynomial

$$\theta'_i = \theta_{i,0} + \theta_{i,1}x + \cdots + \theta_{i,r_i}x^{r_i}, \qquad i = 1, \ldots, t,$$

where $\theta_{i,j} \in B$. We may assume that $\theta_{i,0} \notin \mathfrak{m}^2$ for all $i = 1, \ldots, t$.

We have that

$$\min\{\overline{\nu}_{\mathfrak{m}'}(\theta'_1), \ldots, \overline{\nu}_{\mathfrak{m}'}(\theta'_t)\} \leq \min\{\overline{\nu}_{\mathfrak{m}}(\theta_{1,0}), \ldots, \overline{\nu}_{\mathfrak{m}}(\theta_{t,0})\}. \tag{5.1}$$

If the classes of $\theta_{1,0}, \ldots, \theta_{t,0}$ in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent then $\{\theta_{1,0}, \ldots, \theta_{t,0}\}$ is a $\lambda_{\mathfrak{m}}$-sequence and we conclude that $\mathcal{S}\text{-sl}(B) \geq \mathcal{S}\text{-sl}(B')$.

If the classes of $\theta_{1,0}, \ldots, \theta_{t,0}$ are linearly dependent, there are $\mu_1, \ldots, \mu_t \in B$, not all zero in $B/\mathfrak{m}$, such that

$$\mu_1\theta_{1,0} + \cdots + \mu_t\theta_{t,0} \in \mathfrak{m}^2.$$

Since $\theta_{i,0} \notin \mathfrak{m}^2$ for $i = 1, \ldots, t$, there are at least two indices $i \neq j$ such that $\mu_i, \mu_j \notin \mathfrak{m}$. Let i_0 be such that

$$\overline{\nu}_{\mathfrak{m}'}(\theta'_{i_0}) = \min\{\overline{\nu}_{\mathfrak{m}'}(\theta'_1), \ldots, \overline{\nu}_{\mathfrak{m}'}(\theta'_t)\}.$$

Then, either $i_0 \neq i$ or $i_0 \neq j$. Assume that $i_0 \neq i$ and define

$$\theta''_\ell := \theta'_\ell, \quad \ell \neq i \qquad \text{and} \qquad \theta''_i := x^{-1}\big(\mu_1\theta'_1 + \cdots + \mu_t\theta'_t - (\mu_1\theta_{1,0} + \cdots + \mu_t\theta_{t,0})\big).$$

We have that $\theta''_1, \ldots, \theta''_t$ is a $\lambda_{\mathfrak{m}'}$-sequence and

$$\min\{\overline{\nu}_{\mathfrak{m}'}(\theta'_1), \ldots, \overline{\nu}_{\mathfrak{m}'}(\theta'_t)\} = \min\{\overline{\nu}_{\mathfrak{m}'}(\theta''_1), \ldots, \overline{\nu}_{\mathfrak{m}'}(\theta''_t)\},$$

so that inequality (5.1) also holds and the degree of the polynomial for θ_i'' is smaller. After finitely many steps we arrive to the case where $\theta_{1,0}, \dots, \theta_{t,0}$ form a $\lambda_{\mathfrak{m}}$-sequence. □

Definition 5.2 [6, Definition 4.4]. Let S be a regular ring and let $\mathfrak{q} \subset S$ be a prime such that the quotient $S/\mathfrak{q}$ is a regular ring. Let $f(z) \in S[z]$ be a monic polynomial of degree m in z:

$$f(z) = z^m + a_1 z^{m-1} + \cdots + a_{m-1} z + a_m, \quad a_i \in S,\ j = 1, \dots, m.$$

Set $r_j = \nu_{\mathfrak{q}}(a_j)$ for $j = 1, \dots m$, and set

$$q := \min\left\{ \frac{r_j}{j} : j = 1, \dots, m \right\} = \min\left\{ \frac{\nu_{\mathfrak{q}}(a_j)}{j} : j = 1, \dots, m \right\}.$$

For every $j = 1, \dots, m$, if $jq = r_j$ then set $A_j := \mathrm{In}_{\mathfrak{q}}(a_j) \in \mathfrak{q}^{jq}/\mathfrak{q}^{jq+1}$, and if $jq < r_j$, set $A_j := 0$.

We define the *weighted initial form* of f at $\mathfrak{q}$ as the polynomial:

$$\mathrm{w{-}in}_{\mathfrak{q}}(f(z)) := z^m + \sum_{j=1}^{m} A_j z^{m-j} \in \mathrm{Gr}_{\mathfrak{q}}(S)[z], \tag{5.2}$$

where $\mathrm{Gr}_{\mathfrak{q}}(S) = \bigoplus_{i \geq 0} \mathfrak{q}^i/\mathfrak{q}^{i+1}$. Note that $\mathrm{w{-}in}_{\mathfrak{q}}(f(z))$ is a weighted polynomial of degree mq, where the degree of z is q and the degree of elements in $\mathfrak{q}/\mathfrak{q}^2$ is one.

Remark 5.3. Let $(S, \mathfrak{n}) \to (B, \mathfrak{m})$ be a finite-transversal projection. Let $\theta \in B$, and set $q = \overline{\nu}_{\mathfrak{m}}(\theta)$. If $f(z) = z^m + a_1 z^{m-1} + \cdots + a_{m-1} z + a_m \in S[z]$ is the minimal polynomial of θ over S, we can associate with θ the weighted initial form of f at $\mathfrak{n} = \mathfrak{m} \cap S$, $\mathrm{w{-}in}_{\mathfrak{n}}(f)$. Note that $\mathrm{w{-}in}_{\mathfrak{n}}(f(z))$ is a monic polynomial on z of degree m different from z^m since B is reduced. In particular, there is some j with $A_j \neq 0$.

In fact, there exists some $s \in S$ such that $\overline{\nu}_{\mathfrak{m}}(\theta - s) > \overline{\nu}_{\mathfrak{m}}(\theta)$ if and only if $\mathrm{w{-}in}_{\mathfrak{n}}(f)$ is an mth power. See [6, Remark 4.6] for a discussion on the context of algebraic varieties.

Remark 5.4. Let $(S, \mathfrak{n}) \to (B, \mathfrak{m})$ be a finite-transversal projection of equicharacteristic local rings. Assume that $B = S[\theta]$ for some $\theta \in B$. By Corollary 4.6, there exists some $s \in S$ such that $\overline{\nu}_{\mathfrak{m}}(\theta - s) = \mathcal{S}{-}\mathrm{sl}(B)$. Remark 5.3 gives us an iterative procedure to find $s \in S$:

If $\mathrm{w{-}in}_{\mathfrak{n}}(f(z))$ is not an mth power then $\overline{\nu}_{\mathfrak{m}}(\theta) = \mathcal{S}{-}\mathrm{sl}(B)$.

If $\mathrm{w{-}in}_{\mathfrak{n}}(f(z))$ is an mth power then choose $s_1 \in S$ such that $\mathrm{w{-}in}_{\mathfrak{n}}(f(z)) = (z - \mathrm{In}_{\mathfrak{n}}(s_1))^m$. Note that in this case $\overline{\nu}_{\mathfrak{m}}(\theta)$ must be an integer. Set $\theta_1 = \theta - s_1$. We know that $\overline{\nu}_{\mathfrak{m}}(\theta_1) > \overline{\nu}_{\mathfrak{m}}(\theta)$, and then we can repeat the procedure with θ_1. Observe that $f_1(z) = f(z + s)$ is the minimal polynomial of θ_1.

Now, since $\mathcal{S}{-}\mathrm{sl}(B) < \infty$ (Theorem 4.5), it is clear that, after finitely many steps, the weighted initial form is not an mth power.

Theorem 5.5. *Let $(B, \mathfrak{m}, k)$ be an equicharacteristic equidimensional excellent local ring, and let $(B, \mathfrak{m}) \to (B', \mathfrak{m}')$ be a local-étale extension. Then, $\mathcal{S}{-}\mathrm{sl}(B) = \mathcal{S}{-}\mathrm{sl}(B')$.*

Proof. If B is regular or if $\mathcal{S}{-}\mathrm{sl}(B) = 1$ there is nothing to prove. Otherwise let $k = B/\mathfrak{m}$ and let $k' = B'/\mathfrak{m}'$. If $k = k'$ then the result is [2, Proposition 3.10]. After ruling over the previous cases, by Theorem 4.7, we can assume that both B and B' are reduced, in the extremal case, and that $k \subsetneq k'$ is a separable finite extension. Set $d = \dim(B)$. Since B is in the extremal case, by Lemma 4.4, $\mathfrak{m}$ has a reduction generated by d elements, $\langle y_1, \dots, y_d \rangle$. Observe that $\langle y_1, \dots, y_d \rangle$ also expands to a reduction of $\mathfrak{m}'$ in B', and similarly, expands to reductions of $\widehat{\mathfrak{m}}$ in $\widehat{B}$ and $\widehat{\mathfrak{m}'}$ in $\widehat{B'}$, respectively.

Since $B \to B'$ is étale, by [18, Proposition 17.6.3], $\widehat{B'}$ is formally étale over $\widehat{B}$ for the $\mathfrak{m}$-adic topologies, furthermore, $\widehat{B'}$ is faithfully flat over $\widehat{B}$ and finite. To ease notation, let us denote again by k some coefficient field of $\widehat{B}$. By the natural map $\widehat{B} \to \widehat{B'}$, the image of k maps into some coefficient field of $\widehat{B'}$ which for simplicity we denote by k'. Then, we have the following commutative diagram:

$$\begin{array}{ccc}
\widehat{B} & \longrightarrow & \widehat{B}' \\
\uparrow & & \uparrow \\
S = k[[y_1, \dots, y_d]] & \longrightarrow & S' = k'[[y_1, \dots, y_d]],
\end{array}$$

where the lower horizontal map is local étale, the vertical maps are finite-transversal, and we write $y_1, \dots, y_d$ for the images of these elements in both $\widehat{B}$ and $\widehat{B}'$.

Write $\widehat{B} = S[\theta_1, \dots, \theta_e]$ for some $\theta_i \in B$ (see 2.5). Then, it can be checked that $\widehat{B}' = S'[\theta_1, \dots, \theta_e]$ (here we use the fact that $\widehat{B}'$ is finite over $\widehat{B}$).

Denote by $\mathfrak{n}$ (resp. $\mathfrak{n}'$) the maximal ideal of S (resp. of S'). For $i = 1, \dots, e$, let $\mathrm{w}\!-\!\mathrm{in}_{\mathfrak{n}}(f_i)$ be the weighted initial form of $f_i(z_i)$, the minimal polynomial of θ_i over S. Note that $f_i(z_i)$ is also the minimal polynomial of θ_i over S'. To justify this consider the following diagram:

$$\begin{array}{ccc}
\widehat{B} & \longrightarrow & \widehat{B}' \\
\uparrow & & \uparrow \\
S[\theta_i] & \longrightarrow & S'[\theta_i] \\
\uparrow & & \uparrow \\
S & \longrightarrow & S'.
\end{array}$$

By Proposition 2.3, $S[\theta_i] \cong S[z_i]/f_i(z_i)$, and $S'[\theta_i] = S[\theta_i] \otimes_S S'$.

The image of $\mathrm{w}\!-\!\mathrm{in}_{\mathfrak{n}}(f_i) \in \mathrm{Gr}_{\mathfrak{n}}(S)$ in $\mathrm{Gr}_{\mathfrak{n}'}(S')$ is $\mathrm{w}\!-\!\mathrm{in}_{\mathfrak{n}'}(f_i)$. Now we conclude, since $\mathrm{w}\!-\!\mathrm{in}_{\mathfrak{n}}(f_i)$ is an m-power in $\mathrm{Gr}_{\mathfrak{n}}(S)$ if and only if $\mathrm{w}\!-\!\mathrm{in}_{\mathfrak{n}'}(f_i)$ is an m-power in $\mathrm{Gr}_{\mathfrak{n}'}(S')$. Here, we are using the fact that $\mathrm{Gr}_{\mathfrak{n}'}(S') = \mathrm{Gr}_{\mathfrak{n}}(S) \otimes_k k'$ and the extension $k \to k'$ is étale (see [18, Proposition 16.2.2], in fact flatness is enough to guarantee the isomorphism). Now, the result follows from Remark 5.3 and Corollary 4.6, because:

$$\mathcal{S}\!-\!\mathrm{sl}(\widehat{B}) = \min\{\mathcal{S}\!-\!\mathrm{sl}(S[\theta_i]) \mid i = 1, \dots, e\}. \qquad \square$$

6 | COMPARING SLOPES AT PRIME IDEALS

As indicated in Section 1, for a Noetherian ring B, the function

$$\begin{array}{cccc}
\mathcal{S}\!-\!\mathrm{sl}: & \mathrm{Spec}(B) & \longrightarrow & \mathbb{Q} \cup \{\infty\} \\
& \mathfrak{p} & \mapsto & \mathcal{S}\!-\!\mathrm{sl}(B_{\mathfrak{p}})
\end{array}$$

is not upper semicontinuous in general, even after restricting to the top multiplicity locus of B. This can be checked in the following example:

Example 6.1. Let $p \in \mathbb{Z}_{>0}$ be a prime number, and let $B := \mathbb{F}_p[x, y_1, y_2]/\langle f \rangle$ where $f = x^p - y_1^p y_2$. Observe that $\mathfrak{p} = \langle \overline{x}, \overline{y_1} \rangle$ determines a non-closed point in $\mathrm{Spec}(B)$ of maximum multiplicity p. It can be checked that $\mathcal{S}\!-\!\mathrm{sl}(B_{\mathfrak{p}}) = \overline{\nu}_{\mathfrak{p}B_{\mathfrak{p}}}(\overline{x}) = 1$. However for every maximal ideal $\mathfrak{m} \supset \mathfrak{p}$ we have that $\mathcal{S}\!-\!\mathrm{sl}(B_{\mathfrak{m}}) = \overline{\nu}_{\mathfrak{m}}(\overline{x}) = (p+1)/p$.

Observe that in the example $\mathcal{S}\!-\!\mathrm{sl}(B_{\mathfrak{p}}) \leq \mathcal{S}\!-\!\mathrm{sl}(B_{\mathfrak{m}})$ for all maximal ideals $\mathfrak{m} \supset \mathfrak{p}$. In fact this will happen quite generally, as the following result states.

Theorem 6.2. *Let B be an equidimensional excellent ring containing a field and let $\mathfrak{p} \in \mathrm{Spec}(B)$. Then, there is a dense open set $U \subset \mathrm{MaxSpec}(B/\mathfrak{p})$ such that*

$$\mathcal{S}\!-\!\mathrm{sl}(B_{\mathfrak{p}}) \leq \mathcal{S}\!-\!\mathrm{sl}(B_{\mathfrak{m}}) \qquad \textit{for all} \qquad \mathfrak{m}/\mathfrak{p} \in U.$$

Before addressing the proof of the theorem we need an auxiliary result.

Proposition 6.3. *Let $(S, \mathfrak{n}) \to (B, \mathfrak{m})$ be a finite-transversal projection of equicharacteristic local rings. Suppose that $B = S[\theta]$ for some $\theta \in B$. Let $\mathfrak{p}$ be a prime in B such that $B/\mathfrak{p}$ is regular, and $m = e_{B_\mathfrak{p}}(\mathfrak{p}B_\mathfrak{p}) = e_B(\mathfrak{m}) > 1$. Then, there is some $s \in S$ such that:*

$$\overline{\nu}_\mathfrak{p}(\theta - s) = \overline{\nu}_{\mathfrak{p}B_\mathfrak{p}}(\theta - s) = \mathcal{S}\text{−sl}(B_\mathfrak{p}).$$

Proof. Set $\mathfrak{q} = \mathfrak{p} \cap S$, and let $f(z) \in S[z]$ be the minimal polynomial of θ over S,

$$f(z) = z^m + a_1 z^{m-1} + \cdots + a_{m-1} z + a_m, \qquad a_i \in S.$$

By Proposition 2.3(1), $f(z)$ is also the minimal polynomial of θ over $S_\mathfrak{q}$. By Proposition 2.3(2), $B = S[\theta] \cong S[z]/\langle f(z)\rangle$, therefore the generic rank of the finite-transversal extension $S \subset B$ is $m = e_B(\mathfrak{m})$. Since $e_{B_\mathfrak{p}}(\mathfrak{p}B_\mathfrak{p}) = m$, and $S_q \subset B_\mathfrak{p}$ is finite-transversal (Proposition 2.7(i)) the generic rank is also m and also $B_\mathfrak{p} = S_\mathfrak{q}[\theta] \cong S_\mathfrak{q}[z]/\langle f(z)\rangle$.

By Corollary 4.6, there is some $\tilde{s} \in S_\mathfrak{q}$ with $\overline{\nu}_{\mathfrak{p}B_\mathfrak{p}}(\theta - \tilde{s}) = \mathcal{S}\text{−sl}(B_\mathfrak{p})$. Remark 5.4 indicates that $\tilde{s}$ can be obtained looking at the weighted initial form $\text{w−in}_{\mathfrak{q}S_\mathfrak{q}}(f(z))$.

Since $\mathfrak{p}$ is a regular prime in B, $\mathfrak{q}$ is a regular prime in S (see Proposition 2.7(ii)). Consider the natural map

$$\text{Gr}_\mathfrak{q}(S) = \bigoplus_{i\geq 0} \mathfrak{q}^i/\mathfrak{q}^{i+1} \to \text{Gr}_{\mathfrak{q}S_\mathfrak{q}}(S_\mathfrak{q}) = \text{Gr}_\mathfrak{q}(S) \otimes_{S/\mathfrak{q}} K(S/\mathfrak{q}). \tag{6.1}$$

Note that $\text{w−in}_{\mathfrak{q}S_\mathfrak{q}}(f(z))$ is the image of $\text{w−in}_\mathfrak{q}(f(z))$ by the map in (6.1). Then, both rings in (6.1) are regular, in particular they are Unique Factorization Domain (UFDs) and the second is a localization of the first. Now, it follows that $\text{w−in}_\mathfrak{q}(f(z))$ is an mth power if and only if $\text{w−in}_{\mathfrak{q}S_\mathfrak{q}}(f(z))$ is an mth power. Hence, there is some s in S such that

$$\overline{\nu}_\mathfrak{p}(\theta - s) = \overline{\nu}_{\mathfrak{p}B_\mathfrak{p}}(\theta - s) = \mathcal{S}\text{−sl}(B_\mathfrak{p}). \qquad \square$$

Proof of Theorem 6.2. Since B is excellent, by [11, Theorem 2.33], there exists a dense open set U in $\text{Spec}(B/\mathfrak{p})$ such that $e_{B_\mathfrak{p}}(\mathfrak{p}B_\mathfrak{p}) = e_{B_\mathfrak{m}}(\mathfrak{m}B_\mathfrak{m})$ for every $\mathfrak{m}/\mathfrak{p} \in U$. After shrinking U if needed, we can also assume that $B/\mathfrak{p}$ is regular at all maximal ideals in U. Hence, we can assume to be in the case where B is the localization at some maximal ideal $\mathfrak{m}/\mathfrak{p} \in U$.

By Proposition 5.1, we may assume that the residue field of B is infinite, and by Theorem 4.7 we may assume that B is reduced. By Proposition 4.1 and the arguments in Section 2.5 we may assume that B is complete: here we use the fact that $\mathcal{S}\text{−sl}(B_\mathfrak{p}) \leq \mathcal{S}\text{−sl}(\hat{B}_{\mathfrak{p}\hat{B}})$ and $\mathcal{S}\text{−sl}(\hat{B}) = \mathcal{S}\text{−sl}(B)$, hence it suffices to prove that $\mathcal{S}\text{−sl}(\hat{B}_{\mathfrak{p}\hat{B}}) \leq \mathcal{S}\text{−sl}(\hat{B})$.

Again, by the arguments if Section 2.5 we have a finite-transversal extension $S \subset B$. Set $\mathfrak{q} := \mathfrak{p} \cap S$. By Proposition 2.7 we have that $S_\mathfrak{q} \to B_\mathfrak{p}$ is finite-transversal and $S/\mathfrak{q} = B/\mathfrak{p}$.

By Proposition 2.8, $\mathfrak{q}B$ is a reduction of $\mathfrak{p} \subset B$ and there are $\theta_1, \ldots, \theta_e \in \mathfrak{p}$ such that $B = S[\theta_1, \ldots, \theta_e]$,

$$\mathfrak{m} = \mathfrak{n}B + \langle \theta_1, \ldots, \theta_e\rangle, \qquad \text{and} \qquad \mathfrak{p} = \mathfrak{q} + \langle \theta_1, \ldots, \theta_e\rangle.$$

Now, consider the commutative diagram:

$$\begin{array}{ccc}
B = S[\theta_1, \ldots, \theta_e] & \longrightarrow & S_\mathfrak{q}[\theta_1, \ldots, \theta_e] = B_\mathfrak{p} \\
\uparrow & & \uparrow \\
S[\theta_i] & \longrightarrow & S_\mathfrak{q}[\theta_i] \\
\uparrow & & \uparrow \\
S = k[|y_1, \ldots, y_d|] & \longrightarrow & S_\mathfrak{q}.
\end{array}$$

Set $\mathfrak{m}_i = \mathfrak{m} \cap S[\theta_i]$ and $\mathfrak{p}_i = \mathfrak{p} \cap S[\theta_i]$, for $i = 1, \ldots, e$.

By Proposition 6.3, there are some $s_i \in \mathfrak{p}$ such that for $i = 1 \ldots, e$,

$$\overline{\nu}_\mathfrak{p}(\theta_i - s_i) = \overline{\nu}_{\mathfrak{p}_i}(\theta_i - s_i) = \mathcal{S}\text{−sl}(S_\mathfrak{q}[\theta_i]).$$

Therefore, after translation by elements of S, we can assume that $\overline{\nu}_{\mathfrak{p}}(\theta_i) = \mathcal{S}\text{-sl}(S_{\mathfrak{q}}[\theta_i])$. On the other hand, note that

$$\mathcal{S}\text{-sl}(B_{\mathfrak{p}}) = \min\{\mathcal{S}\text{-sl}(S_{\mathfrak{q}}[\theta_i]) \mid i = 1, \dots, e\}.$$

Now the result follows since for $i = 1, \dots, e$ we have:

$$\mathcal{S}\text{-sl}(S[\theta_i]) \geq \overline{\nu}_{\mathfrak{m}}(\theta_i) \geq \overline{\nu}_{\mathfrak{p}}(\theta_i) = \mathcal{S}\text{-sl}(S_{\mathfrak{q}}[\theta_i])$$

and

$$\mathcal{S}\text{-sl}(B) = \min\{\mathcal{S}\text{-sl}(S[\theta_i]) \mid i = 1, \dots, e\}. \qquad \square$$

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CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

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