

Bisymmetric Nonnegative Jacobi Matrix Realizations

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ABSTRACT

Within the symmetric inverse eigenvalue problem, the case of bisymmetric Jacobi matrices occupies a central place, since for any strictly monotone list of n real numbers there exists a unique bisymmetric Jacobi matrix realizing the list. Apart from their meaning in several issues such physics, mechanics, statistics, to cite some of them, the families of this kind of matrices whose spectrum is known are used as models for testing the different algorithms to recover the entries of matrices from spectra data. However, the spectrum is known only for a few families of bisymmetric Jacobi matrices and the examples mainly refer to the case when the spectrum is given by a linear or quadratic function of the order and of the row index. In the first part of the paper, we join all known cases by proving a general result about bisymmetric Jacobi realizations of strictly monotone sequences that are quadratic at most. In the second part we focus on the nonnegative bisymmetric realizations, obtaining new necessary conditions for a given list to be realized by a nonnegative bisymmetric Jacobi matrix. The main novelty in our techniques is considering the gaps between the eigenvalues instead of focusing on the eigenvalues themselves. In the last part of the paper, we explicitly obtain the bisymmetric realization of any list for order less or equal to six.

KEYWORDS

Jacobi matrix, Nonnegative matrix, Realization, Bisymmetric matrix

1. Introduction

Throughout the paper, for given $n \in \mathbb{N}^*$, $\mathbf{e}_n \in \mathbb{R}^n$ denotes the vector of all ones, whereas $\mathbf{I}_n$ is the identity matrix of order n . As a general rule, we drop the subindex n when it does not lead to confusion. Given $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$, $\mathbf{v}'$ denotes the decreasing rearrangement of $\mathbf{v}$; that is $\mathbf{v}' = (v'_1, \dots, v'_n)$ with $v'_1 \geq \dots \geq v'_n$. We write $\mathbf{v} \geq 0$, respectively $\mathbf{v} > 0$, when $v'_n \geq 0$, respectively $v'_n > 0$. Analogous notations are in force for $\mathbf{v} \leq 0$ and $\mathbf{v} < 0$ or even for square matrices of order n with real entries. If $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, then $\mathbf{v} < \mathbf{w}$ (respectively $\mathbf{v} \leq \mathbf{w}$) means that $\mathbf{w} - \mathbf{v} > 0$ (respectively $\mathbf{w} - \mathbf{v} \geq 0$).

Here we deal with $n \times n$ real symmetric matrices. If $\mathbf{M}$ is such a matrix, it is well-known that its eigenvalues are real and moreover the matrix is orthogonally similar to a diagonal one. The set of eigenvalues of $\mathbf{M}$, considering multiplicities, is denoted by $\Lambda(\mathbf{M})$. We always assume that the list $\Lambda(\mathbf{M}) = \{\lambda_1(\mathbf{M}), \dots, \lambda_n(\mathbf{M})\}$ is ordered in decreasing order; that is, $\lambda_1(\mathbf{M}) \geq \dots \geq \lambda_n(\mathbf{M})$.

The symmetric inverse eigenvalue problem aims to determine if, for a given list of n ordered real values, say $\Lambda = \{\lambda_1, \dots, \lambda_n\}$, there exists an n -order symmetric matrix M such that $\Lambda = \Lambda(M)$, in which case we say that M *realizes* Λ . Clearly, M realizes Λ iff $Q_\Lambda(x) = \prod_{k=1}^n (x - \lambda_k)$ is the characteristic polynomial of M , $\det(xI - M)$. Stated in these terms, the problem becomes trivial, because any diagonal matrix whose entries are the λ 's solves it. Therefore additional properties on the family of the matrix realizing the given list are considered, as irreducibility and/or nonnegativeness, or specifying the structure of the family, as for instance demanding the realization by tridiagonal matrices.

Many partial results around this problem have been obtained along the last fifty years, given either necessary or sufficient conditions on the list and focusing the problem on specific families of matrices (and hence specializing the corresponding given list to this kind of matrices). For instance, Perron-Frobenius' theory assures that $\lambda_1 \geq |\lambda_j|$, $j = 2, \dots, n$ and λ_1 simple are necessary conditions for Λ to be the spectrum of a some non-negative and irreducible matrix. In this case, λ_1 is called the *dominant eigenvalue* or the *Perron eigenvalue*. Moreover, if $\lambda_1 + \lambda_n = 0$ then $\lambda_i + \lambda_{n+1-i} = 0$ for $i = 2, \dots, \lceil \frac{n}{2} \rceil$. Therefore, when $\lambda_1 + \lambda_n = 0$ any realization of Λ by a non-negative and irreducible matrix has all its diagonal entries null.

The essential monograph [21], edited in Russian at the beginning of the forties of the past century, coined the term *Jacobi matrix* to refer to a *tridiagonal matrix* with real coefficients; that is, a square matrix $J = (a_{ij})$ satisfying that $a_{ij} = 0$ for $|i - j| > 1$. Then, J is irreducible iff $a_{kk+1}a_{k+1k} \neq 0$, $k = 1, \dots, n - 1$. In addition, when $a_{kk+1}a_{k+1k} \geq 0$, $k = 1, \dots, n - 1$, then J is isospectral with the symmetric Jacobi matrix $\hat{J} = (\hat{a}_{ij})$, where $\hat{a}_{kk} = a_{kk}$, $k = 1 \dots, n$, and $\hat{a}_{kk+1} = \hat{a}_{k+1,k} = \sqrt{a_{kk+1}a_{k+1k}}$, $k = 1, \dots, n - 1$, see [18, Lemma 0.1.1], which implies that each eigenvalue of J is real and simple. As a conclusion of the above properties, most of the real lists Λ realizable by irreducible tridiagonal matrices are realizable by a symmetric tridiagonal matrix with positive secondary diagonal. In fact, due to the above properties, nowadays, and specially in the context of the reconstruction matrices from spectral data, it is commonly accepted that the term Jacobi matrix means a real, symmetric, tridiagonal matrix, $J = (a_{ij})$ such that $a_{kk+1} = a_{k+1k} > 0$, $k = 1, \dots, n - 1$, see [4,5,13,22,24,26–29,41]. This will be the point of view in this work so, in what follows, we denote it by $J(\mathbf{a}, \mathbf{b})$, where $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $\mathbf{b} = (b_1, \dots, b_{n-1}) \in \mathbb{R}^{n-1}$, $\mathbf{b} > 0$, the Jacobi matrix

$$J(\mathbf{a}, \mathbf{b}) = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_{n-1} & b_{n-1} \\ 0 & \cdots & 0 & b_{n-1} & a_n \end{bmatrix}.$$

We remark that we always assume that $\mathbf{b} > 0$, and hence that $J(\mathbf{a}, \mathbf{b})$ is irreducible. Therefore, if $J(\mathbf{a}, \mathbf{b})$ realizes a list Λ , then there exist infinite nonsymmetric Jacobi matrices $J = (a_{ij})$ with the same diagonal $\mathbf{a}$ and realizing the same list Λ : For any $a_{kk+1} > 0$, $k = 1, \dots, n - 1$ the choice $a_{k+1k} = \frac{b_k^2}{a_{kk+1}}$, $k = 1, \dots, n - 1$, leads to an irreducible tridiagonal matrix realizing Λ . Notice that the nonnegativeness of the

off-diagonal entries; that is $\mathbf{b} > 0$, is a necessary condition to achieve some kind of uniqueness of the realization, since when $\mathbf{b} > 0$, $J(\mathbf{a}, \mathbf{b})$ and $J(\mathbf{a}, \hat{\mathbf{b}})$ where $\hat{b}_k = \pm b_k$, $k = 1, \dots, n$ are isospectral *via* signature matrices. In particular, we could choose $a_{kk+1} = a_{k+1k} = \pm b_k$, $k = 1, \dots, n-1$, so the signs of the b 's is irrelevant for realizability and there exist 2^{n-1} different irreducible and symmetric realizations with the same diagonal $\mathbf{a}$. This kind of hypotheses on $\mathbf{b}$ are also assumed in the literature, see for instance [22, 38–40] where the considered symmetric Jacobi matrices have $\mathbf{b} < 0$ and hence they are Z -matrices. It is the case in papers related with combinatorial issues, where Jacobi matrices are interpreted as Laplacians or more generally Schrödinger operators on a path, see for instance [2]. Besides, in [18] the Jacobi matrix $J(\mathbf{a}, \mathbf{b})$ is called *normal Jacobi matrix* when it is a Z -matrix; that is, $\mathbf{b} \leq 0$ (however, in [21] the term normal refers to the case $\mathbf{b} > 0$). We hope that this type of possible misunderstandings does not occur here, where we always assume the hypothesis $\mathbf{b} > 0$.

Tridiagonal or Jacobi matrices appear in many areas of applied mathematics, see previous references and other mentions that we make later in the paper. Many of them are closely related to the interpretation of the linear systems whose coefficient matrix is a Jacobi matrix as a discrete self-adjoint boundary value problem, specifically a Sturm-Liouville problem; thus linking Jacobi matrices with the so-called *three-terms recurrences* and hence with orthogonal polynomials, some combinatorial sequences, and so on. Of course, keeping the three-term recurrence, more general boundary conditions could be considered. In particular, periodic boundary conditions lead to the so-called *periodic Jacobi matrices*, see [16, 17] and references therein for a general treatment of this theory. Therefore, $\lambda \in \mathbb{R}$ is an eigenvalue for $J(\mathbf{a}, \mathbf{b})$ iff there exists $\mathbf{z} = (z_1, \dots, z_n)$ non null satisfying

$$\left. \begin{aligned} (a_1 - \lambda)z_1 + b_1z_2 &= 0, \\ b_{k-1}z_{k-1} + (a_k - \lambda)z_k + b_kz_{k+1} &= 0, \quad k = 2, \dots, n-1, \\ b_{n-1}z_{n-1} + (a_n - \lambda)z_n &= 0 \end{aligned} \right\} \quad (1)$$

which, in the boundary value problems terminology, means that the Sturm-Liouville problem (1) is not regular. This point of view already appeared in the well-known book [42], see the formula 48.1, 48.2 and 48.3 there. Irreducibility implies that for any data $\mathbf{w} \in \mathbb{R}^n$, any *initial value problem*, $z_k = \alpha$, $z_{k+1} = \beta$, where $1 \leq k \leq n-1$, has a unique solution. Moreover, standard arguments from Sturm-Liouville problems show that the space of solutions of (1) has dimension 1 at most, or equivalently, that the nullity of $J(\mathbf{a} - \lambda \mathbf{e}, \mathbf{b})$ equals 1, see [18, Lemma 0.1.1] for an alternative proof, based on matrix theory.

Fixed n and a list $\Lambda \subset \mathbb{R}$ with n different values, if we ask for the possible Jacobi matrices of order n realizing Λ , we see that we need to determine $2n-1$ entries from n data. So, in order to obtain a unique solution, we need to give $n-1$ additional data at least. This additional data could be focus either on $n-1$ additional spectra data, see [4, 5, 24, 26–29, 40, 41] or in the inner structure of the matrix. This last option is the one we adopted in this paper.

The Jacobi matrix $J(\mathbf{a}, \mathbf{b})$ is called *bisymmetric* when it is symmetric about both of its main diagonals; that is, when its entries satisfy that $a_k = a_{n+1-k}$, $k = 1, \dots, n$ and $b_k = b_{n-k}$, $k = 1, \dots, n-1$, or equivalently

$$a_k = a_{n+1-k} \quad \text{and} \quad b_k = b_{n-k}, \quad k = 1, \dots, \left\lceil \frac{n}{2} \right\rceil. \quad (2)$$

The structure of both general bisymmetric matrices and their spectrum were described

by A. Cantoni and P. Butler in [9]. In the third section, we specify these results for Jacobi matrices, the kind of matrices we are dealing with in this work.

Notice that bisymmetry reduces to n the number of entries to determine, so it is reasonable to expect that there is only one Jacobi bisymmetric matrix realizing a given list with n different values. The interpretation of bisymmetry in terms of the related mechanical systems can be found for instance in the monography [22, Section 4.6], see also [39].

In the 1970s, H. Hochstadt proved that given an ordered list Λ there exists at most one bisymmetric Jacobi matrix $J(a, b)$ with $b > 0$ realizing Λ , see [27, Theorem 3], and O. Hald showed that such a matrix exists, see [26, Theorem 3]. Both authors obtained their results as consequence of their reasonings to solve the Jacobi Inverse Eigenvalue Problem (JIEP), that are strongly based on *interlacing*, so imposing $n - 1$ additional spectral data. They also proved the continuous dependence of $J(a, b)$ from the data, describing an algorithm for its construction which however is unstable, see also the work by L. J. Gray and D. G. Wilson [24, Theorem 1] for an alternative proof. Just a couple of years later, C. de Boor and G. H. Golub developed an efficient algorithm to obtain $J(a, b)$, based on orthogonal polynomials, and hence in the known three-term recurrence, see [4, Section 4]. In this paper, the problem of finding a bisymmetric Jacobi matrix, called there Problem C, is also considered, incorporating some valuable comments about the *intriguing* choice of the weights, that already appeared in [27], see also [22, section 4.3] for an explanation.

Summing up, in this paper we are interested in the spectral properties of bisymmetry Jacobi matrices, specially in the nonnegative inverse eigenvalue problem. We remark that apart from their physical meaning, Jacobi bisymmetric matrices are important because each list characterizes a unique Jacobi bisymmetric matrix realizing it.

Since we are interested in (non-negative) realizability characterizations, it is important to have examples of bisymmetric Jacobi matrices for which the spectrum is known. In fact, many of these kinds of matrices are used as test problems to verify the efficiency of some algorithms to compute eigenvalues and also in regard to its stability, see the comments in [23, Section 1]. However, we also remark that, despite having a very simple structure, in general Jacobi matrices, or even bisymmetric Jacobi matrices, with closed form eigenvalues are rare.

The paper is organized as follows. In Section 2, we survey the known realizations of ordered lists from bisymmetric Jacobi matrices. Perhaps the main novelty consists in presenting the known results in a fairly unified way. It is surprising that most of the known realizable lists correspond to two specific types of bisymmetric matrices whose entries either have a slight periodicity (generally of period 2 at most) or are linear or quadratic functions of the order of the matrix and of the row index, which leads to lists of eigenvalues of the same kind, quadratic at most. It is also surprising how the same results appear rediscovered several times in the numerous publications in this field. In Section 3, we present the specialization of the JNIEP to bisymmetric matrices. The main novelty in our treatment is that we focus the reasonings on the gaps between eigenvalues instead on the eigenvalues themselves. The idea of considering the sequence of gaps appears from our results in low order bisymmetric Jacobi matrices and, in fact, was implicit in many of the mentioned references, see for instance [23], where a more than implicit mention to gaps is made. We first translate into terms of gaps the necessary conditions on the given list to be realizable by a nonnegative Jacobi matrix obtained in [20]. For the case of bisymmetric realizations, we show that, for a given gap sequence, there exists a threshold value for the lower eigenvalue, assuring the non-negativeness. Moreover, the adaptation of the results by Cantoni and

Butler to bisymmetric Jacobi matrices allows us to obtain explicitly two entries of the realizing matrix and hence a new necessary condition on the gaps for nonnegative bisymmetric Jacobi realizability. The last section of the paper is devoted to obtaining the unique (nonnegative) bisymmetric Jacobi realization for an ordered list with low order, meaning $n \leq 6$.

2. Known bisymmetric Jacobi realizations

The purpose of this section is to survey the (few) families of bisymmetric Jacobi matrices for which the spectrum is known. Basically, two kinds of matrices $J(\mathbf{a}, \mathbf{b})$ have been analyzed: the first one is related to vectors $\mathbf{a}$ and $\mathbf{b}$ with periodic entries and the second corresponds to vectors $\mathbf{a}$ and $\mathbf{b}$ whose entries are given by linear or quadratic functions of the order and the row index.

2.1. Periodic entries

We consider here Jacobi matrices $J(\mathbf{a}, \mathbf{b})$ whose coefficients $\mathbf{a}$ and $\mathbf{b}$ are periodic; that is, there exists $p \in \mathbb{N}^*$ such that $a_k = a_{k+p}$, $k = 1, \dots, n-p$ and $b_k = b_{k+p}$, $k = 1, \dots, n-1-p$. The known cases are reduced to very low periods, $p = 1, 2$. Of course, the case $p = 1$ corresponds to constant diagonals and in this case $J(\mathbf{a}, \mathbf{b})$ is known as *Jacobi-Toeplitz* matrix.

We analyze first a slight variation of the 1-periodic case, that appears in the discretization of boundary value problems for second order linear differential equations, see for instance [7], specially Section 5. In this case, $\mathbf{a} = (\alpha, a, \dots, a, \alpha) \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$ and $\mathbf{b} = b \mathbf{e}_{n-1} \in \mathbb{R}^{n-1}$, $b > 0$. Notice that the choice $\alpha = a$ becomes the 1-periodic case. Applying techniques from discrete Sturm-Liouville problems, see [2], we obtain that $\lambda \in \mathbb{R}$ is an eigenvalue of the bisymmetric Jacobi matrix $J(\mathbf{a}, \mathbf{b})$ iff λ is a root of the n degree polynomial

$$P(x) = [(\alpha - a)^2 - b^2]U_{n-2}\left(\frac{x - a}{2b}\right) - b[2\alpha - a - x]U_{n-1}\left(\frac{x - a}{2b}\right),$$

where U_k denotes the Chebyshev polynomial of second kind, see [33] for the definition and the main properties and also [2]. In general, no closed formulas to obtain the roots of the above Chebyshev polynomials are available. Therefore, this is a good example of how complicated the computation of the eigenvalues of bisymmetric Jacobi matrices is, even when the matrix has a nice and simple structure. However, since the roots $U_m(z)$ are $\cos\left(\frac{j\pi}{m+1}\right)$, $j = 1, \dots, m$, see [33, Section 2.2], when P becomes a multiple of U_m , then we have a closed formula for the eigenvalues of the given Jacobi matrix. Next, we list these cases that have appeared periodically in the literature (we newly remark that we are only considering the bisymmetric examples), see [43,44].

1. $\alpha = a$, see [25, Example 7.4] and also [34, page 514]. In this case,

$$P(x) = -b^2 U_n\left(\frac{x - a}{2b}\right),$$

which implies that $\Lambda(J(\mathbf{a}, \mathbf{b})) = \left\{a + 2b \cos\left(\frac{k\pi}{n+1}\right)\right\}_{k=1}^n$.

2. $\alpha = a + b$, see [25, Example 7.7]. In this case,

$$P(x) = -b(a + 2b - x)U_{n-1}\left(\frac{x - a}{2b}\right),$$

which implies that $\Lambda(J(\mathbf{a}, \mathbf{b})) = \left\{a + 2b \cos\left(\frac{\pi}{n}(k-1)\right)\right\}_{k=1}^n$.

3. $\alpha = a - b$, see [44, Theorem 4]. In this case, we obtain that

$$P(x) = -b[a - 2b - x]U_{n-1}\left(\frac{x - a}{2b}\right),$$

which implies that $\Lambda(J(\mathbf{a}, \mathbf{b})) = \left\{a + 2b \cos\left(\frac{k\pi}{n}\right)\right\}_{k=1}^n$.

The particular choice $a = 0$ was considered in [7, Identity (25)]. All the three cases are also considered in [12, Theorem 2], where the main motivation was the discretization of some boundary value problems as an intermediate step to study Navier-Stokes equations with non-slip boundary conditions.

When the period $p = 2$, to the best of our knowledge, the only bisymmetric example in the literature corresponds to n odd, which implies that $\mathbf{b} = b\mathbf{e}_{n-1}$, $b > 0$, see for instance [31,32]. Then, the ordered eigenvalues are, see [25, Example 7.9],

$$\lambda_k = \frac{1}{2}\left[a_1 + a_2 + \sqrt{(a_1 - a_2)^2 + 16b^2 \cos^2\left(\frac{k\pi}{n+1}\right)}\right], \quad k = 1, \dots, \lfloor \frac{n}{2} \rfloor,$$

$$\lambda_{\lceil \frac{n}{2} \rceil} = a_1,$$

$$\lambda_{k+\lceil \frac{n}{2} \rceil} = \frac{1}{2}\left[a_1 + a_2 - \sqrt{(a_1 - a_2)^2 + 16b^2 \cos^2\left(\frac{k\pi}{n+1}\right)}\right], \quad k = 1, \dots, \lfloor \frac{n}{2} \rfloor.$$

When $a_2 = a_1$ we recover the 1-periodic case above.

The following variant, where only the main diagonal has 2-periodic entries and $\mathbf{b} \in \mathbb{R}^{n-1}$ is given by $b_k = b\sqrt{k(n-k)}$, $b > 0$, $k = 1, \dots, n-1$, was analyzed in [32, Section 2]. Then, the ordered eigenvalues are

$$\lambda_k = \frac{1}{2}\left[a_1 + a_2 + \sqrt{(a_1 - a_2)^2 + 16b^2(\lfloor \frac{n}{2} \rfloor - k)^2}\right], \quad k = 1, \dots, \lfloor \frac{n}{2} \rfloor,$$

$$\lambda_{\lceil \frac{n}{2} \rceil} = a_1,$$

$$\lambda_{k+\lceil \frac{n}{2} \rceil} = \frac{1}{2}\left[a_1 + a_2 - \sqrt{(a_1 - a_2)^2 + 16b^2 k^2}\right], \quad k = 1, \dots, \lfloor \frac{n}{2} \rfloor.$$

When $a_2 = -a_1$ we recover the alternating main diagonal entries case, studied in [31].

2.2. Eigenvalues as a strictly monotone quadratic sequence

The known examples of Jacobi matrices $J(\mathbf{a}, \mathbf{b})$, whose entries are given by linear or quadratic functions of the order and the row index, correspond to Jacobi matrices

whose eigenvalues are linear or quadratic functions of the order and the row index of the matrix.

For a *quadratic sequence*, we understand $\{\nu_k\}$ given by

$$\nu_k = \gamma k^2 + \beta k + \alpha, \quad k \in \mathbb{N}^*,$$

where $\alpha, \beta, \gamma \in \mathbb{R}$. Clearly, quadratic sequences encompass *linear sequences*, that correspond to taking $\gamma = 0$. Given $m \geq 3$, then $\nu_{m-k} = \nu_k$ for some $k = 1, \dots, m-1$ such that $2k \neq m$ iff $\beta = -m\gamma$ and hence we can express the sequence as

$$\nu_k = -\gamma(\rho + k)(\rho + m - k) + \gamma\rho(\rho + m) + \alpha, \quad k = 1, \dots, m-1,$$

for any $\rho \in \mathbb{R}$. Therefore, the only linear sequences $\mathbf{a}$ or $\mathbf{b}$ that could appear in a bisymmetric Jacobi matrix $J(\mathbf{a}, \mathbf{b})$ are constant.

For the reason that will be clear in the next sections, we prefer to tackle the problem from the structure of the eigenvalues. So, given $n \in \mathbb{N}$, $n \geq 2$, the non constant sequences with n terms we deal with here are expressed as

$$\mu_k = ck^2 + (2b - c)k + a - 2b, \quad k = 1, \dots, n, \quad (3)$$

where $a, b, c \in \mathbb{R}$ and $|c| + |b| > 0$, since otherwise $\mu_k = a - 2b$ and hence the sequence is constant. It is clear that the sequence is linear when $c = 0$, quadratic when $c \neq 0$ and in both cases describes the most general sequence of these types. Of course, the above sequence can also be expressed in the flipped way as

$$\mu_{n+1-k} = (n - k)[c(n + 1 - k) + 2b] + a, \quad k = 1, \dots, n,$$

and $\{\mu_k\}_{k=1}^n$ is strictly increasing iff $\{\mu_{n+1-k}\}_{k=1}^n$ is strictly decreasing. Obviously, each μ_k is an integer when a, b, c are integers.

On the other hand, $\mu_k - \mu_{k+1} = -2(b + ck)$, $k = 1, \dots, n-1$, which implies that $\{\mu_k\}_{k=1}^{n-1}$ is strictly decreasing iff either $b < c(1 - n)$ when $c \geq 0$ or $b < -c$ when $c < 0$, whereas $\{\mu_k\}_{k=1}^{n-1}$ is strictly increasing iff either $b > -c$ when $c \geq 0$ or $b > c(1 - n)$ when $c < 0$. Summing up and defining $\frac{b}{0} = -\infty$ when $b < 0$ and $\frac{b}{0} = +\infty$ when $b > 0$, the sequence $\{\mu_k\}_{k=1}^{n-1}$ is strictly monotone iff $c^{-1}b \notin [1 - n, -1]$.

Next, we join in a common statement all known results about the realizability of the strictly monotone quadratic sequence $\{\mu_k\}_{k=1}^n$. Since, to the best of our knowledge, such a result does not appear explicitly in the literature, we include here a proof for the sake of completeness. It is strongly based on the proof given in [39, Theorem 1] for a particular case.

Theorem 2.1. *Given $n \in \mathbb{N}$, $n \geq 2$, $a, b, c \in \mathbb{R}$, such that $|c| + |b| > 0$ and $c^{-1}b \notin [1 - n, -1]$, consider the sequence*

$$\mu_k = ck^2 + (2b - c)k + a - 2b, \quad k = 1, \dots, n,$$

and the vectors $\mathbf{a}^n \in \mathbb{R}^n$ and $\mathbf{b}^{n-1} \in \mathbb{R}^{n-1}$ whose entries are

$$\begin{aligned} a_k^n &= -c[(n-k)^2 + (k-1)^2] + (n-1)(b+cn) + a, \quad k = 1, \dots, n, \\ b_k^{n-1} &= \sqrt{k(n-k)(b+ck)(b+c(n-k))}, \quad k = 1, \dots, n-1. \end{aligned}$$

Then $J(\mathbf{a}^n, \mathbf{b}^{n-1})$ is bisymmetric and

$$\Lambda(J(\mathbf{a}^n, \mathbf{b}^{n-1})) = \begin{cases} \{\mu_{n+1-k}\}_{k=1}^n, & \text{if either } c \geq 0 \text{ and } b > -c, \\ & \text{or } c < 0 \text{ and } b > c(1-n), \\ \{\mu_k\}_{k=1}^n, & \text{if either } c \geq 0 \text{ and } b < c(1-n), \\ & \text{or } c < 0 \text{ and } b < -c. \end{cases}$$

Proof. We proceed by induction on n to prove that the spectrum of the bisymmetric Jacobi matrix $J(\mathbf{a}^n, \mathbf{b}^{n-1})$ is $\{\mu_k\}_{k=1}^n$. First, for $n = 2$ we have that $c^{-1}b \neq -1$, so $b \neq -c$ and hence $|b+c| > 0$. Moreover,

$$J(\mathbf{a}^2, \mathbf{b}^1) = \begin{bmatrix} a+b+c & |b+c| \\ |b+c| & a+b+c \end{bmatrix}$$

whose eigenvalues are $\mu_1 = a$ and $\mu_2 = a + 2(b+c)$.

Assume that the claim is true for $n \geq 2$ and take $a, b, c \in \mathbb{R}$ such that $c^{-1}b \notin [-n, -1]$, which implies that $c^{-1}b \notin [1-n, -1]$. Consider the values

$$\begin{aligned} h_k &= \sqrt{(n+1-k)|b+c(n+1-k)|}, \quad k = 1, \dots, n+1, \\ r_k &= \sqrt{k|b+ck|}, \quad k = 0, \dots, n. \end{aligned}$$

Notice that the hypothesis on b and c implies that either $b+ck > 0$ for any $k = 1, \dots, n$ or $b+ck < 0$ for any $k = 1, \dots, n$ and hence, $h_k r_k = b_k^n$, $k = 1, \dots, n$ and also $h_{k+1} r_k = b_k^{n-1}$, $k = 1, \dots, n-1$.

Define the matrix $\mathbf{M} = (m_{ij}) \in \mathcal{M}_{(n+1) \times (n+1)}(\mathbb{R})$ whose entries are given by $m_{jj} = h_j$, $j = 1, \dots, n+1$, by $m_{j+1,j} = r_j$, $j = 1, \dots, n$ and by 0 otherwise. Then, $\mathbf{M}^\top \mathbf{M} = \begin{bmatrix} J(\hat{\mathbf{d}}, \mathbf{b}^{n-1}) & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ where $\hat{d}_k = h_k^2 + r_k^2$, $k = 1, \dots, n$, whereas $\mathbf{M} \mathbf{M}^\top = J(\mathbf{d}, \mathbf{b}^n)$ where $d_k = h_k^2 + r_{k-1}^2$, $k = 1, \dots, n+1$. On the other hand $\mu_{n+1} = cn^2 + n(2b+c) + a$.

If $b+ck > 0$, $k = 1, \dots, n$, then

$$\begin{aligned} d_k &= c[(n+1-k)^2 + (k-1)^2] + nb = \mu_{n+1} - a_k^{n+1}, \quad k = 1, \dots, n+1, \\ \hat{d}_k &= c[(n+1-k)^2 + k^2] + (n+1)b \\ &= c[(n-k)^2 + (k-1)^2] + 2nc + (n+1)b = \mu_{n+1} - a_k^n, \quad k = 1, \dots, n \end{aligned}$$

which implies that

$$\mathbf{M}^\top \mathbf{M} = \mu_{n+1} \mathbf{I}_{n+1} - \begin{bmatrix} J(\mathbf{a}^n, -\mathbf{b}^{n-1}) & \mathbf{0}^\top \\ \mathbf{0} & \mu_{n+1} \end{bmatrix}, \quad \mathbf{M} \mathbf{M}^\top = \mu_{n+1} \mathbf{I}_{n+1} - J(\mathbf{a}^{n+1}, -\mathbf{b}^n).$$

Taking into account that $J(\mathbf{a}^n, -\mathbf{b}^{n-1})$ and $J(\mathbf{a}^n, \mathbf{b}^{n-1})$ are isospectral and also that $J(\mathbf{a}^{n+1}, -\mathbf{b}^n)$ and $J(\mathbf{a}^{n+1}, \mathbf{b}^n)$ are isospectral, from the first identity, and applying the induction hypothesis, the eigenvalues of $\mathbf{M}^\top \mathbf{M}$ are $\mu_{n+1} - \mu_k$, $k = 1, \dots, n+1$. Since $\mathbf{M}\mathbf{M}^\top$ has the same eigenvalues as $\mathbf{M}^\top \mathbf{M}$, from the second identity, we conclude that the eigenvalues of $J(\mathbf{a}^{n+1}, \mathbf{b}^n)$ are μ_k , $k = 1, \dots, n+1$.

If $b + ck < 0$, $k = 1, \dots, n$, then

$$\begin{aligned} d_k &= -c[(n+1-k)^2 + (k-1)^2] - nb = a_k^{n+1} - \mu_{n+1}, & k = 1, \dots, n+1, \\ \hat{d}_k &= -c[(n+1-k)^2 + k^2] - (n+1)b \\ &= -c[(n-k)^2 + (k-1)^2] - 2nc - (n+1)b = a_k^n - \mu_{n+1}, & k = 1, \dots, n \end{aligned}$$

which implies that

$$\mathbf{M}^\top \mathbf{M} = \begin{bmatrix} J(\mathbf{a}^n, \mathbf{b}^{n-1}) & 0^\top \\ 0 & \mu_{n+1} \end{bmatrix} - \mu_{n+1} \mathbf{I}_{n+1}, \quad \mathbf{M}\mathbf{M}^\top = J(\mathbf{a}^{n+1}, \mathbf{b}^n) - \mu_{n+1} \mathbf{I}_{n+1}.$$

From the first identity, and applying the induction hypothesis, the eigenvalues of $\mathbf{M}^\top \mathbf{M}$ are $\mu_k - \mu_{n+1}$, $k = 1, \dots, n+1$ and from the second identity, we conclude that the eigenvalues of $J(\mathbf{a}^{n+1}, \mathbf{b}^n)$ are μ_k , $k = 1, \dots, n+1$. $\square$

Observe that the entries of $\mathbf{a}^n$ can also be expressed as

$$a_k^n = 2c(k-1)(n-k) + a + (b+c)(n-1), \quad k = 1, \dots, n,$$

and also as

$$a_k^n = 2ck(n+1-k) + n(b-c) + a - b - c, \quad k = 1, \dots, n.$$

As by-products of the previous general theorem, some specific results that appear in the literature can be obtained. The first one refers to the Jacobi realizability of a list given in arithmetic progression; that is, when $c = 0$. The result is well known and basically corresponds to the symmetric version of the so called *Kac-Sylvester matrix*, also named *Clement matrix*, whose origins go back to the nineteenth century, see [36], but appears recurrently in the literature, see [3,6,8,11,35] and also [15,19,30,37] for some applications of this kind of matrices in the analysis of random walks and some problems from statistical mechanics or quantum physics.

Corollary 2.2. *Given $n \in \mathbb{N}$, $n \geq 2$, $a, b \in \mathbb{R}$, such that $b \neq 0$, consider the vectors $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^{n-1}$ whose entries are*

$$\begin{aligned} a_k &= b(n-1) + a, & k = 1, \dots, n, \\ b_k &= |b|\sqrt{k(n-k)}, & k = 1, \dots, n-1. \end{aligned}$$

Then

$$\Lambda(J(\mathbf{a}, \mathbf{b})) = \begin{cases} \left\{ 2b(n-k) + a \right\}_{k=1}^n, & \text{if } b > 0, \\ \left\{ 2b(k-1) + a \right\}_{k=1}^n, & \text{if } b < 0. \end{cases}$$

The following cases refer to the realizability of non linear, quadratic and monotone lists; that is, when $c \neq 0$. They are a sample of how quadratic lists are used as test examples, and they reappear from time to time even in recent literature. We rename the other parameters in Theorem 2.1 as cb and $a - 2cb$ instead of b and a , respectively.

Corollary 2.3 ([8, Section 5]). *Given $n \in \mathbb{N}$, $n \geq 2$, $a, b, c \in \mathbb{R}$, such that $c \neq 0$ and $b \notin [1 - n, -1]$, consider the vectors $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^{n-1}$ whose entries are*

$$\begin{aligned} a_k &= 2ck(n+1-k) + c(b-1)n(n-1) + a, & k = 1, \dots, n, \\ b_k &= |c|\sqrt{k(n-k)(b+k)(b+n-k)}, & k = 1, \dots, n-1. \end{aligned}$$

Then

$$\Lambda(J(\mathbf{a}, \mathbf{b})) = \begin{cases} \left\{ c(n+1-k)(n+2b-k) + a \right\}_{k=1}^n, & \text{if } c > 0 \text{ and } b > -1, \\ & \text{or } c < 0 \text{ and } b < 1-n, \\ \\ \left\{ ck(k+2b-1) + a \right\}_{k=1}^n, & \text{if } c > 0 \text{ and } b < 1-n, \\ & \text{or } c < 0 \text{ and } b > -1. \end{cases}$$

The objective in [8] was to relate tridiagonal matrices with some orthogonal polynomials. In fact, the above result appears in relation with the so-called *Dual Hahn Polynomials*. We newly remark that we are considering here only bisymmetric matrices.

The choice $b = 1$ in the above Corollary was considered by W. Chu in [10, Section 2] with the objective of introducing a family of tridiagonal matrices whose entries depend on three parameters, really two, since one of them corresponds to a constant diagonal value, all three being quadratic functions of the row index and the order. He identified only six pairs of these values for which it is possible to compute their eigenvalues. However, only one of these pairs, corresponding to $b = 1$, produces a Jacobi bisymmetric matrix.

Considering $-c$ and $a + cn(n-1)$ instead of c and a in the above corollary, the choice $b = 0$ leads to the following result.

Corollary 2.4 ([14, Identity (2)]). *Given $n \in \mathbb{N}$, $n \geq 2$, $a, c \in \mathbb{R}$, such that $c \neq 0$, consider the vectors $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^{n-1}$ whose entries are*

$$\begin{aligned} a_k &= c[(k-1)^2 + (n-k)^2] + a, \\ b_k &= |c|k(n-k), & k = 1, \dots, n-1. \end{aligned}$$

Then

$$\Lambda(J(\mathbf{a}, \mathbf{b})) = \begin{cases} \left\{ c[(2n-1)^2 - (2(n-k)+1)^2] + a \right\}_{k=1}^n, & \text{if } c < 0, \\ \\ \left\{ c[(2n-1)^2 - (2k-1)^2] + a \right\}_{k=1}^n, & \text{if } c > 0. \end{cases}$$

We remark again that we are considering bisymmetric realizations and with the notation of [14], the above corollary corresponds to taking $s = 0$ there. The choice $c = 1$ was treated in [6, Sections 3-5].

The most recent example corresponds, in a somewhat simplified form, to taking $b = (2m - 1)\frac{c}{2}$, $m > -\frac{1}{2}$, in Theorem 2.1.

Corollary 2.5 ([1, Section 2]). *Given $n \in \mathbb{N}$, $n \geq 2$, $a, c \in \mathbb{R}$, such that $c \neq 0$, consider the vectors $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^{n-1}$ whose entries are*

$$a_k = a + c[(2m + 1)(n - 1) + 4(k - 1)(n - k)], \quad k = 1, \dots, n,$$

$$b_k = |c|\sqrt{k(n - k)(2(m + k) - 1)(2(m + n - k) - 1)}, \quad k = 1, \dots, n - 1.$$

Then

$$\Lambda(J(\mathbf{a}, \mathbf{b})) = \begin{cases} \left\{ a + 2c(m + n - k)^2 \right\}_{k=1}^n, & \text{if } c > 0, \\ \left\{ a + 2c(m + k - 1)^2 \right\}_{k=1}^n, & \text{if } c < 0. \end{cases}$$

The case $m = 0$ in the above corollary was solved in [39, Theorem 1]. Observe that when $a = 0$ and $m \in \mathbb{N}$, the eigenvalues are perfect squares, which is one of the main motivations to consider this kind of matrices.

Of course many similar examples can be obtained by choosing an appropriate relation between c and b .

3. Bisymmetric nonnegative Jacobi matrices

Our aim in this section is to study the spectral properties of irreducible and bisymmetric Jacobi matrices of order n . So, we consider $J(\mathbf{a}, \mathbf{b})$, where $\mathbf{b} > 0$ and $\Lambda(J(\mathbf{a}, \mathbf{b})) = \{\lambda_1, \dots, \lambda_n\}$ is its (ordered) list of eigenvalues. We also study conditions on the list Λ assuring the nonnegativeness of $J(\mathbf{a}, \mathbf{b})$.

At the end of the 1970s, S. Friedland and A.A. Melkman gave in [20] a very complete list of necessary conditions on the given list Λ to be realized by an irreducible and nonnegative Jacobi matrix. These conditions were shown sufficiently to solve the JNIEP for order less than or equal to 6, see the last remark [20, pg. 253]. Below, we mention these conditions, expressing them according to our techniques. However, except for the easy case when the lowest element in the list Λ is nonnegative, neither of the above conditions guarantees that the realizing nonnegative matrix is additionally bisymmetric. Recently, R. Vaia and L. Spadini and also Andelić, da Fonseca, Kiliç and Stanić, have treated the case in which the elements of the list Λ are proportional to a sequence of squares of n successive integers. In this case, the realizing Jacobi matrix is explicitly obtained and it is used to design arbitrarily long perfectly periodic mass-spring chains, see [39, Theorem 2], returning to the initial motivation to study bisymmetric Jacobi matrices in [21].

Our strategy is based on considering the gaps in the list Λ instead of considering the elements of the list. Specifically, if $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ is an ordered list, then the *gap sequence* is $g_j = \lambda_j - \lambda_{j+1}$, $j = 1, \dots, n - 1$. Clearly, $g_j > 0$ for $j = 1, \dots, n - 1$ and $\lambda_k = \lambda_n + \sum_{j=k}^{n-1} g_j$, for any $k = 1, \dots, n$. For instance, the gap sequence for the list in [39] is $g_j = \omega^2(2(n - j) - 1)$, $j = 1, \dots, n - 1$, where $\omega > 0$.

We call any positive vector in $\mathbb{R}^{n-1}$ *gap vector*; that is, $\mathbf{g} \in \mathbb{R}^{n-1}$ such that $\mathbf{g} > 0$. If $\mathbf{g} = (g_1, \dots, g_{n-1})$ is a gap vector, then $g_1, \dots, g_{n-1}$ is its associated gap sequence and we define the (flipped) *accumulated gap sequence* as

$$\mu_k(\mathbf{g}) = \sum_{j=k}^{n-1} g_j, \quad k = 1, \dots, n. \quad (4)$$

Clearly, $\{\mu_k(\mathbf{g})\}_{k=1}^n$ is strictly decreasing and moreover $\mu_n(\mathbf{g}) = 0$, because we adopt here the common agreement that empty sums equal 0. Since $g_k = \mu_k(\mathbf{g}) - \mu_{k+1}(\mathbf{g})$, $k = 1, \dots, n-1$, the accumulated gap sequence determines the gap sequence or equivalently the gap vector.

Given a gap vector $\mathbf{g} > 0$, for any real value $\lambda \in \mathbb{R}$ we consider the ordered list $\Lambda(\mathbf{g}; \lambda) = \{\lambda + \mu_k(\mathbf{g})\}_{k=1}^n$ and $J(\mathbf{g}; \lambda)$, the unique bisymmetric Jacobi matrix realizing the list $\Lambda(\mathbf{g}; \lambda)$. Therefore, λ is the lowest eigenvalue of $J(\mathbf{g}; \lambda)$ and moreover

$$J(\mathbf{g}; \lambda) = \lambda I_n + J(\mathbf{g}), \quad (5)$$

where $J(\mathbf{g}) = J(\mathbf{g}; 0)$ is the unique bisymmetric Jacobi matrix realizing the accumulated gap list $\Lambda(\mathbf{g}) = \Lambda(\mathbf{g}; 0) = \{\mu_k(\mathbf{g})\}_{k=1}^n$. The results mentioned in Section 1 ensure that, given $n \geq 2$, for any $\mathbf{g} \in (0, +\infty)^{n-1}$, there exist unique vectors $\mathbf{a}(\mathbf{g}) \in \mathbb{R}^{\lceil \frac{n}{2} \rceil}$ and $\mathbf{b}(\mathbf{g}) \in \mathbb{R}^{\lfloor \frac{n}{2} \rfloor}$ such that $\mathbf{b}(\mathbf{g}) > 0$ and

$$J(\mathbf{g}) = \begin{bmatrix} a_1(\mathbf{g}) & b_1(\mathbf{g}) & 0 & \cdots & 0 \\ b_1(\mathbf{g}) & a_2(\mathbf{g}) & b_2(\mathbf{g}) & \ddots & \vdots \\ 0 & b_2(\mathbf{g}) & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_2(\mathbf{g}) & b_1(\mathbf{g}) \\ 0 & \cdots & 0 & b_1(\mathbf{g}) & a_1(\mathbf{g}) \end{bmatrix}. \quad (6)$$

The extreme property of the Rayleigh quotient determines the inequalities $\mu_n(\mathbf{g}) = 0 < a_k(\mathbf{g}), b_k(\mathbf{g}) < \mu_1(\mathbf{g})$, $k = 1, \dots, \lceil \frac{n}{2} \rceil$, and hence $\mathbf{a}(\mathbf{g}) > 0$, see [26, Lemma 2]. In particular, $J(\mathbf{g}) > 0$ and hence, $J(\mathbf{g}; \lambda)$ is nonnegative for $\lambda \geq 0$. Moreover, for any $\mathbf{g} > 0$, the set of values $\lambda \in \mathbb{R}$, such that the bisymmetric Jacobi matrix $J(\mathbf{g}; \lambda)$ is also nonnegative, is the closed unbounded interval $[-\Phi(\mathbf{g}), +\infty)$, where the *threshold value* $\Phi(\mathbf{g})$ equals $a'_{\lceil \frac{n}{2} \rceil}(\mathbf{g})$, since $a'_{\lceil \frac{n}{2} \rceil}(\mathbf{g}) = \min_{j=1, \dots, \lceil \frac{n}{2} \rceil} \{a_j(\mathbf{g})\}$, and hence $\Phi(\mathbf{g}) > 0$.

If $\Theta \in \mathbb{R}$ satisfies that $\Theta \leq \Phi(\mathbf{g})$ and $\lambda \geq -\Theta$, then $J(\mathbf{g}; \lambda) \geq 0$, whereas when $\Theta \geq \Phi(\mathbf{g})$ if $J(\mathbf{g}; \lambda) \geq 0$, necessarily $\lambda \geq -\Theta$. Therefore, any lower bound of the threshold value leads to a sufficient condition for bisymmetric realizability and any upper bound produces a necessary condition. As immediate consequences of the identity (4) and the above reasonings we have the following result.

Proposition 3.1. *Given $\mathbf{g} > 0$, $J(\mathbf{g}; \lambda)$ is nonnegative when $\lambda \geq 0$. Moreover, if for some λ the matrix $J(\mathbf{g}; \lambda)$ is nonnegative, then it has null trace iff $\mathbf{a}(\mathbf{g}) = a\mathbf{e}$, $a > 0$ and then $\Phi(\mathbf{g}) = -\lambda = a$.*

Later, in Proposition 3.5 we characterize when the conditions of the above proposition hold; that is, when $J(\mathbf{g}; \lambda)$ has null diagonal entries. As we will see, this corresponds to a certain symmetry of the gap sequence.

Remark 1. Instead of basing our reasonings on gap sequence and the lowest eigenvalue, we could also consider $\mu = \lambda_1$ and hence, $\lambda_k = \mu - \mu_1(\mathbf{g}) + \mu_k(\mathbf{g})$, $k = 1, \dots, n$ and $\widehat{\mathbf{J}}(\mathbf{g}; \mu)$ the unique bisymmetric Jacobi matrix realizing the list $\{\mu - \mu_1(\mathbf{g}) + \mu_k(\mathbf{g})\}_{k=1}^n$. Therefore, $\widehat{\mathbf{J}}(\mathbf{g}; \mu) = (\mu - \mu_1(\mathbf{g}))\mathbf{I}_n + \mathbf{J}(\mathbf{g})$ and hence $\widehat{\mathbf{J}}(\mathbf{g}; \mu)$ is nonnegative iff $\mu \in [\mu_1(\mathbf{g}) + \Phi(\mathbf{g}), +\infty)$. Notice that if we define $\hat{\mathbf{a}}(\mathbf{g}) = \mathbf{a}(\mathbf{g}) - \mu_1(\mathbf{g})\mathbf{e}$, then $\hat{\mathbf{a}}(\mathbf{g}) < 0$ and $\widehat{\mathbf{J}}(\mathbf{g}; \mu) = \mu\mathbf{I}_n + \mathbf{J}(\hat{\mathbf{a}}(\mathbf{g}), \mathbf{b}(\mathbf{g}))$. Moreover, $\mathbf{J}(-\hat{\mathbf{a}}(\mathbf{g}), \mathbf{b}(\mathbf{g}))$ is the unique bisymmetric Jacobi matrix realizing the list $\left\{\sum_{j=1}^{n-k} g_j\right\}_{k=1}^n$ and it is nonnegative.

Summing up, the raised problem is reduced to obtaining the functions $\mathbf{a}: (0, +\infty)^{\lceil \frac{n}{2} \rceil} \rightarrow (0, +\infty)$ and $\mathbf{b}: (0, +\infty)^{\lfloor \frac{n}{2} \rfloor} \rightarrow (0, +\infty)$ such that for any $\mathbf{g} > 0$, $\mathbf{J}(\mathbf{g})$ determined by (6) is the unique bisymmetric Jacobi matrix realizing the accumulated gap list $\Lambda(\mathbf{g}) = \{\mu_1(\mathbf{g}), \dots, \mu_n(\mathbf{g})\}$.

The nonnegativeness of $\mathbf{J}(\mathbf{g}; \lambda)$ depends only on $\mathbf{a}(\mathbf{g})$. In fact, the real problem is to estimate the size of the interval $[-\Phi(\mathbf{g}), 0)$.

The results of Section 2 determine that the problem is completely solved for linear sequences of gaps; that is, for gaps forming an arithmetic progression.

Assume that the entries of the gap vector $\mathbf{g} \in \mathbb{R}^{n-1}$ are given in arithmetic progression, specifically $g_k = -2(ck + b)$, $k = 1, \dots, n-1$, where $b, c \in \mathbb{R}$. Clearly, the gap sequence is strictly increasing if $c < 0$, strictly decreasing when $c > 0$ and constant when $c = 0$. Since the gaps must be positive, it is necessary that $|b| + |c| > 0$ and also that $b < -c$ when $c < 0$ and $b < c(1-n)$ when $c \geq 0$. So, assuming that $b < 0$ when $c = 0$, we have that

$$g_k = -2(ck + b), \quad k = 1, \dots, n-1, \quad |b| + |c| > 0 \text{ and } c^{-1}b \notin [1-n, -1]. \quad (7)$$

The accumulated gap sequence is given by

$$\mu_k(\mathbf{g}) = ck^2 + (2b - c)k + a - 2b, \quad k = 1, \dots, n,$$

where $a = (1-n)(nc + 2b) = \frac{1}{2}(n-1)(g_1 + g_{n-1})$. Notice that $a > 0$, $\mu_1 = a$, $\mu_n = 0$ and moreover, $\{\mu_k(\mathbf{g})\}_{k=1}^n$ is strictly decreasing.

Theorem 2.1 shows the entries of the bisymmetric realization in terms of the parameters b and c . We now translate its results in terms of the gaps.

Corollary 3.2. *Given $n \in \mathbb{N}$, $n \geq 2$, $b, c \in \mathbb{R}$, such that $|c| + |b| > 0$ and $c^{-1}b \notin [1-n, -1]$, and $\mathbf{g} \in \mathbb{R}^{n-1}$ whose components are $g_k = -2(ck + b)$, $k = 1, \dots, n-1$, then the entries of vectors $\mathbf{a}(\mathbf{g}) \in \mathbb{R}^{\lceil \frac{n}{2} \rceil}$ and $\mathbf{b}(\mathbf{g}) \in \mathbb{R}^{\lfloor \frac{n}{2} \rfloor}$ are*

$$\begin{aligned} a_k(\mathbf{g}) &= \frac{1}{2}(n-1)g_{n-1} + (k-1)(g_{n-k} - g_{k+1} + g_k - g_1), \quad k = 1, \dots, \lceil \frac{n}{2} \rceil, \\ b_k(\mathbf{g}) &= \frac{1}{4}\sqrt{k(n-k)g_k g_{n-k}}, \quad k = 1, \dots, \lfloor \frac{n}{2} \rfloor. \end{aligned}$$

Moreover, the threshold value is $\Phi(\mathbf{g}) = \frac{1}{2}(n-1)g_{n-1}$ when $c \geq 0$ and

$$\Phi(\mathbf{g}) = \frac{1}{2}(n-1)g_{n-1} + \frac{(n-2)(g_{\lceil \frac{n}{2} \rceil} - g_1)}{4(g_1 - g_{n-1})^2} ((n-2)(g_1 - g_{\lfloor \frac{n}{2} \rfloor}) + g_1 - g_{n-1}),$$

when $c < 0$.

For periodic gaps related to 1-periodic eigenvalues, we know the following cases: Given $b > 0$,

1. If $g_k = b \sin((2k+1)\theta)$, $k = 1, \dots, n-1$, where $\theta = \frac{\pi}{2(n+1)}$, then the accumulated gap sequence is

$$\mu_k(\mathbf{g}) = \frac{b}{\sin(\theta)} [\cos^2(k\theta) - \cos^2(n\theta)], \quad k = 1, \dots, n.$$

On the other hand, $\mathbf{a}(\mathbf{g}) = 0$, $\mathbf{b}(\mathbf{g}) = \frac{b}{4 \sin(\theta)} \mathbf{e}_{\lfloor \frac{n}{2} \rfloor}$ and hence $\Phi(\mathbf{g}) = 0$. Notice that the gap sequence is symmetric in the sense that $g_{n-k} = g_k$, $k = 1, \dots, n-1$.

2. If $g_k = b \sin((2k-1)\phi)$, $k = 1, \dots, n-1$, where $\phi = \frac{\pi}{2n}$, then the accumulated gap sequence is

$$\mu_k(\mathbf{g}) = \frac{b}{2 \sin(\phi)} [\cos(2(k-1)\phi) + \cos(2\phi)], \quad k = 1, \dots, n,$$

On the other hand, $\mathbf{a}(\mathbf{g}) = \frac{b}{4 \sin(\phi)} (1, 0, \dots, 0)$, $\mathbf{g} = \frac{b}{4 \sin(\phi)} \mathbf{e}_{\lfloor \frac{n}{2} \rfloor}$, and hence $\Phi(\mathbf{g}) = 0$. The symmetry of the gap sequence is given by $g_{n+1-k} = g_k$, $k = 2, \dots, n-1$.

3. If $g_k = b \sin((2k+1)\phi)$, $k = 1, \dots, n-1$, where $\phi = \frac{\pi}{2n}$, then the accumulated gap sequence is

$$\mu_k(\mathbf{g}) = \frac{b \cos^2(k\phi)}{\sin(\phi)}, \quad k = 1, \dots, n,$$

On the other hand, $\mathbf{a}(\mathbf{g}) = \frac{-b}{4 \sin(\phi)} (1, 0, \dots, 0)$, $\mathbf{g} = \frac{b}{4 \sin(\phi)} \mathbf{e}_{\lfloor \frac{n}{2} \rfloor}$ and $\Phi(\mathbf{g}) = \frac{-b}{4 \sin(\phi)}$. Now the symmetry presented by the gap sequence is given by $g_{n-1-k} = g_k$, $k = 1, \dots, n-2$.

In general, obtaining $\mathbf{a}(\mathbf{g})$ is a very difficult problem and the same applies to the threshold value. However, we can obtain an upper bound for $\Phi(\mathbf{g})$ by formulating, in terms of the gap sequence, the necessary conditions for nonnegative realizability given in [20]. Observe first that, for any $k = 1, \dots, \lceil \frac{n}{2} \rceil$, we have that

$$\mu_{n+1-k}(\mathbf{g}) = \mu_k(\mathbf{g}) - \sum_{j=k}^{n-k} g_j \quad \text{and hence} \quad \mu_k(\mathbf{g}) + \mu_{n+1-k}(\mathbf{g}) = 2\mu_k(\mathbf{g}) - \sum_{j=k}^{n-k} g_j.$$

Lemma 3.3. ([20, Theorem 2]) *Given $\mathbf{g} > 0$, if $J(\mathbf{g}; \lambda)$ is nonnegative, then for any $k = 1, \dots, \lceil \frac{n}{2} \rceil$ it is satisfied that*

$$2\Phi(\mathbf{g}) \leq a'_n(\mathbf{g}) + a'_{n-1}(\mathbf{g}) \leq 2\mu_k(\mathbf{g}) - \sum_{j=k}^{n-k} g_j \leq a'_1(\mathbf{g}) + a'_2(\mathbf{g}) < 2\mu_1(\mathbf{g})$$

and hence, $\lambda \geq \frac{1}{2} \sum_{j=k}^{n-k} g_j - \mu_k(\mathbf{g})$, $k = 1, \dots, \lceil \frac{n}{2} \rceil$.

From the above inequalities we directly conclude that when $\lambda \geq -\Phi(\mathbf{g})$; that is, when $J(\mathbf{g}; \lambda)$ is nonnegative, necessarily $\lambda_k(\mathbf{g}) = \lambda + \mu_k \geq \sum_{j=k}^{n-k} g_j$ and hence $\lambda_k(\mathbf{g}) > 0$ for any $k = 1, \dots, \lceil \frac{n}{2} \rceil - 1$ and $\lambda_{\lceil \frac{n}{2} \rceil}(\mathbf{g}) \geq 0$ with strict inequality when n is even. In addition, the inequalities in Lemma 3.3, suggest considering for any $k = 1, \dots, \lceil \frac{n}{2} \rceil$, the function $\psi_k: (0, +\infty)^{n-1} \rightarrow (0, +\infty)$ defined for $\mathbf{g} > 0$ as

$$\psi_k(\mathbf{g}) = \frac{1}{2}(\mu_k(\mathbf{g}) + \mu_{n+1-k}(\mathbf{g})) = \mu_k(\mathbf{g}) - \frac{1}{2} \sum_{j=k}^{n-k} g_j = \sum_{j=k}^{n-1} g_j - \frac{1}{2} \sum_{j=k}^{n-k} g_j \quad (8)$$

and also the function $\Psi: (0, +\infty)^{n-1} \rightarrow (0, +\infty)$,

$$\Psi = \min_{k=1, \dots, \lceil \frac{n}{2} \rceil} \{\psi_k\}. \quad (9)$$

Notice that $\psi_1(\mathbf{g}) = \frac{1}{2} \mu_1(\mathbf{g})$, whereas $\psi_{\lceil \frac{n}{2} \rceil}(\mathbf{g}) = \mu_{\lceil \frac{n}{2} \rceil}(\mathbf{g}) - \frac{1}{4}(1 + (-1)^n)g_{\lceil \frac{n}{2} \rceil}$. In addition, $\Phi(\mathbf{g}) \leq \Psi(\mathbf{g}) < \mu_1(\mathbf{g})$, so $\Psi(\mathbf{g})$ is an upper bound of the threshold value. Unlike the value $\Phi(\mathbf{g})$, the value $\Psi(\mathbf{g})$ is easy to obtain, since it depends only on the gap sequence. Moreover, when $\lambda > -\Psi(\mathbf{g})$ we know, see [20, Theorem 3], that the list $\Lambda(\mathbf{g}; \lambda)$ is realizable by a nonnegative, in fact positive, Jacobi matrix, but we do not know if any of the realizations are bisymmetric. Most of the paper [20] is devoted to obtaining necessary conditions under which $\Psi(\mathbf{g}) \leq \Phi(\mathbf{g})$, or equivalently $\Psi(\mathbf{g}) = \Phi(\mathbf{g})$. We remark that $\Psi(\mathbf{g}) \leq \Phi(\mathbf{g})$; that is, $\Psi(\mathbf{g}) = \Phi(\mathbf{g})$, which means that there exists $\ell = 1, \dots, \lceil \frac{n}{2} \rceil$ such that $\psi_\ell(\mathbf{g}) \leq \Phi(\mathbf{g})$, or equivalently $J(\mathbf{g}, -\psi_\ell(\mathbf{g}))$ is nonnegative. When this happens, then $\Phi(\mathbf{g}) = \Psi(\mathbf{g}) = \psi_\ell(\mathbf{g})$ and for any $k = 1, \dots, \lceil \frac{n}{2} \rceil$ such that $\psi_\ell(\mathbf{g}) < \psi_k(\mathbf{g})$, we know that some diagonal entry of the bisymmetric matrix $J(\mathbf{g}, -\psi_k(\mathbf{g}))$ is negative.

The following result, easy to prove, establishes when $\psi_\ell(\mathbf{g}) \leq \psi_k(\mathbf{g})$.

Lemma 3.4. *Given $\mathbf{g} > 0$, for any $k, \ell = 1, \dots, \lceil \frac{n}{2} \rceil$ we have that $\psi_\ell(\mathbf{g}) \leq \psi_k(\mathbf{g})$ iff*

$$\text{sign}(\ell - k) \sum_{j=\min\{k, \ell\}}^{\max\{k, \ell\}-1} (g_j - g_{n-j}) \geq 0$$

and the inequality is strict when $\psi_\ell(\mathbf{g}) < \psi_k(\mathbf{g})$. In particular, it is satisfied that $\psi_1(\mathbf{g}) = \dots = \psi_{\lceil \frac{n}{2} \rceil}(\mathbf{g})$ iff $g_{n-j} = g_j$, $j = 1, \dots, \lceil \frac{n}{2} \rceil$ in which case $\psi_1(\mathbf{g}) =$

$$\sum_{j=1}^{\lceil \frac{n}{2} \rceil - 1} g_j + \frac{1}{4}(1 + (-1)^n)g_{\lceil \frac{n}{2} \rceil}.$$

The following result is the gap version of [20, Theorem 5] and characterizes when $\psi_1(\mathbf{g}) = \Phi(\mathbf{g})$. It completes the result in Proposition 3.1.

Proposition 3.5. *Given $\mathbf{g} > 0$, we have that $\psi_1(\mathbf{g}) = \Phi(\mathbf{g})$ iff $g_{n-j} = g_j$, $j = 1, \dots, \lceil \frac{n}{2} \rceil$ and then $\mathbf{J}(\mathbf{g}, -\psi_1(\mathbf{g}))$ has null trace and hence, $\mathbf{a}(\mathbf{g}) = \psi_1(\mathbf{g}) \mathbf{e}$.*

Next we obtain necessary conditions for $\Psi(\mathbf{g}) = \Phi(\mathbf{g})$ when $\psi_1(\mathbf{g}) > \Phi(\mathbf{g})$, which represent the gap version of Lemmas 2 and 3 in [20].

Proposition 3.6. *Given $\mathbf{g} > 0$, assume that $\psi_1(\mathbf{g}) > \Phi(\mathbf{g}) = \Psi(\mathbf{g})$ and consider $\ell = 2, \dots, \lceil \frac{n}{2} \rceil$ satisfying that $\Psi(\mathbf{g}) = \psi_\ell(\mathbf{g})$. Then $g_{\ell-1} > g_{n+1-\ell}$, $g_{n-\ell} > g_\ell$ and $\sum_{j=1}^{\ell-1} (g_j - g_{n-j}) > 0$. In particular, when n is even then $\ell < \lceil \frac{n}{2} \rceil$ and $\sum_{j=\ell}^{\lceil \frac{n}{2} \rceil - 1} (g_j - g_{n-j}) < 0$. Moreover, $\mathbf{a}_k(\mathbf{g}) \geq \psi_\ell(\mathbf{g})$, $k = 1, \dots, n$, with equality for at least $n + 1 - \ell$ indices, $a_1(\mathbf{g}) = a_n(\mathbf{g}) = \psi_\ell(\mathbf{g})$ and furthermore $a_2(\mathbf{g}) = a_{n-1}(\mathbf{g}) = \psi_\ell(\mathbf{g})$ except when n is odd and $\ell = \lceil \frac{n}{2} \rceil$.*

Since the diagonal entries of a bisymmetric matrix are symmetric with respect to the secondary diagonal, we obtain the following consequence, that is more precise than the result in [20, Theorem 7].

Corollary 3.7. *Given $\mathbf{g} > 0$, assume that $\psi_2(\mathbf{g}) = \Phi(\mathbf{g}) < \psi_1(\mathbf{g})$. Then, n is odd, $g_1 > g_{n-1}$, $g_{n-2} > g_2$, $\sum_{j=2}^{k-1} (g_j - g_{n-j}) \leq 0$ for $k \geq 4$, and moreover $\mathbf{a}(\mathbf{g}) = \psi_2(\mathbf{g}) \mathbf{e} + (0, \dots, \overset{\lceil \frac{n}{2} \rceil}{a}, \dots, 0)$, where $a > 0$.*

Remark 2. If $n > 4$, there exist $2 \leq \ell_1 < \dots < \ell_k \leq \lceil \frac{n}{2} \rceil$, $k \geq 2$, such that $\psi_{\ell_j}(\mathbf{g}) \leq \Phi(\mathbf{g}) < \psi_1(\mathbf{g})$, $j = 1, \dots, k$, then $\psi_{\ell_1}(\mathbf{g}) = \dots = \psi_{\ell_k}(\mathbf{g})$, $g_{\ell_j-1} > g_{n+1-\ell_j}$, $g_{n-\ell_j} > g_{\ell_j}$, $j = 1, \dots, k$, and

$$\sum_{j=1}^{\ell_1-1} (g_j - g_{n-j}) > 0, \quad \sum_{j=\ell_1}^{\ell_2-1} (g_j - g_{n-j}) = \dots = \sum_{j=\ell_{k-1}}^{\ell_k-1} (g_j - g_{n-j}) = 0.$$

In particular, when n is even, then $\ell_k < \lceil \frac{n}{2} \rceil$ and $\sum_{j=\ell_k}^{\lceil \frac{n}{2} \rceil - 1} (g_j - g_{n-j}) < 0$. Besides, $\mathbf{a}_k(\mathbf{g}) \geq \psi_{\ell_1}(\mathbf{g})$, $k = 1, \dots, n$, with equality for at least $n + 1 - \ell_1$ indices and moreover, $a_1(\mathbf{g}) = a_2(\mathbf{g}) = a_{n-1}(\mathbf{g}) = a_n(\mathbf{g}) = \psi_{\ell_1}(\mathbf{g})$.

In addition to considering the gap sequence to describe realizability, we also use the A. Cantoni and P. Butler characterization of the spectra of bisymmetric matrices, see [9], to reduce the given problem to an equivalent one but of half the size. In fact, this treatment goes back to the original one given by F.R. Gantmacher and M.G. Krein [21].

To do this, given $n \geq 2$, $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^{n-1}$, we consider the vectors $\tilde{\mathbf{e}} = (0, \dots, 0, 1) \in \mathbb{R}^{\lfloor \frac{n}{2} \rfloor - 1}$, $\tilde{\mathbf{a}} = (a_1, \dots, a_{\lfloor \frac{n}{2} \rfloor})$ and $\tilde{\mathbf{b}} = (b_1, \dots, b_{\lfloor \frac{n}{2} \rfloor - 1})$ and the values $a = a_{\lceil \frac{n}{2} \rceil}$, $b = b_{\lfloor \frac{n}{2} \rfloor}$. Then the Jacobi matrix $J(\tilde{\mathbf{a}}, \tilde{\mathbf{b}})$ is the principal submatrix of $J(\mathbf{a}, \mathbf{b})$ with order $\lfloor \frac{n}{2} \rfloor$. Clearly, when $J(\mathbf{a}, \mathbf{b})$ is nonnegative, then $J(\tilde{\mathbf{a}}, \tilde{\mathbf{b}})$ is also nonnegative.

Lemma 3.8 ([9, Lemmas 2 and 3, Theorems 5 and 6]). *Consider $n \geq 2$, $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^{n-1}$ and assume that the Jacobi matrix $J(\mathbf{a}, \mathbf{b})$ is bisymmetric. Then, the following properties hold:*

(i) *If n is even, then*

$$\lambda_{2k-1}(J(\mathbf{a}, \mathbf{b})) = \lambda_k(J(\tilde{\mathbf{a}} + b\tilde{\mathbf{e}}, \tilde{\mathbf{b}})) \text{ and } \lambda_{2k}(J(\mathbf{a}, \mathbf{b})) = \lambda_k(J(\tilde{\mathbf{a}} - b\tilde{\mathbf{e}}, \tilde{\mathbf{b}})),$$

for any $k = 1, \dots, \lceil \frac{n}{2} \rceil$.

(ii) *If n is odd, then $\lambda_{2k}(J(\mathbf{a}, \mathbf{b})) = \lambda_k(J(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}))$ for any $k = 1, \dots, \lfloor \frac{n}{2} \rfloor$, whereas*

$$\lambda_{2k-1}(J(\mathbf{a}, \mathbf{b})) = \lambda_k \left(\begin{bmatrix} J(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}) & \sqrt{2}b\tilde{\mathbf{e}}^\top \\ \sqrt{2}b\tilde{\mathbf{e}} & a \end{bmatrix} \right),$$

for any $k = 1, \dots, \lceil \frac{n}{2} \rceil$.

The spectrum of any bisymmetric Jacobi matrix appears as the union of the spectra of two symmetric Jacobi matrices, so the order of the initial matrix is reduced to a half. We end this section applying the results in the above lemma, to determine the value $a_{\lceil \frac{n}{2} \rceil}(\mathbf{g})$ of the diagonal entry of the matrix $J(\mathbf{g})$. Since $\Phi(\mathbf{g}) \leq a_{\lceil \frac{n}{2} \rceil}(\mathbf{g})$, we obtain a new necessary condition for bisymmetric nonnegative realizability: if $J(\lambda; \mathbf{g}) \geq 0$, then $\lambda \geq -a_{\lceil \frac{n}{2} \rceil}(\mathbf{g})$. Moreover, from Proposition 3.5, if the gap sequence is symmetric, that is $g_{n-k} = g_k$, $k = 1, \dots, n-1$, then the above inequality becomes an equality.

Theorem 3.9. *Given $n \in \mathbb{N}^*$ and $\mathbf{g} \in \mathbb{R}^{n-1}$ such that $\mathbf{g} > 0$, then*

$$a_{\lceil \frac{n}{2} \rceil}(\mathbf{g}) = \begin{cases} \left(2 \sum_{j=1}^m g_{2j-1} \right)^{-1} \sum_{j=1}^m \left(g_{2j-1}^2 + 2g_{2j-1} \sum_{i=2j}^{n-1} g_i \right), & n = 2m, \\ \sum_{j=1}^m g_{2j-1}, & n = 2m + 1, \end{cases}$$

$$b_{\lfloor \frac{n}{2} \rfloor}(\mathbf{g}) = \begin{cases} \frac{1}{2} \sum_{j=1}^m g_{2j-1}, & n = 2m, \\ \sqrt{\frac{1}{2} \sum_{j=1}^m g_{2j-1} \sum_{i=j}^m g_{2i}}, & n = 2m + 1. \end{cases}$$

Proof. Consider $m = \lfloor \frac{n}{2} \rfloor$ and for any $k = 1, \dots, m$, the following values related to the values of the symmetric functions at the accumulated gaps:

$$O_k(\mathbf{g}) = \sum_{1 \leq i_1 < \dots < i_k \leq m} \mu_{2i_1-1}(\mathbf{g}) \cdots \mu_{2i_m-1}(\mathbf{g}),$$

$$E_k(\mathbf{g}) = \sum_{1 \leq i_1 < \dots < i_k \leq m} \mu_{2i_1}(\mathbf{g}) \cdots \mu_{2i_m}(\mathbf{g}).$$

Notice that as $\mu_n(\mathbf{g}) = 0$, then when n is even $E_m(\mathbf{g}) = 0$ and more generally

$$E_k(\mathbf{g}) = \sum_{1 \leq i_1 < \dots < i_k \leq m-1} \mu_{2i_1}(\mathbf{g}) \cdots \mu_{2i_k}(\mathbf{g}),$$

whereas, when n is odd then

$$O_k(\mathbf{g}) = \sum_{1 \leq i_1 < \dots < i_k \leq m+1} \mu_{2i_1-1}(\mathbf{g}) \cdots \mu_{2i_k-1}(\mathbf{g}).$$

On the other hand, we denote by $Q_{\mathbf{g}}(x)$ the characteristic polynomial of the Jacobi matrix

$$J_m(\mathbf{g}) = \begin{bmatrix} a_1(\mathbf{g}) & b_1(\mathbf{g}) & 0 & \cdots & 0 \\ b_1(\mathbf{g}) & a_2(\mathbf{g}) & b_2(\mathbf{g}) & \ddots & \vdots \\ 0 & b_2(\mathbf{g}) & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_{m-1}(\mathbf{g}) & b_{m-1}(\mathbf{g}) \\ 0 & \cdots & 0 & b_{m-1}(\mathbf{g}) & a_m(\mathbf{g}) \end{bmatrix}$$

and by $P_{\mathbf{g}}(x)$ the characteristic polynomial of $J_{m-1}(\mathbf{g})$, the matrix obtained from $J_m(\mathbf{g})$ by removing its last row and its last column. Throughout the proof we assume that

$$Q_{\mathbf{g}}(x) = \sum_{k=0}^m \alpha_k(\mathbf{g}) x^{m-k} \quad \text{and} \quad P_{\mathbf{g}}(x) = \sum_{k=0}^m \beta_k(\mathbf{g}) x^{m-k},$$

where $\alpha_0(\mathbf{g}) = 1$, $\beta_0(\mathbf{g}) = 0$ and $\beta_1(\mathbf{g}) = 1$. In addition,

$$\beta_2(\mathbf{g}) = -(a_1(\mathbf{g}) + \cdots + a_{m-1}(\mathbf{g})) \quad \text{and} \quad \alpha_1(\mathbf{g}) = -(a_1(\mathbf{g}) + \cdots + a_m(\mathbf{g})),$$

which implies that $a_m(\mathbf{g}) = \beta_2(\mathbf{g}) - \alpha_1(\mathbf{g})$.

When n is even, from part (i) of Lemma 3.8, we know that

$$\begin{aligned} \prod_{j=1}^m (x - \mu_{2j-1}(\mathbf{g})) &= \sum_{k=0}^m (\alpha_k(\mathbf{g}) - b_m(\mathbf{g})\beta_k(\mathbf{g})) x^{m-k}, \\ \prod_{j=1}^m (x - \mu_{2j}(\mathbf{g})) &= \sum_{k=0}^m (\alpha_k(\mathbf{g}) + b_m(\mathbf{g})\beta_k(\mathbf{g})) x^{m-k}, \end{aligned}$$

which implies that

$$\alpha_k(\mathbf{g}) - b_m(\mathbf{g})\beta_k(\mathbf{g}) = (-1)^k O_k(\mathbf{g}), \quad k = 1, \dots, m,$$

$$\alpha_k(\mathbf{g}) + b_m(\mathbf{g})\beta_k(\mathbf{g}) = (-1)^k E_k(\mathbf{g}), \quad k = 1, \dots, m,$$

and hence that

$$2b_m(\mathbf{g})\beta_k(\mathbf{g}) = (-1)^k (E_k(\mathbf{g}) - O_k(\mathbf{g})), \quad k = 1, \dots, m,$$

$$2\alpha_k(\mathbf{g}) = (-1)^k (E_k(\mathbf{g}) + O_k(\mathbf{g})), \quad k = 1, \dots, m,$$

Since $\beta_1(\mathbf{g}) = 1$, we obtain

$$2b_m(\mathbf{g}) = O_1(\mathbf{g}) - E_1(\mathbf{g}) = \sum_{j=1}^m (\mu_{2j-1}(\mathbf{g}) - \mu_{2j}(\mathbf{g})) = \sum_{j=1}^m g_{2j-1},$$

and hence,

$$\begin{aligned} a_m(\mathbf{g}) &= \beta_2(\mathbf{g}) - \alpha_1(\mathbf{g}) = \frac{E_2(\mathbf{g}) - O_2(\mathbf{g})}{O_1(\mathbf{g}) - E_1(\mathbf{g})} + \frac{1}{2}(E_1(\mathbf{g}) + O_1(\mathbf{g})) \\ &= \frac{2E_2(\mathbf{g}) - 2O_2(\mathbf{g}) + O_1(\mathbf{g})^2 - E_1(\mathbf{g})^2}{2(O_1(\mathbf{g}) - E_1(\mathbf{g}))} \\ &= \left(2 \sum_{j=1}^m g_{2j-1}\right)^{-1} \sum_{j=1}^m (\mu_{2j-1}^2(\mathbf{g}) - \mu_{2j}^2(\mathbf{g})) \\ &= \left(2 \sum_{j=1}^m g_{2j-1}\right)^{-1} \sum_{j=1}^m (g_{2j-1}^2 + 2g_{2j-1}\mu_{2j}(\mathbf{g})). \end{aligned}$$

When n is odd, the characteristic polynomial of the Jacobi matrix

$$\begin{bmatrix} a_1(\mathbf{g}) & b_1(\mathbf{g}) & 0 & \cdots & 0 & 0 \\ b_1(\mathbf{g}) & a_2(\mathbf{g}) & b_2(\mathbf{g}) & \ddots & \vdots & \vdots \\ 0 & b_2(\mathbf{g}) & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & a_{m-1}(\mathbf{g}) & b_{m-1}(\mathbf{g}) & 0 \\ 0 & \cdots & 0 & b_{m-1}(\mathbf{g}) & a_m(\mathbf{g}) & \sqrt{2}b_m(\mathbf{g}) \\ 0 & \cdots & 0 & 0 & \sqrt{2}b_m(\mathbf{g}) & a_{m+1}(\mathbf{g}) \end{bmatrix}$$

is

$$(x - a_{m-1}(\mathbf{g}))Q_{\mathbf{g}}(x) - 2b_m^2(\mathbf{g})P_{\mathbf{g}}(x) = x^{m+1} + \sum_{k=0}^m \gamma_{k+1}(\mathbf{g})x^{m-k},$$

$$(x - a_{m-1}(\mathbf{g}))Q_{\mathbf{g}}(x) - 2b_m^2(\mathbf{g})P(x) = x^{m+1} + \sum_{k=1}^{m+1} \gamma_k(\mathbf{g})x^{m+1-k},$$

where $\gamma_{m+1}(\mathbf{g}) = -a_{m+1}(\mathbf{g})\alpha_m(\mathbf{g}) - 2b_m^2(\mathbf{g})\beta_m(\mathbf{g})$ and

$$\gamma_k(\mathbf{g}) = \alpha_k(\mathbf{g}) - a_{m+1}(\mathbf{g})\alpha_{k-1}(\mathbf{g}) - 2b_m^2(\mathbf{g})\beta_{k-1}(\mathbf{g}), \quad k = 1, \dots, m$$

From part (ii) of Lemma 3.8, we know that

$$\begin{aligned} \prod_{j=1}^m (x - \mu_{2j}(\mathbf{g})) &= Q_{\mathbf{g}}(x) = \sum_{k=0}^m \alpha_k(\mathbf{g})x^{m-k}, \\ \prod_{j=1}^{m+1} (x - \mu_{2j-1}(\mathbf{g})) &= x^{m+1} + \sum_{k=1}^{m+1} \gamma_k(\mathbf{g})x^{m+1-k}, \end{aligned}$$

and hence $\gamma_{m+1}(\mathbf{g}) = 0$ and

$$\alpha_k(\mathbf{g}) = (-1)^k E_k(\mathbf{g}), \quad k = 1, \dots, m,$$

$$\gamma_k(\mathbf{g}) = (-1)^k O_k(\mathbf{g}), \quad k = 1, \dots, m;$$

which imply that

$$2b_m^2(\mathbf{g})\beta_0(\mathbf{g}) = O_1(\mathbf{g}) - E_1(\mathbf{g}) - a_{m+1}(\mathbf{g})\alpha_0(\mathbf{g}),$$

$$2b_m^2(\mathbf{g})\beta_{k-1}(\mathbf{g}) = (-1)^k [E_k(\mathbf{g}) - O_k(\mathbf{g}) + a_{m+1}(\mathbf{g})E_{k-1}(\mathbf{g})], \quad k = 2, \dots, m,$$

$$2b_m^2(\mathbf{g})\beta_m(\mathbf{g}) = (-1)^{m+1} a_{m+1}(\mathbf{g})E_m(\mathbf{g}).$$

Taking into account that $\beta_0(\mathbf{g}) = 0$ and that $\alpha_0(\mathbf{g}) = 1$, from the first equation we obtain that

$$a_{m+1}(\mathbf{g}) = O_1(\mathbf{g}) - E_1(\mathbf{g}) = \sum_{j=1}^m g_{2j-1}.$$

On the other hand, bearing in mind that $\beta_1(\mathbf{g}) = 1$, taking $k = 2$ we obtain that

$$2b_m^2(\mathbf{g}) = E_2(\mathbf{g}) - O_2(\mathbf{g}) + E_1(\mathbf{g})(O_1(\mathbf{g}) - E_1(\mathbf{g})).$$

which determines the value of $b_m(\mathbf{g})$. □

As a consequence, we have the following upper bound of $\Phi(\mathbf{g})$, that can be seen as a necessary condition for nonnegativeness.

Corollary 3.10. *Given $n \in \mathbb{N}^*$ and $\mathbf{g} \in \mathbb{R}^{n-1}$ such that $\mathbf{g} > 0$, then*

$$\Phi(\mathbf{g}) \leq \begin{cases} \left(2 \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} g_{2j-1}\right)^{-1} \left(\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} g_{2j-1}^2 + 2 \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} g_{2j-1} \sum_{i=2j}^{n-1} g_i\right), & \text{when } n \text{ is even,} \\ \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} g_{2j-1}, & \text{when } n \text{ is odd.} \end{cases}$$

More precise results are available when the gap vector is symmetric.

Corollary 3.11. *Given $n \in \mathbb{N}^*$ and $\mathbf{g} \in \mathbb{R}^{n-1}$ such that $\mathbf{g} > 0$ and $g_{n-k} = g_k$, $k = 1, \dots, \lfloor \frac{n}{2} \rfloor$, then*

$$\Phi(\mathbf{g}) = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} g_j - \frac{g_{\lfloor \frac{n}{2} \rfloor}}{4} (1 + (-1)^n).$$

Moreover $\mathbf{a}(\mathbf{g}) = \Phi(\mathbf{g}) \mathbf{e}$, whereas

$$b_{\lfloor \frac{n}{2} \rfloor}(\mathbf{g}) = \begin{cases} \frac{gm}{4}(1 - (-1)^m) + \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} g_{2j-1}, & n = 2m, \\ \sqrt{\frac{1}{2} \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} g_{2j-1}^2 + \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} g_{2j-1} \sum_{i=2j}^m g_i}, & n = 2m + 1. \end{cases}$$

In particular, when the gap sequence is constant, $\mathbf{g} = g\mathbf{e}$, then the above formula becomes

$$\Phi(\mathbf{g}) = \frac{g}{2}(n-1) \quad \text{and} \quad b_{\lfloor \frac{n}{2} \rfloor} = \frac{g}{2} \sqrt{\lfloor \frac{n}{2} \rfloor \left(n - \lfloor \frac{n}{2} \rfloor \right)}.$$

In fact, from Corollary 3.2, we know that $b_k(\mathbf{g}) = \frac{g}{4} \sqrt{k(n-k)}$, $k = 1, \dots, \lfloor \frac{n}{2} \rfloor$

4. Low order bisymmetric realizations

As is shown in proof of Theorem 3.9, it would be possible to obtain the coefficients of the bisymmetric Jacobi matrix $\mathbf{J}(\mathbf{g})$ either from the identities

$$\begin{aligned} 2b_m(\mathbf{g})\beta_k(\mathbf{g}) &= (-1)^k (E_k(\mathbf{g}) - O_k(\mathbf{g})), \quad k = 1, \dots, m, \\ 2\alpha_k(\mathbf{g}) &= (-1)^k (E_k(\mathbf{g}) + O_k(\mathbf{g})), \quad k = 1, \dots, m, \end{aligned} \tag{10}$$

when $n = 2m$, in which case $E_m(\mathbf{g}) = 0$, or

$$\begin{aligned} \alpha_k(\mathbf{g}) &= (-1)^k E_k(\mathbf{g}), \quad k = 1, \dots, m, \\ 2b_m^2(\mathbf{g})\beta_0(\mathbf{g}) &= O_1(\mathbf{g}) - E_1(\mathbf{g}) - a_{m+1}(\mathbf{g})\alpha_0(\mathbf{g}), \\ 2b_m^2(\mathbf{g})\beta_{k-1}(\mathbf{g}) &= (-1)^k [E_k(\mathbf{g}) - O_k(\mathbf{g}) + a_{m+1}(\mathbf{g})E_{k-1}(\mathbf{g})], \quad k = 2, \dots, m. \\ 2b_m^2(\mathbf{g})\beta_m(\mathbf{g}) &= (-1)^{m+1} a_{m+1}(\mathbf{g})E_m(\mathbf{g}), \end{aligned} \tag{11}$$

when $n = 2m + 1$. In both cases

$$\det(x\mathbf{I} - \mathbf{J}_m(\mathbf{g})) = \sum_{k=0}^m \alpha_k(\mathbf{g})x^{m-k} \quad \text{and} \quad \det(x\mathbf{I} - \mathbf{J}_{m-1}(\mathbf{g})) = \sum_{k=0}^m \beta_k(\mathbf{g})x^{m-k},$$

which imply that $\alpha_0(\mathbf{g}) = 1$, $\beta_0(\mathbf{g}) = 0$ and $\beta_1(\mathbf{g}) = 1$. However, the high non-linearity relation between a 's and b 's with α 's and β 's, which grows with m , makes it impossible in practice to determine the vectors $\mathbf{a}(\mathbf{g}) \in \mathbb{R}^{\lfloor \frac{n}{2} \rfloor}$ and $\mathbf{g} \in \mathbb{R}^{\lfloor \frac{n}{2} \rfloor}$.

We end this paper obtaining these vectors for low orders, $n \leq 6$, when the complexity of the above mentioned relations still allow us to solve them. We proceed by increasing the order, going from the easy cases to very complex ones, thus showing the difficulties we must face in the general case.

4.1. Nonnegative Bisymmetric Jacobi Matrices of order $n = 2$

In this case, $m = 1$, and Theorem 3.9 gives us all the information. For any $\mathbf{g} = g > 0$, we have that $\Lambda(\mathbf{g}) = \{g, 0\}$, $\psi_1(\mathbf{g}) = \frac{g}{2}$, $a_1(g) = b_1(g) = \frac{g}{2}$. Moreover, $\Phi(g) = \Psi(\mathbf{g}) = \frac{g}{2}$. Of course, this case could have been solved directly without any of the previous results.

The only bisymmetric Jacobi matrices are given by $J(a, b) = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$, where $a, b \in \mathbb{R}$ and $b > 0$. Since $\Lambda(J(a, b)) = \{a + b, a - b\}$, then $\lambda_1 = a + b$, $\lambda_2 = a - b$, which implies that $g = 2b$. Moreover, if $\lambda = \lambda_2$, then $a = \lambda + \frac{g}{2}$, $b = \frac{g}{2}$ and hence

$$J(a, b) = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{g}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad J(\mathbf{g}) = \frac{g}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Therefore, $J(a, b)$ is nonnegative iff $a \geq 0$; that is, iff $\lambda \geq -\frac{g}{2}$.

4.2. Nonnegative Bisymmetric Jacobi Matrices of order $n = 3$

Given $\mathbf{g} = (g_1, g_2) > 0$, then $\Lambda(\mathbf{g}) = \{g_1 + g_2, g_2, 0\}$, $\psi_1(\mathbf{g}) = \frac{1}{2}(g_1 + g_2)$, $\psi_2(\mathbf{g}) = g_2$, which implies that $\psi_1(\mathbf{g}) = \psi_2(\mathbf{g})$ iff $g_1 = g_2$.

From Theorem 3.9 we have $a_2(\mathbf{g}) = g_1$ and $b_1(\mathbf{g}) = \sqrt{\frac{g_1 g_2}{2}}$. To obtain $a_1(\mathbf{g})$ we use the identity $a_m(\mathbf{g}) = \beta_{m-2}(\mathbf{g}) - \alpha_{m-1}(\mathbf{g})$. Taking into account that $\beta_{-1}(\mathbf{g}) = 0$ and $\alpha_0 = -E_1(\mathbf{g}) = -g_2$, we obtain that $a_1(\mathbf{g}) = g_2$ and hence,

$$J(\mathbf{g}) = \begin{bmatrix} g_2 & \sqrt{\frac{g_1 g_2}{2}} & 0 \\ \sqrt{\frac{g_1 g_2}{2}} & g_1 & \sqrt{\frac{g_1 g_2}{2}} \\ 0 & \sqrt{\frac{g_1 g_2}{2}} & g_2 \end{bmatrix}.$$

Moreover, $\Phi(\mathbf{g}) = \min\{g_1, g_2\} \leq \Psi(\mathbf{g}) = \frac{1}{2} \min\{g_1 + g_2, 2g_2\}$ and the equality holds iff $g_2 \leq g_1$. In this case, $\Phi(\mathbf{g}) = \psi_1(\mathbf{g})$ iff $g_2 = g_1$, or equivalently $a_1(\mathbf{g}) = a_2(\mathbf{g})$; whereas $\Phi(\mathbf{g}) = \psi_2(\mathbf{g})$ iff $g_2 < g_1$, in which case we have

$$a(\mathbf{g}) = \psi_2(\mathbf{g})(1, 1) + a(0, 1), \quad \text{where } a = g_1 - g_2 > 0,$$

which is in agreement with the results in Proposition 3.5 and Corollary 3.7. Notice that the case $g_2 > g_1$ shows that that $\Phi(\mathbf{g}) < \Psi(\mathbf{g})$ can happen.

4.3. Nonnegative Bisymmetric Jacobi Matrices of order $n = 4$

Given $\mathbf{g} = (g_1, g_2, g_3) > 0$, then $\Lambda(\mathbf{g}) = \{g_1 + g_2 + g_3, g_2 + g_3, g_3, 0\}$ and hence, $\psi_1(\mathbf{g}) = \frac{1}{2}(g_1 + g_2 + g_3)$, $\psi_2(\mathbf{g}) = g_3 + \frac{g_2}{2}$, which implies that $\psi_1(\mathbf{g}) = \psi_2(\mathbf{g})$ iff $g_1 = g_3$. From Theorem 3.9, we have

$$a_2(\mathbf{g}) = \frac{g_1^2 + 2g_1(g_2 + g_3) + g_3^2}{2(g_1 + g_3)} = \frac{1}{2}(g_1 + g_3) + \frac{g_1 g_2}{g_1 + g_3} \quad \text{and} \quad b_2(\mathbf{g}) = \frac{1}{2}(g_1 + g_3).$$

Since $Q_{\mathbf{g}}(x) = (x - a_1(\mathbf{g}))(x - a_2(\mathbf{g})) - b_1^2(\mathbf{g})$ and $P_{\mathbf{g}}(x) = (x - a_1(\mathbf{g}))$, we get $\beta_0(\mathbf{g}) = -a_1(\mathbf{g})$, $\alpha_0(\mathbf{g}) = a_1(\mathbf{g})a_2(\mathbf{g}) - b_1^2(\mathbf{g})$ and $\alpha_1(\mathbf{g}) = -a_1(\mathbf{g}) - a_2(\mathbf{g})$. Moreover,

from (10),

$$\begin{aligned} a_1(\mathbf{g})a_2(\mathbf{g}) - b_1^2(\mathbf{g}) &= \alpha_0(\mathbf{g}) = \frac{1}{2}(E_2(\mathbf{g}) + O_2(\mathbf{g})) = \frac{g_3}{2}(g_1 + g_2 + g_3), \\ a_1(\mathbf{g}) + a_2(\mathbf{g}) &= -\alpha_1(\mathbf{g}) = \frac{1}{2}(E_1(\mathbf{g}) + O_1(\mathbf{g})) = \frac{1}{2}(g_1 + 2g_2 + 3g_3), \end{aligned}$$

which imply that

$$\begin{aligned} a_1(\mathbf{g}) &= \frac{1}{2}(g_1 + 2g_2 + 3g_3) - \frac{1}{2}(g_1 + g_3) - \frac{g_1g_2}{g_1 + g_3} = \frac{g_3(g_1 + g_2 + g_3)}{g_1 + g_3}, \\ b_1^2(\mathbf{g}) &= \frac{g_3(g_1 + g_2 + g_3)}{2(g_1 + g_3)}(2a_2(\mathbf{g}) - (g_1 + g_3)) = \frac{g_1g_2g_3(g_1 + g_2 + g_3)}{(g_1 + g_3)^2}. \end{aligned}$$

In addition,

$$\begin{aligned} \Phi(\mathbf{g}) &= \frac{1}{2(g_1 + g_3)} \min \left\{ (g_1 + g_3)^2 + 2g_1g_2, 2g_3(g_1 + g_2 + g_3) \right\}, \\ \Psi(\mathbf{g}) &= \frac{1}{2} [g_2 + g_3 + \min \{g_1, g_3\}]. \end{aligned}$$

Therefore, $\Psi(\mathbf{g}) = \psi_1(\mathbf{g})$ iff $g_1 \leq g_3$ and then

$$\Psi(\mathbf{g}) - \Phi(\mathbf{g}) = \frac{1}{2}(g_1 + g_2 + g_3) - \frac{g_1 + g_3}{2} - \frac{g_1g_2}{g_1 + g_3} = \frac{g_2(g_3 - g_1)}{2(g_1 + g_3)} \geq 0,$$

whereas when $g_3 < g_1$, then

$$\Psi(\mathbf{g}) - \Phi(\mathbf{g}) = g_3 + \frac{g_2}{2} - g_3 - \frac{g_2g_3}{g_1 + g_3} = \frac{g_2(g_1 - g_3)}{2(g_1 + g_3)} > 0.$$

So, $\Psi(\mathbf{g}) = \Phi(\mathbf{g})$ iff $g_1 = g_3$ in which case,

$$a_1(\mathbf{g}) = a_2(\mathbf{g}) = g_1 + \frac{g_2}{2}, \quad b_1(\mathbf{g}) = \frac{\sqrt{g_2(2g_1 + g_2)}}{2}, \quad b_2(\mathbf{g}) = g_1.$$

4.4. Nonnegative Bisymmetric Jacobi Matrices of order $n = 5$

Given $\mathbf{g} = (g_1, g_2, g_3, g_4) > 0$, then

$$\Lambda(\mathbf{g}) = \{g_1 + g_2 + g_3 + g_4, g_2 + g_3 + g_4, g_3 + g_4, g_4, 0\}$$

and $\psi_1(\mathbf{g}) = \frac{1}{2}(g_1 + g_2 + g_3 + g_4)$, $\psi_2(\mathbf{g}) = g_4 + \frac{1}{2}(g_2 + g_3)$ and $\psi_3(\mathbf{g}) = g_3 + g_4$, which implies that $\psi_1(\mathbf{g}) = \psi_2(\mathbf{g}) = \psi_3(\mathbf{g})$ iff $g_1 = g_4$ and $g_2 = g_3$ in which case

$\Psi(\mathbf{g}) = g_1 + g_2$. From Theorem 3.9, we have

$$\begin{aligned} a_3(\mathbf{g}) &= g_1 + g_3, \\ b_2(\mathbf{g}) &= \sqrt{\frac{g_1(g_2 + g_4) + g_3g_4}{2}}, \end{aligned}$$

and the remaining equations from the system (11) are

$$\begin{aligned} -(a_1(\mathbf{g}) + a_2(\mathbf{g})) &= \alpha_1(\mathbf{g}) = -E_1(\mathbf{g}) = -(g_2 + g_3 + 2g_4), \\ a_1(\mathbf{g})a_2(\mathbf{g}) - b_1^2(\mathbf{g}) &= \alpha_2(\mathbf{g}) = E_2(\mathbf{g}) = g_4(g_2 + g_3 + g_4), \\ -2b_2^2(\mathbf{g})a_1(\mathbf{g}) &= 2b_2^2(\mathbf{g})\beta_2(\mathbf{g}) = -a_3(\mathbf{g})g_4(g_2 + g_3 + g_4). \end{aligned}$$

Then we easily obtain that

$$\begin{aligned} a_1(\mathbf{g}) &= \frac{g_4(g_1 + g_3)(g_2 + g_3 + g_4)}{g_1(g_2 + g_4) + g_3g_4}, \\ a_2(\mathbf{g}) &= \frac{g_4^2(g_1 + g_3) + g_1g_2(g_2 + g_3 + 2g_4)}{g_1(g_2 + g_4) + g_3g_4}, \\ b_1(\mathbf{g}) &= \frac{\sqrt{g_1g_2g_3g_4(g_1 + g_2 + g_3)(g_2 + g_3 + g_4)}}{g_1(g_2 + g_4) + g_3g_4}, \end{aligned}$$

and hence

$$\Phi(\mathbf{g}) = \frac{1}{g_1(g_2 + g_4) + g_3g_4} \min \left\{ \begin{aligned} &g_4(g_1 + g_3)(g_2 + g_3 + g_4), \\ &g_4^2(g_1 + g_3) + g_1g_2(g_2 + g_3 + 2g_4), \\ &(g_1 + g_3)(g_1(g_2 + g_4) + g_3g_4) \end{aligned} \right\}.$$

When $g_4 = g_1$, then $\psi_1(\mathbf{g}) = g_1 + \frac{1}{2}(g_2 + g_3)$, $\psi_2(\mathbf{g}) = g_1 + \frac{1}{2}(g_2 + g_3)$, $\psi_3(\mathbf{g}) = g_1 + g_3$ and $\Phi(\mathbf{g}) = g_1 + \min \{g_2, g_3\}$. If $g_3 \neq g_2$, then

$$\begin{aligned} g_1 + \frac{1}{2}(g_2 + g_3) &= \Psi(\mathbf{g}) > \Phi(\mathbf{g}) = g_1 + g_2, \quad \text{when } g_3 > g_2, \\ g_1 + g_3 &= \Psi(\mathbf{g}) = \Phi(\mathbf{g}) = g_1 + g_3, \quad \text{when } g_3 < g_2. \end{aligned}$$

4.5. Nonnegative Bisymmetric Jacobi Matrices of order $n = 6$

Given $\mathbf{g} = (g_1, g_2, g_3, g_4, g_5) > 0$, then

$$\Lambda(\mathbf{g}) = \{g_1 + g_2 + g_3 + g_4 + g_5, g_2 + g_3 + g_4 + g_5, g_3 + g_4 + g_5, g_4 + g_5, g_5, 0\}$$

and

$$\begin{aligned} \psi_1(\mathbf{g}) &= \frac{1}{2}(g_1 + g_2 + g_3 + g_4 + g_5) \\ \psi_2(\mathbf{g}) &= g_5 + \frac{1}{2}(g_2 + g_3 + g_4) \\ \psi_3(\mathbf{g}) &= g_4 + g_5 + \frac{g_3}{2}, \end{aligned}$$

which implies that $\psi_1(\mathbf{g}) = \psi_2(\mathbf{g}) = \psi_3(\mathbf{g})$ iff $g_1 = g_5$ and $g_2 = g_4$ in which case $\Psi(\mathbf{g}) = g_1 + g_2 + \frac{g_3}{2}$. From Theorem 3.9, we have

$$a_3(\mathbf{g}) = \frac{g_1^2 + g_3^2 + g_5^2 + 2(g_1(g_2 + g_3 + g_4 + g_5) + g_3(g_4 + g_5))}{2(g_1 + g_3 + g_5)},$$

$$b_3(\mathbf{g}) = \frac{1}{2}(g_1 + g_3 + g_5),$$

and the remaining equations from the system (10) are

$$\begin{aligned} 2b_3(\mathbf{g})(a_1(\mathbf{g}) + a_2(\mathbf{g})) &= O_2(\mathbf{g}) - E_2(\mathbf{g}), \\ 2b_3(\mathbf{g})(a_1(\mathbf{g})a_2(\mathbf{g}) - b_1^2(\mathbf{g})) &= O_3(\mathbf{g}), \\ 2(a_1(\mathbf{g})a_2(\mathbf{g}) + a_1(\mathbf{g})a_3(\mathbf{g}) + a_2(\mathbf{g})a_3(\mathbf{g}) - b_1^2(\mathbf{g}) - b_2^2(\mathbf{g})) &= E_2(\mathbf{g}) + O_2(\mathbf{g}), \\ 2(a_1(\mathbf{g})a_2(\mathbf{g})a_3(\mathbf{g}) - a_3(\mathbf{g})b_1^2(\mathbf{g}) - a_1(\mathbf{g})b_2^2(\mathbf{g})) &= O_3(\mathbf{g}). \end{aligned}$$

Substituting the first and the second identities in the third one, we obtain

$$\begin{aligned} b_1^2(\mathbf{g}) &= a_1(\mathbf{g})a_2(\mathbf{g}) - \frac{O_3(\mathbf{g})}{2b_3(\mathbf{g})}, \\ b_2^2(\mathbf{g}) &= \frac{O_3(\mathbf{g})}{2b_3(\mathbf{g})} + \frac{a_3(\mathbf{g})}{2b_3(\mathbf{g})}(O_2(\mathbf{g}) - E_2(\mathbf{g})) - \frac{1}{2}(E_2(\mathbf{g}) + O_2(\mathbf{g})). \end{aligned}$$

After some calculations we finally obtain that,

$$\begin{aligned} a_1(\mathbf{g}) &= \frac{g_5(g_3 + g_4 + g_5)(g_1 + g_2 + g_3 + g_4 + g_5)(g_1g_2 + g_1g_4 + g_3g_4)}{g_1(g_1 + g_2 + g_3)(g_2g_3 + g_2g_5 + g_4g_5) + g_5(g_3 + g_4 + g_5)(g_1g_2 + g_1g_4 + g_3g_4)}, \\ a_2(\mathbf{g}) &= \frac{\rho(\mathbf{g})}{g_1(g_1 + g_2 + g_3)(g_2g_3 + g_2g_5 + g_4g_5) + g_5(g_3 + g_4 + g_5)(g_1g_2 + g_1g_4 + g_3g_4)}, \end{aligned}$$

where

$$\begin{aligned} \rho(\mathbf{g}) &= g_1^3g_2g_3(g_3 + g_4 + 2g_5) \\ &+ (g_5^2(g_2 + g_4)(g_1^3 + 2g_2 + 3g_3 + 2g_4 + 2g_5 + g_5^2 + 2g_5(g_2 + 2g_3 + g_4)) \\ &+ g_1^2g_2g_3g_5(4g_2 + 6g_3 + 4g_4 + 3g_5 + g_2g_3 + (g_2 + 2g_3)(g_3 + g_4)) \\ &+ ((g_2 + 6g_3 + 3g_4)g_2^2 + (6g_3^2 + 7g_3g_4 + 3g_4^2)g_2 + g_4(g_3 + g_4)(3g_3 + g_4))g_5^2 \\ &+ g_1g_2g_3((g_3 + 2g_5)(g_2 + g_3)(g_2 + g_3 + g_4) + 2g_5(g_2 + g_3)) \\ &+ g_3g_4g_5^2(g_3 + g_4 + g_5)^2. \end{aligned}$$

4.6. Some examples

We finish this work showing some specific examples in low dimension that to the best of our knowledge have not been considered in the literature on the subject. Given $n \in \mathbb{N}$, $n = 2, \dots, 6$, and $r > 0$, we will consider two types of gap sequences, determined by

r : $g_k = r^{n-k}$ and $\hat{g}_k = (n-k)^r$, $k = 1, \dots, n-1$. Note that if $r = 1$, then $\{g_k\}$ is constant whereas $\{\hat{g}_k\}$ is linear, cases that have been treated and completely solved in section 2.2.

In order to simplify the presentation we will only study the cases corresponding to $r = 2$. Therefore, both gap sequences are decreasing and the accumulated gap sequences are

$$\mu_k = 2(2^{n-k} - 1) \text{ and } \hat{\mu}_k = \frac{1}{6}(n-k)(n+1-k)(2(n-k)+1), \quad k = 1, \dots, n-1. \quad (12)$$

For the sake of simplicity, in this section we denote as J_n and $\hat{J}_n$ the unique bisymmetric matrices realizing the lists

$$\Lambda_n = \{2(2^{n-k} - 1)\}_{k=1}^n \text{ and } \hat{\Lambda}_n = \left\{ \frac{1}{6}(n-k)(n+1-k)(2(n-k)+1) \right\}_{k=1}^n,$$

respectively. Note that $\Lambda_{n+1} = \{2(2^n - 1)\} \cup \Lambda_n$ and $\hat{\Lambda}_{n+1} = \{\frac{1}{6}n(n+1)(2n+1)\} \cup \hat{\Lambda}_n$. In both cases we also denote by Φ_n and $\hat{\Phi}_n$ the corresponding threshold values and we use the identities obtained in the preceding subsection. The results are the following ones:

- 1) When $n = 2$, $g_1 = 2$; $\hat{g}_1 = 1$, $\Lambda_2 = \{2, 0\}$, $\hat{\Lambda}_2 = \{1, 0\}$ and hence,

$$J_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } \hat{J}_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Moreover, $\Phi = 1$ and $\hat{\Phi} = \frac{1}{2}$.

- 2) When $n = 3$, $g_1 = 2$, $g_2 = 4$, $\hat{g}_1 = 4$, $\hat{g}_2 = 1$, $\Lambda_3 = \{6, 2, 0\}$, $\hat{\Lambda}_3 = \{5, 1, 0\}$ and hence,

$$J_3 = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 2 \end{bmatrix} \text{ and } \hat{J}_3 = \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 4 & \sqrt{2} \\ 0 & \sqrt{2} & 1 \end{bmatrix}.$$

Moreover, $\Phi = 2$ and $\hat{\Phi} = 1$.

- 3) When $n = 4$, $g_1 = 2$, $g_2 = 4$, $g_3 = 8$, $\hat{g}_1 = 9$, $\hat{g}_2 = 4$, $\hat{g}_3 = 1$, $\Lambda_4 = \{14, 6, 2, 0\}$, $\hat{\Lambda}_4 = \{14, 5, 1, 0\}$ and hence,

$$J_4 = \frac{1}{5} \begin{bmatrix} 14 & 4\sqrt{14} & 0 & 0 \\ 4\sqrt{14} & 41 & 25 & 0 \\ 0 & 25 & 41 & 4\sqrt{14} \\ 0 & 0 & 4\sqrt{14} & 14 \end{bmatrix} \text{ and } \hat{J}_4 = \frac{1}{5} \begin{bmatrix} 7 & 3\sqrt{14} & 0 & 0 \\ 3\sqrt{14} & 43 & 25 & 0 \\ 0 & 25 & 43 & 3\sqrt{14} \\ 0 & 0 & 3\sqrt{14} & 7 \end{bmatrix}.$$

Moreover, $\Phi = \frac{14}{5}$ and $\hat{\Phi} = \frac{7}{5}$.

- 4) When $n = 5$, $g_1 = 16$, $g_2 = 8$, $g_3 = 4$, $g_4 = 2$, $\hat{g}_1 = 16$, $\hat{g}_2 = 9$, $\hat{g}_3 = 4$, $\hat{g}_4 = 1$,

$\Lambda_5 = \{30, 14, 6, 2, 0\}$, $\hat{\Lambda}_5 = \{30, 14, 5, 1, 0\}$ and hence,

$$J_5 = \frac{1}{3} \begin{bmatrix} 10 & 8\sqrt{2} & 0 & 0 & 0 \\ 8\sqrt{2} & 38 & 6\sqrt{21} & 0 & 0 \\ 0 & 6\sqrt{21} & 60 & 6\sqrt{21} & 0 \\ 0 & 0 & 6\sqrt{21} & 38 & 8\sqrt{2} \\ 0 & 0 & 0 & 8\sqrt{2} & 10 \end{bmatrix}$$

and

$$\hat{J}_5 = \frac{1}{41} \begin{bmatrix} 70 & 6\sqrt{406} & 0 & 0 & 0 \\ 6\sqrt{406} & 545 & 41\sqrt{82} & 0 & 0 \\ 0 & 41\sqrt{82} & 820 & 41\sqrt{82} & 0 \\ 0 & 0 & 41\sqrt{82} & 545 & 6\sqrt{406} \\ 0 & 0 & 0 & 6\sqrt{406} & 70 \end{bmatrix}.$$

Moreover, $\Phi = \frac{10}{3}$ and $\hat{\Phi} = \frac{70}{41}$.

- 5) When $n = 6$, $g_1 = 32$, $g_2 = 16$, $g_3 = 8$, $g_4 = 4$, $g_5 = 2$, $\hat{g}_1 = 25$, $\hat{g}_2 = 16$, $\hat{g}_3 = 9$, $\hat{g}_4 = 4$, $\hat{g}_5 = 1$, $\Lambda_6 = \{62, 30, 14, 6, 2, 0\}$, $\hat{\Lambda}_6 = \{55, 30, 14, 5, 1, 0\}$. As we have seen, this case is more complex than the previous ones, and clearly shows that $n = 6$ is, in general, the limiting dimension for effective calculations. Since the coefficients of the realizing matrices are expressed in terms of both the gaps and the symmetric functions, we give here these values for this particular cases. According to section 4.5 we have that $O_2 = 1020$, $O_3 = 1736$, $E_2 = 180$, $\hat{O}_2 = 839$, $\hat{O}_3 = 770$, $\hat{E}_2 = 150$, $a_3 = 37$, $b_3 = 21$, $\hat{a}_3 = \frac{2297}{70}$ and $\hat{b}_3 = \frac{35}{2}$. Then,

$$J_6 = \frac{1}{51} \begin{bmatrix} 186 & 16\sqrt{186} & 0 & 0 & 0 & 0 \\ 16\sqrt{186} & 834 & 68\sqrt{102} & 0 & 0 & 0 \\ 0 & 68\sqrt{102} & 1887 & 1071 & 0 & 0 \\ 0 & 0 & 1071 & 1887 & 68\sqrt{102} & 0 \\ 0 & 0 & 0 & 68\sqrt{102} & 834 & 16\sqrt{186} \\ 0 & 0 & 0 & 0 & 16\sqrt{186} & 186 \end{bmatrix}$$

and

$$\hat{J}_6 = \begin{bmatrix} \frac{51590}{26563} & \frac{5250\sqrt{319}}{26563} & 0 & 0 & 0 & 0 \\ \frac{5250\sqrt{319}}{26563} & \frac{16496257}{929705} & \frac{2\sqrt{53126}}{35} & 0 & 0 & 0 \\ 0 & \frac{2\sqrt{53126}}{35} & \frac{2297}{70} & \frac{35}{2} & 0 & 0 \\ 0 & 0 & \frac{35}{2} & \frac{2297}{70} & \frac{2\sqrt{53126}}{35} & 0 \\ 0 & 0 & 0 & \frac{2\sqrt{53126}}{35} & \frac{16496257}{929705} & \frac{5250\sqrt{319}}{26563} \\ 0 & 0 & 0 & 0 & \frac{5250\sqrt{319}}{26563} & \frac{51590}{26563} \end{bmatrix}.$$

Moreover, $\Phi = \frac{62}{17}$ and $\hat{\Phi} = \frac{51590}{26563}$.

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