Is multichannel retail marketing integration a panacea?*

Guiomar Martín-Herrán† and Simon-Pierre Sigué‡

November 1, 2024

Abstract

Channel integration/centralization understood as joint pricing of multiple channels is touted as the ideal organization to maximize the profitability of multichannel retailers. This study challenges this claim and analytically examines with two two-period models whether multi-channel retailers should centralize or decentralize online and offline pricing decisions when vertical channel interactions and consumer reference price effects are considered. We found that under certain conditions (which depend on several factors, including the manufacturer's advertising strategies over the two periods, the intensity of price competition between channels, and the consumers' sensitivity to price changes over time), the retailer may find it optimal to centralize or decentralize online and offline pricing decisions. Therefore, our findings support the idea that multichannel retailing integration is not a panacea, especially in a context where complex vertical interactions with manufacturers are taken into account and where consumers compare current market prices to recent past prices at the time of purchase.

*We would like to thank the associate editor and two anonymous referees for their useful comments. Guiomar Martín-Herrán gratefully acknowledges financial support from the Spanish Ministry of Innovation and Sciences (AEI) under projects PID2020-112509GB-I00 and TED2021-130390B-I00.

†Corresponding author: Guiomar Martín-Herrán IMUVA, Universidad de Valladolid, Valladolid, Spain. Facultad de Ciencias Económicas y Empresariales. Depto. Economía Aplicada. Avda. Valle Esgueva, 6. 47011-Valladolid, Spain. email: guiomar.martin-herran@uva.es ORCID: 0000-0002-9161-2349

‡Athabasca University, Athabasca, Canada. Faculty of Business, Athabasca University and Visiting Professor School of Business Sciences, University of Witwatersrand, Sotu Africa. Athabasca University, 1 University Drive, Athabasca, T9S 3A3, Canada. email: simons@athabascau.ca ORCID: 0000-0002-4061-0853

Is multichannel retail marketing integration a panacea?

November 1, 2024

Abstract

Channel integration/centralization understood as joint pricing of multiple channels is touted as the ideal organization to maximize the profitability of multichannel retailers. This study challenges this claim and analytically examines with two two-period models whether multi-channel retailers should centralize or decentralize online and offline pricing decisions when vertical channel interactions and consumer reference price effects are considered. We found that under certain conditions (which depend on several factors, including the manufacturer's advertising strategies over the two periods, the intensity of price competition between channels, and the consumers' sensitivity to price changes over time), the retailer may find it optimal to centralize or decentralize online and offline pricing decisions. Therefore, our findings support the idea that multichannel retailing integration is not a panacea, especially in a context where complex vertical interactions with manufacturers are taken into account and where consumers compare current market prices to recent past prices at the time of purchase.

1 Introduction

Multichannel retailing, where retailers operate multiple channels to sell products to consumers, has attracted a lot of research over the past three decades, mainly due to the advent of digital commerce (Zhang et al., 2010). Lookdowns and social distancing measures related to COVID-19 have further accelerated the growth of digital commerce to the point where even some of the most skeptical traditional retailers who have long resisted online expansion now operate multichannel structures, or at least combine their online and offline operations. Despite the near-generalization of multichannel retailing, it poses several management challenges for practitioners and academics that are far from being resolved. One of such challenges, identified earlier by Neslin et al. (2006), Rangaswamy and Van Bruggen (2005), and Zhang et al. (2010), is the selection of the best organizational structure so as to maximize market performance and profitability.

The theoretical debate over choosing the appropriate organizational structure of multichannel retailing has traditionally pitted centralized and decentralized structures against each other. The decentralized structure in which channels are run as separate businesses is typically adopted by offline retailers in the early stages of their e-commerce expansion. For instance, Walmart started a new online division to develop its online operations with the goal not only to attract and retain executives with appropriate experience in e-commerce, but also to best respond to competitors and market expectations (Zhang et al., 2010). This was also the case of Barnes and Noble, which originally only operated physical stores and launched its online store, BarnesAndNoble.com, as a completely independent division (Berger et al., 2006; Gulati and Garino, 2000). Most recently, Saks Fifth Avenue, a U.S-based retailer, successfully split its e-commerce and offline businesses into two companies, Saks.com for online operations and SFA to operate a fleet of 40 brick-and-mortar stores (Latona, 2022; Scott, 2021). Generally, in a decentralized multichannel structure, channels are managed by different teams, who aim to maximize channel specific profits, leading to cannibalization when they compete for the same customer base (Verhoef, 2021). Implementing a decentralized structure implicitly assumes that its benefits outweigh its known drawbacks - specifically the duplication and inefficiency of business processes, the lack of coordination of marketing activities, and inconsistent customer experiences across channels.

The centralized retail structure, which is also known as channel integration, is credited to attenuate the disadvantages of the decentralized structure listed above. All channels are managed by a central authority, who is in charge of setting marketing decisions for each so as to maximize their combined profits. A growing number of retailers such as Zara, Best Buy, and Staples are opting for this organizational structure, which often results in identical prices for online and offline channels (Cavallo, 2017; Wang et al., 2023). It is believed that, despite some level of marketing coordination - the main weakness of the centralized structure is the fact that the channels continue to be treated as separate entities who serve their own customers and whose individual performance can be identified and assessed. Specifically, the customers' abilities to use all the channels at the same time and the synergy that can be created between them to provide a seamless experience and gain efficiency are overlooked (Verhoef et al., 2015).

The conventional wisdom in both the business and academic literature is that retail integration helps achieve better market performance and better profits than retail decentralization (e.g., Berger et al., 2006; Gulati and Garino, 2000; Verhoef, 2021; Wang et al., 2023). However, discussions and research leading to such a statement usually take place outside the context of the vertical marketing channel where the retailer is only one member of the value chain. They also minimized or overlooked critical aspects of consumer behavior and the dynamic nature of some key business decisions. This raises questions about the relevance and generalizability of this conventional wisdom, especially in the context where some of these critical factors are known to impact channel decisions at the retail level. For instance, Martín-Herrán et al. (2014) found that vertical phenomena such as double marginalization and vertical free-riding affect the dealers' decision to horizontally integrate or not. Moreover, in the context of dealership networks, Kalnins and Lafontaine (2004) argued that under certain conditions manufacturers may prefer some degree of intranetwork competition between dealers and set their marketing decisions accordingly. Karray and Sigué (2018) also showed that channel power dynamics change with actions taken at the retail level, such as adding an online channel to a traditional offline store.

On the other hand, the importance of consumer behavior in marketing decisions, such as those related to pricing and advertising, no longer needs to be emphasized. In multichannel retailing, depending on the organizational structure chosen, channels may compete or not on prices, but they must consider customer expectations. One way of incorporating customer expectations in analytical marketing models has been through the concept of reference price, which is the customer price expectation against which market prices are compared to assess whether customers view market prices positively or negatively (Li et al., 2021, Mazumdar et al., 2005; Prakash and Spann, 2022; Thaler, 1985). Ignoring these customer expectations can lead to suboptimal retail strategies.

Finally, there is a growing body of dynamic marketing research which shows that shortsighted strategies that ignore the long-term effects of marketing activities hurt profitability

(Jørgensen and Zaccour, 2014; Karray et al., 2017, 2021, & 2022; Martín-Herrán and Sigué, 2023). For example, how channel members schedule their advertising decisions over time has been shown to affect channel outcomes (Martín-Herrán and Sigué, 2017a&b). As a result, Zhang et al. (2010) called for multichannel retailing research which goes beyond short-term sales growth and profitability and takes a long-term perspective.

Therefore, there is a need to develop a comprehensive framework for choosing the organizational structure of multichannel retailing, which recognizes the critical role that vertical interactions play on what happens at the retail level, and which takes into account the dynamic nature of certain key marketing decisions such as pricing and advertising. Such a framework should help establish the conditions under which multi-channel retailers should consider centralizing or decentralizing online and offline retail pricing decisions.

A two-period model in which a single manufacturer deals with a multichannel retailer that operates both online and offline channels is proposed to generate such a framework. Specifically, we consider a situation where, from the outset, the retailer announces to centralize or decentralize retail prices over a planning horizon of two periods. The manufacturer then sets the wholesale price and the level of investment in the product advertising, while the multichannel retailer sets (jointly or separately depending on the announcement) the retail prices for each channel in each period.

Three important features of this model are worth noting. The first feature is that the multichannel retailer has the freedom to set different prices for the two channels. This implies that if the two channels end up having the same price, it will be the result of an endogenous derivation, not an exogenous decision. The second feature is that the advertising carried out in the first period extends into the second period, but its effects decrease exponentially (Martín-Herrán and Sigué, 2017a). This takes into account the dynamic nature of manufacturer advertising, which contributes to building her brand image over time. The third characteristic is that customers use the price of the first period of a channel as their internal reference price for this channel in the second period. Consequently, the model captures customer sensitivity to price variations over time (Martín-Herrán and Sigué, 2023).

By considering two scenarios where the retailer jointly (Scenario 1) or separately (Scenario 2) sets channel prices, we solve two Stackelberg games. Each of the two games has four endogenous equilibrium solutions where the two players set their respective prices and the manufacturer chooses one of the following four advertising arrangements: The manufacturer advertises in each period of the game (Equilibrium I); the manufacturer exclusively advertises in the first period (Equilibrium II); the manufacturer exclusively advertises in the second period (Equilibrium III); and the manufacturer does not advertise at any time (Equilibrium IV). Depending on the model parameters, the manufacturer can implement each of these four equilibrium solutions, which correspond to different advertising schedules, in each scenario to maximize her profits.

We further compare the players' profits for the different equilibria of the two scenarios and find that, the retailer can favor setting prices jointly or separately depending on how sensitive customers are to price variations over time, the intensity of price competition between channels, and the effects of manufacturer advertising. These findings imply that the general belief that the integration of multichannel retailers leads to better performance is misleading. In fact, these findings confirm our initial thinking that choosing a multichannel retailer organizational structure is not as simple as claimed in the current literature. It requires not only taking into account vertical marketing interactions between the retailer and the manufacturer, but also analyzing how customers react to price changes and how manufacturers advertise for their products over time. This work therefore offers a new perspective that helps understand why certain multichannel retailers such as Barnes and Noble have opted for decentralization, while others such as Staples and Zara have integrated their offline and online pricing decisions.

The remainder of the paper is organized as follows: Section 2 provides the background literature; Section 3 describes the model and discusses its main assumptions; Section 4 briefly describes the derivation of the equilibrium solutions for the two scenarios and discusses their implementation; Section 5 discusses player's preferences for the two organizational scenarios; Section 6 concludes and discusses the theoretical and managerial implications of this research. The detailed derivation of the equilibrium solutions is presented in the Appendix.

2 Overview of the background literature

This paper builds on an extensive literature in marketing. For the sake of brevity, we quickly cover three important research areas, namely vertical marketing channels, multichannel retailing, and reference price to better position our work.

2.1 Vertical marketing channels

Central to vertical marketing channel research is the idea that the members of a channel are interdependent so that channel output is the product of the members' interactions and activities. Based on this premise, the main objective of marketing channel research has been to find ways to help channel members organize or coordinate their marketing activities to avoid conflicts or improve market performance and member profits.

For instance, in marketing channel models involving pricing decisions, the primary goal has been to try to overcome the negative effects on channel outcomes of double marginalization which occurs when the manufacturer and its retailer can each charge a price containing a monopolistic markup on its own marginal cost in maximizing own profit. Otherwise, any increase in wholesale price at the manufacturer level can be passed on to customers in the form of a higher retail price, resulting in lower consumer demand and lower profits for channel members (Jeuland and Shugan, 2008). Another vertical externality within marketing channels that occurs when the manufacturer or/and the retailer conduct non-price marketing activities such as advertising, which impact consumer demand, is vertical free-riding. In such a case, it has been demonstrated that the channel member that does not invest in the activity can take advantage of the other partner's investment by adjusting its own strategies to improve its profit (Sigué and Chintagunta, 2009).

Some work has also investigated the impact of online expansion of traditional offline retailers on channel outcomes, strategies, and profits. The key finding of this research is that, even if the online expansion is exclusively done at the retailer level, manufacturers also adjust their marketing strategies to cope with this new reality, which generally allows retailers to expand their market coverage and increase consumer demand (e.g., Cheng and Xiong, 2015; Yoo and Lee, 2011).

This paper builds on this research stream by considering the choice of the organizational structure of a multichannel retailer as a game-changer in its interactions with manufacturers. Therefore, while acknowledging that both organizational structures considered have inherent advantages and disadvantages, the assessment of their profitability for multichannel retailers must take into account how manufacturers respond to each.

2.2 Multichannel retailing

The term "multichannel retailing" is used in this paper to refer to situations where a retailer, such as Costco, Walmart, Staples, and Zara, sells products to consumers through a least two different channels. This precision is worthwhile because the term "multichannel" is also used in the literature for situations where a manufacturer uses multiple distribution channels.

These days, most retailers operate both offline and online stores. This has generated a great deal of work and discussion on the benefits of this form of organization for retailers (Neslin et al., 2006; Rangaswamy and Van Bruggen, 2005; Verhoef, 2021; Zhang et al., 2010). A few other works have analytical investigated whether or not it pays to move from a single-channel retailer to a multi-channel retailer (e.g., Cheng and Xiong, 2015; Karray and Sigué, 2018;Yoo and Lee, 2011; Zhang, 2009). This research generally discusses the conditions under which multichannel retailing should be adopted.

A growing number of academics have recently turned their attention to how to manage multichannel retailing to improve performance (Verhoef, 2021). Among others, operating multiple channels comes with the question of whether these channels should be managed as separate entities or coordinated/integrated (Zhang et al., 2010). Conventional wisdom favors channel coordination, culminating in full integration through the concept of an omnichannel structure, which is touted as the ideal multichannel setup where management relies on the synergy between channels to deliver a better and seamless experience to consumers (Verhoef et al., 2015).

In terms of research, the article by Berger et al. (2006) is one of the first attempts to analytically study advertising integration and separation in the context of multichannel retailing. Taking the example the organizational structure of the book retailer Barnes and Noble, these authors study how cooperative advertising should be managed between the Barnes and Noble head office, its offline retail stores, and its online store, and find that advertising integration is better than advertising separation. In this study, the head office acts as the manufacturer in many cooperative advertising models available in the literature. In contrast, this study overlooks the vertical interactions between the head office and book suppliers, which can fundamentally change the strategies of all channel members.

Several other papers have empirically investigated the relationship between cross-channel integration and firm performance (e.g., Avery et al., 2012; Cao and Li, 2015; Oh et al., 2012; Tagasshira and Minami, 2019). For instance, Cao and Li (2015) report a positive influence of cross-channel integration on sales growth through five different mechanisms, including improved trust, increased customer loyalty, higher consumer conversion rates, greater opportunities to cross-sell, and the loss of specific channel features. Tagasshira and Minami (2019) find a positive association between cross-channel integration and cost efficiency.

In this paper, we consider that a multichannel retailer adopts a decentralized (centralized) organizational structure when the prices of online and offline channels are separately (jointly) set. Having channel prices set jointly to maximize a combined profit function can be considered partial centralization (integration), since many other aspects of the business may not be affected by such an arrangement as one would expect in an omnichannel structure.

2.3 Reference price

The last major topic associated with this article is consumer reference price. Consumer reference price is defined as the price at which a consumer compares the actual price before committing to purchase a product. Situations in which reference prices are higher (lower) than actual prices are considered favorable (unfavorable) to consumers and lead to an increase (decrease) in consumer demand (Thaler, 1985). It is now well established in the marketing literature that reference prices play a critical role in consumer purchasing decisions (Mazumdar et al., 2005).

Related to this research, there is a growing literature that documents the impact of reference prices on marketing strategies and profits within marketing channels and between competitive retailers. Consider research on marketing channels (e.g., Lin, 2016; Malekian and Rasti-Barzoki, 2019; Martín-Herrán and Sigué, 2023; Sun et al., 2022; Zhang et al., 2013

& 2014). For instance, a key finding in Zhang et al. (2014) is that channel members earn more profits when consumers have a higher initial reference price. Martín-Herrán and Sigué (2023), who used the first-period retail price as the second-period reference price in a two-period game, found that channel strategies and profits critically depend on how sensitive consumers are to price variations over time. Lin (2016) reported that, depending on the model parameters, channel profits may increase or decrease with the reference price effect. In the same line, Malekian and Rasti-Barzoki (2019) found that the reference price effect and the memory factor largely impacted the strategies and profits of channel members.

Research on competitive retailers also acknowledges the critical role of reference price on the players' strategies and profits (e.g., Li et al., 2023; Wang et al., 2021). Specifically, Wang et al. (2021), who studied price competition between online and offline retailers in the context where the price of one channel is used as the reference price of the two channels, found that each retailer would prefer consumers to use his price as the reference price for the two channels. Also, Li et al. (2023) studied price competition between online and offline retailers. In a strategic game where players have the choice to use or not to use an external reference price, it was found that the retailer who used the reference price alone earned more profits. In the case where the two retailers used reference prices, the one with a higher reference price earned more profits.

The operationalization of the concept of reference price varies considerably in the literature. In this paper, we follow Martín-Herrán and Sigué (2023) and consider that consumers use an internal reference price, which is the price of the product memorized from their last purchase. This basically means that for our two-period game, the reference price of a given channel in the second period is the price of the first period of that channel. Beyond the operationalization of the reference price concept, the major innovation of our paper is to consider the reference price effect in the choice of the organizational structure of a multichannel retailer.

3 Model

We consider a supply chain configuration where a manufacturer sells a product to a single retailer who can resell it to consumers through integrated online and offline channels or through two independent online and offline channels. The scenario in which the retailer operates two independent online and offline channels is identical to dealing with two retailers as the management of each channel aims at maximizing its profit without any coordination with the other. On the other hand, online and offline channels are horizontally integrated when marketing decisions are set so as to maximize the joint profit of the two channels. The manufacturer sets her marketing decision over a two-period planning horizon to take into account its long-term effects. Particularly, for each period i, $i \in \{1, 2\}$, the manufacturer sells the product to the retailer at the wholesale price of w_i and advertises at the rate of a_{Mi} to promote the product. In response, the retailer sells the product in each channel j, $j \in \{1, 2\}$ at the retail price, p_{ij}, to consumers. For simplicity, we consider the effects of other marketing activities that may be undertaken at the retail level, such as the service provided and local advertising, to be negligible.

As in many other papers in the marketing literature (e.g., Martín-Herrán and Sigué, 2017 & 2023), we assume linear demand functions for both online and offline channels. Specifically, the first-period demand functions are:

$$\text{First retailer or first channel: } q_{11} = g - p_{11} + \delta p_{12} + \phi a_{M1},$$
$$\text{Second retailer or second channel: } q_{12} = g - p_{12} + \delta p_{11} + \phi a_{M1}.$$

The second-period demand functions are:

$$\text{First retailer or first channel: } q_{21} = g - p_{21} + \delta p_{22} - \gamma(p_{21} - p_{11}) + \phi a_{M2} + \phi^2 a_{M1},$$
$$\text{Second retailer or second channel: } q_{22} = g - p_{22} + \delta p_{21} - \gamma(p_{22} - p_{12}) + \phi a_{M2} + \phi^2 a_{M1},$$

where the positive parameters g, δ, ϕ, and γ, respectively, represent the baseline demand of each channel at the start of the game, the cross-channel price effect on demand, the effect of manufacturer advertising on demand, and the reference price effect. The above demand functions are based on several assumptions. The major assumptions are discussed below.

First, both the online and offline channels have the same baseline demand. This assumption is used to obtain more compact analytical results given the complexity of the current model, but does not affect the tractability of the model and the qualitative results discussed in this paper. In reality, the baseline demand of online and offline channels heavily depends on the type of product sold. Some products, such as grocery products, have a higher market base offline, while other, such as books and music, have a greater market base online (Verhoef, 2021).

Second, the direct price effect in each channel on demand is normalized to 1. In addition, we also assume that the following two conditions on the price effects are met: $0 < \delta < 1$ and $0 < \gamma < 1$. These assumptions are very common in the marketing literature and aim at simplifying the model specification. They imply that a channel direct price effect at any moment has a greater impact on its demand than the price of the competitive channel and the variation of prices over time within the same channel.

Third, manufacturer advertising in a given period impacts current sales and post-advertising sales. The current effect is captured by parameter ϕ, whose values lie between 0 and 1, while the post-advertising effect is represented by ϕ^2. Our specification recognizes that advertising has carry-over effects, but these effects decrease exponentially over time (Martín-Herrán and Sigué, 2017a). Consequently, despite these carry-over effects, the manufacturer may have to continuously invest in advertising to sustain or grow sales over time as consumers tend to forget past advertising activities. This specification is similar to the use of the decay rate in Nerlove-Arrow type advertising models used in some differential game advertising models (Jørgensen et al. 2001).

Fourth, the component $\gamma(p_{2i} - p_{1i})$ in the second-period demand function of channel i, $i \in \{1, 2\}$, represents the internal reference price effect on the demand. It suggests that at the time of purchase in the second period, consumers compare the prices of the first and second periods. If the price of the second period is higher (perceived loss), consumers see it unfavorably and purchase less in the second period, while the reverse occurs when the second-period price is lower (perceived gain). In the case of a new product, for example, a penetration (skimming) pricing has a negative (positive) impact on the second-period sales if consumers use the first-period price as their internal reference. Our formulation of the internal reference price is a special case of Nerlove's (1958) adaptative expectations framework, in which the reference price (r_t) is a weighted average of past prices as follows: $r_t = \alpha r_{t-1} + (1 - \alpha)p_{t-1}$, $\alpha \in [0, 1)$, where α is considered as a memory factor. Therefore, by assuming that consumers have very short memories, $\alpha = 0$, we can see that they can only remember the price of the immediate past period, $r_t = p_{t-1}$ (Martín-Herrán and Sigué, 2023).

Fifth, another underlying assumption of our model is that the effect of the reference price on demand is symmetric. Despite suggestions from behavioral theories that perceived losses have greater impacts than perceived gains on consumer demand, empirical evidence in this regard is mixed, indicating that, in some cases, such an asymmetry does not exist or the asymmetry works the other way round (Kopalle et al., 1996; Krishnamurthi et al., 1992). In addition, asymmetric reference price effects add non trivial complications to marketing channel models. Therefore, we focus on the symmetric scenario as in Malekian and Rasti-Barzoki (2019), Martín-Herrán and Sigué (2023), and Zhang et al. (2013).

Sixth, in the same line, we consider that consumers react identically to prices and advertising in online and offline channels. This is another very common assumption used to simplify multichannel models and allow the derivation of insightful results (e.g., Karray et al., 2021). A more general specification should be able to handle situations where consumer reactions to prices and advertising are channel-specific.

Finally, as in most marketing channel models, we normalize all channel production, inventory, and administrative costs to zero. Only the manufacturer's advertising cost is factored in her profit function. Particularly, the manufacturer faces the following quadratic cost function in each period j, $\frac{1}{2}a_{Mj}^2$, $j \in \{1, 2\}$. This cost function suggests that as the manufacturer increases her advertising activities, advertising becomes more expensive. Taking that into account, the manufacturer sets the wholesale price and the level of advertising in each period j, $j \in \{1, 2\}$, to maximize her total profit function over the two periods:

$$\max_{w_j, a_{Mj}} \sum_{j=1}^{2} \left[w_j q_{j1} + w_j q_{j2} - \frac{1}{2} a_{Mj}^2 \right].$$

We assume that the manufacturer sets a single wholesale price for both channels, regardless of how they are managed by the retailer. Indeed, the product sold is identical and the manufacturer bears identical costs to serve the two channels.

On the other hand, depending on how the retailer manages both channels, his profit functions differ in each scenario. In the case where the two channels are horizontally integrated (Scenario 1), the retailer aims at maximizing the following profit function over the two periods, $j \in \{1, 2\}$:

$$\max_{p_{j1}, p_{j2}} \sum_{j=1}^{2} \left[(p_{j1} - w_j) q_{j1} + (p_{j2} - w_j) q_{j2} \right].$$

The retailer simultaneously sets retail prices for both channels to maximize their joint profit. The two channels may continue to compete on retail prices, but a central authority ensures that if different prices are set, they first aim at better serving the customers in each channel. Competition remains healthy and does not heavily damage the retailer's bottom line.

Alternatively, if the retailer runs her two channels as two separate divisions (Scenario 2), each channel will maximize its own profit over the two periods, $j \in \{1, 2\}$. The problem that channel i, $i \in \{1, 2\}$ is facing is as follows:

$$\max_{p_{ji}} \sum_{j=1}^{2} (p_{ji} - w_j) q_{ji}.$$

Each channel sets its price separately and therefore, horizontal competition can be intense as each channel cares exclusively about its own profit. As a result, channels are more likely to keep retail prices low to remain competitive. The retailer's total profit for the two channels over the two periods is therefore:

$$\sum_{j=1}^{2} (p_{j1} - w_j) q_{j1} + \sum_{j=1}^{2} (p_{j2} - w_j) q_{j2}.$$

Channel members' total profits over the two periods are mere additions of their profits from the two periods. As is very common in the literature, players do not discount their second-period profits (e.g., Martín-Herrán and Sigué, 2017 & 2023).

4 Equilibria

The retailer's decision to centralize or decentralize is based on the outcomes of the equilibrium strategies of the two scenarios. These equilibria are derived using the Stackelberg equilibrium concept. The sequence of moves for the entire game is summarized in Table 1, considering that for each of the two scenarios the players go through Stages 2 to 5.

Table 1: Sequence of moves

	Player	Decision
Stage 1	Retailer	Centralize/Decentralize
Stage 2	Manufacturer	w_1, a_{M1}
Stage 3	Retailer	p_{11}, p_{12}
Stage 4	Manufacturer	w_2, a_{M2}
Stage 5	Retailer	p_{21}, p_{22}

First, the game begins with the retailer announcing to centralize or decentralize pricing decisions. Second, the manufacturer reacts to the retailer's move and announces her first-period wholesale price and advertising strategies. Third, the retailer reacts to the manufacturer's first-period announcement and sets first-period retail prices for the online and offline channels as previously announced. Fourth, considering the retailer first-period pricing strategies, the manufacturer announces her second-period wholesale price and advertising strategies. Finally, in the fifth stage, the retailer sets the second-period retail prices for two channels as originally announced.

Subgame-perfect equilibrium solutions are obtained for the two scenarios by solving the games backwards (See Appendix). As a result, the retailer's second-period equilibrium strategies are first obtained before the derivation of the manufacturer's second-period strategies in each game. By considering the two players' second-period equilibrium strategies in either game, the retailer's first-period strategies are derived, allowing them to be incorporated into the manufacturer's first-period problem. Finally, the retailer decides whether or not to centralize knowing the players' strategies and profits for the two scenarios.

We have imposed the conditions ensuring that the strict concavity of the retailer's profit function with respect to his decision variable in each scenario, period, and equilibrium are satisfied. However, depending on the values of the model parameters, the manufacturer's profit function with respect to her first-period and second-period decision variables could be concave or not. As a result, the conditions that ensure that the problem admits an interior solution were identified (Equilibrium I). The corner solutions corresponding to situations where the manufacturer does not advertise at all or does only advertises in a single period have been characterized (Equilibria $II, III, \& IV$). The following two propositions summarize these four possible Stackelberg equilibrium solutions for Scenarios 1 and 2.

Proposition 1 *When the retailer sets retail prices jointly (Scenario 1) the game may have the following four equilibria depending on the game parameters:*

- *Equilibrium I^C: The manufacturer advertises in the two periods ($a_{M1} > 0,\ a_{M2} > 0$).*

  The second-period retail prices, p_{21} and p_{22}, the wholesale price, w_2, and the manufacturer's advertising rate in the second period, as functions of the first-period retail prices, p_{11} and p_{12}, and the manufacturer's advertising rate in the first period, a_{M1}, read:

$$p_{21} = \frac{6(\gamma+\delta+1)\left(a_{M1}\phi^2+g\right)+\gamma p_{11}\left(5\gamma+\delta-\phi^2+5\right)+\gamma p_{12}\left(\gamma+5\delta+\phi^2+1\right)}{4(\gamma+\delta+1)\left(2(\gamma-\delta+1)-\phi^2\right)}, \quad (1)$$

$$p_{22} = \frac{6(\gamma+\delta+1)\left(a_{M1}\phi^2+g\right)+\gamma p_{12}\left(5\gamma+\delta-\phi^2+5\right)+\gamma p_{11}\left(\gamma+5\delta+\phi^2+1\right)}{4(\gamma+\delta+1)\left(2(\gamma-\delta+1)-\phi^2\right)}, \quad (2)$$

$$w_2 = \frac{2\left(a_{M1}\phi^2+g\right)+\gamma(p_{11}+p_{12})}{4(\gamma-\delta+1)-2\phi^2}, \quad (3)$$

$$a_{M2} = \phi\frac{2\left(a_{M1}\phi^2+g\right)+\gamma(p_{11}+p_{12})}{4(\gamma-\delta+1)-2\phi^2}. \quad (4)$$

  The first-period retail prices, p_{11} and p_{12}, as functions of the wholesale price, w_1, and the manufacturer's advertising rate in the first period, a_{M1}, read:

$$p_{11} = p_{12} = \frac{-\gamma(\gamma-\delta+1)\left(a_{M1}\phi^2+g\right)-2\left(2(\gamma-\delta+1)-\phi^2\right)^2\left(a_{M1}\phi+g+(1-\delta)w_1\right)}{\gamma^2(\gamma-\delta+1)+4(\delta-1)\left(2(\gamma-\delta+1)-\phi^2\right)^2}. \quad (5)$$

  The first-period wholesale price, w_1, and manufacturer's advertising rate, a_{M1}, read:

$$w_1 = -g\frac{Numw_1}{Denw_1},$$
$$a_{M1} = -g\phi\frac{Numa_{M1}}{Dena_{M1}},$$

  where $Denw_1 = 2Dena_{M1}$. The expressions of $Numw_1$, $Numa_{M1}$ and $Dena_{M1}$ are long and omitted for brevity, and they are collected in the Appendix in expressions (51), (52) and (53), respectively.

- *Equilibrium II^C: The manufacturer exclusively advertises in the first period ($a_{M1} > 0,\ a_{M2} = 0$).*

  The second-period retail prices, p_{21} and p_{22}, the wholesale price, w_2, and the manufacturer's advertising rate in the second period, as functions of the first-period prices,

p_{11} and p_{12}, and the manufacturer's advertising rate in the first period, a_{M1}, read:

$$p_{21} = \frac{6(\gamma+\delta+1)\left(a_{M1}\phi^2+g\right)+\gamma(p_{11}(5\gamma+\delta+5)+p_{12}(\gamma+5\delta+1))}{8(\gamma-\delta+1)(\gamma+\delta+1)}, \quad (6)$$

$$p_{22} = \frac{6(\gamma+\delta+1)\left(a_{M1}\phi^2+g\right)+\gamma(p_{12}(5\gamma+\delta+5)+p_{11}(\gamma+5\delta+1))}{8(\gamma-\delta+1)(\gamma+\delta+1)}, \quad (7)$$

$$w_2 = \frac{2a_{M1}\phi^2+2g+\gamma(p_{11}+p_{12})}{4(\gamma-\delta+1)}, \quad (8)$$

$$a_{M2} = 0. \quad (9)$$

The first-period retail prices, p_{11} and p_{12}, as functions of the wholesale price, w_1, and the manufacturer's advertising rate in the first period, a_{M1}, read:

$$p_{11} = p_{12} = \frac{(a_{M1}\phi+g)(8(1-\delta)-\gamma(\phi+8))-\gamma g(1-\phi)-8(1-\delta)w_1(\gamma-\delta+1)}{\gamma^2-16(1-\delta)(\gamma-\delta+1)}. \quad (10)$$

The first-period wholesale price, w_1, and manufacturer's advertising rate, a_{M1} read:

$$w_1 = -g(\gamma-\delta+1)\frac{Numw_1}{Denw_1},$$
$$a_{M1} = -g\phi(\gamma-\delta+1)\frac{Numa_{M1}}{Dena_{M1}},$$

where $Denw_1 = 2Dena_{M1}$. The expressions of $Numw_1$, $Numa_{M1}$ and $Dena_{M1}$ are given by expressions (59), (60) and (61), respectively, in the Appendix.

- *Equilibrium III^C: The manufacturer exclusively advertises in the second period ($a_{M1} = 0,\ a_{M2} > 0$).*

  The second-period retail prices, p_{21} and p_{22}, the wholesale price, w_2, and the manufacturer's advertising rate in the second period, as functions of the first-period retail prices, p_{11} and p_{12}, and the manufacturer's advertising rate in the first period, a_{M1} are given by (1), (2), (3) and (4). The first-period retail prices, p_{11} and p_{12}, as functions of the wholesale price, w_1, and the manufacturer's advertising rate in the first period, a_{M1}, are given by (5).

  The first-period wholesale price, w_1, and manufacturer's advertising rate, a_{M1} read:

$$w_1 = -g\frac{Numw_1}{Denw_1},$$
$$a_{M1} = 0,$$

where the expressions of $Numw_1$ and $Denw_1$ are given by expressions (55) and (56), respectively, in the Appendix.

- *Equilibrium IV^C: The manufacturer does not advertise at any time ($a_{M1} = 0$, $a_{M2} = 0$).*

  The second-period retail prices, p_{21} and p_{22}, the wholesale price, w_2, and the manufacturer's advertising rate in the second period, as functions of the first-period retail prices, p_{11} and p_{12}, and the manufacturer's advertising rate in the first period, a_{M1} are given by (6), (7), (8) and (9). The first-period retail prices, p_{11} and p_{12}, as functions of the wholesale price, w_1, and the manufacturer's advertising rate in the first period, a_{M1}, are given by (10).

  The first-period wholesale price, w_1, and manufacturer's advertising rate, a_{M1} read:

$$\begin{aligned} w_1 &= -\frac{g\left(\gamma^3 + 8\gamma^2(\delta-1) + 144\gamma(\delta-1)^2 - 128(\delta-1)^3\right)}{32(\delta-1)^2\left(\gamma^2 + 8(\delta-1)(\gamma-\delta+1)\right)}, \\ a_{M1} &= 0. \end{aligned}$$

Proof. See the Appendix. In the Appendix we present the conditions ensuring the strict concavity of the objective functions of the retailer and the manufacturer in each period. ■

Proposition 2 *When the retailer sets retail prices separately (Scenario 2), the game may have the following four equilibria depending on the game parameters:*

- *Equilibrium I: The manufacturer advertises in the two periods ($a_{M1} > 0$, $a_{M2} > 0$).*

  The second-period retail prices, p_{21} and p_{22}, the wholesale price, w_2, and the manufacturer's advertising rate in the second period, as functions of the first-period retail prices, p_{11} and p_{12}, and the manufacturer's advertising rate in the first period, a_{M1}, read:

$$p_{21} = \frac{Nump_{21}}{4(2(\gamma+1)+\delta)\left((\gamma-\delta+1)(2(\gamma+1)-\delta)-(\gamma+1)\phi^2\right)}, \tag{11}$$

$$p_{22} = \frac{Nump_{22}}{4(2(\gamma+1)+\delta)\left((\gamma-\delta+1)(2(\gamma+1)-\delta)-(\gamma+1)\phi^2\right)}, \tag{12}$$

$$w_2 = \frac{(2\gamma-\delta+2)\left(2a_{M1}\phi^2 + 2g + \gamma(p_{11}+p_{12})\right)}{4(\gamma-\delta+1)(2\gamma-\delta+2) - 4(\gamma+1)\phi^2}, \tag{13}$$

$$a_{M2} = \frac{(\gamma+1)\phi\left(2a_{M1}\phi^2 + 2g + \gamma(p_{11}+p_{12})\right)}{2(\gamma-\delta+1)(2\gamma-\delta+2) - 2(\gamma+1)\phi^2}, \tag{14}$$

where

$$Nump_{21} = 2(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)\left(a_{M1}\phi^2+g\right)+\gamma\left((\gamma+1)(p_{11}-p_{12})\left(10(\gamma+1)-7\delta-2\phi^2\right)\right.$$
$$\left.+2p_{12}(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)\right),$$
$$Nump_{22} = 2(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)\left(a_{M1}\phi^2+g\right)+\gamma\left((\gamma+1)(p_{12}-p_{11})\left(10(\gamma+1)-7\delta-2\phi^2\right)\right.$$
$$\left.+2p_{11}(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)\right).$$

The first-period retail prices, p_{11} and p_{12}, as functions of the wholesale price, w_1, and the manufacturer's advertising rate in the first period, a_{M1}, read:

$$p_{11} = p_{12} = \frac{Nump_{11}}{Denp_{11}}, \tag{15}$$

where

$$Nump_{11} = -2(\gamma+1)\phi^2(\gamma-\delta+1)\left(a_{M1}(\gamma+1)\phi(\gamma(\phi+16)+16)-4a_{M1}\delta^2\phi+(17\gamma+16)(\gamma+1)g\right.$$
$$\left.-4\delta^2 g+16(\gamma+1)^2 w_1-4\delta^2 w_1\right)+(\gamma-\delta+1)^2\left(a_{M1}\phi\left(\gamma(\gamma+1)\phi(6\gamma-\delta+6)\right.\right.$$
$$\left.+4(2\gamma+\delta+2)(-2\gamma+\delta-2)^2\right)+g\left(-8(\gamma+1)\delta^2-(\gamma+1)(17\gamma+16)\delta+2(\gamma+1)^2(19\gamma+16)+4\delta^3\right)$$
$$\left.+4w_1(2\gamma+\delta+2)(-2\gamma+\delta-2)^2\right)+4(\gamma+1)^2\phi^4(2\gamma+\delta+2)(a_{M1}\phi+g+w_1),$$
$$Denp_{11} = 2(\gamma+1)\phi^2(\gamma-\delta+1)\left(\gamma^2(\gamma+1)-4(\delta-2)(-2\gamma+\delta-2)(2\gamma+\delta+2)\right)$$
$$-(\gamma-\delta+1)^2\left((\gamma+1)\gamma^2(6\gamma-\delta+6)+4(\delta-2)(2\gamma+\delta+2)(-2\gamma+\delta-2)^2\right)$$
$$-4(\gamma+1)^2(\delta-2)\phi^4(2\gamma+\delta+2).$$

The closed-form expressions of the first-period wholesale price, w_1, and manufacturer's advertising rate, a_{M1} have been obtained with the help of Mathematica 12.3 and are omitted here because these expressions are cumbersome.

- *Equilibrium II: The manufacturer exclusively advertises in the first period ($a_{M1} > 0,\ a_{M2} = 0$).*

  The second-period retail prices, p_{21} and p_{22}, the wholesale price, w_2, and the manufacturer's advertising rate in the second period, as functions of the first-period retail prices, p_{11} and p_{12}, and the manufacturer's advertising rate in the first period, a_{M1}, *read:*

$$
\begin{aligned}
p_{21} &= \frac{Nump_{21}}{4(\gamma-\delta+1)\left(4(\gamma+1)^2-\delta^2\right)}, & (16)\\
p_{22} &= \frac{Nump_{22}}{4(\gamma-\delta+1)\left(4(\gamma+1)^2-\delta^2\right)}, & (17)\\
w_2 &= \frac{2\left(a_{M1}\phi^2+g\right)+\gamma(p_{11}+p_{12})}{4(\gamma-\delta+1)-2\phi^2}, & (18)\\
a_{M2} &= \phi\frac{2\left(a_{M1}\phi^2+g\right)+\gamma(p_{11}+p_{12})}{4(\gamma-\delta+1)-2\phi^2}, & (19)
\end{aligned}
$$

where

$$
\begin{aligned}
Nump_{21} &= 2(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)\left(a_{M1}\phi^2+g\right)+\gamma\left((\gamma+1)p_{11}(10(\gamma+1)-7\delta)\right.\\
&\quad\left.+p_{12}\left(5(\gamma+1)\delta+2(\gamma+1)^2-4\delta^2\right)\right),\\
Nump_{22} &= 2(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)\left(a_{M1}\phi^2+g\right)+\gamma\left((\gamma+1)p_{12}(10(\gamma+1)-7\delta)\right.\\
&\quad\left.+p_{11}\left(5(\gamma+1)\delta+2(\gamma+1)^2-4\delta^2\right)\right).
\end{aligned}
$$

The first-period retail prices, p_{11} and p_{12}, as functions of the wholesale price, w_1, and the manufacturer's advertising rate in the first period, a_{M1}, read:

$$
p_{11} = p_{12} = \frac{Nump_{11}}{Denp_{11}}, \qquad (20)
$$

where

$$
\begin{aligned}
Nump_{11} &= a_{M1}\phi\left(\gamma(\gamma+1)\phi(-6\gamma+\delta-6)-4(-2\gamma+\delta-2)^2(2\gamma+\delta+2)\right)+g\left(8(\gamma+1)\delta^2\right.\\
&\quad\left.+(\gamma+1)(17\gamma+16)\delta-2(\gamma+1)^2(19\gamma+16)-4\delta^3\right)-4w_1(2\gamma+\delta+2)(-2\gamma+\delta-2)^2,\\
Denp_{11} &= (\gamma+1)\gamma^2(6\gamma-\delta+6)+4(\delta-2)(2\gamma+\delta+2)(-2\gamma+\delta-2)^2.
\end{aligned}
$$

The closed-form expressions of the first-period wholesale price, w_1, and manufacturer's advertising rate, a_{M1} are omitted due to their length.

- *Equilibrium III: The manufacturer exclusively advertises in the second period ($a_{M1} = 0,\ a_{M2} > 0$).*

The second-period retail prices, p_{21} and p_{22}, the wholesale price, w_2, and the manufacturer's advertising rate in the second period, as functions of the first-period retail prices, p_{11} and p_{12}, and the manufacturer's advertising rate in the first period, a_{M1} are given by (11), (12), (13) and (14). The first-period retail prices, p_{11} and p_{12}, as *functions of the wholesale price, w_1, and the manufacturer's advertising rate in the first period, a_{M1}, are given by (15).*

In this case, $a_{M1} = 0$ and the first-period wholesale price, w_1, is also omitted due to its length.

- *Equilibrium IV: The manufacturer does not advertise at any time ($a_{M1} = 0$, $a_{M2} = 0$).*

  The second-period retail prices, p_{21} and p_{22}, the wholesale price, w_2, and the manufacturer's advertising rate in the second period, as functions of the first-period retail prices, p_{11} and p_{12}, and the manufacturer's advertising rate in the first period, a_{M1}, are given by (16), (17), (18) and (19). The first-period retail prices, p_{11} and p_{12}, as functions of the wholesale price, w_1, and the manufacturer's advertising rate in the first period, a_{M1}, are given by (20).

  The first-period wholesale price, w_1, and manufacturer's advertising rate, a_{M1} read:

$$\begin{aligned} w_1 &= \frac{gNumw_1}{Denw_1}. \\ a_{M1} &= 0, \end{aligned}$$

  where the expressions of $Numw_1$ and $Denw_1$ are given by (85) and (86).

Proof. See the Appendix. In the Appendix we present the conditions ensuring the strict concavity of the objective functions of the retailer and the manufacturer in each period. ■

Propositions 1 and 2 suggest that, at the equilibrium, the manufacturer may adopt four different advertising schedules. Equilibrium I or I^C corresponds to a continuous advertising schedule known to be effective when advertising produces a limited carryover effect (Martín-Herrán and Sigué, 2017a). Equilibrium II or II^C corresponds to the pulsing schedule where the advertiser alternates between high and zero levels of advertising. This advertising schedule is known to be more effective than the continuous schedule for specific demand functions and when the advertising carryover effect is important as the advertiser can reduce advertising costs by taking an advertising break, while still benefitting of the effects of previous advertising on sales (Martín-Herrán and Sigué, 2017a). In Equilibrium III or III^C, the manufacturer postpones the start of advertising to the second period when advertising has a minimal impact on current sales (Martín-Herrán and Sigué, 2017b). The adoption of Equilibrium IV or IV^C means that the manufacturer does not need advertising to achieve her profit goals. They can be achieved by properly setting the wholesale price which will induce the retailer to set retail prices to reach the level of sales that guarantees maximum profits to the manufacturer.

Additionally, we further explore how prices and advertising in the second period vary with prices and advertising in the first period. The following proposition summarizes the results of this analysis.

Proposition 3 *When the retailer sets retail prices either jointly (Scenario 1) or separately (Scenario 2), wholesale and retail prices and advertising in the second period change with retail prices and advertising in the first period as follows:*

- *Regardless of the equilibrium played (equilibrium I^C, II^C, III^C, or IV^C in Scenario 1 or equilibrium I, II, III, or IV in Scenario 2), the second-period retail and wholesale prices increase with the first-period retail prices and advertising.*

$$\frac{\partial p_{21}}{\partial p_{11}} = \frac{\partial p_{22}}{\partial p_{12}} > 0, \quad \frac{\partial p_{21}}{\partial p_{12}} = \frac{\partial p_{22}}{\partial p_{11}} > 0, \quad \frac{\partial p_{21}}{\partial a_{M1}} = \frac{\partial p_{22}}{\partial a_{M1}} > 0,$$
$$\frac{\partial w_2}{\partial p_{11}} = \frac{\partial w_2}{\partial p_{12}} > 0, \quad \frac{\partial w_2}{\partial a_{M1}} > 0.$$

- *When either Equilibrium I^C or Equilibrium III^C is played in Scenario 1 or either Equilibrium I or Equilibrium III is played in Scenario 2, second-period advertising increases with first-period retail prices and advertising (when possible).*

$$\frac{\partial a_{M2}}{\partial a_{M1}} > 0, \quad \frac{\partial a_{M2}}{\partial p_{11}} = \frac{\partial a_{M2}}{\partial p_{12}} > 0.$$

- *Regardless of the equilibrium played (equilibrium I^C, II^C, III^C, or IV^C in Scenario 1 or equilibrium I, II, III, or IV in Scenario 2),*
  - *The change in second-period retail prices with first-period advertising is greater than the change in second-period wholesale prices with first-period advertising:*

$$\frac{\partial p_{21}}{\partial a_{M1}} = \frac{\partial p_{22}}{\partial a_{M1}} > \frac{\partial w_2}{\partial a_{M1}}.$$

- *The change in second-period retail prices with first-period retail prices is greater than the change in second-period wholesale prices with first-period retail prices:*

$$\frac{\partial p_{21}}{\partial p_{11}} = \frac{\partial p_{22}}{\partial p_{12}} > \frac{\partial w_2}{\partial p_{11}} = \frac{\partial w_2}{\partial p_{12}}.$$

- *The effect on the second-period retail price of a channel of a change in the first-period retail price of this channel is stronger than the effect of a change in the first-period retail price of the other channel:*

$$\frac{\partial p_{21}}{\partial p_{11}} = \frac{\partial p_{22}}{\partial p_{12}} > \frac{\partial p_{21}}{\partial p_{12}} = \frac{\partial p_{22}}{\partial p_{11}}.$$

- *When either Equilibrium I^C or Equilibrium III^C in Scenario 1 is played:*

  - *The effect of a change in the first-period retail price of a channel on the second-period retail price of the other channel can be stronger or weaker than the effect on the second-period wholesale price depending on whether ϕ^2 is greater or lower than $1 + \gamma - 3\delta$:*

$$\frac{\partial p_{21}}{\partial p_{12}} = \frac{\partial p_{22}}{\partial p_{11}} > \frac{\partial w_2}{\partial p_{11}} = \frac{\partial w_2}{\partial p_{12}} \quad \textit{if and only if} \quad \phi^2 > 1 + \gamma - 3\delta.$$

  - *The effect of a change in the first-period retail prices or in the manufacturer's advertising rate in the first period on the second-period wholesale price is stronger than the effect on manufacturer's advertising rate in the second-period:*

$$\frac{\partial w_2}{\partial p_{11}} = \frac{\partial w_2}{\partial p_{12}} > \frac{\partial a_{M2}}{\partial p_{11}} = \frac{\partial a_{M2}}{\partial p_{12}}, \quad \frac{\partial w_2}{\partial a_{M1}} > \frac{\partial a_{M2}}{\partial a_{M1}}.$$

- *When either Equilibrium I or Equilibrium III in Scenario 2 is played:*

  - *The effect of a change in the first-period retail price of a channel on the second-period retail price of the other channel can be stronger or weaker than the effect* *on the second-period wholesale price depending on whether ϕ^2 is greater or lower than $1+\gamma+\delta(3\delta-5)/(2(\gamma+1))$:*

$$\frac{\partial p_{21}}{\partial p_{12}} = \frac{\partial p_{22}}{\partial p_{11}} > \frac{\partial w_2}{\partial p_{11}} = \frac{\partial w_2}{\partial p_{12}} \quad \text{if and only if} \quad \phi^2 > 1+\gamma+\frac{\delta(3\delta-5)}{2(\gamma+1)}.$$

- *The effect of a change in the first-period retail prices or in the manufacturer's advertising rate in the first period on the second-period wholesale price is stronger or softer than the effect on manufacturer's advertising rate in the second-period depending on whether δ is lower or greater than $2(1+\gamma)(1-\phi)$:*

$$\frac{\partial w_2}{\partial p_{11}} = \frac{\partial w_2}{\partial p_{12}} > \frac{\partial a_{M2}}{\partial p_{11}} = \frac{\partial a_{M2}}{\partial p_{12}}, \quad \frac{\partial w_2}{\partial a_{M1}} > \frac{\partial a_{M2}}{\partial a_{M1}} \quad \text{if and only if} \quad \delta < 2(1+\gamma)(1-\phi).$$

- *When either Equilibrium II^C or Equilibrium IV^C in Scenario 1 is played, the effect of a change in the first-period retail price of a channel on the second-period retail price of the other channel can be stronger or weaker than the effect on the second-period wholesale price depending on whether δ is greater or lower than $(1+\gamma)/3$:*

$$\frac{\partial p_{21}}{\partial p_{12}} = \frac{\partial p_{22}}{\partial p_{11}} > \frac{\partial w_2}{\partial p_{11}} = \frac{\partial w_2}{\partial p_{12}} \quad \text{if and only if} \quad \delta > \frac{1+\gamma}{3}.$$

- *When either Equilibrium II or Equilibrium IV in Scenario 2 is played, the effect of a change in the first-period retail price of a channel on the second-period retail price of the other channel can be stronger or weaker than the effect on the second-period wholesale price depending on whether δ is greater or lower than $2(1+\gamma)/3$:*

$$\frac{\partial p_{21}}{\partial p_{12}} = \frac{\partial p_{22}}{\partial p_{11}} > \frac{\partial w_2}{\partial p_{11}} = \frac{\partial w_2}{\partial p_{12}} \quad \text{if and only if} \quad \delta > \frac{2(1+\gamma)}{3}.$$

Proposition 3 reveals that in most cases, increases in first-period decisions (retail prices and advertising), either on the part of the manufacturer or retailer, positively affect second-period decisions. However, increases in first-period advertising affect more second-period retails prices than the second-period wholesale price. All things being equal, this means that the manufacturer's advertising benefits the retailer more in the long term in terms of higher margins than the manufacturer itself. Similarly, a change in first-period retail prices affects retail prices more than the wholesale price in the second period. Changes in a channel's retail price in the first period have a stronger impact on its own retail price in the second period than on its rival channel's second-period retail price, but this change can be stronger or weaker than the impact on the manufacturer's second-period wholesale price. Consequently, channel members' first-period decisions play an important role on their subsequent decisions. This is expected given that we model both the reference price effect and the advertising carryover effect, two phenomena that take into account the long-term effects of retail prices and advertising of the first period.

The existence of four possible equilibria poses the problem of the choice of the equilibrium to be adopted or preferred by the players. The next two subsections focus on this problem.

4.1 Centralized pricing decisions

Figures 1-3 compare channel members' profits for the four equilibria when the retailer centralizes retail prices for different values of the parameter ϕ, $\phi \in \{0.1, 0.5, 0.9\}$, while the values of the parameters δ and γ vary between 0 and 1. An area in a figure is labelled as i, $i \in \{I, II, III, IV\}$, to indicate that the manufacturer (retailer) attains the greatest profits in the area by playing Equilibrium i. These figures reveal that the preferences of the two players with respect to these four equilibria change considerably with the values of the three parameters.

Because the decision to advertise or not rests exclusively with the manufacturer, the following development focuses on her preferences. In particular, Figure 1 (left) shows that when the effect of manufacturer advertising on demand is very small, $\phi = 0.1$, the manufacturer exclusively implements Equilibrium I in which she advertises in both periods, regardless of price competition between channels and the reference price effect. This result is consistent with previous studies, which acknowledge the supremacy of the continuous advertising schedule when the carryover effect is small (e.g., Martín-Herrán and Sigué, 2017a). In the context of this research, a small current advertising effect leads to an even smaller carryover effect, as advertising decays exponentially over time.

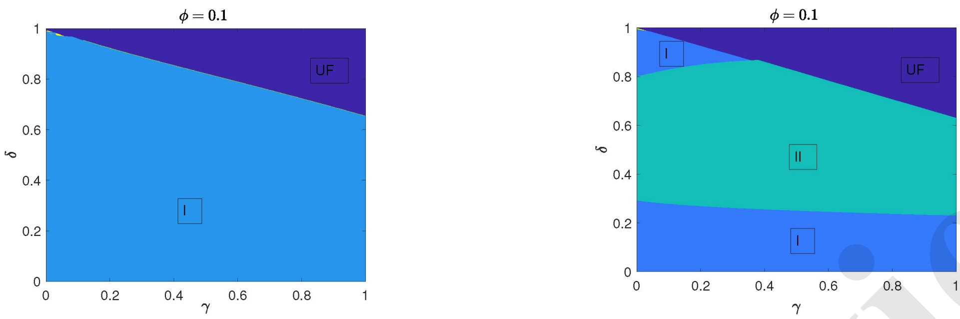


Figure 1: Scenario 1. Comparison of manufacturer's (left) and retailers' (right) profits in the different equilibria. $\phi = 0.1$

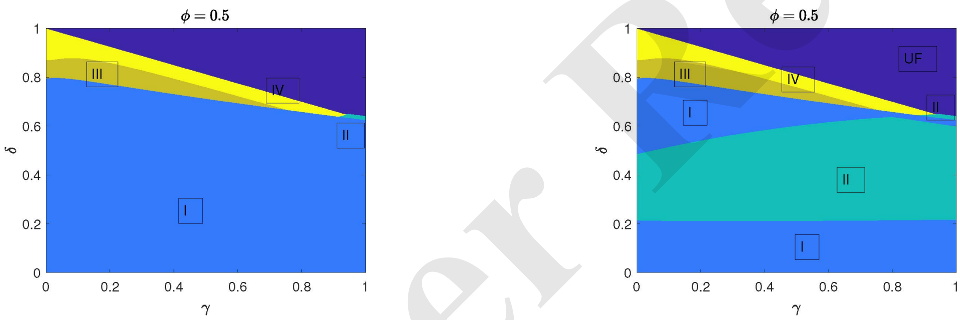


Figure 2: Scenario 1. Comparison of manufacturer's (left) and retailers' (right) profits in the different equilibria. $\phi = 0.5$

On the other hand, it can be easily seen from the other figures that, as the current effect of the manufacturer's advertising as well as its carryover effect (albeit at a low rate) on demand increases, the area in the parameter space where the manufacturer prefers Equilibrium I decreases to make room for the other equilibria. For example, in Figure 2 (left) when $\phi = 0.5$, Equilibrium I is still one of the favorites, but the other equilibria are also chosen, particularly, when the intensity of price competition between channels (δ) is relatively high. The manufacturer can choose either to not advertise at all (Equilibrium IV) or to advertise only in the first period (Equilibrium II) or only in the second period (Equilibrium III). Relative to equilibrium I, thanks to high advertising effectiveness, any of these three equilibria can help reduce the manufacturer's advertising investments and increase its

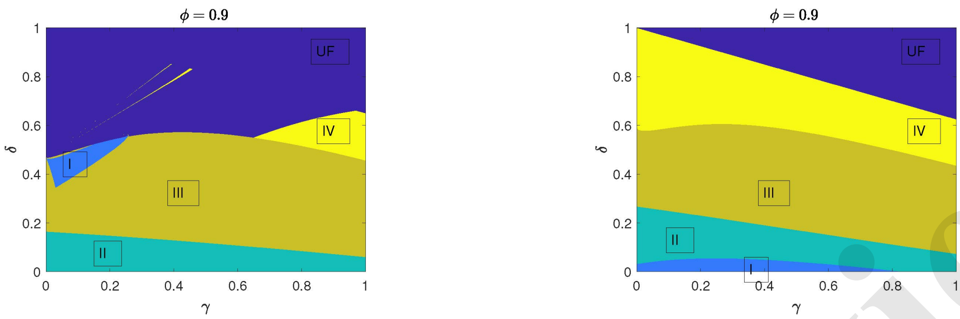


Figure 3: Scenario 1. Comparison of manufacturer's (left) and retailers' (right) profits in the different equilibria. $\phi = 0.9$

profitability if the retailer sets appropriate prices that maintain or increase total sales. In Figure 3 (left), when $\phi = 0.9$, the manufacturer only implements Equilibrium I in a very small area where the reference price effect (γ) is small and the intensity of price competition between channels (δ) is moderate. The manufacturer implements Equilibrium II when the intensity of price competition between channels is relatively small and Equilibrium IV when the intensity of price competition between channels is moderate and the reference price effect is very high, meaning that the impact on a channel second-period demand is higher (lower) when the first-period price is higher (lower) than that of the second-period. Between the areas occupied by these previous three equilibria, the largest space is reserved for Equilibrium III where the manufacturer exclusively advertises in the second period of the game. Equilibrium III ensures that only retail prices affect the first period's demand and therefore there is no advertising carryover effect in the second period.

The findings in Figures 1, 2, and 3 (left) support the view that, if the choice of an advertising schedule by the manufacturer is generally linked to the advertising carryover effect, in this configuration, the carryover effect alone may not be enough, especially when the advertising impact on both current and future sales becomes significant. In such a context, when setting pricing and advertising strategies, the manufacturer should consider not only how to take advantage of high advertising effectiveness but also how consumers react to price changes over time and the price competition between the retailer's online and offline channels.This is so important that the manufacturer's advertising decisions can go against the interests of the retailer, as can be seen in some areas of figures 1, 2, and 3 (left and right), and generate channel conflicts.

4.2 Decentralized pricing decisions

Figures 4-6 compare channel members' profits for the four equilibria when the retailer decentralizes retail pricing for different values of the parameter ϕ, $\phi \in \{0.1, 0.5, 0.9\}$. They show that players' preferences with respect to these four equilibria change considerably with the values of the three parameters considered.

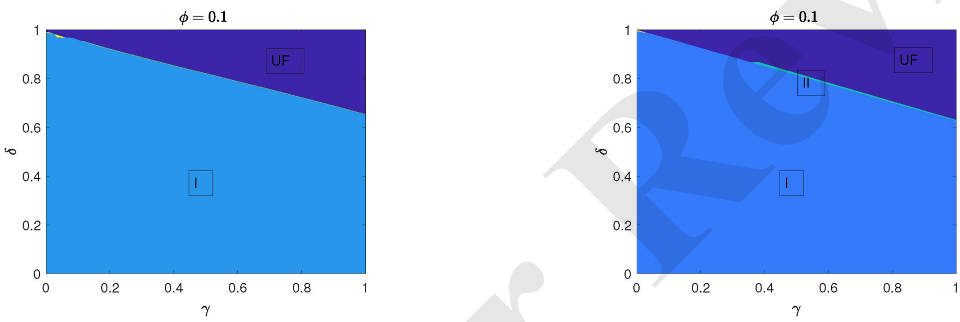


Figure 4: Scenario 2. Comparison of manufacturer's (left) and retailers' (right) profits in the different equilibria. $\phi = 0.1$

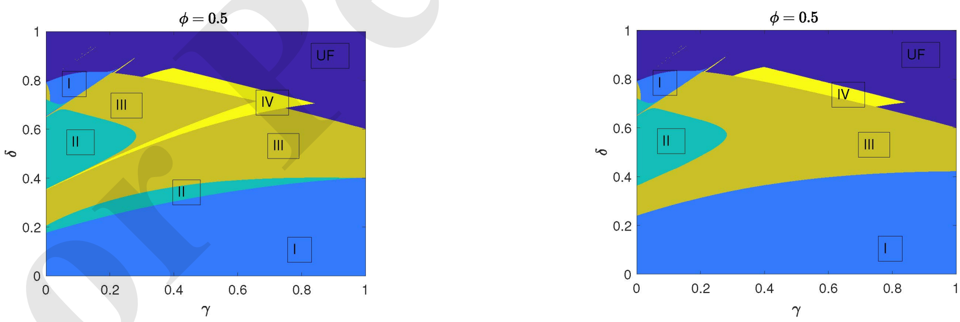


Figure 5: Scenario 2. Comparison of manufacturer's (left) and retailers' (right) profits in the different equilibria. $\phi = 0.5$

The results reported in Figures 4-6 are qualitatively similar to those of Figures 1-3 discussed previously. Specifically, the manufacturer exclusively implements Equilibrium I,

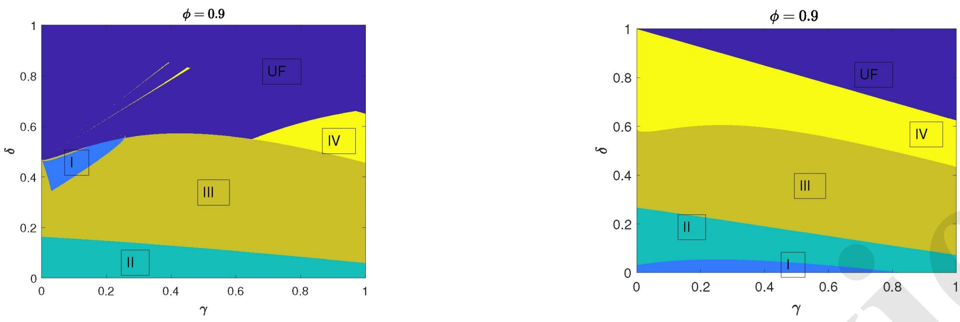


Figure 6: Scenario 2. Comparison of manufacturer's (left) and retailers' (right) profits in the different equilibria. $\phi = 0.9$

which corresponds to a continuous advertising schedule where advertising is carried out over both periods, regardless of price competition between channels and the reference price effect when advertising minimally impacts current and future sales ($\phi = 0.1$). The manufacturer cannot rely on the brand image built with previous advertising to stop advertising and reduce advertising investments. As the advertising impacts on both current and future sales become significant ($\phi = 0.50$ *and* $\phi = 0.9$), the manufacturer may adopt any of the four equilibria depending on the reference price effect and the price competition between online and offline channels, but Equilibrium *I* becomes marginal in the parameter space. Indeed, increasing the effectiveness of advertising may allow the manufacturer to completely or temporarily stop advertising to improve her profitability, taking into account the retailer's pricing strategies. Additionally, as in the context of centralized channels, the preferences of the manufacturer and retailer are not always aligned.

5 Choosing a multichannel retail organization

This section focuses on the first stage of the game, which is the retailer's decision to centralize or decentralize pricing decisions for both channels. As noted previously, the retailer knows the outcome of both scenarios at this point and selects the one that provides the greatest profits. The resulting question therefore is: Given the players' strategies for each of the two scenarios, how should the retailer set his retail prices? To answer this question, we first simultaneously compare the eight possible equilibria of the two scenarios where the retailer jointly or separately sets retail prices for both online and offline channels to identify the preferences of the two players, then examine the equilibria to be implemented. Hereafter, Equilibria I, II, III, and IV in Scenario 1 are denoted I^C, II^C, III^C, and IV^C, while in Scenario 2, they are denoted I, II, III, and IV. Figures 7-12 summarize this analysis for the following fixed values of the parameter ϕ, $\phi \in \{0.1, 0.5, 0.9\}$, while the values of the parameters δ and γ vary between 0 and 1.

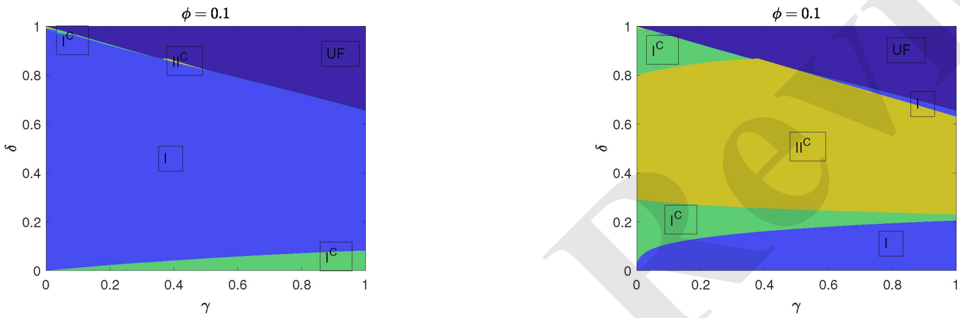


Figure 7: Scenarios 1 and 2. Comparison of manufacturer's (left) and retailers' (right) profits in the different equilibria. $\phi = 0.1$

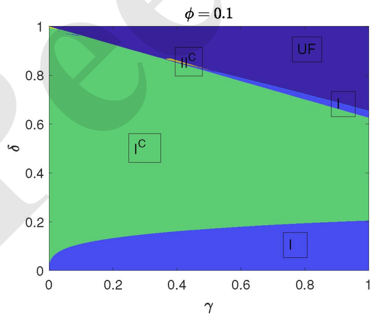


Figure 8: Different implemented equilibria. $\phi = 0.1$.

Figures 7, 9, and 11 show that the preferences of the players for the eight equilibria can converge or diverge and depend on the values of the model parameters. For instance, when the effect of manufacturer advertising on current demand is very small ($\phi = 0.1$), the manufacturer predominately prefers I, while the retailer prefers II^C in most areas of the parameter space. Given that the choice of the advertising schedule is out of the retailer's

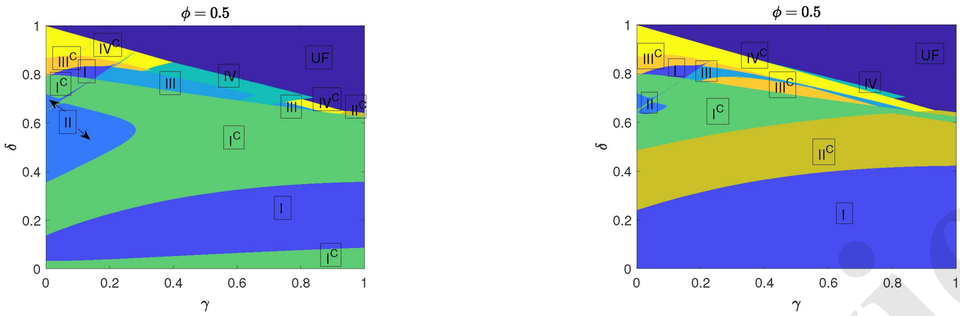


Figure 9: Scenarios 1 and 2. Comparison of manufacturer's (left) and retailers' (right) profits in the different equilibria. $\phi = 0.5$

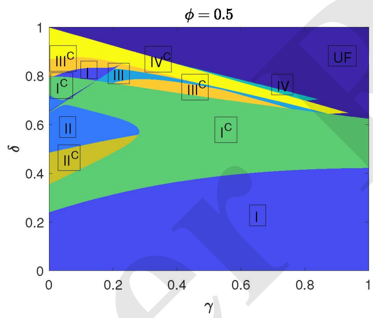


Figure 10: Different implemented equilibria. $\phi = 0.5$.

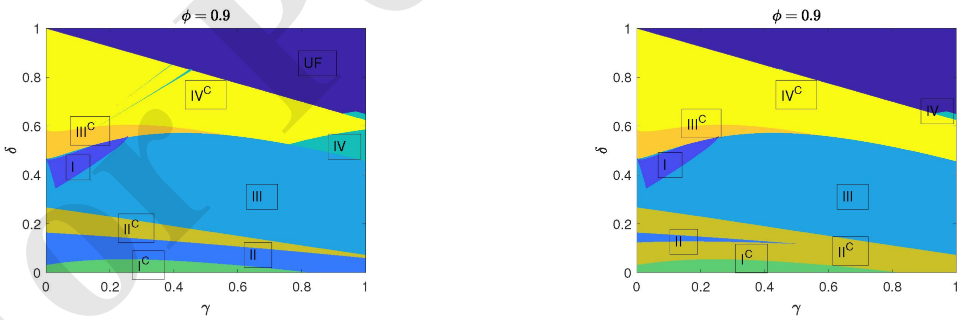


Figure 11: Scenarios 1 and 2. Comparison of manufacturer's (left) and retailers' (right) profits in the different equilibria. $\phi = 0.9$

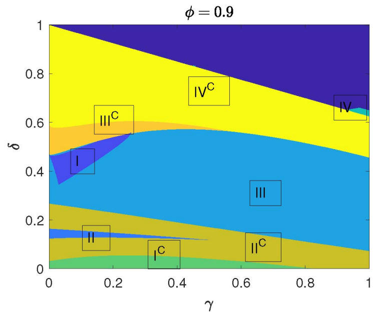


Figure 12: Different implemented equilibria. $\phi = 0.9$.

control, in areas where the advertising schedule preferences between the two players diverge, the manufacturer implements its preferred choice (in this case I), leaving the retailer with only the choice to implement either I or I^C as shown in Figure 8. Observe that in this case the retailer's choice to jointly or separately set retail prices is exclusively intended to maximize his own profits and may not meet the expectations of the manufacturer. On the other hand, in the small area where the manufacturer's preference for the pulsing schedule (II^C) meets the expectations of the retailer, the latter implements it to their mutual benefit.

The cases where the effects of manufacturer advertising are relatively ($\phi = 0.5$) or very ($\phi = 0.9$) large reveal are situations in which all eight equilibria can be implemented (Figures 10 and 12). Focusing on $\phi = 0.9$, one can easily see that the manufacturer's advertising schedule preferences are aligned with those of the retailer who then chooses to price jointly or separately in accordance with or against the manufacturer's expectations. In fact, when price competition between channels is relatively high, the two players prefer IV and IV^C where the manufacturer does not advertise at any time. The manufacturer also supports jointly set retail prices in most parts, except where the reference price effect is relatively high. In this case, players have areas of divergence where each of them favors prices set separately. When price competition between channels is moderate, player preferences for I, III, and III^C converge. Equilibria I and III^C are preferred in small areas where the reference price effect is relatively small, while III is adopted in all other areas, regardless of the value of the reference price effect parameter. Finally, when price competition between channels is relatively small, II, I^C, and II^C are implemented, but in some areas, the retailer's choice to jointly or separately set the retail prices differs from the preferences of the manufacturer who expects the opposite. Unfortunately for the manufacturer, after deciding which advertising schedule to adopt, she has no control over how the retailer sets his retail prices. The equilibria to be implemented in Figure 12 are those which maximize the profit of the retailer.

6 Conclusion

This paper analyzes whether a multichannel retailer should centralize or decentralize retail pricing decisions for his online and offline channels over a two-period planning horizon in a vertical channel context, where the manufacturer controls both the wholesale price and advertising decisions. It has been shown that whether the retailer centralizes (Scenario 1) or decentralizes (Scenario 2) retail pricing decisions for the two channels, the manufacturer can endogenously adopt the following four advertising arrangements: The manufacturer advertises in each period of the game (Equilibrium I); the manufacturer exclusively advertises in the first period (Equilibrium II); the manufacturer exclusively advertises in the second period (Equilibrium III); and the manufacturer does not advertise at any time (Equilibrium IV). The manufacturer's decision to implement either one of the four equilibria in a given scenario essentially depends on the manufacturer advertising effects, the intensity of price competition between channels, and the reference price effect or consumer sensitivity to price variations over time. As to whether or not the retailer should centralize or decentralize retail pricing across channels, the answer is not straightforward and depends on the advertising arrangement chosen by the manufacturer and the parameters of the model.

The findings of this research expand the existing literature on the organization of multichannel retailing to include vertical interactions between channel members and consumer sensitivity to price variations over time (reference price effect). In particular, they show that the conventional wisdom, which recommends cross-channel integration rather than channel decentralization as a way to improve multichannel retailer performance, is not a panacea. Although the context of multichannel retailing is different in several aspects, this finding is consistent with research findings on dealer integration in marketing channels (e.g., Martín-Herrán et al., 2014; Kalnins and Lafontaine, 2004). In some circumstances, marketing decentralization is the best organizational structure for multichannel retailers. As a matter of fact, according to Barnes & Noble CEO James Daunt cited by Bomey (2023), the recent success of this company has been its strategy to run a retail chain as a sum of independent retailers, where local managers, instead of corporate staff, are empowered to make decisions about their stores to better respond to the needs of local customers. This also explains the company's commitment to separate online and offline operations. Saks Fifth Avenue's decision in 2021 to move from an integrated channel structure to two separately managed online and offline companies is also part of a desire to better meet the expectations of different stakeholders, including investors (Latona, 2022; Scott, 2021).

The findings of our research also support the view that, rather than viewing the choice of organizational form as something over which multichannel retailers have full control, one should be aware that any organization form they choose generates responses from manufacturers that influence its profitability. Ignoring some of these potential responses, as in Berger et al. (2006), normally leads to suboptimal strategies and profits. This is well demonstrated in this research where, beyond the wholesale price adjustments that the manufacturer can make, she also plans her advertising investments over time according to the organizational choice expected from the retailer.

Another important takeaway of this research is that the reference price effect also plays a role in the choice of the organizational structure of multichannel retailers. In particular, this implies that the decision to decentralize or centralize offline and online pricing decisions should take into account how customers react to price variations over time. This finding is consistent with previous research in the channel marketing literature which concludes that the reference price effect impacts both channel marketing strategies and profitability (e.g., Lin, 2016; Malekian and Rasti-Barzoki, 2019; Martín-Herrán and Sigué, 2023; Sun et al., 2022).

Finally, channel centralization (integration) and decentralization are used very narrowly in this paper to mean that a multichannel retailer jointly and separately sets retail prices for his online and offline channels. Therefore, the results of this research cannot be generalized to situations where full integration of marketing channels, as envisioned in the omnichannel literature, is expected. In addition, our model relies on several other simplifying assumptions that could be relaxed in future work. In particular, we consider the online and offline channels to be symmetric, meaning that pricing and advertising decisions have similar effects on both channels, while it has been established that consumer price expectations on channels often differ (Zhang et al., 2010). This assumption explains why retail channel marketing activities are limited to pricing. One can consider scenarios where retail channels engage in other marketing activities, such as advertising and services, to further differentiate themselves or generate horizontal free-riding considerations (Rangaswamy and Van Bruggeen (2005)). Other formulations of reference price can also be considered. For instance, the price of one channel can serve as the reference price for both channels (e.g., Wang et al. 2021).

References

[1] Avery, J., Steenburgh, T.J., Deighton, J., & Caravella, M. (2012). Adding Bricks to Clicks: Predicting the patterns of cross-channel elasticities over time. *Journal of Marketing,* 76(3), 96-111.

[2] Bomey, N. (2023). How Barnes & Noble CEO rescued the bookstore chain. https://www.axios.com/2023/03/01/barnes-and-noble-james-duant-ceo

[3] Berger, P.D., Lee, J., & Weinberg, B.D. (2006). Optimal cooperative advertising integration strategy for organizations adding a direct online channel. *Journal of the Operational Research Society*, 57, 920-927.

[4] Cao, L., & Li, L. (2015). The impact of cross-channel integration on retailer's sales growth. *Journal of Retailing*, 91(2), 198-216.

[5] Cavallo, A. (2017). Are online and offline prices similar? Evidence from large multi-channel retailers. *American Economic Review*, 107(1), 283-303.

[6] Cheng, Y., & Xiong, Z. (2015). Strategic conditions for opening an Internet store and pricing policies in a retailer-dominant supply chain. *Mathematical Problems in Engineering*. https://doi.org/10.1155/2015/640719

[7] Gulati, R., & Garino, J. (2000). Get the right mix of bricks & clicks. *Harvard Business Review*, May-June, 107-114.

[8] Jeuland, A.P., & Shugan, S.M. (2008). Managing channel profits. *Marketing Science*, 27(1), 52-69.

[9] Jørgensen, S., Sigué, S. P., & Zaccour, G. (2001). Stackelberg leadership in a marketing channel. *International Game Theory Review*, 3(1), 1–14.

[10] Jørgensen, S., & Zaccour, G. (2014). A survey of game-theoretic models of cooperative advertising. *European Journal of Operational Research*, 237, 1–14.

[11] Kalnins, A., & Lanfontaine, F. (2004). Multi-Unit ownership in franchising: Evidence from the fast-food industry in Texas. *The RAND Journal of Economics,* 35(4), 747-761.

[12] Karray, S., Martín-Herrán, G., & Sigué, S. P. (2017). Cooperative advertising for competing manufacturers: The impact of long-term promotional effects. *International Journal of Production Ecomomics*, 184, 21-32.

[13] Karray, S., Martín-Herrán, G., & Sigué, S. P. (2021). Cooperative advertising in competing supply chains and the long-term effects of retailer advertising. J*ournal of the Operational Research Society*. https://doi.org/10.1080/01605682.2021.1973350

[14] Karray, S., Martín-Herrán, G., & Sigué, S. P. (2022). Managing advertising investments in marketing channels. *Journal of Retailing and Consumer Services*, 65. Art. No. 102852. https://doi.org/10.1016/j.jretconser.2021.102852.

[15] Karray, S., & Sigué, S.P. (2018). Offline retailers expanding online to compete with manufacturers: Strategies and channel power. *Industrial Marketing Management*, 71, 203-214.

[16] Kopalle, P., Rao, A.G., & Assunção, J. (1996). Asymmetric reference price effects and dynamic pricing policies. *Marketing Science*, 15 (1), 60–85.

[17] Krishnamurthi, L., Mazumdar, T., & Raj, S.P. (1992). Asymmetric response to price in consumer brand choice and purchase quantity decisions. *Journal of Consumer Research*, 19 (3), 387–400.

[18] Latona, D. (2022). Online vs. store operations: Breaking up is hard to do. https://imanetwork.org/online-vs-store-operations-breaking-up-is-hard-to-do/ (Accessed October 28, 2024)

[19] Li, H., Qian, Y., & Peng, T. (2021). How does the reference price mechanism affect competition? *IEEE Access*, 10, 26637-26653. DOI: 10.1109/ACCESS.2021.3131469.

[20] Li, H., Yang, K., & Zhang, G. (2023). Optimal pricing strategy in a dual-channel supply chain: A two-period game analysis. *Journal of Industrial and Management Optimization*, 19(4), 2955-2985. Doi: 10.3934/jimo.2022072.

[21] Lin, Z. (2016). Price promotion with reference price effects in supply chain. *Transportation Research Part E*, 85, 52-68.

[22] Malekian, Y., & Rasti-Barzoki, M. (2019). A game theoretic approach to coordinate price promotion and advertising policies with reference price effects in a two-echelon supply chain. *Journal of Retailing and Consumer Services*, 51, 114-128.

[23] Martín-Herrán, G., Sigué, S. P., & Zaccour, G. (2014). Downstream horizontal integration and multiunit dealership. *International Transactions in Operational Research*, 21, 81-101.

[24] Martín-Herrán, G., & Sigué, S. P. (2017a). Retailer and manufacturer advertising scheduling in a channel. *Journal of Business Research*, 70, 67-73.

[25] Martín-Herrán, G., & Sigué, S. P. (2017b). An integrative framework of cooperative advertising: Should manufacturers continuously support retailer advertising? *Journal of Business Research*, 70, 67-73.

[26] Martín-Herrán, G., & Sigué, S. P. (2023). An integrative framework of cooperative advertising with reference price effects. *Journal of Retailing and Consumer Services*, 70, 103166, https://doi.org/10.1016/j.jretconser.2022.103166.

[27] Mazumdar, T., Raj, S.P., & Sinha, I. (2005). Reference price research: Review and propositions. *Journal of Marketing*, 69,84-102.

[28] Nerlove, M. (1958). Adaptive expectations and cobweb phenomena. *Quarterly Journal of Economics*, 72, 227-240.

[29] Neslin, S.A., Grewal, D., Leghorn, R., Shankar, V., Teerling, M.L., Thomas, J.S., Verhoef, P.C. (2006). Challenges and opportunities in multichannel management. *Journal of Service Research*, 9(2), 95-112.

[30] Oh, L.-B., Teo, H.-H., & Sambamurthy, V. (2012). The effects of retail channel integration through the use of information technologies on retailer performance. *Journal of Operations Management*, 30(5), 368-381.

[31] Prakash, D., & Spann, M. (2022). Dynamic pricing and reference price effects. *Journal of Business Research*, 152, 300-314.

[32] Rangaswamy, A., & Van Bruggen, G.H. (2005). Opportunities and challenges in multichannel marketing: An introduction to the special issue. *Journal of Interactive Marketing*, 19(2), 5-11.

[33] Scott, I. (2021). Separating stores and online operations. IWD Magazine. https://magazine.iwd.io/separating-stores-and-online-operations.(Accessed October 28, 2024).

[34] Sigué, S.P., & Chintagunta, P. (2009). Advertising strategies in a franchise system. *European Journal of Operational Research*, 198(2), 655-665.

[35] Sun, L., Jiao, X., Guo, X., & Yu, Y. (2022). Pricing policies in dual distribution channels: The reference effect of official prices. *European Journal of Operational Research*, 296, 146-157.

[36] Tagashira, T., & Minami, C. (2019). The effect of cross-channel integration on cost efficiency. *Journal of Interactive Marketing*, 47, 68-83.

[37] Thaler, R.H. (1985). Mental accounting and consumer choice. *Marketing Science*, 4(3), 199-214.

[38] Verhoef, P.C. (2021). Omni-channel retailing: some reflections. *Journal of Strategic Marketing*, 29(7), 608-617.

[39] Verhoef, P.C., Kannan, P.K., & Inman, J.J. (2015). From multichannel retailing to omni-channel retailing: Introduction to the special issue on multi-channel retailing. *Journal of Retailing*, 91(2), 174-181.

[40] Wang, R., Nan, G., Kou, G., Li, M. (2023). Separation or integration: The game between retailers with online and offline channels. *European Journal of Operational Research*, 307, 1348-1359.

[41] Wang, N., Zhang, T., Zhu, X., & Li, P. (2021). Online-offline competitive pricing with reference price effect. *Journal of the Operational Research Society*, 72(3), 642-653.

[42] Yoo, W.S., & Lee, E. (2011). Internet Channel Entry: A strategic analysis of mixed channel structures. *Marketing Science*, 30(1), 29-41.

[43] Zhang, X. (2009). Retailers' multichannel and price advertising strategies. *Marketing Science*, 28(6), 1080-1094.

[44] Zhang, J., Chiang, W-Y. K., & Liang, L. (2014). Strategic pricing with reference effects in a competitive supply chain. *Omega*, 44, 126-135.

[45] Zhang, J., Farris, P.W., Irvin, J.W., Kushawaha, T., Steenburgh, T.J., & Weitz, B.A. (2010). Crafting integrated multichannel retailing strategies. *Journal of Interactive Marketing*, 24, 168-180.

[46] Zhang, J., Gou, Q., Liang, L., & Huang, Z. (2013). Supply chain coordination through cooperative advertising with reference price effect. *Omega*, 41, 345–353.

Appendix A. Proof of Propositions 1 and 2

In this appendix we prove Propositions 1 and 2, when the retailer centralizes (Scenario 1) or decentralizes (Scenario 2) retail prices. In both cases the two-period Stackelberg game between the manufacturer (leader) and the retailer (follower) is played in four stages. The subgame-perfect equilibrium solutions are obtained by solving backwards the game. The game is solved in four stages.

Scenario 1: Centralized pricing decisions. Proof of Proposition 1

Stage 4: At this stage of the game, the retailer manages both channels and looks at maximizing his second-period profits, and with this aim chooses the second-period prices for both channels, p_{21}, and p_{22}. Therefore, the retailer's problem reads:

$$\max_{p_{21},p_{22}} R_2 \tag{21}$$

where the retailer's second-period profits, R_2, and the second-period demand functions for channels, q_{21}, and q_{22}, are given by

$$\begin{aligned} R_2 &= (p_{21}-w_1)q_{21}+(p_{22}-w_2)q_{22}, && (22)\\ q_{21} &= g-p_{21}+\delta p_{22}-\gamma(p_{21}-p_{11})+\phi a_{M2}+\phi^2 a_{M1}, && (23)\\ q_{22} &= g-p_{22}+\delta p_{21}-\gamma(p_{22}-p_{12})+\phi a_{M2}+\phi^2 a_{M1}. && (24)\end{aligned}$$

The retailer's second-period profits is a strictly concave function of his decision variables in this period, p_{21}, p_{22} for any $\delta,\gamma,\phi\in(0,1)$, because the entries of the Hessian matrix are $-2(\gamma+1)$ and 2δ in the first row and 2δ and $-2(\gamma+1)$ in the second row, and hence, the quadratic form associated with the Hessian matrix is negative definite for any $\delta,\gamma,\phi\in(0,1)$. The first-order optimality conditions for an interior solution for problem (21) allow us to obtain the retailer's reaction functions. These functions express p_{21} and p_{21}, as functions of the manufacturer's second-period decision variables, the wholesale price, w_2 and the manufacturer's advertising rate, a_{M2}, as well as of the retailer's first-period decision variables, retailer's prices, p_{11} and p_{12}, and manufacturer's advertising rate in the first period, a_{M1}. The retailer's reaction functions read:

$$p_{21} = \frac{(\gamma+\delta+1)(\phi(a_{M1}\phi+a_{M2})+g+w_2(\gamma-\delta+1))+\gamma((\gamma+1)p_{11}+\delta p_{12})}{2(\gamma-\delta+1)(\gamma+\delta+1)}, \quad (25)$$

$$p_{22} = \frac{(\gamma+\delta+1)(\phi(a_{M1}\phi+a_{M2})+g+w_2(\gamma-\delta+1))+\gamma((\gamma+1)p_{12}+\delta p_{11})}{2(\gamma-\delta+1)(\gamma+\delta+1)}. \quad (26)$$

Stage 3: At this stage of the game the manufacturer looks at maximizing her second-period profits, and with this aim chooses the second-period wholesale price, w_2, and her advertising rate in the second period, a_{M2}. Therefore, the manufacturer's problem reads:

$$\max_{w_2, a_{M2}} M_2, \quad (27)$$

where the manufacturer's second-period profits, M_2, are given by

$$M_2 = w_2 q_{21} + w_2 q_{22} - \frac{1}{2} a_{M2}^2, \quad (28)$$

with q_{21} and q_{22} the demand functions for channels, in this period given in (23) and (24). At this stage of the game, the manufacturer as the Stackelberg leader knows the retailer's (follower's) reaction functions derived in Stage 4, and hence, takes into account these functions when making her optimal decisions on pricing and advertising in the second-period. Hence, the manufacturer substitutes the reaction functions in (25) and (26), in her objective function (27).

Solving this problem one can get the manufacturer's second-period decision variables, the wholesale price, w_2, and the advertising rate, a_{M2}, as functions of the first-period retailer's prices, p_{11} and p_{12}, and manufacturer's advertising rate in the first period, a_{M1}.

The manufacturer's second-period profits is a strictly concave function of her decision variables in this period if the quadratic form associated with the Hessian matrix is negative definite. The entries of the Hessian matrix are $-2(\gamma-\delta+1)$ and ϕ in the first row, and ϕ and -1 in the second row. Therefore, the manufacturer's second-period profits is a strictly concave function of her decision variables in this period if and only if $2(\gamma-\delta+1)-\phi^2 > 0$. Assuming that last inequality is satisfied, the first-order conditions for problem (27) lead to the following interior solution (case 1, $a_{M2} > 0$)

$$w_2 = \frac{2\left(a_{M1}\phi^2+g\right)+\gamma(p_{11}+p_{12})}{4(\gamma-\delta+1)-2\phi^2}, \quad (29)$$

$$a_{M2} = \phi\frac{2\left(a_{M1}\phi^2+g\right)+\gamma(p_{11}+p_{12})}{4(\gamma-\delta+1)-2\phi^2}. \quad (30)$$

The second-period retail prices as functions of the first-period prices, p_{11}, and p_{12}, and manufacturer's advertising rate in the first period, a_{M1}, can be obtained substituting the expressions above into the retailer's reaction functions in (25) and (26):

$$
\begin{aligned}
p_{21} &= \frac{6(\gamma+\delta+1)\left(a_{M1}\phi^2+g\right)+\gamma p_{11}\left(5\gamma+\delta-\phi^2+5\right)+\gamma p_{12}\left(\gamma+5\delta+\phi^2+1\right)}{4(\gamma+\delta+1)\left(2(\gamma-\delta+1)-\phi^2\right)}, && (31)\\
p_{22} &= \frac{6(\gamma+\delta+1)\left(a_{M1}\phi^2+g\right)+\gamma p_{12}\left(5\gamma+\delta-\phi^2+5\right)+\gamma p_{11}\left(\gamma+5\delta+\phi^2+1\right)}{4(\gamma+\delta+1)\left(2(\gamma-\delta+1)-\phi^2\right)}. && (32)
\end{aligned}
$$

Substituting expressions (29), (30), (31) and (32) in (22) and (28), respectively, the second-period retailer's and manufacturer's optimal profits are obtained:

$$
R_2=\frac{(\gamma-\delta+1)(\gamma+\delta+1)\left(2(a_{M1}\phi^2+g)+\gamma(p_{11}+p_{12})\right)^2+\gamma^2(p_{11}-p_{12})^2\left(2(\gamma-\delta+1)-\phi\right)^2}{8(\gamma+\delta+1)\left(2(\gamma-\delta+1)-\phi\right)^2}, \quad (33)
$$

$$
M_2=\frac{\left(2(a_{M1}\phi^2+g)+\gamma(p_{11}+p_{12})\right)^2}{16(\gamma-\delta+1)-8\phi^2}. \quad (34)
$$

Alternatively, we can analyze case 2, $a_{M2}=0$, and the second-period wholesale price can be derived when the manufacturer does not advertise in the second period, $a_{M2}=0$. In this case, the expression of w_2 is obtained from the optimality condition from the maximization of the manufacturer's second-period profits with respect to w_2 taking into account that $a_{M2}=0$. Therefore, in this case

$$
\begin{aligned}
w_2 &= \frac{2a_{M1}\phi^2+2g+\gamma(p_{11}+p_{12})}{4(\gamma-\delta+1)}, && (35)\\
a_{M2} &= 0. && (36)
\end{aligned}
$$

The second-period retail prices as functions of the first-period prices, p_{11}, and p_{12}, and manufacturer's advertising rate in the first period, a_{M1}, can be obtained substituting the expressions above into the retailer's reaction functions in (25) and (26):

$$
\begin{aligned}
p_{21} &= \frac{6(\gamma+\delta+1)\left(a_{M1}\phi^2+g\right)+\gamma(p_{11}(5\gamma+\delta+5)+p_{12}(\gamma+5\delta+1))}{8(\gamma-\delta+1)(\gamma+\delta+1)}, && (37)\\
p_{22} &= \frac{6(\gamma+\delta+1)\left(a_{M1}\phi^2+g\right)+\gamma(p_{12}(5\gamma+\delta+5)+p_{11}(\gamma+5\delta+1))}{8(\gamma-\delta+1)(\gamma+\delta+1)}. && (38)
\end{aligned}
$$

Substituting expressions (35), (36), (37) and (38) in (22) and (28), respectively, the second-period retailer's and manufacturer's optimal profits are obtained:

$$
\begin{aligned}
R_2 &= \frac{(\gamma+\delta+1)\left(2(a_{M1}\phi^2+g)+\gamma(p_{11}+p_{12})\right)^2+4\gamma^2(\gamma-\delta+1)(p_{11}-p_{12})^2}{32(\gamma-\delta+1)(\gamma+\delta+1)}, && (39)\\
M_2 &= \frac{\left(2(a_{M1}\phi^2+g)+\gamma(p_{11}+p_{12})\right)^2}{16(\gamma-\delta+1)}. && (40)
\end{aligned}
$$

Stage 2: In the first period the retailer looks at maximizing his total profits over the two periods for both channels $R = R_1 + R_2$, and with this aim chooses the retail prices, p_{11} and p_{12}.

Two different possibilities arise depending on whether the manufacturer does advertise (Case 1, $a_{M2} > 0$) or she does not advertise (Case 2, $a_{M2} = 0$) in the second period.

First possibility: $a_{M2} > 0$

The retailer's total profits read:

$$R = (p_{11} - w_1)q_{11} + (p_{12} - w_1)q_{12} + R_2, \tag{41}$$

where the second-period retailer's profits R_2 should be replaced by their expression in (33).

The entries of the Hessian matrix $H = (h_{ij})$ associated with the retailer's profits with respect to his decision variables in the first period, p_{11}, p_{12}, are as follows:

$$\begin{aligned} h_{11} = h_{22} &= \frac{1}{4}\gamma^2 \left(\frac{\gamma - \delta + 1}{(-2\gamma + 2\delta + \phi^2 - 2)^2} + \frac{1}{\gamma + \delta + 1} \right) - 2, \\ h_{12} = h_{21} &= \frac{1}{4} \left(\frac{\gamma^2(\gamma - \delta + 1)}{(-2\gamma + 2\delta + \phi^2 - 2)^2} - \frac{\gamma^2}{\gamma + \delta + 1} + 8\delta \right). \end{aligned}$$

The retailer's profits is a strictly concave function in the retailer's first-period decision variables if and only if the following two conditions are satisfied:

$$h_{11} < 0, \quad h_{11}^2 - h_{12}^2 > 0. \tag{42}$$

Assuming that conditions in (42) are satisfied, the first-order conditions for problem (80) lead to the following interior solution:

$$p_{11} = p_{12} = \frac{-\gamma(\gamma-\delta+1)\left(a_{M1}\phi^2+g\right)-2\left(2(\gamma-\delta+1)-\phi^2\right)^2\left(a_{M1}\phi+g+(1-\delta)w_1\right)}{\gamma^2(\gamma-\delta+1)+4(\delta-1)\left(2(\gamma-\delta+1)-\phi^2\right)^2}. \tag{43}$$

Second possibility: $a_{M2} = 0$

In this case the retailer's total profits read:

$$R = (p_{11} - w_1)q_{11} + (p_{12} - w_1)q_{12} + R_2, \tag{44}$$

where the second-period retailer's profits R_2 should be replaced by their expression in (39).

The entries of the Hessian matrix $H' = (h'_{ij})$ associated with the retailer's total profits with respect to his decision variables in the first period, p_{11}, p_{12} are as follows:

$$h'_{11} = h'_{22} = \frac{1}{16}\gamma^2\left(\frac{4}{\gamma+\delta+1} + \frac{1}{\gamma-\delta+1}\right) - 2,$$
$$h'_{12} = h'_{21} = \frac{1}{16}\gamma^2\left(\frac{1}{\gamma-\delta+1} - \frac{4}{\gamma+\delta+1}\right) + 2\delta.$$

It can be easily proved that $h'_{11} < 0$ for any $\gamma, \delta \in (0,1)$, and therefore, the retailer's total profits is a strictly concave function in the retailer's first-period decision variables if and only if the following condition is satisfied:

$$(h'_{11})^2 - (h'_{12})^2 > 0. \tag{45}$$

Assuming that condition in (45) is satisfied, the first-order conditions for problem (44) lead to the following interior solution:

$$p_{11} = p_{12} = \frac{(a_{M1}\phi + g)(8(1-\delta) - \gamma(\phi+8)) - \gamma g(1-\phi) - 8(1-\delta)w_1(\gamma-\delta+1)}{\gamma^2 - 16(1-\delta)(\gamma-\delta+1)}. \tag{46}$$

Stage 1: At this stage of the game, the manufacturer looks at maximizing her total profits $M = M_1 + M_2$ and with this aim she chooses the first-period wholesale price, w_1, and her advertising rate in the first period, a_{M1}.

Therefore, the manufacturer's problem reads:

$$\max_{w_1, a_{M1}} M_1 + M_2. \tag{47}$$

The two different possibilities already analyzed in Stage 2 that depend on whether the manufacturer does advertise (Case 1, $a_{M2} > 0$) or she does not advertise (Case 2, $a_{M2} = 0$) in the second period emerge in this stage too.

First possibility: $a_{M2} > 0$

In this case, the manufacturer's total profits $M = M_1 + M_2$ are obtained substituting the second-period manufacturer's profits given by (34). The manufacturer as Stackelberg leader also knows the retailer's reaction functions given by (43) and takes into account these functions when maximizing her total profits over the two periods. The first-order optimality conditions for the maximization of the manufacturer's total profits with respect to w_1 and

a_{M1} lead to a unique feasible interior solution given by:

$$w_1 = -g\frac{Numw_1}{Denw_1}, \tag{48}$$

$$a_{M1} = -g\phi\frac{Numa_{M1}}{Dena_{M1}}, \tag{49}$$

where $Denw_1 = 2Dena_{M1}$, and

$$\begin{aligned} Numw_1 = {} & 32(\delta-1)\phi^2(\gamma-\delta+1)^2\left(\gamma^2(\phi-1)-\gamma(\delta-1)\left(2\phi^2-\phi+8\right)+2(\delta-1)^2\left(\phi^2-\phi+4\right)\right) \\ & +2(\delta-1)\phi^6\left(\gamma^2(12\phi-11)-2\gamma(\delta-1)\left(12\phi^2-6\phi+11\right)+8(\delta-1)^2\left(3\phi^2-3\phi+4\right)\right) \qquad (50) \\ & -2(\delta-1)\phi^4(\gamma-\delta+1)\left(3\gamma^2(8\phi-7)-2\gamma(\delta-1)\left(24\phi^2-12\phi+41\right)+48(\delta-1)^2\left(\phi^2-\phi+2\right)\right) \\ & +\left(\gamma^3+8\gamma^2(\delta-1)+144\gamma(\delta-1)^2-128(\delta-1)^3\right)(\gamma-\delta+1)^3-4(\delta-1)\phi^8((\phi-1)(\gamma-2(\delta-1)\phi) \\ & -2\delta+2), \end{aligned} \tag{51}$$

$$\begin{aligned} Numa_{M1} = {} & 2(\delta-1)\phi^6\left(2\gamma^2-\gamma(\delta-1)(3\phi+19)+4(\delta-1)^2(\phi+4)\right)-2(\delta-1)\phi^4(\gamma-\delta+1)\left(10\gamma^2\right. \\ & \left.-\gamma(\delta-1)(17\phi+65)+24(\delta-1)^2(\phi+2)\right)+32(\delta-1)\phi^2(\gamma-\delta+1)^2\left(\gamma^2-2\gamma(\delta-1)(\phi+3)\right. \\ & \left.+(\delta-1)^2(3\phi+4)\right)+(\gamma-\delta+1)^3\left(\gamma^3-\gamma^2(\delta-1)(\phi+16)+8\gamma(\delta-1)^2(5\phi+13)\right. \\ & \left.-64(\delta-1)^3(\phi+1)\right)+4(\delta-1)^2\phi^8, \end{aligned} \tag{52}$$

$$\begin{aligned} Dena_{M1} = {} & 4(\delta-1)\phi^8\left(\gamma^2-\gamma(\delta-1)(3\phi+8)+2(\delta-1)^2\left(\phi^2+5\right)\right)+(\delta-1)\phi^6\left(-20\gamma^3+\gamma^2(\delta-1)(68\phi+115)\right. \\ & \left.-4\gamma(\delta-1)^2(\phi(12\phi+17)+64)+16(\delta-1)^3\left(3\phi^2+10\right)\right)+8(\delta-1)\phi^4(\gamma-\delta+1)\left(4\gamma^3-\gamma^2(\delta-1)(16\phi+19)\right. \\ & \left.+4\gamma(\delta-1)^2(\phi(3\phi+4)+14)-4(\delta-1)^3\left(3\phi^2+10\right)\right)+\phi^2(\gamma-\delta+1)^2\left(\gamma^4-2\gamma^3(\delta-1)(\phi+8)\right. \\ & \left.+\gamma^2(\delta-1)^2(\phi(\phi+80)+60)-16\gamma(\delta-1)^3(\phi(4\phi+5)+24)+64(\delta-1)^4\left(\phi^2+5\right)\right)+4(\delta-1)^2\phi^{10} \\ & +16(\delta-1)^2\left(\gamma^2+8(\delta-1)(\gamma-\delta+1)\right)(\gamma-\delta+1)^3. \end{aligned} \tag{53}$$

Substituting (48) and (49) in (43), (29), (30), (25) and (26), Equilibrium I ($a_{M1} > 0, a_{M2} > 0$) in Proposition 1 when the retailer centralizes pricing decisions is completely characterized.

We compute the entries of the Hessian matrix of function M with respect to w_1 and a_{M1}. These entries are cumbersome and omitted for brevity. Using these entries we derive the first and second minors of the Hessian matrix whose signs characterize the strictly concavity of function M. When Equilibrium I in Proposition 1 has been considered, in the numerical simulations we have checked that the conditions ensuring that M is a strictly concave function are satisfied.

A second case arises corresponding to a corner solution at which $a_{M1} = 0$. In this case the manufacturer's wholesale price in the first period is given by the following expression:

$$w_1 = -g\frac{Numw_1}{Denw_1}, \tag{54}$$

$$\begin{aligned} Numw_1 = {} & \left(\gamma^3+8\gamma^2(\delta-1)+144\gamma(\delta-1)^2-128(\delta-1)^3\right)(\gamma-\delta+1)^3 \\ & -32(\delta-1)^2\phi^2(9\gamma-8(\delta-1))(\gamma-\delta+1)^2+8(\delta-1)^2\phi^8+2(\delta-1)\phi^6\left(\gamma^2-34\gamma(\delta-1)+32(\delta-1)^2\right) \\ & -2(\delta-1)\phi^4\left(3\gamma^2-106\gamma(\delta-1)+96(\delta-1)^2\right)(\gamma-\delta+1) \end{aligned} \tag{55}$$

$$\begin{aligned} Denw_1 = {} & 2(\delta-1)^2\left(-2\gamma+2(\delta-1)+\phi^2\right)^2\Big(4\gamma^3+\gamma^2\left(28(\delta-1)-\phi^2\right)-32\gamma(\delta-1)\left(2(\delta-1)+\phi^2\right) \\ & +8(\delta-1)\left(2\delta+\phi^2-2\right)^2\Big). \end{aligned} \tag{56}$$

Substituting (54) and $a_{M1} = 0$ in (43), (29), (30), (25) and (26), Equilibrium III ($a_{M1} = 0, a_{M2} > 0$) in Proposition 1 when the retailer centralizes pricing decisions is completely characterized.

Second possibility: $a_{M2} = 0$

In this case, the manufacturer's total profits $M = M_1 + M_2$ are obtained substituting the second-period manufacturer's profits given by (40). The manufacturer as Stackelberg leader also knows the retailer's reaction functions given by (46) and takes into account these functions when maximizing her total profits over the two periods. The first-order optimality conditions for the maximization of the manufacturer's total profits with respect to w_1 and a_{M1} lead to a unique feasible interior solution given by:

$$w_1 = -g(\gamma-\delta+1)\frac{Numw_1}{Denw_1}, \tag{57}$$

$$a_{M1} = -g\phi(\gamma-\delta+1)\frac{Numa_{M1}}{Dena_{M1}}, \tag{58}$$

where $Denw_1 = 2Dena_{M1}$ and

$$Numw_1 = \gamma^3+32(\delta-1)(\phi-1)\phi^2(\gamma-2(\delta-1)\phi)+8\gamma(\delta-1)(\gamma+2\delta-2)+128(\delta-1)^2(\gamma-\delta+1), \tag{59}$$

$$Numa_{M1} = \gamma^3-\gamma^2(\delta-1)(\phi+16)+8\gamma(\delta-1)^2(5\phi+13)-64(\delta-1)^3(\phi+1), \tag{60}$$

$$\begin{aligned} Dena_{M1} = {} & 16(\delta-1)^2\left(\gamma^2+8(\delta-1)(\gamma-\delta+1)\right)(\gamma-\delta+1)+\phi^2\Big(\gamma^4-2\gamma^3(\delta-1)(\phi+8) \\ & +\gamma^2(\delta-1)^2(\phi(\phi+80)+80)-16\gamma(\delta-1)^3(\phi(4\phi+5)+8)+64(\delta-1)^4\left(\phi^2+1\right)\Big). \end{aligned} \tag{61}$$

Substituting (57) and (58) in (46), (35), (30), (25) and (26), Equilibrium II ($a_{M1} > 0, a_{M2} = 0$) in Proposition 1 when the retailer centralizes pricing decisions is completely characterized.

We compute the entries (J_{ij}) of the Hessian matrix J of function M with respect to w_1 and a_{M1}. These entries are given by:

$$j_{11} = \frac{64(\delta-1)^2\left(\gamma^2+8(\delta-1)(\gamma-\delta+1)\right)(\gamma-\delta+1)}{\left(\gamma^2+16(\delta-1)(\gamma-\delta+1)\right)^2},$$

$$j_{12} = j_{21} = \frac{2\phi\left(\gamma^4-\gamma^3(\delta-1)(\phi-8)+8\gamma^2(\delta-1)^2(2\phi+15)-16\gamma(\delta-1)^3(\phi+16)+128(\delta-1)^4\right)}{\left(\gamma^2+16\gamma(\delta-1)-16(\delta-1)^2\right)^2},$$

$$j_{22} = \frac{Numj_{22}}{2\left(\gamma^2+16\gamma(\delta-1)-16(\delta-1)^2\right)^2(\gamma-\delta+1)},$$

where

$$Numj_{22} = -\gamma\phi^2(\gamma(\phi+8)-8(\delta-1))\left(\gamma^2(\phi-8)+8\gamma(\delta-1)(4\phi+1)-32(\delta-1)^2\phi\right)$$
$$\left(\gamma^2+16\gamma(\delta-1)-16(\delta-1)^2\right)^2\left(\phi^4-2(\gamma-\delta+1)\right)$$

When Equilibrium II in Proposition 1 has been considered, in the numerical simulations we have checked that the conditions $j_{11} < 0, j_{11}j_{22} - j_{12}^2 > 0$ ensuring that M is a strictly concave function are satisfied.

A second case arises corresponding to a corner solution at which $a_{M1} = 0$. In this case the manufacturer's wholesale price in the first period is given by the following expression:

$$w_1 = -\frac{g\left(\gamma^3+8\gamma^2(\delta-1)+144\gamma(\delta-1)^2-128(\delta-1)^3\right)}{32(\delta-1)^2\left(\gamma^2+8(\delta-1)(\gamma-\delta+1)\right)}. \qquad (62)$$

Substituting (62) and $a_{M1} = 0$ in (46), (35), (30), (25) and (26), Equilibrium IV ($a_{M1} = 0, a_{M2} = 0$) in Proposition 1 when the retailer centralizes pricing decisions is completely characterized.

Scenario 2: Decentralized pricing decisions. Proof of Proposition 2

This subsection follows the same steps as the previous subsection with the exception that now the retailer decentralizes retail pricing and hence, each channel sets its price separately, and therefore, there is horizontal competition between channels. As a result, at Stages 4 and 2 where the optimal decisions of the retailer are characterized, these decisions are obtained as a Nash equilibrium between both channels. There is a Nash game between channels at the retailer level, and a Stackelberg game between the manufacturer and the retailer. As it could be expected, this more complex and richer specification leads to more complex computations for the characterization of the four equilibria. In what follows we present the four stages of the game and present the corresponding optimal solutions, except when these solutions are cumbersome expressions that do not provide any insight.

Stage 4: At this stage of the game, the retailer manages both channels separately, and each channel cares exclusively about its own profits. Channel i looks at maximizing its second-period profits, and with this aim chooses the second-period prices, p_{2i}. Therefore, channel i's problem reads:

$$\max_{p_{2i}} R_{2i} \tag{63}$$

where channel i's second-period profits, R_{2i}, and the second-period demand functions for channels, q_{21}, and q_{22}, are given by (23) and (24).

$$R_{2i} = (p_{2i} - w_2)q_{2i}. \tag{64}$$

Each channel second-period profits are a strictly concave function of its decision variable in this period, p_{2i} for channel i for any $\delta, \gamma, \phi \in (0, 1)$, because $\frac{\partial^2 R_{2i}}{\partial p_{2i}^2} = -2(\gamma+1) < 0$. Taking into account the first-order optimality conditions for an interior solution for problem (63) allow us to obtain the retailer's reaction functions for both channels as the Nash equilibrium of the game representing the horizontal competition among both channels. These functions express p_{21} and p_{21} as functions of the manufacturer's second-period decision variables, the wholesale price, w_2, and the manufacturer's advertising rate, a_{M2}, as well as of the retailer's first-period decision variables, retailer's prices, p_{11} and p_{12}, and manufacturer's advertising rate in the first period, a_{M1}. The retailer's reaction functions read:

$$p_{21} = \frac{(2(\gamma+1)+\delta)(\phi(a_{M1}\phi + a_{M2}) + g + (\gamma+1)w_2) + \gamma(2(\gamma+1)p_{11} + \delta p_{12})}{4(\gamma+1)^2 - \delta^2}, \tag{65}$$

$$p_{22} = \frac{(2(\gamma+1)+\delta)(\phi(a_{M1}\phi + a_{M2}) + g + (\gamma+1)w_2) + \gamma(2(\gamma+1)p_{12} + \delta p_{11})}{4(\gamma+1)^2 - \delta^2}. \tag{66}$$

Stage 3: At this stage of the game the manufacturer looks at maximizing her second-period profits, and with this aim chooses the second-period wholesale price, w_2, and her advertising rate in the second period, a_{M2}. Therefore, the manufacturer's problem reads as in (27), with manufacturer's second-period profits given by (28). The manufacturer as the

Stackelberg leader knows the retailer's (follower's) reaction functions derived in Stage 4, and hence, takes into account these functions when making her optimal decisions on pricing and advertising in the second-period. The manufacturer substitutes the reaction functions in (65) and (66), in her objective function (27).

Solving this problem one can get the manufacturer's second-period decision variables, the wholesale price, w_2, and the advertising rate, a_{M2}, as functions of the first-period retailer's prices, p_{11} and p_{12}, and manufacturer's advertising rate in the first period, a_{M1}.

The manufacturer's second-period profits is a strictly concave function of her decision variables in this period if the quadratic form associated with the Hessian matrix is negative definite. The entries of the Hessian matrix are $-\frac{4(\gamma+1)(\gamma-\delta+1)}{2\gamma-\delta+2}$ and $\frac{2(\gamma+1)\phi}{2\gamma-\delta+2}$ in the first row, and $\frac{2(\gamma+1)\phi}{2\gamma-\delta+2}$ and -1 in the second row. Therefore, the manufacturer's second-period profits is a strictly concave function of her decision variables in this period if and only if $(\gamma-\delta+1)(2\gamma-\delta+2)-(\gamma+1)\phi^2 > 0$. Assuming that last inequality is satisfied, the first-order conditions for problem (27) lead to the following interior solution (case 1, $a_{M2} > 0$)

$$w_2 = \frac{(2\gamma-\delta+2)\left(2a_{M1}\phi^2+2g+\gamma(p_{11}+p_{12})\right)}{4(\gamma-\delta+1)(2\gamma-\delta+2)-4(\gamma+1)\phi^2}, \tag{67}$$

$$a_{M2} = \frac{(\gamma+1)\phi\left(2a_{M1}\phi^2+2g+\gamma(p_{11}+p_{12})\right)}{2(\gamma-\delta+1)(2\gamma-\delta+2)-2(\gamma+1)\phi^2}. \tag{68}$$

The second-period retail prices as functions of the first-period prices, p_{11}, and p_{12}, and manufacturer's advertising rate in the first period, a_{M1} can be obtained substituting the expressions above into the retailer's reaction functions in (65) and (66):

$$p_{21} = \frac{Nump_{21}}{4(2(\gamma+1)+\delta)\left((\gamma-\delta+1)(2(\gamma+1)-\delta)-(\gamma+1)\phi^2\right)}, \tag{69}$$

$$p_{22} = \frac{Nump_{22}}{4(2(\gamma+1)+\delta)\left((\gamma-\delta+1)(2(\gamma+1)-\delta)-(\gamma+1)\phi^2\right)}, \tag{70}$$

where

$$\begin{aligned}Nump_{21} &= 2(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)\left(a_{M1}\phi^2+g\right)+\gamma\left((\gamma+1)(p_{11}-p_{12})\left(10(\gamma+1)-7\delta-2\phi^2\right)\right.\\ &\quad \left.+2p_{12}(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)\right),\\ Nump_{22} &= 2(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)\left(a_{M1}\phi^2+g\right)+\gamma\left((\gamma+1)(p_{12}-p_{11})\left(10(\gamma+1)-7\delta-2\phi^2\right)\right.\\ &\quad \left.+2p_{11}(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)\right).\end{aligned}$$

Substituting expressions (67), (68), (69) and (70) in (64) and (28), respectively, the second-period channels' and manufacturer's optimal profits are obtained:

$$R_{21} = \frac{NumR_{21}}{16(2(\gamma+1)+\delta)^2\left((\gamma-\delta+1)(2(\gamma+1)-\delta)-(\gamma+1)\phi^2\right)^2}, \tag{71}$$

$$R_{22} = \frac{NumR_{22}}{16(2(\gamma+1)+\delta)^2\left((\gamma-\delta+1)(2(\gamma+1)-\delta)-(\gamma+1)\phi^2\right)^2}, \tag{72}$$

$$M_2 = \frac{(\gamma+1)\left(2a_{M1}\phi^2+2g+\gamma(p_{11}+p_{12})\right)^2}{8(\gamma-\delta+1)(2(\gamma+1)-\delta)-8(\gamma+1)\phi^2}, \tag{73}$$

where

$$\begin{aligned} NumR_{21} &= (\gamma+1)\left((\gamma-\delta+1)\left(2(2(\gamma+1)+\delta)\left(a_{M1}\phi^2+g+\gamma p_{12}\right)\right)\right.\\ &\quad \left.+\gamma(p_{11}-p_{12})\left((\gamma-\delta+1)(6(\gamma+1)-\delta)-2(\gamma+1)\phi^2\right)\right)^2, \\ NumR_{22} &= (\gamma+1)\left((\gamma-\delta+1)\left(2(2(\gamma+1)+\delta)\left(a_{M1}\phi^2+g+\gamma p_{11}\right)\right)\right.\\ &\quad \left.+\gamma(p_{12}-p_{11})\left((\gamma-\delta+1)(6(\gamma+1)-\delta)-2(\gamma+1)\phi^2\right)\right)^2. \end{aligned}$$

Alternatively, we can analyze case 2, $a_{M2}=0$, and the second-period wholesale price can be derived when the manufacturer does not advertise in the second period, $a_{M2}=0$. In this case, the expression of w_2 is obtained from the optimality condition from the maximization of the manufacturer's second-period profits with respect to w_2 taking into account that $a_{M2}=0$, and w_2 is given by (35).

The second-period retail prices as functions of the first-period prices, p_{11}, and p_{12}, and manufacturer's advertising rate in the first period, a_{M1} can be obtained substituting $a_{M2}=0$ and w_2 by its expression in (35) into the retailer's reaction functions in (69) and (70):

$$p_{21} = \frac{Nump_{21}}{4(\gamma-\delta+1)\left(4(\gamma+1)^2-\delta^2\right)}, \tag{74}$$

$$p_{22} = \frac{Nump_{22}}{4(\gamma-\delta+1)\left(4(\gamma+1)^2-\delta^2\right)}, \tag{75}$$

where

$$\begin{aligned} Nump_{21} &= 2(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)\left(a_{M1}\phi^2+g\right)+\gamma\left((\gamma+1)p_{11}(10(\gamma+1)-7\delta)\right.\\ &\quad \left.+p_{12}\left(5(\gamma+1)\delta+2(\gamma+1)^2-4\delta^2\right)\right), \\ Nump_{22} &= 2(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)\left(a_{M1}\phi^2+g\right)+\gamma\left((\gamma+1)p_{12}(10(\gamma+1)-7\delta)\right.\\ &\quad \left.+p_{11}\left(5(\gamma+1)\delta+2(\gamma+1)^2-4\delta^2\right)\right). \end{aligned}$$

Substituting expressions (35), (36), (74) and (75) in (64) and (28), respectively, the second-period channels' and manufacturer's optimal profits are obtained:

$$R_{21} = \frac{(\gamma+1)\left(2(2(\gamma+1)+\delta)\left(a_{M1}\phi^2+g\right)+\gamma p_{11}(6(\gamma+1)-\delta)+\gamma p_{12}(3\delta-2(\gamma+1))\right)^2}{16\left(\delta^2-4(\gamma+1)^2\right)^2}, \quad (76)$$

$$R_{22} = \frac{(\gamma+1)\left(2(2(\gamma+1)+\delta)\left(a_{M1}\phi^2+g\right)+\gamma p_{12}(6(\gamma+1)-\delta)+\gamma p_{11}(3\delta-2(\gamma+1))\right)^2}{16\left(\delta^2-4(\gamma+1)^2\right)^2}, \quad (77)$$

$$M_2 = \frac{(\gamma+1)\left(2a_{M1}\phi^2+2g+\gamma(p_{11}+p_{12})\right)^2}{8(\gamma-\delta+1)(2\gamma-\delta+2)}. \quad (78)$$

Stage 2: In the first period the retailer manages both channels separately. Channel i looks at maximizing its total profits over the two periods $R_i = R_{1i} + R_{2i}$, and with this aim chooses its first-period price, p_{1i}. Therefore, channel i's problem reads:

$$\max_{p_{1i}} R_i = R_{1i} + R_{2i} \quad (79)$$

Two different possibilities arise depending on whether the manufacturer does advertise (Case 1, $a_{M2} > 0$) or she does not advertise (Case 2, $a_{M2} = 0$) in the second period.

First possibility: $a_{M2} > 0$

Channel i's total profits read:

$$R_i = (p_{1i} - w_1)q_{1i} + +R_{2i}, \quad (80)$$

where the second-period channel i's profits, R_{2i}, should be replaced by their expression in (71) or in (72) for channel 1 and 2, respectively.

We have

$$\frac{\partial^2 R_i}{\partial p_{1i}^2} = \frac{(\gamma+1)\left(\gamma(\gamma-\delta+1)(6(\gamma+1)-\delta)-2\gamma(\gamma+1)\phi^2\right)^2}{8(2(\gamma+1)+\delta)^2\left(-3(\gamma+1)\delta-(\gamma+1)\phi^2+2(\gamma+1)^2+\delta^2\right)^2} - 2.$$

If the expression above is negative, we have that each channel total profits are a strictly concave function of its decision variable in the first period period, p_{1i}, for channel i. Assuming that this condition is satisfied and taking into account the first-order optimality conditions for an interior solution for problem (79) allow us to obtain the retailer's reaction functions for both channels as the Nash equilibrium of the game representing the horizontal competition among both channels. These functions express p_{11} and p_{12}, as functions of the manufacturer's first-period decision variables, the wholesale price, w_1, and the manufacturer's advertising rate, a_{M1}. The retailer's reaction functions read:

$$p_{11} = p_{12} = \frac{Nump_{11}}{Denp_{11}}, \tag{81}$$

where

$$\begin{aligned}
Nump_{11} &= -2(\gamma+1)\phi^2(\gamma-\delta+1)\left(a_{M1}(\gamma+1)\phi(\gamma(\phi+16)+16) - 4a_{M1}\delta^2\phi + (17\gamma+16)(\gamma+1)g\right.\\
&\left. -4\delta^2 g + 16(\gamma+1)^2 w_1 - 4\delta^2 w_1\right) + (\gamma-\delta+1)^2\left(a_{M1}\phi\left(\gamma(\gamma+1)\phi(6\gamma-\delta+6)\right.\right.\\
&\left.\left. +4(2\gamma+\delta+2)(-2\gamma+\delta-2)^2\right) + g\left(-8(\gamma+1)\delta^2 - (\gamma+1)(17\gamma+16)\delta + 2(\gamma+1)^2(19\gamma+16) + 4\delta^3\right)\right.\\
&\left. +4w_1(2\gamma+\delta+2)(-2\gamma+\delta-2)^2\right) + 4(\gamma+1)^2\phi^4(2\gamma+\delta+2)(a_{M1}\phi+g+w_1),\\
Denp_{11} &= 2(\gamma+1)\phi^2(\gamma-\delta+1)\left(\gamma^2(\gamma+1) - 4(\delta-2)(-2\gamma+\delta-2)(2\gamma+\delta+2)\right)\\
&-(\gamma-\delta+1)^2\left((\gamma+1)\gamma^2(6\gamma-\delta+6) + 4(\delta-2)(2\gamma+\delta+2)(-2\gamma+\delta-2)^2\right)\\
&-4(\gamma+1)^2(\delta-2)\phi^4(2\gamma+\delta+2).
\end{aligned}$$

Second possibility: $a_{M2} = 0$

In this case channel i's total profits read as (80) where the second-period channel i's profits, R_{2i}, should be replaced by their expression in (76) or in (77) for channel 1 and 2, respectively.

In this case, condition

$$\frac{\partial^2 R_i}{\partial p_{1i}^2} = \frac{\gamma^2(\gamma+1)(-6\gamma+\delta-6)^2}{8(-2\gamma+\delta-2)^2(2\gamma+\delta+2)^2} - 2 < 0, \tag{82}$$

ensures that each channel total profits is a strictly concave function of its decision variable in the first period period, p_{1i}, for channel i. Assuming that condition in (82) is satisfied, following the same procedure as in the previous case, but using $a_{M2} = 0$, the first-order conditions lead to the retailer's reaction functions:

$$p_{11} = p_{12} = \frac{Nump_{11}}{Denp_{11}}, \tag{83}$$

where

$$\begin{aligned}
Nump_{11} &= a_{M1}\phi\left(\gamma(\gamma+1)\phi(-6\gamma+\delta-6) - 4(-2\gamma+\delta-2)^2(2\gamma+\delta+2)\right) + g\left(8(\gamma+1)\delta^2\right.\\
&\left. +(\gamma+1)(17\gamma+16)\delta - 2(\gamma+1)^2(19\gamma+16) - 4\delta^3\right) - 4w_1(2\gamma+\delta+2)(-2\gamma+\delta-2)^2,\\
Denp_{11} &= (\gamma+1)\gamma^2(6\gamma-\delta+6) + 4(\delta-2)(2\gamma+\delta+2)(-2\gamma+\delta-2)^2.
\end{aligned}$$

Stage 1: At this stage of the game, the manufacturer looks at maximizing her total profits $M = M_1 + M_2$ and with this aim she chooses the first-period wholesale price, w_1, and her advertising rate in the first period, a_{M1}.

Therefore, the manufacturer's problem reads as in (47).

The two different possibilities already analyzed in Stage 2 that depend on whether the manufacturer does advertise (Case 1, $a_{M2} > 0$) or she does not advertise (Case 2, $a_{M2} = 0$) in the second period emerge in this stage too.

First possibility: $a_{M2} > 0$

In this case, the manufacturer's total profits $M = M_1 + M_2$ are obtained substituting the second-period manufacturer's profits given by (73). The manufacturer as Stackelberg leader also knows the retailer's reaction functions given by (81) and takes into account these functions when maximizing her total profits over the two periods. The first-order optimality conditions for the maximization of the manufacturer's total profits with respect to w_1 and a_{M1} lead to a unique feasible interior solution. We do not write the expressions of w_1 and a_{M1} because there are really messy and have been obtained with the help of Mathematica 12.3.

Substituting these expressions into (81), (67), (68), (69) and (70) Equilibrium I ($a_{M1} > 0, a_{M2} > 0$) in Proposition 2 when the retailer decentralizes pricing decisions is completely characterized.

We have also computed the entries of the Hessian matrix of function M with respect to w_1 and a_{M1}. These entries are cumbersome and omitted for brevity. Using these entries we derive the first and second minors of the Hessian matrix whose signs characterize the strictly concavity of function M. When Equilibrium I in Proposition 2 has been considered, in the numerical simulations we have checked that the conditions ensuring that M is a strictly concave function are satisfied.

A second case arises corresponding to a corner solution at which $a_{M1} = 0$. In this case the manufacturer's wholesale price in the first period is also omitted because of its length. Substituting this expression and $a_{M1} = 0$ into (81), (67), (68), (69) and (70) Equilibrium III ($a_{M1} = 0, a_{M2} > 0$) in Proposition 2 when the retailer decentralizes pricing decisions is completely characterized.

Second possibility: $a_{M2} = 0$

In this case, the manufacturer's total profits $M = M_1 + M_2$ are obtained substituting the second-period manufacturer's profits given by (78). The manufacturer as Stackelberg leader also knows the retailer's reaction functions given by (83) and takes into account these functions when maximizing her total profits over the two periods. The first-order optimality conditions for the maximization of the manufacturer's total profits with respect to w_1 and a_{M1} lead to a unique feasible interior solution. These expressions are omitted because their length.

Substituting these expressions in (83), (35), (36), (65) and (66), Equilibrium II ($a_{M1} > 0, a_{M2} = 0$) in Proposition 3 when the retailer decentralizes pricing decisions is completely characterized. When this Equilibrium II has been considered, in the numerical simulations we have checked that the conditions ensuring that M is a strictly concave function are satisfied.

A second case arises corresponding to a corner solution at which $a_{M1} = 0$. In this case the manufacturer's wholesale price in the first period is given by the following expression:

$$w_1 = \frac{gNumw_1}{Denw_1}. \tag{84}$$

$$\begin{aligned} Numw_1 = {} & -4(\gamma(\gamma+21)+28)\delta^7+8(\gamma+1)(\gamma(5\gamma+7)+20)\delta^6-4(\gamma+1)(\gamma(\gamma(25\gamma-67)-235)-144)\delta^5 \\ & -(\gamma+1)^2(\gamma(\gamma(31\gamma+644)+2552)+1920)\delta^4+2(\gamma+1)^3(\gamma(\gamma(201\gamma+128)+368)+384)\delta^3 \\ & -(\gamma+1)^4(\gamma(\gamma(451\gamma-96)-4160)-3584)\delta^2+4(\gamma+1)^5(\gamma(\gamma(27\gamma+80)-1360)-1280)\delta \\ & +4(\gamma+1)^6(\gamma(\gamma(9\gamma-80)+544)+512)+16\delta^8, \end{aligned} \tag{85}$$

$$\begin{aligned} Denw_1 = {} & 8(-2\gamma+\delta-2)^2(2\gamma+\delta+2)\big(12(\gamma+2)\delta^5+4(\gamma(2\gamma-5)-9)\delta^4-(\gamma+1)(\gamma(47\gamma+120)+48)\delta^3 \\ & +(\gamma+1)^2(\gamma(25\gamma+208)+192)\delta^2+(\gamma+1)^2(\gamma(\gamma(6\gamma-83)-288)-192)\delta-2(\gamma+1)^3(\gamma(5\gamma-32)-32)-4\delta^6\big) \end{aligned} \tag{86}$$

Substituting (84) and $a_{M1} = 0$ in (83), (35), (36), (65) and (66), Equilibrium IV ($a_{M1} = 0, a_{M2} = 0$) in Proposition 2 when the retailer decentralizes pricing decisions is completely characterized.

Appendix B. Proof of Proposition 3

Scenario 1

- If equilibrium I^C or equilibrium III^C in Scenario 1 is played, from expressions (1), (2), (3), (4) one gets:

$$\begin{aligned}
\frac{\partial p_{21}}{\partial p_{11}} &= \frac{\partial p_{22}}{\partial p_{12}} = \frac{\gamma(5\gamma+\delta-\phi^2+5)}{4(\gamma+\delta+1)\left(2(\gamma-\delta+1)-\phi^2\right)}, \\
\frac{\partial p_{21}}{\partial p_{12}} &= \frac{\partial p_{22}}{\partial p_{11}} = \frac{\partial p_{22}}{\partial p_{12}} = \frac{\gamma(\gamma+5\delta+\phi^2+1)}{4(\gamma+\delta+1)\left(2(\gamma-\delta+1)-\phi^2\right)}, \\
\frac{\partial p_{21}}{\partial a_{M1}} &= \frac{\partial p_{22}}{\partial a_{M1}} = \frac{3\phi^2}{2\left(2(\gamma-\delta+1)-\phi^2\right)}, \\
\frac{\partial w_2}{\partial p_{11}} &= \frac{\partial w_2}{\partial p_{12}} = \frac{\gamma}{2\left(2(\gamma-\delta+1)-\phi^2\right)}, \\
\frac{\partial w_2}{\partial a_{M1}} &= \frac{\phi^2}{2(\gamma-\delta+1)-\phi^2}, \\
\frac{\partial a_{M2}}{\partial a_{M1}} &= \frac{\phi^3}{2(\gamma-\delta+1)-\phi^2}, \\
\frac{\partial a_{M2}}{\partial p_{11}} &= \frac{\partial a_{M2}}{\partial p_{12}} = \frac{\gamma\phi}{2\left(2(\gamma-\delta+1)-\phi^2\right)}.
\end{aligned}$$

In stage 4 in the proof of Proposition 1 we have shown that the strict concavity of the manufacturer's second-period profits requires condition $2(\gamma-\delta+1)-\phi^2>0$, and therefore, all the partial derivatives above are positive under this condition.

Furthermore,

$$\begin{aligned}
\frac{\partial p_{21}}{\partial a_{M1}} &= \frac{\partial p_{22}}{\partial a_{M1}} = \frac{3}{2}\frac{\partial w_2}{\partial a_{M1}}, \\
\frac{\partial p_{21}}{\partial p_{11}} &= \frac{\partial p_{22}}{\partial p_{12}} = \frac{5\gamma+\delta-\phi^2+5}{2(\gamma+\delta+1)}\frac{\partial w_2}{\partial p_{11}} = \frac{5\gamma+\delta-\phi^2+5}{2(\gamma+\delta+1)}\frac{\partial w_2}{\partial p_{12}}, \\
\frac{\partial p_{21}}{\partial p_{12}} &= \frac{\partial p_{22}}{\partial p_{11}} = \frac{\gamma+5\delta+\phi^2+1}{2(\gamma+\delta+1)}\frac{\partial w_2}{\partial p_{11}} = \frac{\gamma+5\delta+\phi^2+1}{2(\gamma+\delta+1)}\frac{\partial w_2}{\partial p_{12}}, \\
\frac{\partial w_2}{\partial p_{11}} &= \frac{\partial w_2}{\partial p_{12}} = \frac{1}{\phi}\frac{\partial a_{M2}}{\partial p_{11}} = \frac{1}{\phi}\frac{\partial a_{M2}}{\partial p_{12}}, \\
\frac{\partial w_2}{\partial a_{M1}} &= \frac{1}{\phi}\frac{\partial a_{M2}}{\partial a_{M1}}.
\end{aligned}$$

Because

$$\frac{5\gamma+\delta-\phi^2+5}{2(\gamma+\delta+1)}>1,$$

under the concavity condition $2(\gamma-\delta+1)-\phi^2>0$, the following inequality applies

$$\frac{\gamma(5\gamma+\delta-\phi^2+5)}{4(\gamma+\delta+1)(2(\gamma+1-\delta)-\phi^2)}>\frac{\gamma(\gamma+5\delta+\phi^2+1)}{4(\gamma+\delta+1)(2(\gamma+1-\delta)-\phi^2)}$$

and

$$\frac{\gamma+5\delta+\phi^2+1}{2(\gamma+\delta+1)}>1 \text{ if and only if } \phi^2>1+\gamma-3\delta$$

the results of the comparisons in Proposition 3 easily follow.

- If equilibrium II^C or equilibrium IV^C in Scenario 1 is played, from expressions (6), (7), (8) one gets:

$$\begin{aligned}
\frac{\partial p_{21}}{\partial p_{11}} &= \frac{\partial p_{22}}{\partial p_{12}}=\frac{\gamma(5\gamma+\delta+5)}{8(\gamma+\delta+1)(\gamma-\delta+1)},\\
\frac{\partial p_{21}}{\partial p_{12}} &= \frac{\partial p_{22}}{\partial p_{11}}=\frac{\gamma(\gamma+5\delta+1)}{8(\gamma+\delta+1)(\gamma-\delta+1)},\\
\frac{\partial p_{21}}{\partial a_{M1}} &= \frac{\partial p_{22}}{\partial a_{M1}}=\frac{3\phi^2}{4(\gamma-\delta+1)},\\
\frac{\partial w_2}{\partial p_{11}} &= \frac{\partial w_2}{\partial p_{12}}=\frac{\gamma}{4(\gamma-\delta+1)},\\
\frac{\partial w_2}{\partial a_{M1}} &= \frac{\phi^2}{2(\gamma-\delta+1)}.
\end{aligned}$$

All the expressions above are positive.

Furthermore,

$$\begin{aligned}
\frac{\partial p_{21}}{\partial a_{M1}} &= \frac{\partial p_{22}}{\partial a_{M1}}=\frac{3}{2}\frac{\partial w_2}{\partial a_{M1}},\\
\frac{\partial p_{21}}{\partial p_{11}} &= \frac{\partial p_{22}}{\partial p_{12}}=\frac{5\gamma+\delta+5}{2(\gamma+\delta+1)}\frac{\partial w_2}{\partial p_{11}}=\frac{5\gamma+\delta+5}{2(\gamma+\delta+1)}\frac{\partial w_2}{\partial p_{12}},\\
\frac{\partial p_{21}}{\partial p_{12}} &= \frac{\partial p_{22}}{\partial p_{11}}=\frac{\gamma+5\delta+1}{2(\gamma+\delta+1)}\frac{\partial w_2}{\partial p_{11}}=\frac{\gamma+5\delta+1}{2(\gamma+\delta+1)}\frac{\partial w_2}{\partial p_{12}}.
\end{aligned}$$

Because

$$\frac{5\gamma+\delta+5}{2(\gamma+\delta+1)}>1$$

and

$$\frac{\gamma+5\delta+1}{2(\gamma+\delta+1)}>1 \text{ if and only if } \delta>\frac{1+\gamma}{3}$$

the results of the comparisons in Proposition 3 easily follow.

Scenario 2

- If equilibrium I or equilibrium III in Scenario 2 is played, from expressions (11), (12), (13), (14) one gets:

$$\begin{aligned}
\frac{\partial p_{21}}{\partial p_{11}} &= \frac{\partial p_{22}}{\partial p_{12}} = \frac{\gamma(\gamma+1)(10(\gamma+1)-7\delta-2\phi^2)}{4(2(\gamma+1)+\delta)\left((\gamma+1-\delta)(2(\gamma+1)-\delta)-(\gamma+1)\phi^2\right)},\\
\frac{\partial p_{21}}{\partial p_{12}} &= \frac{\partial p_{22}}{\partial p_{11}} = \gamma\frac{2(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)-(\gamma+1)(10(\gamma+1)-7\delta-2\phi^2)}{4(2(\gamma+1)+\delta)\left((\gamma+1-\delta)(2(\gamma+1)-\delta)-(\gamma+1)\phi^2\right)},\\
\frac{\partial p_{21}}{\partial a_{M1}} &= \frac{\partial p_{22}}{\partial a_{M1}} = \frac{(3(\gamma+1)-2\delta)\phi^2}{2\left((\gamma+1-\delta)(2(\gamma+1)-\delta)-(\gamma+1)\phi^2\right)},\\
\frac{\partial w_2}{\partial p_{11}} &= \frac{\partial w_2}{\partial p_{12}} = \frac{\gamma(2(\gamma+1)-\delta)}{4\left((\gamma+1-\delta)(2(\gamma+1)-\delta)-(\gamma+1)\phi^2\right)},\\
\frac{\partial w_2}{\partial a_{M1}} &= \frac{(2(\gamma+1)-\delta)\phi^2}{2\left((\gamma+1-\delta)(2(\gamma+1)-\delta)-(\gamma+1)\phi^2\right)},\\
\frac{\partial a_{M2}}{\partial a_{M1}} &= \frac{(\gamma+1)\phi^3}{(\gamma+1-\delta)(2(\gamma+1)-\delta)-(\gamma+1)\phi^2},\\
\frac{\partial a_{M2}}{\partial p_{11}} &= \frac{\partial a_{M2}}{\partial p_{12}} = \frac{\gamma\phi(\gamma+1)}{2\left((\gamma+1-\delta)(2(\gamma+1)-\delta)-(\gamma+1)\phi^2\right)}.
\end{aligned}$$

In stage 4 in the proof of Proposition 2 we have shown that the strict concavity of the manufacturer's second-period profits requires the following condition

$$(\gamma+1-\delta)(2(\gamma+1)-\delta)-(\gamma+1)\phi^2>0, \tag{87}$$

and therefore, all the partial derivatives above are positive under this condition.

Furthermore,

$$\begin{aligned}
\frac{\partial p_{21}}{\partial a_{M1}} &= \frac{\partial p_{22}}{\partial a_{M1}} = \frac{3(\gamma+1)-2\delta}{2(\gamma+1)-\delta}\frac{\partial w_2}{\partial a_{M1}}, \\
\frac{\partial p_{21}}{\partial p_{11}} &= \frac{\partial p_{22}}{\partial p_{12}} = \frac{(\gamma+1)(10(\gamma+1)-7\delta-2\phi^2)}{(2(\gamma+1)+\delta)(2(\gamma+1)-\delta)}\frac{\partial w_2}{\partial p_{11}} = \frac{(\gamma+1)(10(\gamma+1)-7\delta-2\phi^2)}{(2(\gamma+1)+\delta)(2(\gamma+1)-\delta)}\frac{\partial w_2}{\partial p_{12}}, \\
\frac{\partial p_{21}}{\partial p_{12}} &= \frac{\partial p_{22}}{\partial p_{11}} = \frac{2(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)-(\gamma+1)(10(\gamma+1)-7\delta-2\phi^2)}{(2(\gamma+1)+\delta)(\gamma+1-\delta)}\frac{\partial w_2}{\partial p_{11}} \\
&= \frac{2(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)-(\gamma+1)(10(\gamma+1)-7\delta-2\phi^2)}{(2(\gamma+1)+\delta)(\gamma+1-\delta)}\frac{\partial w_2}{\partial p_{12}}, \\
\frac{\partial w_2}{\partial p_{11}} &= \frac{\partial w_2}{\partial p_{12}} = \frac{2(\gamma+1)-\delta}{2(\gamma+1)\phi}\frac{\partial a_{M2}}{\partial p_{11}} = \frac{2(\gamma+1)-\delta}{2(\gamma+1)\phi}\frac{\partial a_{M2}}{\partial p_{12}}, \\
\frac{\partial w_2}{\partial a_{M1}} &= \frac{2(\gamma+1)-\delta}{2(\gamma+1)\phi}\frac{\partial a_{M2}}{\partial a_{M1}}.
\end{aligned}$$

Because

$$\frac{3(\gamma+1)-2\delta}{2(\gamma+1)-\delta} > 1,$$

$$\frac{\partial p_{21}}{\partial a_{M1}} = \frac{\partial p_{22}}{\partial a_{M1}} > \frac{\partial w_2}{\partial a_{M1}}.$$

Under the strict concavity condition given in (87), we next show that

$$\frac{(\gamma+1)(10(\gamma+1)-7\delta-2\phi^2)}{(2(\gamma+1)+\delta)(2(\gamma+1)-\delta)} > 1, \tag{88}$$

and therefore,

$$\frac{\partial p_{21}}{\partial p_{11}} = \frac{\partial p_{22}}{\partial p_{12}} > \frac{\partial w_2}{\partial p_{11}} = \frac{\partial w_2}{\partial p_{12}}.$$

The concavity condition (87) can be rewritten as

$$\phi^2 < \frac{(\gamma+1-\delta)(2(\gamma+1)-\delta)}{\gamma+1}. \tag{89}$$

Condition in (88) can be rewritten as

$$\phi^2 < \frac{(\gamma+1)(6(\gamma+1)-7\delta)+\delta^2}{2(\gamma+1)}.$$

If

$$\frac{(\gamma+1-\delta)(2(\gamma+1)-\delta)}{\gamma+1} < \frac{(\gamma+1)(6(\gamma+1)-7\delta)+\delta^2}{2(\gamma+1)},$$

the proof is finished. Last inequality can be rewritten as

$$2(\gamma+1-\delta)(2(\gamma+1)-\delta) < (\gamma+1)(6(\gamma+1)-7\delta)+\delta^2$$

or equivalently,

$$-2(\gamma+1)^2+\delta(\gamma+1)+\delta^2 < 0.$$

The analysis of this quadratic polynomial in variable $\gamma+1$ shows that last inequality is satisfied if either $1+\gamma < -\delta/2$ or $1+\gamma > \delta/2$, and last condition always holds for any $\gamma, \delta \in (0,1)$.

To prove that

$$\frac{\partial p_{21}}{\partial p_{11}} = \frac{\partial p_{22}}{\partial p_{12}} > \frac{\partial p_{21}}{\partial p_{12}} = \frac{\partial p_{22}}{\partial p_{11}},$$

it is equivalent to prove that the following inequality holds:

$$\frac{(\gamma+1)(10(\gamma+1)-7\delta-2\phi^2)}{2(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)-(\gamma+1)(10(\gamma+1)-7\delta-2\phi^2)} > 1$$

or equivalently,

$$\frac{2(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)-(\gamma+1)(10(\gamma+1)-7\delta-2\phi^2)}{(\gamma+1)(10(\gamma+1)-7\delta-2\phi^2)} < 1,$$

that is,

$$\frac{2(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)}{(\gamma+1)(10(\gamma+1)-7\delta-2\phi^2)} - 1 < 1$$

and simplifying

$$\frac{(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)}{(\gamma+1)(10(\gamma+1)-7\delta-2\phi^2)} < 1.$$

Last inequality can be rewritten as:

$$(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta) < (\gamma+1)(10(\gamma+1)-7\delta-2\phi^2)$$

and rearranging terms:

$$-4(\gamma+1)^2+6\delta(\gamma+1)-2\delta^2+2\phi^2(\gamma+1) < 0.$$

Last inequality is equivalent to the following condition in terms of an upper bound for ϕ^2:

$$\phi^2 < \frac{2(\gamma+1)^2-3\delta(\gamma+1)+\delta^2}{\gamma+1}.$$

This last condition coincides with the concavity condition in (89).

From the expressions above, one has that

$$\frac{\partial p_{21}}{\partial p_{12}} = \frac{\partial p_{22}}{\partial p_{11}} > \frac{\partial w_2}{\partial p_{11}} = \frac{\partial w_2}{\partial p_{12}},$$

if and only if the following inequality holds:

$$\frac{2(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)-(\gamma+1)(10(\gamma+1)-7\delta-2\phi^2)}{(2(\gamma+1)-\delta)(2(\gamma+1)+\delta)} > 1.$$

Last inequality can be rewritten as

$$2(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)-(\gamma+1)(10(\gamma+1)-7\delta-2\phi^2) > (2(\gamma+1)-\delta)(2(\gamma+1)+\delta),$$

and rearranging terms:

$$-2(\gamma+1)^2+5(\gamma+1)\delta+2(\gamma+1)\phi^2-3\delta^2 > 0.$$

Therefore,

$$\frac{\partial p_{21}}{\partial p_{12}} = \frac{\partial p_{22}}{\partial p_{11}} > \frac{\partial w_2}{\partial p_{11}} = \frac{\partial w_2}{\partial p_{12}} \text{ if and only if } \phi^2 > 1+\gamma+\frac{\delta(3\delta-5)}{2(\gamma+1)}.$$

To prove that

$$\frac{\partial w_2}{\partial p_{11}} = \frac{\partial w_2}{\partial p_{12}} > \frac{\partial a_{M2}}{\partial p_{11}} = \frac{\partial a_{M2}}{\partial p_{12}}, \; \frac{\partial w_2}{\partial a_{M1}} > \frac{\partial a_{M2}}{\partial a_{M1}}$$

is equivalent to prove that

$$\frac{2(\gamma+1)-\delta}{2(\gamma+1)\phi} > 1,$$

or equivalently,

$$\delta < 2(\gamma+1)(1-\phi).$$

- If equilibrium II or equilibrium IV in Scenario 2 is played, from expressions (16), (17), (18) one gets:

$$\begin{aligned}
\frac{\partial p_{21}}{\partial p_{11}} &= \frac{\partial p_{22}}{\partial p_{12}} = \frac{\gamma(\gamma+1)(10(\gamma+1)-7\delta)}{4(\gamma-\delta+1)(4(\gamma+1)^2-\delta^2)},\\
\frac{\partial p_{21}}{\partial p_{12}} &= \frac{\partial p_{22}}{\partial p_{11}} = \frac{\gamma(5(\gamma+1)\delta+2(\gamma+1)^2-4\delta^2)}{4(\gamma-\delta+1)(4(\gamma+1)^2-\delta^2)},\\
\frac{\partial p_{21}}{\partial a_{M1}} &= \frac{\partial p_{22}}{\partial a_{M1}} = \frac{(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)\phi^2}{2(\gamma-\delta+1)(4(\gamma+1)^2-\delta^2)},\\
\frac{\partial w_2}{\partial p_{11}} &= \frac{\partial w_2}{\partial p_{12}} = \frac{\gamma}{4(\gamma-\delta+1)},\\
\frac{\partial w_2}{\partial a_{M1}} &= \frac{\phi^2}{2(\gamma-\delta+1)}.
\end{aligned}$$

All the expressions above are positive.

Furthermore,

$$\begin{aligned}
\frac{\partial p_{21}}{\partial a_{M1}} &= \frac{\partial p_{22}}{\partial a_{M1}} = \frac{(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)}{4(\gamma+1)^2-\delta^2}\frac{\partial w_2}{\partial a_{M1}},\\
\frac{\partial p_{21}}{\partial p_{11}} &= \frac{\partial p_{22}}{\partial p_{12}} = \frac{(\gamma+1)(10(\gamma+1)-7\delta)}{4(\gamma+1)^2-\delta^2}\frac{\partial w_2}{\partial p_{11}} = \frac{(\gamma+1)(10(\gamma+1)-7\delta)}{4(\gamma+1)^2-\delta^2}\frac{\partial w_2}{\partial p_{12}},\\
\frac{\partial p_{21}}{\partial p_{12}} &= \frac{\partial p_{22}}{\partial p_{11}} = \frac{5(\gamma+1)\delta+2(\gamma+1)^2-4\delta^2}{4(\gamma+1)^2-\delta^2}\frac{\partial w_2}{\partial p_{11}} = \frac{5(\gamma+1)\delta+2(\gamma+1)^2-4\delta^2}{4(\gamma+1)^2-\delta^2}\frac{\partial w_2}{\partial p_{12}},\\
\frac{\partial p_{21}}{\partial p_{11}} &= \frac{\partial p_{22}}{\partial p_{12}} = \frac{(\gamma+1)(10(\gamma+1)-7\delta)}{5(\gamma+1)\delta+2(\gamma+1)^2-4\delta^2}\frac{\partial p_{21}}{\partial p_{12}} = \frac{(\gamma+1)(10(\gamma+1)-7\delta)}{5(\gamma+1)\delta+2(\gamma+1)^2-4\delta^2}\frac{\partial p_{22}}{\partial p_{11}}.
\end{aligned}$$

To prove that

$$\frac{\partial p_{21}}{\partial a_{M1}} = \frac{\partial p_{22}}{\partial a_{M1}} > \frac{\partial w_2}{\partial a_{M1}},$$

it is equivalent to prove that the following inequality holds:

$$\frac{(2(\gamma+1)+\delta)(3(\gamma+1)-2\delta)}{4(\gamma+1)^2-\delta^2} > 1.$$

Last inequality simplifies as:

$$2(\gamma+1)^2-\delta(\gamma+1)-\delta^2 > 0.$$

This condition is satisfied if and only if either $\gamma+1 < -1/(2\delta)$ or $\gamma+1 > \delta$. Last condition is always satisfied for any $\delta, \gamma \in (0,1)$.

To prove that

$$\frac{\partial p_{21}}{\partial p_{11}} = \frac{\partial p_{22}}{\partial p_{12}} > \frac{\partial w_2}{\partial p_{11}} = \frac{\partial w_2}{\partial p_{12}},$$

it is equivalent to prove that the following inequality holds:

$$\frac{(\gamma+1)(10(\gamma+1)-7\delta)}{4(\gamma+1)^2-\delta^2} > 1.$$

Last inequality simplifies as:

$$6(\gamma+1)^2-7\delta(\gamma+1)+\delta^2 > 0.$$

This condition is satisfied if and only if either $\gamma+1 < -1/(2\delta)$ or $\gamma+1 > \delta$. Last condition is always satisfied for any $\delta, \gamma \in (0,1)$.

To prove that

$$\frac{\partial p_{21}}{\partial p_{11}} = \frac{\partial p_{22}}{\partial p_{12}} > \frac{\partial p_{21}}{\partial p_{12}} = \frac{\partial p_{22}}{\partial p_{11}},$$

it is equivalent to prove that the following inequality holds:

$$\frac{(\gamma+1)(10(\gamma+1)-7\delta)}{5(\gamma+1)\delta+2(\gamma+1)^2-4\delta^2} > 1.$$

Last inequality simplifies as:

$$2(\gamma+1)^2 - 3\delta(\gamma+1) + \delta^2 > 0.$$

This condition is satisfied if and only if either $\gamma+1 < \delta/2$ or $\gamma+1 > \delta$. Last condition is always satisfied for any $\delta, \gamma \in (0,1)$.

From the expressions above, one has that

$$\frac{\partial p_{21}}{\partial p_{12}} = \frac{\partial p_{22}}{\partial p_{11}} > \frac{\partial w_2}{\partial p_{11}} = \frac{\partial w_2}{\partial p_{12}},$$

if and only if the following inequality holds:

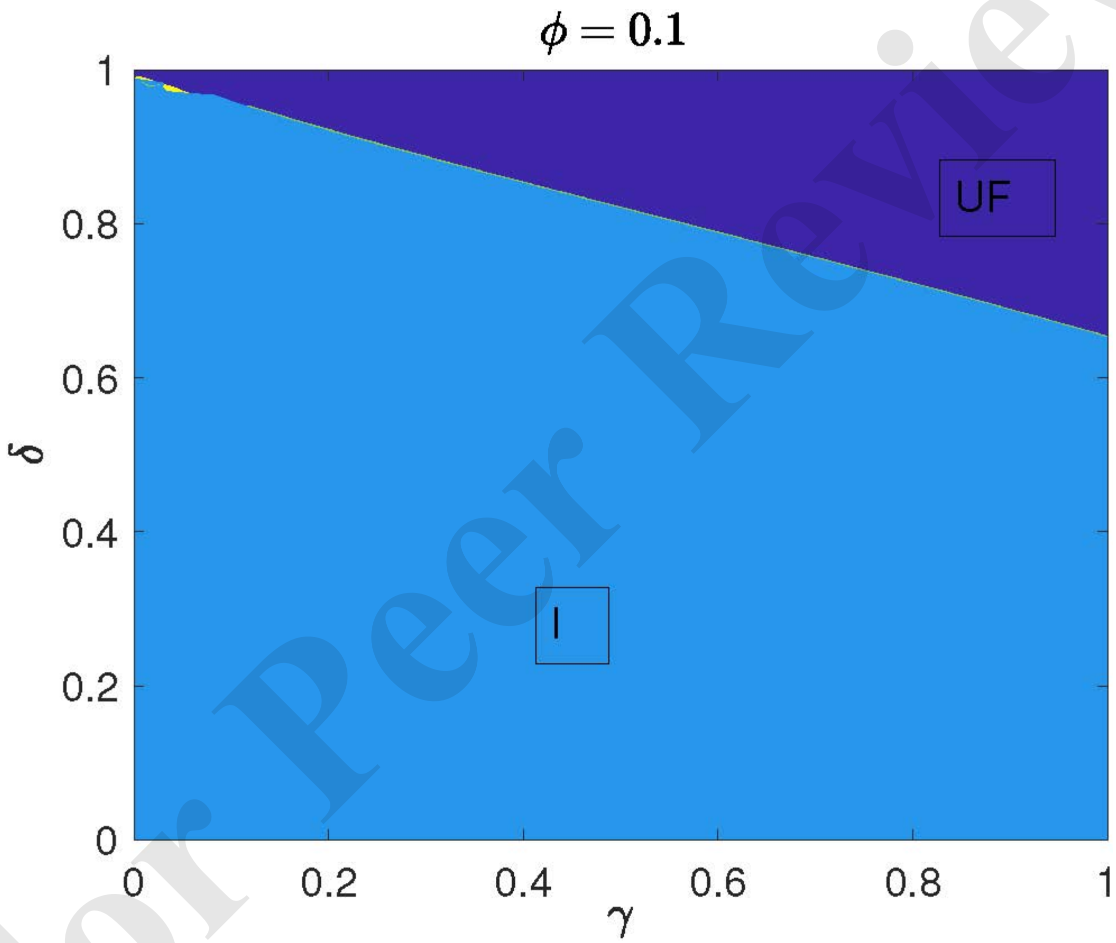
$$\frac{5(\gamma+1)\delta+2(\gamma+1)^2-4\delta^2}{4(\gamma+1)^2-\delta^2} > 1.$$

Last inequality simplifies as: as

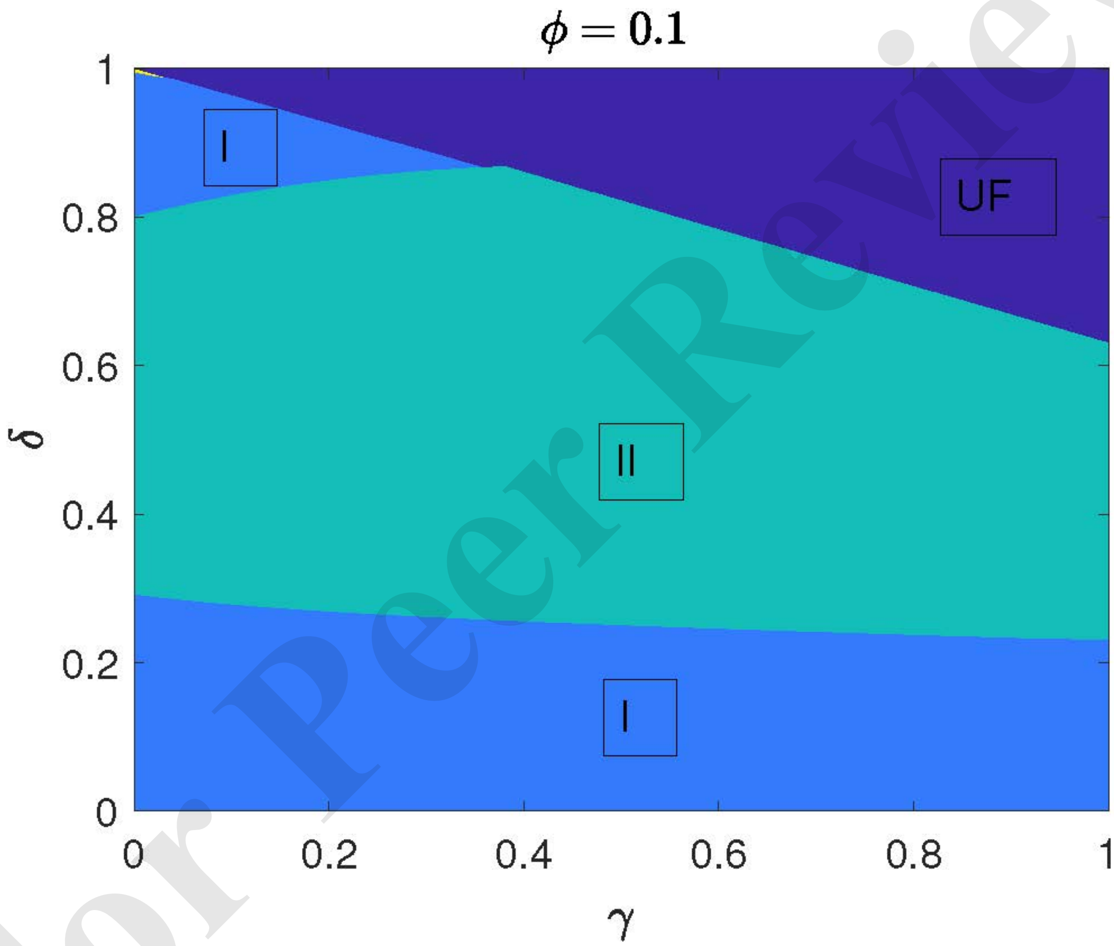
$$-2(\gamma+1)^2 + 5(\gamma+1)\delta - 3\delta^2 > 0.$$

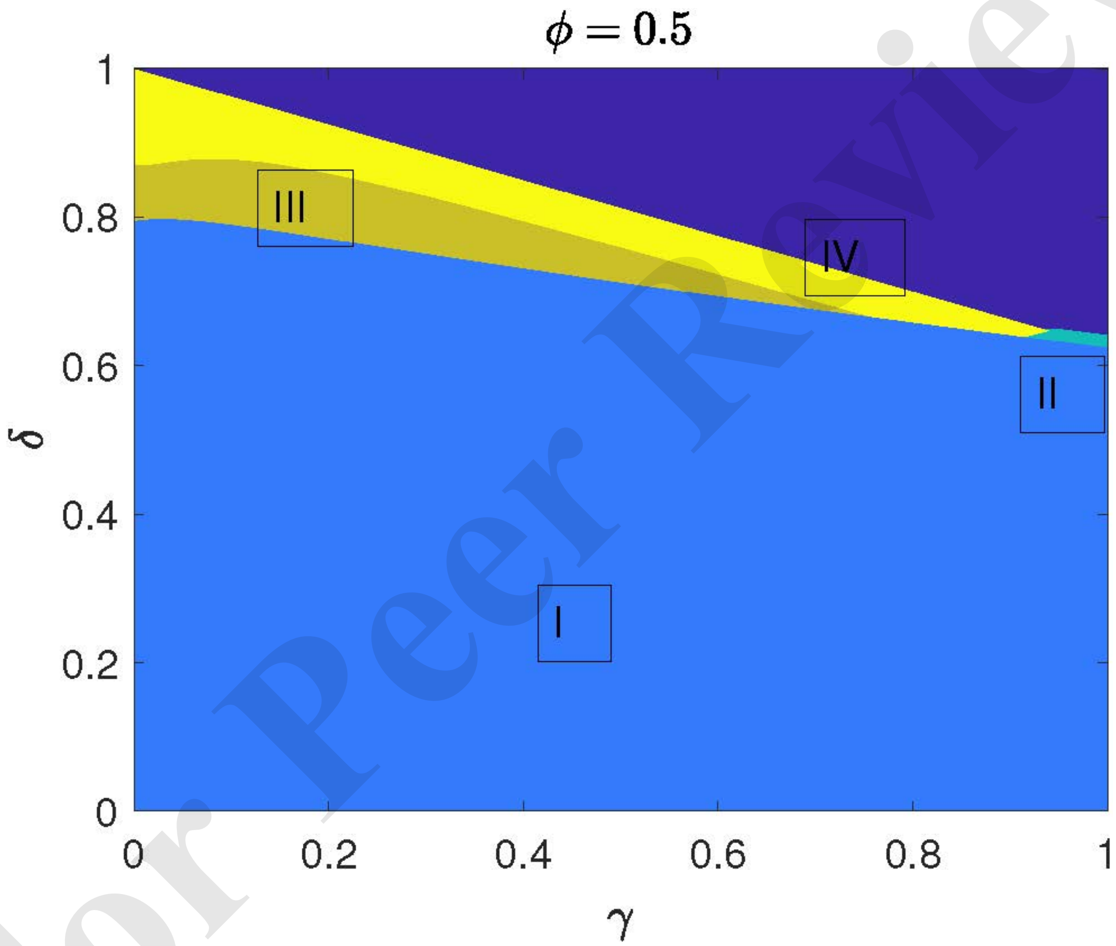
This condition is satisfied if and only if $\delta < \gamma+1 < 3\delta/2$. Therefore,

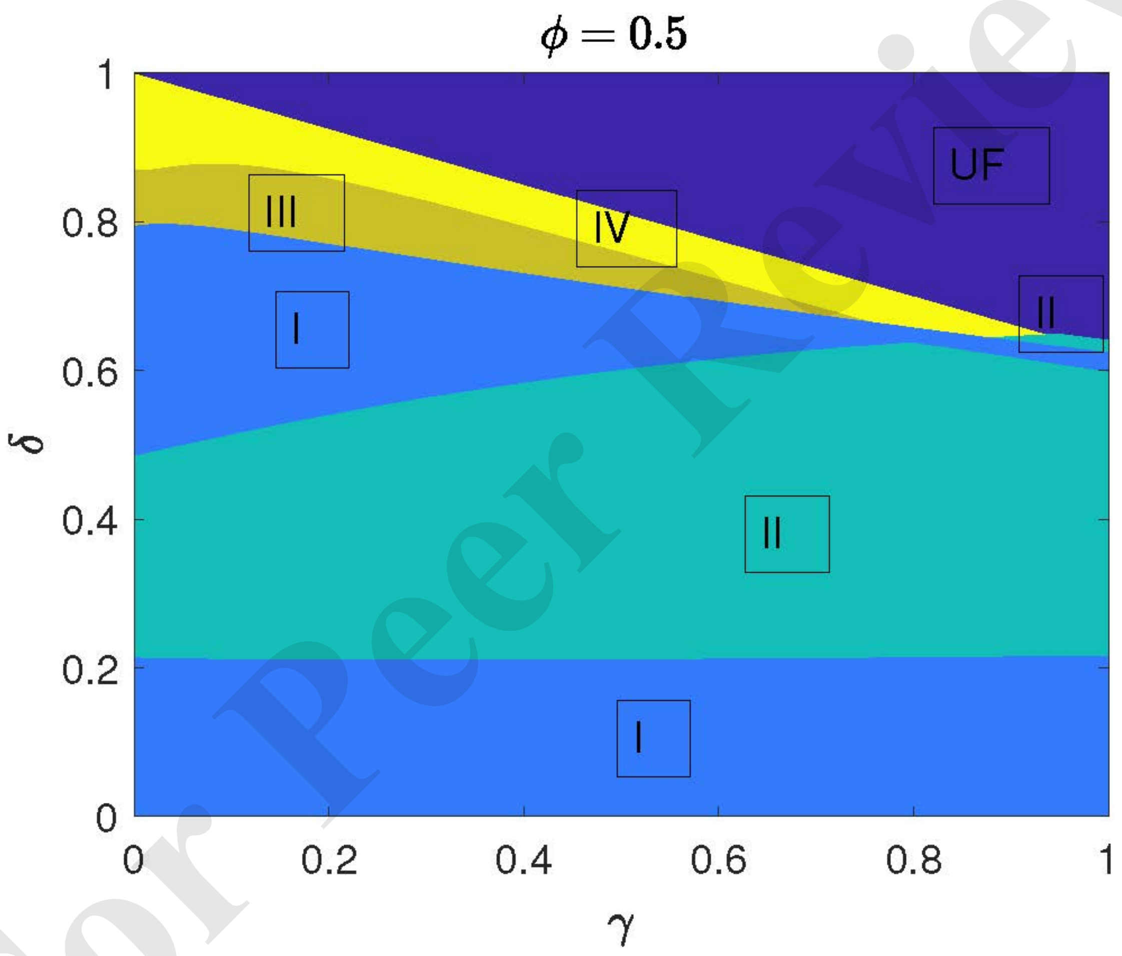
$$\frac{\partial p_{21}}{\partial p_{12}} = \frac{\partial p_{22}}{\partial p_{11}} > \frac{\partial w_2}{\partial p_{11}} = \frac{\partial w_2}{\partial p_{12}} \text{ if and only if } \delta > \frac{2}{3}(\gamma+1).$$



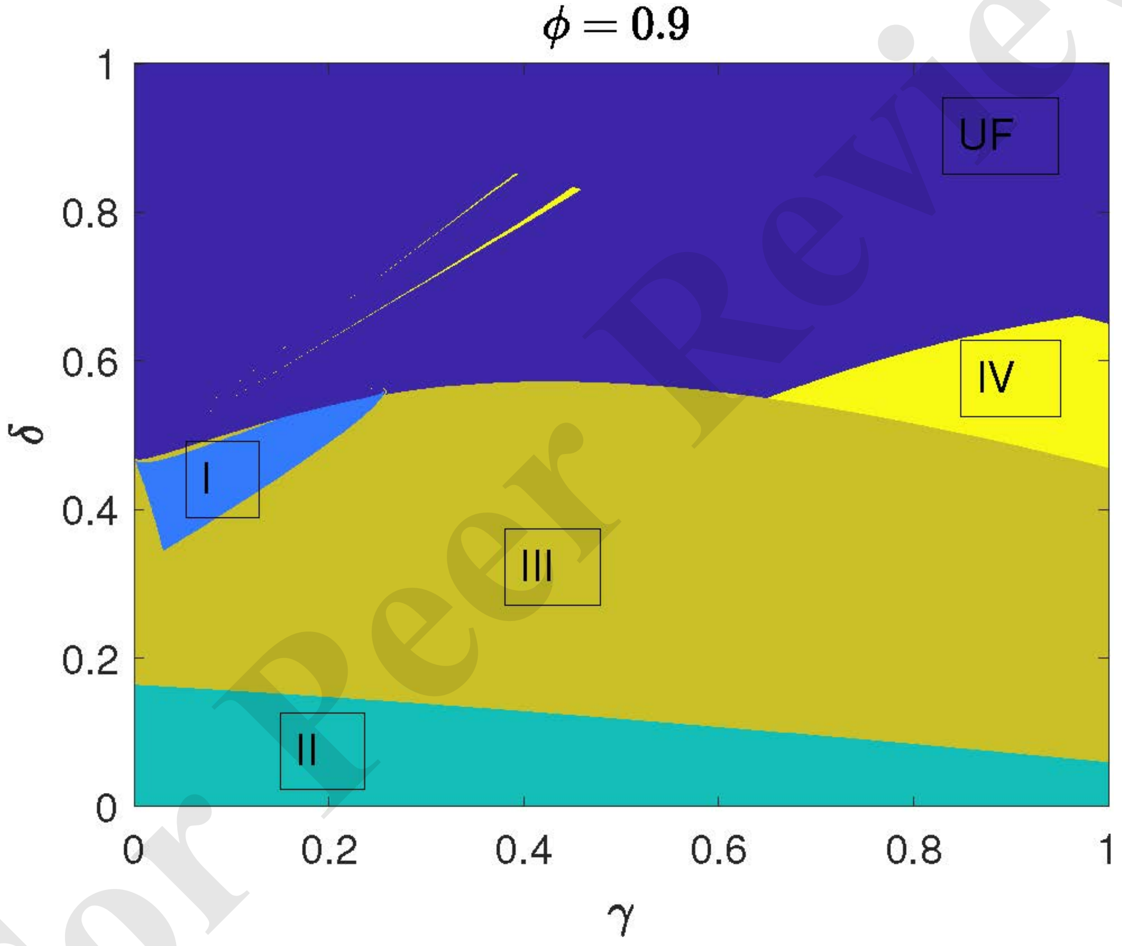
$\phi = 0.1$
UF
I
δ
γ
0
0.2
0.4
0.6
0.8
1

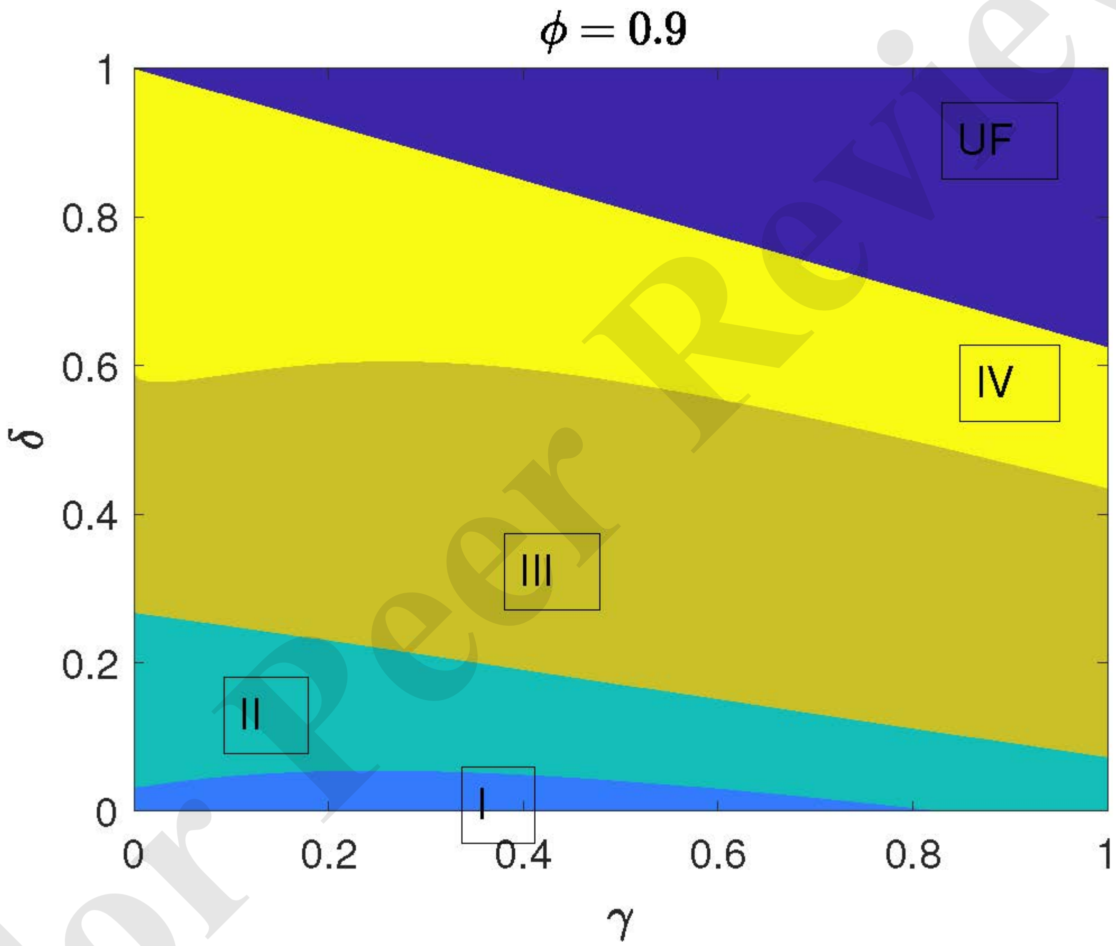

$\phi = 0.1$
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
δ
γ
I
UF
II
I

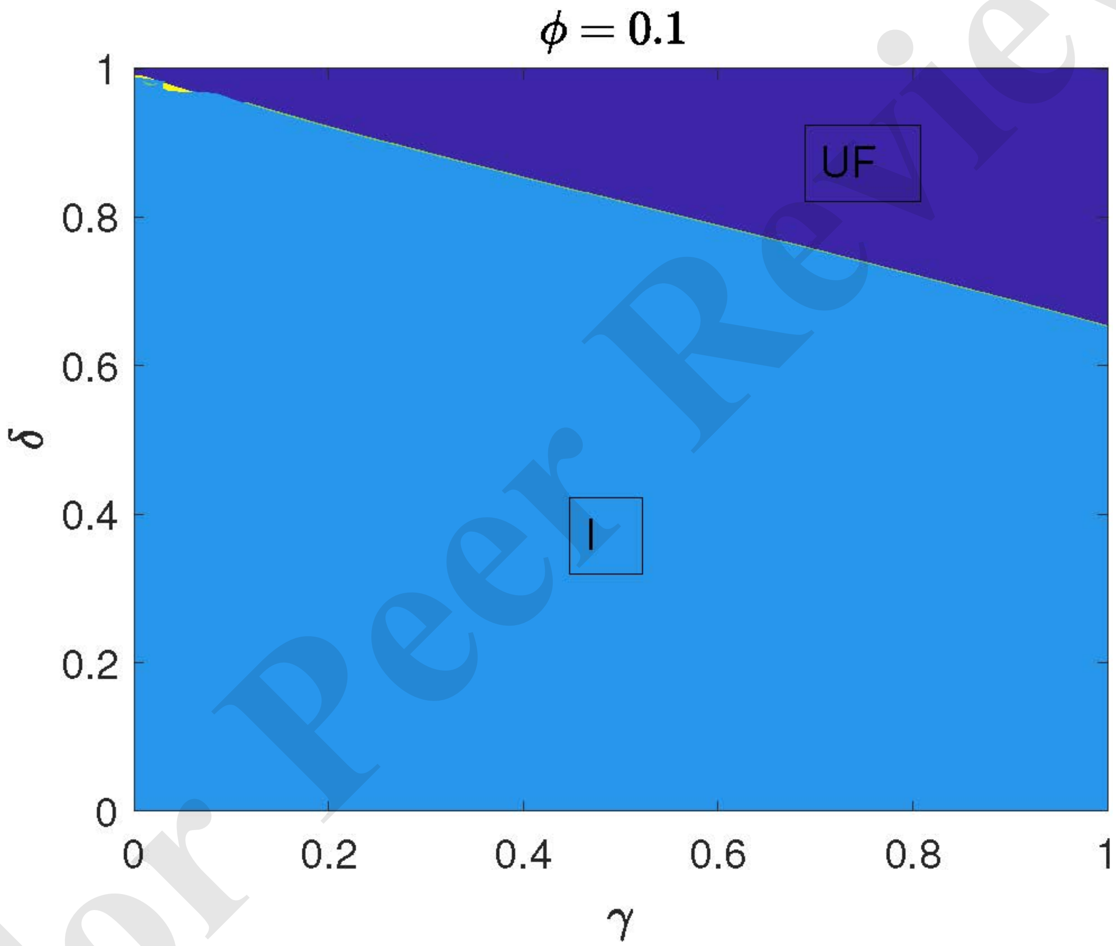

$\phi = 0.5$
1
0.8
0.6
0.4
0.2
0
δ
0
0.2
0.4
0.6
0.8
1
γ
III
IV
II
I

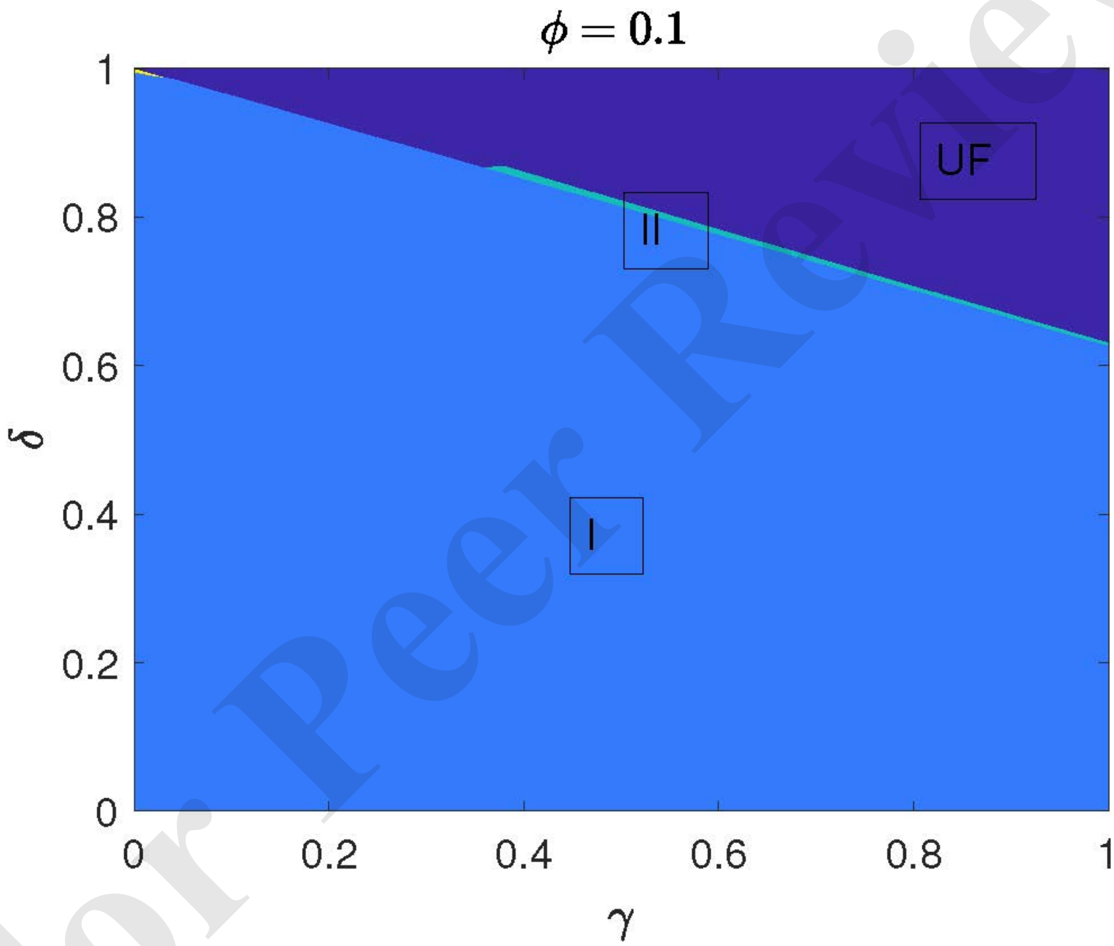


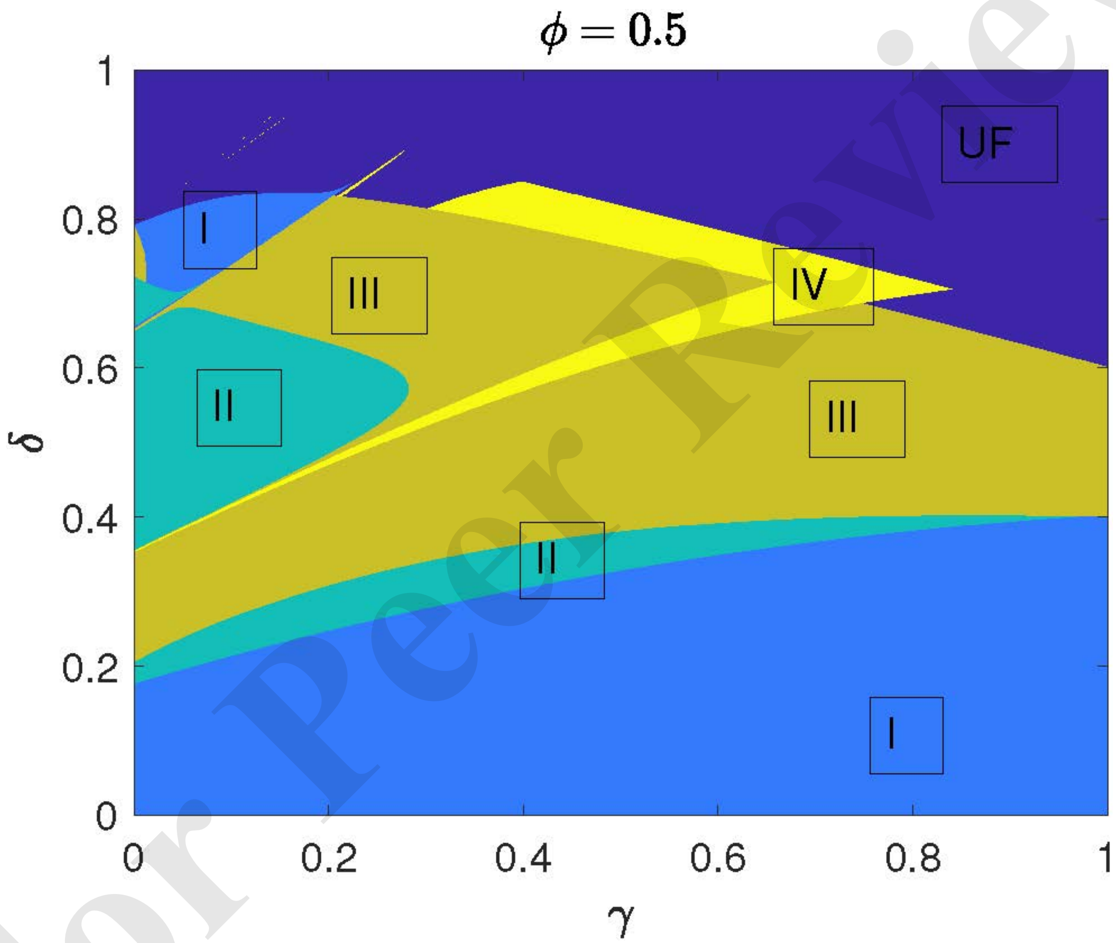
$\phi = 0.5$
UF
III
IV
I
II
II
I
δ
γ
0
0.2
0.4
0.6
0.8
1


$\phi = 0.9$
UF
IV
I
III
II
δ
γ
0
0.2
0.4
0.6
0.8
1

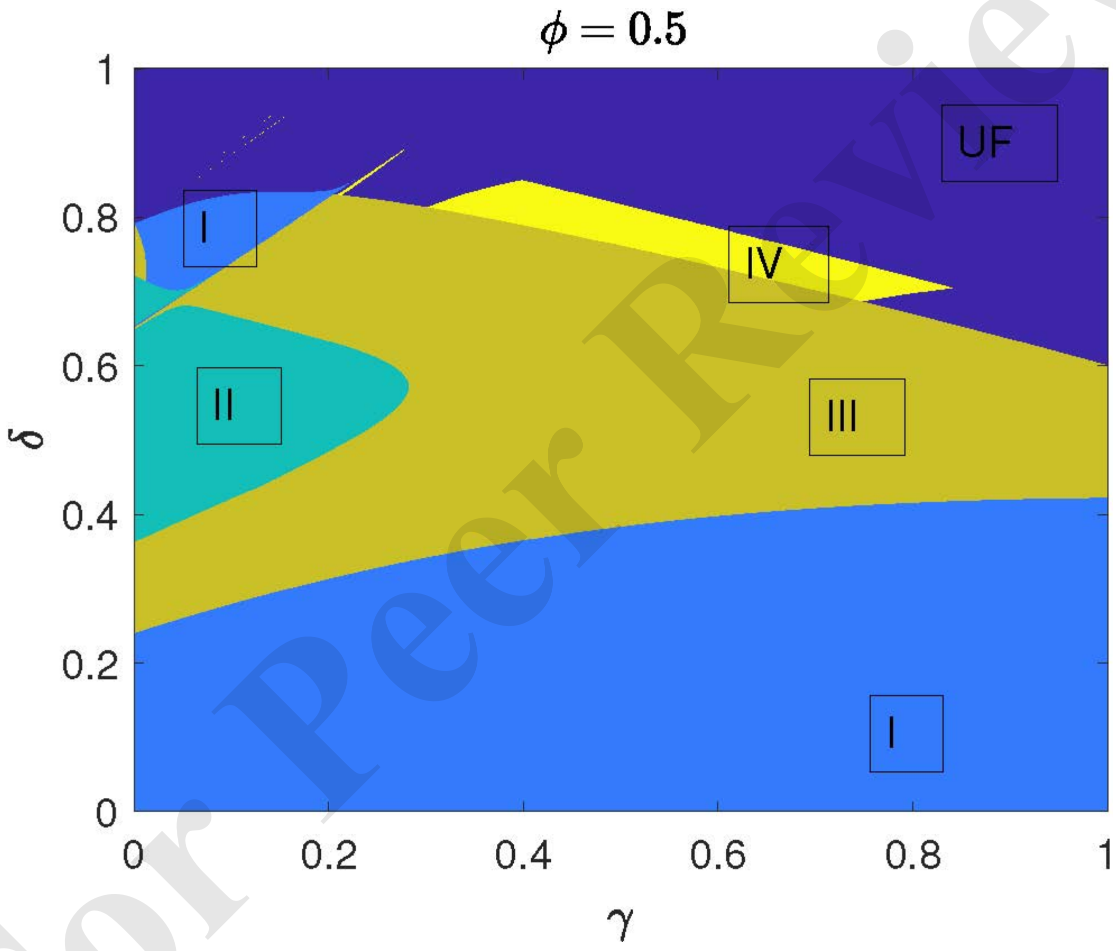

$\phi = 0.9$
UF
IV
III
II
I
δ
γ
0
0.2
0.4
0.6
0.8
1


$\phi = 0.1$
UF
I
δ
γ
0
0.2
0.4
0.6
0.8
1

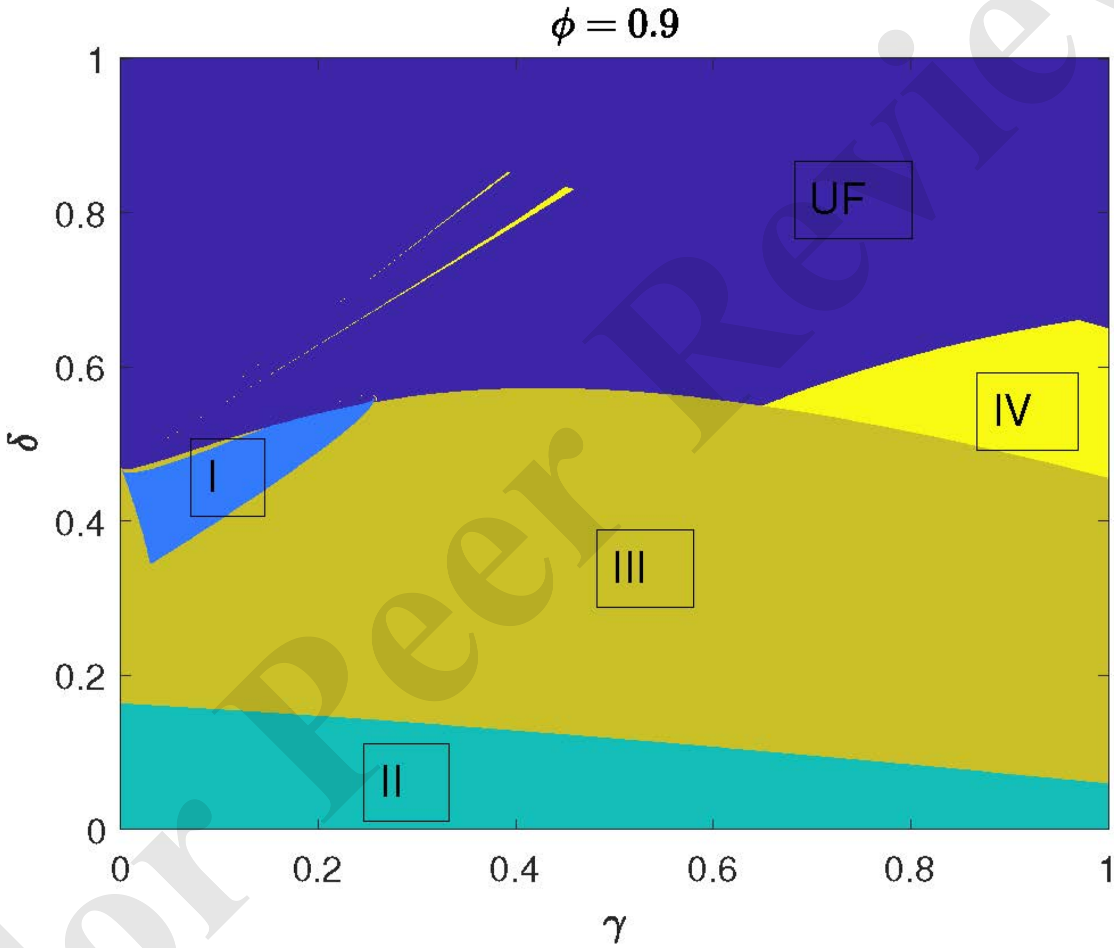

$\phi = 0.1$
UF
II
I
δ
γ
0
0.2
0.4
0.6
0.8
1



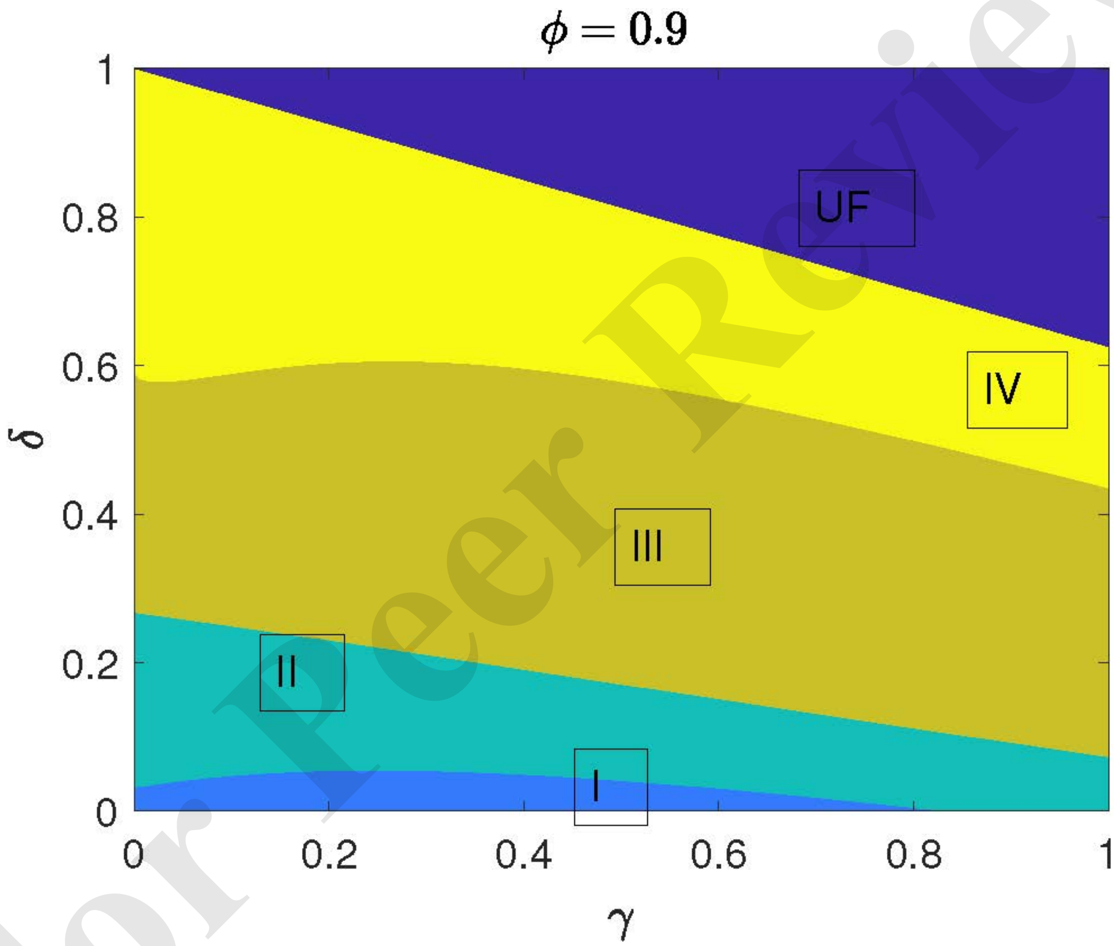
$\phi = 0.5$
UF
I
III
IV
II
III
II
I
δ
γ
0
0.2
0.4
0.6
0.8
1

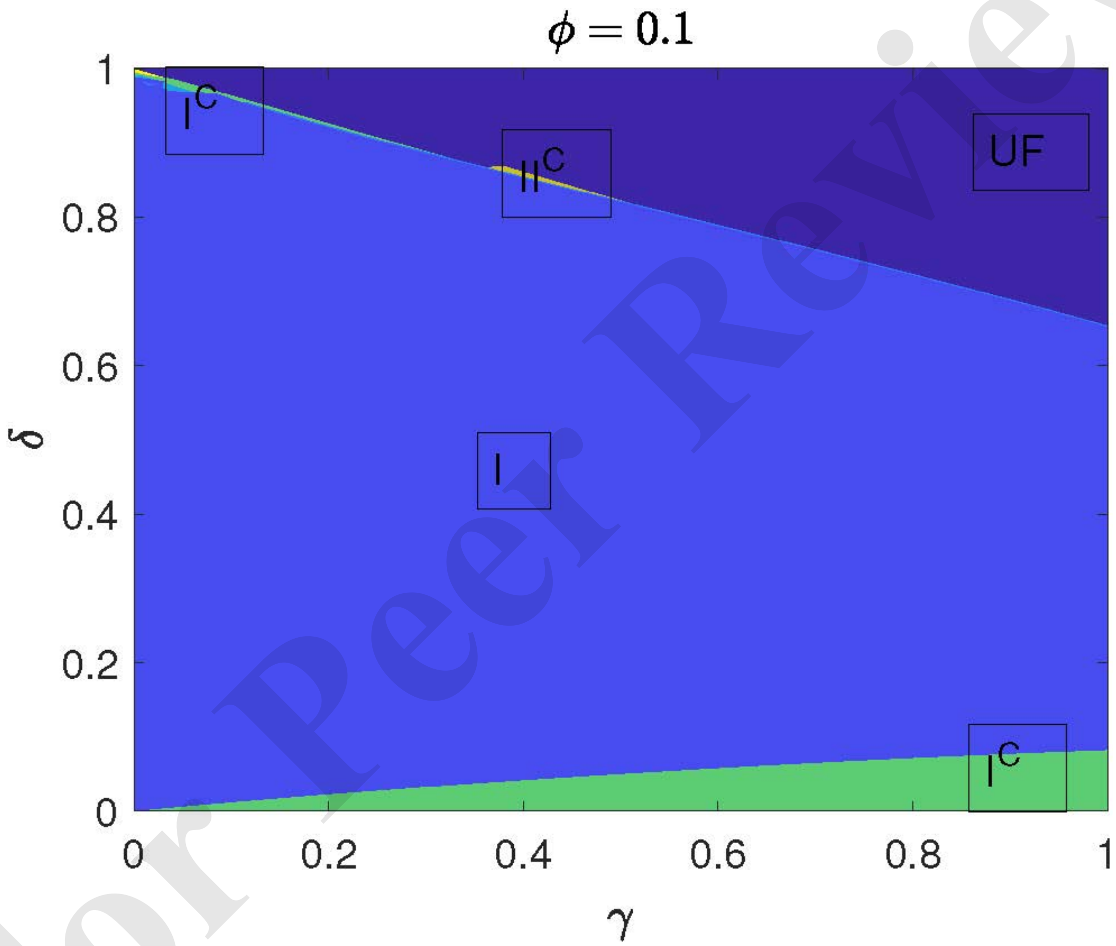


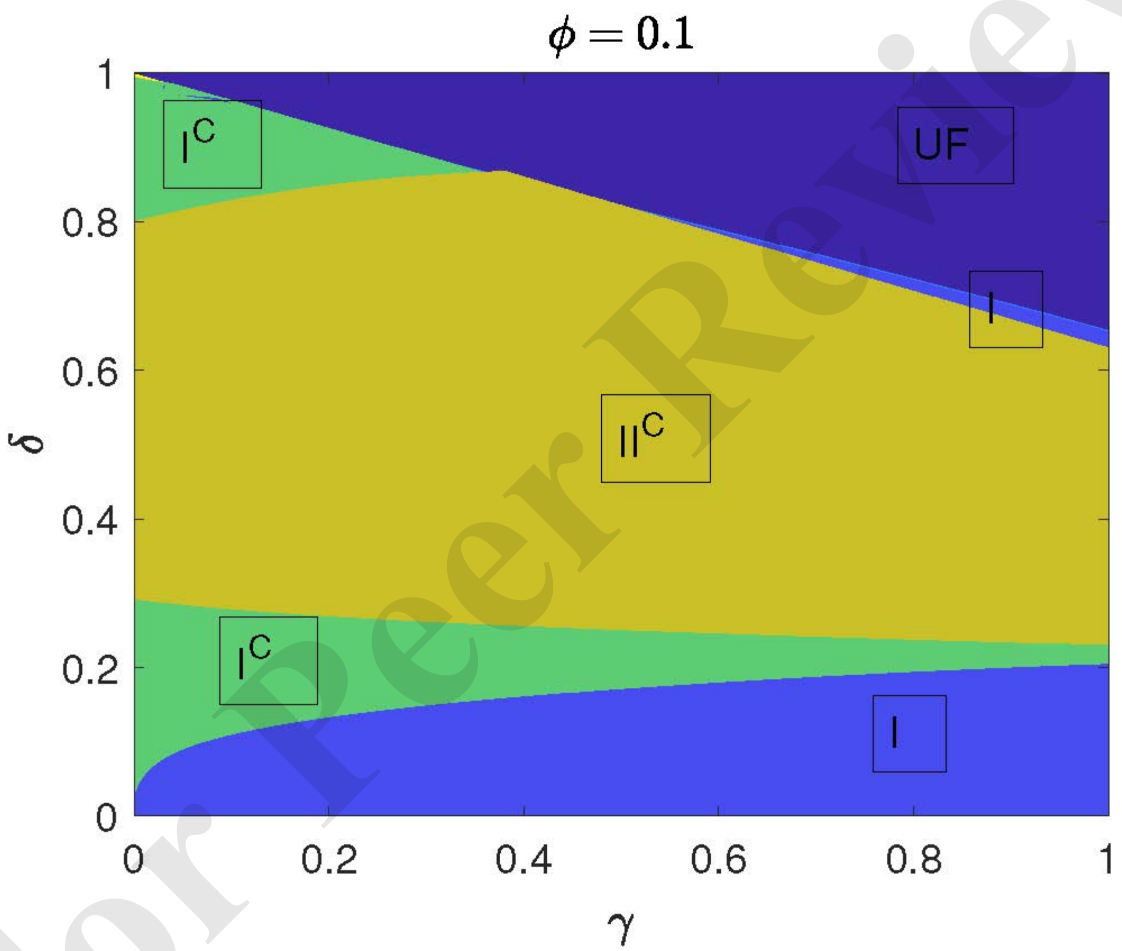
$\phi = 0.5$
1
0.8
0.6
0.4
0.2
0
δ
0
0.2
0.4
0.6
0.8
1
γ
UF
I
IV
II
III
I

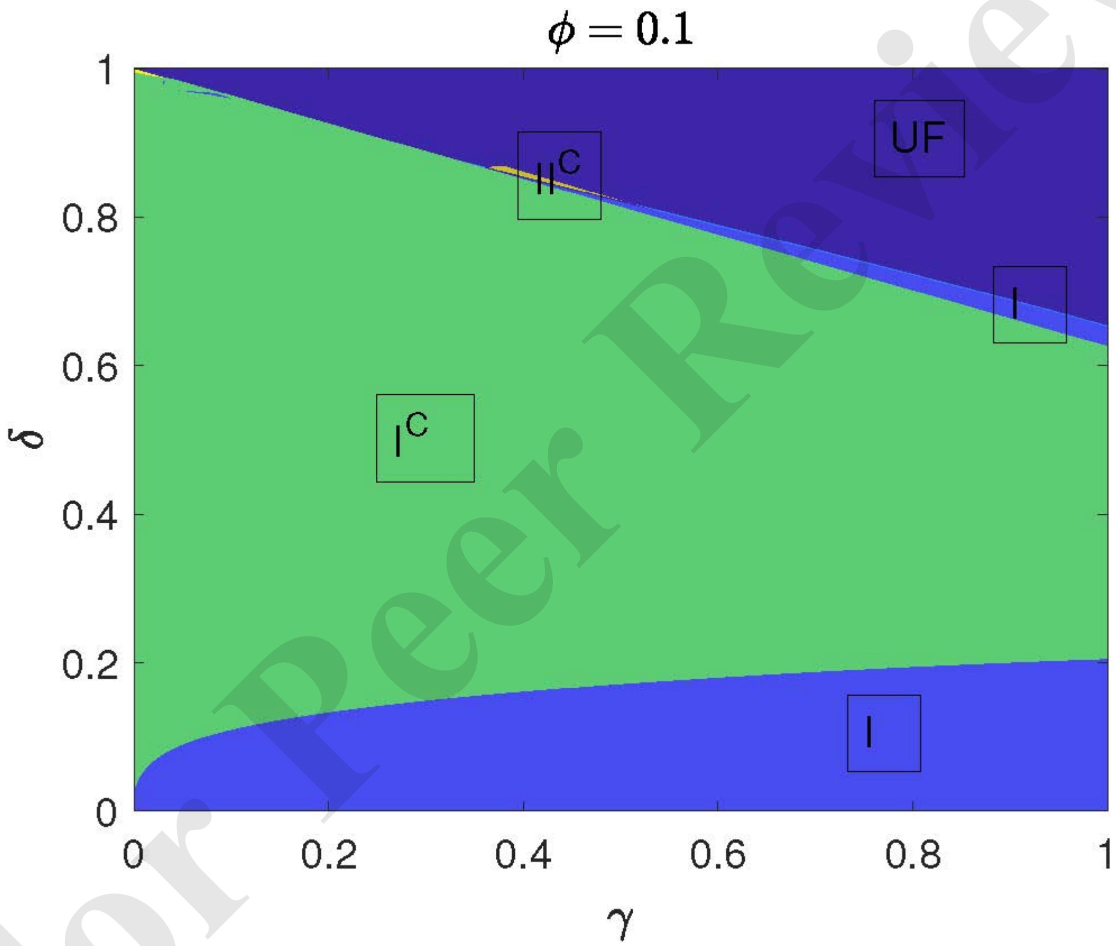


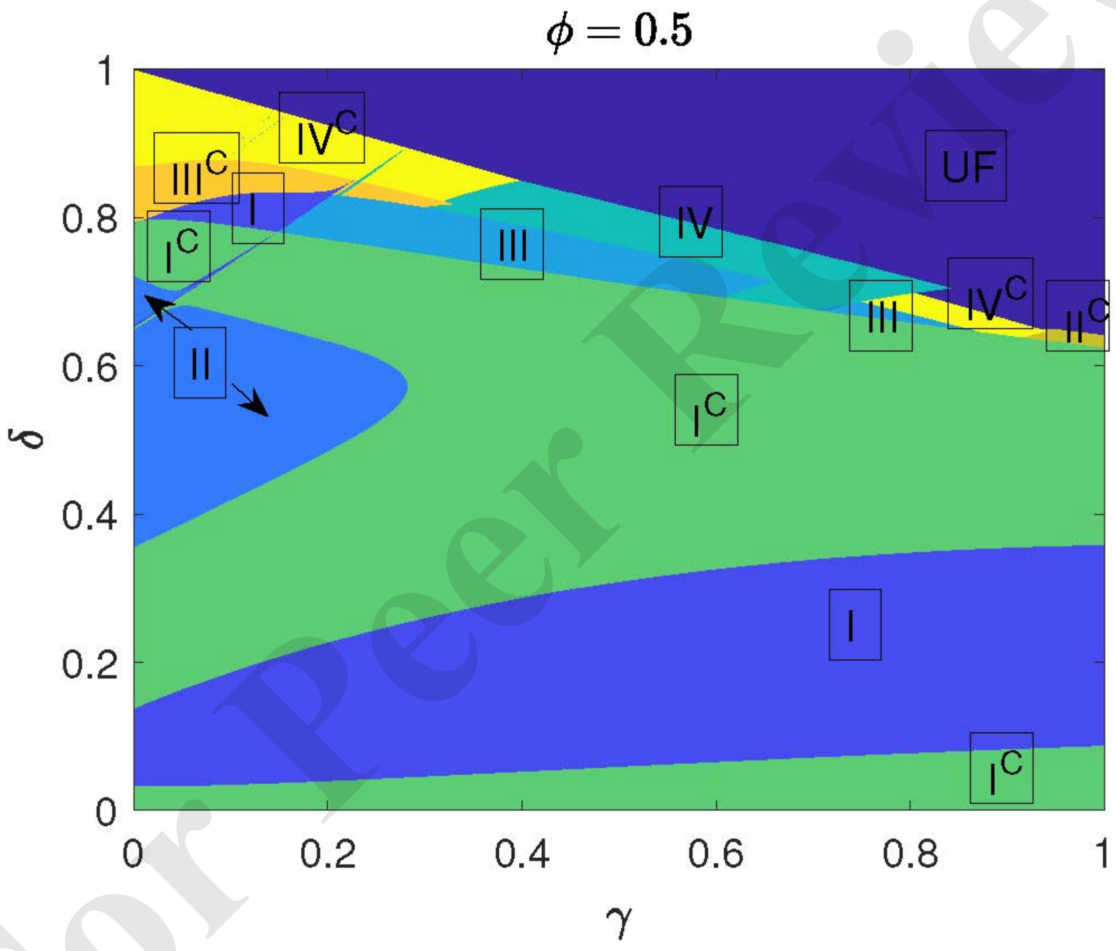
$\phi = 0.9$
1
0.8
0.6
0.4
0.2
0
δ
UF
IV
I
III
II
0
0.2
0.4
0.6
0.8
1
γ

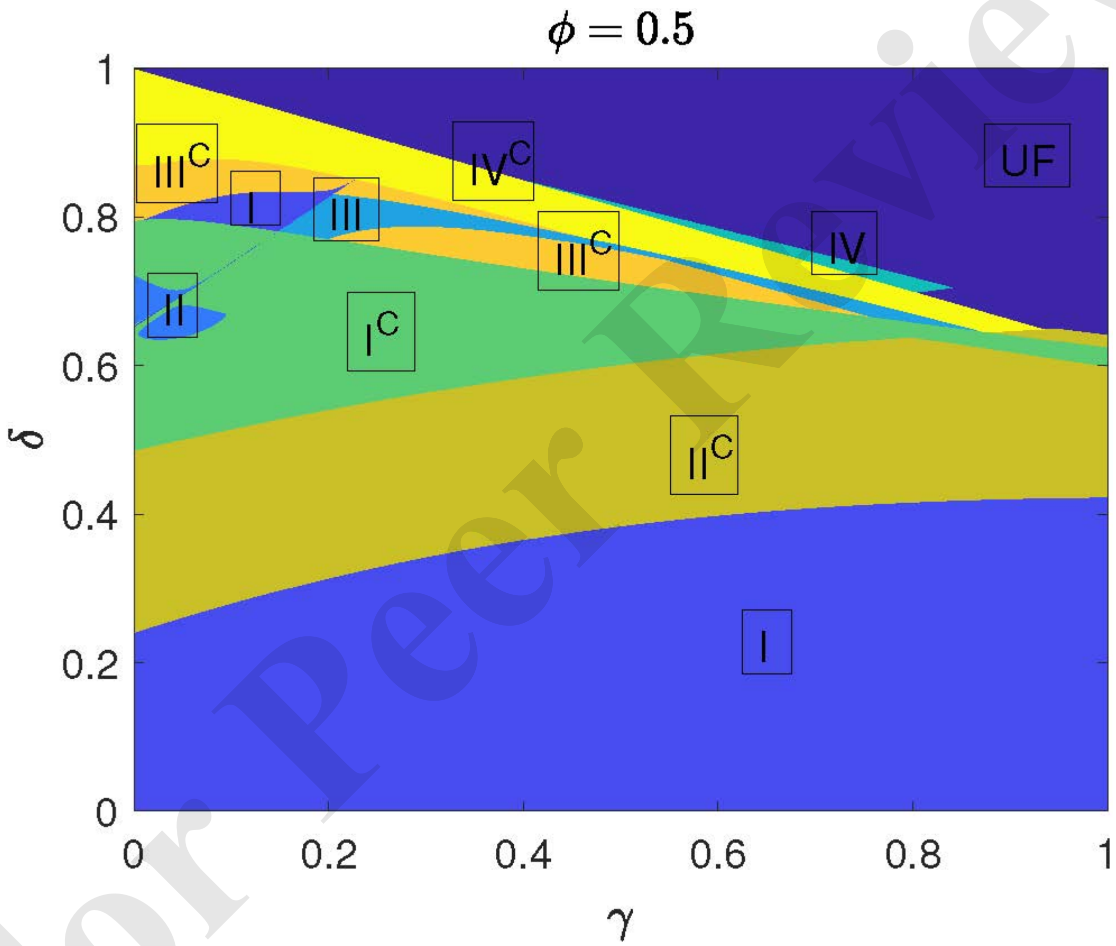

$\phi = 0.9$
UF
IV
III
II
I
δ
γ
0
0.2
0.4
0.6
0.8
1


$\phi = 0.1$
I^C
II^C
UF
I
I^C
δ
γ
0
0.2
0.4
0.6
0.8
1

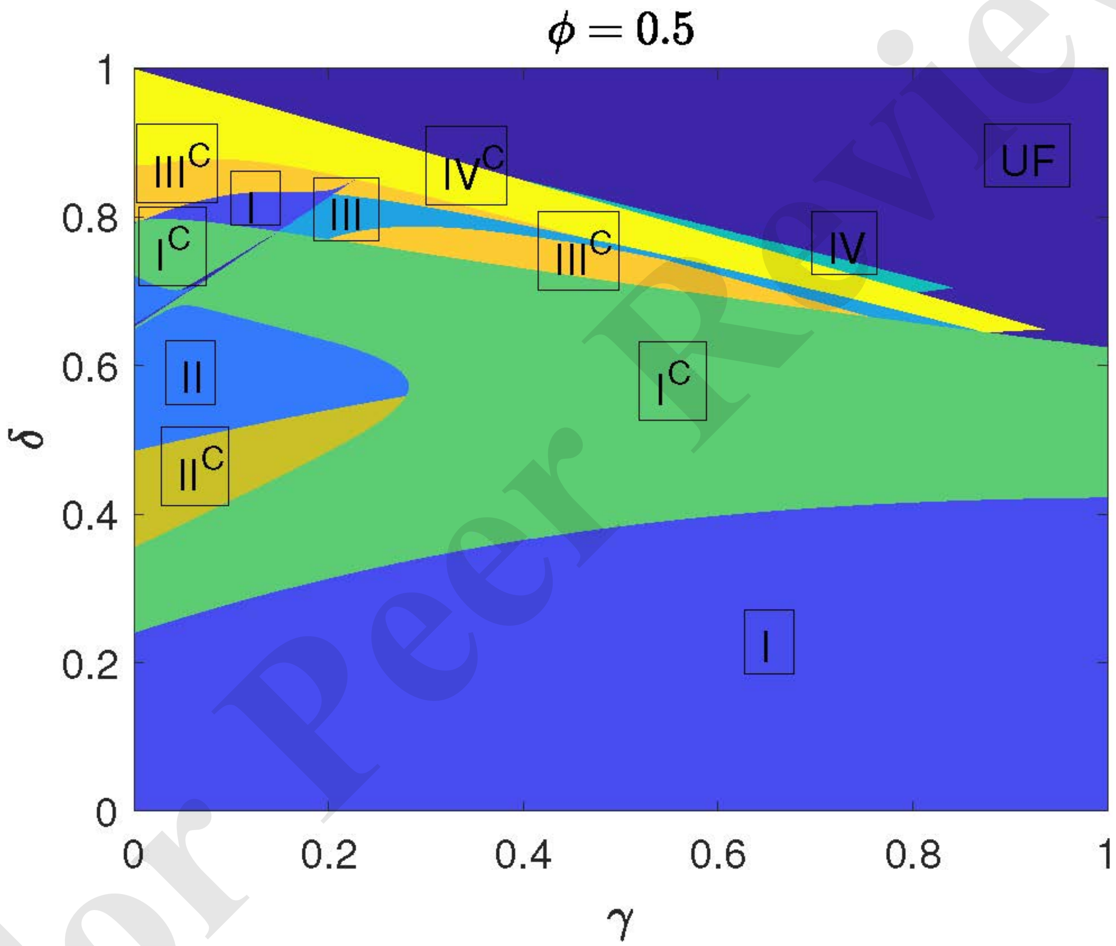

$\phi = 0.1$
$\mathrm{I^C}$
UF
I
$\mathrm{II^C}$
$\mathrm{I^C}$
I
δ
γ
0
0.2
0.4
0.6
0.8
1

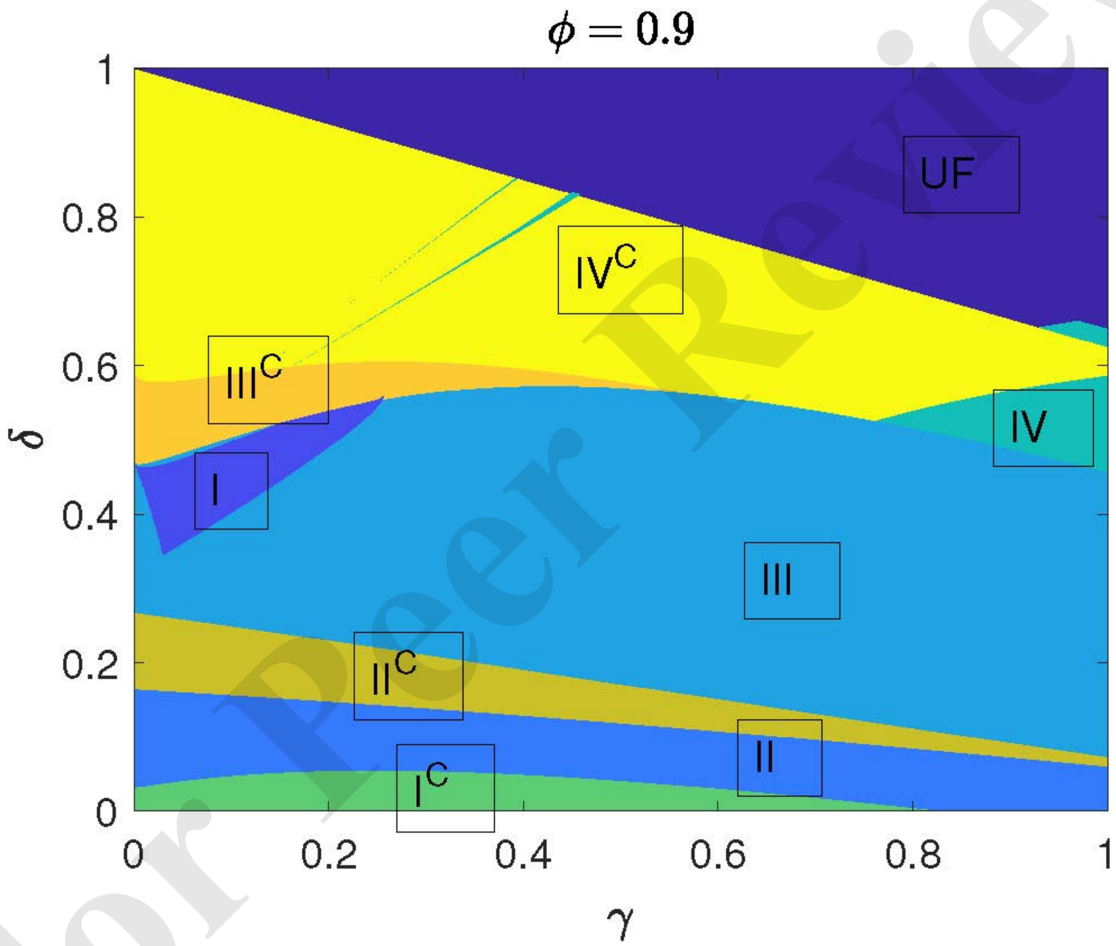

$\phi = 0.1$
UF
II^C
I
I^C
I
δ
γ
0
0.2
0.4
0.6
0.8
1


$\phi = 0.5$
1
0.8
0.6
0.4
0.2
0
δ
0
0.2
0.4
0.6
0.8
1
γ
IV^C
III^C
I
I^C
III
IV
UF
III
IV^C
II^C
II
I^C
I
I^C

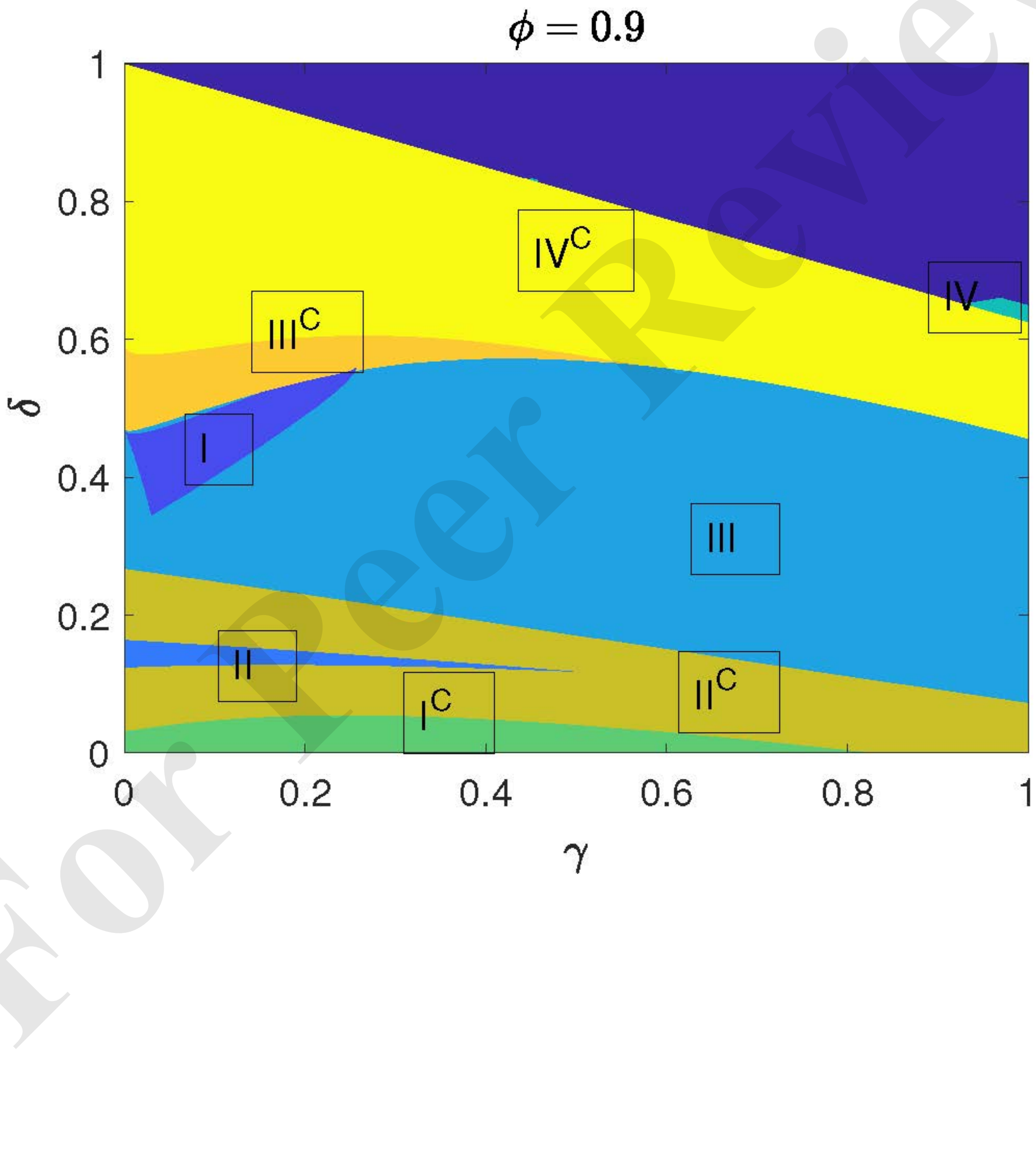


$\phi = 0.5$
1
0.8
0.6
0.4
0.2
0
δ
0
0.2
0.4
0.6
0.8
1
γ
III^C
I
III
IV^C
UF
III^C
IV
II
I^C
II^C
I

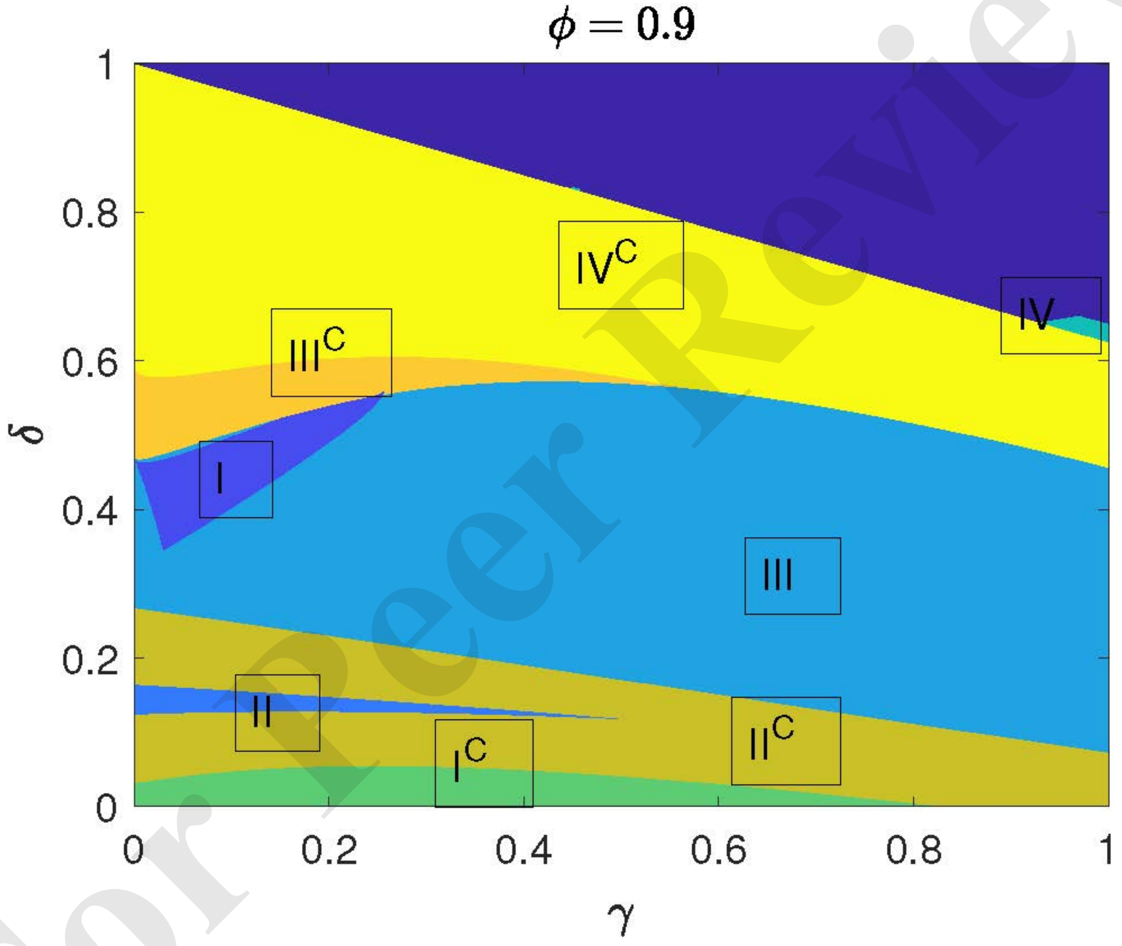

$\phi = 0.5$
III^C
I
III
IV^C
UF
I^C
III^C
IV
II
I^C
II^C
I
δ
γ
0
0.2
0.4
0.6
0.8
1



$\phi = 0.9$
1
0.8
0.6
0.4
0.2
0
δ
0
0.2
0.4
0.6
0.8
1
γ
UF
IV^C
III^C
IV
I
III
II^C
II
I^C



$\phi = 0.9$
1
0.8
0.6
0.4
0.2
0
δ
0
0.2
0.4
0.6
0.8
1
γ
IV^C
IV
III^C
I
III
II
II^C
I^C


$\phi = 0.9$
1
0.8
0.6
0.4
0.2
0
δ
0
0.2
0.4
0.6
0.8
1
γ
IV^C
IV
III^C
I
III
II
I^C
II^C