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Combining the Borda Count with Approval and Disapproval Voting

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Abstract
In this paper, we extend preference-approval structures to a more general situation, where voters can sort the alternatives in three disjoint classes instead of two (for instance, acceptable, neutral and unacceptable). We propose a parameterized family of voting systems related to the Borda count, where positive (negative) individual scores are assigned to acceptable (unacceptable) alternatives in a decreasing way from best to worst, while neutral alternatives obtain null scores. We analyze the role of parameters and provide some properties that satisfy the proposed voting systems.

Keywords Voting systems · Borda count · Approval voting · Preference-approval

1 Introduction

Voting systems aggregate individual opinions on a set of alternatives to generate an outcome, usually a single winning alternative, a subset of winning or acceptable alternatives or a weak order on the set of alternatives (some surveys on voting systems can be found in Nurmi (1983) and Brams and Fishburn (2002)).

Individual opinions can be of different nature: the best alternative, as in plurality rule; the worst alternative, as in antiplurality rule; the best and worst alternatives, as in best-worst voting systems (see García-Lapresta et al. 2010); a subset of acceptable alternatives, as in approval voting (see Brams and Fishburn 1978); a weak or linear order on the set of alternatives, as in the Borda rule; an evaluation of each alternative through an ordinal scale, as in Majority Judgment (Balinski and Laraki 2007, 2011); etc.

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Next we focus on several types of inputs: voters either rank all alternatives, or sort the alternatives in several classes, or sort the alternatives into two or three classes and also rank them. In all cases, voters' opinions are aggregated and a collective weak order on the set of alternatives is obtained.

1.1 Ranking

The Borda rule (Borda 1784) requires voters to provide linear orders on the set of alternatives; individual scores are assigned to the alternatives in arithmetic progression, and the alternatives are ranked according to the sum of individual scores. Scoring rules extend the Borda rule by allowing general decreasing scores (see Chebotarev and Shamis 1998).

Black (1976) adjusts the Borda rule to weak orders in two equivalent ways. First, assigning the average score that alternatives in the same indifference class should have if they were linearized, that is equivalent to assign to each alternative the number of alternatives worse than it plus half the number of alternatives indifferent to it (see also Smith 1973; Cook and Seiford 1982). And, secondly, assigning to each alternative a score that is the number of alternatives worse than it, minus the number of alternatives better than it; obviously, in this case, some alternatives can obtain null or negative scores.

1.2 Sorting

Approval voting (Brams and Fishburn 1978) only considers which alternatives are acceptable to each voter; alternatives are ranked according to the total number of approvals. It is important to note that approval voting does not take into account the voters' preferences neither over acceptable nor over unacceptable alternatives.

Balinski and Laraki introduce and analyze the Majority Judgment voting system (Balinski and Laraki 2007, 2011, 2014, 2020, 2022). It requires that voters assess the alternatives through an ordinal scale[1]; alternatives are ranked according to the lower medians of the individual assessments and a tie-breaking procedure.

In some sense, Majority Judgment can be considered as a generalization of approval voting, where only two grades are used. However, Balinski and Laraki (2022) point out: "*Approve* is not the opposite of *Disapprove*. Two grades are simply too few to adequately express voters' opinions. At least three grades are necessary". The authors prove that Majority Judgment with three grades excludes the no-show paradox (Moulin 1988).

Different voting systems using three grades can be found in Felsenthal (1989); Felsenthal and Machover (1997); Yılmaz (1999); Hillinger (2004); Aleskerov and Yakuba (2007); Aleskerov et al. (2007, 2010); Alcantud and Laruelle (2014); Grandi et al. (2016); Gonzalez et al. (2019); Balinski and Laraki (2022); Ye et al. (2024), among others.

[1] The authors consider {'to reject', 'poor', 'acceptable', 'good', 'very good', 'excellent'} to evaluate political candidates.

1.3 Sorting and Ranking

Voters that rank the alternatives from best to worst through linear or weak orders do not declare which alternatives are either acceptable or unacceptable. In fact, several voters may share the same linear or weak order having very different opinions about the alternatives they rank in the same way. Consider three individuals that rank three destinations for a trip, **A**thens, **B**udapest and **C**openhagen, in the same way: $\mathbf{B} \succ \mathbf{A} \succ \mathbf{C}$. Nonetheless, it is possible they may have different feelings. For instance, the first individual may think that the three destinations are acceptable, the second one may think that only **B** and **A** are acceptable, and the third one may consider that only **B** is acceptable.

On the other hand, when voters only sort the alternatives into different classes and are not forced to rank them in order of preference, some of them may share the same classification of the alternatives and have different feelings. Taking into account the previous example, imagine that three individuals sort **B** and **C** as acceptable, and **A** as unacceptable. It could happen that the first individual prefers **B** to **C**, the second one prefers **C** to **B**, and the third one be indifferent between **B** and **C**.

Preference-approval structures consider preferences over the alternatives, through a weak order, specifying which of them are acceptable (see (Brams 2008, Chapter 3)), Brams and Sanver (2009) and Sanver (2011)).

In preference-approval structures, voters can pay attention to which alternatives are acceptable and rank them[2]. Voters may either rank unacceptable alternatives or avoid to declare their preferences about them[3] by (implicitly) showing indifference between these alternatives.

Different contributions in the setting of preference-approvals can be found in Erdamar et al. (2014); Kamwa (2019, 2023); Dong et al. (2021); Kruger and Sanver (2021); Barokas (2022, 2023); Barokas and Sprumont (2022); Albano et al. (2023, 2024); Ye et al. (2024); Santos-García and Alcantud (2025), among others.

1.4 Our Proposal

In this paper, we focus on ternary preference structures. They generalize preference-approval structures by allowing voters to sort the alternatives within three classes (e.g., acceptable, neutral and unacceptable). We shall assume that voters mainly focus on the first and third classes and they only rank the alternatives belonging to these two classes[4]. This implicitly means that voters are indifferent between the alternatives of the second class. This threefold approach is an extension of the inputs managed by the best-worst voting systems, where voters only show their best and worst alternatives (see García-Lapresta et al. 2010).

[2] This is the case of the Borda rules on top-truncated preferences analyzed by Terzopoulou and Endriss (2021).

[3] This is the case of fallback voting in Brams and Sanver (2009) (see also Kamwa 2023).

[4] According to (Dummett 1984, p. 243): "If there are, say, twenty possible outcomes, the task of deciding the precise order of preference in which he ranks them may induce a kind of psychological paralysis in the voter". Then, if voters do not rank all the alternatives in order of preference and are allowed to discard those that belong to the second class, they will be relieved in their decision processes.

In this scenario, we propose a parameterized family of voting systems where, taking into account individual preferences, decreasing positive scores are assigned, from best to worst, to the alternatives in the first class; a null score is assigned to each alternative in the second class; and decreasing negative scores are assigned to the alternatives in the third class. These scores are inspired by Black s second Borda count in the setting of weak orders (see Black 1976), but adapted to our new scenario[5].

As in the Borda count, the alternatives are ranked collectively by adding up the individual scores. Furthermore, we suggest how to generate a collective ternary preference from the collective scores (Remark 7). We also analyze the role of parameters and justify some properties of the voting systems devised.

1.5 Related Litetature

Although the Borda count and the approval voting procedures deal with different informational bases from the agents when considering the alternatives (pairwise comparisons and acceptance assessments one by one, respectively), García-Lapresta and Martínez-Panero (2002) outlined a fuzzy approach between these different points of view.

It is important to mention that our proposal has some similarities with the decision-making procedure proposed by Grandi et al. (2016) in the context of sentimental analysis. These authors consider that individuals provide a positive, neutral or negative polarity to some items and, furthermore, may rank positive and negative items. Through a parameterized Borda count, a collective sentiment over multiple issues is generated and some properties of the proposed procedure are justified within the framework of Social Choice.

It is also interesting to note that, more recently, Barokas and Sprumont (2022) have proposed what they call the *broken Borda rule* in a preference-approval context, assigning Borda scores plus a large positive constant to acceptable alternatives and just the Borda scores to unacceptable ones. These authors justify that this broken Borda rule is equivalent to a lexicographic combination of approval voting and the Borda count: first, the alternatives are ranked collectively based on the number of voters who find them acceptable (approval voting); a tie-breaking procedure is then proposed using Borda scores. As we shall show, the broken Borda rule is related to a particular case in our more general setting of ternary preferences.

The rest of the paper is organized as follows. Section 2 includes the basic notation, preference-approvals and their extension to ternary preferences. Section 3 introduces the family of Borda-ternary voting systems and provide examples, properties, and an extension to the case of multiple criteria. Finally, Section 4 concludes the paper with some remarks.

[5] In his analysis of the Borda count, Gärdenfors (1973) considers that “political scientists [have] unjustly accused [it] of being a (stupid) preference intensity amalgamating method". Note that our procedure manages a richer information than the Borda count.

2 From Preference-Approvals to Ternary Preferences

In this section, we introduce the basic notation, preference-approvals and ternary preferences.

2.1 Notation

A *weak order* (or *complete preorder*) on X is a complete and transitive binary relation on X. A *linear order* on X is an antisymmetric weak order on X. With $\mathcal{W}(X)$ and $\mathcal{L}(X)$ we denote the set of weak and linear orders on X, respectively. Given $R \in \mathcal{W}(X)$, with P and I we denote the asymmetric and the symmetric parts of R, respectively: $x_i \, P \, x_j$ if not $x_j \, R \, x_i$; and $x_i \, I \, x_j$ if $(x_i \, R \, x_j$ and $x_j \, R \, x_i)$; the inverse of R is defined as $x_i \, R^{-1} \, x_j$ if $x_j \, R \, x_i$.

Given a set Y, with $\mathcal{P}(Y)$ we denote its power set, i.e., $I \in \mathcal{P}(Y) \Leftrightarrow I \subseteq Y$. In turn, with $\#Y$ we denote the cardinality of Y.

2.2 Preference-Approvals

Consider that a set of voters $V = \{1, 2, \ldots, m\}$, with $m \geqslant 2$, have to show their opinions over a finite set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$, with $n \geqslant 2$. In the setting of preference-approvals ((Brams 2008, Chapter 3) Brams and Sanver (2009) and Sanver (2011)), each voter ranks the alternatives in X by means of a weak order and, additionally, assesses each alternative as either acceptable or unacceptable by partitioning X into A, the set of *acceptable* alternatives, and $U = X \setminus A$, the set of *unacceptable* alternatives, where A and U can be the empty set. The following consistency condition is considered: Given two alternatives x_i and x_j, if x_j is acceptable and x_i is at least as good as x_j, then x_i should be acceptable as well.

Definition 1 A *preference-approval* on X is a pair $(R, A) \in \mathcal{W}(X) \times \mathcal{P}(X)$ satisfying the following condition: if $x_i \, R \, x_j$ and $x_j \in A$, then $x_i \in A$.

It is easy to see that for any preference-approval (R, A) on X, if x_i is acceptable and x_j is unacceptable, then x_i should be preferred to x_j; and if x_i is at least as good as x_j and x_i is unacceptable, then x_j should also be unacceptable.

2.3 Ternary Preferences

Hereinafter, we consider that each voter $v \in V$ arranges the set of alternatives through a weak order $R_v \in \mathcal{W}(X)$ and, additionally, makes a partition of X in three categories: A_v (acceptable), N_v (neutral) and U_v (unacceptable)[6]. Any of these categories can be empty, but not all of them simultaneously[7].

[6] As mentioned in the Introduction, voters do not need to rank the alternatives belonging to the second category (implicitly, they are indifferent to each other).

[7] We note that if $N_v = \emptyset$, then we fall in the case of preference-approvals.

Although the words used to name these three categories have been different depending on the authors (see Table 1), in all cases the alternatives of A_v are preferred to those of N_v and U_v, and the alternatives of N_v are preferred to those of U_v.

We now formally introduce the notion of ternary preference.

Definition 2 A *ternary preference* on X is a 4-tuple

$$(R, A, N, U) \in \mathcal{W}(X) \times \mathcal{P}(X) \times \mathcal{P}(X) \times \mathcal{P}(X)$$

satisfying the following conditions:

1. $A \cap N = A \cap U = N \cap U = \emptyset$.
2. $A \cup N \cup U = X$.
3. If $x_i \in A$ and $x_j \in (N \cup U)$, then $x_i \, P \, x_j$.
4. If $x_i \in N$ and $x_j \in U$, then $x_i \, P \, x_j$.

The set of ternary preferences on X is denoted by $\mathcal{T}(X)$. With $\mathcal{T}_0(X)$ we denote the set of ternary preferences on X that satisfies the condition

5. If $x_i, x_j \in N$, then $x_i \, I \, x_j$.

From a behavioural point of view, it is reasonable to think that voters first identify which alternatives are either acceptable or unacceptable[8], being the rest neutral. In a

[8] Individuals can usually easily choose which are the best and worst alternatives. Finn and Louviere (1992) proposed, and studied, a discrete choice procedure in which an individual selects both the best and the worst alternatives (see also Marley and Louviere (2005) and García-Lapresta et al. (2010)).

Table 1 Wording of three categories

Reference	A	N	U
Felsenthal (1989)	Favor	Abstain	Against
Felsenthal and Machover (1997)	Yes	Abstention	No
Yılmaz (1999)	Favorite	Acceptable	Disapproved
Hillinger (2004)	Good	Indifferent	Bad
Aleskerov and Yakuba (2007)	Good	Average	Bad
Alcantud and Laruelle (2014)	Approved	?	Disapproved
Grandi et al. (2016)	Positive	Neutral	Negative
Gonzalez et al. (2019)	Approve	Neutral	Disapprove
Balinski and Laraki (2022)	Approve	Neither	Disapprove
Ye et al. (2024)	Acceptance	Hesitation	Refusal

second stage, voters may rank the set of acceptable alternatives and, if they wish, the set of unacceptable alternatives.

To visualize the three categories in ternary preferences, we use two lines: alternatives above the upper line belong to A, alternatives between the two lines belong to N, and alternatives below the lower line belong to U.

Example 1 Consider $(R, A, N, U) \in \mathcal{T}_0(\{x_1, x_2, \ldots, x_9\})$ represented by

$$\begin{array}{c} x_2 \\ x_1\ x_5 \\ \hline \\ x_3\ x_6 \\ \hline \\ x_8 \\ x_4\ x_7\ x_9 \end{array}$$

Then, $A = \{x_1, x_2, x_5\}$, $N = \{x_3, x_6\}$ and $U = \{x_4, x_7, x_8, x_9\}$.

It is clear that the expressiveness of voters with ternary preferences increases with respect to possible approvals and preference-approvals[9].

For instance, in the case of two alternatives, $X = \{x_1, x_2\}$, there are 4 possible approvals: $\emptyset$, $\{x_1\}$, $\{x_2\}$ and $\{x_1, x_2\}$, and 8 possible preference-approvals (see Fig. 1).

Fig. 2 includes the 13 possible ternary preferences of $\mathcal{T}_0(\{x_1, x_2\})$.

Definition 3 Given $T_v = (R_v, A_v, N_v, U_v) \in \mathcal{T}_0(X)$, its *inverse* is defined as $T_v^{-1} = (R_v^{-1}, A_v^{-1}, N_v, U_v^{-1})$, where $A_v^{-1} = U_v$ and $U_v^{-1} = A_v$.

Example 2 The inverse of the ternary preference of Example 1 is

$$\begin{array}{c} x_4\ x_7\ x_9 \\ x_8 \\ \hline \\ x_3\ x_6 \\ \hline \\ x_1\ x_5 \\ x_2 \end{array}$$

[9](Albano et al. 2023, Table 1) show the number of possible approvals, linear orders, weak orders and preference-approvals when the number of alternatives is $n = 2, 3, \ldots, 10$. The exact number of weak orders and preference-approval structures, for any n value, can be obtained from Santos-García and Alcantud (2025).

x_1	x_1	——	x_2	x_2	——	$x_1\ x_2$	——
x_2	——	x_1	x_1	——	x_2	——	$x_1\ x_2$
——	x_2	x_2	——	x_1	x_1		

Fig. 1 Preference-approvals for 2 alternatives

Fig. 2 Ternary preferences of $\mathcal{T}_0(\{x_1, x_2\})$

x_1	x_1	x_1	—	—
x_2	—	—	x_1	—
—	x_2	—	—	x_1
—	—	x_2	x_2	x_2

x_2	x_2	x_2	—	—
x_1	—	—	x_2	—
—	x_1	—	—	x_2
—	—	x_1	x_1	x_1

$x_1\ x_2$	—	—
—	$x_1\ x_2$	—
—	—	$x_1\ x_2$

3 Borda-Ternary Voting Systems

Early in 1770, Borda (1784) orally proposed what he called “election by order of merit" as follows[10]:

> Suppose that there are just three candidates and [...] the voter ranked *A* first, *B* second and *C* third. Now we must assume that the degree of superiority which this voter gave *A* over *B* is the same as that he gave *B* over *C*. As candidate *B* is no more likely to be ranked in one particular place on the scale between *A* and *C* than in any other, we have no reason to say that the voter who ranked the candidates *ABC* wanted to place *B* nearer *A* than *C* or vice versa; no reason to say, that is, that he accorded the first more superiority over the second than he accorded the second over the third. Furthermore, because of the supposed equality between the voters, each rank must be assumed to have the same value and to represent the same degree of merit as the same rank assigned to another candidate, or even by another voter.
>
> If we take a to be the degree of merit which each voter attributes to last place and $a + b$ the degree of merit attributed to second place, we can represent first place by $a + 2b$.

In this way, according to Sen (1976), Borda considered an “equidistanced cardinalization of an ordering" which only takes into account ordinal information and is therefore less demanding than using preference intensities.

Borda pointed out that the values of a and b “can be anything one wishes", and he (and also Morales (1797)) used $a = b = 1$. Later on, Morales (1805) took $a = 0$ and

[10] See (McLean and Urken 1995, pp. 84-85).

$b = 1$ in order to show that, in this case, the score assigned by a voter for each alternative corresponds to the number of those considered worse than such alternative.

Borda's approach requires lineal orders to arrange the alternatives, but here we follow Black's first adjusted count (see (Black 1958, pp. 61–64) and Black (1976)) extending the original scheme by considering weak orders, and taking $b = 1$ and $a = \varepsilon > 0$. Also, as in Black's second count, we allow negative scores for disapproval purposes[11].

Remark 1 Our proposal essentially differs from Borda's classic approach. In our ternary scenario, each voter considers acceptable, neutral or unacceptable alternatives and, paraphrasing Borda's quotation above (italics ours),

> the degree of superiority which this elector has accorded *A (acceptable)* over *B (neutral) might not necessarily* be counted the same as the degree of superiority which he accords *B* over *C (unnaceptable)*.

3.1 Individual Borda-Ternary Scores

The following definition is intended to combine the symmetry of the Borda rule in the usual preference context (Young's cancellation (Young 1974) and Saari's reversal property (Saari 1994; Saari and Barney 2003)) with the duality of acceptable and unacceptable alternatives in the preference-approval setting.

In the framework of ternary preferences, we distinguish between acceptable, neutral and unacceptable alternatives. The score of an acceptable alternative is the Borda score in the subset of acceptable alternatives, plus the number of neutral alternatives multiplied by a parameter α, plus the number of unacceptable alternatives multiplied by a parameter β, plus a discriminant parameter ε. All neutral alternatives have a null score. And the score of an unacceptable alternative is the opposite of the Borda score in the subset of unacceptable alternatives, minus the number of neutral alternatives multiplied by α, minus the number of acceptable alternatives multiplied by β, minus ε.

Definition 4 Given a voter $v \in V$, $T_v = (R_v, A_v, N_v, U_v) \in \mathcal{T}_0(X)$ and $\alpha, \beta, \varepsilon > 0$, the individual count of voter *v* is defined as

$$B_v(x_i) = \begin{cases} \#\{x_j \in A_v \mid x_i \, P_v \, x_j\} + \frac{1}{2} \cdot \#\{x_j \in A_v \setminus \{x_i\} \mid x_j \, I_v \, x_i\} + \\ \alpha \cdot \#N_v + \beta \cdot \#U_v + \varepsilon, \quad \text{if} \;\; x_i \in A_v, \\ 0, \quad \text{if} \;\; x_i \in N_v, \\ -\#\{x_j \in U_v \mid x_j \, P_v \, x_i\} - \frac{1}{2} \cdot \#\{x_j \in U_v \setminus \{x_i\} \mid x_j \, I_v \, x_i\} - \\ \alpha \cdot \#N_v - \beta \cdot \#A_v - \varepsilon, \quad \text{if} \;\; x_i \in U_v. \end{cases}$$

[11] Duddy and Piggins (2013) consider in an approval context the following possibility: if an alternative is acceptable, then it should obtain a score equal to the number of unacceptable alternatives; and conversely, with zero minus the number of acceptable alternatives. Our approach, in a preference-approval context, will take into account a richer informational basis.

The role of the parameters $\alpha > 0$ and $\beta > 0$ will be discussed in Subsection 3.2. The positivity of ε will ensure that all acceptable (unacceptable) alternatives have positive (negative) scores. Its role is discussed in this subsection.

Note that $B_v(x_i)$ coincides with the score given by the Borda count in the setting of weak orders in the case of $x_i \in A_v$ whenever $\alpha = \beta = 1$ and $\varepsilon = 0$. With the same values of the parameters $\alpha, \beta, \varepsilon$, we have that $B_v(x_i)$ coincides with the opposite of the Borda score in the setting of weak orders in the case of $x_i \in U_v$ for R^{-1}.

Remark 2 Note that the individual count of $v \in V$ in T_v^{-1} is the opposite of the individual count of *v* in T_v.

Remark 3 The sign of the the individual count $B_v(x_i)$ is determined by the class to which x_i belongs:

1. $x_i \in A_v \;\Leftrightarrow\; B_v(x_i) > 0.$
2. $x_i \in N_v \;\Leftrightarrow\; B_v(x_i) = 0.$
3. $x_i \in U_v \;\Leftrightarrow\; B_v(x_i) < 0.$

This distinction of signs is ensured by the fact that $\alpha, \beta, \varepsilon > 0$. Otherwise, if $\alpha, \beta, \varepsilon$ were allowed to be null, it would be possible $B_v(x_i) = 0$ when one of the following situations occurs:

1. x_i is the single worst alternative of A_v and $N_v = U_v = \emptyset$.
2. x_i is the single worst alternative of A_v, $N_v = \emptyset$ and $\beta = 0$.
3. x_i is the single worst alternative of A_v, $U_v = \emptyset$ and $\alpha = 0$.
4. x_i is the single worst alternative of A_v and $\alpha = \beta = 0$.
5. x_i is the single best alternative of U_v and $A_v = N_v = \emptyset$.
6. x_i is the single best alternative of U_v and $N_v = \emptyset$ and $\beta = 0$.
7. x_i is the single best alternative of U_v, $A_v = \emptyset$ and $\alpha = 0$.
8. x_i is the single best alternative of U_v and $\alpha = \beta = 0$.

We note that Grandi et al. (2016) also assign positive, null and negative individual scores to the alternatives sorted in the three classes they consider (positive, neutral and negative, respectively). These authors allow individuals to provide incomplete information (not all alternatives must be sorted or ranked).

In contrast, in their broken Borda rule, Barokas and Sprumont (2022) assign positive individual scores to alternatives in a preference-approval framework assuming that voters rank alternatives through linear orders. Specifically, they assign Borda scores to acceptable alternatives plus *mn*, and simply Borda scores to unacceptable alternatives. Taking into account this large gap between the scores of the worst acceptable alternative and the best unacceptable one (for each voter), the authors justify that the broken Borda rule is a lexicographic combination of approval voting and the Borda count.

If we restrict our proposal of individual Borda-ternary scores (Definition 4) to the setting of the broken Borda rule, i.e., $N_v = \emptyset$ and $R_v \in \mathcal{L}(X)$, and add the opposite of the minimum Borda-ternary score, $\#U_v - 1 + \beta \cdot \#A_v + \varepsilon$, to the original scores, we obtain the following individual scores:

$$B'_v(x_i) = \begin{cases} \#\{x_j \in X \mid x_i \, P_v \, x_j\} + \beta n + 2\varepsilon - 1, & \text{if } \ x_i \in A_v, \\ \#\{x_j \in X \mid x_i \, P_v \, x_j\}, & \text{if } \ x_i \in U_v. \end{cases}$$

Note a difference between $B'_v(x_i)$ with respect to the individual score obtained with the broken Borda rule: now, acceptable alternatives have the Borda score plus $\beta n + 2\varepsilon - 1$ instead of *mn*.

Example 3 Following with Example 1, we have the following scores:

$$\begin{aligned} &B_v(x_2) = 2 + 2\alpha + 4\beta + \varepsilon \\ &B_v(x_1) = B_v(x_5) = 0.5 + 2\alpha + 4\beta + \varepsilon \\ &B_v(x_3) = B_v(x_6) = 0 \\ &B_v(x_8) = -2\alpha - 3\beta - \varepsilon \\ &B_v(x_4) = B_v(x_7) = B_v(x_9) = -2 - 2\alpha - 3\beta - \varepsilon. \end{aligned}$$

A gap can be observed between the lowest positive and highest negative scores: $0.5 + 2\alpha + 4\beta + \varepsilon - (-2\alpha - 3\beta - \varepsilon) = 0.5 + 4\alpha + 7\beta + 2\varepsilon$.

The following result quantifies the minimum gap between positive and negative individual scores in the general case.

Proposition 1 *If both $A_v, U_v \neq \emptyset$ for voter $v \in V$, then the gap between v's scores of the worst acceptable alternative(s) and the best unacceptable one(s) is*

$$g_v = \frac{t}{2} + (2\alpha - \beta) \cdot \#N_v + \beta n + 2\varepsilon,$$

where

$$t = \#\{x_i \in A_v \mid \forall x_j \in A_v \;\; x_j \, R_v \, x_i\} + \#\{x_i \in U_v \mid \forall x_j \in U_v \;\; x_i \, R_v \, x_j\} - 2.$$

Proof Since the score of the worst acceptable alternative(s) is

$$\frac{1}{2} \cdot (\#\{x_i \in A_v \mid \forall x_j \in A_v \;\; x_j \, R_v \, x_i\} - 1) + \alpha \cdot \#N_v + \beta \cdot \#U_v + \varepsilon$$

and the score of the best unacceptable alternative(s) is

$$-\frac{1}{2} \cdot (\#\{x_i \in U_v \mid \forall x_j \in U_v \;\; x_i \, R_v \, x_j\} - 1) - \alpha \cdot \#N_v - \beta \cdot \#A_v - \varepsilon,$$

we have that the gap is

$$g_v = \frac{t}{2} + 2\alpha \cdot \# N_v + \beta \cdot (\# U_v + \# A_v) + 2\varepsilon =$$
$$\frac{t}{2} + 2\alpha \cdot \# N_v + \beta \cdot (n - \# N_v) + 2\varepsilon =$$
$$\frac{t}{2} + (2\alpha - \beta) \cdot \# N_v + \beta n + 2\varepsilon.$$

Remark 4 Notice that the minimum positive score reachable for an alternative is ε. It occurs when a voter declares that all alternatives are acceptable and that alternative is the single worst alternative for that voter. In turn, the maximum negative score is $-\varepsilon$. It occurs when another voter declares that all alternatives are unacceptable and that alternative is the single best alternative for that voter. Therefore, 2ε represents the shortest jump between the positive and negative scores corresponding to acceptable and unacceptable alternatives (for different voters), respectively (see Fig. 3).

3.2 Adjusting the Parameters

In order to analyze the role of the parameters α and β in the process leading to determine the individual scores, let $T_v = (R_v, A_v, N_v, U_v) \in \mathcal{T}_0(X)$ be the ternary preference of a voter $v \in V$. Consider that this voter changes their opinion about an alternative $x_i \in X$, and the new ternary preference $T'_v = (R'_v, A'_v, N'_v, U'_v) \in \mathcal{T}_0(X)$ does not change with respect to the other alternatives.

We now present four conditions on how some changes in the situation of one alternative affect another[12].

Condition 1: Given $T_v = (R_v, A_v, N_v, U_v), T'_v = (R'_v, A'_v, N'_v, U'_v) \in \mathcal{T}_0(X)$, if there exist $x_i, x_j \in X$ such that T_v and T'_v coincide in $X \setminus \{x_i\}$, $x_i \in A_v \cap U'_v$, $x_j \in A_v \cap A'_v$ and $x_i \, P_v \, x_j$ (see Fig. 4), then $B'_v(x_j) > B_v(x_j)$.

This condition assumes that if an acceptable alternative x_i is preferred to another acceptable alternative x_j, and x_i becomes unacceptable, *ceteris paribus*, then the score of x_j should increase.

[12] These conditions do not apply to the Borda rule, where scores come from victories in pairwise tournaments, regardless of preference intensities. However, in our ternary setting we have more information, and individual scores are assigned in a different way depending on the nature of the alternatives to be compared (acceptable, neutral or unacceptable).

$\cdots\cdots\cdots$

$x_i \;\to\; \varepsilon$

$x_j \;\to\; -\varepsilon$

$\cdots\cdots\cdots$

Fig. 3 Shortest jump between positive and negative scores for different voters

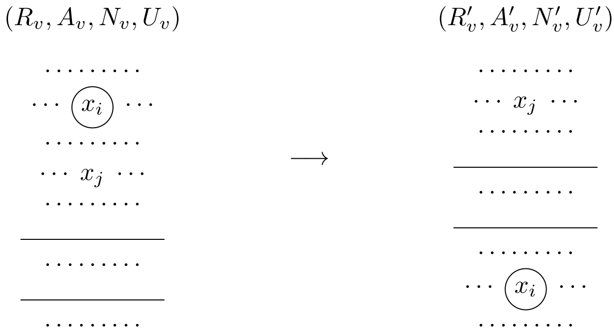

Fig. 4 Acceptable alternative x_i that is preferred to alternative x_j becomes unacceptable

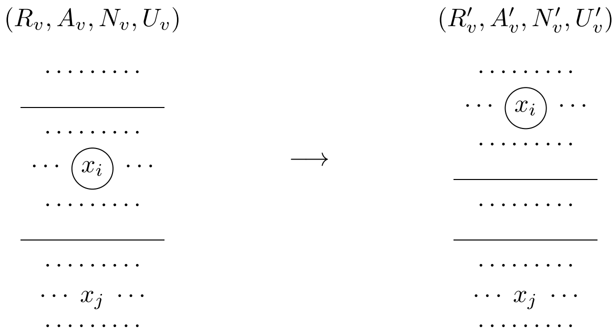

Fig. 5 Neutral alternative x_i becomes acceptable and alternative x_j remains unacceptable

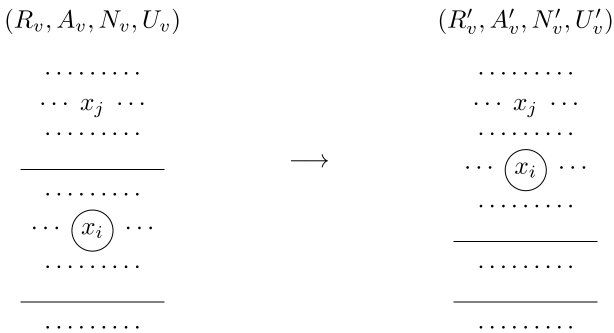

Fig. 6 Neutral alternative x_i becomes acceptable and it is preferred by alternative x_j

Condition 2: Given $T_v = (R_v, A_v, N_v, U_v), T'_v = (R'_v, A'_v, N'_v, U'_v) \in \mathcal{T}_0(X)$, if there exist $x_i, x_j \in X$ such that T_v and T'_v coincide in $X \setminus \{x_i\}$, $x_i \in N_v \cap A'_v$ and $x_j \in U_v \cap U'_v$ (see Fig. 5), then $B'_v(x_j) > B_v(x_j)$.

This condition assumes that it is less disgraceful for an unacceptable alternative to be defeated by an acceptable alternative than by a neutral one, *ceteris paribus*.

Condition 3: Given $T_v = (R_v, A_v, N_v, U_v), T'_v = (R'_v, A'_v, N'_v, U'_v) \in \mathcal{T}_0(X)$, if there exist $x_i, x_j \in X$ such that T_v and T'_v coincide in $X \setminus \{x_i\}$, $x_i \in N_v \cap A'_v$, $x_j \in A_v \cap A'_v$ and $x_j \, P'_v \, x_1$ (see Fig. 6), then $B'_v(x_j) > B_v(x_j)$.

This condition assumes that it is more meritorious to beat an acceptable alternative than to a neutral one, *ceteris paribus*.

Condition 4: Given $T_v = (R_v, A_v, N_v, U_v), T'_v = (R'_v, A'_v, N'_v, U'_v) \in \mathcal{T}_0(X)$, if there exist $x_i, x_j \in X$ such that T_v and T'_v coincide in $X \setminus \{x_i\}$, $x_i \in N_v \cap A'_v$, $x_j \in A_v \cap A'_v$ and $x_i \, I'_v \, x_j$ (see Fig. 7), then $B'_v(x_j) > B_v(x_j)$.

This condition assumes that it is more meritorious to tie with an acceptable alternative than to beat a neutral one, *ceteris paribus*.

We now show how Conditions 1, 2, 3 and 4 affect the parameters values.

Remark 5 1. Condition 1 is equivalent to $\beta > 0$:

$$B'_v(x_j) = B_v(x_j) + \beta > B_v(x_j) \;\Leftrightarrow\; \beta > 0.$$

2. Condition 2 is equivalent to $\beta < \alpha$:

$$B'_v(x_j) = B_v(x_j) + \alpha - \beta > B_v(x_j) \;\Leftrightarrow\; \beta < \alpha.$$

3. Condition 3 is equivalent to $\alpha < 1$:

$$B'_v(x_j) = B_v(x_j) - \alpha + 1 > B_v(x_j) \;\Leftrightarrow\; \alpha < 1.$$

4. Condition 4 is equivalent to $\alpha < 0.5$:

$$B'_v(x_j) = B_v(x_j) + \frac{1}{2} - \alpha > B_v(x_j) \;\Leftrightarrow\; \alpha < 0.5.$$

Thus, under Conditions 1, 2 and 3, we have $0 < \beta < \alpha < 1$. These restrictions can be represented in the triangle with vertices (0, 0), (1, 0) and (1, 1) of Fig. 8.

Conditions 1, 2 and 3 seem incontestable and we will assume them from now on. Note that under Conditions 1, 2 and 4, we have $0 < \beta < \alpha < 0.5$. These restrictions

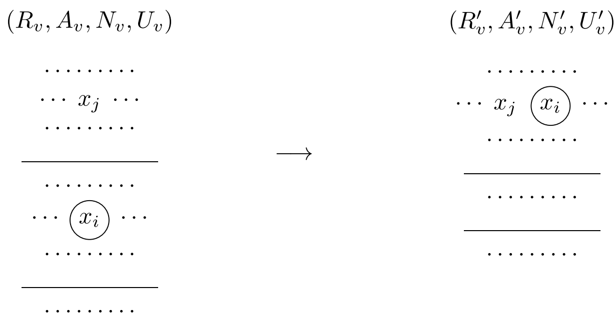

Fig. 7 Neutral alternative x_i becomes acceptable and it is indifferent to alternative x_j

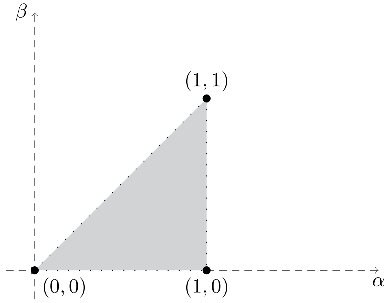

Fig. 8 Map of parameter values under Conditions 1, 2 and 3

Fig. 9 Voting paradox

R_1	R_2	R_3
x_1	x_2	x_3
x_2	x_3	x_1
x_3	x_1	x_2

can be represented in a triangle similar to that in Fig. 8, but now with vertices (0, 0), (0.5, 0) and (0.5, 0.5).

It is easy to check that, applying the same reasoning to the rest of the cases in which the alternative x_i changes from the initial position to others, no stronger restrictions on parameter values are obtained.

3.3 Collective Borda-Ternary Scores

Once the individual scores have been defined, a collective score is assigned to each alternative by adding the individual scores.

Definition 5 A *profile* is a vector

$$\boldsymbol{T} = \big((R_1, A_1, N_1, U_1), (R_2, A_2, N_2, U_2), \ldots, (R_m, A_m, N_m, U_m)\big) \in \mathcal{T}_0(X)^m,$$

where (R_v, A_v, N_v, U_v) is the ternary preference of voter $v \in V$.

Example 4 It is well-known that simple majority can produce cycles. In Fig. 9 we show a profile of three linear orders on $\{x_1, x_2, x_3\}$ that produce a cycle when applying simple majority (and a tie between the three alternatives when applying the Borda rule).

Each of the three linear orders included in Fig. 9 induces 10 different ternary preferences. Fig. 10 includes those of R_1. Note that only 7 of them belong to $\mathcal{T}_0(\{x_1, x_2, x_3\})$ (T_1^3, T_1^4 and T_1^7 do not, because different neutral alternatives are not indifferent). Consequently, there are $7^3 = 343$ profiles of ternary preferences of $\mathcal{T}_0(\{x_1, x_2, x_3\})$ based on the profile of Fig. 10.

Definition 6 Given a profile $\boldsymbol{T} \in \mathcal{T}_0(X)^m$, the *collective Borda-ternary score* of the alternative $x_i \in X$ is defined as

Fig. 10 Ternary preferences for R_1

T_1^1	T_1^2	T_1^3	T_1^4	T_1^5
x_1	x_1	x_1	—	x_1
x_2	x_2	—	x_1	x_2
x_3	—	x_2	x_2	—
—	x_3	x_3	x_3	—
—	—	—	—	x_3

T_1^6	T_1^7	T_1^8	T_1^9	T_1^{10}
x_1	—	x_1	—	—
—	x_1	—	x_1	—
x_2	x_2	—	—	x_1
—	—	x_2	x_2	x_2
x_3	x_3	x_3	x_3	x_3

Fig. 11 Profile of ternary preferences in Example 5

T_1	T_2	T_3	T_4	T_5
x_1	x_1	x_1	x_2	x_2
—	—	—	—	x_1
x_2	x_2	x_2	x_1	—
—	—	—	—	—

$$B^t(x_i) = \sum_{v=1}^{m} B_v(x_i).$$

Definition 7 Given $\alpha, \beta, \varepsilon > 0$ in Def. 4, the *Borda-ternary voting system associated with* $\alpha, \beta, \varepsilon$ is the mapping $F : \mathcal{T}_0(X)^m \longrightarrow \mathcal{W}(X)$ defined as $F(\boldsymbol{T}) = R^t$, where

$$x_i \, R^t \, x_j \;\Leftrightarrow\; B^t(x_i) \geqslant B^t(x_j).$$

We note that the Borda-ternary voting system generates the same outcomes as the Borda count whenever either $N_v = U_v = \emptyset$ for every $v \in V$ or $A_v = N_v = \emptyset$ for every $v \in V$.

Example 5 The value of ε can be determinant in the result. For instance, in the profile $(T_1, T_2, T_3, T_4, T_5) \in \mathcal{T}_0(\{x_1, x_2\})^5$ included in Fig. 11, we obtain $B^t(x_1) = 4\varepsilon + 3\alpha$ and $B^t(x_2) = 1 + 2\varepsilon + \alpha$. Since $4\varepsilon + 3\alpha > 1 + 2\varepsilon + \alpha \;\Leftrightarrow\; \varepsilon > 0.5 - \alpha$, for $\alpha = 0.4$, we have $x_1 \, P^t \, x_2 \;\Leftrightarrow\; \varepsilon > 0.1$, $x_2 \, P^t \, x_1 \;\Leftrightarrow\; \varepsilon < 0.1$, and $x_1 \, I^t \, x_2 \;\Leftrightarrow\; \varepsilon = 0.1$.

Example 6 Following with Example 4, suppose that voters not only rank the alternatives, but also can sort them as acceptable, neutral and unacceptable. In Fig. 12, we

Fig. 12 Profile of ternary preferences in Example 6

T_1	T_2	T_3
x_1	x_2	x_3
——	x_3	——
x_2	——	x_1
——	x_1	——
x_3	——	x_2

Fig. 13 Profile of ternary preferences in Example 7

T_1	T_2	T_3	T_4	T_5
$x_1\ x_2$	x_3	x_4	x_2	$x_1\ x_3$
——	——	——	x_4	——
x_3	x_4	$x_1\ x_3$	——	x_4
——	——	——	x_1	——
x_4	x_2	x_2	——	x_2
	x_1		x_3	

consider one of the 343 possible profiles of ternary preferences of $\mathcal{T}_0(\{x_1, x_2, x_3\})^3$ based on the original profile of linear orders.

We obtain the following scores: $B^t(x_1) = \varepsilon + \alpha + \beta$, $B^t(x_2) = 1 - \beta$ and $B^t(x_3) = \varepsilon + \alpha$. If we take $\alpha = 0.4$, $\beta = 0.3$ and $\varepsilon = 0.1$, then $B^t(x_1) = 0.8 > 0.7 = B^t(x_2) > 0.5 = B^t(x_3)$. Thus, $x_1 \, P^t \, x_2 \, P^t \, x_3$. Therefore, the cycle and the tie that appeared when using simple majority and the Borda rule, respectively, in the original profile of linear orders are broken. Obviously, maintaining the original profile of linear orders, the outcome could change depending on which ternary preferences and values of $\alpha, \beta, \varepsilon$ are considered.

Example 7 Consider the profile $(T_1, T_2, T_3, T_4, T_5) \in \mathcal{T}_0(\{x_1, x_2, x_3, x_4\})^5$ included in Fig. 13.

We obtain the following collective Borda-ternary scores:

$$
\begin{aligned}
B^t(x_1) &= \varepsilon + \alpha + \beta \\
B^t(x_2) &= 1.5 - \alpha \\
B^t(x_3) &= 0.5 + \varepsilon + \alpha + \beta \\
B^t(x_4) &= \varepsilon + 2\alpha.
\end{aligned}
$$

Taking into account the restrictions appearing in Remark 5, $0 < \beta < \alpha < 0.5$, we have $B^t(x_3) > B^t(x_4)$, $B^t(x_4) > B^t(x_1)$ and $B^t(x_4) > B^t(x_2)$. On the other hand, $B^t(x_2) > B^t(x_1) \Leftrightarrow \varepsilon + 2\alpha + \beta < 1.5$. Then, depending on the values of the parameters $\alpha, \beta, \varepsilon$, we obtain the orders included in Table 2. For instance, if $\alpha = 0.4$, $\beta = 0.3$ and $\varepsilon = 0.1$, then $x_2 \, P^t \, x_1$; if $\alpha = 0.45$, $\beta = 0.4$ and $\varepsilon = 0.2$, then $x_1 \, I^t \, x_2$; and if $\alpha = 0.45$, $\beta = 0.4$ and $\varepsilon = 0.3$, then $x_1 \, P^t \, x_2$.

Table 2 Collective orders generated by the Borda-ternary voting systems in Example 7

ε+2α+β<1.5	ε+2α+β=1.5	ε+2α+β>1.5
x_3	x_3	x_3
x_4	x_4	x_4
x_2	x_1 x_2	x_1
x_1		x_2

Table 3 Collective orders generated by the Borda rule, the Majority Judgment voting system and the Broken Borda rule in Example 7

Borda rule	Majority Judgment	Broken Borda rule
x_4	x_1 x_3 x_4	x_3 x_4
x_1	x_2	x_1
x_3		x_2
x_2		

We note that approval voting generates a total tie between the four alternatives (all of them are acceptable to two voters). In Table 3 we show the orders generated by the Borda rule, the Majority Judgment voting system and the Broken Borda rule (in the last case we have considered two categories: acceptable and neutral-unacceptable). Note that all the outcomes are different.

If in the aggregation process we consider the social outcome as an entity that represents individuals opinions in the same way as individuals do, then that social outcome should be a ternary preference of $\mathcal{T}_0(X)$ with the same features as in the individual case. Under this assumption, by Remark 3, the sign of the collective Borda-ternary scores would determine which alternatives are socially acceptable, neutral of unacceptable.

In the following two results we provide necessary and sufficient conditions, involving the value of ε, that relate social acceptability (unacceptability) to positive (negative) collective scores.

Proposition 2 *Given $0 < \alpha, \beta < 1$, any alternative acceptable to a majority of voters obtains a positive collective score if and only if*

$$\varepsilon > \frac{mn - m - n + 1}{2}.$$

Proof If m is odd, then the minimum collective score of an alternative x_i that is acceptable to a majority of voters is obtained whenever $(m+1)/2$ voters consider x_i the single worst acceptable alternative and all other alternatives are also acceptable, while $(m-1)/2$ voters consider x_i the single worst unacceptable alternative and, as $\alpha, \beta < 1$, all other alternatives are also unacceptable:

$$B^t(x_i) = \frac{m+1}{2} \cdot \varepsilon + \frac{m-1}{2} \cdot \big(-(n-1+\varepsilon) \big).$$

After some computations, we obtain

$$B^t(x_i) = \varepsilon - \frac{(m-1)\cdot(n-1)}{2}.$$

Consequently, $B^t(x_i) > 0$ if and only if $2\varepsilon > mn - m - n + 1$.

If *m* is even, following arguments similar to those of the odd case, the minimum collective score of an alternative x_i that is acceptable to a majority of voters is:

$$B^t(x_i) = \left(\frac{m}{2}+1\right)\cdot\varepsilon + \left(\frac{m}{2}-1\right)\cdot\big(-(n-1+\varepsilon)\big).$$

After some computations, we obtain

$$B^t(x_i) = 2\varepsilon - \left(\frac{m}{2}-1\right)\cdot(n-1).$$

Consequently, $B^t(x_i) > 0$ if and only if $2\varepsilon > \left(\frac{m}{2}-1\right)\cdot(n-1)$.

Hence, as

$$\max\left\{(m-1)(n-1), \left(\frac{m}{2}-1\right)\cdot(n-1)\right\} =$$
$$(m-1)(n-1) = mn - m - n + 1,$$

we can assure that, in any case, an alternative acceptable to a majority of voters obtains a positive collective score if and only if $2\varepsilon > mn - m - n + 1$, i.e., $\varepsilon > \dfrac{mn-m-n+1}{2}$. □

In the next proposition we establish a similar result to the previous one concerning unacceptable alternatives.

Proposition 3 *Given $0 < \alpha, \beta < 1$, any alternative unacceptable to a majority of voters obtains a negative collective score if and only if*

$$\varepsilon > \frac{mn-m-n+1}{2}.$$

Proof Taking into account that alternatives that are unacceptable to a majority of voters became acceptable in the inverse profile, the result follows from Remark 2 and Proposition 2. □

Remark 6 According to Barokas and Sprumont (2022) in their preference-approval framework, for each voter "replacing the number *mn* in the definition of the broken Borda rule by any number $K > mn - m - n$ produces the very same rule". Therefore, taking this new reduced value, the minimum gap between the scores of the worst acceptable alternative and the best unacceptable one changes from $mn+1$ to $K+1$, becoming $K+1 > mn - m - n + 1$.

It is interesting to note that in our ternary preference approach, for different voters, we have obtained the same value for the shortest jump between positive and negative scores corresponding to acceptable and unacceptable alternatives, i.e., $2\varepsilon > mn - m - n + 1$. In this way, although ε was initially intended for discriminant purposes, that only required $\varepsilon > 0$ (see Remark 3), its magnitude can be relevant, reaching a large value.

Remark 7 Note that the outcomes of the Borda-ternary voting systems are not ternary preferences (as the inputs are), but weak orders. It is possible to generate a collective ternary preference from the obtained weak order by considering two thresholds γ and δ, with $\gamma < \delta$, in the following way[13]:

1. x_i is collectively acceptable if $\dfrac{B^t(x_i)}{m} > \delta$.
2. x_i is collectively neutral if $\gamma \leqslant \dfrac{B^t(x_i)}{m} \leqslant \delta$.
3. x_i is collectively unacceptable if $\dfrac{B^t(x_i)}{m} < \gamma$.

Furthermore, taking into account Remark 2, when $\gamma = -\delta$, a collectively acceptable alternative cannot remain so if all individual ternary preferences are inverted.

Depending on the specific problem, the values of the thresholds γ and δ can be chosen according to the case.

Let us imagine that the human resources team of a company has to cover some positions and each member of the team evaluates the candidates through ternary preferences. Relevant positions may be recruited by considering only collectively acceptable candidates. However, other less important or temporary positions may be recruited by considering collectively neutral candidates as well. In both cases, the order in which the candidates are collectively ranked is relevant.

In Remark 7, δ does not necessarily have to be 0, i.e., not all alternatives that have positive collective Borda-ternary scores necessarily must be considered collectively acceptable[14]. For instance, if an alternative is considered acceptable by only one voter and the rest of the voters declare that the alternative is neutral, it will have a positive collective Borda-ternary score. In fact, δ could be positive, as large as convenient. Furthermore, if necessary, it could be required that, for an alternative to be considered collectively acceptable, a certain proportion of voters would have to declare that alternative acceptable.

[13] To allow thresholds to be set independently of the number of voters, we divide the collective Borda-ternary score by the number of voters, *m*, i.e., we have considered the average of the individual Borda-ternary scores.

[14] This is a clear difference from the case of individual counts, where the sign of Borda-ternary scores determines the class to which the alternatives belong (see Remark 3).

3.4 Properties

To introduce some properties of the Borda-ternary voting systems, we need some additional pieces of notation.

Given a permutation $\pi : V \longrightarrow V$ and a profile $\boldsymbol{T} \in \mathcal{T}_0(X)^m$, with $\boldsymbol{T}_\pi$ we denote the profile $(T_{\pi(1)}, T_{\pi(2)}, \ldots, T_{\pi(m)})$.

If $\sigma : \{1,2,\ldots,n\} \longrightarrow \{1,2,\ldots,n\}$ is a permutation and $T = (R, A, N, U) \in \mathcal{T}_0(X)$, with T^σ we denote $(R^\sigma, A^\sigma, N^\sigma, U^\sigma)$, where $x_i \, R \, x_j \;\Leftrightarrow\; x_{\sigma(i)} \, R^\sigma \, x_{\sigma(j)}$, $x_i \in A^\sigma \;\Leftrightarrow\; x_{\sigma(i)} \in A$, $x_i \in N^\sigma \;\Leftrightarrow\; x_{\sigma(i)} \in N$ and $x_i \in U^\sigma \;\Leftrightarrow\; x_{\sigma(i)} \in U$. Given a profile $\boldsymbol{T} \in \mathcal{T}_0(X)^m$, with $\boldsymbol{T}^\sigma$ we denote the profile $(T_1^\sigma, T_2^\sigma, \ldots, T_m^\sigma)$, where $T_v^\sigma = (R_v^\sigma, A_v^\sigma, N_v^\sigma, U_v^\sigma)$.

If $\boldsymbol{T} \in \mathcal{T}_0(X)^m$, with $\boldsymbol{T}^{-1}$ we denote the profile $(T_1^{-1}, T_2^{-1}, \ldots, T_m^{-1})$.

We now show some properties that the Borda-ternary voting systems satisfy[15].

Proposition 4 *The Borda-ternary voting systems satisfy the following properties:*

1. *Anonymity: For all permutation* $\pi : V \longrightarrow V$ *and* $\boldsymbol{T} \in \mathcal{T}_0(X)^m$, *it holds* $F(\boldsymbol{T}_\pi) = F(\boldsymbol{T})$.
2. *Neutrality: For all permutation* $\sigma : \{1,2,\ldots,n\} \longrightarrow \{1,2,\ldots,n\}$ *and* $\boldsymbol{T} \in \mathcal{T}_0(X)^m$, *it holds* $F(\boldsymbol{T}^\sigma) = F(\boldsymbol{T})^\sigma$.
3. *Reciprocity:* $F\left(\boldsymbol{T}^{-1}\right) = F(\boldsymbol{T})^{-1}$.
4. *R-unanimity: If* $x_i \, R_v \, x_j$ *for every* $v \in V$, *then* $x_i \, R^t \, x_j$.
5. *P-unanimity: If* $x_i \, P_v \, x_j$ *for every* $v \in V$, *then* $x_i \, P^t \, x_j$.
6. *I-unanimity: If* $x_i \, I_v \, x_j$ *for every* $v \in V$, *then* $x_i \, I^t \, x_j$.
7. *Strict Pareto: If* $x_i \, R_v \, x_j$ *for every* $v \in V$ *and* $x_i \, P_u \, x_j$ *for some* $u \in V$, *then* $x_i \, P^t \, x_j$.
8. *Cancellation: If* $\boldsymbol{T} = (T_1, T_2) \in \mathcal{T}_0(X)^2$ *is such that* $T_2 = T_1^{-1}$, *then* $x_i \, I^t \, x_j$ *for all* $x_i, x_j \in X$.

Proof 1. It is straightforward.

2. It is straightforward.
3. For each $v \in V$, let $T_v = (R_v, A_v, N_v, U_v) \in \mathcal{T}_0(X)$ and $T'_v = T_v^{-1} = (R_v^{-1}, U_v, N_v, A_v)$. Taking into account Remark 2, as $B'_v(x_i) = -B_v(x_i)$, we have $(B^t)'(x_i) = -B^t(x_i)$, for every $x_i \in X$. Hence, $F\left(\boldsymbol{T}^{-1}\right) = F(\boldsymbol{T})^{-1}$.
4. For every $v \in V$, if $x_i \, R_v \, x_j$ then $B_v(x_i) \geqslant B_v(x_j)$. Consequently, $B^t(x_i) \geqslant B^t(x_j)$ and $x_i \, R^t \, x_j$.
5. For every $v \in V$, if $x_i \, P_v \, x_j$, then $B_v(x_i) > B_v(x_j)$. Consequently, $B^t(x_i) > B^t(x_j)$ and $x_i \, P^t \, x_j$.
6. If $x_i \, I_v \, x_j$, then $x_i \, R_v \, x_j$ and $x_j \, R_v \, x_i$. Then, *I*-unanimity is a consequence of *R*-unanimity.

[15] Some of them appear in Maniquet and Mongin (2015) in a related scenario.

Fig. 14 Original profile of ternary preferences in Remark 8

T_1	T_2
x_1	x_2
x_3	x_3
———	———
x_2	x_1
———	———
x_4	x_4

Fig. 15 Second profile of ternary preferences in Remark 8

T_1'	T_2
x_1	x_2
x_3	x_3
x_2	———
———	x_1
———	———
x_4	x_4

7. If $x_i \, R_v \, x_j$ for every $v \in V$ and $x_i \, P_u \, x_j$ for some $u \in V$, then $B_v(x_i) \geqslant B_v(x_j)$ for every $v \in V$ and $B_u(x_i) > B_u(x_j)$. Hence, $B^t(x_i) > B^t(x_j)$ and $x_i \, P^t \, x_j$.
8. From Remark 2, we have $B^t(x_i) = B_1(x_i) + B_2(x_i) = B_1(x_i) - B_1(x_i) = 0$ for every $x_i \in X$.□

A consequence of reciprocity is the immunity to the *inversion paradox*: if $x_i \in X$ is the unique winner (i.e., $x_i \, P^t x_j$ for every $x_j \in X \setminus \{x_i\}$), and all voters invert their preferences, then x_i must not be a winner (i.e., at least another alternative is collectively preferred to x_i).

Cancellation implies that if $\boldsymbol{T} = \left(T_1, \ldots, T_{\frac{m}{2}}, T_1^{-1}, \ldots, T_{\frac{m}{2}}^{-1}\right) \in \mathcal{T}_0(X)^m$ (*m* is even), then $x_i \, I^t \, x_j$ for all $x_i, x_j \in X$.

Remark 8 The proposed Borda-ternary voting systems are not monotonic: a greater support of a voter for an alternative does not necessarily favor that alternative. Consider the profile of ternary preferences of $\mathcal{T}_0(\{x_1, x_2, x_3, x_4\})^2$ included in Fig. 14.

The collective score obtained by x_1 and x_2 is $B^t(x_1) = B^t(x_2) = 1 + \alpha + \beta + \varepsilon$, then x_1 and x_2 are tied.

Now, consider that the first voter gives more support to the alternative x_2, becoming acceptable instead of neutral, while it is still worse than x_1 and x_3 and nothing changes for the second voter (see Fig. 15).

In this new situation, the collective scores obtained by x_1 and x_2 are $B^t(x_1) = 2 + \beta + \varepsilon$ and $B^t(x_2) = 1 + \alpha + 2\beta + 2\varepsilon$, respectively. If we take $\alpha = 0.4$, $\beta = 0.3$ and $\varepsilon = 0.1$, then $B^t(x_1) = 2.4 > 2.2 = B^t(x_2)$ and now x_2 is defeated by x_1.

In terms of manipulability, the non-monotonicity of the proposed Borda-ternary voting systems implies that strategic behavior could be counterproductive. In monotonic voting systems, voters know that if they give a higher opinion of an alternative, that alternative has a better chance of being chosen. In our case, that possibility could disappear, as the previous example shows: other alternatives may be favored more than the one the voter wishes to favor.

By their very nature, multi-stage and hybrid voting methods generally violate monotonicity in some of its versions. Therefore, since the proposed Borda-ternary voting systems fall under the latter type of procedures, such unfulfillment must not be considered unexpected.

According to (Felsenthal and Nurmi 2017, p. 87) "it is generally agreed among social choice theorists that a voting method that is susceptible to non-monotonicity suffers from a particularly serious defect. So why are some of these methods [...] actually used? The answer is probably that, if instances of non-monotonicity arise in actual elections, voters or analysts would generally not know that the outcome of the election exemplified some type of non-monotonicity or a closely related paradox, because they would generally not have access to all voters ballots (and hence would not be able to verify how all other voters ranked the competing candidates)". The previous comment also applies for possible strategy aspects arising in non-monotonic voting systems.

3.5 Multiple Criteria

In this subsection, we extend our proposal to the case of multiple criteria, weighting the scores obtained by the alternatives in each criterion.

Let $C = \{c_1, c_2, \ldots, c_q\}$ be the set of criteria under which voters evaluate the alternatives. For each criterion $c_k \in C$, voters' opinions are collected in a profile of ternary preferences

$$\boldsymbol{T}^k = \big((R_1^k, A_1^k, N_1^k, U_1^k), (R_2^k, A_2^k, N_2^k, U_2^k), \ldots, (R_m^k, A_m^k, N_m^k, U_m^k)\big) \in \mathcal{T}_0(X)^m.$$

Following the same pattern of Def. 4, B_v^k is the individual count of voter $v \in V$ for criterion $c_k \in C$.

Since the criteria may have different importance, a weighting vector is usually considered: $(w_1, w_2, \ldots, w_q) \in [0,1]^q$ such that $w_1 + w_2 + \cdots + w_q = 1$, where w_k is the weight corresponding to criterion c_k, $k = 1, 2, \ldots, q$. Then, the collective Borda-ternary score of the alternative $x_i \in X$ under criterion $c_k \in C$ is defined as

$$B_k^t(x_i) = \sum_{v=1}^{m} B_v^k(x_i).$$

Finally, the collective overall Borda-ternary score of the alternative $x_i \in X$ is defined as

$$B^t(x_i) = \sum_{k=1}^{q} w_k \cdot B_k^t(x_i).$$

Similarly to Def. 7, now alternatives are ranked through the collective overall Borda-ternary scores:

$$x_i \, R^t \, x_j \;\Leftrightarrow\; B^t(x_i) \geqslant B^t(x_j).$$

4 Concluding Remarks

The proposed voting systems extend both the Borda count and approval voting, allowing voters a higher expressiveness. Voters not only sort the alternatives into three classes, but they may also rank acceptable and unacceptable alternatives. The scores that voters implicitly assign to the alternatives follow a rational procedure: positive (negative) scores to acceptable (unacceptable) alternatives in decreasing order, from best to worst, and null scores to neutral alternatives. As in the Borda count and in approval voting, the alternatives are ordered according to the sum of the individual scores.

The parameters used in the voting systems allow greater or lesser importance to be given to acceptable, neutral and unacceptable alternatives. This flexibility is an important feature of the proposal, but as we have discussed in Subsection 3.2, some restrictions are needed. As is usual when using parameters in different scenarios (multi-criteria decision making, scoring rules, welfare economics, etc.), several problems arise: who chooses the values of the parameters, and when and how are they determined?

For each specific decision-making problem, a decision maker or a group of experts can establish these values taking into account the importance given to acceptable, neutral and unacceptable alternatives. In any case, a sensitive analysis can be done.

Depending on the specific decision-making scenario, the values of the parameters α, β and ε can be fixed before or after voters express their opinions on the alternatives. For instance, if a company wants to know the honest opinions of a group of experts to make a decision, it is not necessary to set parameter values before the experts provide their opinions. However, in other situations where transparency of the process is important, it might be appropriate to announce parameter values before voters express their opinions. In that case, some voters might act strategically, especially if they know the opinions of other voters and the number of voters is small (for instance, in a committee).

In the Majority Judgment voting system (Balinski and Laraki 2007, 2011), voters do not have to rank the alternatives. However, a voter that sorts several alternatives in the same category (for instance, 'acceptable'), can prefer some alternatives to others. The ternary preference approach can be generalized to more than three categories, which allows voters to express even greater expressiveness.

The proposed voting systems can be extended to the family of scoring rules following the same pattern, changing the Borda scores for new ones.

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Declarations

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