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Optimal investment and benefit strategies for a target benefit pension plan where the risky assets are jump diffusion processes ☆

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ABSTRACT

In this paper, we study the optimal management of a target benefit pension plan. The fund manager adjusts the benefit to guarantee the plan stability. The fund can be invested in a riskless asset and several risky assets, where the uncertainty comes from Brownian and Poisson processes. The aim of the manager is to maximize the expected discounted utility of the benefit and the terminal fund wealth. A stochastic control problem is considered and solved by the programming dynamic approach. Optimal benefit and investment strategies are analytically found and analyzed, both in finite and infinite horizons. A numerical illustration shows the effect of some parameters on the optimal strategies and the fund wealth.

1. Introduction

Demographic changes have been observed in many developed countries, such as the increase in life expectancy and the reduction in the birth rate. This increases wealth awareness and concern after retirement. For this reason, it is of interest to find a type of pension system seeking financial sustainability and sharing risk between the fund manager and the participants. On the other hand, due to unexpected news, the evolution of prices in the financial markets may be affected in the form of sudden changes. All this must be taken into account when designing the pension plan model to be analyzed. The objective of this paper is to study a dynamic model of a risk sharing pension plan that takes into account those demographic and financial changes.

There are two major types of pension plan, defined benefit (DB) and defined contribution (DC). In a DB plan, the benefits are fixed in advance and the contributions are designed to maintain the fund in balance, that is, to fund employees' promised benefits. Usually, benefits are linked to salaries, and the contributions are shared by employer and employee. The fund manager bears the risk of funding the pension fund to assure future benefits, and the employee does not suffer possible investment losses. In contrast, in a DC plan, the individual builds his/her own pension fund, selecting a fixed contribution rate and an investment strategy across assets, such as equities and bonds. Benefits are not fixed anymore, but the inherent risk is entirely borne by the individual. The target benefit plan (TBP) is a new type of collective pension plan that blends elements of the DB and DC plans to provide benefits at retirement that are linked to how well the pension plan performs. The contributions are fixed in advance and the benefits must be selected. To do so, the fund manager can invest the fund in a financial market. This pension plan can provide better risk sharing for participants, adequate benefits, while also maintaining the stability of the plan.

The basic framework has already been explored by many authors with dynamic programming methods, such as Battocchio et al. (2007), Baltas et al. (2022), Berkelaar and Kouwenberg (2003), Cairns (2000), Chang (1999), Chang et al. (2003), Haberman and Sung (1994, 2005), Haberman et al. (2000), Josa-Fombellida and Rincón-Zapatero (2001, 2004, 2008a, 2008b, 2010, 2012), Josa-Fombellida et al. (2018), Taylor (2002), Wang et al. (2018) or Zhang and Guo (2020).

Some hybrid pension plans that combine the features of DB and DC pension plans are proposed, such as the target benefit plan in Canada (see CIA (2015)) and the collective DC plans in the Netherlands (Kortleve (2013)). Wang et al. (2018) proposed a continuous investment and intergenerational risk sharing model for Canadian target benefit pension plans. In their model setting, TBPs are collective pension schemes with fixed contributions, and the corresponding target benefit level is calculated according to a formula usually linked to the partici-

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pants' annual salaries. At the same time, all the risks are shared among different generations of plan participants. Wang et al. (2019) investigated the optimal investment and benefit payment problem of TBP for loss-averse participants with S-shape utility. Except TBPs, other investment problems for hybrid pension plans have been investigated recently. The hybrid pension plan whose contribution and benefit levels are adjusted simultaneously was considered in Wang and Lu (2019). Wang et al. (2021) considered a robust optimal problem for TBP with exponential function maximization of wealth and benefit excess or minimization of wealth and benefit gap. Roch (2022) considered a pay-as-you-go pension system where the aim of the fund manager is to minimize the deviations of benefit and fund with respect to its target levels in a financial market with a riskless asset and several risky assets. Previously, Haberman and Zimbidis (2002) considered a similar model, but minimizing the deviation of contribution with respect to its target instead the deviation of fund with respect to its target.

Some pension plan models consider the utility maximization criterion. Vigna (2014) analyzed the portfolio selection problem in the accumulation phase of a DC pension scheme according to the criterion of maximizing the expected utility function of the terminal wealth. Josa-Fombellida and Rincón-Zapatero (2019), Guan et al. (2022) and Josa-Fombellida and López-Casado (2023) studied the optimal management of an aggregated overfunded DB pension plan game, where the aim of the participants is to maximize a utility of the extra benefits. Zhao and Wang (2022) analyze the optimal investment and benefit problem where the manager maximizes a Coob-Douglas and Epstein-Zin recursive utility, when the fund is invested in a financial market with one risk free bond and one stock.

In this paper, we are interested in aggregated pension plans of the TBP type, where the risky assets are stochastic and include both Brownian motions and Poisson jumps. The inclusion of jumps in risky assets is motivated by the possible sudden rise or fall in their price. Other aim is to obtain a closed–form solution that allows us to isolate the effects of the jumps in both optimal investment and benefit strategies. This inclusion of Poisson uncertainty requires the use of a more general Hamilton–Jacobi–Bellman equation (HJB) and verification theorems, as in Fleming and Soner (2006). See, for instance, Guo and Xu (2004) or Hanson (2007), and Oksendal and Sulem (2005), for more general Lévy processes. The first paper considering Poisson jumps in dynamic asset allocation in continuous time was Merton (1971), considering the investment and consumption model with a riskless bond and several risky assets, whose uncertainty is modeled separately by a Brownian motion and a Poisson process. Wu (2003) considered and calibrated a model where the risky asset is a jump diffusion process in a dynamic asset allocation problem. In the pension funding framework, Ngwira and Gerrard (2007) and Josa-Fombellida and Rincón-Zapatero (2012) considered the optimal management of a DB pension plan where jumps appear in the risky assets and in the benefits. Zhang and Guo (2020) considered the management of a defined contribution pension plan, where the salary and the risky asset are both jump diffusion processes. Josa-Fombellida and López-Casado (2023) considered a defined benefit pension plan game between the firm and the union of the participants, where jumps appear in the risky assets.

The main contributions with respect to other papers are described as follows. 1) The model includes Poisson jumps, whose effects on the optimal policies are analyzed. 2) Bounded and unbounded horizons are considered. 3) Closed-form expressions for the optimal benefit and numerical solution for the optimal investment are obtained. 4) The contribution proportion can be increasing or decreasing. 5) Several risky assets are considered. 6) A weighting coefficient is added to the utility function of the terminal fund wealth.

Our findings indicate a linear relationship between the optimal investment and the optimal benefit. The presence of Poisson jumps significantly influences both the optimal solutions and the trajectory of the fund over time. Notably, optimality can be achieved in the infinite horizon scenario when the contribution remains constant. Numerical illustration shows how higher risk aversion leads to a more conservative investment approach, reducing the optimal proportion invested. The impact of risk aversion is consistent across various jump intensities, with a more pronounced effect during negative jumps. The analytic solution reveals the effect of jumps on the growth of the fund, depending on the market regime. Overall, the findings emphasize the intricate interplay between risk, contribution, jump uncertainty, and market dynamics in pension fund management.

The paper is organized as follows. Section 2 defines the elements of the pension plan, describes the financial market and shows the fund wealth evolution. Section 3 analyzes the management of the TBP plan as a stochastic optimal control problem with the objective of maximizing the power instantaneous utility of the benefit and the power terminal utility of the fund. The optimal benefit and the optimal investment strategy are provided, by means of dynamic programming techniques, together with the optimal fund and some properties. The model without jumps and the infinite horizon case are also considered. Section 4 includes a numerical illustration of previous results. Finally, Section 5 establishes some conclusions. All proofs are relegated to Appendix A.

2. The pension model

Consider an aggregated target benefit pension plan where, at every instant of time, active participants coexist with retired participants.

We suppose that the contribution rate is proportional to the size of the fund. The benefit is a control variable for the fund manager that also adjusts the investment.

The main elements intervening in the TBP are the following:

- T: Planning horizon or date of the end of the pension plan, with $0 < T \leq \infty$;
- $F(t)$: value of fund assets at time t;
- $P(t)$: benefits promised to the participants at time t; they are related to the salary at the moment of retirement;
- $C(t)$: contribution proportion of the fund wealth made by the manager at time t to the funding process; it is a deterministic function;
- ρ: positive constant rate of discount or time preference of the manager;
- r: constant risk-free market interest rate.

An interesting case appears when the contribution proportion has an exponential form: $C(t) = c_1 e^{c_2 t}$, with $c_1 > 0$. We consider three interesting particular cases along the paper. When $c_2 = 0$ the contribution is constantly indexed to the fund wealth: $C(t) = c_1$, with $c_1 > 0$. When $c_2 > 0$, we assume a salary growth that is materialized in the contribution, see Roch (2022). When $c_2 < 0$, the manager allows a reduction in the contribution proportion without fund increase, in order to make the pension plan more attractive to the participants, see Zhao and Wang (2022).

The main objective of the manager is to increase the benefits as much as possible in order to make the pension plan more attractive to the participants.

The fund surplus is invested in a financial market composed of one riskless asset and several risky assets. In order to include the sudden variations of the market, the uncertainty is modeled by Brownian motions and Poisson processes.

2.1. The financial market

Following Josa-Fombellida and Rincón-Zapatero (2012), we suppose that the risky assets are jump diffusion processes where the uncertainty is given by Brownian motions and Poisson processes. To model the pension game, we consider a probability space $(\Omega^w, \mathscr{F}^w, \mathbb{P}^w)$, where $\mathbb{P}^w$ is a probability measure on Ω^w and $\mathscr{F}^w = \{\mathscr{F}_t^w\}_{t\geq 0}$ is a complete and right continuous filtration generated by the l-dimensional standard Brownian motion $w = (w_1, \dots, w_l)^\top$, that is to say, $\mathscr{F}_t^w =$

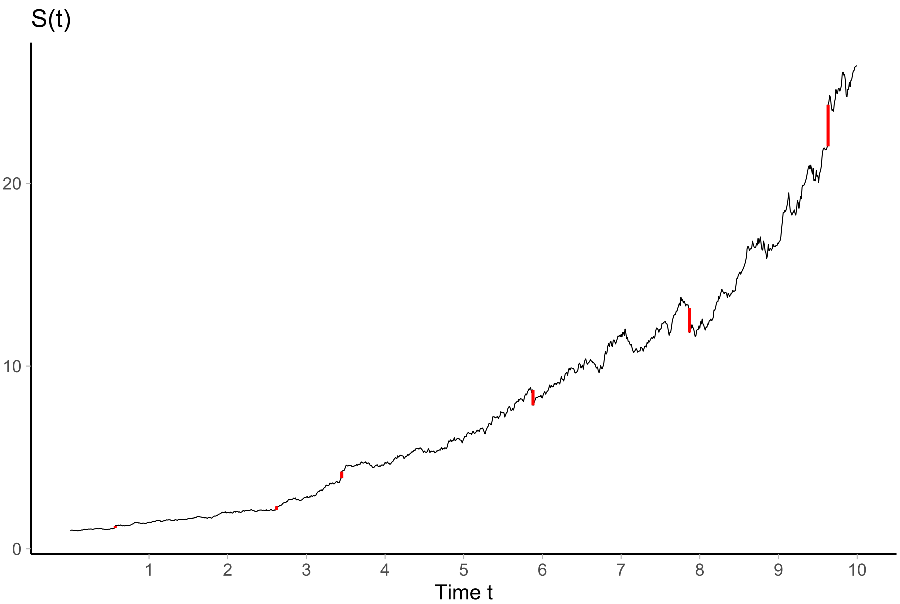

Fig. 1. Risky asset evolution for $b = 0.15$, $\sigma = 0.10$, $r = 0.01$, $\lambda_1 = 0.5$, $\lambda_2 = 0.25$, $\varphi_1 = 0.1$, $\varphi_2 = -0.1$ and $S(0) = 1$.

$\sigma\left\{(w_1(s), \ldots, w_l(s)); 0 \le s \le t\right\}$, $t \ge 0$. We also consider an m-dimensional Poisson process $N = (N_1, \ldots, N_m)^\top$ with constant intensity $\lambda = (\lambda_1, \ldots, \lambda_m)^\top$, $\lambda_1, \ldots, \lambda_m \in \mathbb{R}_+$, defined on a complete probability space $(\Omega^N, \mathscr{F}^N, \mathbb{P}^N)$, where $\mathscr{F}_t^N = \sigma\left\{(N_1(s), \ldots, N_m(s)); 0 \le s \le t\right\}$, $t \ge 0$. Note that the process $H_i(t) = N_i(t) - \lambda_i t$, $i = 1, \ldots, m$, is an $\mathscr{F}^N$-martingale, which is called the compensated Poisson process; see Jeanblanc-Picqué and Pontier (1990) and García and Griego (1994). This fact facilitates the stochastic calculus and the use of the dynamic programming method. Let $(\Omega, \mathscr{F}, \mathbb{P}) = (\Omega^w \times \Omega^N, \mathscr{F}^w \otimes \mathscr{F}^N, \mathbb{P}^w \otimes \mathbb{P}^N)$ denote the product probabilistic space. We suppose w and N are independent processes on this space.

The plan sponsor manages the fund in a bounded planning horizon by means of a portfolio formed by n risky assets $S^1, \ldots, S^n$, which are extended geometric Brownian motions (GBM henceforth, stochastic processes extending the deterministic exponential function), and a riskless asset or bond S^0 (its price is an exponential function), as proposed Guo and Xu (2004), that is, whose evolutions are given by the equations:

$$dS^0(t) = rS^0(t)dt, \quad S^0(0) = 1, \tag{1}$$

$$dS^i(t) = S^i(t-)\Big(b_i dt + \sum_{j=1}^{l} \sigma_{ij} dw_j(t) + \sum_{k=1}^{m} \varphi_{ik} dN_k(t)\Big),$$

$$S^i(0) = s_i, \quad i = 1, \ldots, n. \tag{2}$$

Here, $r > 0$ denotes the short risk-free rate of interest, $b_i > 0$ the mean rate of return of the risky asset S^i, and $\sigma_{ij} > 0$ and $\varphi_{ik} > -1$ the uncertainty parameters, for each i, j. It is usual to assume that $b_i + \sum_{k=1}^m \lambda_k \varphi_{ik} > r$, for each $i = 1, \ldots, n$, so the manager has incentives to invest with risk. The matrix (σ_{ij}) is denoted by σ, the matrix (φ_{ik}) is denoted by φ, b is the (column) vector $(b_1, \ldots, b_n)^\top$, and $\bar{1}$ is the (column) vector of 1's. We will suppose that the symmetric matrix $\Sigma = \sigma\sigma^\top$ and the matrix $(\sigma|\varphi)(\sigma|\varphi)^\top$ are positive definite.

As an example, Fig. 1 shows the time evolution of a risky asset $S(t)$ with two Poisson processes. We observe the effect of the parameters φ_1 an φ_2. The vertical segments represent the time and the magnitude of the jump on the same axis scale. The price of the risky asset shows a significant increase with the upward jumps, but only a slight increase with the downward jumps. We observe 4 positive jumps at the times 0.6, 2.6, 3.4 and 9.6 years, and 2 negative jumps at times 5.6 and 7.8.

2.2. The fund wealth

In order to provide the promised benefits at retirement, the fund manager adopts an amortization scheme and proceeds actively in the financial market to form suitable portfolios. In this risk sharing scheme, the contributions are fixed and the benefit is a control variable. The fund wealth $F > 0$ is invested in the riskless asset S^0 and the n risky assets $S^1, \ldots, S^n$. Let $\Pi = (\pi_1, \ldots, \pi_n)^\top$, where each π_i is the proportion of the fund to be invested in S^i, for each $i = 1, \ldots, n$, so that $1 - \sum_{i=1}^n \pi_i$ is invested in S^0. Borrowing and shortselling are allowed. A negative value of π_i means that the manager sells a part of her/his risky asset S^i short while, if π_i is larger than 1, he or she then gets into debt to purchase the corresponding stock, borrowing money at the riskless interest rate r.

Under the chosen investment/benefit policy, the dynamics of the fund F is driven by

$$dF(t) = \sum_{i=1}^{n} \pi_i(t) F(t) \frac{dS^i(t)}{S^i(t)} + \Big(1 - \sum_{i=1}^{n} \pi_i(t)\Big) F(t) \frac{dS^0(t)}{S^0(t)} + (C(t)F(t) - P(t))dt, \tag{3}$$

with $F(0) = F_0 > 0$. By substituting (1) and (2) in (3), the dynamic fund wealth evolution under the investment policy Π is

$$dF(t) = \Big(rF(t) + \Pi^\top(t)(b - r\bar{1})F(t) + C(t)F(t) - P(t)\Big)dt + \Pi^\top(t)F(t)\sigma\, dw(t) + \Pi^\top(t)F(t)\varphi\, dN(t), \tag{4}$$

with the initial condition $F(0) = F_0$.

We assume admissible strategies, that is to say, strategies to fulfill some technical conditions. A strategic profile (P, Π) is called admissible if the extra benefits strategy $\{P(t) : t \ge 0\}$ and the investment strategy $\{\Pi(t) : t \ge 0\}$ are Markovian processes and stationary, $P = P(t, F)$ and $\Pi = \Pi(t, F)$, adapted to filtration $\{\mathscr{F}_t\}_{t\ge 0}$, and $P(t)$ and $\Pi(t)$ are $\mathscr{F}_t$-measurable, $\forall t > 0$, and such that they satisfy the integrability condition

$$\mathbb{E}\int_0^T P(t)dt + \mathbb{E}\int_0^T \Pi^\top(t)\Pi(t)dt < \infty. \tag{5}$$

Thus, the stochastic differential equation (SDE) (4) admits a unique solution for every initial condition $F(0) = F_0$. We assume $P(t) > 0$. We denote by $\mathcal{A}$ the set of admissible strategy profiles.

3. The optimal strategies when the isoelastic utility is maximized

In this section, we analyze how the manager selects the optimal benefit and investment strategies. As explained in the Introduction, we model the manager's preferences with a power instantaneous utility of the benefit and a power final utility of the terminal fund to be maximized. In the optimization process, the manager faces one element of randomness due to the risky assets of the financial market.

The objective functional to be maximized over the class of admissible controls $\mathcal{A}$, is given by

$$J((t,F);(P,\pi)) = \mathbb{E}_{t,F}\left\{ \int_t^T e^{-\rho(s-t)}U(P(s))ds + e^{-\rho(T-t)}\alpha U(F(T)) \right\},$$

where U is a utility function of the benefit and the fund. Similarly to Josa-Fombellida et al. (2023) and He et al. (2020), $\alpha > 0$ is a weighting factor indicating the importance of maximizing the final utility of the fund versus the instantaneous utility of the benefit. The time preference of the manager is given by $\rho > 0$. The aim is to maximize the expected utility of the benefit along the planning interval and of the fund wealth at the end of the plan. Note that we are considering P and Π as control variables. Here, $\mathcal{A}$ denotes the set of Markovian processes (P,Π), adapted to the filtration $\{\mathscr{F}_t\}_{t\geq 0}$ where (P,Π) satisfies (5), and where F satisfies (4). In the above, $\mathbb{E}_{t,F}$ denotes conditional expectation with respect to the initial condition (t,F).

We consider a CRRA utility function:

$$U(x) = \frac{x^{1-\gamma}-1}{1-\gamma}, \quad \gamma > 0, \gamma \neq 1.$$

We call γ the risk aversion parameter. This utility function is increasing and strictly concave. When $\gamma = 1$, the utility function considered is of logarithmic type: $U(x) = \ln x$.

The value function is defined as

$$\hat{V}(t,F) = \max_{(P,\Pi)\in\mathcal{A}} \left\{ J((t,F);(P,\Pi)) : \text{ s.t. (4) and } F(t) = F \right\}.$$

It is clear that the value function so defined is non-negative and strictly concave. The connection between value functions and optimal feedback controls in stochastic control theory under Poisson–diffusion setting is accomplished by the HJB; see Sennewald (2007).

In order to simplify the length of some equations along the paper, we denote

$$\Psi(\Pi,\nu) := r + \Pi^\top(b - r\bar{1}) - \frac{1}{2}\nu\Pi^\top\Sigma\Pi + \frac{1}{1-\nu}\sum_{k=1}^m \lambda_k\left((1+\Pi^\top\varphi_k)^{1-\nu} - 1\right),$$

where $\nu > 0$, $\nu \neq 1$, and φ_k is the column k of the matrix φ, $k = 1,\dots,m$. When $\nu = 1$, applying the L'Hôpital rule, we have

$$\Psi(\Pi,1) = \lim_{\nu\to 1}\Psi(\Pi,\nu) = r + \Pi^\top(b - r\bar{1}) - \frac{1}{2}\Pi^\top\Sigma\Pi + \sum_{k=1}^m \lambda_k \ln(1+\Pi^\top\varphi_k).$$

We assume the technical conditions: $1 + \Pi^\top\varphi_k > 0$, for all k.

The following result provides the optimal strategies and the evolution of the optimal fund wealth.

Theorem 3.1. *The optimal benefit and the optimal investment proportions in the risky assets are given by*

$$P^* = e^{-\int_t^T a(s)ds}\left(\alpha^{1/\gamma} + \int_t^T e^{-\int_s^T a(u)du}ds\right)^{-1} F^*, \tag{6}$$

where

$$a(t) = -\frac{\rho}{\gamma} + \frac{1-\gamma}{\gamma}\left(C(t) + \Psi(\Pi^*,\gamma)\right), \tag{7}$$

where Π^* is a constant solution of the system of algebraic equations

$$b_i - r - \gamma\sum_{j=1}^l a_{ij}\pi_j + \sum_{k=1}^m \lambda_k\left(1+\Pi^\top\varphi_k\right)^{-\gamma}\varphi_{ik} = 0, \quad i = 1,\dots,n, \tag{8}$$

where $a_{ij} = \sum_{p=1}^l \sigma_{ip}\sigma_{jp}$, that is to say, the element (i,j) of the matrix $\Sigma = \sigma\sigma^\top$, and the optimal fund wealth is given by

$$\begin{aligned} dF^*(t) =& \Big(r + \Pi^{*\top}(b - r\bar{1}) + C(t) \\ &- e^{-\int_t^T a(s)ds}\left(\alpha^{1/\gamma} + \int_t^T e^{-\int_s^T a(u)du}ds\right)^{-1}\Big)F^*(t)dt \\ &+ \Pi^{*\top}\sigma F^*(t)\,dw(t) + \Pi^{*\top}\varphi F^*(t)\,dN(t), \end{aligned} \tag{9}$$

with $F^*(0) = F_0 > 0$.

The optimal benefit P^* is a linear function of the fund assets F^*. The optimal investment proportions Π^* are constants, but it can not be explicitly obtained from (8) when $\lambda_k \neq 0$ for some k. We observe that this implies a linear relation between the optimal benefit and the optimal investment. Both optimal strategies and the optimal fund wealth depend on the parameters of the financial market (including the jumps), the risk aversion, and the contribution rate. The optimal benefit also depends on the weighting factor. Depending on the jump parameters, shortselling or borrowing could be necessary. The optimal benefit proportion P^*/F^* is a positive deterministic function and, depending on the level of the contribution proportion, the risk aversion, the weighting factor and the financial market, it can be an increasing function over time. We also observe that $P^*(T) = \alpha^{1/\gamma}F^*(T)$. Note that, when $\alpha = 1$, then the optimal terminal benefit matches the terminal fund, $P^*(T) = F^*(T)$.

Note that, in the scalar case, where $n = l = m = 1$, it is possible to check if a solution to (8) exists. A necessary condition for a solution to exist is $1 + \pi\varphi > 0$, that is, the uncertainty of the Poisson processes and the investment strategies are positively compensated. If we define $f(\pi) = b - r - \gamma\sigma^2\pi + \lambda(1+\pi\varphi)^{-\gamma}\varphi$, then we have $\lim_{\pi\to-\infty} f(\pi) = \infty$ and $\lim_{\pi\to\infty} f(\pi) = -\infty$, because $\gamma > 0$. Thus, applying Bolzano's Theorem, an investment strategy π such that $f(\pi) = 0$ exists, that is, it is a solution of (8). On the other hand, as $f'(\pi) = -\gamma(\sigma^2 + \lambda\varphi^2(1+\pi\varphi)^{-\gamma-1}) < 0$, then $f(\pi)$ is strictly decreasing and this implies uniqueness. If we assume $f(0) = b - r + \lambda\varphi$ to be positive, then we have a unique positive investment strategy if the condition $1 + \pi\varphi > 0$ holds (for instance, when $\varphi \geq 0$). Note that negative investments, that is, allowing shortselling, can be found for negative diffusion jump parameters. Given a solution π of (8), it is straightforward to obtain the optimal benefit P^* and the evolution of the optimal fund F^* from (9).

From (9), the expected optimal fund is given by

$$\begin{aligned} \mathbb{E}F^*(t) =& F_0\exp\Big\{\Big(r + \Pi^{*\top}(b - r\bar{1}) + \sum_{k=1}^m \lambda_k\Pi^{*\top}\varphi_k\Big)t \\ &+ \int_0^t \Big(C(v) - e^{-\int_v^T a(s)ds}\Big(\alpha^{1/\gamma} + \int_v^T e^{-\int_s^T a(u)du}ds\Big)^{-1}\Big)dv\Big\}. \end{aligned}$$

Remark 3.1 *(Exponential contribution proportion).* When $C(t) = c_1e^{c_2t}$, with $c_1 > 0$, then $a(t) = -\frac{\rho}{\gamma} + \frac{1-\gamma}{\gamma}\left(c_1e^{c_2t} + \Psi(\Pi^*,\gamma)\right)$ and $(P^*/F^*)(t) = e^{-\int_t^T a(s)ds}\left(\alpha^{1/\gamma} + \int_t^T e^{-\int_s^T a(u)du}ds\right)^{-1}$. The optimal fund wealth evolution is given by

$$\begin{aligned} dF^*(t) =& \Big(r + \Pi^{*\top}(b - r\bar{1}) + c_1e^{c_2t} \\ &- e^{-\int_t^T a(s)ds}\left(\alpha^{1/\gamma} + \int_t^T e^{-\int_s^T a(u)du}ds\right)^{-1}\Big)F^*(t)dt \\ &+ \Pi^{*\top}\sigma F^*(t)\,dw(t) + \Pi^{*\top}\varphi F^*(t)\,dN(t), \end{aligned}$$

with $F^*(0) = F_0 > 0$. When C is constant the expressions are simplified a little more.

Remark 3.2 *(Model without jumps)*. When $\varphi = 0$ and $\lambda = 0$, the optimal investment proportions can be explicitly obtained, $\Pi^* = \frac{1}{\gamma}\Sigma^{-1}(b - r\overline{1})$, that is to say, the well known maximum portfolio growth rule. Thus, $\pi_i^* > 0$, for all i, because $\gamma > 0$, that is to say, shortselling is not necessary. However, $\pi_i^* > 1$ if and only if $\gamma > \overline{e_i}\Sigma^{-1}(b - r\overline{1})$, where $\overline{e_i} = (0, ..., 1^{i)}, 0, ...0)$, that is to say, for risk aversion high enough that the manager must borrow money at rate r to invest in risky asset S^i. Now, P^* is given by (6), but with $a(t) = -\frac{\rho}{\gamma} + \frac{1-\gamma}{\gamma}\left(r + C(t) + \frac{\theta^\top\theta}{2\gamma}\right)$, where $\theta = \sigma^{-1}(b - r\overline{1})$ is the market price of risk or Sharpe ratio of the portfolio. The optimal fund wealth evolution is given by

$$dF^*(t) = \left(r + \frac{1}{\gamma}\theta^\top\theta + C(t) - e^{\int_t^T a(s)ds}\left(\alpha^{1/\gamma} + \int_t^T e^{-\int_s^T a(u)du}ds\right)^{-1}\right)F^*(t)dt + \frac{1}{\gamma}\theta^\top\sigma F^*(t)\,dw(t),$$

with $F^*(0) = F_0 > 0$ and the expected optimal fund is given by

$$\mathbb{E}F^*(t) = F_0 \exp\left\{\left(r + \frac{1}{\gamma}\theta^\top\theta\right)t + \int_0^t \left(C(v) - e^{-\int_v^T a(s)ds}\left(\alpha^{1/\gamma} + \int_v^T e^{-\int_s^T a(u)du}ds\right)^{-1}\right)dv\right\}.$$

In the extreme case where there is no contribution, $C = 0$, and the utility functions have the same weight, $\alpha = 1$, we are considering the portfolio selection model of Merton (1971), where we are maximizing an instantaneous utility function of the consumption (now the benefit) and a utility function of the terminal wealth (now the fund).

When only one risky asset is considered, it is a special case of the Coob-Douglas utility maximization analyzed in Zhao and Wang (2022).

Remark 3.3 *(Infinite horizon)*. When $T = \infty$, we assume that the contribution proportion C is a constant. The objective functional to be maximized is given by

$$J(F;(P,\Pi)) = \mathbb{E}_F\left\{\int_t^\infty e^{-\rho s}U(P(s))ds\right\},$$

and the value function is time independent, $V = V(F)$. The optimal benefit is given by

$$P^* = \mu^{-1/\gamma}F^*,$$

where

$$\mu = \left(\frac{\rho}{\gamma} - \frac{1-\gamma}{\gamma}\left(C + \Psi(\Pi^*,\gamma)\right)\right)^{-\gamma}, \tag{10}$$

the vector of the optimal investment proportions in the risky assets Π^* is a constant solution of the system of algebraic equations (8) and the optimal fund wealth is given by

$$dF^*(t) = \left(r + \Pi^{*\top}(b - r\overline{1}) + C - \mu^{-1/\gamma}\right)F^*(t)dt + \Pi^{*\top}\sigma F^*(t)\,dw(t) + \Pi^{*\top}\varphi F^*(t)\,dN(t), \tag{11}$$

with $F^*(0) = F_0 > 0$. We assume that the optimal benefit proportion $\mu^{-1/\gamma}$ is positive in order to obtain positive optimal benefit P^*. See proof in Appendix A.

Remark 3.4 *(Logarithmic utility)*. When $U(x) = \ln x$ then $a(t) = -\rho$. The optimal benefit is given by $P^* = e^{\rho(T-t)}\left(\alpha^{1/\gamma} - \frac{1}{\rho}(1 - e^{\rho(T-t)})\right)^{-1}F^*$, the vector of the optimal investment proportions Π^* is the solution of the system

$$b_i - r - \sum_{j=1}^{l} a_{ij}\pi_j + \sum_{k=1}^{m}\lambda_k\left(1 + \Pi^\top\varphi_k\right)^{-1}\varphi_{ik} = 0, \quad i = 1, \ldots, n,$$

and the optimal fund wealth evolution is given by

$$dF^*(t) = \Big(r + \Pi^{*\top}(b - r\overline{1}) + C(t) - e^{\rho(T-t)}\left(\alpha^{1/\gamma} - \frac{1}{\rho}(1 - e^{\rho(T-t)})\right)^{-1}\Big)F^*(t)dt + \Pi^{*\top}\sigma F^*(t)\,dw(t) + \Pi^{*\top}\varphi F^*(t)\,dN(t),$$

with $F^*(0) = F_0 > 0$, that is to say, the expressions (6), (8) and (9) for $\gamma = 1$.

4. A numerical illustration

In this section, a sensitivity analysis is provided of the optimal benefit and investment proportion strategies, as well as the optimal fund wealth with respect to the contribution, jump and risk aversion parameters, and the economic regime. In order to simplify the development, we consider the scalar case, $l = m = n = 1$. The equations are solved numerically with the standard R Stats package.

In the bull regime, the economy is booming, and in the bear regime, it is in recession. We select the data characterizing both regimes, following the recommendations in Zou and Cadenillas (2017). The risk premium is greater in boom periods than in recession periods, $b_1 - r_1 > b_2 - r_2$, the stock volatility is greater when the economy is in recession, $\sigma_2 > \sigma_1$, and we assume that the risk premium by unit of volatility is higher in the boom periods than under recession, $\frac{b_1-r_1}{\sigma_1^2} > \frac{b_2-r_2}{\sigma_2^2}$. We have denoted the bull regime with subscript 1 and the bear regime with 2.

We consider the following values for the parameters:

- The initial fund wealth F_0 is set to be 0.67 billion, which is the same as that in Sanders (2016).
- The parameters of the financial market used to illustrate the simulations in a bull regime are $b = 0.2615$, $\sigma = 0.1301$, with a final time $T = 10$ years and initial asset price $S_0 = 1$. They have been estimated, as in Josa-Fombellida and Rincón-Zapatero (2019), based on data of the S&P 500 index from June 2021 to June 2022. The value of the risk-free interest rate is $r = 0.01$ which coincides with that considered in Zhao and Wang (2022). In a bear regime of the financial market, we consider data of the S&P 500 index from June 2022 to May 2023, and then the estimated values of the parameters are $b = 0.0449$, $\sigma = 0.2076$.
- We vary the values of the jump intensity $\lambda = 0, 0.25, 0.5$ along the graphical analysis. The case without jumps is covered for $\lambda = 0$. In order to cover two types of jump, upward and downward, the uncertainty Poisson process takes two values $\varphi = -0.1, 0.1$.
- For the aggregate contribution proportion to the fund wealth $C(t) = c_1e^{c_2t}$, we assume that $c_1 = 0.08$ and $c_2 = -0.05$, as in Zhao and Wang (2022). We also consider another alternative value $c_2 = 0.03$, for an increasing contribution, as in Roch (2022), and the constant contribution case where $c_2 = 0$.
- The discount rate or time preference ρ is set to be 0.03 per year based on Kraft and Weiss (2019).
- Following Mehra and Prescott (1985), we consider moderate and high risk aversion values $\gamma \in [1, 10]$. Note that the moderate $\gamma = 1$ is the logarithmic case. An interesting value is $\gamma = 3$, which coincides with the literature on life-cycle portfolio choice, as indicated in Zhao and Wang (2022).
- We consider several weights to the terminal utility of the fund, $\alpha = 0.01, 0.5, 1, 5, 100$. When $\alpha = 1$, the same importance is given to maximizing the utility of the benefits as to the utility of the fund. A

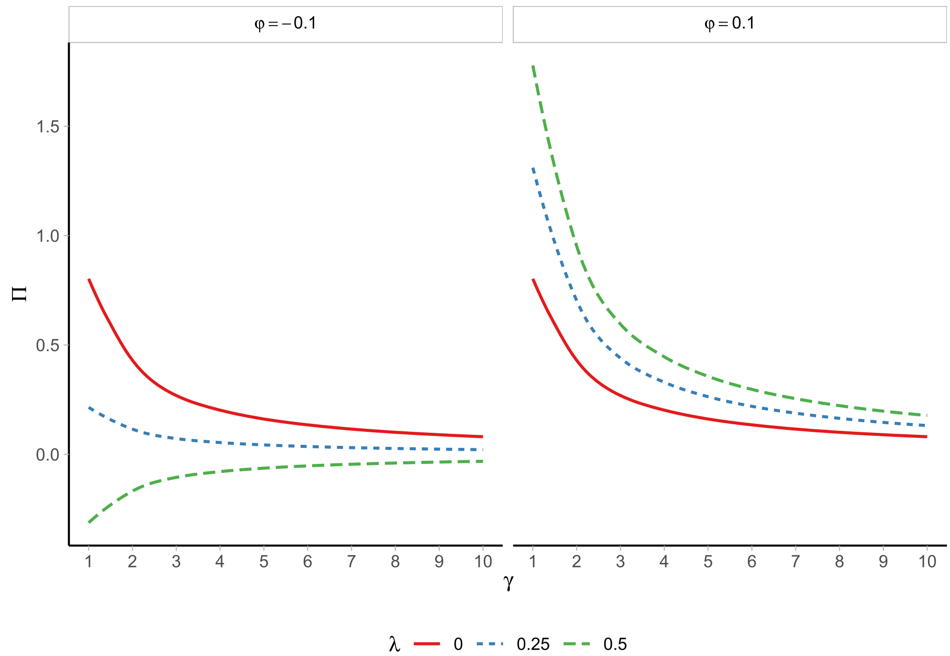

Fig. 2. Optimal investment proportion by risk aversion, jump intensity and uncertainty Poisson parameter for $c_2 = -0.05$ and $\alpha = 1$ under a bear regime.

very high value α indicates that only the maximization of the final utility of the fund is taken into account. An α value close to zero indicates that the utility of the fund is hardly taken into account in the maximization process.

The technical conditions imposed in Theorem 3.1, $1+\Pi^\top \varphi_k > 0$, for all k, are satisfied.

Fig. 2 shows the optimal investment proportion Π^* by the risk aversion γ, the jump intensity λ, and uncertainty Poisson parameter φ under the bear market. We assume the same weight factor $\alpha = 1$ to the utility functions. We consider the decreasing contribution function where $c_2 = -0.05$. We observe the optimal investment proportion approach to 0 when the risk aversion goes to 10. The investment proportion decreases with the risk aversion, with upward and downward jumps, but with high jump intensity. For any intensity of the jump, higher levels of risk aversion among fund managers result in a more conservative investment strategy. Specifically, for a low risk aversion ($\gamma < 4$), negative jumps derive in a lower investment policy. This reflects a cautious approach to risk, where decision-makers prioritize stability and are willing to sacrifice potential returns for reduced exposure to risk. A greater intensity jump increases the investment with upward jumps. For instance, when $\gamma = 3$, $\Pi^* = 0.3$, for $\lambda = 0$ or, when there are no jumps, $\Pi^* = 0.5$, for $\lambda = 0.25$, or when the intensity of the jumps is moderate, and $\Pi^* = 0.6$, for $\lambda = 0.5$ or high intensity. In particular, in the case without jumps, less investment is needed. Borrowing is necessary with low or moderate risk aversion. With downward jumps, we observe the opposite behavior. The investment decreases with jump intensity and borrowing is not necessary. Thus, the jumps impact the optimal investment strategy. Similar results can be obtained with a bull market.

The evolution of the optimal relative benefit P^*/F^* by the jump intensity λ is illustrated in Fig. 3. We consider a moderate risk aversion $\gamma = 3$, an upward jump parameter $\varphi = 0.1$, a negative rate of contribution $c_2 = -0.05$, a same weight factor $\alpha = 1$ to the utilities and a bull regime. This figure shows the growing trend of optimal benefit over time and the match between the optimal terminal relative benefit and the terminal fund, $P^*(T) = F^*(T)$. There is a significant upsurge in the benefit for participants rewarded in the final two years of the plan. Note that the contribution proportion C declines over time because $c_2 = -0.05$, which reduces the inflow to the fund. However, the fund itself continues to grow due to the investments made by the fund manager. The increasing benefit payments toward the end are a result of the fund manager's strategy to optimize the final utility of the participants. Higher fund levels achieved through optimal investment over time allow for more generous benefits towards the plan's conclusion, ensuring that participants are rewarded for longer-term contributions and risk exposure. Additionally, as the planning horizon approaches its end, the optimization becomes more focused on benefit payments, leading to the observed final surge.

The Poisson jumps incorporated into the model simulate sudden changes in asset prices. For moderate jump intensities, these jumps play a pivotal role in shaping the fund's trajectory, adding a layer of uncertainty, but also creating opportunities for higher returns. In particular, upward jumps in a bull market can cause a significant rise in the fund, supporting a substantial increase in benefit payments near the end of the plan.

Fig. 4 shows the optimal relative benefit P^*/F^* over time by the jump intensity λ, and the rate of contribution c_2 under the bull market. We consider a moderate risk aversion $\gamma = 3$, same weight factor $\alpha = 1$ to the utilities and upward jumps with $\varphi = 0.1$. The optimal relative benefit always exhibits a positive trend for any jump intensity. Notably, there is a significant upsurge in the benefit for participants rewarded in the final two years of the plan. When contribution c_2 takes a negative value, the optimal benefit experiences a decline in the middle of the plan, followed by a recovery in the concluding years. Conversely, in the case of a positive contribution, the optimal benefit demonstrates a continual upward trajectory over time, avoiding any periods of decline. Additionally, the data indicates that larger jump intensities correlate with a lower proportion of the benefit. For instance, at the initial time $t = 0$ and for $c_2 = -0.05$, $P^* = 0.81$, for $\lambda = 0$ or, in the without jumps case, $P^* = 0.77$ for $\lambda = 0.25$ and $P^* = 0.73$, for $\lambda = 0.5$; thus the relative benefit decreases with the jump intensity. Then, the jumps also impact the optimal benefit strategy.

On the other hand, we illustrate the expected fund evolution $\mathbb{E}F^*$ to analyze the impact of jumps under two distinct economic regimes, bull and bear. Fig. 5 yields two primary insights. We consider a moderate risk aversion $\gamma = 3$, a high jump intensity $\lambda = 0.5$, a same weight factor $\alpha = 1$ to the utilities and a negative rate of contribution $c_2 = -0.05$. The expected fund evolution shows four distinct trends, depending on the economic regime (bull or bear market) and the direction of jumps (positive or negative). In the bull market, the fund grows rapidly due to higher returns from risky assets, which have a greater risk premium

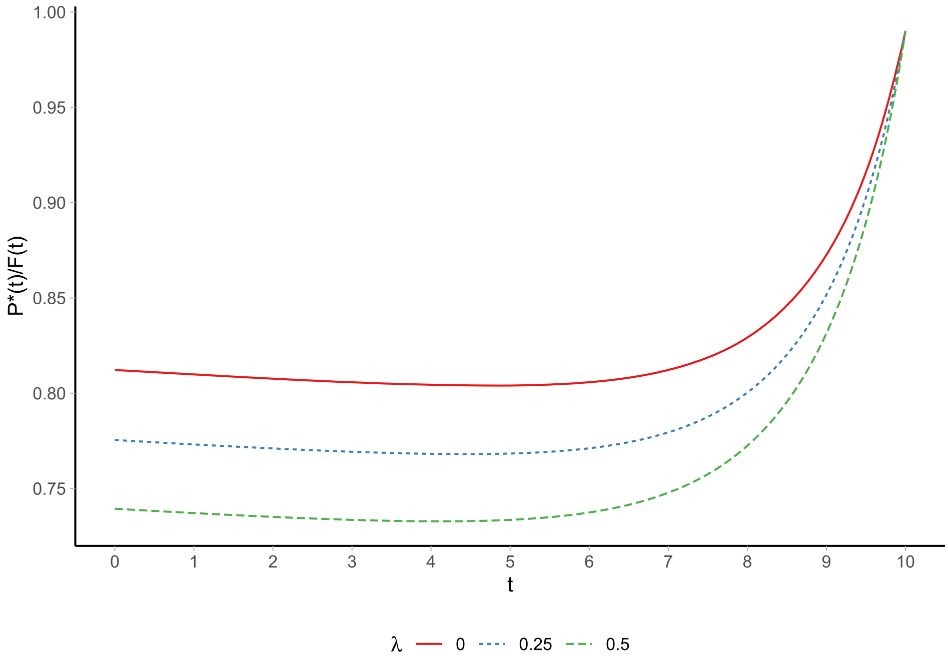

Fig. 3. Optimal relative benefit over time by the jump intensity over time for $\gamma = 3$, $c_2 = -0.05$, $\alpha = 1$ and $\varphi = 0.1$ under a bull regime.

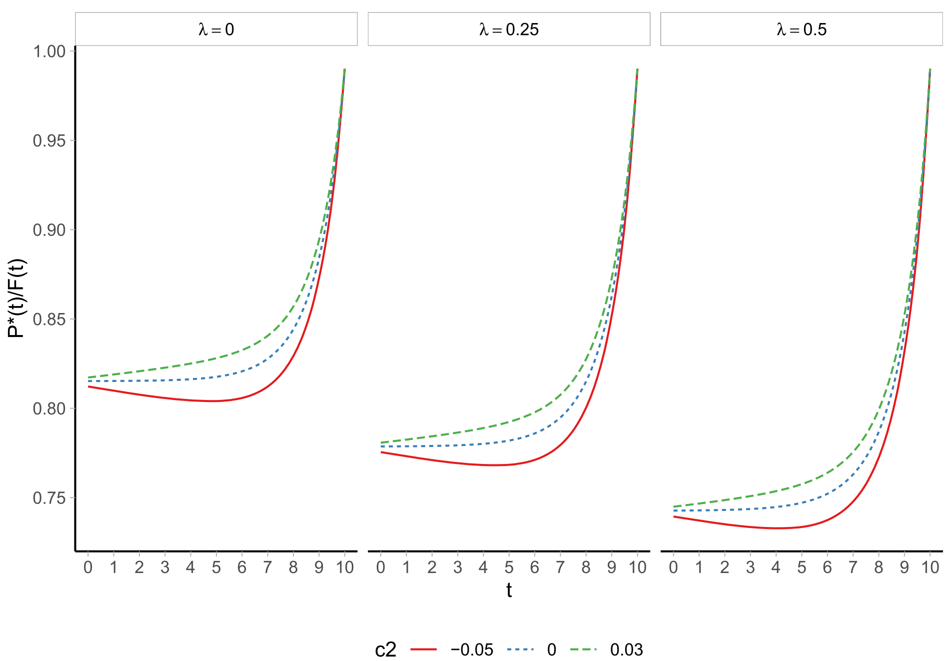

Fig. 4. Optimal relative benefit over time by the rate of contribution and the jump intensity over time for $\gamma = 3$, $\alpha = 1$ and $\varphi = 0.1$ under a bull regime.

compared to the bear market. This is reflected in the steeper upward trend of the expected fund values over time. When positive jumps occur ($\varphi = 0.1$), they accelerate this growth, particularly in the final years of the plan, where the fund can reach a value close to $\mathbb{E}F^*(T) = 3829.755$, as seen by the significant divergence between the curves for positive and negative jumps. This large terminal fund highlights the impact of favorable market conditions and sudden positive asset price increases. On the other hand, negative jumps ($\varphi = -0.1$) reduce the growth rate significantly, with the fund barely increasing over the plan's duration, ending with $\mathbb{E}F^*(T) = 1.829$. Despite a negative contribution rate ($c2 = -0.05$), the fund continues to grow, demonstrating the robustness of the investment strategy, which efficiently offsets the contribution decay through capital market gains. This makes the pension plan attractive even when the contributions are diminishing.

Conversely, in the bear market, the growth of the fund is much more subdued. The negative jumps ($\varphi = -0.1$) further reduce the growth potential, with the expected fund value remaining low throughout the time horizon. It dwindles to values close to 0.067, with upward jumps, and 0.045, with downward jumps.

The divergence in trends between positive and negative jumps under different market regimes highlights the effectiveness of the fund manager's risk management strategy. The model accounts for varying market dynamics, where positive jumps allow for higher returns, and negative jumps serve as a conservative scenario that tests the resilience of the fund.

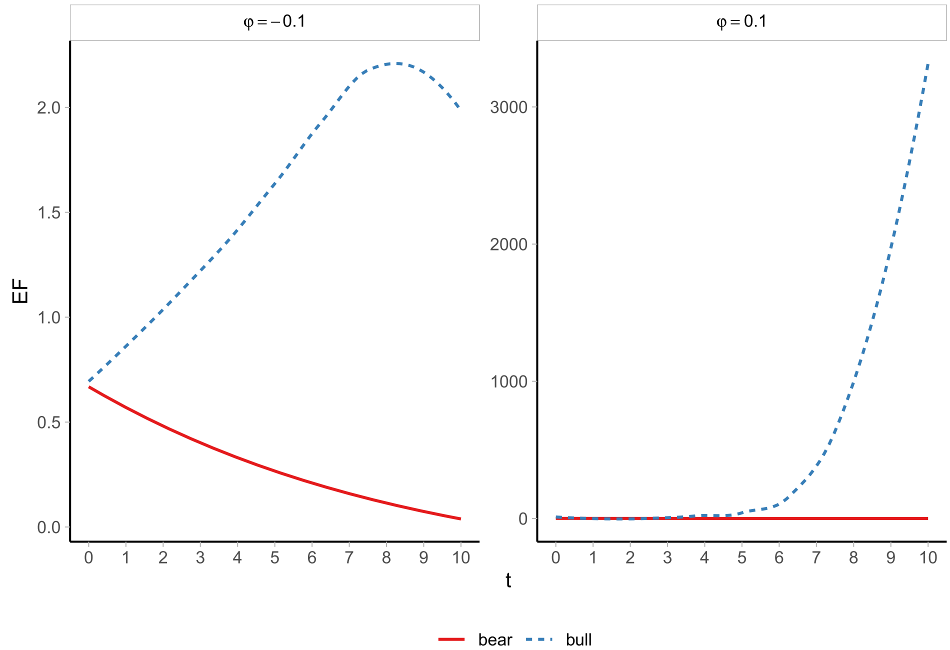

Fig. 5. Expected fund evolution by jump uncertainty parameter for $\gamma = 3$, $\lambda = 0.5$, $\alpha = 1$ and $c_2 = -0.05$ under bear and bull regimes.

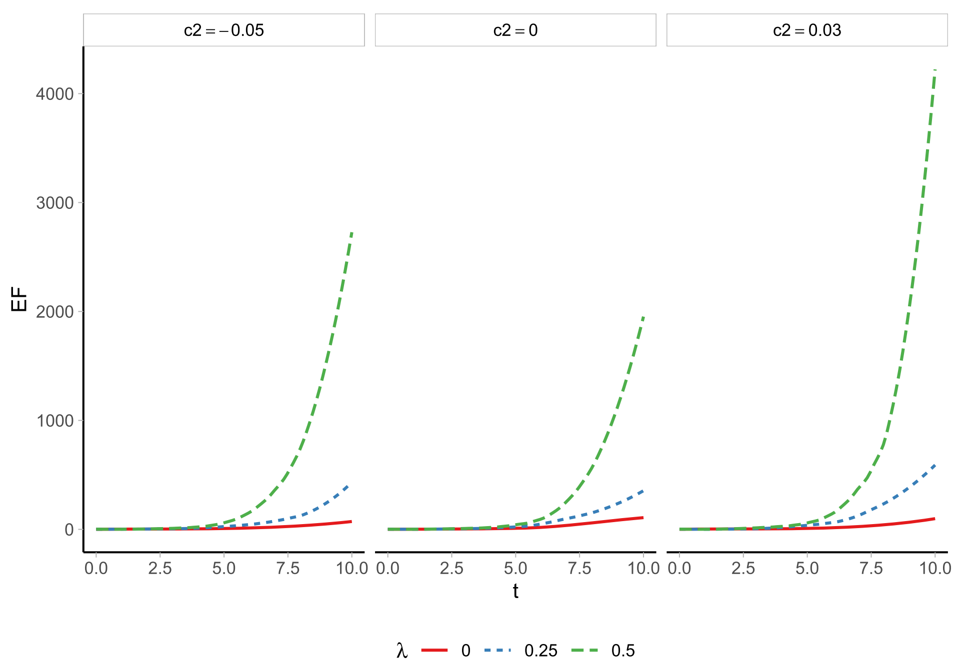

Fig. 6. Expected fund evolution by rate of contribution and jump intensity for $\varphi = 0.1$ and $\alpha = 1$ under bull regime.

In the following, we analyze the effect of the rate of contribution c_2 on the expected fund evolution $\mathbb{E}F^*$ and on the terminal fund $\mathbb{E}F^*(T)$. First, little effect is observed in the bear case. However, in the bull regime, some notable effects are observed. Fig. 6 analyzes the expected fund evolution for upward jumps, with a same weight factor $\alpha = 1$ to the utilities, with $\varphi = 0.1$ by the jump intensity λ and the rate of contribution c_2 under the bull market. The expected value of the fund increases with the time. The higher the intensity of the jump, the greater the expected fund. For instance, for $c_2 = -0.05$ and at the terminal time $T = 10$, $\mathbb{E}F^* = 70$, for $\lambda = 0$, $\mathbb{E}F^* = 400$, for $\lambda = 0.25$ and $\mathbb{E}F^* = 2650$, for $\lambda = 0.5$; thus the relative benefit decreases with the jump intensity. So, according to this checking, the intensity of the jumps also has an effect on the optimal fund wealth. A positive contribution increases the expected value of the fund. However, a negative rate makes the fund improve, but to a lesser extent.

Finally, we study the effect of the weight factor α on the optimal relative benefit strategy P^*/F^* and the optimal expected fund wealth $\mathbb{E}F^*$. First, note that the investment strategy Π^* does not depend on α. Fig. 7 analyzes the optimal relative benefit for $\gamma = 3$, $\lambda = 0.5$, $c_2 = -0.05$, with upward jumps and bull market by the weight factor α. We consider several cases, where $\alpha = 0.01, 0.5, 1, 5, 100$, including the case where the final utility of the fund it is not important, $\alpha = 0.01$, the case where this utility function has almost all the weight of the objective function, $\alpha = 100$, and the case where both utilities have the same importance, $\alpha = 1$. We see that α does not influence the benefit in the first half of the time period. However, an effect of α on the optimal relative ben-

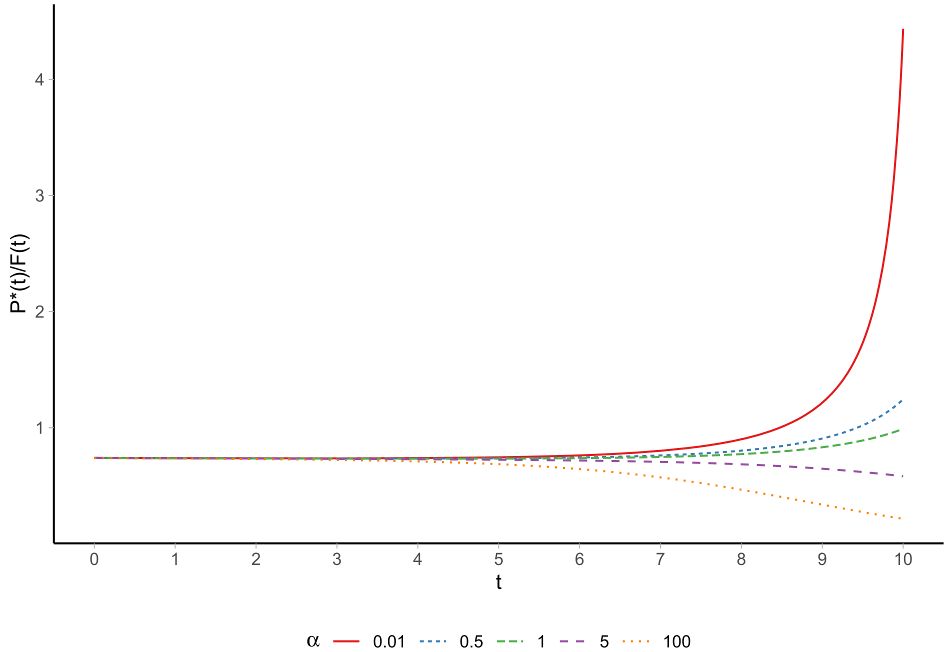

Fig. 7. Optimal relative benefit over time by the weight factor α for $\gamma = 3$, $\lambda = 0.5$, $c_2 = -0.05$ and $\varphi = 0.1$ under a bull market.

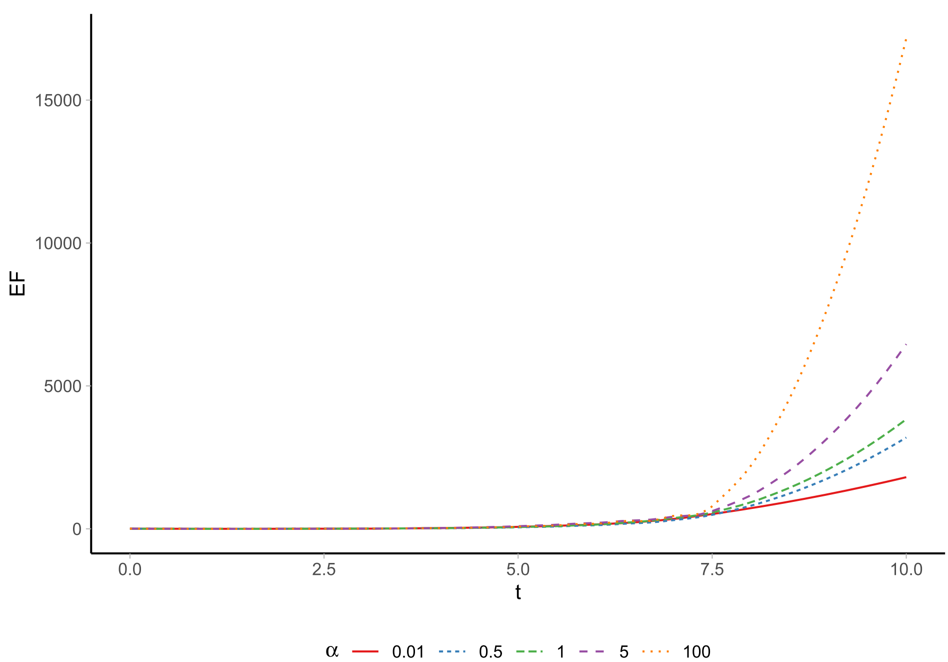

Fig. 8. Expected fund evolution over time by the weight factor α for $\gamma = 3$, $\lambda = 0.5$, $c2 = -0.05$ and $\varphi = 0.1$ under a bull market.

efit is observed mainly from the middle of the planning period. When the importance given to the final utility of the fund compared to the instantaneous utility of the benefit increases, the optimal relative benefit decreases to a greater extent. When the final utility of the fund is not important, that is α is approaching 0, the final value of the optimal relative contribution multiplies its value by more than 4. When all the importance is given to the final utility of the fund, that is α is very high, the final value of the optimal relative contribution is below 30%. Thus, it can be said that an increase in α slows down a sharp increase in the benefit.

Fig. 8 analyzes the optimal fund wealth over time for $\gamma = 3$, $\lambda = 0.5$ and $c2 = -0.05$, with upward jumps and bull market by the weight factor α. An increase in the fund is observed over time. An effect of α on the optimal relative benefit is observed only and mainly in the last years of the planning period. The increase of the fund is greater in later years, especially when α is higher.

5. Conclusions

In this paper, we apply the programming dynamic approach to analyze the optimal management of a target benefit pension plan with the objective of maximizing the expected discounted utility of benefits and terminal utility of fund wealth. The analytical solutions obtained yield several conclusions:

Firstly, the presence of jumps in the financial market has an impact on the optimal investment and benefit strategies and on the optimal fund wealth. With upward jumps and bull regime, the greater the intensity of jumps, the greater the fund wealth, but at the cost of reducing the benefit and increasing the investment. Secondly, there is an impact of risk aversion on investment allocation. Higher levels of risk aversion among fund managers result in a more conservative investment approach, with a reduced proportion allocated to risky assets. This reflects a risk-averse behavior consistent with the desire to ensure the stability of the pension plan. On the other hand, the growing trend in optimal benefits over time suggests a positive outlook for participants, indicating potential for increased financial security during their retirement years. The significant upsurge in benefits for participants rewarded in the final two years of the plan implies a strategic alignment of incentives to encourage long-term commitment.

The study also highlights the influence of contributions on the trajectory of optimal benefits. A negative contribution growth rate leads to a temporary decline in benefits, followed by recovery; while positive contributions result in a continuous upward trajectory. This underscores the economic implications of contribution policies on the financial well-being of plan participants.

The analysis of fund performance under different market regimes provides insights into the jump effect on the market dynamics. Positive jumps contribute significantly to the rapid growth of the fund in a bull market, while negative jumps can impede fund ascent. This sensitivity emphasizes the importance of considering and managing market uncertainties in the optimal solution of a target benefit pension plan.

Finally, with a bull regime, the optimal fund wealth increases over time, even if the rate of contribution is negative, which makes the pension plan more attractive. A major weight in the utility function of the terminal fund increases the fund wealth more in the last years of the planning period. However, the optimal relative benefit can decrease over time well below 1 when the weight factor is high.

For the future, a CARA utility function can be considered. On the other hand, rather than considering a contribution proportion of the fund wealth as a deterministic function, it could be valuable to add more complexity, considering a stochastic function of the contribution. This approach not only enhances the model's flexibility, but also brings it closer to the complexities of real-world scenarios. In order to provide the members with a better life after retirement, some restrictions on trading strategies that do not allow borrowing or short selling can be considered. A model where the aim of the fund manager is to minimize the deviations of the benefit and the fund wealth with respect to targets can also be considered.

CRediT authorship contribution statement

Ricardo Josa-Fombellida: Writing – original draft, Supervision, Investigation. **Paula López-Casado:** Writing – original draft, Software, Investigation.

Declaration of competing interest

There is no competing interest.

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Appendix A

Proof of Theorem 3.1. For the problem of Section 3, the HJB equation is

$$-\rho V+\max_{P,\Pi}\Bigg\{V_t+\frac{P^{1-\gamma}-1}{1-\gamma}+\big(rF+\Pi^\top(b-r\bar{1})F+CF-P\big)V_F+\frac{1}{2}\Pi^\top\Sigma\Pi F^2V_{FF}+\sum_{k=1}^m\lambda_k\Big(V\big(F+\sum_{j=1}^n\pi_jF\varphi_{jk}\big)-V(F)\Big)\Bigg\}=0, \quad (12)$$

for all (t,F), with the final condition $V(T,F)=\alpha\frac{F^{1-\gamma}-1}{1-\gamma}$, for all F.

If there is a smooth solution V of the equation (12), strictly concave, then the maximizer values of the benefit and the investments are given by

$$P^{-\gamma}-V_F=0\Rightarrow P=V_F^{-1/\gamma}, \quad (13)$$

$$(b_i-r)FV_F(x)+\sum_{j=1}^l a_{ij}\pi_jF^2V_{FF}+\sum_{k=1}^m\lambda_kV_F\Big(F+\sum_{j=1}^n\pi_jF\varphi_{jk}\Big)\varphi_{ik}F=0, \quad (14)$$

for all $i=1,\dots,n$. The structure of the HJB equation obtained, once we have substituted these values for P and Π in (12), suggests a power function $V(t,F)=g(t)\frac{F^{1-\gamma}}{1-\gamma}+\frac{u(t)}{1-\gamma}$, with g and u suitable functions. From (13), we get that the benefit P is explicitly found in terms of the fund F, $P=g^{-1/\gamma}F$, where the function g must be determined with the HJB equation. From (14), we get that the vector of investment proportions Π is the constant proportion of fund that solves the algebraic equation (8). Plugging into the HJB equation (12), function u satisfies $u'(t)=\rho u(t)+1$, with $u(T)=-\alpha$, that is to say, $u(t)=-\frac{1}{\rho}+(-\alpha+\frac{1}{\rho})e^{-\rho(T-t)}$, and the following nonlinear differential equation for g is obtained

$$g_t+\big(-\rho+(1-\gamma)\big(C+\Psi(\Pi,\gamma)\big)\big)g-\gamma g^{1-1/\gamma}=0,$$

with $g(T)=\alpha$. In order to linearize it, we consider the transformation $h=g^{1/\gamma}$, and then h is determined by the linear differential equation

$$h'(t)+a(t)h(t)-1=0,$$

with $h(T)=\alpha^{1/\gamma}$, where $a(t)$ is given by (7). The solution h is given by

$$h(t)=e^{\int_t^T a(s)ds}\Big(\alpha^{1/\gamma}+\int_t^T e^{-\int_s^T a(u)du}ds\Big),$$

and then, in terms of h, the benefit is given by (13), $F=h^{-1}F$.

Finally, by substituting in (4) we obtain (9). □

Proof of Remark 3.3. For the problem of Section 3, the HJB equation is

$$-\rho V+\max_{P,\Pi}\Bigg\{\frac{P^{1-\gamma}-1}{1-\gamma}+\big(rF+\Pi^\top(b-r\bar{1})F+CF-P\big)V_F+\frac{1}{2}\Pi^\top\Sigma\Pi F^2V_{FF}+\sum_{k=1}^m\lambda_k\Big(V\big(F+\sum_{j=1}^n\pi_jF\varphi_{ik}\big)-V(F)\Big)\Bigg\}=0. \quad (15)$$

If there is a smooth solution V of the equation (12), strictly concave, then the maximizer values of the benefit and the investments are given by

$$P^{-\gamma}-V_F=0\Rightarrow P=V_F^{-1/\gamma}, \quad (16)$$

$$(b_i-r)FV_F(x)+\sum_{j=1}^l a_{ij}\pi_jF^2V_{FF}+\sum_{k=1}^m\lambda_kV_F\Big(F+\sum_{j=1}^n\pi_jF\varphi_{jk}\Big)\varphi_{ik}F=0, \quad (17)$$

for all $i=1,\dots,n$. The structure of the HJB equation obtained, once we have substituted these values for P and Π in (12), suggests a power function $V(F)=\mu\frac{F^{1-\gamma}}{1-\gamma}+\frac{\eta}{1-\gamma}$, with μ and η suitable constants. From (16), we get that the benefit P is explicitly found in terms of the fund F,

$P = \mu^{-1/\gamma} F$, where the constants μ and η must be determined with the HJB equation. From (17), we get that the vector of investments Π is the constant proportion of surplus that solves the algebraic equation (17). Plugging into the HJB equation (15), the following algebraic equation for μ is obtained

$$\frac{\rho}{1-\gamma} = \frac{\gamma}{1-\gamma}\mu^{-1/\gamma} + c + \Psi(\Pi,\gamma),$$

which allows us to obtain (10), and $\eta = -\rho$.

By substituting in (4), we obtain the evolution of the optimal fund wealth, (11), that is to say, F^* is an extended GBM to Poisson jumps.

By Theorem 8.5 of Dockner et al. (2000), the proof of optimality concludes when the transversality condition in infinite

$$\lim_{t\to\infty} e^{-\rho t}\mathbb{E}_{F_0} V(F^*(t)) = \frac{1}{1-\gamma}\Big(\mu \lim_{t\to\infty} e^{-\rho t}\mathbb{E}_{F_0} F^*(t)^{1-\gamma} + \eta \lim_{t\to\infty} e^{-\rho t}\Big) = 0$$

is checked. By Ito's formula (see García and Griego (1994) or Hanson (2007)), we obtain

$$\begin{aligned} dF^*(t)^{1-\gamma} =& (1-\gamma)\Big(C - \mu^{-1/\gamma} + \Psi(\Pi^*,\gamma)\Big)F^*(t)^{1-\gamma}dt \\ &+ (1-\gamma)\Pi^{*\top}\sigma F^*(t)^{1-\gamma}\,dw(t) \\ &+ \sum_{k=1}^{m}\Big(\big(1+\Pi^{*\top}\varphi_k\big)^{1-\gamma} - 1\Big)F^*(t)^{1-\gamma}\,dH_k(t), \end{aligned}$$

where H is the compensated Poisson process and $F^*(0) = F_0 > 0$. Then

$$\mathbb{E}_{F_0} F^*(t)^{1-\gamma} = F_0^{1-\gamma}\exp\Big\{(1-\gamma)\big(C - \mu^{-1/\gamma} + \Psi(\Pi^*,\gamma)\big)t\Big\}$$

and, by (10),

$$\mathbb{E}_{F_0} F^*(t)^{1-\gamma} = F_0^{1-\gamma}\exp\Big\{\frac{1-\gamma}{\gamma}\Big(-\rho + C + \Psi(\Pi^*,\gamma)\Big)t\Big\}.$$

It is immediate to check that the transversality condition is

$$\rho > \frac{1-\gamma}{\gamma}\Big(-\rho + C + \Psi(\Pi^*,\gamma)\Big),$$

which is equivalent to $\mu^{-1/\gamma} > 0$. □

Data availability

No data was used for the research described in the article.

References

Baltas, I., Dopierala, L., Kolodziejczyk, K., Szczepański, M., Weber, G.-W., Yannacopoulos, A.N., 2022. Optimal management of defined contribution pension funds under the effect of inflation, mortality and uncertainty. Eur. J. Oper. Res. 298, 1162–1174.

Battocchio, P., Menoncin, F., Scaillet, O., 2007. Optimal asset allocation for pension funds under mortality risk during the accumulation and decumulation phases. Ann. Oper. Res. 152, 141–165.

Berkelaar, A., Kouwenberg, R., 2003. Retirement saving with contribution payments and labor income as a benchmark for investments. J. Econ. Dyn. Control 27, 1069–1097.

Cairns, A.J.G., 2000. Some notes on the dynamics and optimal control of stochastic pension fund models in continuous time. ASTIN Bull. 30, 19–55.

Chang, S.C., 1999. Optimal pension funding through dynamic simulations: the case of Taiwan public employees retirement system. Insur. Math. Econ. 24, 187–199.

Chang, S.C., Tzeng, L.Y., Miao, J.C.Y., 2003. Pension funding incorporating downside risks. Insur. Math. Econ. 32, 217–228.

CIA, 2015. Report of the task force on target benefit plans. https://www.cia-ica.ca/docs/default-source/2015/215043e.pdf.

Dockner, E., Jørgensen, S., Van Long, N., Sorger, G., 2000. Differential Games in Economics and Management Sciences. Cambridge University Press, Cambridge, UK.

Fleming, W.H., Soner, H.M., 2006. Controlled Markov Processes and Viscosity Solutions. Springer Verlag, New York.

García, M.A., Griego, R.J., 1994. An elementary theory of stochastic differential equations driven by a Poisson process. Stoch. Models 10, 335–363.

Guan, G., Hu, J., Liang, Z., 2022. Robust equilibrium strategies in a defined benefit pension plan game. Insur. Math. Econ. 106, 193–217.

Guo, W., Xu, C., 2004. Optimal portfolio selection when stock prices follow an jump–diffusion process. Math. Methods Oper. Res. 60, 485–496.

Haberman, S., Butt, Z., Megaloudi, C., 2000. Contribution and solvency risk in a defined benefit pension scheme. Insur. Math. Econ. 27, 237–259.

Haberman, S., Sung, J.H., 1994. Dynamics approaches to pension funding. Insur. Math. Econ. 15, 151–162.

Haberman, S., Sung, J.H., 2005. Optimal pension funding dynamics over infinite control horizon when stochastic rates of return are stationary. Insur. Math. Econ. 36, 103–116.

Haberman, S., Zimbidis, A., 2002. An investigation of the pay-as-you-go financing method using a contingency fund and optimal control techniques. N. Am. Actuar. J. 6 (2), 60–75.

Hanson, F.B., 2007. Applied Stochastic Processes and Control for Jump-Diffusions: Modeling, Analysis and Computation. Society for Industrial and Applied Mathematics (SIAM).

He, L., Liang, Z., Yuan, F., 2020. Optimal DB-PAYGO pension management towards a habitual contribution rate. Insur. Math. Econ. 94, 125–141.

Jeanblanc-Picqué, M., Pontier, M., 1990. Optimal portfolio for a small investor in a market model with discontinuous prices. Appl. Math. Optim. 22, 287–310.

Josa-Fombellida, R., López-Casado, P., 2023. A defined benefit pension plan game with Brownian and Poisson jumps uncertainty. Eur. J. Oper. Res. 310, 1294–1311.

Josa-Fombellida, R., López-Casado, P., Navas, J., 2023. A defined benefit pension plan model with stochastic salary and heterogeneous discounting. ASTIN Bull. 53, 62–83.

Josa-Fombellida, R., López-Casado, P., Rincón-Zapatero, J.P., 2018. Portfolio optimization in a defined benefit pension plan where the risky assets are processes with constant elasticity of variance. Insur. Math. Econ. 82, 73–86.

Josa-Fombellida, R., Rincón-Zapatero, J.P., 2001. Minimization of risks in pension funding by means of contribution and portfolio selection. Insur. Math. Econ. 29, 35–45.

Josa-Fombellida, R., Rincón-Zapatero, J.P., 2004. Optimal risk management in defined benefit stochastic pension funds. Insur. Math. Econ. 34, 489–503.

Josa-Fombellida, R., Rincón-Zapatero, J.P., 2008a. Funding and investment decisions in a stochastic defined benefit pension plan with several levels of labor-income earnings. Comput. Oper. Res. 35, 47–63.

Josa-Fombellida, R., Rincón-Zapatero, J.P., 2008b. Mean–variance portfolio and contribution selection in stochastic pension funding. Eur. J. Oper. Res. 187, 120–137.

Josa-Fombellida, R., Rincón-Zapatero, J.P., 2010. Optimal asset allocation for aggregated defined benefit pension funds with stochastic interest rates. Eur. J. Oper. Res. 201, 211–221.

Josa-Fombellida, R., Rincón-Zapatero, J.P., 2012. Stochastic pension funding when the benefit and the risky asset follow jump diffusion processes. Eur. J. Oper. Res. 220, 404–413.

Josa-Fombellida, R., Rincón-Zapatero, J.P., 2019. Equilibrium strategies in a defined benefit pension plan game. Eur. J. Oper. Res. 275, 374–386.

Kortleve, N., 2013. The "defined ambition" pension plan: a Dutch interpretation. Rotman Int. J. Pension Manag. 6 (1), 6–11.

Kraft, H., Weiss, F., 2019. Consumption-portfolio choice with preferences for cash. J. Econ. Dyn. Control 98, 40–59.

Mehra, R., Prescott, E., 1985. The equity premium: a puzzle. J. Monet. Econ. 15, 145–161.

Merton, R.C., 1971. Optimal consumption and portfolio rules in a continuous-time model. J. Econ. Theory 3, 373–413.

Ngwira, B., Gerrard, R., 2007. Stochastic pension fund control in the presence of Poisson jumps. Insur. Math. Econ. 40, 283–292.

Oksendal, B., Sulem, A., 2005. Applied Stochastic Control of Jump Diffusions. Springer, Berlin.

Roch, O., 2022. Continuous-time optimal pension indexing in pay-as-you-go systems. Appl. Stoch. Models Bus. Ind. 38, 458–474.

Sanders, B., 2016. Analysis of target benefit plan design options. Society of Actuaries.

Sennewald, K., 2007. Controlled stochastic differential equations under Poisson uncertainty and with unbounded utility. J. Econ. Dyn. Control 31, 1106–1131.

Taylor, G., 2002. Stochastic control of funding systems. Insur. Math. Econ. 30, 323–350.

Vigna, E., 2014. On efficiency of mean-variance based portfolio selection in defined contribution pension schemes. Quant. Finance 14, 237–258.

Wang, S., Lu, Y., 2019. Optimal investment strategies and risk-sharing arrangements for a hybrid pension plan. Insur. Math. Econ. 89, 40–62.

Wang, S., Lu, Y., Sanders, B., 2018. Optimal investment strategies and intergenerational risk sharing for target benefit pension plans. Insur. Math. Econ. 80, 1–14.

Wang, S., Rong, X., Zhao, H., 2019. Optimal investment and benefit payment strategy under loss aversion for target benefit pension plans. Appl. Math. Comput. 346, 205–218.

Wang, P., Rong, X., Zhao, H., Wang, S., 2021. Robust optimal investment and benefit payment adjustment strategy for target benefit pension plans under default risk. J. Comput. Appl. Math. 391, 113382.

Wu, L., 2003. Jumps and dynamic asset allocation. Rev. Quant. Finance Account. 20, 207–243.

Zhang, X., Guo, J., 2020. Optimal defined contribution pension management with salary and risky assets following jump diffusion processes. East Asian J. Appl. Math. 10, 22–39.

Zhao, H., Wang, S., 2022. Optimal investment and benefit adjustment problem for a target benefit pension plan with Cobb-Douglas utility and Epstein-Zin recursive utility. Eur. J. Oper. Res. 301 (3), 1166–1180.

Zou, B., Cadenillas, A., 2017. Optimal investment and liability ratio policies in a multidimensional regime switching model. Risks 5 (1), 6. https://doi.org/10.3390/risks5010006. 1–22.