Capacity Coordination in a Duopoly with Perfect Cournot Complements

Revised Version

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Abstract

We study capacity coordination in a dynamic duopoly with perfect Cournot complements where firms face investment adjustment costs when choosing their production capacities. We show that optimal capacities must become identical from a certain date on. We perform a numerical analysis of the optimal investment and capital paths and the optimal date at which capacities become equal. We find that eliminating overcapacity does not always mean decreasing the highest initial capacity (this is especially true when the firm with the lowest initial capacity has the highest depreciation rate). Moreover, it is possible that the investment of the firm with the lowest initial capacity is non-monotonic.

Key Words : Capacity investment, Cournot Model of Complements, Complementors, Capacity Coordination, Overcapacity.

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1 Introduction

We study cooperative capacity investment strategies in a dynamic version of Cournot oligopoly model of complements. In this model, consumers have a downward-sloping demand for a final product which is made out of n different components, each of which is being produced by a monopoly supplier. These n components are perfect complements in the sense that one unit of the final good requires one unit of each of the complement goods (think of a computer and its operating system, an electric guitar and its amplifier, a console and a video game and so on). Therefore, when consumers demand one unit of the final good, they actually demand one unit of each of the complement goods and as a consequence the price of this final good is the sum of the prices of the complement goods.

Cooperation through capacity investment is especially important in international transport. First of all, there has been a wave of mergers/acquisitions among global carriers and a rise in the number of alliances. In these alliances carriers share vessels in order to decrease their costs and get broader service coverage. In particular, as carriers are often stronger in certain regions or lanes, mergers and alliances are worthwhile in complementary regions (Mitsuhashi and Greve, 2009 and Ghorbani *et al.*, 2022). Mergers and alliances also avoid free-rider effects of carriers benefiting from reduction of fleet overcapacity without scrapping their own capacity themselves (see Merk *et al.*, 2018, p. 41).

More generally, capacity decisions in markets for complements pertain to capacity decisions in revenue-sharing joint ventures where resources shared are complementary (the final product or the effective capacity being constrained by the scarcest resource). As an example, consider the joint ventures that took place between Imax, an entertainment technology company, with theaters. In these joint ventures, IMAX contributes the projection systems (at a reduced charge) while the theaters provide the physical space and labor (see, Levi *et al.*, 2020, for a careful analysis of these joint ventures).

Capacity decisions in sharing consortia or joint ventures are, of course, of a different nature than capacity decisions made in merged companies. That is because, when firms agree to a partnership, they pay more attention to their own returns which can be harmful for the cooperation outcome.[1] In this connection, aligning the stakeholders' interests is crucial. One way to do that is to make sure that the capacity decisions result in a profit that is close to the profit achieved when the decisions are made centrally (as would happen if the participating partners had merged). Therefore, to design

[1] According to a study by Bamford *et al.* (2004), only half of the join ventures in their study succeeded.

efficient sharing consortia and joint ventures it is important to study coordination by merged firms as a benchmark. In this study it is also important to unravel the impact of a number of different operational factors, such as the market size, the shape of adjustment costs and the depreciation rate of firms' capacities. This is what this paper aims to contribute.

For the sake of clarity, we concentrate on a simple setup, i.e., a duopoly, where firms maximize the sum of their profits with respect to capacity investments. Time is continuous and the firms' decision horizon is infinite. To produce one unit of each complementary product, one needs one unit of a specific capacity. Moreover, to build up capacities, firms face direct investment costs as well as linear-quadratic adjustment costs and take into account the fact that capacities depreciate at fixed rates.

A key issue regarding capacity building for complementary products is how to avoid overcapacity. That is because, when firms start aligning their decisions, they almost always face an overcapacity problem, as there is no a priori reason that capacities are equal. In this connection, our research questions are as follows

- Does cooperation always entail reducing the greatest initial capacity?
- Should the cooperators always increase the lowest initial capacity?
- If both capacities must be downsized, how downsizing must be organized?
- How the answers to the previous questions depend on key parameters like the unit investment cost, the adjustment cost of capacity, the capacity depreciation rate, as well as the demand parameters (the choke price and the sensitivity of final demand to its price)?

We show that there is a finite date at which firms capacities must become equal. Before this date, firms follow specific investment policies which aim at achieving the equality of the capacities. Thereafter, under certain conditions, firms' capacities remain equal. Moreover, the common value of the capacities evolves in a monotonically way and converges to a steady state.

We enrich our theoretical study with a numerical analysis of the optimal decisions. We find that if the lowest initial production capacity is relatively small, aligning capacity investments entails increasing this capacity. Furthermore, it may happen that it is optimal to decrease initially the largest capacity and then begin to increase this capacity while ensuring that the lowest capacity catches up with the largest (this may occur when the firm with the lowest initial capacity has the highest depreciation rate). Moreover, it is possible that the investment of the firm with the lowest initial capacity is non-monotonic. Finally, we also observe that when producing becomes more profitable, overcapacity is removed earlier. It is noteworthy that the same conclusion applies when the direct cost of investment increases.

The remainder of the paper unfolds as follows. In the next section, we present a brief literature review. We set up the model in Section 3. Section 4 studies the capacity choices focusing on the case where optimal capacities are equal. Section 5 concentrates on the date at which optimal capacities become equal from that date on. Section 6 presents numerical comparative statics results with respect to a change in different operational factors. Section 7 concludes. Some proofs and detailed simulations results are relegated to the appendix.

2 Literature Review

Cooperation between firms is ubiquitous and applies to many different topics.[2] To save space, however, we shall concentrate on cooperation between firms producing complementary products. The approach to producing and marketing complementary products differs from that for substitutable products. That is because, complementary products benefit from each other's sales instead of losing sales to the other products. In this connection, Amir and Gama (2019) generalize Cournot (1838) result that integrating n-different monopoly suppliers of complementary products is Pareto-improving. To wit, the price of the final good made out of the complementary products is lower, its quantity higher, and the profits of the monopoly suppliers are higher as well.

Cooperation is also useful when competition is not viable. Dobson (1992) argues that with certain types of "rational" non-cooperative behavior complementary monopolists will produce zero output. He suggests that cooperation between firms makes the market function effectively; for example,

[2] In the case where firms face dynamic problems, one such a topic is advertising (see, *e.g.*, Cellini and Lambertini (2003, a,b), Jørgensen and Zacccour (2014), and Jørgensen and Gromova (2016)). Another issue on which cooperation is expected is R&D (concerning either process innovation or product innovation). See, *e.g*, Lambertini (2018), chapter 6, for an overview of R&D in differential games. See also Colombo and Labrecciosa (2018), section 5. Yet another issue in which cooperation is specially fruitful is the management of natural resources and the environment. For instance, cooperation (through collusive behavior) reduces production, and thus emissions (see, *e.g.*, Benchekroun and Chaudhury (2011)). Cooperation also reduces the over-exploitation of natural resources (at the cost of raising their prices). See Lambertini, (2018), chapter 7, for a lucid presentation of this literature.

through the complementary firms aiming for joint profit maximization (*e.g.*, by a merger or an overt agreement such as establishing a "joint venture" or a tacit understanding).

Yet, adopting a dynamic viewpoint, Casadesus-Masanell and Yoffie (2007) challenge the view that two tight complements (like Intel and Microsoft) will generally have well-aligned incentives. They demonstrate that natural conflicts emerge over pricing, or the timing of new product releases, and over who captures the greatest value at different phases of product generations. Conflicts occur when one firm wants to serve the installed base of goods (*e.g.*, computers), whereas the other firm favors selling new goods (*e.g.*, software). While both firms have similar incentives to invest in R&D, they conflict over prices (Microsoft favors low prices initially, whereas Intel favors high prices).

In a similar vein, Yalcin *et al.* (2013) provide a two-stage analysis of what they called value-capture and value-creation problems. In their approach, the demands of complement goods depend on their prices *and* their qualities. Quality choices are made before the price decisions. Improving the quality of one complement enhances the demand for the others. But quality improvement is costly. Consequently, there is a risk of quality underprovision for all products (this is the value-creation problem). They show that relying on a royalty fee (for rewarding the quality choice made by the first firm) does not solve the problem (allowing more competition is a better way to address the problem of quality understand).[3]

Mantovani and Ruiz-Aliseda (2015) propose an explanation of the burst in the number of collaborative activities among firms selling complementary products. They assume that firms first form pairwise collaboration, then invest cooperatively in order to improve the quality of the match and finally make price decisions non-cooperatively. They find that while firms end up forming as many collaboration ties as it is possible, they would all prefer a scenario where collaboration is forbidden.

Another aspect of cooperation between firms refers to coordination in supply chains.[4] Coordination refers to a mechanism (*e.g.*, a contract) that allows all firms within a chain to be economically better off (Cachon, 2003), notably to avoid the double marginalization effect (that is, the fact that all firms charge a high price, which decreases the demand for the final good, and eventually firms' profits). The literature that investigates successful coordination in supply chains includes two

[3] Dobson and Chakraborty (2020) study the effect of managers remuneration contracts on the innovation efforts of complementors. They show that these kinds of contract alleviate the problem of value creation mentioned above.

[4] Interestingly, in this setting, competition is not always viable. Crettez *et al.* (2025) show that because the final good price may be a discontinuous function of production capacities, open-loop equilibria for the game played by the manufacturers do not always exist.

groups: non-price-based and price-based mechanisms (for a brief and thoughtful overview of the literature, see Buratto *et al.* (2019), Introduction). Non-price-based coordination mechanisms refer to the establishment of contracts based on operational issues (*i.e.*, inventory, advertising, and so on).[5] A second stream of research focuses on pricing-based contracts in which coordination relies on both the pricing strategies and the sharing mechanisms (see, *e.g.*, Cachon and Lariviere (2005)).

In relation to these mechanisms, the strategies used to coordinate decisions within and between supply chains for complementary products are different from those for substitutable products. Bundling and joint selling partnerships are instances of these strategies (see, *e.g.*, Granot and Yin (2008), for in chain coordination). He and Yin (2015) study how competition, either at the suppliers' level, or the retailers' level affects the choice to joint selling complementary goods (competition between chains at the suppliers' level discourages this choice; the opposite effect occurs, however, if competition between chains applies at the retailers' level).

Coordination may also be realized at different levels of the supply chains. In this regard, Wei *et al.* (2019) consider the integration of two supply chains with complementary products and they pay attention to the effects of downstream, upstream and vertical integration on the supply chain members' decisions and profits.[6] Considering centralized and decentralized decision models as benchmarks, they show that the total profit of the supply chain increases with the number of integrated players and that vertical integration can be more profitable than upstream and downstream integrations.

This paper focuses on cooperative capacity buildings and adopts a dynamic viewpoint.[7] As was mentioned above, a key issue regarding capacity building for complementary products is how to avoid overcapacity. In this respect, a key issue is the determination of the first date at which capacities become equal. This issue will be tackled in a latter section.

[5] A relatively new issue, in this regard, is closed-loop supply chains. This issue is tackled in Hamed *et al.* (2020). In their approach, a manufacturer first sets his prices (for two complements) and the return rate, then a retailer makes a price decision. They show that the choice among the different options for the closed-loop supply chain is especially difficult when the manufacturer sells complementary products.

[6] Upstream integration means integration of the suppliers; downstream integration means integration of the retailers and vertical integration means integration of a supplier and a retailer.

[7] In contrast to Wang and Gerchak (2003), demand is certain.

3 Model

Consider a final product that is made out of two complementary products supplied by two firms i and j, and let P be its price. Let $K_h(t)$ denote firm h's production capacity. In the spirit of Cournot model of complements assume that the inverse demand function for the final good is given by:

$$P(t) = a - b \min\{K_i(t), K_j(t)\},$$

where a and b are positive real numbers. In the equation above a represents the maximum willingness to pay whereas b is a parameter showing how much the price of the final good changes for each unit of change in the quantity of the complementary products. For the sake of simplicity, we further assume that firms have zero production cost. Without loss of generality, we assume that the initial values K_{i0} and K_{j0} of the firms' capacities are such that $\min\{K_{i0}, K_{j0}\} < \frac{a}{2b}$.[8]

The law of motion of firm h's capacity reads

$$\dot{K}_h(t) = I_h(t) - \delta_h K_h(t),\ K_h(0) = K_{h0},$$

where I_h is firm h's investment and δ_h ($\delta_h > 0$) is the rate of depreciation of firm h's capacity. Investing is costly. We let $C_h(I_h)$ be the investment cost borne by firm h and we suppose that

$$C_h(I_h) = \alpha_h I_h + \frac{\beta_h}{2} I_h^2,$$

where $\alpha_h > 0$ and $\beta_h > 0$. The parameter α_h represents the unit cost of investment, and the parameter β_h measures the adjustment cost of the capacity. We allow for different investment costs. Namely, $\alpha_i \neq \alpha_j$, $\beta_i \neq \beta_j$. This is a relevant assumption if the production processes are genuinely different (recall the Microsoft-Intel example: it is likely that producing software requires different machines than producing chips).

In order to obtain non-trivial (i.e. non-nil) steady-state capacity, we shall also assume that

$$a - \alpha_i \delta_i - \alpha_j \delta_j - r(\alpha_i + \alpha_j) > 0,$$

[8] Under this assumption, the minimum capacity is always fully used since production cost are nil and because it is lower than the quantity that would be chosen by a static monopoly.

where $r > 0$ is the instantaneous discount rate.

To interpret this condition, rewrite it as follows: $\frac{a-\alpha_i\delta_i-\alpha_j\delta_j}{r} - (\alpha_i + \alpha_j) > 0$. Assume that the initial value of the investment rates and the capacities are nil. Now increase I_i and I_j by one unit, at time zero so that the two capacities approximately increase by one unit as well. The increase in the investment costs is approximately $\alpha_i + \alpha_j$. Further, assume that we maintain forever the new capacities. This brings about a permanent net income equal to $a - \alpha_i\delta_i - \alpha_j\delta_j$. In this expression a is the marginal receipt and the terms $\alpha_k\delta_k$ correspond to the costs of maintaining the capacities. Notice that the present value of the permanent net income is equal to $\frac{a-\alpha_i\delta_i-\alpha_j\delta_j}{r}$. The inequality above means that the present value of the permanent net income is higher than the initial investment costs, so that operating the two firms is worthwhile.

Let $\underline{K}(t) = \min\{K_i(t), K_j(t)\}$ and $R(\underline{t}) = P(t) \times \underline{K}(t)$ be the sales revenues. That is,

$$R\big(\underline{K}(t)\big) = \big(a - b\underline{K}(t)\big)\underline{K}(t), \ \text{ if } \underline{K}(t) \leq \frac{a}{2b}.$$

Firms maximize the sum of their discounted payoffs. Formally, they solve the following problem

$$\max_{I_i(t),\, I_j(t)} \int_0^{\infty} e^{-rt} \left(R\big(\underline{K}(t)\big) - \alpha_i I_i(t) - \frac{\beta_i}{2}(I_i(t))^2 - \alpha_j I_j(t) - \frac{\beta_j}{2}(I_j(t))^2 \right) dt$$

for all $I_i(\cdot)$ and $I_j(\cdot)$ such that

$$\dot{K}_i(t) = I_i(t) - \delta_i K_i(t), \ K_i(0) = K_{i0},$$
$$\dot{K}_j(t) = I_j(t) - \delta_j K_j(t), \ K_j(0) = K_{j0},$$

A solution to the problem above is a *cooperative solution.*

Before analyzing this solution a comment on non-cooperative solutions is in order. Assume that at each time, the two firms first compete in price, and then choose their capacity in a non-coordinated way. That is, there is a price game, and then a capacity game. The reason why this paper focuses on the cooperative solution is given in the following result.

Proposition 1 *There is no open-loop Nash equilibrium for the dynamic game where at each time firms first play a price game, and then a capacity game.*

Proof. Under our assumptions, the dynamic game corresponds formally to the game played by two supply chain that is analyzed in Crettez et al. (2025) where $\omega_i = 0$, and $\phi_i = 1/2$, $\alpha_i = \alpha$, $\beta_i = \beta$, $i = 1, 2$. Then the conclusion results from Theorem 3 in Crettez et al. (*ibid*). ■

Thus, because there might be no equilibrium when firms do not coordinate their decisions, it is worth considering what occurs when they cooperate.

Reformulation of the problem

To avoid using the variable $\underline{K}$, the minimum value of two state variables, and technical intricacies, we next reformulate the problem above. Specifically, we consider the following problem

$$\max_{I_i(t), I_j(t), K(t)} \int_0^\infty e^{-rt} \left(R\big(K(t)\big) - \alpha_i I_i(t) - \frac{\beta_i}{2}(I_i(t))^2 - \alpha_j I_j(t) - \frac{\beta_j}{2}(I_j(t))^2 \right) dt \tag{1}$$

for all $I_i(\cdot)$ and $I_j(\cdot)$ such that

$$\dot{K}_i(t) = I_i(t) - \delta_i K_i(t),\ K_i(0) = K_{i0}, \tag{2}$$

$$\dot{K}_j(t) = I_j(t) - \delta_j K_j(t),\ K_j(0) = K_{j0}, \tag{3}$$

$$K(t) \leq K_i(t), \tag{4}$$

$$K(t) \leq K_j(t). \tag{5}$$

In this problem above, K is a control variable (and *not* a state variable).[9] The set of admissible paths comprises the paths $K(\cdot)$, $I_h(\cdot)$, $K_h(\cdot)$, where K, I_h are piecewise continuous, K_h is continuous, piecewise continuously differentiable and such that $\dot{K}_h(t) = I_h(t) - \delta_h K_h(t)$ whenever I_h is continuous at t. We shall assume that there exists an admissible solution. Since the integrand is strictly concave with respect to the controls K and I_h, if there is a solution, it is unique.

To state optimality conditions, define the Hamiltonian

$$H(I_i, I_j, K, K_i, K_j, \lambda_i, \lambda_j, \eta_i, \eta_j) = e^{-rt} \left((a - bK)K - \alpha_i I_i - \frac{\beta_i}{2} I_i^2 - \alpha_j I_j - \frac{\beta_j}{2} I_j^2 \right)$$
$$+ \lambda_i (I_i - \delta_i K_i) + \lambda_j (I_j - \delta_j K_j) + \eta_i (K_i - K) + \eta_j (K_j - K). \tag{6}$$

Notice that the Hamiltonian is regular in the terminology of Grass et al. (2010). That is, the controls that maximize the Hamiltonian are always unique. That is because, the Hamiltonian is

[9] Notice that the inequality constraints are *mixed* in the sense that they include both control and state variables.

strictly concave with respect to K and I_h. Therefore, the optimal commands K, I_h are everywhere continuous (see Grass et al. (2010), Proposition 3.62).

Denoting with a star as superscript the optimal solution, the necessary conditions read for all t

$$e^{-rt}(a - 2bK^*) - \eta_i - \eta_j = 0, \tag{7}$$

$$-e^{-rt}\left(\alpha_i + \beta_i I_i^*\right) + \lambda_i = 0, \tag{8}$$

$$-e^{-rt}\left(\alpha_j + \beta_j I_j^*\right) + \lambda_j = 0, \tag{9}$$

$$\dot{\lambda}_i = \delta_i \lambda_i - \eta_i, \tag{10}$$

$$\dot{\lambda}_j = \delta_j \lambda_j - \eta_j, \tag{11}$$

$$K^* \leq K_i^*, \tag{12}$$

$$K^* \leq K_j^*, \tag{13}$$

$$\eta_i(K_i^* - K^*) = 0, \tag{14}$$

$$\eta_i \geq 0, \tag{15}$$

$$\eta_j(K_j^* - K^*) = 0, \tag{16}$$

$$\eta_j \geq 0. \tag{17}$$

The adjoint variables λ_i and λ_j can be interpreted as the shadow values of the capacities of firm i and j, respectively. Equations (8) and (9) provides the rules for making investment decisions. These equations state that at each date, the marginal cost of investment of capacity h is equal to its shadow value. The term η_h is the Lagrange multiplied appended to the capacity constraint $K \leq K_h$. This multiplier can be interpreted as the static shadow value of capacity h. That is, η_h is the marginal revenue that the cooperators could obtain if capacity h were increased by one unit. If capacities are large enough, of course, $\eta_h = 0$, since the cooperators can maximize instant profits without operating at full capacity. This property is reflected in the conditions (14) and (16), i.e., the complementary slackness conditions. Equations (10) and (11) give the dynamics of the shadow values of the capacities. The instantaneous change in the shadow value of firm h's capacity at t reads $\dot{\lambda}_h(t) = -(\eta_h - \delta_h \lambda_h)$. The decrease in the shadow value includes the difference of two terms. The first term, $\eta_h(t)$, is what could have been earned at t if the capacity of firm h had been increased by one unit. As time goes by, this is no longer possible (what was available is irremediably lost). The second term, $\delta_h \lambda_h(t)$ refers to the instantaneous depreciation of firm h's capacity K_h. More precisely, increasing firm h's capacity by one unit implies that there is more instantaneous depreciated capital. As the instantaneous depreciated capital can no longer be put at used and be profitable, there is a corresponding instantaneous loss of future incomes that must be subtracted from $\eta_h(t)$.

4 Capacity choices

Considering the problem (1)-(5), we first establish that in a cooperative solution capacities necessarily meet.

Proposition 2 *Suppose that $K_{i0} < K_{j0}$ and that there exists a cooperative solution. Then there is a date $\underline{t} \in \mathbb{R}_+$ such that $K_j^*(\underline{t}) = K_i^*(\underline{t})$.*

Proof. The proof is by way of a contradiction. By assumption, $K_{i0} < K_{j0}$. If $K_i^*(t) < K_j^*(t)$ for all $t \in \mathbb{R}_+$, then $K^*(t) = K_i^*(t) < K_j^*(t)$ for all t and using the first-order conditions above we get that $\eta_j(t) = 0$ for all t and thus it holds that

$$\begin{aligned} \lambda_j(t) &= (\alpha_j + \beta_j I_j^*(t))e^{-rt}, \qquad (18) \\ \dot{\lambda}_j(t) &= \delta_j \lambda_j(t), \\ \dot{K}_j^*(t) &= I_j^*(t) - \delta_j K_j^*(t),\ K_j(0) = K_{j0}. \end{aligned}$$

The second equation yields $\lambda_j(t) = c_1 e^{\delta_j t}$ where $c_1 \in \mathbb{R}$. Substituting this value in equation (18) we get $I_j^*(t) = \frac{1}{\beta_j}[c_1 e^{(\delta_j + r)t} - \alpha_j]$ and the last equation becomes $\dot{K}_j^*(t) = \frac{1}{\beta_j}[c_1 e^{(\delta_j+r)t} - \alpha_j] - \delta_j K_j^*(t)$. The solution to this equation reads $K_j^*(t) = c_2 e^{-\delta_j t} + \frac{c_1}{\beta_j(r+2\delta_j)} e^{(\delta_j+r)t} - \frac{\alpha_j}{\beta_j \delta_j}$. Since $K_j^*(t)$ is bounded $c_1 = 0$ and we have $K_j^*(t) = c_2 e^{-\delta_j t} - \frac{\alpha_j}{\beta_j \delta_j}$. It is clear, however, that $K_j^*(t)$ goes to a negative value which is impossible. ■

The intuition of the result above is as follows. If capacities never meet, then there is always overcapacity. Yet it pays for the two firms to let the highest capacity always depreciate. This, of course, is impossible because even if the lowest capacity decreases over time, it never goes to zero as producing is always profitable. But then, there must be a date at which the highest capacity meets the lowest.

While capacities must become equal at some date, once they meet, they could become different at some future date. However, we shall now concentrate on the case where capacities remain equal forever once they meet. For the sake of simplicity, we shall suppose that capacities are equal from date zero on.

Proposition 3 *Assume that the optimal capacities are equal from date* 0 *on and that* $K^*(t) = K_i^*(t) = K_j^*(t)$ *for all* t. *Then we have*

$$K^*(t) = (K_0 - K_\infty)e^{z_1 t} + K_\infty, \tag{19}$$

where

$$K_\infty = \frac{a - (\alpha_i\delta_i + \alpha_j\delta_j + r(\alpha_i + \alpha_j))}{\beta_i\delta_i(\delta_i + r) + \beta_j\delta_j(\delta_j + r) + 2b}$$

and z_1 *is the negative root of the following polynomial*

$$z^2 - rz - \frac{\beta_i\delta_i(\delta_i + r) + \beta_j\delta_j(\delta_j + r) + 2b}{\beta_i + \beta_j} = 0.$$

Proof. By assumption, we have for all t

$$I_i^*(t) = \dot{K}^*(t) + \delta_i K^*(t),$$
$$I_j^*(t) = \dot{K}^*(t) + \delta_j K^*(t).$$

Then using the first-order conditions (8) and (9), we also get

$$\lambda_i = e^{-rt}(\alpha_i + \beta_i(\dot{K}^* + \delta_i K^*)),$$
$$\lambda_j = e^{-rt}(\alpha_j + \beta_j(\dot{K}^* + \delta_j K^*)),$$

and thus (since I_h and $\dot{K}_h$ are differentiable)

$$\dot{\lambda}_i = -r\lambda_i + e^{-rt}\beta_i(\ddot{K}^* + \delta_i\dot{K}^*),$$
$$\dot{\lambda}_j = -r\lambda_j + e^{-rt}\beta_j(\ddot{K}^* + \delta_j\dot{K}^*).$$

Using equations (7), (10) and (11) we also have

$$\begin{aligned} e^{-rt}\left(a - 2bK^*\right) &= \eta_i + \eta_j \\ &= -\dot{\lambda}_i + \delta_i\lambda_i - \dot{\lambda}_j + \delta_j\lambda_j. \end{aligned}$$

Using the expressions of λ_h and $\dot{\lambda}_h$ above and rearranging, we arrive at

$$\ddot{K}^* - r\dot{K}^* - \frac{\beta_i\delta_i(\delta_i + r) + \beta_j\delta_j(\delta_j + r) + 2b}{\beta_i + \beta_j}K^* + \frac{a - (\alpha_i\delta_i + \alpha_j\delta_j + r(\alpha_i + \alpha_j))}{\beta_i + \beta_j} = 0.$$

Considering the polynomial equation

$$z^2 - rz - \frac{\beta_i\delta_i(\delta_i + r) + \beta_j\delta_j(\delta_j + r) + 2b}{\beta_i + \beta_j} = 0 \tag{20}$$

associated with the above second-order differential equation, the negative root is given by

$$z_1 = \frac{r - \sqrt{\triangle}}{2}$$

where

$$\triangle = r^2 + 4\frac{\beta_i\delta_i(\delta_i + r) + \beta_j\delta_j(\delta_j + r) + 2b}{\beta_i + \beta_j}.$$

The value of K_∞ is easily obtained. ■

According to Proposition 3, the common value of the capacity goes to a finite value in the long run. This value increases with the choke price a, and decreases with the depreciation rates δ_h, as well as the parameters of the investment cost function α_h and β_h, the interest rate r and the parameter b describing the sensitivity of the demand for the final product to its price. The speed $e^{z_1 t}$ at which convergence occurs decreases with the interest rate, the rates of depreciation, the parameter β_h associated with the quadratic part of the investment cost, and the parameter b.

In the Proposition above, we have obtained the optimal investment decisions under the assumption that firms' capacities are always equal. In the next result, we provide sufficient conditions under which this assumption holds.

Proposition 4 *Assume that $K_{i0} = K_{j0} = K_0$. Furthermore, assume that either $K_0 < K_\infty$ or*

$$K_\infty < K_0 < \frac{(r+\delta_h)\alpha_h + \beta_h K_\infty \delta_h (r+\delta_h)}{-\beta_h z_1 (r - z_1)} + K_\infty, \quad h = i, j. \tag{21}$$

Then, $K^*(t) = K_i^*(t) = K_j^*(t)$ for all t.

Proof. Consider the candidate optimal solution for the capacities given in Proposition 3. From these values we can use equations (2) and (3) to get candidate optimal values for the investment decisions. In turn, we can use equations (8) to (11) and the optimal values of I_h and K_h to get candidate optimal values for λ_h and $\dot{\lambda}_h$. Moreover, because the Hamiltonian is concave the first-order conditions (7)-(17) are sufficient. Thus the candidate optimal solution (for I_h and K_h) is indeed the optimal solution if $\eta_h(t) \geq 0$ for all t and all h (see inequalities (15) and (17)). Using the condition $\eta_h(t) = \delta_h \lambda_h(t) - \dot{\lambda}_h(t)$, and substituting for the expressions of $\lambda_h(t)$ and $\dot{\lambda}_h(t)$ respectively, we obtain that

$$\begin{aligned}
\eta_h(t) &= e^{-rt}\left(\alpha_h(r+\delta_h) + (r+\delta_h)\beta_h\delta_h K^* + r\beta_h \dot{K}^* - \beta_h \ddot{K}^*\right) \\
&= e^{-rt}\left(\alpha_h(r+\delta_h) + (r+\delta_h)\beta_h\delta_h\left(K_\infty + (K_0 - K_\infty)e^{z_1 t}\right)\right. \\
&\left. + r\beta_h z_1 (K_0 - K_\infty)e^{z_1 t} - \beta_h z_1^2 (K_0 - K_\infty)e^{z_1 t}\right).
\end{aligned} \tag{22}$$

Assume first that $K_0 < K_\infty$. Inspecting equation (22) one can check that $\eta_h(t) > 0$ for all h. Suppose instead that $K_0 > K_\infty$. Then, we have

$$\begin{aligned}
\eta_h(t) &\geq e^{-rt}\left(\alpha_h(r+\delta_h) + (r+\delta_h)\beta_h\delta_h K_\infty\right. \\
&\left. + r\beta_h z_1 (K_0 - K_\infty) - \beta_h z_1^2 (K_0 - K_\infty)\right) \\
&= e^{-rt}((r+\delta_h)(\alpha_h + \beta_h\delta_h K_\infty) + \beta_h z_1 (K_0 - K_\infty)(r - z_1)).
\end{aligned}$$

Now,

$$\begin{aligned}
(r+\delta_h)\alpha_h + \beta_h\delta_h(r+\delta_h)K_\infty - \beta_h z_1 K_\infty (r - z_1) &> -\beta_h z_1 K_0 (r - z_1) \\
\iff K_0 &< \frac{(r+\delta_h)\alpha_h}{-\beta_h z_1 (r - z_1)} + \beta_h \frac{K_\infty[\delta_h(r+\delta_h) - z_1(r - z_1)]}{-\beta_h z_1 (r - z_1)} \\
i.e.\ K_0 &< \frac{(r+\delta_h)\alpha_h + \beta_h K_\infty \delta_h (r+\delta_h)}{-\beta_h z_1 (r - z_1)} + K_\infty
\end{aligned}$$

By assumption, the inequality above always holds, and thus $\eta_h(t) \geq 0$ for all h and t. As the candidate solution satisfies all the necessary and sufficient conditions, it is indeed the optimal solution (the latter being unique). ■

5 Study of the Date $\underline{t}$ at which Capacities Meet

In the preceding section, we have studied the optimal path of the capacities assuming that there are always equal. We have also provided some sufficient conditions under which optimal capacities do remain equal if they are so initially. We now study the optimal paths assuming that the capacities are different initially, and that they become forever equal at a finite positive date $\underline{t}$. Our objective is to determine $\underline{t}$ but also to have a better understanding of the investment decisions made *before* $\underline{t}$.

Proposition 5 *Assume that when the optimal capacities meet for the first time they remain forever equal. Then, the date $\underline{t}$ at which firms' capacities become equal forever is a solution to the following equation*

$$K^{*,i}(\underline{t}) = K^{*,j}(\underline{t}), \tag{23}$$

where

$$K^{*,h}(\underline{t}) = \frac{\left(z_{2h}e^{z_{2h}\underline{t}} - z_{1h}e^{z_{1h}\underline{t}}\right)\left(K_{h0}e^{z_{1h}\underline{t}} + K_{h\infty}(1 - e^{z_{1h}\underline{t}})\right) - \left(z_1 K_\infty + (K_{h0} - K_{h\infty})z_{1h}e^{z_{1h}\underline{t}}\right)\left(e^{z_{2h}\underline{t}} - e^{z_{1h}\underline{t}}\right)}{z_{2h}e^{z_{2h}\underline{t}} - z_{1h}e^{z_{1h}\underline{t}} - z_1\left(e^{z_{2h}\underline{t}} - e^{z_{1h}\underline{t}}\right)}. \tag{24}$$

Moreover, the dynamics of firm h's capacity up to date $\underline{t}$ is given by

$$K_h^*(t) = A_h e^{z_{1h}t} + B_h e^{z_{2h}t} + K_{h\infty}, \tag{25}$$

where A_h and B_h satisfy

$$\begin{aligned}
A_h &= K_{h0} - B_h - K_{h\infty}, \\
B_h &= \frac{z_1(K^*(\underline{t}) - K_\infty) - (K_{h0} - K_{h\infty}) z_{1h} e^{z_{1h}\underline{t}}}{z_{2h} e^{z_{2h}\underline{t}} - z_{1h} e^{z_{1h}\underline{t}}}, \\
K_{i\infty} &= \frac{a - \alpha_i (r + \delta_i)}{2b + \beta_i \delta_i (r + \delta_i)}, \\
K_{j\infty} &= -\frac{\alpha_j}{\beta_j \delta_j},
\end{aligned}$$

with

$$\begin{aligned}
z_{1i} &= \frac{r - \sqrt{\triangle_i}}{2} \\
z_{2i} &= \frac{r + \sqrt{\triangle_i}}{2}
\end{aligned}$$

where

$$\begin{aligned}
\triangle_i &= r^2 + 4\left(\frac{2b}{\beta_i} + \delta_i (r + \delta_i)\right), \\
z_{1j} &= -\delta_j, \\
z_{2i} &= r + \delta_j,
\end{aligned}$$

and z_1 is the negative root of the polynomial in (39).

Proof. See the appendix. ■

In view of the preceding Proposition, a necessary condition for the optimal path to be such that capacities meet forever at date $\underline{t}$ when they differ initially, is that there is a solution $K^{*,i}(\underline{t}) = K^{*,j}(\underline{t})$ to the equation (23) . The next result ensures that this equation always has a solution.

Lemma 1 *There exists a solution to the equation (23) $K^{*,i}(\underline{t}) = K^{*,j}(\underline{t})$.*

Proof. Consider the function $\varphi : (0, \infty) \to \mathbb{R}$, defined by $\varphi(t) = K^{*,i}(t) - K^{*,j}(t)$, where $K^{*,h}(t)$ is given by the right-hand side of equation (24). Observe that

$$\lim_{t \to 0+} \varphi(t) = K_{i0} - K_{j0} < 0. \tag{26}$$

$$\lim_{t\to\infty}\varphi(t) = \frac{z_{2i}K_{i\infty} - z_1 K_\infty}{z_{2i} - z_1} - \frac{z_{2j}K_{j\infty} - z_1 K_\infty}{z_{2j} - z_1} \tag{27}$$

$$= \frac{z_{2i}K_{i\infty}}{z_{2i} - z_1} - \frac{z_{2j}K_{j\infty}}{z_{2j} - z_1} - z_1 K_\infty \left(\frac{1}{z_{2i} - z_1} - \frac{1}{z_{2j} - z_1} \right) \tag{28}$$

$$= \frac{z_{2i}K_{i\infty}}{z_{2i} - z_1} - \frac{z_{2j}K_{j\infty}}{z_{2j} - z_1} - z_1 K_\infty \frac{z_{2j} - z_{2i}}{(z_{2i} - z_1)(z_{2j} - z_1)} > 0, \tag{29}$$

because $z_{1h} < 0, z_{2h} > 0$ for $h = 1, 2$, $z_1 < 0$, $K_{i\infty} > 0$, $K_{j\infty} < 0$, $K_\infty > 0$, and $z_{2j} - z_{2i} = \frac{r+\sqrt{\triangle_i}}{2} + \delta_j > 0$.

Since $\varphi(\cdot)$ is continuous, there is a date $\underline{t}$ such that $\varphi(\underline{t}) = 0$. ■

In the previous Lemma, we have assumed that the optimal capacities are equal forever after they meet for the first time. We now provide conditions under which this assumption is satisfied. That is, we provide conditions under which the optimal values of the capacities are given by equations (25) for $t \leq \underline{t}$, $\underline{t}$ solves equation (24), and the common optimal value for the capacity after $\underline{t}$ is provided by equation (19).[10]

Proposition 6 *Assume either that*

$$K_{i0} < K_{j0} < K_\infty \tag{30}$$

or that

$$K_\infty < \max_h K_{h0} < \min_h \frac{(r + \delta_h)\alpha_h + \beta_h K_\infty \delta_h (r + \delta_h)}{-\beta_h z_1 (r - z_1)} + K_\infty, \quad h = i, j. \tag{31}$$

Then there is a date $\underline{t}$ at which optimal capacities meet, remain forever equal and converge to K_∞.

Proof. See the appendix. ■

Notice that the result above also ensures that there exists an optimal solution since the first-order conditions are sufficient.

We shall assume from now on that one of the sufficient conditions provided in the Proposition above hold. We shall also build on the results obtained heretofore to perform a sensitivity analysis of the optimal solution. Unfortunately, because the date $\underline{t}$ at which firms' capacities meet is determined

[10] In equation (19), we must substitute $K^*(\underline{t})$ for K_0, and take care of the fact that the dynamics is presented assuming that $\underline{t}$ is the starting date.

only implicitly through equation (24), it is difficult to carry out an analytical study of the sensitivity of the optimal solution to changes in the values of the parameters or the initial conditions. We shall instead rely on a numerical approach that we present next.[11]

6 Numerical Illustrations

In this section we follow a numerical approach to address the ensuing questions: How do the date $\underline{t}$ at which capacities meet for the first time change with the key operational factors faced by the firms? When the initial value of the lowest capacity is lower than its stationary value, should we reduce overcapacity by decreasing the capacity which is the highest initially? When the initial value of the lowest capacity is higher than its stationary value, how should we organize the downsizing of these capacities?

To address these questions we now perform numerical simulations for three different scenarios, depending on how the initial conditions of the capital stocks, K_{i0}, K_{j0}, and the steady state of the capital, K_∞, compare.[12] These scenarios are as follows.

- Scenario I: $K_{i0} < K_{j0} < K_\infty$.
- Scenario II: $K_{i0} < K_\infty < K_{j0}$.
- Scenario III: $K_\infty < K_{i0} < K_{j0}$.

The effect on the date $\underline{t}$ at which capacities first meet of changes in the operational factors in the completely symmetric case, that is, where firms only differ in the initial values of their capacities, are collected in Table 1.[13]

	$\alpha \uparrow$	$\beta \uparrow$	$\delta \uparrow$	$r \uparrow$	$a \uparrow$	$b \uparrow$	$K_{i0} \uparrow$	$K_{j0} \uparrow$
$\underline{t}$	$\downarrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$

Table 1: Sensitivity analysis of $\underline{t}$ with respect to changes in parameters or initial conditions.

[11] In all of our numerical results, the parameters satisfy one of the conditions provided in Proposition 6.

[12] To simplify the analysis, we consider that the parameters pertaining to both firms are the same, except for the depreciation rates and the initial values of the capacities. As a benchmark case we retain the following parameter values: $\alpha_i = \alpha_j = \alpha = 1$, $\beta_i = \beta_j = \beta = 1$, $\delta_i = \delta_j = \delta = 0.01$, $r = 0.05$, $a = 1$, $b = 1$, and the following initial conditions: $K_{i0} = 0.25$, $K_{j0} = 0.3$.

[13] See the appendix B for a detailed presentation of the simulations.

That a rise in K_{i0} or decrease in K_{j0} have a negative effect on $\underline{t}$ is expected since these changes make the initial values of the capacities closer. We also observe that if the choke price a increases or if the demand for the final good is less sensible to its price (b takes lower values) then the meeting date $\underline{t}$ is advanced.[14] When the lowest capacity is lower than its steady state value, a rise in a or a decrease in b makes production more profitable. It thus pays to increase the lowest capacity at a higher rate which allows to catching up at an earlier date with the highest capacity (the highest capacity might increase as well, however). When the lowest capacity is higher than the steady state value, a rise in a or a decrease in b makes the steady state common capacity closer to the lowest capacity. Thus, both capacities decrease at a slower rate, but the lowest capacity decreases at a rate which is lower than that of the highest capacity.[15] Interestingly, a rise in α and β, the parameters of the cost function have opposite effects on $\underline{t}$, even though in both cases the steady state value K_∞ decreases. Recall that under or symmetry assumption the marginal cost of investment for each firm is $\alpha + \beta I$. A rise in α refers to a rise in the unit cost of investment and has an effect that is similar to that of a decrease in the choke price a. A rise in β affects the quadratic part of the investment cost. Its effect depends on the value of investment. The higher this value the higher the marginal cost (this is in contrast with what occurs with a rise in α which has the same effect, whatever the value of the investment rates). When β increases, there is both a short run and a long-run effects. In the long run, the steady state value of investment is lower because it is costlier to maintain a given value of capacity. In the short run, since the adjustment cost is higher, it is worthwhile to slow the building (or the downsizing) of the capacity. For instance, suppose that capacities are lower than their steady-state value. A rise in β results in a decrease in this steady-state value. As the long-run common capacity value is lower, it pays to build less capacity, and thus overcapacity should be eliminated earlier. But on the other hand, as the marginal cost of investment is higher, firms are better off by scaling down their investments and thus it takes a longer time to build capacity. It turns out, however, that this positive effect dominates the negative one.

All the preceding effects are also obtained when firms differ with respect to the depreciation rates of their capacities. Specifically, the date $\underline{t}$ at which capacities become equal increases when the depreciation rate of firm i's capacity is greater than the depreciation rate of firm j's capacity, and vice versa. To understand this effect, suppose that K_{i0} is lower than the steady state value. On

[14] Moreover, under these assumptions we see that the long-term value of the stationary capacity $K_\infty = \frac{a - \left(\alpha_i \delta_i + \alpha_j \delta_j + r(\alpha_i + \alpha_j)\right)}{\beta_i \delta_i (\delta_i + r) + \beta_j \delta_j (\delta_j + r) + 2b}$ is higher.

[15] The negative effect of a rise in r or in the depreciation rate δ on $\underline{t}$ can be understood in a similar way.

the one hand, if firm i's depreciation rate is higher than firm j's, it takes a longer time to build capacity and thus to catch up with firm j's capacity. On the other hand, the steady state value is lower (since it is costlier to maintain a given common capacity) and thus firms need to build less capacity so that overcapacity can be eliminated earlier. As the date $\underline{t}$ at which capacities become equal increases, this second negative effect is compensated by the first positive one.

Let us turn to the investment decisions and the evolution of capacities. Our simulations illustrate the fact that K_j does not necessarily decrease initially to get rid of overcapacity. Indeed, figure 1 shows that when the long-run value of the common capacity is higher than the initial values of the capacities, then as both capacities must eventually increase, it is not optimal to decrease firm j's capacity. Yet even if the initial capacities are lower than the long-run common values, it can be that it is optimal to first decrease firm j's capacity and then let it grow afterwards. Figure 2 illustrates this case. Such case occurs when the depreciation rate of firm i is much higher than firm j's ($\delta_i = 0.2$, $\delta_i = 0.01$). In that case, it takes more time to build firm i's capacity and it this then worthwhile to reduce firm j's overcapacity initially (moreover, building firm j's capacity is cheaper than building firm i's).

Now assume that firms must downsize their capacities. Interestingly, it is not always the case that their investments are monotonic. Figure 3 displays a case where firm i's investment first increases, and then decreases after having reached a maximum value (in this case, the unit cost of investment takes its highest value, i.e., $\alpha = 5$). Afterwards, the common value of investment increases again (it must go to a positive value in order to maintain the common value of the capacity). Notice that firm i's investment always grows at a slower rate than firm j's before date $\underline{t}$. The reason why firm i's investment rate is non-monotonic is that because firms' depreciation rates are equal, when their capacities meet at date $\underline{t}$ their investment must be equal at this date as well (otherwise capacities would evolve differently at $\underline{t}$). Since firm j decreases its capacity at a higher rate than firm i, firm i must downsize at a higher rate at some time to catching up with firm j' investment. Firm i's downsizing rate cannot be monotonic, however, because that would imply decreasing its capacity at a higher rate at the starting date. But doing so is costly (the investment cost would be higher) and would imply that firms' investments become equal at an earlier date than $\underline{t}$ while their capacities would be different at this date. Yet it is not optimal to choose the same investment rates if the capacities are not identical.

investment_benchmark_scenario_I.png

capital_benchmark_scenario_I.png

Figure 1: Scenario I: Investment (left) and capacity (right) optimal paths. Benchmark case.

investment_benchmark_scenario_I_deltai_02mayor_deltaj.png

capital_benchmark_scenario_I_deltai_02mayor_deltaj.

Figure 2: Scenario I: Investment (left) and capacity (right) optimal paths. $\alpha = 1$, $\delta_i = 0.2$, $\delta_j = 0.01$

7 Conclusion

This paper has advanced the analysis of the dynamic version of Cournot duopoly with perfect complements. We have analyzed the cooperative outcome that is obtained when firms behave like a monopoly selling complementary goods and coordinate their capacity investment decisions and we have studied how this outcome changed with different operational factors, like the market size, the shape of adjustment cost and the depreciation rate. We have highlighted the fact that optimal firms' capacities must become identical in a finite time (and, under some conditions, remain equal forever). Indeed, cooperation is useful as long as it eliminates overcapacity, and overcapacity always holds whenever initial capacities are different. Yet, our numerical results highlight the fact that investment_benchmark_scenario_III_alpha_5.png capital_benchmark_scenario_III_alpha_5.png

Figure 3: Scenario III: Investment (left) and capital (right) optimal paths. $b = 2, K_{i0} = 0.24, \alpha = 5$.

eliminating overcapacity does not always mean decreasing the highest initial capacity. We have seen that when the stationary capacity is relatively high, cooperation entails increasing both production capacities. However, when the depreciation rate of the firms with the lowest capacity is the highest, the best policy is to start downsizing the highest capacity initially. These findings are useful in order to assess the optimal investment policy of firms involved in a merger or in a joint venture.

There are at least four avenues for future research, besides considering a setting with more than two firms. First, it would be worthwhile to pay attention to different forms of adjustment cost. While we have considered a quadratic cost function, one could also include a term that depends on the value of the capacity (*e.g.*, *ratio* of investment to the capacity). Second, it would also be worthwhile to take into account a demand saturation effect, as in Ngendakuriyo and Taboubi (2017). Third, it would be interesting to consider the case where investment is not always reversible (that is, when the investment decisions I_i are always non-negative). Fourth, it would also be interesting to consider pollution emissions and the choice of abatement policies as in El Ouardighi (2016, 2021).

Data Availability

All the data analyzed are included in the paper.

Conflicts of interest

The authors declare that they have no conflicts of interest.

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A Proofs

Proof of Proposition 5

Using the optimality conditions (7)-(17) for all $t < \underline{t}$, we have

$$\begin{aligned}
e^{-rt}(a - 2bK^*) - \eta_i &= 0 \\
-e^{-rt}\left(\alpha_i + \beta_i I_i^*\right) + \lambda_i &= 0 \\
-e^{-rt}\left(\alpha_j + \beta_j I_j^*\right) + \lambda_j &= 0 \\
\dot{\lambda}_i &= -\eta_i + \delta_i \lambda_i \\
\dot{\lambda}_j &= \delta_j \lambda_j \\
K^* &\leq K_i^* \\
K^* &\leq K_j^* \\
\eta_i(K_i^* - K^*) &= 0 \\
\eta_i &\geq 0 \\
\eta_j(K_j^* - K^*) &= 0 \\
\eta_j &= 0
\end{aligned}$$

as well as

$$K_h^*(\underline{t}) = \underline{K}^*(\underline{t}) = K^*(\underline{t}).$$

Since the controls K^* and I_h^* are continuous using Proposition 3 we obtain that

$$\begin{aligned}
I_h^*(\underline{t}) &= \dot{K}^*(\underline{t}) + \delta_h K^*(\underline{t}) \\
&= z_1\big(K^*(\underline{t}) - K_\infty\big) + \delta_h K^*(\underline{t}).
\end{aligned}$$

Let us concentrate of firm i's decisions before date $\underline{t}$.

Using the first-order conditions, we get

$$\begin{aligned}
e^{-rt}\big(a - 2bK^*(t)\big) &= \eta_i(t) \\
&= \delta_i \lambda_i(t) - \dot{\lambda}_i(t).
\end{aligned}$$

As we have already observed, the differential equations are well-defined everywhere (the controls being continuous) so that both $\dot{\lambda}_h$ and $\dot{K}_h$ are differentiable. Using this fact in the expression above we have

$$\begin{aligned} e^{-rt}\big(a - 2bK^*(t)\big) &= \eta_i(t) \\ &= \delta_i \lambda_i(t) - \dot{\lambda}_i(t) \\ &= (\delta_i + r)e^{-rt}\big(\alpha_i + \beta_i I_i^*\big) - e^{-rt}\beta_i \dot{I}_i^*. \end{aligned}$$

Rearranging, we get

$$\ddot{K}^* - r\dot{K}^* - \frac{2b + \beta_i\delta_i(r+\delta_i)}{\beta_i}K^* + \frac{a - \alpha_i(r+\delta_i)}{\beta_i} = 0.$$

The characteristic roots are given by

$$\begin{aligned} z_{1i} &= \frac{r - \sqrt{\triangle_i}}{2} \\ z_{2i} &= \frac{r + \sqrt{\triangle_i}}{2} \end{aligned}$$

where

$$\triangle_i = r^2 + 4\big(\frac{2b}{\beta_i} + \delta_i(r+\delta_i)\big).$$

Then we get

$$K_i^*(t) = K^*(t) = A_i e^{z_{1i}t} + B_i e^{z_{2i}t} + \frac{a - \alpha_i\big(r+\delta_i\big)}{2b + \beta_i\delta_i\big(r+\delta_i\big)},$$

where A_i and B_i are such that

$$\begin{aligned} K_{i0} = K_i^*(0) &= A_i + B_i + \frac{a - \alpha_i\big(r+\delta_i\big)}{2b + \beta_i\delta_i(r+\delta_i)} \\ K_i^*(\underline{t}) = K_j^*(\underline{t}) &= A_i e^{z_{1i}\underline{t}} + B_i e^{z_{2i}\underline{t}} + \frac{a - \alpha_i\big(r+\delta_i\big)}{2b + \beta_i\delta_i\big(r+\delta_i\big)}. \end{aligned}$$

Notice that $K_i^*(\underline{t}) = K_j^*(\underline{t})$ are still to be determined. To proceed, let us turn to firm j's program.

Proceeding as for firm i, we arrive at the following second-order differential equation

$$\ddot{K}_j^*(t) - r\dot{K}_j^*(t) - (r+\delta_j)\delta_j K_j^*(t) - \frac{\alpha_j(r+\delta_j)}{\beta_j} = 0.$$

The characteristic roots are given by

$$\begin{aligned} z_{1j} &= -\delta_j \\ z_{2j} &= r+\delta_j. \end{aligned}$$

We thus have

$$K_j^*(t) = A_j e^{-\delta_j t} + B_j e^{(r+\delta_j)t} - \frac{\alpha_j}{\beta_j\delta_j}$$

where A_j and B_j are such that

$$\begin{aligned} K_{j0} = K_j^*(0) &= A_j + B_j - \frac{\alpha_j}{\beta_j\delta_j} \\ K_j^*(\underline{t}) = K^*(\underline{t}) &= A_j e^{-\delta_j \underline{t}} + B_j e^{(r+\delta_j)\underline{t}} - \frac{\alpha_j}{\beta_j\delta_j}. \end{aligned}$$

From

$$\begin{aligned} I_h^*(\underline{t}) &= \dot{K}^*(\underline{t}) + \delta_h K^*(\underline{t}) \\ &= z_1\big(K^*(\underline{t}) - K_\infty\big) + \delta_h K^*(\underline{t}), \end{aligned}$$

we see that the following relations must also hold

$$\begin{aligned} & z_{1i}A_i e^{z_{1i}\underline{t}} + z_{2i}B_i e^{z_{2i}\underline{t}} + \delta_i K^*(\underline{t}) = z_1\big(K^*(\underline{t}) - K_\infty\big) + \delta_i K^*(\underline{t}) \\ \iff\ & z_{1i}A_i e^{z_{1i}\underline{t}} + z_{2i}B_i e^{z_{2i}\underline{t}} = z_1\big(K^*(\underline{t}) - K_\infty\big), \end{aligned}$$

$$\begin{aligned} & z_{1j}A_j e^{z_{1j}\underline{t}} + z_{2j}B_j e^{z_{2j}\underline{t}} + \delta_j K^*(\underline{t}) = z_1\big(K^*(\underline{t}) - K_\infty\big) + \delta_j K^*(\underline{t}) \\ \iff\ & z_{1j}A_j e^{z_{1j}\underline{t}} + z_{2j}B_j e^{z_{2j}\underline{t}} = z_1\big(K^*(\underline{t}) - K_\infty\big), \end{aligned}$$

which implies that

$$z_{1i} A_i e^{z_{1i}\underline{t}} + z_{2i} B_i e^{z_{2i}\underline{t}} = z_1\big(K^*(\underline{t}) - K_\infty\big) = z_{1j} A_j e^{z_{1j}\underline{t}} + z_{2j} B_j e^{z_{2j}\underline{t}}.$$

So we have 6 equations for 6 unknowns (that is A_i, B_i, A_j, B_j, $K^*(\underline{t})$ and $\underline{t}$).

Define

$$\begin{aligned} K_{i\infty} &= \frac{a - \alpha_i\big(r+\delta_i\big)}{2b + \beta_i\delta_i(r+\delta_i)}, \\ K_{j\infty} &= -\frac{\alpha_j}{\beta_j\delta_j}. \end{aligned}$$

Using the relation $A_h = K_{h0} - B_h - K_{h\infty}$ and $z_{1h}A_h e^{z_{1h}\underline{t}} + z_{2h}B_h e^{z_{2h}\underline{t}} = z_1(K^*(\underline{t}) - K_\infty)$, we get

$$B_h = \frac{z_1(K^*(\underline{t}) - K_\infty) - (K_{h0} - K_{h\infty}) z_{1h} e^{z_{1h}\underline{t}}}{z_{2h} e^{z_{2h}\underline{t}} - z_{1h} e^{z_{1h}\underline{t}}}.$$

From the relation $A_h = K_{h0} - B_h - K_{h\infty}$ and $A_h e^{z_{1h}\underline{t}} + B_h e^{z_{2h}\underline{t}} + K_{h\infty} = K^*(\underline{t})$, we obtain

$$K^*(\underline{t}) = \frac{\big(z_{2h} e^{z_{2h}\underline{t}} - z_{1h} e^{z_{1h}\underline{t}}\big)\left(K_{h0} e^{z_{1h}\underline{t}} + K_{h\infty}(1 - e^{z_{1h}\underline{t}})\right) - \big(z_1 K_\infty + (K_{h0} - K_{h\infty}) z_{1h} e^{z_{1h}\underline{t}}\big)\left(e^{z_{2h}\underline{t}} - e^{z_{1h}\underline{t}}\right)}{z_{2h} e^{z_{2h}\underline{t}} - z_{1h} e^{z_{1h}\underline{t}} - z_1\big(e^{z_{2h}\underline{t}} - e^{z_{1h}\underline{t}}\big)} \tag{32}$$

Denote by $K^{*,h}(\underline{t})$ the right-hand side of the equation above. Clearly, $\underline{t}$ is the solution of the following equation

$$K^{*,i}(\underline{t}) = K^{*,j}(\underline{t}). \tag{33}$$

and the proof is complete.

Proof of Proposition 6

a) Assume that $K_{i0} < K_{j0} < K_\infty$.

In view of Proposition 4 it suffices to establish that the date $\underline{t}$ at which capacities meet (whose existence is asserted by Proposition 1) is such that $K_i^*(\underline{t}) = K_j^*(\underline{t}) < K_\infty$. Let us establish this result. We have:

$$K^{*,j}(\underline{t}) = \frac{\big(z_{2j} e^{z_{2j}\underline{t}} - z_{1j} e^{z_{1j}\underline{t}}\big)\left(K_{j0} e^{z_{1j}\underline{t}} + K_{j\infty}(1 - e^{z_{1j}\underline{t}})\right) - \big(z_1 K_\infty + (K_{j0} - K_{j\infty}) z_{1j} e^{z_{1j}\underline{t}}\big)\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)}{z_{2j} e^{z_{2j}\underline{t}} - z_{1j} e^{z_{1j}\underline{t}} - z_1\big(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\big)}.$$

Taking out the common factor K_{j0} in the numerator we get

$$K^{*,j}(\underline{t}) = \frac{1}{z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}} - z_1\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)} \left\{\left[\left(z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\right)e^{z_{1j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)\right]K_{j0}\right. \tag{34}$$

$$\left. + \left(z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\right)K_{j\infty}(1 - e^{z_{1j}\underline{t}}) - \left(z_1K_\infty - K_{j\infty}z_{1j}e^{z_{1j}\underline{t}}\right)\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)\right\}. \tag{35}$$

The denominator is positive and the term in brackets multiplying K_{j0} is positive too, because z_{ij} and z_1 are negative and z_{2j} is positive. Therefore, if $K_{j0} < K_\infty$,

$$K^{*,j}(\underline{t}) < \frac{1}{z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}} - z_1\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)} \left\{\left[\left(z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\right)e^{z_{1j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)\right]K_\infty\right.$$
$$\left. + \left(z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\right)K_{j\infty}(1 - e^{z_{1j}\underline{t}}) - \left(z_1K_\infty - K_{j\infty}z_{1j}e^{z_{1j}\underline{t}}\right)\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)\right\}$$

and the fulfillment of the following inequality

$$\frac{1}{z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}} - z_1\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)} \left\{\left[\left(z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\right)e^{z_{1j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)\right]K_\infty\right.$$
$$\left. + \left(z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\right)K_{j\infty}(1 - e^{z_{1j}\underline{t}}) - \left(z_1K_\infty - K_{j\infty}z_{1j}e^{z_{1j}\underline{t}}\right)\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)\right\} < K_\infty, \tag{36}$$

implies the fulfillment of $K^{*,j}(\underline{t}) < K_\infty$. Inequality (36) can be rewritten as

$$\left[\left(z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\right)e^{z_{1j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right) - \left(z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\right) + z_1\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)\right]K_\infty$$
$$+ \left(z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\right)K_{j\infty}(1 - e^{z_{1j}\underline{t}}) - \left(z_1K_\infty - K_{j\infty}z_{1j}e^{z_{1j}\underline{t}}\right)\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right) < 0.$$

Equivalently,

$$\left[\left(z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\right)\left(e^{z_{1j}\underline{t}} - 1\right) - \left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)\left(z_{1j}e^{z_{1j}\underline{t}} - z_1\right)\right]K_\infty$$
$$+ \left[\left(z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\right)(1 - e^{z_{1j}\underline{t}}) + \left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)z_{1j}e^{z_{1j}\underline{t}}\right]K_{j\infty} - z_1K_\infty\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right) < 0$$
$$\left[\left(z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\right)\left(e^{z_{1j}\underline{t}} - 1\right) - \left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)z_{1j}e^{z_{1j}\underline{t}}\right]K_\infty$$
$$+ \left[\left(z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\right)(1 - e^{z_{1j}\underline{t}}) + \left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)z_{1j}e^{z_{1j}\underline{t}}\right]K_{j\infty} < 0$$
$$\left[\left(z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\right)\left(e^{z_{1j}\underline{t}} - 1\right) - \left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)z_{1j}e^{z_{1j}\underline{t}}\right](K_\infty - K_{j\infty}) < 0.$$

Because K_∞ is positive and $K_{j\infty}$ is negative, the inequality above is satisfied if and only if

$$\left(z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\right)\left(e^{z_{1j}\underline{t}} - 1\right) - \left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right) z_{1j}e^{z_{1j}\underline{t}} < 0. \tag{37}$$

Inequality (37) can be rewritten as:

$$\begin{aligned}
&e^{z_{1j}\underline{t}}\left(z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}} - z_{1j}(e^{z_{2j}\underline{t}}) - e^{z_{1j}\underline{t}})\right) - \left(z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\right) < 0. \\
&e^{z_{1j}\underline{t}}\left(z_{2j} - z_{1j}\right) e^{z_{2j}\underline{t}} - \left(z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}\right) < 0.
\end{aligned}$$

Multiplying by $e^{-z_{1j}\underline{t}}$, we have

$$\left(z_{2j} - z_{1j}\right) e^{z_{2j}\underline{t}} - z_{2j}e^{(z_{2j}-z_{1j})\underline{t}} + z_{1j} < 0.$$

The left-hand side of the inequality evaluated at $\underline{t} = 0$ is zero, and the derivative with respect to $\underline{t}$ reads:

$$\left(z_{2j} - z_{1j}\right) z_{2j}e^{z_{2j}\underline{t}} - z_{2j}(z_{2j} - z_{1j})e^{(z_{2j}-z_{1j})\underline{t}}$$

which is equal to

$$z_{2j}(z_{2j} - z_{1j})e^{z_{2j}\underline{t}}(1 - e^{-z_{1j}\underline{t}}).$$

Last expression takes negative values for any positive $\underline{t}$. Therefore,

$$\left(z_{2j} - z_{1j}\right) e^{z_{2j}\underline{t}} - z_{2j}e^{(z_{2j}-z_{1j})\underline{t}} + z_{1j},$$

is a decreasing function of $\underline{t}$, that takes negative values for any positive $\underline{t}$.

b) Assume that

$$K_\infty < \max_h K_{h0} < \min_h \frac{(r+\delta_h)\alpha_h + \beta_h K_\infty \delta_h (r+\delta_h)}{-\beta_h z_1 (r - z_1)} + K_\infty, \quad h = i, j \tag{38}$$

First observe that from equation (35) it holds that

$$K^{*,j}(\underline{t}) = (K_{j0} - K_{j\infty})e^{z_{1j}\underline{t}}\left(\lambda_j(\underline{t}) - \frac{z_{1j}(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}})}{z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}} - z_1\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)}\right) + \lambda_j(\underline{t})K_{j\infty} + (1 - \lambda_j(\underline{t}))K_\infty$$

where

$$\lambda_j(\underline{t}) = \frac{z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}}{z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}} - z_1\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)}. \tag{39}$$

Thus,

$$K^{*,j}(\underline{t}) = (K_{j0} - K_{j\infty})e^{z_{1j}\underline{t}}\left(\frac{(z_{2j} - z_{1j})e^{z_{2j}\underline{t}}}{z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}} - z_1\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)}\right) + \lambda_j(\underline{t})K_{j\infty} + (1 - \lambda_i(\underline{t}))K_\infty.$$

Secondly, let us show that

$$e^{z_{1j}\underline{t}}\left(\frac{(z_{2j} - z_{1j})e^{z_{2j}\underline{t}}}{z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}} - z_1\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)}\right) \leq \lambda_j(\underline{t}) = \frac{z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}}}{z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}} - z_1\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)}.$$

To see this, notice that the above inequality reduces to

$$(r + 2\delta_j)e^{(r+\delta_j)\underline{t}} \leq (r + \delta_j)e^{r+2\delta_j} + \delta_j. \tag{40}$$

The above relation is true for $\underline{t} = 0$. Set

$$f(\underline{t}) = (r + 2\delta_j)e^{(r+\delta_j)\underline{t}}$$
$$g(\underline{t}) = (r + \delta_j)e^{(r+2\delta_j)\underline{t}} + \delta_j.$$

It holds that

$$f'(\underline{t}) = (r + 2\delta_j)(r + \delta_j)e^{(r+\delta_j)\underline{t}} < g'(\underline{t}) = (r + 2\delta_j)(r + \delta_j)e^{(r+2\delta_j)\underline{t}}.$$

Thus the inequality (40) is always satisfied.

Thirdly, we have:

$$K^{*,j}(\underline{t}) = (K_{j0} - K_{j\infty})e^{z_{1j}\underline{t}}\left(\frac{(z_{2j} - z_{1j})e^{z_{2j}\underline{t}}}{z_{2j}e^{z_{2j}\underline{t}} - z_{1j}e^{z_{1j}\underline{t}} - z_1\left(e^{z_{2j}\underline{t}} - e^{z_{1j}\underline{t}}\right)}\right) + \lambda_j(\underline{t})K_{j\infty} + (1 - \lambda_i(\underline{t}))K_\infty \tag{41}$$

$$\leq (K_{j0} - K_{j\infty})\lambda_j(\underline{t}) + \lambda_j(\underline{t})K_{j\infty} + (1 - \lambda_i(\underline{t}))K_\infty \tag{42}$$

$$\leq K_{j0}\lambda_j(\underline{t}) + (1 - \lambda_i(\underline{t}))K_\infty \tag{43}$$

$$\leq K_{j0}. \tag{44}$$

Then, the result directly follows from Proposition 4.

B A Numerical Analysis of the Sensitivity of $\underline{t}$ with Respect to Changes in Parameters or Initial Conditions.

We consider the three scenarios in turn.

Scenario I. $K_{i0} < K_{j0} < K_\infty$

Benchmark parameter values: $\alpha_i = \alpha_j = \alpha = 1$, $\beta_i = \beta_j = \beta = 1$, $\delta_i = \delta_j = \delta = 0.01$, $r = 0.05$, $a = 1$, $b = 1$, and initial conditions: $K_{i0} = 0.25$, $K_{j0} = 0.3$.

In all the simulations in this scenario, we keep the values of all the parameters and the initial conditions as in the benchmark case, except for a single value that we change as indicated in tables 2, 3, 4 and 5.

Tables 6, 7, 8 and 9 collect the results when the completely symmetric case is relaxed, and different depreciation rates of the firms' capacities are considered ($\delta_i = 0.015, \delta_j = 0.01$).

Scenario II. $K_{i0} < K_\infty < K_{j0}$

Benchmark parameter values: $\alpha_i = \alpha_j = \alpha = 1$, $\beta_i = \beta_j = \beta = 1$, $\delta_i = \delta_j = \delta = 0.01$, $r = 0.05$, $a = 1$, $b = 1$, and initial conditions: $K_{i0} = 0.25$, $K_{j0} = 0.5$.

In all the simulations in this scenario, we keep the values of all the parameters and the initial conditions as in the benchmark case, except for a single value that we change as indicated in tables 10, 11, 12 and 13. When the result of a case analyzed in Scenario I is not included in any of the tables below, it is because that case is not feasible in Scenario II.

Tables 14, 15, 16 and 17 present the results for the case $\delta_i > \delta_j$.

Scenario III. $K_\infty < K_{i0} < K_{j0}$

Benchmark parameter values: $\alpha_i = \alpha_j = \alpha = 1$, $\beta_i = \beta_j = \beta = 1$, $\delta_i = \delta_j = \delta = 0.01$, $r = 0.05$, $a = 1$, $b = 2$, and initial conditions: $K_{i0} = 0.24$, $K_{j0} = 0.5$.

In all the simulations in this scenario, we keep the values of all the parameters and the initial conditions as in the benchmark case, except for a single value that we change as indicated in tables 18 and 19. When the result of a case analyzed in Scenario I is not included in any of the tables, it is either because that case is not feasible in Scenario III or for condition $K_{i0} < a/(2b)$ to be fulfilled, more than one parameter value needs to be changed. Tables 20 and 21 present the results for the case $\delta_i > \delta_j$.

	α				β		
	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 2$	$\beta = 0.5$	$\beta = 1$	$\beta = 1.5$
$\underline{t}$	0.55559	0.54185	0.52684	0.52202	0.37238	0.52684	0.64542
K_∞	0.49370	0.46972	0.43974	0.37977	0.43487	0.43974	0.43960

Table 2: Scenario I. Sensitivity with respect to changes in α and β.

	δ			r		
	$\delta = 0.005$	$\delta = 0.01$	$\delta = 0.015$	$r = 0.045$	$r = 0.05$	$r = 0.055$
$\underline{t}$	0.52926	0.52684	0.52446	0.52912	0.52684	0.52462
K_∞	0.44488	0.43974	0.43458	0.44476	0.43974	0.43472

Table 3: Scenario I. Sensitivity with respect to changes in δ and r.

	a			b		
	$a = 0.75$	$a = 1$	$a = 1.5$	$b = 0.75$	$b = 1$	$b = 1.25$
$\underline{t}$	0.75743	0.52684	0.35718	0.45352	0.52684	0.63386
K_∞	0.31481	0.43974	0.68959	0.58619	0.43974	0.35183

Table 4: Scenario I. Sensitivity with respect to changes in a and b.

	K_{i0}			K_{j0}		
	$K_{i0} = 0.2$	$K_{i0} = 0.25$	$K_{i0} = 0.28$	$K_{j0} = 0.28$	$K_{j0} = 0.3$	$K_{j0} = 0.35$
$\underline{t}$	0.73768	0.52684	0.23252	0.38979	0.52684	0.82210

Table 5: Scenario I. Sensitivity with respect to changes in K_{i0} and K_{j0}.

	α				β		
	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 2$	$\beta = 0.5$	$\beta = 1$	$\beta = 1.5$
$\underline{t}$	0.55603	0.54356	0.52985	0.50701	0.37448	0.52985	0.64915
K_∞	0.49336	0.46838	0.43716	0.37471	0.43733	0.43716	0.43698

Table 6: Scenario I. Sensitivity with respect to changes in α and β. $\delta_i = 0.015, \delta_j = 0.01$.

	r		
	$r = 0.045$	$r = 0.05$	$r = 0.055$
$\underline{t}$	0.53218	0.52985	0.52758
K_∞	0.44218	0.43716	0.43213

Table 7: Scenario I. Sensitivity with respect to changes in r. $\delta_i = 0.015, \delta_j = 0.01$.

	a			b		
	$a = 0.75$	$a = 1$	$a = 1.5$	$b = 0.75$	$b = 1$	$b = 1.25$
$\underline{t}$	0.76487	0.52985	0.35831	0.45557	0.52985	0.63855
K_∞	0.31225	0.43716	0.68696	0.58272	0.43716	0.34978

Table 8: Scenario I. Sensitivity with respect to changes in a and b. $\delta_i = 0.015, \delta_j = 0.01$.

	K_{i0}			K_{j0}		
	$K_{i0} = 0.2$	$K_{i0} = 0.25$	$K_{i0} = 0.28$	$K_{j0} = 0.28$	$K_{j0} = 0.3$	$K_{j0} = 0.35$
$\underline{t}$	0.74148	0.52985	0.23399	0.39195	0.52985	0.82708

Table 9: Scenario I. Sensitivity with respect to changes in K_{i0} and K_{j0}. $\delta_i = 0.015, \delta_j = 0.01$.

	α					β		
	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 2$	$\alpha = 5$	$\beta = 0.5$	$\beta = 1$	$\beta = 1.5$
$\underline{t}$	2.73095	1.95456	1.65268	1.37820	1.06301	1.16552	1.65268	2.02800
K_∞	0.49370	0.46972	0.43974	0.37977	0.19988	0.43987	0.43974	0.43960

Table 10: Scenario II. Sensitivity with respect to changes in α and β.

	δ			r		
	$\delta = 0.005$	$\delta = 0.01$	$\delta = 0.015$	$r = 0.045$	$r = 0.05$	$r = 0.055$
$\underline{t}$	1.69031	1.65268	1.61844	1.68747	1.65268	1.62116
K_∞	0.44488	0.43974	0.43458	0.44476	0.43974	0.43472

Table 11: Scenario II. Sensitivity with respect to changes in δ and r.

	a			b		
	$a = 0.75$	$a = 1$	$a = 1.1$	$b = 1$	$b = 1.25$	$b = 1.5$
$\underline{t}$	2.23798	1.65268	1.47404	1.65268	2.04939	2.40274
K_∞	0.31481	0.43974	0.48971	0.43974	0.35183	0.29322

Table 12: Scenario II. Sensitivity with respect to changes in a and b.

	K_{i0}			K_{j0}		
	$K_{i0} = 0.2$	$K_{i0} = 0.25$	$K_{i0} = 0.29$	$K_{j0} = 0.5$	$K_{j0} = 0.6$	$K_{j0} = 0.8$
$\underline{t}$	1.73003	1.65268	1.58057	1.65268	2.20967	3.24450

Table 13: Scenario II. Sensitivity with respect to changes in K_{i0} and K_{j0}.

	α				β		
	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 2$	$\beta = 0.5$	$\beta = 1$	$\beta = 1.5$
$\underline{t}$	2.73779	1.96312	1.66281	1.38991	1.17250	1.66281	2.04067
K_∞	0.49336	0.46838	0.43716	0.37471	0.43733	0.43716	0.43698

Table 14: Scenario II. Sensitivity with respect to changes in α and β. $\delta_i = 0.015$, $\delta_j = 0.01$.

	r		
	$r = 0.045$	$r = 0.05$	$r = 0.055$
$\underline{t}$	1.69823	1.66281	1.63073
K_∞	0.44218	0.43716	0.43213

Table 15: Scenario II. Sensitivity with respect to changes in r. $\delta_i = 0.015$, $\delta_j = 0.01$.

	a			b		
	$a = 0.75$	$a = 1$	$a = 1.1$	$b = 1$	$b = 1.25$	$b = 1.5$
$\underline{t}$	2.25149	1.66281	1.48243	1.66281	2.06067	2.41294
K_∞	0.31225	0.43716	0.48712	0.43716	0.34978	0.29151

Table 16: Scenario II. Sensitivity with respect to changes in a and b. $\delta_i = 0.015$, $\delta_j = 0.01$.

	K_{i0}			K_{j0}		
	$K_{i0} = 0.2$	$K_{i0} = 0.25$	$K_{i0} = 0.29$	$K_{j0} = 0.5$	$K_{j0} = 0.6$	$K_{j0} = 0.8$
$\underline{t}$	1.73971	1.66281	1.59114	1.66281	2.22157	3.25630

Table 17: Scenario II. Sensitivity with respect to changes in K_{i0} and K_{j0}. $\delta_i = 0.015$, $\delta_j = 0.01$.

	α				β		
	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 2$	$\alpha = 5$	$\beta = 0.5$	$\beta = 1$	$\beta = 1.5$
$\underline{t}$	4.10391	2.87916	2.33520	1.33123	2.02108	2.87916	3.54555
K_∞	0.23493	0.21993	0.18994	0.09997	0.219967	0.21993	0.21990

Table 18: Scenario III. Sensitivity with respect to changes in α and β.

	δ			r		
	$\delta = 0.005$	$\delta = 0.01$	$\delta = 0.015$	$r = 0.045$	$r = 0.05$	$r = 0.055$
$\underline{t}$	3.01102	2.87916	2.76335	3.00036	2.87916	2.77257
K_∞	0.22247	0.219993	0.21739	0.22244	0.21993	0.217429

Table 19: Scenario III. Sensitivity with respect to changes in δ and r.

	α				β		
	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 2$	$\alpha = 5$	$\beta = 0.5$	$\beta = 1$	$\beta = 1.5$
$\underline{t}$	4.10946	2.88644	2.05301	1.34371997	0.21187	0.21866	0.218621

Table 20: Scenario III. Sensitivity with respect to changes in α and β. $\delta_i = 0.015$, $\delta_j = 0.01$.

	r		
	$r = 0.045$	$r = 0.05$	$r = 0.055$
$\underline{t}$	3.00798	2.88644	2.77954
K_∞	0.22117	0.21866	0.21616

Table 21: Scenario III. Sensitivity with respect to changes in r. $\delta_i = 0.015$, $\delta_j = 0.01$.