Computers & Industrial Engineering 210 (2025) 111500

Contents lists available at ScienceDirect

Computers & Industrial Engineering

journal homepage: www.elsevier.com/locate/caie

Optimal sustainable inventory policy for items with price-and-time-dependent demand rate

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ARTICLE INFO

Keywords:
Sustainable EOQ model
Carbon emissions
Shortages
Power demand pattern
Algebraic-price demand
Optimal price

ABSTRACT

This paper studies a sustainable inventory model for items whose demand rate is the product of a time-dependent function and a price-dependent function. The inventory system allows shortages during the product management period. Carbon emissions from transportation and storage are included in the model. The consideration of a demand rate that combines the effects of a price-algebraic function and a time-power function, with full backlogging and environmental constraints, is a novel and more realistic hypothesis and it should be studied. To determine the optimal inventory policy for this system can help to improve the efficiency and sustainability practices in inventory control. The objective is to determine a sustainable inventory policy that maximizes the average profit per unit time. We include the following significant components in the objective function: the average revenue, the ordering cost, the purchasing cost, the shipping cost, the holding cost, the shortage cost, and the carbon emissions costs in transportation and storage. To find the solution to this sustainable inventory problem, four scenarios are analyzed and, for each scenario, the optimal inventory policy is obtained. This policy determines the lot size, the optimal selling price, the maximum shortage, and the maximum profit per unit time. Some numerical examples are presented to illustrate the proposed methodology for determining the optimal policy of this sustainable inventory problem. We examine the effects on the best inventory policy when some parameters of the system are changed. Useful managerial insights derived from these results are proposed.

1. Introduction

In a global scenario increasingly conditioned by climate urgency, regulatory pressure, and consumer environmental awareness, it has become essential to reconsider classic business management models from a sustainability-oriented perspective. In particular, the field of inventory management, traditionally focused on economic efficiency, requires a reformulation that explicitly incorporates the real environmental costs derived from logistical operations, including carbon emissions associated with transportation, storage, and product replenishment processes.

As is well known, environmental regulations imposed by governments have forced many companies to take measures to reduce their carbon emissions. These measures affect the entire supply chain and, therefore, also the management of product inventories. Thus, numerous researchers in inventory management have devoted themselves to examining the effect that carbon emissions have on the replenishment policies of products.

One of the first papers that explicitly consider carbon emissions costs in the formulation of the inventory model is Hua et al. (2011). They compared the best inventory policy under the cap-and-trade mechanism with the classical Economic Order Quantity (EOQ) model and investigated the impacts of carbon cap and carbon price on that optimal policy. Chen et al. (2013) used a model analogous to that of Hua et al. (2011) and gave a condition under which carbon emissions can be reduced by modifying lot sizes. Toptal et al. (2014) studied a retailer's joint inventory problem and carbon emission reduction investment under three carbon emission regulation policies. Konur and Schaefer (2014) developed a retailer's integrated inventory system and the transportation decisions of a retailer under four different carbon emissions regulation policies, assuming two common practices

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https://doi.org/10.1016/j.cie.2025.111500
Received 25 February 2025; Received in revised form 11 June 2025; Accepted 23 August 2025
Available online 29 August 2025

of trucking. Hovelaque and Bironneau (2015) studied an EOQ inventory model, taking the carbon emissions into account under a price-and-environmental dependent demand. Yenipazarli (2016) analyzed the effect of emissions taxes on optimal pricing and production policy and studied the economic, environmental and social impacts of remanufacturing. The coordination problem of a two-echelon supply chain system is developed by Xu et al. (2016), where decisions depend on both the sustainability investment and the selling price. Tiwari et al. (2018) studied an integrated single vendor–buyer supply chain for deteriorating items with imperfect quality, assuming carbon emissions due to transporting, warehousing, and keeping deteriorating products. Halat and Hafezalkotob (2019) developed a multi-stage green supply chain under four different carbon emission regulations. A sustainable carbon tax and cap-based economic production quantity model is considered in Mishra et al. (2020), with a controllable carbon emission rate under three different shortage situations. Malleeswaran and Uthayakumar (2020) analyzed an integrated supply chain model with price dependent demand and carbon emission costs, where demand during the lead time is normally distributed and shortages are completely backlogged. A two-plant production system with a warranty period and carbon emission effects during the production process is considered by Manna et al. (2021). Later, Taleizadeh et al. (2022) analyzed a sustainable EOQ model with partial delay in payments and backordering under environmental issues. A two-echelon supply chain is developed in Asadkhani et al. (2022), where the buyer acquires a random fraction of repairable items and the carbon footprint is incorporated into the total cost, using carbon emissions. Ebrahimi et al. (2022) studied a sustainable two-echelon supply chain with stochastic demand under a double-level sustainability effort. Mahato and Mahata (2023) analyzed an EOQ inventory model with carbon emission costs, assuming limited warehouse storage space, all-units discount and backlogging under order-size-dependent trade credit. Khan et al. (2023a) presented a production–inventory system for a manufacturer in a circular economy, where both demand and gross profit per unit depend on the circularity level and carbon emissions from the manufacturer's operations are considered. Khan et al. (2023b) developed an inventory system with prepayment, and time-and-price dependent demand under carbon tax regulations for a growing item. Jain et al. (2023) presented a three-echelon supply chain inventory model that considers carbon emissions due to the activities of manufacturing, transportation and storage. Lok et al. (2023) studied an EOQ model for deteriorating items, including investment in preservation technology under carbon emissions. More recently, Khan et al. (2024) developed a sustainable inventory model for an industrial livestock farm that operates with a single kind of growing item. San-José et al. (2024) studied a sustainable inventory model for non-instantaneous deteriorating items with power demand pattern and backlogged shortages, considering a carbon emissions tax. Sebatjane et al. (2024) studied various inventory models for a three-echelon food supply chain comprising growing items where the demand rate for the items depends on both the selling price and the carbon emissions.

Despite the growing academic interest in sustainable inventory systems, models that rigorously incorporate environmental costs into the objective function remain relatively scarce and often partial. This limitation is accentuated when formulations are required to simultaneously consider realistic demand patterns that are sensitive to both time and price. The present study addresses this research gap by proposing a novel and sustainable inventory system for an article whose demand is the product of a power-time function and an algebraic price-function. This demand function is very versatile and allows the demand of the articles to be adjusted for a wide variety of situations. Note that, in particular, the algebraic-price function is an extension of the well-known isoelastic-price function. Besides, we suppose that shortages are allowed and fully backlogged. Thus, all customer demands are satisfied, but some customer requests may be met with a delay. We also consider environmental constraints for sustainable inventory management. Hence, various taxes to carbon emissions associated with logistics operations are included in the model.

The main innovation of the study lies in integrating a price-dependent demand function with a power pattern adapted to the temporal distribution of demand, and a comprehensive environmental cost structure that simultaneously considers emissions from transport and storage. This formulation generalizes previous models and provides a more accurate and versatile tool for strategic decision-making in environments where consumer behavior is dynamic, price acts as a determining factor in demand configuration, and environmental regulations are increasingly stringent.

The most significant contribution of this work is to help inventory system managers reduce the environmental impact by developing more sustainable inventory models. To the best of our knowledge, this is the first time that the optimal joint pricing, lot size and maximum shortage quantity policy that maximizes average profit per unit time for an inventory system is determined, while also considering a demand rate that is the product of a power-time function and a rational-price function, with shortages completely backordered and carbon tax regulations. We thoroughly study the novel sustainable inventory problem and provide an algorithmic procedure to obtain the optimal inventory policy for all possible inventory system scenarios.

To highlight the differences of this study with respect to the previous papers, Table 1 presents a list of articles that have been cited in this introduction, classified by time demand pattern type, price demand pattern type, shortage type, and if there exists a cost for carbon emissions from stocking. They are shown in chronological order.

The remainder of this article is organized as follows. Section 2 provides the assumptions and notations used to develop the inventory system. The problem is formulated and the mathematical model to determine the objective function and the constraints of the problem is introduced in Section 3. A solution procedure to obtain the optimal sustainable policy is presented in Section 4. Section 5 provides several numerical examples to illustrate the solution procedure previously developed. Section 6 investigates the variation of the optimal inventory policy, when some parameters of the sustainable system are modified and gives some useful managerial insights derived from those results. Finally, some suggestions and conclusions are given in Section 7.

2. Notation and assumptions

Table 2 provides the notation used to establish the proposed inventory model.

The lot size model studied in this paper is developed under the following hypotheses. The inventory is continuously reviewed and replenishment is instantaneous. The item is a single product and the planning horizon is infinite. The lead time is zero or negligible and shortages are allowed and these are fully backordered. There is a procurement of q units when the number of backorders attains the amount b. The ordering cost K is fixed regardless of the lot size. The purchasing cost p is a known constant and the selling price s is a decision variable. The other decision variables are the lot size q and the maximum shortage b. The cost of shipping is an affine function of lot size (that is, $g_0 + g_1 q$). The carbon emissions due to transportation are also an affine function of lot size (that is, $d_0 + d_1 q$). The carbon emissions in the warehouse depend on the average inventory. There are taxes that apply to carbon emissions, so there exists a tax r_1 applied to carbon emissions in transportation and another tax r_2 applied to carbon emissions in inventory storage. The holding cost per unit is a linear function of time in storage and the backordering cost ω per unit and time is known and constant. The demand rate $\lambda(s,t)$ is a bivariate function of price and time. Thus, we assume that $\lambda(s,t)$ multiplies the effects of a decreasing rational price-dependent function $\lambda_1(s)$ and a power time-dependent function $\lambda_2(t)$, that is, we consider that $\lambda(s,t) = \lambda_1(s)\lambda_2(t)$, where $\lambda_1(s)$ is the algebraic price-dependent function defined by

$$\lambda_1(s) = \left(a_0 + a_1 s\right)^{-\gamma}, \text{ with } a_0 \geq 0,\ a_1 > 0 \text{ and } \gamma \geq 1$$

Table 1
Summary of literature on inventory models under carbon emissions.

Authors	Time-dependent demand	Price-dependent demand	Backlogging	Carbon emissions in storage
Hua et al. (2011)	No	No	No	Yes
Chen et al. (2013)	No	No	No	Yes
Toptal et al. (2014)	No	No	No	Yes
Konur and Schaefer (2014)	No	No	No	Yes
Hovelaque and Bironneau (2015)	No	Linear	No	Yes
Yenipazarli (2016)	No	Linear	No	No
Xu et al. (2016)	No	Linear	No	No
Tiwari et al. (2018)	No	No	No	Yes
Halat and Hafezalkotob (2019)	No	No	No	Yes
Mishra et al. (2020)	No	No	Partial	Yes
Malleeswaran and Uthayakumar (2020)	No	Power	Full	No
Manna et al. (2021)	Linear warranty	No	No	No
Taleizadeh et al. (2022)	No	Linear	Partial	Yes
Asadkhani et al. (2022)	No	No	No	Yes
Ebrahimi et al. (2022)	No	No	Partial	No
Mahato and Mahata (2023)	No	No	Partial	Yes
Khan et al. (2023a)	No	No	No	Yes
Khan et al. (2023b)	Power	Power	No	Yes
Jain et al. (2023)	No	No	No	Yes
Lok et al. (2023)	No	No	No	Yes
Khan et al. (2024)	Power	Power	Full	Yes
San-José et al. (2024)	Power	No	Full	Yes
Sebatjane et al. (2024)	No	Linear	No	Yes
This paper	Power	Algebraic	Full	Yes

Table 2
List of notations.

Variables	
q	Lot size per cycle (*decision variable*)
T	Length of the inventory cycle
b	Maximum shortage quantity per cycle (*decision variable*)
τ_1	Time period where the net stock is positive
τ_2	Time period where the net stock is negative
M	Maximum level of the stock
s	Unit selling price (*decision variable*)
Parameters	
g_0	Fixed shipment cost
g_1	Shipment cost per transported unit
d_0	Fixed carbon emissions in transporting
d_1	Variable carbon emissions in transporting
r_1	Tax charged on carbon emissions in transporting ($/per carbon kilogram emission)
e_0	Fixed carbon emissions in holding
e_1	Carbon emissions per unit held in stock
r_2	Tax charged on carbon emissions in storage ($/per carbon kilogram emission)
p	Unit purchasing cost
h	Unit holding cost per unit time
ω	Unit backordering cost per unit time
K	Ordering cost
a_0	Non-centrality parameter of the price-dependent demand rate
a_1	Sensitivity coefficient for the price-dependent demand
γ	Exponent of the price-dependent demand
δ	Index of demand pattern
π_0	Auxiliary parameter given by $\pi_0 = K + g_0 + r_1 d_0 + r_2 e_0$
π_1	Auxiliary parameter given by $\pi_1 = p + g_1 + r_1 d_1$
π_2	Auxiliary parameter given by $\pi_2 = \left(h + \omega + r_2 e_1\right) / (\delta + 1)$
Functions	
$\lambda(s,t)$	Demand rate at time t when the selling price is s
$I(s,t)$	Inventory level at time t when the selling price is s
$P(s,q,b)$	Average profit per unit time

and $\lambda_2(t)$ is the power time-dependent function given by

$$\lambda_2(t) = \frac{1}{\delta}\left(\frac{t}{T}\right)^{(1-\delta)/\delta}, \text{ with } \delta > 0$$

In the function $\lambda_1(s)$, the coefficient a_0 can be interpreted as the non-centrality parameter of the demand rate regarding the selling price (see Pando et al., 2021), and the parameters a_1 and γ are coefficients that represent the sensitivity of demand with respect to the selling price. This algebraic price-dependent demand function was also used by other researchers, such as Huang et al. (2013), Jeuland and Shugan (1988), and Zhu and Cetinkaya (2014). Note that if we set $a_0 = 0$, we obtain the well-known isoelastic price-dependent function (see, e.g., Rubio-Herrero and Baykal-Gürsoy, 2020; Yang and Liu, 2023; Terzi et al., 2024; and Pando et al., 2024).

The function $\lambda_2(t)$ describes the way in which units are taken from stock to cover customer demand, based on the time at which they are requested. Some explanations about the practical utility of the function $\lambda_2(t)$ to describe the demand for certain products can be found in San-José et al. (2021, 2024, 2017, 2020). Thus, the price-and-time-dependent function considered in this work can be useful to describe the real demand for some articles, since it can better fit the empirical data.

The price elasticity of demand is $\varepsilon(s) = -a_1\gamma s/\left(a_0 + a_1 s\right)$ and the price super-elasticity, defined as the elasticity of the function $\varepsilon(s)$ (see, e.g., Kimball, 1995; and Mrázová and Neary, 2017), is given by $\sigma(s) = a_0/\left(a_0 + a_1 s\right)$. Thus, if the parameter a_0 is positive, the price elasticity, which depends on the unit selling price, is strictly decreasing and convex, as is the price super-elasticity. Note that the isoelastic price-function has an elasticity equal to $-\gamma$ and its super-elasticity is 0.

To illustrate the characteristic of the customer's demand as a function of the parameters γ and δ, we have depicted the function $\lambda(s,t)$ for different possible values of these indexes in Figs. 1, 2 and 3.

3. Model formulation

The behavior of the net inventory level $I(s,t)$ is described as follows. At the beginning of the inventory period there are M units stored and that amount decreases during the time period $(0,\tau_1)$ and falls to zero at $t = \tau_1$. Therefore,

$$M = \int_0^{\tau_1} \lambda(s,t)dt$$

Next, during the time period (τ_1, T), shortages occur and demand is completely backordered.

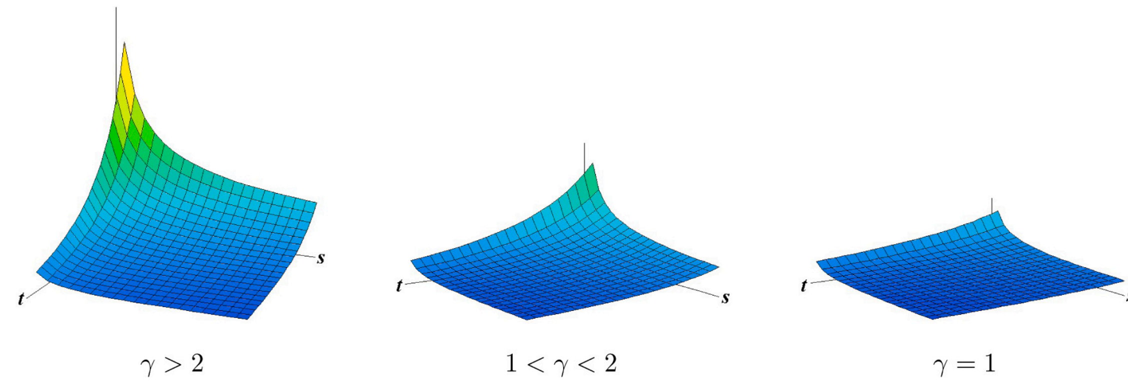

Fig. 1. Demand rate functions $\lambda(s,t)$ when $\delta > 1$.

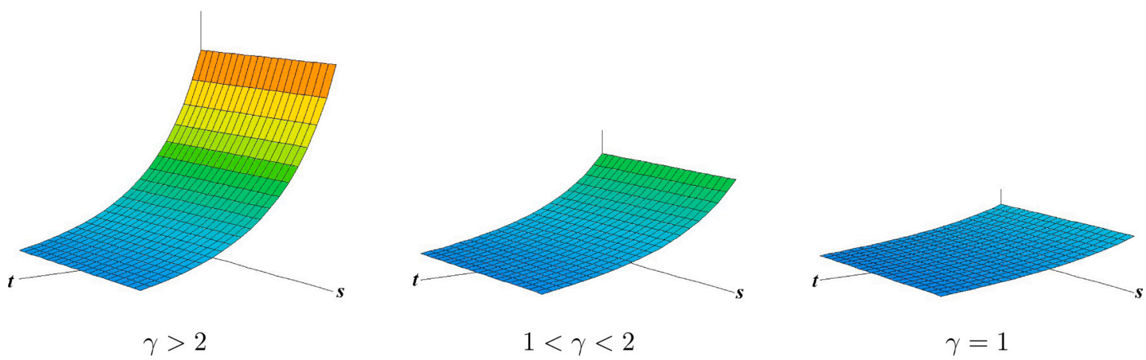

Fig. 2. Demand rate functions $\lambda(s,t)$ when $\delta = 1$.

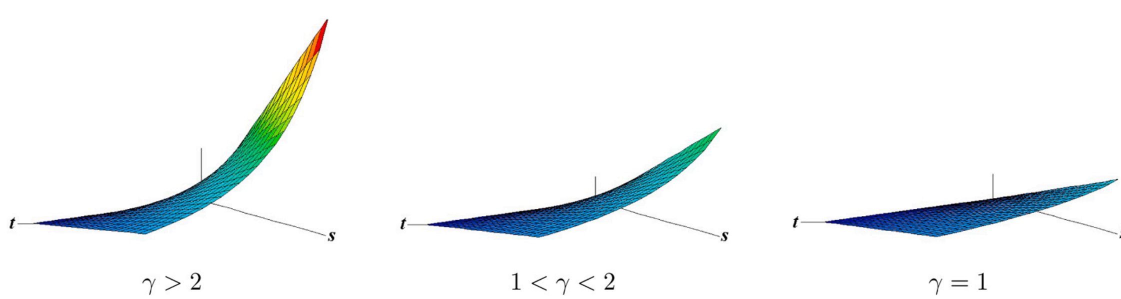

Fig. 3. Demand rate functions $\lambda(s,t)$ when $\delta < 1$.

The net inventory level $I(s,t)$ at time t, with $t \in [0,T)$, is given by

$$I(s,t) = M - \int_0^t \lambda(s,u)du = \int_t^{\tau_1} \lambda(s,u)du \tag{1}$$

Since demand during the stock-out period is completely backlogged, the lot size q must match demand during the inventory cycle. Therefore,

$$q = \int_0^T \lambda(s,t)dt = \left(a_0 + a_1 s\right)^{-\gamma} T \tag{2}$$

The maximum shortage quantity coincides with the demand during the stock-out period, that is,

$$b = \int_{\tau_1}^T \lambda(s,t)dt = \left(a_0 + a_1 s\right)^{-\gamma} \left(1 - \left(\frac{\tau_1}{T}\right)^{1/\delta}\right) T = \left(1 - \left(\frac{\tau_1}{T}\right)^{1/\delta}\right) q \tag{3}$$

Substituting $\lambda_1(s)$ and $\lambda_2(t)$ into Eq. (1) , we have that the net inventory level, for $0 \le t < T$ and $p \le s$, is given by

$$I(s,t) = \left(a_0 + a_1 s\right)^{-\gamma} \left(\left(\frac{\tau_1}{T}\right)^{1/\delta} - \left(\frac{t}{T}\right)^{1/\delta}\right) T$$

$$= \left(1 - \left(\frac{t}{q\left(a_0 + a_1 s\right)^{\gamma}}\right)^{1/\delta}\right) q - b \tag{4}$$

From (2), the length of the inventory cycle T is given by

$$T = q\left(a_0 + a_1 s\right)^{\gamma} \tag{5}$$

and, from (3), the length of the stock-in period is given by

$$\tau_1 = q\left(a_0 + a_1 s\right)^{\gamma} \left(1 - \frac{b}{q}\right)^{\delta} \tag{6}$$

Therefore, the length of the stock-out period is given by

$$\tau_2 = T - \tau_1 = q\left(a_0 + a_1 s\right)^{\gamma} \left(1 - \left(1 - \frac{b}{q}\right)^{\delta}\right) \tag{7}$$

The objective is to maximize the average profit per unit time $P(s,q,b) = B(s,q,b)/T$, where $B(s,q,b)$ is the profit during the inventory cycle T. This profit includes the following significant components: the average revenue, the ordering cost, the purchasing cost, the shipping cost, the holding cost, the average emissions cost and the backordering cost. It is clear that, at each cycle, the revenue is sq, the ordering cost is K, the purchasing cost is pq, the shipping cost is $g_0 + g_1 q$ and the holding cost is

$$\int_0^{\tau_1} hI(s,t)dt = \frac{h}{\delta+1}\left(a_0+a_1 s\right)^{-\gamma}\tau_1 T\left(\frac{\tau_1}{T}\right)^{1/\delta} = \frac{h}{\delta+1}\left(a_0+a_1 s\right)^{\gamma} q^2\left(1-\frac{b}{q}\right)^{\delta+1} \tag{8}$$

Since carbon emissions are due to transportation and storage, the cost of carbon emissions per cycle is the sum of the costs incurred by the carbon emissions in each of these two tasks. That is,

$$r_1\left(d_0+d_1 q\right)+r_2\left(e_0+e_1\int_0^{\tau_1} I(s,t)dt\right) = r_1\left(d_0+d_1 q\right)+r_2\left(e_0+\frac{e_1}{\delta+1}\left(a_0+a_1 s\right)^{\gamma} q^{1-\delta}(q-b)^{\delta+1}\right) \tag{9}$$

Finally, the backorder cost per cycle is

$$\int_{\tau_1}^{T}\omega(-I(s,t))\,dt = \omega\left(a_0+a_1 s\right)^{-\gamma} \times T\left(\frac{\delta}{\delta+1}T-\left(T-\frac{\tau_1}{\delta+1}\right)\left(\frac{\tau_1}{T}\right)^{1/\delta}\right) = \omega q\left(a_0+a_1 s\right)^{\gamma} \times\left(b-\frac{q}{\delta+1}\left(1-\left(1-\frac{b}{q}\right)^{\delta+1}\right)\right) \tag{10}$$

Thus, the profit during the cycle $[0,T)$ is

$$B(s,q,b) = \left(s-\pi_1\right)q-\pi_0-\pi_2 q^2\left(a_0+a_1 s\right)^{\gamma}\left(1-\frac{b}{q}\right)^{\delta+1} - \omega q\left(a_0+a_1 s\right)^{\gamma}\left(b-\frac{q}{\delta+1}\right)$$

where

$$\pi_0 = K+g_0+r_1 d_0+r_2 e_0,\ \pi_1 = p+g_1+r_1 d_1 \text{ and } \pi_2 = \left(h+\omega+r_2 e_1\right)/(\delta+1) \tag{11}$$

Consequently, the average profit per unit time is given by

$$P(s,q,b) = \frac{B(s,q,b)}{T} = \left(s-\pi_1-\frac{\pi_0}{q}\right)\left(a_0+a_1 s\right)^{-\gamma} - \pi_2 q\left(1-\frac{b}{q}\right)^{\delta+1}-\omega\left(b-\frac{q}{\delta+1}\right) \tag{12}$$

The aim is to obtain the values of the variables s, q and b that maximize the function $P(s,q,b)$ given by (12), subject to the constraints $q>0$, $0\le b\le q$ and $p\le s$.

4. Analysis and solution

Since, for a fixed value of s, the function $P_s(q,b) = P(s,q,b)$ is strictly concave (see Lemma 1 in the Appendix), it is easy to deduce that it reaches its maximum value at the point $(q^*(s), b^*(s))$, which is obtained by solving the system of nonlinear equations $\frac{\partial P(s,q,b)}{\partial q} = 0$ and $\frac{\partial P(s,q,b)}{\partial b} = 0$. Thus, we have

$$q^*(s) = \sqrt{\frac{(1+\delta)\pi_0}{\delta\omega\xi_0\left(a_0+a_1 s\right)^{\gamma}}} \tag{13}$$

and

$$b^*(s) = \xi_0 q^*(s) = \sqrt{\frac{(1+\delta)\pi_0\xi_0}{\delta\omega\left(a_0+a_1 s\right)^{\gamma}}} \tag{14}$$

where $\xi_0 = 1-\left(\frac{\omega}{(1+\delta)\pi_2}\right)^{1/\delta} = 1-\left(\frac{\omega}{h+\omega+r_2 e_1}\right)^{1/\delta}$.

Therefore the initial optimization problem can be reduced to the problem of maximizing $G(s) = P_s(q^*(s), b^*(s))$ with the condition $s\ge p$.

4.1. Determining the optimal unit selling price

From (12), (13) and (14), after a few algebraic operations, we obtain

$$G(s) = \frac{s-\pi_1}{\left(a_0+a_1 s\right)^{\gamma}} - 2\sqrt{\frac{\delta\xi_0\pi_0\omega}{(1+\delta)\left(a_0+a_1 s\right)^{\gamma}}} \tag{15}$$

It is clear that: (i) $G(p)<0$; (ii) $\lim_{s\to\infty} G(s) = 1/a_1$ if $\gamma=1$ and $\lim_{s\to\infty} G(s) = 0$ if $\gamma>1$; and (iii) the function $G(s)$ is differentiable, and its first derivative is given by

$$G'(s) = \frac{\beta(s)}{\left(a_0+a_1 s\right)^{\gamma+1}} \tag{16}$$

where

$$\beta(s) = a_0 - a_1(\gamma-1)s + a_1\gamma\left(\pi_1+\sqrt{\frac{\delta\xi_0\pi_0\omega\left(a_0+a_1 s\right)^{\gamma}}{1+\delta}}\right) \tag{17}$$

Thus, we have that sign $\left[G'(s)\right]$ = sign $[\beta(s)]$. For this reason, we now study some properties of the function $\beta(s)$ that will help us determine the optimal selling price.

First of all, it is clear that $\beta(p)>0$, because $\beta(p) = a_0+a_1 p + a_1\gamma\left(g_1+r_1 d_1+\sqrt{\frac{\delta\xi_0\pi_0\omega(a_0+a_1 p)^{\gamma}}{\delta+1}}\right)$. Next, we show the behavior of the function $\beta(s)$ regarding the value of the parameter γ.

Proposition 1. *Let $\beta(s)$ be given by (17). Then:*

1. *If $\gamma=1$, then $\beta(s)$ is a strictly increasing and concave function with $\lim_{s\to\infty}\beta(s)=\infty$.*
2. *If $1<\gamma<2$, then $\beta(s)$ is a strictly concave function and $\lim_{s\to\infty}\beta(s)=-\infty$.*
3. *If $\gamma=2$, then $\beta(s)$ is an affine function. Moreover,*

$$\lim_{s\to\infty}\beta(s) = \begin{cases} -\infty & if\ \sqrt{\frac{1+\delta}{\delta\xi_0\pi_0\omega}} > 2a_1 \\ 2\left(a_0+a_1\pi_1\right) & if\ \sqrt{\frac{1+\delta}{\delta\xi_0\pi_0\omega}} = 2a_1 \\ \infty & if\ \sqrt{\frac{1+\delta}{\delta\xi_0\pi_0\omega}} < 2a_1 \end{cases}$$

4. *If $\gamma>2$, then $\beta(s)$ is a strictly convex function that attains its minimum value at the point s_0 given by*

$$s_0 = \frac{1}{a_1}\left(\left(\frac{4(1+\delta)(\gamma-1)^2}{a_1^2\gamma^4\delta\xi_0\pi_0\omega}\right)^{1/(\gamma-2)} - a_0\right) \tag{18}$$

Moreover, $\lim_{s\to\infty}\beta(s)=\infty$.

Proof. Please, see Appendix. □

Taking into account the above result, we now study separately the scenarios (i) $\gamma=1$, (ii) $\gamma\in(1,2)$, (iii) $\gamma=2$ and (iv) $\gamma>2$.

4.1.1. Scenario $\gamma = 1$

The following theorem ensures the existence, in this scenario, of a unit selling price from which the inventory system is profitable.

Theorem 1. *Let $G(s)$ be given by (15). If $\gamma=1$, then the function $G(s)$ does not reach its maximum at any point $s\ge p$, the supremum of $G(s)$ is $1/a_1$ and it is obtained when s tends to infinity. Moreover, the inventory system is profitable for $s>s_1$, where $s_1 = \pi_1+2a_1\xi_1+2\sqrt{\xi_1\left(a_0+a_1\left(\pi_1+a_1\xi_1\right)\right)}$, with $\xi_1 = \delta\xi_0\pi_0\omega/(1+\delta)$.*

Proof. Please, see Appendix. □

Note that, for each value v, with $v\in\left(0,1/a_1\right)$, there is a unique unit selling price s_v for which the inventory system has an average profit per unit time equal to v.

4.1.2. Scenario $\gamma \in (1,2)$

Next, we provide a criterion to determine the optimal selling price s^* in this scenario.

Theorem 2. *Let $G(s)$ and $\beta(s)$ be given, respectively, by (15) and (17). If $\gamma \in (1,2)$, then the function $G(s)$ reaches its maximum at the point $s^* = \arg_{s\in(p,\infty)}\{\beta(s)=0\}$, with the value*

$$G(s^*) = \left(a_0 + a_1 s^*\right)^{-\gamma/2}\left(\frac{\left(a_0+a_1 s^*\right)^{1-\gamma/2}}{a_1\gamma} - \sqrt{\frac{\delta\xi_0\pi_0\omega}{1+\delta}}\right) \tag{19}$$

Proof. Please, see Appendix. □

Note that, in this scenario, there is always an optimal selling price and, therefore, the sustainable inventory system is always profitable.

4.1.3. Scenario $\gamma = 2$

Now, let us determine the optimal selling price s^* in this other scenario, when $\gamma = 2$.

Theorem 3. *Let $G(s)$ be given by (15). If $\gamma = 2$, then the optimum selling price s^* can be determined in the following way:*

1. *If $\sqrt{\frac{1+\delta}{\delta\xi_0\pi_0\omega}} > 2a_1$, then the function $G(s)$ reaches its maximum at the point*

$$s^* = s_2 = \frac{1}{a_1}\left(\frac{2\left(a_0+a_1\pi_1\right)}{1-2a_1\sqrt{\frac{\delta\xi_0\pi_0\omega}{1+\delta}}} - a_0\right)$$

with the value

$$G(s_2) = \frac{1}{a_0+a_1\pi_1}\left[\frac{1}{4a_1} + \sqrt{\frac{\delta\xi_0\pi_0\omega}{1+\delta}}\left(a_1\sqrt{\frac{\delta\xi_0\pi_0\omega}{1+\delta}} - 1\right)\right]$$

2. *If $\sqrt{\frac{1+\delta}{\delta\xi_0\pi_0\omega}} \le 2a_1$, then the function $G(s)$ does not reach its maximum at any point $s \ge p$. The supremum of $G(s)$ is 0 and is obtained when s tends to infinity.*

Proof. Please, see Appendix. □

4.1.4. Scenario $\gamma > 2$

In the following theorem, the optimal selling price s^* is obtained when it is assumed that $\gamma > 2$.

Theorem 4. *Let $G(s)$, $\beta(s)$ and s_0 be given, respectively, by (15), (17) and (18). The optimum selling price s^* can be determined in the following way:*

A. *If $s_0 \le p$, then the function $G(s)$ does not reach its maximum at any point $s \ge p$. The supremum of $G(s)$ is 0 and is obtained when s tends to infinity.*

B. *If $s_0 > p$ and $\beta\left(s_0\right) \ge 0$, then the function $G(s)$ does not reach its maximum at any point $s \ge p$. The supremum of $G(s)$ is 0 and is obtained when s tends to infinity.*

C. *If $s_0 > p$ and $\beta\left(s_0\right) < 0$, then let $\widetilde{s} = \arg_{s\in(p,s_0)}\{\beta(s)=0\}$.*

   (i) *If $G(\widetilde{s}) < 0$, then the function $G(s)$ does not reach its maximum at any point $s \ge p$. The supremum of $G(s)$ is 0 and is obtained when s tends to infinity.*

   (ii) *If $G(\widetilde{s}) \ge 0$, then the function $G(s)$ reaches its maximum at the point $s^* = \widetilde{s}$, with maximum profit per unit time given by (19).*

Proof. Please, see Appendix. □

4.2. Sustainable inventory without shortages

It is clear that the optimal inventory policy for the system in which shortages are not allowed can be obtained through the previous results, taking the limit as ω tends to ∞.

Thus, from (13), we have that the optimal lot size for a fixed selling price s is now

$$q_w^*(s) = \lim_{\omega\to\infty} q^*(s) = \sqrt{\frac{(1+\delta)\,\pi_0}{\left(h+r_2 e_1\right)\left(a_0+a_1 s\right)^{\gamma}}}$$

and, proceeding in a similar way to the full backlogging case, the optimal unit selling price is the value s_w^* that maximizes the function $G_w(s)$, where

$$G_w(s) = \lim_{\omega\to\infty} G(s) = \frac{s-\pi_1}{\left(a_0+a_1 s\right)^{\gamma}} - 2\sqrt{\frac{\pi_0\left(h+r_2 e_1\right)}{(1+\delta)\left(a_0+a_1 s\right)^{\gamma}}} \tag{20}$$

We can now give a criterion for determining the optimal selling price s^*, which is the analogue of Theorems 1 to 4.

Theorem 5. *Let $\beta_w(s) = a_0 - a_1(\gamma-1)s + a_1\gamma\left(\pi_1 + \sqrt{\frac{\pi_0\left(h+r_2e_1\right)\left(a_0+a_1 s\right)^{\gamma}}{1+\delta}}\right)$ and $G_w(s)$ be given by (20).*

1. *If $\gamma = 1$, then the function $G_w(s)$ does not reach its maximum at any point $s \ge p$ and $\sup_{s\ge p} G_w(s) = 1/a_1$, which is obtained when $s \to \infty$.*
2. *If $\gamma \in (1,2)$, then the optimal selling price is $s_w^* = \arg_{s\in(p,\infty)}\left\{\beta_w(s)=0\right\}$ and the maximum average profit per unit time is $G_w\left(s_w^*\right)$.*
3. *If $\gamma = 2$, then:*

   (a) *When $2a_1 < \sqrt{\frac{1+\delta}{\pi_0\left(h+r_2e_1\right)}}$, the optimal selling price is $s_w^* = \frac{1}{a_1}\left(\frac{2\left(a_0+a_1\pi_1\right)}{1-2a_1\sqrt{\pi_0\left(h+r_2e_1\right)(1+\delta)^{-1}}} - a_0\right)$.*

   (b) *Otherwise, $s_w^* = \infty$ and $G_w(s_w^*) = 0$ (that is, the function $G_w(s)$ does not reach its maximum at any point $s \ge p$ and $\sup_{s\ge p} G_w(s) = 0$, which is obtained when $s \to \infty$).*

4. *If $\gamma > 2$, let $s_w = \frac{1}{a_1}\left(\left(\frac{4(1+\delta)(\gamma-1)^2}{a_1^2\gamma^4\pi_0\left(h+r_2e_1\right)}\right)^{1/(\gamma-2)} - a_0\right)$.*

   (a) *In the cases: (i) $s_w \le p$ and (ii) $s_w > p$ and $\beta_w\left(s_w\right) \ge 0$, we have $s_w^* = \infty$ and $G_w(s_w^*) = 0$.*

   (b) *If $s_w > p$ and $\beta_w\left(s_w\right) < 0$, let $\widetilde{s}_w = \arg_{s\in(p,s_w)}\left\{\beta_w(s)=0\right\}$.*

      i. *The optimal selling price is $s_w^* = \widetilde{s}_w$, when $G_w(\widetilde{s}_w) \ge 0$.*

      ii. *However, $s_w^* = \infty$ and $G_w(s_w^*) = 0$, when $G_w(\widetilde{s}_w) < 0$.*

Proof. Please, see Appendix. □

5. Numerical examples

In this section, we illustrate with some numerical examples the solution procedure developed in Section 4.

Example 1. Consider an inventory system that has the characteristics described in Section 2 and assume the following input parameters: ordering cost $K = \$12$, unit purchasing cost $p = \$7.5$, unit holding cost $h = \$2$ per week, unit backlogging cost $\omega = \$2.9$ per week, fixed shipment cost $g_0 = \$6$, shipment cost per transported unit $g_1 = \$0.02$, fixed carbon emissions in transporting: $d_0 = 25$ kg, variable carbon emissions in transporting: $d_1 = 0.8$ kg per unit, fixed carbon emissions in holding: $e_0 = 16$ kg, carbon emission per unit held in stock and per unit of time: $e_1 = 1$ kg, tax charged on carbon emissions kilogram in transporting: $r_1 = \$1.5$, tax charged on carbon emissions kilogram in storage: $r_2 = \$1$, index of demand pattern: $\delta = 1.25$, non-centrality parameter of the price-dependent demand rate: $a_0 = 0.015$, sensitivity coefficient for the price-dependent demand: $a_1 = 0.01$ and exponent of the price-dependent demand: $\gamma = 2.5$. Following the development given in the previous section, from (18), the value of s_0 is $s_0 = \$212920$. Applying Theorem 4, from (17), we calculate $\beta(s_0) = -638.510$. Then, we obtain $\tilde{s} = \$16.9582$ and $G(\tilde{s}) = \$445.996$. Therefore, the optimal selling price is $s^* = \$16.9582$, with the optimal profit $G^* = \$445.996$. From (13), the optimal lot size is $q^* = 83.6341$ units and, from (14), the maximum number of backorders is $b^* = 36.2514$. Moreover, from (5), the optimal inventory cycle is $T^* = 1.22421$ weeks and, from (6), the optimal stock-in period is $\tau_1^* = 0.601731$ weeks.

Example 2. Use the same data as given in Example 1, but changing the value of γ to $\gamma = 2$. Applying Theorem 3, we calculate $\sqrt{\frac{1+\delta}{\delta\xi_0\pi_0\omega}} = 0.141519$, which is greater than $2a_1$. Thus, the optimal selling price is $s^* = s_2 = \$22.3041$, with maximum profit $G^* = \$180.363$. Therefore, the optimal lot size is $q^* = 42.5078$, the maximum shortage is $b^* = 18.4251$, the optimal inventory cycle is $T^* = 2.40864$ weeks and the optimal stock-in period is $\tau_1^* = 1.18391$ weeks.

Example 3. We now assume the same input parameters as in Example 2, but modify the value of a_1 to $a_1 = 0.1$. We have $2a_1 = 0.2$, which is greater than $\sqrt{(1+\delta)/(\delta\xi_0\pi_0\omega)}$. Therefore, by applying Theorem 3 again, we conclude that the function $G(s)$ does not reach its maximum at any point $s \geq p$. Obviously, in this case, the item should not be stocked.

Example 4. Consider the same data as given in Example 1, but changing the value of γ to $\gamma = 1.5$. From Theorem 2, we conclude that the optimum selling price is $s^* = \$40.1507$. The optimal lot size is $q^* = 19.5166$ units, the optimal inventory cycle is $T^* = 5.24610$ weeks, the maximum number of backorders is $b^* = 8.45949$, the optimal stock-in period is $\tau_1^* = 2.57859$ weeks and the optimal profit per unit time is $G^* = \$89.6702$.

Example 5. Assume the same data as given in Example 1, but modify the value of γ to $\gamma = 1$. From Theorem 1 , we deduce that the function $G(s)$ does not reach its maximum at any point $s \geq p$ and the supremum of $G(s)$ is $1/a_1 = 100$. We calculate $s_1 = \$14.3456$ and we can conclude that the inventory system is profitable for $s > \$14.3456$. Moreover, for example, if we want to obtain a profit per unit of time equal to \$60, we should take a unit selling price equal to $s = \$49.2094$.

Example 6. Suppose the same input parameters as in Example 3, but modify the value of γ to $\gamma = 4$. From (18), we obtain $s_0 = \$5.15695$, which is less than the purchasing cost p. Thus, applying Theorem 4, we deduce that the function $G(s)$ does not reach its maximum at any point $s \geq p$.

Example 7. Assume the same data as given in Example 6, but changing the value of a_1 to $a_1 = 0.06$. Now, we have $s_0 = \$14.4915 > p = \$7.5$ and $\beta(s_0) = 0.826062 > 0$. Therefore, from Theorem 4 again, we conclude that the function $G(s)$ does not reach its maximum at any point $s \geq p$.

Example 8. Consider the same data given in Example 6, but modify the value of a_1 to $a_1 = 0.045$. We calculate $s_0 = \$25.8738$ and $\beta(s_0) = -0.139384$. Next, we obtain $\tilde{s} = \$18.5175$ and $G(\tilde{s}) = -\$0.718532$. Thus, we see that the function $G(s)$ does not reach its maximum at any point $s \geq p$.

Example 9. Let us suppose the same data given in Example 1, but now considering that shortages are not allowed. Applying Theorem 5, we calculate $s_w = \$58406.9$, which is greater than p. Since $\beta(s_w) = -174.970 < 0$, we obtain $\tilde{s}_w = \$17.5863$ and $G_w(\tilde{s}_w) = \$402.314 > 0$. Thus, the optimal unit selling price is $s_w^* = \$17.5863$, with maximum profit per unit time $G_w^* = \$402.314$. Therefore, the optimal lot size is $q_w^* = 58.0474$ units, with an inventory cycle equal to $T_w^* = 0.923814$ weeks.

Note that in Examples 3, 6, 7 and 8, as the supremum of $G(s)$ is 0, it means that the profit per unit time is always negative and the inventory system is unprofitable.

5.1. Effects of the sustainable costs in the inventory system

Next, in this subsection, we compare the optimal inventory policy previously obtained with the one achieved from a model where carbon emission costs are not considered.

Let G^* denote the maximum value of the objective function of the model with carbon emission costs. That is, G^* is the value of the average profit per unit time linked with the optimal inventory policy developed in this paper. Let us denote by $G^\#$ the average profit per unit time of the best policy for the inventory model without considering sustainable costs. To compare both inventory policies, it is necessary to calculate the last profit $G^\#$. To do this, we must first calculate the objective function $\hat{P}(s,q,b)$ to be maximized in the model that does not consider carbon emission costs. It is clear that this function $\hat{P}(s,q,b)$ is obtained from (12), but now considering $\pi_0 = K + g_0$, $\pi_1 = p + g_1$ and $\pi_2 = (h+\omega)/(\delta+1)$. Thus, we get the following average profit per unit time

$$\hat{P}(s,q,b) = \left(s - (p+g_1) - \frac{K+g_0}{q}\right)(a_0 + a_1 s)^{-\gamma} - \frac{h+\omega}{\delta+1} q \left(1 - \frac{b}{q}\right)^{\delta+1} - \omega\left(b - \frac{q}{\delta+1}\right)$$

Applying Theorems 1 to 4 to the above function $\hat{P}(s,q,b)$, we obtain the best inventory policy $(\hat{s},\hat{q},\hat{b})$ for the system without considering sustainable costs. Thus, the profit $G^\#$ per unit time related with that policy $(\hat{s},\hat{q},\hat{b})$ is determined as $G^\# = P(\hat{s},\hat{q},\hat{b})$, with the function $P(s,q,b)$ given by (12).

Next, we define as a measure of the difference between the two solutions, the value Gap given by

$$Gap = \begin{cases} 100\left(\frac{G^\# - G^*}{G^*}\right) & \text{if } G^* > 0 \\ -\infty & \text{if } G^* = 0 \text{ and } G^\# < 0 \\ 0 & \text{if } G^* = 0 \text{ and } G^\# = 0 \end{cases}$$

Table 3 shows the results obtained for the previously solved numerical examples in which shortages are allowed. Note that, for Example 2, the selling price for the model without considering the sustainable costs is $\hat{s} = 17.7540$, which is 20.4003% lower than the optimal selling price s^*. The lot size for this model is $\hat{q} = 36.7273$, which is 13.5987% lower than the optimal lot size q^*, and the maximum shortage quantity is $\hat{b} = 12.5866$, which is 31.6878% lower than the optimal maximum shortage quantity q^*. This leads to the relative gap of more than 37%. However, for Example 7 , the best inventory policy obtained for the model without considering sustainability costs is $(\hat{s},\hat{q},\hat{b}) = (12.6142, 9.58405, 3.28448)$; while Theorem 4 indicates that, considering sustainable costs, the profit per unit of time is always negative and the inventory system is not profitable, because the supremum of $G(s)$ is 0. These two examples clearly show that, applying the optimal policy obtained for the inventory model without considering sustainable costs, can lead to a considerable decrease in the maximum profit (or, equivalently, a high additional cost) corresponding to the optimal solution deduced considering sustainable costs.

Table 3
Comparison of the inventory of numerical examples.

Example	s^*	q^*	b^*	G^*	$\hat{s}$	$\hat{q}$	$\hat{b}$	$G^\#$	Gap (%)
1	16.9582	83.6341	36.2514	445.996	14.0462	58.4901	20.0448	388.243	−12.9494
2	22.3041	42.5078	18.4251	180.363	17.7540	36.7273	12.5866	113.200	−37.2375
3	∞	0	0	0	41.3660	1.37530	0.47132	−2.1194	−∞
4	40.1507	19.5166	8.45949	89.6702	29.4880	13.7473	4.71126	80.2760	−10.4763
5	∞	0	0	100	∞	0	0	100	0
6	∞	0	0	0	∞	0	0	0	0
7	∞	0	0	0	12.6142	9.58405	3.28448	−16.9965	−∞
8	∞	0	0	0	11.2872	20.8804	7.15578	−26.6012	−∞

6. Sensitivity analysis and managerial insights

6.1. Impact of some parameters

In this section, we study the variation of the best inventory policy of the system presented when some values of the parameters of the system are modified. To do this, we consider the parameters of Example 1 of Section 5. Then, we obtain the percentage variations of the optimal policies, assuming that the value of each input parameter considered varies by ±5%, ±10%, ±20% and ±30%. Table 4 shows the computational results assuming these percentage variations in the parameters related to the demand rate function; while Table 5 displays the effects, with these same percentage variations, of the parameters p, r_1 and r_2 on the optimal policy.

Hence, Table 4 reveals that the sensitivity coefficient a_1 for the price-dependent demand and the exponent of the price-dependent demand γ are, of the four parameters associated with the demand rate, the parameters that have the greatest influence on the optimal policy. Thus, for changes in γ between −30% and 30%, the maximum profit varies between −73% and 342%. The effect of this parameter on the optimal unit selling price and the optimal inventory cycle is always negative, since both s^* and T^* increase when γ decreases. However, the optimal lot size q^*, the maximum shortage b^* and the initial stock level M^* are strictly increasing as the parameter γ increases. The parameter a_1 also has a significant influence on the maximum profit per unit time and on the optimal values of the decision variables q and b, but not on the optimal selling price s^*. This parameter has a positive effect on the optimal selling price s^* and the optimal cycle T^*, while the maximum profit G^*, the lot size q^*, the maximum shortage b^* and the maximum stock level M^* decrease when a_1 increases. Of the four parameters analyzed in this table, the one that has the least influence on the maximum profit G^* and the optimal unit selling price s^* is the index of demand pattern δ. Thus, for changes in range between −30% and +30%, the maximum benefit only varies between −1% and 1.2%. Finally, the parameter a_0 has a rather contained influence with respect to the optimal policy and the maximum benefit. Thus, when a_0 varies between −30% and +30%, the optimal price s^* varies between −2% and 2%, the optimal lot size between +6% and −5% (the same as the maximum shortage and the initial stock level), and the maximum profit between +7% and −7%. Note that the type of effect on the optimal price s^* and the maximum profit G^* is always inverse, that is, if the optimal price increases when one of the parameters decreases, then the maximum profit decreases, and if the optimal price decreases, the maximum profit increases. It is also interesting to note that when changes occur in the parameter γ, the magnitudes of the changes generated in the optimal lot size q^*, the maximum shortage b^* and the initial stock level M^* coincide. The same occurs with the changes in the parameters a_0 and a_1.

The results shown in Table 5 indicate that, as expected, the parameter p has a notable influence on both the optimal inventory policy (s^*, q^*, b^*) and the maximum profit G^*. Thus, for changes in p between −30% and +30%, the maximum profit G^* varies between +48% and −27%, the optimal selling price s^* fluctuates between −24% and +25%, and the lot size q^* and the maximum shortage b^* change between +37% and −22%. Note that p always has a positive effect on the optimal selling price s^*, while it is negative on lot size q^*, the maximum shortage b^* and the optimal profit G^*. Also, the tax charged on carbon emissions in storage r_2 has a relatively small influence on the optimal selling price s^*, the lot size q^* and the maximum profit per unit time G^*. Thus, for changes in range between −30% and +30%, the lot size q^* varies between −0.01% and +0.14%, the optimal selling price s^* between −0.57% and +0.55%, and the maximum profit G^* between +1.61% and −1.54%. It is also interesting to note that the effect of this parameter r_2 on the optimal lot size q^* is not monotonic, unlike what happens with the other parameters analyzed. Finally, with respect to the tax charged on carbon emissions in transporting r_1, we can say that its influence is not noticeable and that its effect is always positive with respect to the optimal policy (the optimal selling price, the optimal lot size and the maximum shortage increase when r_1 increases), but it is always negative with respect to the maximum profit per unit time.

Table 4
Effects of the parameters a_0, a_1, γ and δ on the optimal policy.

	Δ	Δs^* (%)	Δq^* (%)	Δb^* (%)	ΔM^* (%)	ΔT^* (%)	ΔG^* (%)
a_0	−30%	−2.23841	5.91665	5.91665	5.91665	−5.58614	7.30172
	−20%	−1.49294	3.87668	3.87668	3.87668	−3.73200	4.77373
	−10%	−0.746800	1.90556	1.90556	1.90556	−1.86993	2.34148
	−5%	−0.373482	0.944782	0.944782	0.944782	−0.935939	1.15969
	+5%	0.373645	−0.929168	−0.929168	−0.929168	0.937882	−1.13816
	+10%	0.747453	−1.84309	−1.84309	−1.84309	1.87770	−2.25535
	+20%	1.49555	−3.62662	−3.62662	−3.62662	3.76309	−4.42890
	+30%	2.24429	−5.35332	−5.35332	−5.35332	5.65611	−6.52473
a_1	−30%	−0.277227	50.0998	50.0998	50.0998	−33.3777	143.372
	−20%	−0.441462	29.5333	29.5333	29.5333	−22.7998	75.8839
	−10%	−0.325996	13.2220	13.2220	13.2220	−11.6779	31.0275
	−5%	−0.186159	6.28057	6.28057	6.28057	−5.90942	14.1576
	+5%	0.227738	−5.70707	−5.70707	−5.70707	6.05249	−11.9719
	+10%	0.493226	−10.9135	−10.9135	−10.9135	12.2504	−22.1655
	+20%	1.12560	−20.0598	−20.0598	−20.0598	25.0936	−38.4272
	+30%	1.87851	−27.8256	−27.8256	−27.8256	38.5534	−50.6266
γ	−30%	65.2000	−64.8075	−64.8075	−64.8075	184.152	−72.6123
	−20%	31.5240	−49.1741	−49.1741	−49.1741	96.7502	−59.5595
	−10%	12.2731	−28.2085	−28.2085	−28.2085	39.2923	−37.3323
	−5%	5.51265	−15.1490	−15.1490	−15.1490	17.8536	−21.0599
	+5%	−4.56601	17.5685	17.5685	17.5685	−14.9432	27.2389
	+10%	−8.39850	37.9360	37.9360	37.9360	−27.5026	62.4602
	+20%	−14.4430	88.8832	88.8832	88.8832	−47.0572	166.889
	+30%	−18.9634	157.204	157.204	157.204	−61.1203	341.565
δ	−30%	0.354166	−4.04468	23.0622	−24.7834	−3.26221	−0.991361
	−20%	0.256578	−2.96322	13.8321	−15.8129	−2.39035	−0.719410
	−10%	0.134756	−1.57850	6.29200	−7.60003	−1.27359	−0.378635
	−5%	0.0684866	−0.808515	3.01124	−3.73091	−0.652410	−0.192654
	+5%	−0.0699584	0.839613	−2.77525	3.60526	0.677659	0.197268
	+10%	−0.140802	1.70434	−5.34265	7.09582	1.37575	0.397520
	+20%	−0.283390	3.49055	−9.94537	13.7700	2.81824	0.802070
	+30%	−0.425118	5.32926	−13.9594	20.0866	4.30380	1.20617

6.2. Managerial insights

From the computational results and the above comments, we can deduce the following managerial insights:

Table 5
Effects of the parameters p, r_1 and r_2 on the optimal policy.

	Δ	Δs^* (%)	Δq^* (%)	Δb^* (%)	ΔM^* (%)	ΔT^* (%)	ΔG^* (%)
p	−30%	−24.3973	37.3331	37.3331	37.3331	−27.1844	47.5070
	−20%	−16.2827	22.4526	22.4526	22.4526	−18.3357	28.1780
	−10%	−8.14998	10.2175	10.2175	10.2175	−9.27029	12.6666
	−5%	−4.07708	4.88784	4.88784	4.88784	−4.66007	6.02544
	+5%	4.08116	−4.49691	−4.49691	−4.49691	4.70865	−5.48629
	+10%	8.16631	−8.64635	−8.64635	−8.64635	9.46470	−10.4982
	+20%	16.3482	−16.0472	−16.0472	−16.0472	19.1146	−19.3114
	+30%	24.5449	−22.4455	−22.4455	−22.4455	28.9416	−26.7931
r_1	−30%	−4.63581	−3.07105	−3.07105	−3.07105	−13.0644	8.03214
	−20%	−3.09297	−1.91860	−1.91860	−1.91860	−8.73857	5.24896
	−10%	−1.54780	−0.899302	−0.899302	−0.899302	−4.38489	2.57372
	−5%	−0.774242	−0.435405	−0.435405	−0.435405	−2.19654	1.27455
	+5%	0.774960	0.408297	0.408297	0.408297	2.20508	−1.25064
	+10%	1.55067	0.790785	0.790785	0.790785	4.41903	−2.47804
	+20%	3.10441	1.48314	1.48314	1.48314	8.87475	−4.86573
	+30%	4.66144	2.08596	2.08596	2.08596	13.3694	−7.16796
r_2	−30%	−0.567649	0.0457796	−5.52908	4.31096	−1.25352	1.61458
	−20%	−0.376537	0.00616920	−3.65030	2.80365	−0.856485	1.06743
	−10%	−0.187349	−0.00817471	−1.80778	1.36866	−0.437895	0.529363
	−5%	−0.0934500	−0.00674452	−0.899640	0.676388	−0.221231	0.263616
	+5%	0.0930106	0.0117687	0.891310	−0.661146	0.225561	−0.261529
	+10%	0.185591	0.0282895	1.77446	−1.30766	0.455229	−0.521014
	+20%	0.369499	0.0745831	3.51690	−2.55905	0.926060	−1.03401
	+30%	0.551794	0.137020	5.22855	−3.75838	1.41097	−1.53929

1. The largest increase in average profit per unit of time is obtained when the exponent of the price-dependent demand γ increases. Note that, from the point of view of inventory managers, it is not possible to act directly on this parameter, since it is obtained by fitting demand to the best curve that represents the price-dependent function. However, the price-dependent demand rate could be stimulated by increasing advertising or marketing campaigns (for example, by increasing the number of advertisements about the product's benefits in social networks, television, radio or press), or by encouraging customers to increase their purchases (for example, giving away an additional free unit of the product with the purchase of multiple units of that article).
2. The average profit per unit time can also be increased by reducing the unit purchase price p of the product. Thus, the person responsible for purchasing the article should obtain a price reduction by negotiating with suppliers (for example, by agreeing to a minimum purchase volume over a period of time).
3. Another way to increase the profit per unit time would be to decrease the tax r_1 on carbon emissions in shipping. To achieve this, tax authorities should be convinced of the negative impact that very high environmental taxes can have (for example, on employment).
4. The computational results show that an increase in the potential demand pattern index also leads to a small increase in the profit per unit time. Thus, this increase must be quite large (around 30%) to achieve a significant increase in profit per unit of time. As indicated in the first point, it is not possible for inventory managers to act directly on this parameter. However, they can act indirectly by modifying customer demand through marketing campaigns, as noted in the first point above.

The model demonstrates high applicability in sectors such as e-commerce, characterized by dynamic price variations; the food industry, with high turnover cycles; sustainable pharmaceutical and cosmetics, where demand is influenced by environmental impact and expiration dates; or consumer-oriented startups, which operate under strict regulatory frameworks and sustainability values. In all these cases, the model allows for the characterization of optimal policies that balance profitability and ecological responsibility, showing that sustainable inventory management can be viable and potentially more profitable when pricing, replenishment, and cycle duration decisions are efficiently coordinated. Numerical simulations and sensitivity analysis reinforce the applicability of the model, allowing for the precise characterization of the existence of the conditions for optimal solutions under different price sensitivity scenarios. Taken together, these contributions significantly advance the state of the art in sustainable inventory management, offering researchers and practitioners an effective tool for responsible, informed decision-making aligned with global sustainability challenges.

7. Conclusions and future research

In this paper, we have developed a sustainable inventory model with price-and-time dependent demand under full backlogging, considering taxes applied to carbon emissions. The demand rate is the product of an algebraic price-dependent function and a power time-dependent function, which may be useful to describe the real demand for some items, since it can fit the empirical data well.

We have formulated the mathematical model of profit maximization per inventory cycle and have developed theoretical results to obtain the optimal inventory policy (lot size, maximum shortage quantity and unit selling price). Several numerical examples have been solved to show the solution's procedure and confirm that the policies obtained indicate significant increases in benefits compared to models in which carbon emission rates are not considered.

The results of the numerical sensitivity analysis reflect that, of the four parameters associated with the demand rate, those that have the greatest influence on the optimal policy are the sensitivity coefficient for the price-dependent demand and the exponent of the price-dependent demand; while the least influential parameter on the maximum profit is the index of demand pattern. Thus, it is advisable to stimulate the price-dependent demand rate by increasing advertising or marketing campaigns.

With respect to the purchasing cost, changes in this parameter have a positive effect on the optimal selling price, while it is negative on the lot size, the maximum shortage and the optimal profit per unit time. The tax charged on carbon emissions in transporting has a greater influence on the optimal selling price, the lot size and the maximum profit than the tax charged on carbon emissions in storage. From an environmental perspective, it is advisable to reduce carbon emissions in transportation by using, for example, electric or hybrid vehicles.

This proposal represents a significant advance in the field of sustainable logistics management, offering an analytical model that addresses the challenges faced by organizations immersed in the transition toward more responsible and resilient business models. The work contributes to global sustainability goals, providing operational tools that enable the design of more efficient, environmentally conscious, and economically viable logistics systems. Due to its integrative approach and practical applicability, it constitutes a valuable contribution to the development of academic research, public policy design, and informed business decision-making in contexts of high uncertainty across the supply chain.

The proposed model can be extended in several ways. One possible extension is to suppose different dependence functions of demand with respect to the selling price and/or time. Another possibility is to consider that replenishment is not instantaneous and, therefore, a finite replenishment rate is considered. Furthermore, another interesting study could be to admit the possibility of the deterioration of product items.

CRediT authorship contribution statement

Luis A. San-José: Validation, Formal analysis, Supervision, Investigation, Conceptualization, Writing – original draft, Project administration, Writing – review & editing, Methodology. **Joaquín Sicilia:** Funding acquisition, Conceptualization, Writing – review & editing, Methodology, Formal analysis, Validation, Supervision, Investigation. **Manuel González-de-la-Rosa:** Conceptualization, Writing – review & editing, Investigation, Supervision, Validation, Methodology. **Jaime Febles-Acosta:** Methodology, Validation, Supervision, Conceptualization, Writing – review & editing, Investigation.

Acknowledgments

This work is partially supported by Grant PID2021-125042NB-I00 funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe".

Appendix

This appendix includes the proofs of the results given in the paper.

Lemma 1. *For a fixed value of s, the function $P_s(q,b) = P(s,q,b)$ given by (12) is strictly concave on the set $\Gamma = \{(q,b) : q > 0, 0 \le b \le q\}$.*

Proof. Since Γ can be expressed as $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \mathring{\Gamma}$, with $\Gamma_1 = \{(q,b) : q > 0, b = 0\}$, $\Gamma_2 = \{(q,b) : q > 0, b = q\}$ and $\mathring{\Gamma}$ the set of interior points of Γ, we now prove that the function $P_s(q,b)$ is strictly concave on each of these three convex sets.

First, the restriction of $P_s(q,b)$ on Γ_1 and the restriction of $P_s(q,b)$ on Γ_2 are univariate functions given, respectively, by $\varphi_0(q) = \left(s - \pi_1 - \frac{\pi_0}{q}\right)\left(a_0 + a_1 s\right)^{-\gamma} - \frac{h + r_2 e_1}{\delta+1} q$ and $\varphi_q(q) = \left(s - \pi_1 - \frac{\pi_0}{q}\right)\left(a_0 + a_1 s\right)^{-\gamma} - \frac{\delta\omega}{\delta+1} q$. Thus, it is immediate to check that these two functions have the same second derivative given by $\varphi_0''(q) = -\frac{2\pi_0}{q^3}\left(a_0 + a_1 s\right)^{-\gamma} < 0$. Therefore, the function $P_s(q,b)$ is strictly concave on both Γ_1 and Γ_2.

Next, we analyze the function $P_s(q,b)$ on the set $\mathring{\Gamma}$. Taking into account the fact that the function $P_s(q,b)$ is twice-differentiable on $\mathring{\Gamma}$; we now prove that the Hessian matrix is negative definite on $\mathring{\Gamma}$.

Since the first partial derivatives of $P_s(q,b)$ are given by

$$\frac{\partial}{\partial q} P_s(q,b) = \frac{\pi_0}{\left(a_0 + a_1 s\right)^{\gamma} q^2} - \pi_2 \left(1 - \frac{b}{q}\right)^{\delta}\left(1 + \frac{\delta b}{q}\right) + \frac{\omega}{\delta+1}$$

$$\frac{\partial}{\partial b} P_s(q,b) = \pi_2 (\delta+1)\left(1 - \frac{b}{q}\right)^{\delta} - \omega$$

we have that the second partial derivatives of the average profit are

$$\begin{aligned}\frac{\partial^2}{\partial q^2} P_s(q,b) &= \frac{-2\pi_0}{\left(a_0 + a_1 s\right)^{\gamma} q^3} - \pi_2\left[\frac{\delta b}{q^2}\left(1 - \frac{b}{q}\right)^{\delta-1}\left(1 + \frac{\delta b}{q}\right)\right] \\ &\quad - \left(1 - \frac{b}{q}\right)^{\delta}\frac{\delta b}{q^2} \\ &= -\left(\frac{2\pi_0}{\left(a_0 + a_1 s\right)^{\gamma} q^3} + \delta(\delta+1)\pi_2 \frac{b^2}{q^3}\left(1 - \frac{b}{q}\right)^{\delta-1}\right) \\ \frac{\partial^2}{\partial b \partial q} P_s(q,b) &= -\pi_2\left(\frac{-\delta}{q}\left(1 - \frac{b}{q}\right)^{\delta-1}\left(1 + \frac{\delta b}{q}\right) + \left(1 - \frac{b}{q}\right)^{\delta}\frac{\delta}{q}\right) \\ &= \delta(\delta+1)\pi_2 \frac{b}{q^2}\left(1 - \frac{b}{q}\right)^{\delta-1} \\ \frac{\partial}{\partial b^2} P_s(q,b) &= -\frac{\delta(\delta+1)\pi_2}{q}\left(1 - \frac{b}{q}\right)^{\delta-1}\end{aligned}$$

Therefore, the Hessian matrix is

$$H = \begin{pmatrix} -\left(\frac{2\pi_0}{\left(a_0+a_1 s\right)^{\gamma} q^3} + \delta(\delta+1)\pi_2 \frac{b^2}{q^3}\left(1 - \frac{b}{q}\right)^{\delta-1}\right) & \delta(\delta+1)\pi_2 \frac{b}{q^2}\left(1 - \frac{b}{q}\right)^{\delta-1} \\ \delta(\delta+1)\pi_2 \frac{b}{q^2}\left(1 - \frac{b}{q}\right)^{\delta-1} & -\frac{\delta(\delta+1)\pi_2}{q}\left(1 - \frac{b}{q}\right)^{\delta-1} \end{pmatrix}$$

Thus, we obtain $H_{11} = \partial^2 P_s(q,b)/\partial q^2 < 0$, $H_{22} = \partial^2 P_s(q,b)/\partial b^2 < 0$ and $\det(H) = \frac{2\delta(\delta+1)\pi_0\pi_2}{\left(a_0+a_1 s\right)^{\gamma} q^4}\left(1 - \frac{b}{q}\right)^{\delta-1} > 0$, for all $(q,b) \in \mathring{\Gamma}$. Hence the matrix H is negative definite on $\mathring{\Gamma}$ and, consequently, the function $P_s(q,b)$ is strictly concave on $\mathring{\Gamma}$. This completes the proof. □

Proof of Proposition 1. From (17), the two first derivatives of the function $\beta(s)$ are

$$\beta'(s) = a_1(1-\gamma) + \frac{a_1^2\gamma^2}{2\left(a_0 + a_1 s\right)}\sqrt{\frac{\delta\xi_0\pi_0\omega\left(a_0 + a_1 s\right)^{\gamma}}{\delta+1}} \tag{21}$$

and

$$\beta''(s) = \frac{a_1^3\gamma^2(\gamma-2)}{4}\sqrt{\frac{\delta\xi_0\pi_0\omega\left(a_0 + a_1 s\right)^{\gamma-4}}{\delta+1}}$$

Therefore:

(a) If $\gamma = 1$, then we have $\beta(s) = a_0 + a_1\left(\pi_1 + \sqrt{\frac{\delta\xi_0\pi_0\omega\left(a_0+a_1 s\right)}{1+\delta}}\right)$ and, it is easy to check that $\lim_{s\to\infty}\beta(s) = \infty$. Moreover, from (21), it follows that $\beta'(s) > 0$ and $\beta(s)$ is a strictly increasing function. Also, as $\beta''(s) < 0$, then $\beta(s)$ is a strictly concave function.

(b) If $1 < \gamma < 2$, then $\beta''(s) < 0$ and $\beta(s)$ is a strictly concave function. Furthermore, a trivial verification shows that $\lim_{s\to\infty}\beta(s) = -\infty$.

(c) If $\gamma = 2$, then $\beta(s) = a_0 + 2a_1\left(\pi_1 + a_0\sqrt{\frac{\delta\xi_0\pi_0\omega}{\delta+1}}\right) + \left(2a_1\sqrt{\frac{\delta\xi_0\pi_0\omega}{\delta+1}} - 1\right)a_1 s$, and it is an affine function. Thus, we must consider three cases depending on whether $2a_1\sqrt{\frac{\delta\xi_0\pi_0\omega}{\delta+1}} - 1$ is positive, negative or zero to obtain the limit of the function $\beta(s)$ when s tends to infinity.

(d) If $\gamma > 2$, then $\beta''(s) > 0$ and $\beta(s)$ is a strictly convex function. From (17), it is clear that $\lim_{s\to\infty}\beta(s) = \infty$. Moreover, from (21), it is obvious that $\beta'(s) = a_1(1-\gamma) + \frac{a_1^2\gamma^2}{2}\sqrt{\frac{\delta\xi_0\pi_0\omega\left(a_0+a_1 s\right)^{\gamma-2}}{\delta+1}}$, hence, s_0 given by (18) is the unique root of the equation $\beta'(s) = 0$, and $\beta(s)$ attains its minimum at the point s_0. □

Proof of Theorem 1. From Proposition 1, we have $\beta(s) > \beta(p)$ for $s > p$ and, since $\beta(p) > 0$, it follows that $G(s)$ is a strictly increasing function. Therefore, $\sup_{s\ge p} G(s) = \lim_{s\to\infty} G(s) = 1/a_1$.

Taking into account that $G(p) < 0$, $\lim_{s\to\infty} G(s) > 0$ and $G(s)$ is a strictly increasing function, there exists a unique root of the function $G(s)$ in the interval (p,∞). Thus, after some algebraic manipulations, from (15) and $\gamma = 1$, we deduce that s_1 is the largest of the roots of the equation $\left(s - \pi_1\right)^2 = \frac{4\delta\xi_0\pi_0\omega}{(1+\delta)}\left(a_0 + a_1 s\right)$, that is, $s^2 - 2\left(\pi_1 + 2a_1\xi_1\right)s + \pi_1^2 - 4a_0\xi_1 = 0$, with $\xi_1 = \delta\xi_0\pi_0\omega/(1+\delta)$. Consequently, $s_1 = \pi_1 + 2a_1\xi_1 + 2\sqrt{\xi_1\left(a_0 + a_1\left(\pi_1 + a_1\xi_1\right)\right)}$. The rest of the proof is obvious, because $G(s) > G(s_1) = 0$, for $s > s_1$. □

Proof of Theorem 2. Since $\beta(p) > 0$, $\lim_{s\to\infty}\beta(s) = -\infty$ and $\beta(s)$ is a strictly concave function, there exists a unique root s^* of $\beta(s)$, such that $\beta(s) > 0$ for $s \in [p, s^*)$ and $\beta(s) < 0$ for $s \in (s^*, \infty)$. Therefore, the function $G(s)$ is strictly increasing on the interval $[p, s^*)$ and strictly decreasing on (s^*, ∞). Thus, $G(s)$ reaches its maximum at the point s^*, with $G(s^*) > \lim_{s\to\infty} G(s) = 0$.

On the other hand, since $\beta(s^*) = 0$, we have $\pi_1 = \frac{(\gamma-1)s^*}{\gamma} - \frac{a_0}{a_1\gamma} - \sqrt{\frac{\delta\xi_0\pi_0\omega\left(a_0+a_1 s^*\right)^{\gamma}}{1+\delta}}$ and substituting this value in the expression for $G(s)$ given in (15), we obtain $G(s^*) = \frac{\left(a_0+a_1 s^*\right)^{1-\gamma}}{a_1\gamma} - \sqrt{\frac{\delta\xi_0\pi_0\omega\left(a_0+a_1 s^*\right)^{-\gamma}}{1+\delta}} = \left(a_0 + a_1 s^*\right)^{-\gamma/2}\left(\frac{\left(a_0+a_1 s^*\right)^{1-\gamma/2}}{a_1\gamma} - \sqrt{\frac{\delta\xi_0\pi_0\omega}{1+\delta}}\right)$. □

Proof of Theorem 3. From Proposition 1, we have:

1. If $\sqrt{\frac{1+\delta}{\delta\xi_0\pi_0\omega}} > 2a_1$, then $\beta(s)$ has a unique root, since $\beta(p) > 0$, $\lim_{s\to\infty}\beta(s) = -\infty$ and $\beta(s)$ is an affine function. It is trivial to check that the root is s_2. The rest is immediate.
2. If $\sqrt{\frac{1+\delta}{\delta\xi_0\pi_0\omega}} = 2a_1$, then $\beta(s) = \beta(p) > 0$ for $s > p$ and, therefore, $G(s)$ is a strictly increasing function. Thus, $\sup_{s\ge p} G(s) = \lim_{s\to\infty} G(s) = 0$.

3. If $\sqrt{\frac{1+\delta}{\delta\xi_0\pi_0\omega}} < 2a_1$, then $\beta(s)$ is a strictly increasing function. Therefore, $\beta(s) > \beta(p) > 0$ for $s > p$ and the rest of the proof runs as in the previous case. □

Proof of Theorem 4. Taking into account Proposition 1, we can consider the following two cases:

1. $s_0 \leq p$: since $\beta(s)$ is a strictly convex function that attains its minimum value at s_0, we have $\beta'(s) > 0$ for $s > p$. Hence, $\beta(s) > \beta(p) > 0$ for $s > p$, therefore, $G(s)$ is a strictly increasing function, and we obtain the desired conclusion.
2. $s_0 > p$ and $\beta(s_0) \geq 0$: then $\beta(s) > \beta(s_0) \geq 0$ for $s > p$ and $s \neq s_0$, and the rest of the proof runs as in the previous case.
3. $s_0 > p$ and $\beta(s_0) < 0$: then there exist two roots $\widetilde{s}$ and s_3 of the equation $\beta(s) = 0$, with $p < \widetilde{s} < s_0 < s_3$, such that $\beta(s) > 0$ for $s \in (p, \widetilde{s}) \cup (s_3, \infty)$ and $\beta(s) < 0$ for $s \in (\widetilde{s}, s_3)$. Thus, the function $G(s)$ is strictly increasing for $s \in (p, \widetilde{s}) \cup (s_3, \infty)$ and it is strictly decreasing for $s \in (\widetilde{s}, s_3)$. In consequence, as $\lim_{s\to\infty} G(s) = 0$, it follows that if $G(\widetilde{s}) \geq 0$, then $G(s)$ reaches its maximum at $\widetilde{s}$. However, if $G(\widetilde{s}) < 0$, then $G(s)$ does not reach its maximum at any point $s \geq p$ and $\sup_{s\geq p} G(s) = \lim_{s\to\infty} G(s) = 0$. □

Proof of Theorem 5. This follows by the same method as in the proofs of Theorems 1 to 4, taking into account that $\lim_{\omega\to\infty} \omega\xi_0 = (h + r_2e_1)/\delta$ and, therefore: (i) $\lim_{\omega\to\infty} \beta(s) = \beta_w(s)$, (ii) $\lim_{\omega\to\infty} s_2 = \frac{1}{a_1}\left(\frac{2(a_0+a_1\pi_1)}{1-2a_1\sqrt{\pi_0(h+r_2e_1)(1+\delta)^{-1}}} - a_0\right)$ and (iii) $\lim_{\omega\to\infty} s_0 = s_w$. □

Data availability

No data was used for the research described in the article.

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