



Two-Variable Domination Structures and Applications in Vector Optimization

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Abstract

In this paper, we introduce and study domination structures in real topological Hausdorff linear spaces that take into account the two involved points at each comparison. These binary relations are then applied to define notions of minimizer of a set and optimality concepts for vector optimization problems in the usual way, and their basic properties are obtained. Results on nonlinear scalarization to characterize them are also stated, which can be applied to vector optimization problems with variable ordering structures where the known ones do not work. Comparisons with results of the literature and illustrative examples are given as well.

Keywords Variable domination structure · Vector optimization · Nondominated solutions · Minimal solutions · Nonlinear scalarization functions

1 Introduction

In vector optimization, the concept of optimal solution is one of the key points. This concept is often based on the assumption that the image space Y of the involved vector optimization problem is partially ordered by a nontrivial pointed convex cone. The most common ordering cone used in the finite-dimensional setting is the nonnegative orthant, which leads to the notion of Pareto optimal solution. However, modeling preferences in this way, i.e. via a constant ordering cone, has some drawbacks, which were recognized in the frameworks of *decision-making processes* by Karasakal and Michalowski [26] and Engau [17], and *multiobjective optimization* by Baatar and

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Wiecek [2] and Engau [16]. These limitations were also found concerning *image registration* in medical engineering, as was indicated by Wacker [31].

To deal with these problems, researchers used variable domination structures instead of constant ones, where every element in the image space Y has its own ordering cone provided by a cone-valued mapping $\mathcal{D} : Y \rightrightarrows Y$. The idea of variable domination structure was introduced in 1974 by Yu [32]. Later, Chen et al. introduced a nonlinear scalarization function with respect to a variable domination structure and established its basic properties. They showed that the solutions of a vector optimization problem, a vector variational inequality and a generalized quasi-vector equilibrium problem can be characterized by using this function, see [7–9]. After that, Eichfelder and her collaborators constructed a general and relatively comprehensive framework on variable domination structures and their applications in vector optimization, including discussions on linear and nonlinear scalarization, optimality conditions in nonsmooth problems via subdifferentials and coderivatives, duality, and numerical methods, see [10–15]. The optimality conditions, scalarizations and variational principles were further examined by Bao et al., see [3–6]. Notably, vector equilibrium problems involving variable preferences were analyzed in [18]. Very recently, Anh and Tam [1] and Peng et al. [28] characterized approximate efficiency in vector optimization problems with a variable domination structure by using linear and nonlinear scalarization.

So far, despite of different names, there are two main concepts of optimal solution involving a variable domination structure: one takes into account when the candidate is not dominated by any others w.r.t. their corresponding ordering, and the other considers when the candidate is not dominated w.r.t. its own ordering. Moreover, all relevant properties are also stated in two separate but rather related ways. Our aim in this paper is to introduce a more general preference structure that covers both of the above notions, whose properties can be applied to such two particular cases, and use it to define optimality concepts. This work is inspired by the research of Gutiérrez [21], in which quasi efficient solutions of a multiobjective optimization problem are defined via a set-valued mapping $C : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$, where $\mathbb{R}^n$ is the decision space and $\mathbb{R}^p$ is the image space of the problem. Here, the image space Y is assumed to be a general real topological linear space, and it is partially ordered such that each comparison of two arbitrary points in Y takes into account both involved points via a set-valued mapping $C : Y \times Y \rightrightarrows Y$.

The content of the paper is as follows. Section 2 recalls some concepts and usual notations in variational analysis and vector optimization. Section 3 is devoted to defining two-variable domination structures in an infinite-dimensional setting and studying their basic properties: reflexive, transitive, antisymmetric and compatibility with addition and scalar multiplication. These properties are then applied to two particular cases that correspond to the usual variable ordering structure contexts, which allows us to make comparisons with results in [10, 13]. In Section 4, after giving definitions to minimizers of a set and optimal solutions of vector optimization problems at the beginning of each subsection, we will deeply investigate the relationships between the given concepts and align them with existing results in the literature. Section 5 works with a version of the so-called Gerstewitz scalarization function to get nonlinear scalarization characterizations for the new concepts. These results can be applied to lots of vector optimization problems with variable ordering where the known ones do not work.

Examples are also provided to analyze and illustrate the main obtained results along the paper.

2 Preliminaries

Throughout this paper, let X, Y be two real Hausdorff topological linear spaces, $\emptyset \neq S \subset X$ and $\emptyset \neq M \subset Y$. Denote by $\text{int}M$, $\text{cl}M$, ∂M , M^c , $\text{cone}M$ the topological interior, the closure, the boundary, the complement and the cone generated by M , respectively. In addition, $\mathbb{R}_+^n$ stands for the nonnegative orthant of $\mathbb{R}^n$, $\mathbb{R}_+ := \mathbb{R}_+^1$ and $\mathbb{R}_- := -\mathbb{R}_+$.

Recall that the directional interior of M with respect to $q \in Y \setminus \{0\}$ (see [30]) and the algebraic interior of M (see [24, 25]), denoted by $\text{int}_q M$ and $\text{cor}M$, respectively, are the sets

$$\begin{aligned}\text{int}_q M &:= \{y \in M : \exists \varepsilon > 0, y + [0, \varepsilon]q \subset M\}, \\ \text{cor}M &:= \bigcap_{q \in Y \setminus \{0\}} \text{int}_q M.\end{aligned}\quad (1)$$

The set of all elements in Y which do not belong to $\text{cor}M$ and $\text{cor}(Y \setminus M)$ is called the algebraic boundary of M . It is known that if M is convex with $\text{int}M \neq \emptyset$ then $\text{cor}M = \text{int}M$ (see [25, Lemma 1.32]).

Consider a set-valued mapping $F : X \rightrightarrows Y$. The domain and the graph of F are, respectively, the sets

$$\begin{aligned}\text{dom } F &:= \{x \in X : F(x) \neq \emptyset\}, \\ \text{gph } F &:= \{(x, y) \in X \times Y : y \in F(x)\}.\end{aligned}$$

We denote the outer limit of F at a point $x_0 \in X$ by $\limsup_{x \rightarrow x_0} F(x)$. Recall that

$$y \in \limsup_{x \rightarrow x_0} F(x) \iff \exists X \supset (x_i)_i \rightarrow x_0, \exists Y \supset (y_i)_i \rightarrow y, (x_i, y_i) \in \text{gph } F \ \forall i.$$

One says that F is outer semicontinuous at x_0 if

$$\limsup_{x \rightarrow x_0} F(x) \subset F(x_0).$$

It is claimed that F is outer semicontinuous everywhere if and only if $\text{gph } F$ is closed (see [20, Proposition 2.7.12]). In the sequel, F^c , $\text{cor}F$ stand for the set-valued mappings $F^c(x) := F(x)^c$, $\text{cor}F(x) := \text{cor}(F(x))$, for all $x \in X$.

If $\varphi : Y \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is an extended real-valued function, then the domain of φ is defined as follows:

$$\text{dom } \varphi := \{y \in Y : \varphi(y) < +\infty\}.$$

Moreover, for each nonempty set $M \subset Y$, the next sets will be considered:

$$\begin{aligned}\operatorname{argmin}_M \varphi &:= \{\bar{y} \in M : \varphi(\bar{y}) \leq \varphi(y), \forall y \in M\}, \\ \operatorname{argsmin}_M \varphi &:= \{\bar{y} \in M : \varphi(\bar{y}) < \varphi(y), \forall y \in M \setminus \{\bar{y}\}\}.\end{aligned}$$

Recall that a subset R of the Cartesian product $Y \times Y$ is called a binary relation on Y . We denote $y_1 \leq^R y_2$ whenever $(y_1, y_2) \in R$. Relation R is said to be

- (a) reflexive if $y \leq^R y$, for all $y \in Y$.
- (b) transitive if $y_1 \leq^R y_2, y_2 \leq^R y_3$ implies $y_1 \leq^R y_3$.
- (c) antisymmetric if

$$y_1 \leq^R y_2, y_2 \leq^R y_1 \Rightarrow y_1 = y_2. \quad (2)$$

Notice that statement (2) is equivalent to

$$y_1 \leq^R y_2, y_1 \neq y_2 \Rightarrow y_2 \not\leq^R y_1. \quad (3)$$

- (d) compatible with addition if

$$y_1 \leq^R y_2 \Rightarrow y_1 + y \leq^R y_2 + y, \text{ for all } y \in Y.$$

- (e) compatible with nonnegative scalar multiplication if

$$y_1 \leq^R y_2 \Rightarrow \alpha y_1 \leq^R \alpha y_2, \text{ for all } \alpha > 0.$$

- (f) a partial order if R is reflexive, transitive and antisymmetric.

The image space of a vector optimization problem is often assumed to be partially ordered by a pointed convex cone K (recall that cone K is said to be pointed if $K \cap (-K) = \{0\}$). Specifically, one defines the binary relation

$$y_1, y_2 \in Y, \quad y_1 \leq_K y_2 : \Longleftrightarrow y_2 - y_1 \in K, \quad (4)$$

which is a partial order compatible with addition and nonnegative scalar multiplication. In dealing with the case where K is not constant, a set-valued mapping $\mathcal{D} : Y \rightrightarrows Y$ is considered, and the value $\mathcal{D}(y)$ is usually supposed to be a nonempty pointed convex cone for every $y \in Y$. Then, one can define two different variable domination structures on Y based on $\mathcal{D}$ as follows:

$$\begin{aligned}y_1 \leq_1^{\mathcal{D}} y_2 &: \Longleftrightarrow y_2 - y_1 \in \mathcal{D}(y_1), \\ y_1 \leq_2^{\mathcal{D}} y_2 &: \Longleftrightarrow y_2 - y_1 \in \mathcal{D}(y_2).\end{aligned}$$

These two binary relations suggest different notions of nondomination and minimality, which satisfy lots of interesting properties and results (see [4–6, 10, 13]). Next we introduce an approach to unify these two variable domination structures.

3 Two-Variable Domination Structures

In this section, we introduce and study a two-variable domination structure in Y . As it will be noticed later in Remarks 3.1 and 4.1(ii), it allows dealing with not only the well-known Eichfelder's (one) variable domination structures and their associated optimality notions, but also problems whose preferences require taking into account the two involved points in each pairwise comparison. Practical examples of such problems in decision-making and game theory are shown in Remark 3.1.

Namely, consider a set-valued mapping $C : Y \times Y \rightrightarrows Y$ such that $\text{dom}C = Y \times Y$, i.e., $C(y_1, y_2)$ is a nonempty set for every pair $(y_1, y_2) \in Y \times Y$.

Definition 3.1 Let $y_1, y_2 \in Y$. One defines

$$y_1 \leq^C y_2 : \Longleftrightarrow y_2 \in y_1 + C(y_1, y_2). \quad (5)$$

The set-valued mapping C is called a two-variable domination mapping, and $\leq^C$ is called a two-variable domination structure on Y .

Some motivations to consider two-variable domination structures are collected in the following remark.

Remark 3.1 (i) The above definition covers the usual variable ordering structures $\leq_1^{\mathcal{D}}$ and $\leq_2^{\mathcal{D}}$ introduced by Eichfelder in [10]. Indeed, if $C_1^{\mathcal{D}}(y_1, y_2)$ is defined as $\mathcal{D}(y_1)$, then $\leq^{C_1^{\mathcal{D}}}$ coincides with $\leq_1^{\mathcal{D}}$. Analogously, if we take $C_2^{\mathcal{D}}(y_1, y_2) := \mathcal{D}(y_2)$ then $\leq^{C_2^{\mathcal{D}}}$ coincides with $\leq_2^{\mathcal{D}}$. Thus, the properties and optimality concepts related to $\leq_1^{\mathcal{D}}$ and $\leq_2^{\mathcal{D}}$ will be also encompassed within the corresponding ones based on relation (5) and the two-variable domination mappings $C_1^{\mathcal{D}}$ and $C_2^{\mathcal{D}}$.

(ii) One can find in the literature real-world problems requiring to be solved of domination structures that depend on the two points involved in each pairwise comparison. For instance, in [19] a kind of reference-dependent preferences is introduced and applied to deal with behavioral traps and Nash equilibrium in games. Namely, in an arbitrary set A and via a real-valued function $\Delta : A \times A \rightarrow \mathbb{R}$, the following preference relation is considered:

$$a_1 \leq^{\Delta} a_2 : \Longleftrightarrow \Delta(a_1, a_2) \geq \Delta(a_1, a_1),$$

which depends on the two involved points a_1, a_2 . If A is a linear space, then $\leq^{\Delta} = \leq^C$ for any two-variable domination mapping $C : A \times A \rightrightarrows A$ satisfying $a_2 - a_1 \notin C(a_1, a_2)$ if $\Delta(a_1, a_2) < \Delta(a_1, a_1)$ and $a_2 - a_1 \in C(a_1, a_2)$ otherwise.

Equitability preferences are another interesting example (see [2, 12, 13, 27]), which model multiobjective optimization problems whose criteria are comparable, anonymous and satisfy the principle of transfer. They have been successfully applied to location problems and portfolio optimization, where those assumptions are fulfilled. In the biobjective case, equitability preferences are formulated by the following binary

relation $\leq_e$ in $\mathbb{R}^2$. Consider the sets

$$\begin{aligned} S_1 &:= \{(y_1, y_2) \in \mathbb{R}^2 : y_1 - y_2 \geq 0\}, \\ S_2 &:= \{(y_1, y_2) \in \mathbb{R}^2 : y_2 - y_1 \geq 0\}, \\ D_1 &:= \{(d_1, d_2) \in \mathbb{R}^2 : d_1 \geq 0, d_1 + d_2 \geq 0\}, \\ D_2 &:= \{(d_1, d_2) \in \mathbb{R}^2 : d_2 \geq 0, d_1 + d_2 \geq 0\}, \end{aligned}$$

and the permutation function $p : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $p(y_1, y_2) = (y_2, y_1)$, for all $(y_1, y_2) \in \mathbb{R}^2$. Then,

$$y^1 \leq_e y^2 : \Longleftrightarrow \begin{cases} y^2 - y^1 \in -D_1 & \text{if } y^1, y^2 \in S_1, \\ y^2 - y^1 \in -D_2 & \text{if } y^1, y^2 \in S_2, \\ p(y^2) - y^1 \in -D_1 & \text{if } y^1 \in S_1, y^2 \in S_2, \\ p(y^2) - y^1 \in -D_2 & \text{if } y^1 \in S_2, y^2 \in S_1 \end{cases} \quad (6)$$

and the scenario $y^1 \leq_e y^2$ depends not only on the difference $y^2 - y^1$ or $p(y^2) - y^1$, but also on the location of the two involved points y^1, y^2 in the sets S_1, S_2 . A way to formulate (6) in our framework is to consider $Y = \mathbb{R}^2 \times \mathbb{R}^2$ and the next set-valued mapping $C : Y \times Y \rightrightarrows Y$:

$$C((y^1, v^1), (y^2, v^2)) = \begin{cases} \{(d, e) \in Y : d \in -D_1\} & \text{if } y^1, y^2 \in S_1, \\ \{(d, e) \in Y : d \in -D_2\} & \text{if } y^1, y^2 \in S_2, \\ \{(d, e) \in Y : e \in -D_1\} & \text{if } y^1 \in S_1, y^2 \in S_2, \\ \{(d, e) \in Y : e \in -D_2\} & \text{if } y^1 \in S_2, y^2 \in S_1. \end{cases}$$

Then, it is easy to check that $y^1 \leq_e y^2$ if and only if $(y^1, y^1) \leq^C (y^2, p(y^2))$.

Concerning the two-variable domination mapping C , we define two set-valued mappings $D_C, I_C : Y \rightrightarrows Y$:

$$\begin{aligned} D_C(y) &:= \{d \in Y : y \leq^C y + d\}, \\ I_C(y) &:= \{d \in Y : y - d \leq^C y\}. \end{aligned}$$

The following properties hold true.

Lemma 3.1 *We have that:*

(i) *For each $y, z \in Y$,*

$$y \leq^C z \Longleftrightarrow z \in y + D_C(y) \Longleftrightarrow y \in z - I_C(z).$$

(ii) *$D_C(y)$ and $I_C(y)$ can be expressed in the forms*

$$D_C(y) = \bigcup_{d \in Y} [\{d\} \cap C(y, y + d)] \quad (7)$$

and

$$I_C(y) = \bigcup_{d \in Y} [\{d\} \cap C(y - d, y)]. \quad (8)$$

In addition,

$$\begin{aligned} d \in D_C(y) &\iff d \in I_C(y + d), \\ d \in I_C(y) &\iff d \in D_C(y - d). \end{aligned}$$

(iii) Consider a set-valued mapping $\mathcal{D} : Y \rightrightarrows Y$. It follows that $D_{C_1^{\mathcal{D}}} = \mathcal{D}$ (resp. $I_{C_2^{\mathcal{D}}} = \mathcal{D}$).

Proof (i) We first show that $y \leq^C z \iff z \in y + D_C(y)$. Indeed, suppose $y \leq^C z$, then $z - y \in C(y, z)$. It means that there exists $d \in C(y, z)$ such that $z = y + d$. Hence, $y \leq^C y + d$ and so $d \in D_C(y)$. To prove the reverse implication, suppose that $z \in y + D_C(y)$, i.e., $z - y \in D_C(y)$. From the definition of $D_C(y)$, it follows that $y \leq^C y + (z - y)$, so $y \leq^C z$. By using similar arguments, we obtain the equivalence $y \leq^C z \iff y \in z - I_C(z)$.

(ii) Clearly, from the definition of D_C , $d \in D_C(y)$ if and only if $y \leq^C y + d$. It means that $y + d \in y + C(y, y + d)$, or, $d \in C(y, y + d)$. This is equivalent to (7). Similarly, we obtain (8).

From (7) and (8) we see that

$$d \in D_C(y) \iff d \in C(y, y + d) = C((y + d) - d, y + d) \iff d \in I_C(y + d)$$

and so

$$d \in I_C(y) = I_C((y - d) + d) \iff d \in D_C(y - d),$$

which finishes the proof of part (ii).

(iii) We have

$$\begin{aligned} D_{C_1^{\mathcal{D}}}(y) &= \{d \in Y : y \leq^{C_1^{\mathcal{D}}} y + d\} \\ &= \{d \in Y : y + d \in y + C_1^{\mathcal{D}}(y, y + d)\} \\ &= \{d \in Y : y + d \in y + \mathcal{D}(y)\} \\ &= \{d \in Y : d \in \mathcal{D}(y)\} \\ &= \mathcal{D}(y) \end{aligned}$$

and hence $D_{C_1^{\mathcal{D}}} = \mathcal{D}$. The proof of the equality $I_{C_2^{\mathcal{D}}} = \mathcal{D}$ is similar and it is omitted. $\square$

Remark 3.2 By Lemma 3.1 we deduce that for each $y_1, y_2 \in Y$,

$$y_1 \leq^C y_2 \iff y_1 \leq_1^{D_C} y_2 \iff y_1 \leq_2^{I_C} y_2.$$

In other words, relation $\leq^C$ can be also reformulated as relations $\leq_1^{\mathcal{D}}$ and $\leq_2^{\mathcal{D}}$ by considering the set-valued mapping $\mathcal{D} = D_C$ and $\mathcal{D} = I_C$, respectively.

We now present some basic properties of a two-variable domination structure.

Theorem 3.1 *Let $C : Y \times Y \rightrightarrows Y$ be a two-variable domination mapping.*

- (i) *The binary relation $\leq^C$ is reflexive if and only if $0 \in C(y, y)$, for all $y \in Y$.*
- (ii) *The binary relation $\leq^C$ is transitive if and only if*

$$\bigcup_{z \in y + D_C(y)} (z + D_C(z)) \subset y + D_C(y), \quad \forall y \in Y \quad (9)$$

or

$$\bigcup_{z \in w - I_C(w)} (z - I_C(z)) \subset w - I_C(w), \quad \forall w \in Y. \quad (10)$$

- (iii) *The binary relation $\leq^C$ is antisymmetric if and only if*

$$\bigcup_{y \in Y} \bigcup_{d \in D_C(y)} [\{d\} \cap (-D_C(y + d))] \subset \{0\}, \quad (11)$$

or

$$\bigcup_{y \in Y} \bigcup_{d \in I_C(y)} [\{d\} \cap (-I_C(y - d))] \subset \{0\}. \quad (12)$$

- (iv) *The binary relation $\leq^C$ is compatible with addition if and only if D_C (or I_C) is a constant mapping.*
- (v) *The binary relation $\leq^C$ is compatible with nonnegative scalar multiplication if and only if*

$$\alpha D_C(y) \subset D_C(\alpha y), \quad \forall y \in Y, \forall \alpha > 0 \quad (13)$$

or

$$\alpha I_C(y) \subset I_C(\alpha y), \quad \forall y \in Y, \forall \alpha > 0.$$

Proof Part (i) is obvious. For the remaining parts, we will only prove the results for D_C , since the arguments for I_C are similar.

(ii) Suppose that relation $\leq^C$ is transitive, and take any element w that belongs to the left-hand side of (9). Then there exists $z \in y + D_C(y)$ such that $w \in z + D_C(z)$. According to Lemma 3.1(i), the former means that $y \leq^C z$, meanwhile the latter means that $z \leq^C w$. By applying the transitivity of $\leq^C$, we deduce that $y \leq^C w$, or equivalently, w belongs to the right-hand side of (9). To prove the reverse implication, suppose that (9) holds true, and suppose that $y \leq^C z$ and $z \leq^C w$. By Lemma 3.1(i), it follows that $z \in y + D_C(y)$ and $w \in z + D_C(z)$. Thus, w belongs to the left-hand side of (9), and so w belongs to the right-hand side of (9), which means $y \leq^C w$. Therefore, relation $\leq^C$ is transitive.

(iii) Suppose that $\leq^C$ is antisymmetric. It is enough to consider the case where the set in the left-hand side of (11) is nonempty, as otherwise the result is obvious. Take any d in this set, which means $d \in D_C(y) \cap (-D_C(y + d))$ for some $y \in Y$. That is,

$y \leq^C y + d$ and $y + d \leq^C y$. Since the relation $\leq^C$ is antisymmetric, $y = y + d$. Thus, $d = 0$.

Conversely, suppose $y \leq^C z$, and $z \leq^C y$. Then $d := z - y \in D_C(y)$ and

$$z - y \in -D_C(z) = -D_C(y + d).$$

Thus, due to (11), it follows that $d = 0$, i.e., $z = y$.

(iv) Suppose that $\leq^C$ is compatible with addition. To prove D_C is a constant mapping, we will show that $D_C(y) \subset D_C(z)$ for all y, z in Y . Indeed, take any $d \in D_C(y)$, then $y \leq^C y + d$. From the compatibility with addition of relation $\leq^C$, we have $y + (z - y) \leq^C y + d + (z - y)$, or equivalently, $z \leq^C z + d$. It follows that $d \in D_C(z)$, and hence $D_C(y) \subset D_C(z)$.

Conversely, consider that $y \leq^C z$, and D_C is a constant mapping. Then, $z - y \in D_C(y)$, and $D_C(y) = D_C(y + w)$ for all $w \in Y$. It implies that $z - y \in D_C(y + w)$ and using the definition of D_C , this means $y + w \leq^C (y + w) + (z - y)$. Thus, $y + w \leq^C z + w$ for all $w \in Y$, i.e., relation $\leq^C$ is compatible with addition.

(v) Suppose that $\leq^C$ is compatible with nonnegative scalar multiplication. For any $y \in Y$ and $\alpha > 0$, take an arbitrary $d \in \alpha D_C(y)$. Then, $\frac{1}{\alpha}d \in D_C(y)$. That is, $y \leq^C y + \frac{1}{\alpha}d$. Since the relation $\leq^C$ is compatible with nonnegative scalar multiplication, $\alpha y \leq^C \alpha y + d$, and it means $d \in D_C(\alpha y)$.

It remains to prove that $\leq^C$ is compatible with nonnegative scalar multiplication provided that (13) holds true. Let $y \leq^C z$ and $\alpha > 0$. Thanks to Lemma 3.1(i), we have $z - y \in D_C(y)$. Then $\alpha z - \alpha y \in \alpha D_C(y) \subset D_C(\alpha y)$ and $\alpha y \leq^C \alpha z$. $\square$

Remark 3.3 Notice that statement (11) can be reformulated as follows:

$$\bigcup_{d \in D_C(y) \setminus \{0\}} [\{d\} \cap (-D_C(y + d))] = \emptyset, \quad \forall y \in Y.$$

In addition, for each $y \in Y$ and $d \in D_C(y) \setminus \{0\}$, $\{d\} \cap (-D_C(y + d)) = \emptyset$ means $y \leq^C y + d$ and $y + d \not\leq^C y$ (compare with (3)). Analogously, statement (12) is equivalent to

$$\bigcup_{d \in I_C(y) \setminus \{0\}} [\{d\} \cap (-I_C(y - d))] = \emptyset, \quad \forall y \in Y.$$

Notice that for each $y \in Y$ and $d \in I_C(y) \setminus \{0\}$, $\{d\} \cap (-I_C(y - d)) = \emptyset$ means $y - d \leq^C y$ and $y \not\leq^C y - d$ (compare with (3)). Obviously, antisymmetric property is satisfied whenever the next equivalent pointedness conditions are satisfied: $D_C(Y) \cap (-D_C(Y)) \subset \{0\}$, $I_C(Y) \cap (-I_C(Y)) \subset \{0\}$.

Concerning the transitive property, the next corollary shows an interesting particular case where statements (9) and (10) are easier to check.

Corollary 3.1 Let $C : Y \times Y \rightrightarrows Y$ be a two-variable domination mapping such that the values of the set-valued mappings D_C and I_C are closed convex cones. Then, the

binary relation $\leq^C$ is transitive if and only if

$$D_C(y + d) \subset D_C(y), \quad \forall y \in Y, d \in D_C(y), \quad (14)$$

or

$$I_C(y - d) \subset I_C(y), \quad \forall y \in Y, d \in I_C(y).$$

Proof Let us only prove equivalence (14) as the other one can be stated similarly. Assume that $\leq^C$ is transitive and consider arbitrary points $y \in Y$ and $d \in D_C(y)$. By (9) it is easy to obtain that $d + D_C(y + d) \subset D_C(y)$. Since the sets $D_C(y + d)$ and $D_C(y)$ are cones we have $\alpha d + D_C(y + d) \subset D_C(y)$, for all $\alpha > 0$. Then, as set $D_C(y)$ is closed, we deduce by considering $\alpha \rightarrow 0$ that $D_C(y + d) \subset D_C(y)$. Indeed, for each $z \in D_C(y + d)$ we have

$$z = \lim_{\alpha \rightarrow 0} (\alpha d + z) \in \text{cl } D_C(y) = D_C(y).$$

Conversely, suppose that statement (14) is true and take $y \in Y$ and $d \in D_C(y)$. Then,

$$y + d + D_C(y + d) \subset y + D_C(y) + D_C(y + d) \subset y + D_C(y) + D_C(y) = y + D_C(y)$$

and statement (9) is proved. Therefore, by Theorem 3.1(ii) we deduce that $\leq^C$ is transitive and the proof finishes. $\square$

Theorem 3.1 encompasses the following properties concerning the binary relation $\leq_1^{\mathcal{D}}$ and $\leq_2^{\mathcal{D}}$.

Corollary 3.2 Consider a set-valued mapping $\mathcal{D} : Y \rightrightarrows Y$. We have that:

- (i) For each $i \in \{1, 2\}$, $\leq_i^{\mathcal{D}}$ is reflexive if and only if $0 \in \mathcal{D}(y)$, for all $y \in Y$.
- (ii) Relation $\leq_1^{\mathcal{D}}$ (resp. $\leq_2^{\mathcal{D}}$) is transitive if and only if

$$d + \mathcal{D}(y + d) \subset \mathcal{D}(y) \quad (15)$$

$$\text{(resp. } d + \mathcal{D}(y - d) \subset \mathcal{D}(y)), \quad \forall y \in Y, \forall d \in \mathcal{D}(y). \quad (16)$$

- (iii) Relation $\leq_1^{\mathcal{D}}$ (resp. $\leq_2^{\mathcal{D}}$) is antisymmetric if and only if

$$d \notin -\mathcal{D}(y + d) \quad (17)$$

$$\text{(resp. } d \notin -\mathcal{D}(y - d)), \quad \forall y \in Y, \forall d \in \mathcal{D}(y) \setminus \{0\}. \quad (18)$$

- (iv) For each $i \in \{1, 2\}$, $\leq_i^{\mathcal{D}}$ is compatible with addition if and only if $\mathcal{D}$ is constant.
- (v) For each $i \in \{1, 2\}$, $\leq_i^{\mathcal{D}}$ is compatible with nonnegative scalar multiplication if and only if

$$\alpha \mathcal{D}(y) \subset \mathcal{D}(\alpha y), \quad \forall y \in Y, \forall \alpha > 0. \quad (19)$$

Proof All parts are easy results of applying Theorem 3.1 and Lemma 3.1(iii) to the set-valued mappings $C_1^{\mathcal{D}}$ and $C_2^{\mathcal{D}}$. Notice by Remark 3.1(i) that $\leq^{C_1^{\mathcal{D}}} = \leq_1^{\mathcal{D}}$ and $\leq^{C_2^{\mathcal{D}}} = \leq_2^{\mathcal{D}}$. $\square$

Remark 3.4 Corollary 3.2 encompasses [13, Lemma 1.10], where the values $\mathcal{D}(y)$ are assumed to be convex cones. Namely, we have the following improvements:

- (i) Properties in Corollary 3.2 are true for any set-valued mapping $\mathcal{D}$.
- (ii) Addressing the transitive property, inclusion (15) is more general than the corresponding condition (1.3) in [13, Lemma 1.10(ii)]: $\mathcal{D}(y + d) \subset \mathcal{D}(y)$, for all $y \in Y, d \in \mathcal{D}(y)$. Indeed, take any $y \in Y, d \in \mathcal{D}(y)$. Then,

$$\begin{aligned} d + \mathcal{D}(y + d) &\subset \mathcal{D}(y) + \mathcal{D}(y + d) \\ &\subset \mathcal{D}(y) + \mathcal{D}(y) \\ &= \mathcal{D}(y) \quad (\text{because } \mathcal{D}(y) \text{ is a convex cone}) \end{aligned}$$

and inclusion (15) holds true whenever condition (1.3) in [13, Lemma 1.10(ii)] is satisfied. When $\mathcal{D}(y)$ is closed, for all $y \in Y$, both conditions are equivalent (see Corollary 3.1).

Analogous remarks can also be argued concerning inclusion (16) and the corresponding condition (1.4) in [13, Lemma 1.10(iii)]: $\mathcal{D}(y - d) \subset \mathcal{D}(y)$, for all $y \in Y, d \in \mathcal{D}(y)$.

- (iii) In [13, Lemma 1.10(v)], the compatibility with nonnegative scalar multiplication was characterized by the condition

$$\mathcal{D}(y) \subset \mathcal{D}(\alpha y), \quad \forall y \in Y, \forall \alpha > 0,$$

which is equivalent to (19) as long as $\mathcal{D}(y)$ is a cone.

- (iv) Regarding to the antisymmetric property of relations $\leq_1^{\mathcal{D}}$ and $\leq_2^{\mathcal{D}}$, it was stated in [13, Lemma 1.10(vi)] that condition $\bigcup_{y \in Y} \mathcal{D}(y)$ being pointed is sufficient for its fulfillment. This assumption obviously implies that conditions (17) and (18) are true. However, the converse implication is not valid in general. For instance, let $Y = \mathbb{R}^2$ and $\mathcal{D} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be defined by

$$\mathcal{D}(y) = \begin{cases} \{0\} \times \mathbb{R}_+ & \text{if } y_2 \geq 0 \\ \{0\} \times \mathbb{R}_- & \text{if } y_2 < 0 \end{cases}$$

for all $y = (y_1, y_2) \in \mathbb{R}^2$. Then $\bigcup_{y \in Y} \mathcal{D}(y)$ is not pointed, but (17) is fulfilled.

4 Optimality Notions

This section is divided into two parts. The notions of minimizers of a set are defined in the first part, and the concepts of optimal solutions for vector optimization problems are introduced in the second part.

4.1 Notions of Minimizers of a Set

Let M be a nonempty subset of Y , and $C : Y \times Y \rightrightarrows Y$ be a two-variable domination mapping.

Definition 4.1 A point $\bar{y} \in M$ is said to be a

(a) $\leq^C$ -minimal point of M , denoted by $\bar{y} \in \text{Min}(M, C)$, if

$$y \in M, y \leq^C \bar{y} \Rightarrow \bar{y} \leq^C y; \quad (20)$$

(b) $\leq^C$ -nondominated point of M , denoted by $\bar{y} \in \text{ND}(M, C)$, if there does not exist any $y \in M \setminus \{\bar{y}\}$ such that $y \leq^C \bar{y}$;

(c) $\leq^C$ -strongly nondominated point of M , denoted by $\bar{y} \in \text{SND}(M, C)$, if $\bar{y} \leq^C y$, for all $y \in M \setminus \{\bar{y}\}$;

(d) Assume $\text{dom}(\text{cor}C) = Y \times Y$. Then, $\bar{y}$ is called a $\leq^C$ -weakly nondominated point of M , denoted by $\bar{y} \in \text{WND}(M, C)$, if $\bar{y} \in \text{ND}(M, \text{cor}C)$.

Remark 4.1 (i) Consider $\bar{y} \in M$. It follows that:

$$\begin{aligned} \bar{y} \in \text{Min}(M, C) &\iff [d \in I_C(\bar{y}) \Rightarrow -d \in D_C(\bar{y})] \\ \bar{y} \in \text{ND}(M, C) &\iff \bar{y} \notin y + C(y, \bar{y}), \quad \forall y \in M \setminus \{\bar{y}\} \\ &\iff I_C(\bar{y}) \subset \{0\} \\ \bar{y} \in \text{SND}(M, C) &\iff y \in \bar{y} + C(\bar{y}, y), \quad \forall y \in M \setminus \{\bar{y}\} \\ &\iff (M - \{\bar{y}\}) \setminus \{0\} \subset D_C(\bar{y}). \end{aligned}$$

(ii) Consider a set-valued mapping $\mathcal{D} : Y \rightrightarrows Y$ whose value $\mathcal{D}(y)$ is a proper (i.e., $\mathcal{D}(y) \neq Y$) cone, for all $y \in Y$. The concept of nondominated (resp. minimal) element of M with respect to $\mathcal{D}$ introduced by Eichfelder [10, Definition 2.1] (resp. [10, Definition 2.2]) is the result of applying Definition 4.1(b) to the two-variable domination mapping $C_1^{\mathcal{D}}$ (resp. $C_2^{\mathcal{D}}$).

Analogously, the concept of weakly nondominated (resp. weakly minimal) element of M with respect to $\mathcal{D}$ introduced by Eichfelder [12, Definition 4.1] (resp. [12, Definition 4.2]) corresponds to the notion of $\leq^{C_1^{\mathcal{D}}}$ -weakly nondominated (resp. $\leq^{C_2^{\mathcal{D}}}$ -weakly nondominated) point of M in Definition 4.1(d).

Finally, if one applies Definition 4.1(c) to $\leq^{C_1^{\mathcal{D}}}$ (resp. $\leq^{C_2^{\mathcal{D}}}$), then the concept of strongly nondominated (resp. strongly minimal) element of M with respect to $\mathcal{D}$ introduced by Eichfelder [12, Definition 4.1] (resp. [12, Definition 4.2]) is obtained.

An interesting simple application of binary relation $\leq^C$ is that one can consider simultaneously both Eichfelder's approaches to deal with minimizers of a set. For instance, given a partition M_1, M_2 of a nonempty set M , suppose that we want to check what points in M_1 are nondominated elements of M and what points in M_2 are minimal elements of M . It is possible by considering $\leq^C$ -nondominated

points of M concerning the following two-variable domination mapping on M :

$$C(y, v) = \begin{cases} \mathcal{D}(y) & \text{if } v \in M_1, \\ \mathcal{D}(v) & \text{if } v \in M_2. \end{cases}$$

Hereafter we denote

$$I_C^M(y) := \{d \in I_C(y) : y - d \in M\}, \quad \forall y \in M.$$

Remark 4.2 Clearly, $\text{ND}(M, C_2) \subset \text{ND}(M, C_1)$ whenever the two-variable domination mappings C_1 and C_2 satisfy $\text{gph} C_1 \subset \text{gph} C_2$.

Relationships between the above concepts of minimizer points are examined in the following theorem.

Theorem 4.1 Let $\bar{y} \in M$.

- (i) If $\bar{y}$ is a $\leq^C$ -nondominated point of M , then $\bar{y}$ is a $\leq^C$ -weakly nondominated point of M .
- (ii) Suppose that $C(y, \bar{y}) \cap (-C(\bar{y}, y)) \subset \{0\}$, for all $y \in M \setminus \{\bar{y}\}$. If $\bar{y}$ is a $\leq^C$ -strongly nondominated point of M , then it is also a $\leq^C$ -nondominated point of M . In particular, there is at most one $\leq^C$ -strongly nondominated point of M .
- (iii) If $\bar{y}$ is a $\leq^C$ -nondominated point of M , then it is a $\leq^C$ -minimal point of M . Conversely, if $\bar{y}$ is a $\leq^C$ -minimal point of M and

$$\bigcup_{d \in I_C^M(\bar{y}) \setminus \{0\}} [d] \cap (-I_C^M(\bar{y} - d)) = \emptyset, \quad (21)$$

then it is a $\leq^C$ -nondominated point of M too.

Proof Part (i) follows from Remark 4.2.

(ii) To prove the first claim, suppose that $\bar{y}$ is a $\leq^C$ -strongly nondominated point of M . From the definition one has

$$\bar{y} \in y - C(\bar{y}, y), \quad \forall y \in M \setminus \{\bar{y}\}.$$

Thus, under the assumption, it follows that $\bar{y} \notin y + C(y, \bar{y})$ for all $y \in M \setminus \{\bar{y}\}$. This means that there is no point $y \in M \setminus \{\bar{y}\}$ such that $y \leq^C \bar{y}$ and so $\bar{y}$ is a $\leq^C$ -nondominated point of M . As a consequence, y cannot be a $\leq^C$ -strongly nondominated point of M , for all $y \in M \setminus \{\bar{y}\}$.

(iii) Suppose that $\bar{y}$ is a $\leq^C$ -nondominated point of M . Then there does not exist any $y \in M \setminus \{\bar{y}\}$ such that $y \leq^C \bar{y}$. Hence, implication (20) holds. This means $\bar{y}$ is a $\leq^C$ -minimal point of M .

For proving the reverse assertion, notice from assumption (21), Remark 3.3 and the definition of the set-valued mapping I_C^M that for each $y \in M \setminus \{\bar{y}\}$, $y \leq^C \bar{y}$, then $\bar{y} \not\leq^C y$. Therefore, if $\bar{y} \in M$ is a $\leq^C$ -minimal point of M , it cannot exist any point $y \in M \setminus \{\bar{y}\}$ satisfying $y \leq^C \bar{y}$. In other words, $\bar{y}$ is also a $\leq^C$ -nondominated point of M . $\square$

Remark 4.3 Some properties in Theorem 4.1 have been stated in the literature for the particular cases $C_1^{\mathcal{D}}$ and $C_2^{\mathcal{D}}$. For example, Theorem 4.1(i) reduces to [12, Lemma 4.2(c)] and [13, Lemma 2.23(ii)] by considering the two-variable domination mappings $C_1^{\mathcal{D}}$ and $C_2^{\mathcal{D}}$ and a set-valued mapping $\mathcal{D}$ whose values are pointed convex cones.

Analogously, Theorem 4.1(iii) encompasses [4, Proposition 2.1] by considering the same two-variable domination mappings $C_1^{\mathcal{D}}$ and $C_2^{\mathcal{D}}$, where the set-valued mapping $\mathcal{D}$ satisfies $0 \in \mathcal{D}(y)$, for all $y \in Y$, and the following pointedness assumption concerning the nominal point $\bar{y}$:

$$\left(\bigcup_{y \in M} \mathcal{D}(y) \right) \cap (-\mathcal{D}(\bar{y})) = \{0\}. \quad (22)$$

It is not hard to check that condition (21) applied to $C_1^{\mathcal{D}}$ and $C_2^{\mathcal{D}}$ results in, respectively,

$$\{d\} \cap \mathcal{D}(\bar{y} - d) \cap (-\mathcal{D}(\bar{y})) = \emptyset, \quad (23)$$

$$\{d\} \cap \mathcal{D}(\bar{y}) \cap (-\mathcal{D}(\bar{y} - d)) = \emptyset, \quad \forall d \in \bar{y} - (M \setminus \{\bar{y}\}). \quad (24)$$

Clearly, assumption (22) implies conditions (23) and (24), but they are not equivalent. Indeed, consider $Y = \mathbb{R}^2$, $M = \mathbb{R}_+^2$, $\bar{y} = (0, 0)$, $\mathcal{D}(\bar{y}) = \mathbb{R}_+^2$ and $\mathcal{D}(y) = -\mathbb{R}_+^2$, for all $y \in Y \setminus \{\bar{y}\}$. Assumption (22) is not fulfilled and both conditions (23) and (24) are true. As a consequence, since $\bar{y} \notin \text{ND}(M, C_1^{\mathcal{D}})$, by Theorem 4.1(iii) we deduce that $\bar{y} \notin \text{Min}(M, C_1^{\mathcal{D}})$ and this conclusion cannot be stated by applying [4, Proposition 2.1].

The next easy result allows us to reduce the search of $\leq^C$ -nondominated points to the boundary of the involved set.

Theorem 4.2 Let $\bar{y} \in Y$.

(i) If $\bar{y}$ is a $\leq^C$ -nondominated point of M , and there exists a net $(y_i) \subset Y \setminus \{\bar{y}\}$ satisfying $y_i \rightarrow \bar{y}$ and

$$\bar{y} - y_i \in C(y_i, \bar{y}), \quad \forall i \quad (25)$$

then $\bar{y} \in \partial M$.

(ii) If $\bar{y}$ is a $\leq^C$ -weakly nondominated point of M , and there exists a net $(y_i) \subset Y \setminus \{\bar{y}\}$ satisfying $y_i \rightarrow \bar{y}$ and

$$\bar{y} - y_i \in \text{cor}C(y_i, \bar{y}), \quad \forall i \quad (26)$$

then $\bar{y} \in \partial M$.

Proof (i) Suppose by contradiction that $\bar{y} \in \text{int}M$. Then there exists i_0 such that $y_{i_0} \in M$. It follows from (25) that $y_{i_0} \leq^C \bar{y}$, a contradiction to assumption that $\bar{y}$ is a $\leq^C$ -nondominated point of M . Therefore, $\bar{y} \in \partial M$ and the proof finishes. $\square$

(ii) Similar to the proof of (i). $\square$

Remark 4.4 (i) Theorem 4.2 does not work by replacing the topological boundary with the algebraic one. Consider, for instance, $Y = \mathbb{R}^2$ and

$$M = \{(y_1, y_2) \in \mathbb{R}^2 : |y_2| \geq y_1^2\} \cup \text{cone}\{(1, 0), (-1, 0)\}.$$

Let $H := \{(y_1, y_2) \in \mathbb{R}^2 : y_2 \geq 0\}$, $\mathcal{D}_1 : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$,

$$\mathcal{D}_1((y_1, y_2)) := \begin{cases} -H & \text{if } (y_1, y_2) \in M, y_2 < 0 \\ \text{cone}\{(1, 1)\} & \text{if } (y_1, y_2) \in M, y_2 = 0 \\ H & \text{if } (y_1, y_2) \in M, y_2 > 0 \\ \mathbb{R}_+^2 & \text{if } (y_1, y_2) \notin M. \end{cases}$$

We have $(-1/n, -1/n^3) \rightarrow \bar{y} := (0, 0)$, $(-1/n, -1/n^3) \leq^{C_1^{\mathcal{D}_1}} \bar{y}$, for all $n \geq 2$ and $\bar{y} \in \text{ND}(M, C_1^{\mathcal{D}_1})$. Clearly, $\bar{y} \in \partial M$, but $\bar{y}$ does not belong to the algebraic boundary of M because $\bar{y} \in \text{cor} M$.

Analogously, if we take $\mathcal{D}_2 : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$,

$$\mathcal{D}_2((y_1, y_2)) := \begin{cases} -H & \text{if } (y_1, y_2) \in M, y_2 \leq 0 \\ H & \text{if } (y_1, y_2) \in M, y_2 > 0 \\ \mathbb{R}_+^2 & \text{if } (y_1, y_2) \notin M, \end{cases}$$

then $(-1/n, -1/n^3) \leq^{\text{cor} C_1^{\mathcal{D}_2}} \bar{y}$, for all $n \geq 2$. We still have $(-1/n, -1/n^3) \rightarrow \bar{y}$ and $\bar{y} \in \partial M$. Note that $\bar{y} \in \text{WND}(M, C_1^{\mathcal{D}_2}) \cap \text{cor} M$.

(ii) Assertion (i) in Theorem 4.2 was stated in [13, Lemma 2.34(ii)] (for the algebraic boundary) and [12, Lemma 4.3(b)(ii)] in the particular case $C_1^{\mathcal{D}}$, where $\mathcal{D} : Y \rightrightarrows Y$, $\mathcal{D}(y)$ is a pointed convex cone for all $y \in Y$, and replacing assumption (25) with $\bigcap_{y \in M} \mathcal{D}(y) \neq \{0\}$. Notice that these assumptions are stronger. Indeed, take $\bar{d} \in \bigcap_{y \in M} \mathcal{D}(y)$, $\bar{d} \neq 0$ and define $y_n := \bar{y} - (1/n)\bar{d}$. Clearly, $y_n \rightarrow \bar{y}$, $y_n \neq \bar{y}$ and

$$\bar{y} - y_n = (1/n)\bar{d} \in \bigcap_{y \in M} \mathcal{D}(y) \subset \mathcal{D}(y_n) = C_1^{\mathcal{D}}(y_n, \bar{y}), \quad \forall n.$$

Therefore, condition (25) is also true. In addition, it is weaker. Indeed, consider the following data: $Y = M = \mathbb{R}^2$, $\mathcal{D}(y) = \text{cone}\{y\}$, for all $y \in \mathbb{R}^2$, and $\bar{y} = (1, 1)$. It is obvious that $\bigcap_{y \in M} \mathcal{D}(y) = \{0\}$ and so [12, Lemma 4.3(b)(ii)] and [13, Lemma 2.34(ii)] cannot be applied. However, the sequence $(y_n)_{n \geq 2}$, $y_n := \bar{y} - (1/n)\bar{y}$ satisfies

$$\bar{y} - y_n = (1/n)\bar{y} \in \text{cone}\{y_n\} = \mathcal{D}(y_n) = C_1^{\mathcal{D}}(y_n, \bar{y}), \quad \forall n \geq 2,$$

and so assumption (25) holds true. Thus, by Theorem 4.2(i) we deduce that $\bar{y}$ is not a $\leq^C$ -nondominated point of M , since $\bar{y} \in \text{int} M$.

(iii) Analogously, in [10, Lemma 2.1(a)] and [13, Lemma 2.34(i)], the version of Theorem 4.2(ii) addressing the algebraic boundary and $\leq^C$ -weakly nondominated points was stated in the particular case $C_1^{\mathcal{D}}$, where $\mathcal{D} : Y \rightrightarrows Y$ and $\mathcal{D}(y)$ is a pointed convex cone for all $y \in Y$. In that result, assumption $\bigcap_{y \in M} \text{cor} \mathcal{D}(y) \neq \emptyset$ is considered instead of (26). It yields that these assumptions are stronger. Indeed, on the one hand, reasoning as in part (ii) it is easy to see that the sequence $y_n := \bar{y} - (1/n)\bar{d}$ satisfies condition (26) provided that $\bar{d} \in \bigcap_{y \in M} \text{cor} \mathcal{D}(y)$. On the other hand, assume $Y = M = \mathbb{R}^2$, $\mathcal{D}(y) = \mathbb{R}_+^2$, for all $y \in \mathbb{R}_+^2$, $\mathcal{D}(y) = -\mathbb{R}_+^2$ otherwise, and $\bar{y} = (1, 1)$. Obviously,

$\bigcap_{y \in M} \text{cor}\mathcal{D}(y) = \emptyset$ and hence [10, Lemma 2.1(a)] and [13, Lemma 2.34(i)] cannot be applied. Whereas, one can find the sequence $y_n := \bar{y} - (1/n)\bar{y}$ such that

$$\bar{y} - y_n = (1/n)\bar{y} \in \text{cor}\mathbb{R}_+^2 = \text{cor}\mathcal{D}(y_n) = \text{cor}C_1^{\mathcal{D}}(y_n, \bar{y}), \quad \forall n \geq 1.$$

Consequently, assumption (26) is satisfied. Therefore, by part (ii) we conclude that $\bar{y}$ is not a $\leq^C$ -weakly nondominated point of M as $\bar{y} \in \text{int}M$.

4.2 Optimality concepts for Vector Optimization Problems

Consider the constrained vector optimization problem

$$\text{Minimize}_C \{f(x) : x \in S\}, \quad (\text{P})$$

where $f : X \rightarrow Y$, S is a nonempty subset of X , and $C : Y \times Y \rightrightarrows Y$ defines a two-variable domination structure $\leq^C$ on Y as in Definition 3.1. From now on, $\mathcal{N}(\bar{x})$ stands for the set of all neighborhoods of $\bar{x} \in X$.

Definition 4.2 A point $\bar{x}$ in S is said to be a

- (a) C -local minimal solution of problem (P), denoted by $\bar{x} \in \text{LMin}(f, S, C)$, if there exists $U \in \mathcal{N}(\bar{x})$ such that the following implication holds:

$$x \in U \cap S, \quad f(x) \leq^C f(\bar{x}) \Rightarrow f(\bar{x}) \leq^C f(x). \quad (27)$$

- (b) C -local nondominated solution of problem (P), denoted by $\bar{x} \in \text{LND}(f, S, C)$, if there exists $U \in \mathcal{N}(\bar{x})$ such that

$$x \in U \cap S, \quad f(x) \leq^C f(\bar{x}) \Rightarrow f(x) = f(\bar{x}). \quad (28)$$

- (c) C -local strongly nondominated solution of problem (P), denoted by $\bar{x} \in \text{LSND}(f, S, C)$, if there exists $U \in \mathcal{N}(\bar{x})$ such that

$$f(\bar{x}) \leq^C f(x), \quad \forall x \in U \cap (S \setminus \{\bar{x}\}). \quad (29)$$

- (d) C -local weakly nondominated solution of problem (P), denoted by $\bar{x} \in \text{LWND}(f, S, C)$, if $\bar{x} \in \text{LND}(f, S, \text{cor}C)$.

If condition (27) (resp. (28), (29)) holds true for every $x \in S \setminus \{\bar{x}\}$, then $\bar{x}$ is said to be a C -global minimal (resp. C -global nondominated, C -global strongly nondominated) solution of problem (P), denoted by $\bar{x} \in \text{GMin}(f, S, C)$ (resp. $\bar{x} \in \text{GND}(f, S, C)$, $\bar{x} \in \text{GSND}(f, S, C)$). Analogously, $\bar{x}$ is called a C -global weakly nondominated solution of problem (P), denoted by $\bar{x} \in \text{GWND}(f, S, C)$, if $\bar{x} \in \text{GND}(f, S, \text{cor}C)$.

Remark 4.5 (i) The following equivalences are easy to check:

$$\begin{aligned}
 \bar{x} \in \text{LMin}(f, S, C) &\iff \bar{x} \in S \text{ and } f(\bar{x}) \in \bigcup_{U \in \mathcal{N}(\bar{x})} \text{Min}(f(U \cap S), C) \\
 &\iff \bar{x} \in \bigcup_{U \in \mathcal{N}(\bar{x})} f^{-1}(\text{Min}(f(U \cap S), C)) \cap S, \\
 \bar{x} \in \text{LND}(f, S, C) &\iff \bar{x} \in S \text{ and } f(\bar{x}) \in \bigcup_{U \in \mathcal{N}(\bar{x})} \text{ND}(f(U \cap S), C) \\
 &\iff \bar{x} \in \bigcup_{U \in \mathcal{N}(\bar{x})} f^{-1}(\text{ND}(f(U \cap S), C)) \cap S, \\
 \bar{x} \in \text{LSND}(f, S, C) &\Rightarrow \bar{x} \in S \text{ and } f(\bar{x}) \in \bigcup_{U \in \mathcal{N}(\bar{x})} \text{SND}(f(U \cap S), C) \quad (30) \\
 &\iff \bar{x} \in \bigcup_{U \in \mathcal{N}(\bar{x})} f^{-1}(\text{SND}(f(U \cap S), C)) \cap S.
 \end{aligned}$$

The converse implication in (30) is also true whenever $0 \in C(f(\bar{x}), f(\bar{x}))$ or $\{x \in S : f(x) = f(\bar{x})\} = \{\bar{x}\}$.

(ii) In [4, Definition 4.1], the following notion of local solution of the unconstrained version of problem (P) (i.e., $S = X$) was defined, where $\mathcal{D} : Y \rightrightarrows Y$ is assumed to satisfy $0 \in \mathcal{D}(y)$, for all $y \in Y$, and also condition (22) by replacing M with a neighborhood V of $f(\bar{x})$: $\bar{x}$ is said to be a local nondominated solution of f w.r.t. $\mathcal{D}$ if there are a neighborhood U of $\bar{x}$ and a neighborhood V of $f(\bar{x})$ such that $f(\bar{x}) \in \text{Min}(f_V(U), C_1^{\mathcal{D}})$, where

$$f_V(U) := f(f^{-1}(V) \cap U) = \{y \in V : f^{-1}(y) \cap U \neq \emptyset\}.$$

The next example shows that this concept could not be suitable. Assume $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $\mathcal{D}(y) = [0, +\infty)$, for all $y \in \mathbb{R}$, and

$$f(x_1, x_2) = \begin{cases} 2 & \text{if } x_1 < 0, x_2 < 0, \\ 0 & \text{if } x_1 < 0, x_2 \geq 0, \\ 1 & \text{if } x_1 \geq 0. \end{cases}$$

Clearly, $y_1 \leq^{C_1^{\mathcal{D}}} y_2$ if and only if $y_2 \geq y_1$ and so problem (P) is a usual unconstrained scalar optimization problem. In addition, it is obvious that $\bar{x} = (0, 0)$ is not a local solution. However, by considering U the unit open ball and $V := (1/2, 3/2)$ we have $f^{-1}(V) = f^{-1}(\{1\})$ and so $f_V(U) = \{1\}$. Thus, $f(\bar{x}) \in \text{Min}(f_V(U), C_1^{\mathcal{D}})$ and $\bar{x}$ is a local nondominated solution of f w.r.t. $\mathcal{D}$. In other words, this notion does not coincide with the usual local solution concept of a scalar optimization problem when problem (P) reduces to that particular case.

C -local minimal solutions and C -local nondominated solutions encompass well-known solution concepts that involve an one-variable domination structure $\mathcal{D} : Y \rightrightarrows Y$

via the two-variable domination structures $C_1^{\mathcal{D}}$ and $C_2^{\mathcal{D}}$. Namely, by replacing $\leq^C$ with $\leq^{C_1^{\mathcal{D}}}$ and $\leq^{C_2^{\mathcal{D}}}$ the following notions are obtained (see Remark 3.1(i)).

Definition 4.3 Consider an one-variable domination mapping $\mathcal{D} : Y \rightrightarrows Y$. For $i = 1, 2$, a point $\bar{x}$ in S is said to be a

- (a) $\leq_i^{\mathcal{D}}$ -local minimal solution of problem (P), denoted by $\bar{x} \in \text{LMin}(f, S, \leq_i^{\mathcal{D}})$, if there exists $U \in \mathcal{N}(\bar{x})$ such that the following implication holds:

$$x \in U \cap S, \quad f(x) \leq_i^{\mathcal{D}} f(\bar{x}) \Rightarrow f(\bar{x}) \leq_i^{\mathcal{D}} f(x).$$

- (b) $\leq_i^{\mathcal{D}}$ -local nondominated solution of problem (P) if there exists $U \in \mathcal{N}(\bar{x})$ such that

$$x \in U \cap S, \quad f(x) \leq_i^{\mathcal{D}} f(\bar{x}) \Rightarrow f(x) = f(\bar{x}).$$

It is denoted by $\bar{x} \in \text{LND}(f, S, \leq_i^{\mathcal{D}})$.

The global versions of the above local solution concepts are denoted, respectively, by $\text{GMin}(f, S, \leq_i^{\mathcal{D}})$ and $\text{GND}(f, S, \leq_i^{\mathcal{D}})$.

Remark 4.6 $\leq_1^{\mathcal{D}}$ -local nondominated solutions were named local nondominated solutions in [5, Definition 4.1]. Meanwhile, the concepts of conventional nondominated (resp. conventional efficient) solution and $\mathcal{D}$ -nondominated (resp. $\mathcal{D}$ -efficient) solution introduced in [6, Definition 2.3] correspond to the set $\text{GMin}(f, X, \leq_1^{\mathcal{D}})$ (resp. $\text{GMin}(f, X, \leq_2^{\mathcal{D}})$) and the set $\text{GND}(f, X, \leq_1^{\mathcal{D}})$ (resp. $\text{GND}(f, X, \leq_2^{\mathcal{D}})$).

Some basic properties of the solutions of problem (P) are collected in the next result. A finite-dimensional global formulation of the first one was stated in [21, Theorem 3.1(iv)] and their proofs are similar. The others are direct consequences of Theorem 4.1 and Remark 4.5(i). For each $y \in Y$, define $C^c|_{f(S)}(y, \cdot) : Y \rightrightarrows Y$ as follows: $C^c|_{f(S)}(y, z) := (C(y, z))^c$ if $z \in f(S)$ and $C^c|_{f(S)}(y, z) := \emptyset$ otherwise.

Theorem 4.3 (i) Assume that a net (x_i) in S converges to $\bar{x}$ and there is $U \in \mathcal{N}(\bar{x})$ such that $x_i \in \text{GND}(f, U \cap S, C)$, for all i . Suppose that S is closed, f is continuous at $\bar{x}$ and $C^c|_{f(S)}(y, \cdot)$ is outer semicontinuous at $f(\bar{x})$, for all $y \in f(S) \setminus \{f(\bar{x})\}$. Then $\bar{x}$ belongs to $\text{LND}(f, S, C)$.

(ii) Any C -local nondominated solution of problem (P) is also a C -local weakly nondominated solution.

(iii) Suppose that there exists $U \in \mathcal{N}(\bar{x})$ such that

$$C(y, f(\bar{x})) \cap (-C(f(\bar{x}), y)) \subset \{0\}, \quad \forall y \in f(U \cap S) \setminus \{f(\bar{x})\}.$$

Then, if $\bar{x}$ is a C -local strongly nondominated solution of problem (P), it is a C -local nondominated solution of problem (P) too.

(iv) Any C -local nondominated solution of problem (P) is also a C -local minimal solution. Conversely, if $\bar{x} \in S$ is a $\leq^C$ -local minimal solution and there exists $U \in \mathcal{N}(\bar{x})$ such that

$$\bigcup_{d \in I_C^{f(S \cap U)}(f(\bar{x})) \setminus \{0\}} [\{d\} \cap (-I_C^{f(S \cap U)}(f(\bar{x}) - d))] = \emptyset, \quad (31)$$

then it is a C -local nondominated solution too.

Remark 4.7 Assume that $\mathcal{D} : Y \rightrightarrows Y$ fulfills $0 \in \mathcal{D}(y)$, for all $y \in Y$. In [6, Proposition 2.4.(ii)] it was stated that $\bar{x} \in X$ is a $\leq^{C^{\mathcal{D}}}_1$ -global nondominated solution of problem (P) with $S = X$ whenever it is a $\leq^{C^{\mathcal{D}}}_1$ -global minimal solution and the pointedness condition (22) holds true for $M = f(X)$ and $\bar{y} = f(\bar{x})$:

$$\mathcal{D}(f(x)) \cap (-\mathcal{D}(f(\bar{x}))) = \{0\}, \quad \forall x \in X. \quad (32)$$

Clearly, in this setting, assumption (31) with $U = X$ can be rewritten as in (23):

$$d \in \mathcal{D}(f(\bar{x}) - d) \setminus \{0\}, f(\bar{x}) - d \in f(X) \implies d \notin -\mathcal{D}(f(\bar{x})),$$

which is more general than (32) (see the example at the end of Remark 4.3).

Analogously, for the set-valued mapping $C^{\mathcal{D}}_2$, in [6, Proposition 2.4.(ii)] it was stated that $\bar{x} \in X$ is a $\leq^{C^{\mathcal{D}}}_2$ -global nondominated solution of problem (P) with $S = X$ whenever it is a $\leq^C$ -global minimal solution and pointedness condition (32) is fulfilled. In this framework, assumption (31) is

$$d \in \mathcal{D}(f(\bar{x})) \setminus \{0\}, f(\bar{x}) - d \in f(X) \implies d \notin -\mathcal{D}(f(\bar{x}) - d),$$

which is also more general than (32).

Next, we establish relationships between C -local nondominated solutions and C -local minimal solutions of problem (P) with their counterparts concerning binary relations that depend on one variable. Consider $\bar{x} \in S$. For every $V \in \mathcal{N}(\bar{x})$, define the set-valued mappings $\mathcal{D}^i_{1,V}$, $\mathcal{D}^u_{1,V}$, $\mathcal{D}^u_{2,V} : Y \rightrightarrows Y$ and $\widehat{C}_V : Y \times Y \rightrightarrows Y$ by

$$\begin{aligned} \mathcal{D}^i_{1,V}(y) &:= \bigcap_{x \in V \cap S} C(f(x), y), \\ \mathcal{D}^u_{1,V}(y) &:= \bigcup_{x \in V \cap S} C(f(x), y), \\ \mathcal{D}^u_{2,V}(y) &:= \bigcup_{x \in V \cap S} C(y, f(x)), \\ \widehat{C}_V(y_1, y_2) &:= (C(y_1, y_2) \setminus (-\mathcal{D}^u_{2,V}(y_2))) \cup \{0\}. \end{aligned}$$

Theorem 4.4 Consider a point $\bar{x} \in S$.

- (i) If $\bar{x} \in \bigcup_{V \in \mathcal{N}(\bar{x})} \text{LND}(f, S, \leq^{\mathcal{D}^u_{1,V}}_2)$ then $\bar{x} \in \text{LND}(f, S, C)$.
- (ii) Suppose that $\text{dom} \mathcal{D}^i_{1,V} = Y$, for all $V \in \mathcal{N}(\bar{x})$. If $\bar{x} \in \text{LND}(f, S, C)$, then $\bar{x} \in \bigcup_{V \in \mathcal{N}(\bar{x})} \text{LND}(f, S, \leq^{\mathcal{D}^i_{1,V}}_2)$.

(iii) If $\bar{x} \in \text{LMin}(f, S, C)$ then we have $\bar{x} \in \bigcup_{V \in \mathcal{N}(\bar{x})} \text{LND}(f, S, \widehat{C}_V)$.

Proof (i) Suppose that $\bar{x} \in \bigcup_{V \in \mathcal{N}(\bar{x})} \text{LND}(f, S, \leq_2^{\mathcal{D}_{1,V}^u})$. It yields that there is $V_1 \in \mathcal{N}(\bar{x})$ such that $\bar{x} \in \text{LND}(f, S, \leq_2^{\mathcal{D}_{1,V_1}^u})$. Thus, there exists $V_2 \in \mathcal{N}(\bar{x})$ such that

$$x \in V_2 \cap S, \quad f(x) \leq_2^{\mathcal{D}_{1,V_1}^u} f(\bar{x}) \Rightarrow f(x) = f(\bar{x})$$

that is equivalent to

$$x \in V_2 \cap S, \quad f(\bar{x}) \in f(x) + \mathcal{D}_{1,V_1}^u(f(\bar{x})) \Rightarrow f(x) = f(\bar{x}). \quad (33)$$

If $x_0 \in V_1 \cap V_2 \cap S$ satisfies $f(x_0) \leq^C f(\bar{x})$, then

$$f(\bar{x}) - f(x_0) \in C(f(x_0), f(\bar{x})) \subset \bigcup_{x \in V_1 \cap S} C(f(x), f(\bar{x})) = \mathcal{D}_{1,V_1}^u(f(\bar{x}))$$

and by (33), we deduce $f(x_0) = f(\bar{x})$. Therefore, $\bar{x}$ is a C -local nondominated solution.

(ii) Suppose that $\bar{x}$ is a C -local nondominated solution, with the corresponding neighborhood $V \in \mathcal{N}(\bar{x})$. If $x_0 \in V \cap S$ satisfies $f(x_0) \leq_2^{\mathcal{D}_{1,V}^i} f(\bar{x})$, then

$$f(\bar{x}) - f(x_0) \in \mathcal{D}_{1,V}^i(f(\bar{x})) = \bigcap_{x \in V \cap S} C(f(x), f(\bar{x})) \subset C(f(x_0), f(\bar{x})).$$

Therefore, $f(x_0) \leq^C f(\bar{x})$. Since $\bar{x}$ is a C -local nondominated solution, we have $f(x_0) = f(\bar{x})$, and it follows that $\bar{x}$ is a $\leq_2^{\mathcal{D}_{1,V}^i}$ -local nondominated solution.

(iii) Since $\bar{x}$ is a C -local minimal solution, there exists a neighborhood V of $\bar{x}$ such that for all $x \in V \cap S$, if $f(x) \leq^C f(\bar{x})$ then $f(\bar{x}) \leq^C f(x)$. Suppose that there exists $x_0 \in V \cap S$ such that $f(x_0) \leq^{\widehat{C}_V} f(\bar{x})$, then $f(\bar{x}) - f(x_0) \in \widehat{C}_V(f(x_0), f(\bar{x}))$, or, equivalently,

$$f(\bar{x}) - f(x_0) \in \left(C(f(x_0), f(\bar{x})) \setminus (-\mathcal{D}_{2,V}^u(f(\bar{x}))) \right) \cup \{0\}. \quad (34)$$

Furthermore, we have that

$$\begin{aligned}
 & C(f(x_0), f(\bar{x})) \setminus (-\mathcal{D}_{2,V}^u(f(\bar{x}))) \\
 &= C(f(x_0), f(\bar{x})) \cap \left[-\bigcup_{x \in V \cap S} C(f(\bar{x}), f(x)) \right]^c \\
 &= C(f(x_0), f(\bar{x})) \cap \left[-\bigcap_{x \in V \cap S} C^c(f(\bar{x}), f(x)) \right] \\
 &= \bigcap_{x \in V \cap S} [C(f(x_0), f(\bar{x})) \cap (-C^c(f(\bar{x}), f(x)))] \\
 &\subset C(f(x_0), f(\bar{x})) \cap (-C^c(f(\bar{x}), f(x_0))).
 \end{aligned}$$

If $f(x_0) \neq f(\bar{x})$, then by (34) and the inclusion above we have

$$\begin{aligned}
 f(\bar{x}) - f(x_0) &\in C(f(x_0), f(\bar{x})), \\
 f(x_0) - f(\bar{x}) &\notin C(f(\bar{x}), f(x_0)),
 \end{aligned}$$

i.e., $f(x_0) \leq^C f(\bar{x})$ and $f(\bar{x}) \not\leq^C f(x_0)$, that is contrary to the C -local minimality of $\bar{x}$. Consequently, $f(x_0) = f(\bar{x})$ and it follows that $\bar{x}$ is a $\widehat{C}_V$ -local nondominated solution. $\square$

Remark 4.8 In the case where $S := X$ and $\bar{x}$ is a global optimal solution of problem (P), the above relationships encompass the ones stated in [6, Proposition 2.5]. More precisely, for a set-valued mapping $\mathcal{D} : Y \rightrightarrows Y$, the next particular cases of Theorem 4.4 are obtained:

- (i) Considering the case $V = X$ and the two-variable domination mapping $C_1^{\mathcal{D}}$ in Theorem 4.4(i), we have $\mathcal{D}_{1,X}^u(y) = \mathcal{D}(f(X))$, for all $y \in Y$, and [6, Proposition 2.5(iv)] is covered.
- (ii) by applying Theorem 4.4(ii) to the particular case $V = X$ and the two-variable domination mapping
 - $C_1^{\mathcal{D}}$, then $\mathcal{D}_{1,X}^i(y) = \bigcap_{x \in X} \mathcal{D}(f(x))$, for all $y \in Y$, and the assertion in [6, Proposition 2.5(iii)] is obtained.
 - $C_2^{\mathcal{D}}$, then $\mathcal{D}_{1,X}^i(y) = \mathcal{D}(y)$, for all $y \in Y$, and we get the statement in [6, Proposition 2.5(ii)].
- (iii) by applying Theorem 4.4(iii) to the case $C_2^{\mathcal{D}}$ and $V = X$, we arrive at [6, Proposition 2.5(i)]. Notice that in this setting, $\mathcal{D}_{2,V}^u$ reduces to the constant set-valued mapping $\mathcal{D}_{2,V}^u(y) = \mathcal{D}(f(X))$, for all $y \in Y$, and $\widehat{C}_V(y_1, y_2) := (\mathcal{D}(y_2) \setminus (-\mathcal{D}(f(X)))) \cup \{0\}$.

5 Nonlinear Scalarization

Scalarization technique is one of the most important mathematical tools to deal with vector optimization problems, not only from a practical point of view, since it allows to solve these problems, but also from a theoretical one as lots of properties of vector optimization problems can be obtained via scalarization approaches (see [7, 20, 25, 30]). Specifically, to scalarize a vector optimization problem is to replace it with an ordinary (scalar) optimization one whose solutions are related with the solutions of the nominal vector problem. Usually, the objective function of the scalarized problem is defined by the composition of the objective function of the vector optimization problem with a real-valued function that satisfies suitable order preserving properties, which allow to relate the solutions of both optimization problems (see [22] and the references therein).

The so-called Gerstewitz scalarization function,

$$\varphi_{K,q}(y) = \inf\{t \in \mathbb{R} : y \in tq - K\},$$

has been intensively employed to develop results regarding nonconvex vector optimization problems whose final space Y is ordered by the partial order $\leq_K$ defined for a fixed pointed convex cone K (see (4) and the references [20, 30] for more detailed investigation on this function). If the domination structure is given by an one-variable ordering mapping $\mathcal{D}$, then more investigations on this scalarization tool can be found, such as in [1, 7–9, 13].

In this section, we characterize $\leq^C$ -nondominated/weakly nondominated points of a set M and C -local nondominated/weakly nondominated solutions of problem (P) via a generalization of the above function $\varphi_{K,q}$ based on allowing a variable direction mapping $q : Y \times Y \rightarrow Y$ in place of a constant direction q . Namely, we have achieved two goals. First, we have established general scalarization results that are applicable to a two-variable domination mapping C . Second, in the particular case of an one-variable ordering structure, we have weakened some conditions imposed on the ordering mapping $\mathcal{D}$ in existing literature in order to characterize solutions of vector optimization problems (see Remark 5.2 and Example 5.3). The generalized Gerstewitz scalarization function is defined as follows.

Definition 5.1 Consider a set-valued mapping $C : Y \times Y \rightrightarrows Y$ such that $\text{dom} C = Y \times Y$, a function $q : Y \times Y \rightarrow Y \setminus \{0\}$ and a point $\bar{y} \in Y$. The nonlinear scalarization function $\varphi_{\bar{y},q}^C : Y \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is defined as follows

$$\varphi_{\bar{y},q}^C(y) := \inf \Lambda_q^C(y, \bar{y}),$$

where $\Lambda_q^C : Y \times Y \rightrightarrows \mathbb{R}$,

$$\Lambda_q^C(y_1, y_2) := \{t \in \mathbb{R} : y_2 + tq(y_1, y_2) - y_1 \in C(y_1, y_2)\}$$

with the convention that $\inf \emptyset = +\infty$.

The function $\varphi_{\bar{y},q}^C$ is said to be proper if $\text{dom } \varphi_{\bar{y},q}^C \neq \emptyset$ and $\varphi_{\bar{y},q}^C$ does not take the value $-\infty$. The basic properties of the scalarization function $\varphi_{\bar{y},q}^C$ to characterize $\leq^C$ -nondominated/weakly nondominated points of a set M and C -local nondominated/weakly nondominated solutions of problem (P) are shown in Lemma 5.2.

The algebraic concept of closure by a direction due to Qiu and He [29] will be required (see also [30, Section 2.3.2] and Gutiérrez et al. [23, Section 3]). It is naturally related to the function $\varphi_{\bar{y},q}^C$ since its sublevel sets actually depend on this notion (see Lemma 5.2(iii)). More precisely, the values of the function $\varphi_{\bar{y},q}^C$ depend on the algebraic closure of the values of the mapping C as $\varphi_{\bar{y},q}^C = \varphi_{\bar{y},q}^{\text{vcl}_q C}$ (see Lemma 5.2(i)).

Given a nonempty set $E \subset Y$ and a nonzero vector $q \in Y$, the vector closure of E in the direction q (in the following, q -vector closure of E) is the set

$$\text{vcl}_q E := \{y \in Y : \forall \lambda > 0 \exists \lambda' \in [0, \lambda] \text{ s.t. } y + \lambda' q \in E\}.$$

We say that set E is q -directionally closed if $\text{vcl}_q E = E$. The directional boundary of E with respect to q (see [30]) is the set $\text{bd}_q E := \text{vcl}_q E \setminus \text{int}_{-q} E$. Recall that the recession cone of a set E is the convex cone

$$0^+ E := \{y \in Y : E + \mathbb{R}_+ y = E\}.$$

Next lemma collects some properties of the vector closure of a set. Although they are known (see, for instance, [30, Section 2.3.2]), we provide the proof for the reader's convenience.

Lemma 5.1 *We have that*

- (i) $y \in \text{vcl}_q E$ if and only if there exists a sequence $(t_n) \subset \mathbb{R}_+$ such that $t_n \rightarrow 0$ and $y + t_n q \in E$, for all n .
- (ii) $E \subset \text{vcl}_q E \subset \text{vcl}_q E + \mathbb{R}_+ q = \text{vcl}_q (E + \mathbb{R}_+ q)$.
- (iii) $\text{vcl}_q (\text{vcl}_q E) = \text{vcl}_q E$ and $\text{vcl}_q (\text{vcl}_q E + \mathbb{R}_+ q) = \text{vcl}_q (E + \mathbb{R}_+ q)$.
- (iv) $Y \setminus \text{vcl}_q E = \text{int}_q (Y \setminus E)$.
- (v) If $q \in 0^+ E$, then

$$E + (0, +\infty)q = \text{int}_{-q} E, \quad (35)$$

$$\text{vcl}_q E + [0, +\infty)q = \text{vcl}_q E. \quad (36)$$

- (vi) If $q \in -\text{cor}(0^+ E) \setminus \{0\}$, then $\text{cor} E = \text{int}_q E$.

Proof Statements (i) and (iv) are straightforward.

(ii) The inclusions

$$E \subset \text{vcl}_q E \subset \text{vcl}_q E + \mathbb{R}_+ q \subset \text{vcl}_q (E + \mathbb{R}_+ q)$$

follow directly from the definition.

In order to stay $\text{vcl}_q (E + \mathbb{R}_+ q) \subset \text{vcl}_q E + \mathbb{R}_+ q$ consider a point $y \in Y$ and a sequence $(t_n) \subset \mathbb{R}_+$, $t_n \rightarrow 0$, such that $y + t_n q \in E + \mathbb{R}_+ q$. Thus, there exists a

sequence $(s_n) \subset \mathbb{R}_+$ satisfying $y + t_n q \in E + s_n q$, for all n . If there exists k such that $s_k \geq t_k$, then $y \in E + (s_k - t_k)q \subset E + \mathbb{R}_+ q \subset \text{vcl}_q E + \mathbb{R}_+ q$. Otherwise, $s_n < t_n$ for all n and so $s_n \rightarrow 0$ and $y + (t_n - s_n)q \in E$, for all $n \in \mathbb{N}$. Therefore, $y \in \text{vcl}_q(E) \subset \text{vcl}_q(E) + \mathbb{R}_+ q$ and part (ii) is stated.

(iii) The inclusion $\text{vcl}_q E \subset \text{vcl}_q(\text{vcl}_q E)$ is obvious. Conversely, suppose that $y \in \text{vcl}_q(\text{vcl}_q E)$ and $y \notin \text{vcl}_q E$. Then, there exists $\lambda > 0$ such that $(y + [0, \lambda]q) \cap E = \emptyset$. Since $y \in \text{vcl}_q(\text{vcl}_q E)$ there exists a sequence $(t_n) \subset \mathbb{R}_+$, $t_n \rightarrow 0$, such that $y + t_n q \in \text{vcl}_q E$, for all n . Since $y \notin \text{vcl}_q E$ we deduce that $t_n > 0$ and then, for each n there exists $s_n \in [0, t_n]$ such that $(y + t_n q) + s_n q \in E$. Thus, $s_n \rightarrow 0$ and we have $\alpha_n := t_n + s_n > 0$, $\alpha_n \rightarrow 0$ and $y + \alpha_n q \in E$, that is a contradiction.

Notice by part (ii) that

$$\text{vcl}_q(\text{vcl}_q E + \mathbb{R}_+ q) = \text{vcl}_q \text{vcl}_q(E + \mathbb{R}_+ q) = \text{vcl}_q(E + \mathbb{R}_+ q)$$

and the second equality in (iii) is also obtained.

(v) Consider $y \in E$ and $q \in 0^+ E$. For each $t > 0$ we have that

$$y + tq + [0, t/2](-q) = y + [t/2, t]q \subset E$$

and $y + tq \in \text{int}_{-q} E$. Therefore, $E + (0, +\infty)q \subset \text{int}_{-q} E$. Conversely, if $y \in \text{int}_{-q} E$, then there exists $t > 0$ such that $y + [0, t](-q) \subset E$. Particularly, $y \in tq + E \subset (0, +\infty)q + E$.

Concerning the second equality in part (v), inclusion $\text{vcl}_q E \subset \text{vcl}_q E + [0, +\infty)q$ is obvious. Conversely, if $y \in \text{vcl}_q E$, then there exists a sequence $(t_n) \subset \mathbb{R}_+$, $t_n \rightarrow 0$ such that $y + t_n q \in E$. Hence, for each $s \geq 0$, as $q \in 0^+ E$, we have

$$y + sq + t_n q = (y + t_n q) + sq \in E + [0, +\infty)q = E.$$

Therefore, $y + sq \in \text{vcl}_q E$ and part (v) is proved.

(vi) By statement (1) we have that $\text{cor} E \subset \text{int}_q E$, for all $q \in Y \setminus \{0\}$. Conversely, assume that $q \in -\text{cor}(0^+ E) \setminus \{0\}$ and consider $y \in \text{int}_q E$ and an arbitrary vector $v \in Y \setminus \{0\}$. There exist $\varepsilon_1, \varepsilon_2 > 0$ such that $y + [0, \varepsilon_1]q \subset E$ and $-q + [0, \varepsilon_2]v \subset 0^+ E$, i.e., $E + \mathbb{R}_+(-q + [0, \varepsilon_2]v) = E$. For each $t \in [0, \varepsilon_1 \varepsilon_2]$ we obtain

$$y + tv = (y + \varepsilon_1 q) + \varepsilon_1((t/\varepsilon_1)v - q) \in E + \mathbb{R}_+(-q + [0, \varepsilon_2]v) = E.$$

Hence, $y \in \text{int}_v E$ and so $y \in \text{cor} E$ as v was arbitrarily chosen. $\square$

We now consider some properties of the scalarization function $\varphi_{\bar{y}, q}^C$. Given $C : Y \times Y \rightrightarrows Y$ and $q : Y \times Y \rightarrow Y \setminus \{0\}$, $\text{vcl}_q C$, $\text{bd}_q C$ and $\text{vcl}_q C + \mathbb{R}_+ q$ stand for the set-valued mappings from $Y \times Y$ into Y given by $(\text{vcl}_q C)(y_1, y_2) = \text{vcl}_{q(y_1, y_2)} C(y_1, y_2)$, $(\text{bd}_q C)(y_1, y_2) = \text{bd}_{q(y_1, y_2)} C(y_1, y_2)$ and $(\text{vcl}_q C + \mathbb{R}_+ q)(y_1, y_2) = (\text{vcl}_q C)(y_1, y_2) + \mathbb{R}_+ q(y_1, y_2)$, respectively.

Lemma 5.2 *It follows that*

(i) $\varphi_{\bar{y},q}^C = \varphi_{\bar{y},q}^{C'}$ for any set-valued mapping $C' : Y \times Y \rightrightarrows Y$ such that

$$\text{gph} C \subset \text{gph} \text{vcl}_q C' \subset \text{gph}(\text{vcl}_q C + \mathbb{R}_+ q). \quad (37)$$

In particular, $\varphi_{\bar{y},q}^C = \varphi_{\bar{y},q}^{\text{vcl}_q C} = \varphi_{\bar{y},q}^{\tilde{C}}$, where $\tilde{C} := \text{vcl}_q C + \mathbb{R}_+ q$.
(ii) For each $y \in Y$ and $s \in \mathbb{R}$,

$$\varphi_{\bar{y},q}^C(y) < s \iff y - \bar{y} \in (-\infty, s)q(y, \bar{y}) - C(y, \bar{y}).$$

(iii) For each $y \in Y$ and $s \in \mathbb{R}$,

$$\varphi_{\bar{y},q}^C(y) \leq s \iff y - \bar{y} \in (-\infty, s]q(y, \bar{y}) - \text{vcl}_{q(y,\bar{y})} C(y, \bar{y}). \quad (38)$$

Proof (i) By statement (37) it is obvious that $\Lambda_q^C(y_1, y_2) \subset \Lambda_q^{\tilde{C}}(y_1, y_2)$, for all $y_1, y_2 \in Y$, and so $\varphi_{\bar{y},q}^{\tilde{C}} \leq \varphi_{\bar{y},q}^C$. In particular, $\varphi_{\bar{y},q}^{\tilde{C}}(y) = \varphi_{\bar{y},q}^C(y)$ whenever $\varphi_{\bar{y},q}^{\tilde{C}}(y) = +\infty$. Consider $\varphi_{\bar{y},q}^{\tilde{C}}(y) < +\infty$ and take $(y, \bar{y}, t) \in \text{gph} \Lambda_q^{\tilde{C}}$. Thus,

$$\bar{y} + tq(y, \bar{y}) - y \in \tilde{C}(y, \bar{y}) = \text{vcl}_{q(y,\bar{y})} C(y, \bar{y}) + \mathbb{R}_+ q(y, \bar{y}).$$

Hence, there exists $\alpha \geq 0$ such that

$$\bar{y} + tq(y, \bar{y}) - y - \alpha q(y, \bar{y}) \in \text{vcl}_{q(y,\bar{y})} C(y, \bar{y}).$$

By Lemma 5.1(i) there exists a sequence $(t_n) \subset \mathbb{R}_+$ such that $t_n \rightarrow 0$ and

$$\bar{y} + tq(y, \bar{y}) - y - \alpha q(y, \bar{y}) + t_n q(y, \bar{y}) \in C(y, \bar{y}), \quad \forall n.$$

Thus, $(y, \bar{y}, t - \alpha + t_n) \in \text{gph} \Lambda_q^C$ and $\varphi_{\bar{y},q}^C(y) \leq t - \alpha + t_n$, for all n . As $t_n \rightarrow 0$ and $\alpha \geq 0$ it follows that $\varphi_{\bar{y},q}^C(y) \leq t$, and since t is an arbitrary element of $\Lambda_q^{\tilde{C}}(y, \bar{y})$ we deduce that $\varphi_{\bar{y},q}^C(y) \leq \inf \Lambda_q^{\tilde{C}}(y, \bar{y}) = \varphi_{\bar{y},q}^{\tilde{C}}(y)$. Therefore, $\varphi_{\bar{y},q}^C(y) = \varphi_{\bar{y},q}^{\tilde{C}}(y)$.

Finally, if C' satisfies (37), then

$$\varphi_{\bar{y},q}^C = \varphi_{\bar{y},q}^{\tilde{C}} \leq \varphi_{\bar{y},q}^{\text{vcl}_q C'} \leq \varphi_{\bar{y},q}^C$$

and we see that $\varphi_{\bar{y},q}^C = \varphi_{\bar{y},q}^{\tilde{C}} = \varphi_{\bar{y},q}^{\text{vcl}_q C'}$. Clearly, since $C' := C$ fulfills (37), we have $\varphi_{\bar{y},q}^C = \varphi_{\bar{y},q}^{\text{vcl}_q C} = \varphi_{\bar{y},q}^{\tilde{C}}$.

(ii) Consider $y \in Y$ and $s \in \mathbb{R}$. Clearly, $\varphi_{\bar{y},q}^C(y) < s$ if and only if there exists $t \in \mathbb{R}$ such that $(y, \bar{y}, t) \in \text{gph} \Lambda_q^C$ and $t < s$, which is equivalent to

$$y - \bar{y} \in (-\infty, s)q(y, \bar{y}) - C(y, \bar{y}).$$

(iii) Take arbitrary points $y \in Y$ and $s \in \mathbb{R}$. If $y - \bar{y} \in (-\infty, s]q(y, \bar{y}) - \text{vcl}_{q(y, \bar{y})}C(y, \bar{y})$, then by part (i) we obtain

$$\varphi_{\bar{y}, q}^C(y) = \varphi_{\bar{y}, q}^{\text{vcl}_q C}(y) \leq s.$$

Conversely, suppose that $\varphi_{\bar{y}, q}^C(y) \leq s$. Then, for each n , $\varphi_{\bar{y}, q}^C(y) < s + 1/n$ and by part (ii) we deduce that

$$y - \bar{y} \in (-\infty, s + 1/n)q(y, \bar{y}) - C(y, \bar{y}).$$

Therefore, there exists a sequence of real numbers (t_n) such that $t_n < s + 1/n$ and $y - \bar{y} \in t_n q(y, \bar{y}) - C(y, \bar{y})$, for all n . If $t_n \leq s$ for some n , then

$$y - \bar{y} \in (-\infty, s]q(y, \bar{y}) - C(y, \bar{y}) \subset (-\infty, s]q(y, \bar{y}) - \text{vcl}_{q(y, \bar{y})}C(y, \bar{y}).$$

Otherwise, $s < t_n$ for all n and we have $s_n := t_n - s > 0$, $s_n \rightarrow 0$ and $y - \bar{y} - s q(y, \bar{y}) \in s_n q(y, \bar{y}) - C(y, \bar{y})$, for all n . Thus,

$$y - \bar{y} \in s q(y, \bar{y}) - \text{vcl}_{q(y, \bar{y})}C(y, \bar{y}) \subset (-\infty, s]q(y, \bar{y}) - \text{vcl}_{q(y, \bar{y})}C(y, \bar{y})$$

and the proof is finished. $\square$

The set $\text{vcl}_{q(y, \bar{y})}C(y, \bar{y})$ cannot be replaced by $C(y, \bar{y})$ in equivalence (38). It is demonstrated by the following example.

Example 5.1 Let $Y := \mathbb{R}^2$, $\bar{y} := (0, 0)$, $C(y, \bar{y}) := \text{int}\mathbb{R}_+^2 \cup \{(0, 0)\}$, and the vector $q(y, \bar{y}) := (1, 1)$, for all $y \in Y$. Then, $q(y, \bar{y}) \in 0^+C(y, \bar{y})$ and by (36) we have that

$$(-\infty, 0]q(y, \bar{y}) - \text{vcl}_{q(y, \bar{y})}C(y, \bar{y}) = -\text{vcl}_{q(y, \bar{y})}C(y, \bar{y}) = -\mathbb{R}_+^2, \quad \forall y \in Y.$$

In addition, by Lemma 5.2(i), $\varphi_{\bar{y}, q}^C(y) = \varphi_{\bar{y}, q}^{\text{vcl}_q C}(y)$ for all $y = (y_1, y_2) \in Y$ and

$$\begin{aligned} \varphi_{\bar{y}, q}^C(y) &= \inf\{t \in \mathbb{R} : \bar{y} + tq(y, \bar{y}) - y \in \text{vcl}_{q(y, \bar{y})}C(y, \bar{y})\} \\ &= \inf\{t \in \mathbb{R} : (t - y_1, t - y_2) \in \mathbb{R}_+^2\} \\ &= \max\{y_1, y_2\}. \end{aligned}$$

Now consider the point $z = (0, -1)$, then we have $\varphi_{\bar{y}, q}^C(z) = 0$. Note that $z - \bar{y} = (0, -1)$ belongs to $-\text{vcl}_{q(z, \bar{y})}C(z, \bar{y})$, but it does not belong to $-C(z, \bar{y})$.

Remark 5.1 (i) In [8], the following nonlinear scalarization function $\xi : Y \times Y \rightarrow \mathbb{R}$ is introduced concerning an one-variable ordering mapping $\mathcal{D} : Y \rightrightarrows Y$, where for each $y \in Y$ the value $\mathcal{D}(y)$ is assumed to be a proper closed convex cone and

$$k \in \text{int} \left(\bigcap_{y \in Y} \mathcal{D}(y) \right) : \quad \xi(y, z) = \inf \{t \in \mathbb{R} : z \in tk - \mathcal{D}(y)\}. \quad (39)$$

It is not hard to obtain that $\xi(y, z) = \varphi_{y-z, q^k}^{C_1^{\mathcal{D}}}(y) = \varphi_{y, q^k}^{C_2^{\mathcal{D}}}(y+z)$ where $q^k : Y \times Y \rightarrow Y$, $q^k(y, z) = k$. By the assumptions on the one-variable ordering mapping $\mathcal{D}$, it is easy to check that $0^+ \mathcal{D}(y) = \mathcal{D}(y)$, $\text{cor}(0^+ \mathcal{D}(y)) = \text{int} \mathcal{D}(y)$ and $\text{vcl}_k \mathcal{D}(y) = \mathcal{D}(y)$. Thus, by applying parts (v) and (vi) of Lemma 5.1 we see that parts (ii) and (iii) of Lemma 5.2 reduce to parts (i) and (ii) of [8, Lemma 2.3]. Notice that Lemma 5.2 has been stated without assuming any hypotheses on the two-variable domination mapping C .

(ii) In [4, Proposition 3.1], a reformulation of the scalarization function ξ was defined to characterize nondominated points of a set M with respect to the one-variable ordering mapping $C_1^{\mathcal{D}}$, where it is assumed that $0 \in \mathcal{D}(y)$, for all $y \in Y$ and $k \in \bigcap_{y \in Y} \mathcal{D}(y)$. Specifically, the authors consider the scalarization function $\xi' : Y \times Y \rightarrow \mathbb{R}$, $\xi'(y, z) = \xi(y, y - z)$, for all $y, z \in Y$. By part (i) above we have that $\xi'(y, z) = \varphi_{z, q^k}^{C_1^{\mathcal{D}}}(y) = \varphi_{y, q^k}^{C_2^{\mathcal{D}}}(2y - z)$ and so the sufficient condition of Lemma 5.2(iii) reduces to [4, Proposition 3.1(i)], the necessary one encompasses [4, Proposition 3.1(ii)] by statement (36) and the necessary condition of Lemma 5.2(ii) reduces to [4, Proposition 3.1(iii)].

Lemmas 5.1 and 5.2 allow us to characterize minimizers of a set by the scalarization function $\varphi_{y, q}^C$. In the next example we illustrate this method for the equitability preference considered in Remark 3.1(ii).

Example 5.2 Recall that for each $y^1, y^2 \in \mathbb{R}^2$, $y^1 \leq_e y^2$ if and only if $(y^1, y^1) \leq^C (y^2, p(y^2))$, where $p((y_1, y_2)) = (y_2, y_1)$ and

$$C((y^1, v^1), (y^2, v^2)) = \begin{cases} (-D_1) \times \mathbb{R}^2 & \text{if } y^1, y^2 \in S_1, \\ (-D_2) \times \mathbb{R}^2 & \text{if } y^1, y^2 \in S_2, \\ \mathbb{R}^2 \times (-D_1) & \text{if } y^1 \in S_1, y^2 \in S_2, \\ \mathbb{R}^2 \times (-D_2) & \text{if } y^1 \in S_2, y^2 \in S_1. \end{cases}$$

Consider $q((y^1, v^1), (y^2, v^2)) = (-1, -1, -1, -1)$, for all $(y^1, v^1), (y^2, v^2) \in \mathbb{R}^2 \times \mathbb{R}^2$. It is not hard to check that the values of the set-valued mapping C are closed convex cones. Thus, we have

$$\begin{aligned} 0^+ C((y^1, v^1), (y^2, v^2)) &= C((y^1, v^1), (y^2, v^2)), \\ \text{vcl}_q C((y^1, v^1), (y^2, v^2)) &= C((y^1, v^1), (y^2, v^2)), \quad \forall (y^1, v^1), (y^2, v^2) \in \mathbb{R}^2 \times \mathbb{R}^2. \end{aligned} \quad (40)$$

In addition, by easy calculations we obtain

$$\varphi_{(y^2, p(y^2)), q}^C((y^1, y^1)) = \begin{cases} \max\{y_1^2 - y_1^1, h(y^1, y^2)\} & \text{if } y^1, y^2 \in S_1, \\ \max\{y_2^2 - y_2^1, h(y^1, y^2)\} & \text{if } y^1, y^2 \in S_2, \\ \max\{y_2^2 - y_1^1, h(y^1, y^2)\} & \text{if } y^1 \in S_1, y^2 \in S_2, \\ \max\{y_1^2 - y_2^1, h(y^1, y^2)\} & \text{if } y^1 \in S_2, y^2 \in S_1, \end{cases}$$

where $h : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $h(y^1, y^2) = \frac{y_1^2 + y_2^2 - (y_1^1 + y_2^1)}{2}$. Hence, by assertions (36), (38) and (40), it follows that

$$\varphi_{(y^2, p(y^2)), q}^C((y^1, y^1)) \leq 0 \iff (y^1, y^1) \leq^C (y^2, p(y^2)).$$

In addition, we have that $\varphi_{(y, p(y)), q}^C((y, y)) = 0$, for all $y \in \mathbb{R}^2$. Therefore, given a nonempty set $M \subset \mathbb{R}^2$, it follows that a point $\bar{y} \in M$ is a equitably nondominated element of M , i.e., $y \not\leq_e \bar{y}$ for all $y \in M \setminus \{\bar{y}\}$, if

$$\varphi_{(\bar{y}, p(\bar{y})), q}^C((y, y)) > 0 = \varphi_{(\bar{y}, p(\bar{y})), q}^C((\bar{y}, \bar{y})), \quad \forall y \in M.$$

Hence, by denoting $g_{\bar{y}} : \mathbb{R}^2 \rightarrow \mathbb{R}$, $g_{\bar{y}}(y) = \varphi_{(\bar{y}, p(\bar{y})), q}^C((y, y))$, we have stated that $\bar{y} \in M$ is a equitably nondominated element of M if and only if $\bar{y} \in \operatorname{argmin}_M g_{\bar{y}}$. This equivalence is proved for a general two-variable domination structure in Theorem 5.1(iv).

Next we characterize $\leq^C$ -nondominated points and $\leq^C$ -weakly nondominated points of a set by the same approach as in the previous example.

Theorem 5.1 *Let M be a nonempty subset of Y and $\bar{y} \in M$.*

- (i) *Assume that $q(y, \bar{y}) \in 0^+C(y, \bar{y})$, for all $y \in Y$ and $0 \in \operatorname{bd}_{q(\bar{y}, \bar{y})}C(\bar{y}, \bar{y})$. Then, $\bar{y} \in \operatorname{ND}(M, \operatorname{int}_{-q}C)$ if and only if $\bar{y} \in \operatorname{argmin}_M \varphi_{\bar{y}, q}^C$.*
- (ii) *Consider $q(y, \bar{y}) \in \operatorname{cor}0^+C(y, \bar{y})$, for all $y \in Y \setminus \{\bar{y}\}$, $q(\bar{y}, \bar{y}) \in 0^+C(\bar{y}, \bar{y})$ and $0 \in \operatorname{bd}_{q(\bar{y}, \bar{y})}C(\bar{y}, \bar{y})$. Then, $\bar{y} \in \operatorname{WND}(M, C)$ if and only if $\bar{y} \in \operatorname{argmin}_M \varphi_{\bar{y}, q}^C$.*
- (iii) *Suppose that $q(y, \bar{y}) \in 0^+C(y, \bar{y})$, for all $y \in Y$ and $0 \in \operatorname{bd}_{q(\bar{y}, \bar{y})}C(\bar{y}, \bar{y})$. Then, $\bar{y} \in \operatorname{ND}(M, \operatorname{vcl}_qC)$ if and only if $\bar{y} \in \operatorname{argmin}_M \varphi_{\bar{y}, q}^C$.*
- (iv) *Assume that $C(y, \bar{y})$ is $q(y, \bar{y})$ -directionally closed, for all $y \in Y \setminus \{\bar{y}\}$, $q(y, \bar{y}) \in 0^+C(y, \bar{y})$, for all $y \in Y$, and $0 \in \operatorname{bd}_{q(\bar{y}, \bar{y})}C(\bar{y}, \bar{y})$. Then, $\bar{y} \in \operatorname{ND}(M, C)$ if and only if $\bar{y} \in \operatorname{argmin}_M \varphi_{\bar{y}, q}^C$.*
- (v) *Assume that $q(y, \bar{y}) \in 0^+C(y, \bar{y})$, for all $y \in Y$ and $0 \in \operatorname{bd}_{q(\bar{y}, \bar{y})}C(\bar{y}, \bar{y})$. If $\bar{y} \in \operatorname{argmin}_M \varphi_{\bar{y}, q}^C \setminus \operatorname{argmin}_M \varphi_{\bar{y}, q}^C$, then*

$$\emptyset \neq \{y \in M \setminus \{\bar{y}\} : \bar{y} \in y + \operatorname{bd}_{q(y, \bar{y})}C(y, \bar{y})\} = \{y \in M \setminus \{\bar{y}\} : \varphi_{\bar{y}, q}^C(y) = 0\}.$$

Proof (i) By statement (35) we deduce that

$$C(y, \bar{y}) + (0, +\infty)q(y, \bar{y}) = \operatorname{int}_{-q(y, \bar{y})}C(y, \bar{y}), \quad \forall y \in Y.$$

Then, by Lemma 5.2(ii), it follows that for each $y \in Y$,

$$\varphi_{\bar{y}, q}^C(y) < 0 \iff \bar{y} \in y + \operatorname{int}_{-q(y, \bar{y})}C(y, \bar{y}).$$

Therefore, $\bar{y} \in \operatorname{ND}(M, \operatorname{int}_{-q}C)$ if and only if $\varphi_{\bar{y}, q}^C(y) \geq 0$, for all $y \in M \setminus \{\bar{y}\}$. Moreover, $\varphi_{\bar{y}, q}^C(\bar{y}) = 0$ since $0 \in \operatorname{bd}_{q(\bar{y}, \bar{y})}C(\bar{y}, \bar{y})$. Indeed, from $0 \in \operatorname{vcl}_{q(\bar{y}, \bar{y})}C(\bar{y}, \bar{y})$,

there exists a sequence $(t_n) \subset \mathbb{R}_+$, $t_n \rightarrow 0$, such that $t_n q(\bar{y}, \bar{y}) \in C(\bar{y}, \bar{y})$ for all n . Hence,

$$\varphi_{\bar{y},q}^C(\bar{y}) = \inf\{t \in \mathbb{R} : tq(\bar{y}, \bar{y}) \in C(\bar{y}, \bar{y})\} \leq 0.$$

In addition, $tq(\bar{y}, \bar{y}) \notin C(\bar{y}, \bar{y})$, for all $t < 0$, since, otherwise, $[-t_0, +\infty)q(\bar{y}, \bar{y}) \subset C(\bar{y}, \bar{y})$ for some $t_0 > 0$, which implies $[0, t_0)(-q(\bar{y}, \bar{y})) \subset C(\bar{y}, \bar{y})$, which is a contradiction as $0 \notin \text{int}_{-q(\bar{y}, \bar{y})} C(\bar{y}, \bar{y})$.

Thus, $\bar{y} \in \text{ND}(M, \text{int}_{-q} C)$ if and only if $\varphi_{\bar{y},q}^C(y) \geq \varphi_{\bar{y},q}^C(\bar{y})$, for all $y \in M$, i.e., if and only if $\bar{y} \in \text{argmin}_M \varphi_{\bar{y},q}^C$ and part (i) is stated.

(ii) By the assumption $q(y, \bar{y}) \in \text{cor}^+ C(y, \bar{y})$ and Lemma 5.1(vi) we obtain that $\text{cor} C(y, \bar{y}) = \text{int}_{-q(y, \bar{y})} C(y, \bar{y})$, for all $y \in Y \setminus \{\bar{y}\}$. Hence, by part (i) it follows that

$$\bar{y} \in \text{ND}(M, \text{int}_{-q} C) = \text{ND}(M, \text{cor} C) = \text{WND}(M, C).$$

Part (iii) can be stated by the same reasoning as part (i) and considering equality (36) and Lemma 5.2(iii) instead of equality (35) and Lemma 5.2(ii), respectively.

Part (iv) is an obvious result of part (iii)

(v) Consider $\bar{y} \in \text{argmin}_M \varphi_{\bar{y},q}^C \setminus \text{argsmin}_M \varphi_{\bar{y},q}^C$. By parts (i) and (iii) we see that $\bar{y} \in \text{ND}(M, \text{int}_{-q} C) \setminus \text{ND}(M, \text{vcl}_q C)$. Therefore, there exists $y \in M \setminus \{\bar{y}\}$ such that

$$\bar{y} - y \in \text{vcl}_{q(y, \bar{y})} C(y, \bar{y}) \setminus \text{int}_{-q(y, \bar{y})} C(y, \bar{y}) = \text{bd}_{q(y, \bar{y})} C(y, \bar{y}).$$

In addition, by (38) and assumption $\bar{y} \in \text{argmin}_M \varphi_{\bar{y},q}^C$ it follows that $\varphi_{\bar{y},q}^C(\bar{y}) \leq \varphi_{\bar{y},q}^C(y) \leq 0$ and so $\varphi_{\bar{y},q}^C(y) = 0$ since $\varphi_{\bar{y},q}^C(\bar{y}) = 0$.

Reciprocally, if $\varphi_{\bar{y},q}^C(y) = 0$, then by (36) and (38) we have $\bar{y} - y \in \text{vcl}_{q(y, \bar{y})} C(y, \bar{y})$. Moreover, $\bar{y} - y \notin \text{int}_{-q(y, \bar{y})} C(y, \bar{y})$ since $\bar{y} \in \text{ND}(M, \text{int}_{-q} C)$. Thus, $\bar{y} - y \in \text{bd}_{q(y, \bar{y})} C(y, \bar{y})$ and the proof is completed. $\square$

Remark 5.2 In [13, Theorem 5.11], the scalarization function ξ in (39) was considered to characterize $\leq^{C_1^D}$ -nondominated and $\leq^{C_1^D}$ -weakly nondominated points of a set M with respect to a mapping $\mathcal{D}$, where $\mathcal{D}(y)$ is assumed to be a nontrivial closed pointed convex cone, for all $y \in Y$. Specifically, the author considers the scalarization function $\chi_{z,k} : Y \rightarrow \mathbb{R}$, $\chi_{z,k}(y) = \xi(y, y - z)$, for all $y, z \in Y$. By part (i) above we have that $\chi_{z,k}(y) = \varphi_{z,q^k}^{C_1^D}(y)$. In addition, by the assumptions on the mapping $\mathcal{D}$ we have that $0 \in \text{bd}_k \mathcal{D}(y)$ for all $y \in Y$ whenever $k \in (\bigcap_{y \in Y} \mathcal{D}(y)) \setminus \{0\}$. Thus, parts (ii) and (iii) of Theorem 5.1 reduce to parts (b) and (a) of [13, Theorem 5.11], respectively.

Analogously, by applying parts (ii) and (iii) of Theorem 5.1 to $\varphi_{z,q^k}^{C_2^D}$ we obtain parts (b) and (a) of [13, Corollary 5.14], respectively.

The nonlinear scalarization results of the literature concerning the characterization of solutions of a vector optimization problem with a variable ordering structure $\mathcal{D}$ usually assume the nonemptiness of the set $(\bigcap_{y \in M} \mathcal{D}(y)) \setminus \{0\}$ (see, for instance, [4, Proposition 3.1] and [13, Theorem 5.11(a)]) or $\text{int}(\bigcap_{y \in M} \mathcal{D}(y))$ (see, for example, [8, Lemma 2.3] and [13, Theorem 5.11(b)]). In the results of this section we drop these assumptions as a result of allowing a variable direction mapping $q : Y \times Y \rightarrow Y$

in place of a constant direction q . Namely, instead of imposing the nonemptiness for $(\cap_{y \in M} \mathcal{D}(y)) \setminus \{0\}$ or for $\text{int}(\cap_{y \in M} \mathcal{D}(y))$, we only require $\mathcal{D}(y) \neq \{0\}$ or $\text{int} \mathcal{D}(y) \neq \emptyset$ at every $y \in M$, which are much easier to be fulfilled. This improvement would broaden the class of problems in application. Let us illustrate this with the following two examples.

Example 5.3 Let $Y := \mathbb{R}^2$, $M := \{(y_1, y_2) \in \mathbb{R}_+^2 : y_1 y_2 = 0\}$ and the two-variable domination mapping $C_1^{\mathcal{D}}$, where $\mathcal{D} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is defined by

$$\mathcal{D}(y) := \begin{cases} \mathbb{R}_+^2 & \text{if } y_1 > 0, y_2 > 0 \\ \text{cone}\{(0, 1)\} & \text{if } y_1 \leq 0, y_2 \geq 0 \\ \text{cone}\{(1, 0)\} & \text{if } y_1 \geq 0, y_2 \leq 0, (y_1, y_2) \neq (0, 0) \\ \text{cone}\{y\} & \text{if } y_1 < 0, y_2 < 0. \end{cases}$$

Now we define $q : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \setminus \{(0, 0)\}$ as follows:

$$q(y, z) := \begin{cases} (1, 1) & \text{if } y_1 > 0, y_2 > 0 \\ (0, 1) & \text{if } y_1 \leq 0, y_2 \geq 0 \\ (1, 0) & \text{if } y_1 \geq 0, y_2 \leq 0, (y_1, y_2) \neq (0, 0) \\ y & \text{if } y_1 < 0, y_2 < 0, \end{cases}$$

and the scalarization function $\varphi_{\bar{y}, q}^{C_1^{\mathcal{D}}}$, where $\bar{y} := (0, 0) \in M$. Then we have

$$\varphi_{\bar{y}, q}^{C_1^{\mathcal{D}}}(y) = \begin{cases} \max\{y_1, y_2\} & \text{if } y_1 > 0, y_2 > 0 \\ y_2 & \text{if } y_1 = 0, y_2 \geq 0 \\ y_1 & \text{if } y_1 > 0, y_2 = 0 \\ +\infty & \text{if } y_1 < 0, y_2 \geq 0 \text{ or } y_1 \geq 0, y_2 < 0 \\ 1 & \text{if } y_1 < 0, y_2 < 0. \end{cases}$$

It is easy to check that the following assertions are true: $C_1^{\mathcal{D}}(y, \bar{y}) = \mathcal{D}(y)$ is a closed pointed convex cone, $q(y, \bar{y}) \in \mathcal{D}(y) = 0^+ C_1^{\mathcal{D}}(y, \bar{y})$, $\text{bd}_{q(\bar{y}, \bar{y})} C_1^{\mathcal{D}}(\bar{y}, \bar{y}) = \{(0, 0)\}$, for all $y \in Y$, and $\varphi_{\bar{y}, q}^{C_1^{\mathcal{D}}}(y) > 0 = \varphi_{\bar{y}, q}^{C_1^{\mathcal{D}}}(\bar{y})$ for all $y \in M \setminus \{\bar{y}\}$. Thus, Theorem 5.1(iv) is applicable to this example and we deduce that $\bar{y}$ is a nondominated point of M w.r.t. the domination structure $\leq^{C_1^{\mathcal{D}}} = \leq_1^{\mathcal{D}}$. However, [4, Proposition 3.1] and [13, Theorem 5.11(a)] do not work because $(\cap_{y \in M} \mathcal{D}(y)) \setminus \{(0, 0)\}$ is empty.

Example 5.4 Consider $Y := \mathbb{R}^2$, $M := \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 0\}$, $\bar{y} := (0, 0) \in M$ and the mappings $\mathcal{D} : Y \rightrightarrows Y$ and $q : Y \times Y \rightrightarrows Y$ given for all $y = (y_1, y_2), z \in \mathbb{R}^2$ by

$$\mathcal{D}(y) = \begin{cases} \mathbb{R}_+^2 & \text{if } y_2 \geq 0 \\ -\mathbb{R}_+^2 & \text{if } y_2 < 0 \end{cases} \quad \text{and} \quad q(y, z) := \begin{cases} (1, 1) & \text{if } y_2 \geq 0 \\ (-1, -1) & \text{if } y_2 < 0. \end{cases}$$

It follows that

$$\varphi_{\bar{y},q}^{C^{\mathcal{D}}}(y) = \begin{cases} \max\{y_1, y_2\} & \text{if } y_2 \geq 0 \\ \max\{-y_1, -y_2\} & \text{if } y_2 < 0. \end{cases}$$

Obviously, $\mathcal{D}(y)$ is a pointed convex cone, $\text{int}\mathcal{D}(y) \neq \emptyset$, $q(y, z) \in \text{int}C_1^{\mathcal{D}}(y, z) = \text{cor}0^+C(y, z)$, for all $y, z \in Y$ and $(0, 0) \in \mathbb{R}_+^2 \setminus \text{int}\mathbb{R}_+^2 = \text{bd}_{q(\bar{y}, \bar{y})}C_1^{\mathcal{D}}(\bar{y}, \bar{y})$. Therefore, Theorem 5.1(ii) can be applied to this example and we obtain that $\bar{y}$ is a weakly nondominated point of M w.r.t. $\leq_{C_1^{\mathcal{D}}}^{\leq_{C_1^{\mathcal{D}}}}$ since $\varphi_{\bar{y},q}^{C^{\mathcal{D}}}(y) \geq 0 = \varphi_{\bar{y},q}^{C^{\mathcal{D}}}(\bar{y})$ for all $y \in M$. While, [8, Lemma 2.3] and [13, Theorem 5.11(b)] cannot be applied since $\text{int}(\cap_{y \in M} \mathcal{D}(y))$ is empty.

Next, an existence result for nondominated points of M with respect to the two-variable domination mapping $\text{int}_{-q}C$ is stated. For each $C : Y \times Y \rightrightarrows Y$, $q : Y \times Y \rightarrow Y$, $\bar{y} \in Y$ and $r \in \mathbb{R}$, $(\text{vcl}_q C + rq)_{\bar{y}}$ stands for the set-valued mapping

$$Y \ni y \rightrightarrows (\text{vcl}_q C + rq)_{\bar{y}}(y) := \text{vcl}_{q(y, \bar{y})}C(y, \bar{y}) + rq(y, \bar{y}). \quad (41)$$

Lemma 5.3 *Consider a point $\bar{y} \in Y$ and suppose that $q(y, \bar{y}) \in 0^+C(y, \bar{y})$ and the graph of $(\text{vcl}_q C + rq)_{\bar{y}}$ is closed, for all $y \in Y$ and $r \in \mathbb{R}$. Then $\varphi_{\bar{y},q}^C$ is lower semicontinuous.*

Proof We claim that the sublevel sets of $\varphi_{\bar{y},q}^C$ are closed sets, which proves the result. Indeed, take $r \in \mathbb{R}$ and a net $(y_i) \subset Y$ such that $\varphi_{\bar{y},q}^C(y_i) \leq r$, for all i , and $y_i \rightarrow y_0$. By Lemma 5.2(iii) we deduce that

$$y_i - \bar{y} \in (-\infty, r]q(y_i, \bar{y}) - \text{vcl}_{q(y_i, \bar{y})}C(y_i, \bar{y}), \quad \forall i.$$

Therefore, there exists $(t_i) \subset \mathbb{R}_+$ such that

$$\begin{aligned} y_i - \bar{y} &\in rq(y_i, \bar{y}) - (t_i q(y_i, \bar{y}) + \text{vcl}_{q(y_i, \bar{y})}C(y_i, \bar{y})) \\ &\subset rq(y_i, \bar{y}) - \text{vcl}_{q(y_i, \bar{y})}C(y_i, \bar{y}), \end{aligned}$$

since $q(y_i, \bar{y}) \in 0^+C(y_i, \bar{y})$, for all i . As $y_i \rightarrow y_0$ and the graph of the set-valued mapping (41) is closed we deduce that

$$y_0 - \bar{y} \in rq(y_0, \bar{y}) - \text{vcl}_{q(y_0, \bar{y})}C(y_0, \bar{y}).$$

Hence, by Lemma 5.2(iii) we have that $\varphi_{\bar{y},q}^C(y_0) \leq r$ and so the sublevel set of $\varphi_{\bar{y},q}^C$ at r is a closed set. This finishes the proof as r was arbitrarily chosen. $\square$

Theorem 5.2 *Let $M \subset Y$ be a nonempty compact set. Assume that $q(y_1, y_2) \in 0^+C(y_1, y_2)$, $0 \in \text{bd}_{q(y, y)}C(y, y)$ and the graph of $(\text{vcl}_q C + rq)_y$ is closed, for all $y_1, y_2, y \in Y$ and $r \in \mathbb{R}$. Suppose furthermore that the function $\varphi_{y,q}^C$ is proper, for all $y \in Y$. Then, $\text{argmin}_M \varphi_{y,q}^C \neq \emptyset$, for all $y \in Y$, and*

$$\text{ND}(M, \text{int}_{-q}C) = \bigcup_{y \in M} [\text{argmin}_M \varphi_{y,q}^C] \cap \{y\}. \quad (42)$$

Proof By Lemma 5.3 we deduce that function $\varphi_{y,q}^C$ is lower semicontinuous, for all $y \in M$. Hence, the Weierstrass theorem can be applied to deduce that $\operatorname{argmin}_M \varphi_{y,q}^C \neq \emptyset$, for all $y \in Y$. Finally, equality (42) is a direct consequence of Theorem 5.1(i) and the proof finishes. $\square$

We finish this section by applying Theorem 5.1 to characterize local solutions of problem (P). We denote the set of local solutions (resp. strict local solutions) of the scalar optimization problem defined by the objective function $g : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ and the feasible set $S \subset X$ by $\operatorname{arglmin}_S g$ (resp. $\operatorname{argslmin}_S g$), i.e.,

$$\begin{aligned}\bar{x} \in \operatorname{arglmin}_S g &: \iff \bar{x} \in \bigcup_{V \in \mathcal{N}(\bar{x})} \operatorname{argmin}_{S \cap V} g, \\ (\text{resp. } \bar{x} \in \operatorname{argslmin}_S g &: \iff \bar{x} \in \bigcup_{V \in \mathcal{N}(\bar{x})} \operatorname{argmin}_{S \cap V} g).\end{aligned}$$

In addition, concerning problem (P), the level set of f at $y \in Y$ is denoted $L(f, y)$, i.e.,

$$L(f, y) := \{x \in X : f(x) = y\}.$$

Theorem 5.3 Consider problem (P) and $\bar{x} \in S$.

- (i) Assume that $q(y, f(\bar{x})) \in 0^+C(y, f(\bar{x}))$, for all $y \in Y$, and also $0 \in \operatorname{bd}_{q(f(\bar{x}), f(\bar{x}))} C(f(\bar{x}), f(\bar{x}))$. Then,

$$\bar{x} \in \operatorname{LND}(f, S, \operatorname{int}_{-q} C) \iff \bar{x} \in \operatorname{arglmin}_S (\varphi_{f(\bar{x}), q}^C \circ f).$$

- (ii) Consider that $q(f(\bar{x}), f(\bar{x})) \in 0^+C(f(\bar{x}), f(\bar{x}))$, $q(y, f(\bar{x})) \in \operatorname{cor} 0^+C(y, f(\bar{x}))$, for all $y \in Y \setminus \{f(\bar{x})\}$ and $0 \in \operatorname{bd}_{q(f(\bar{x}), f(\bar{x}))} C(f(\bar{x}), f(\bar{x}))$. Then,

$$\bar{x} \in \operatorname{LWND}(f, S, C) \iff \bar{x} \in \operatorname{arglmin}_S (\varphi_{f(\bar{x}), q}^C \circ f).$$

- (iii) Suppose that $q(y, f(\bar{x})) \in 0^+C(y, f(\bar{x}))$, for all $y \in Y$, and also $0 \in \operatorname{bd}_{q(f(\bar{x}), f(\bar{x}))} C(f(\bar{x}), f(\bar{x}))$. Then,

$$\bar{x} \in \operatorname{LND}(f, S, \operatorname{vcl}_q C) \iff \bar{x} \in \operatorname{argslmin}_{(S \setminus L(f, f(\bar{x}))) \cup \{\bar{x}\}} (\varphi_{f(\bar{x}), q}^C \circ f).$$

- (iv) Assume that $q(y, f(\bar{x})) \in 0^+C(y, f(\bar{x}))$, for all $y \in Y$, $C(y, f(\bar{x}))$ is $q(y, f(\bar{x}))$ -directionally closed, for all $y \in Y \setminus \{f(\bar{x})\}$, and, in addition, $0 \in \operatorname{bd}_{q(f(\bar{x}), f(\bar{x}))} C(f(\bar{x}), f(\bar{x}))$. Then,

$$\bar{x} \in \operatorname{LND}(f, S, C) \iff \bar{x} \in \operatorname{argslmin}_{(S \setminus L(f, f(\bar{x}))) \cup \{\bar{x}\}} (\varphi_{f(\bar{x}), q}^C \circ f).$$

6 Conclusions

We have introduced and investigated two-variable domination structures, which generalize the well-known variable ordering structures due to Yu [32]. These domination structures allow us to define concepts of minimal and nondominated point of a set and local solution of a vector optimization problem, and to examine their properties. Results on the characterization of them have been obtained via a generalization of the Gerstewitz nonlinear scalarization function. These findings improve several ones of the literature concerning vector optimization problems with variable ordering structures. For future research, it would be of interest to characterize C -local nondominated and minimal solutions through duality assertions and multiplier rules.

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Data Availability The authors confirm that the data supporting the findings of this study are available within the article.

Declarations

Competing Interests The authors declare that they have no conflict of interest.

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