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Decision Support

Characterizing competition ranks within a comprehensive family of position operators

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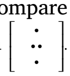
ABSTRACT

There is only one way to assign positions to objects arranged in linear orders: following the sequence of natural numbers (1, 2, 3, 4, …). However, in weak orders, where ties arise, there are different possibilities to assign positions to tied objects. In this paper, we focus mainly on three relevant cases: the standard, modified, and fractional ranks. They are differentiated by the spaces that appear after, before, or on either side of the position values corresponding to the objects that are in a tie. For instance, if two objects are tied and are located immediately below the top object, these ranks assign the positions (1, 2, 2, 4, …), (1, 3, 3, 4, …), and (1, 2.5, 2.5, 4, …), respectively. Collectively, and because of the common properties shown here, we call them "competition ranks". In this paper, we characterize a parameterized family of position operators which includes the competition ranks. We also provide specific axiomatizations of each of them, taking into account the spaces in the sequence of assigned position numbers. It is shown why the dense rank (1, 2, 2, 3, …), another position operator where gaps do not appear, is an essentially different approach. Furthermore, interesting duality relationships are revealed between the competition ranks and between the properties introduced to characterize them, which allow us to understand their internal logic and connections. Different examples, mainly from sports, bibliometrics, etc., illustrate the introduced concepts.

1. Introduction

In the following text[1], which deals with the treatment of ties in the setting of ranking data and related contexts, the principal notions appearing in this paper are introduced in a non-formal pedagogical way:

> It is not always possible to assign rankings uniquely. For example, in a race or competition two (or more) entrants might tie for a place in the ranking […] In this case, one of the strategies below for assigning the rankings may be adopted.
>
> A common shorthand way to distinguish these ranking strategies is by the ranking numbers that would be produced for four items, with the first item ranked ahead of the second and third (which compare equal) which are both ranked ahead of the fourth . These names are also shown below.
>
> ⊳ Standard competition ranking ("1 2 2 4" ranking) […]
>
> ⊳ Modified competition ranking ("1 3 3 4" ranking) […]
>
> ⊳ Ordinal ranking ("1 2 3 4" ranking) […]
>
> ⊳ Fractional ranking ("1 2.5 2.5 4" ranking) […]
>
> ⊳ Dense ranking ("1 2 2 3" ranking) […]

The latter is called "dense" due to the absence of gaps in the sequence of positions[2], while the standard, modified and fractional ranks consider distinct ways of jumping between assigned numbers when ties arise, leaving spaces after, before, or on both sides of repeated positions,

[2] We employ "dense rank", as in García-Lapresta and Martínez-Panero (2024), avoiding "dense ranking". The same stands for the standard, modified, fractional and ordinal ranks. Also, instead of "ranking numbers" assigned to the alternatives under a weak order, we use the term "positions".

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[1] Taken from *Wikipedia*: Strategies for handling ties, in https://en.wikipedia.org/wiki/Ranking.

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respectively. In each of these cases, the jumps depend on the number of ties. On the other hand, the ordinal rank assigns values sequentially, like the dense rank, but it does not allow repetitions, forcing a tiebreaker at random or arbitrarily among shared positions[3]. However, it is worth noting that none of these methods (including ordinal ranking) performs a tiebreaker among alternatives.

Although some authors have used other names[4], the previous terminology, together with self-descriptive mnemonic help, has become common in the literature. Let us mention some examples: in contest theory, Vojnović (2016) provides formal definitions of all these concepts from the same situation involving four objects; Cline (2021) reproduces (with slight variants and without any reference) the quotation above when dealing with rank-based examinations using the R program; and Dunaiski et al. (2018) as well as Orduña-Mallea and Peŕez-Esparrells (2021) consider these possibilities for ranking universities and academic entities depending on the numeric approach adopted. Examples available online are even more common than those appearing in traditional or academic sources.

As the *competition ranks* appearing in the very title of the paper are not coined as such in the literature[5], a comment on this expression is convenient at this point. The term "competition" usually appears jointly with the standard and modified ranks (e.g., in all the references above, except Dunaiski et al., 2018). On the other hand, "competition" is rarely shown alongside the fractional rank[6]. However, as we demonstrate in this paper, all these ranks share connections and similar features, so that we have gathered all of them, yielding the class of "competition ranks".

Within this class, one or another way in which objects in a tie are ranked can be relevant for certain purposes and should be considered in each specific situation. For example, in the recent bibliometric scenario based on the publication of the *Journal Citation Reports* (JCR) from 2023, one decimal place, instead of the three previously considered, is shown in the Journal Impact Factor (JIF) as calculated by *Clarivate Analytics* in the *Web of Science* database. According to Edmunds (2003), "with the move to one decimal, ties will be more common [and] the longstanding approach for JCR is to assign journals with the same JIF in the same category with the same rank position, skipping the ranking position or positions for the journal with the next lower JIF value. This is commonly known as sparse rank[7]". This situation can be observed in the *Operations Research and Management Science* category in 2024. That year, three journals after EJOR, which ranked 13, shared the same JIF. Table 1 shows how, as mentioned above, they shared position 14, and subsequently two places were skipped.

Edmunds (2003) also points out that "rankings for ties can be handled in different ways". In this case, if the fractional rank had been used instead of the standard rank, the tied journals would have received a position rank of 15, and would even have dropped to 16 with the modified rank. It is interesting to note that, as quartiles in each category are calculated from positions as inputs[8], in extreme cases some tied journals could belong to an upper or lower quartile depending on the rank used, the current one (the standard rank) being that which presents the best possible results (among competition ranks) for dealing with ties.

Table 1
First tie in 2024 JIF (Operations research and management science). Source: ooir.org.

Rank	Journal	Impact Factor 2024
...	...	...
13	European Journal of Operational Research	6
14	Production Planning & Control	5.4
–	Socio-Economic Planning Sciences	5.4
–	Safety Science	5.4
17	Production and Operations Management	5.1
...	...	...

Anyway, rather than focusing on practical applications, here we seek to understand the internal logic of these competition ranks, usually used in sports, contests, etc. To this aim, it will be shown how the standard, modified and fractional ranks can be included in a comprehensive family of position operators through an aggregation process (or, equivalently, linear parametric interpolation).

We characterize this parameterized family which extends the competition ranks in several ways, being the common property in all these axiomatizations that of uniform variation under ties. Afterwards, we also provide specific characterizations of each competition rank taking into account the gap structure in the sequence of assigned position numbers, and we explain the obtained recurrence with duality arguments.

We have not considered the ordinal rank as a final result because, as commented in footnote 3, it assigns different positions to objects that compare equal (which is an undesirable property), but we have used it as a provisional output in some processes (see later footnote 12 for a more detailed explanation on this exclusion). Finally, we have extensively considered in this paper the dense rank, already characterized in García-Lapresta and Martínez-Panero (2024), but we have shown that it is essentially different to the previous ranks, and does not belong to the introduced parameterized family nor can be obtained by aggregation in a similar way (moreover, it does not belong to the broader class of representable position operators).

The paper is organized as follows. Section 2 introduces the notation concerning preferences over alternatives, and proposes the codification of weak orders followed throughout the paper. In Section 3, we define position operators, focusing on the competition ranks obtained by aggregation through tie-breaking processes, which allows us to include these and other ranks in a broad class of representable position operators. Section 4 provides characterizations of the competition ranks within a parameterized family of position operators, while Section 5 presents axiomatizations of each particular competition rank attending to the gap structure in the sequence of assigned position numbers. Section 6 sheds some more light on properties and ranks taking into account duality by inversion. Section 7 presents some conclusions and suggests some further research lines. Finally, some technical proofs have been omitted in the main text, although they appear in the Appendix.

2. Codification of weak orders

Consider a finite set of alternatives (or objects) $X = \{x_1, x_2, \dots, x_n\}$, with $n \geqslant 2$. A *weak order* (or *complete preorder*) on X is a complete[9] and

[3] Another interesting possibility, close in some aspects to the ordinal rank, appears in Grzegorzewski (2006), where an extension of Kendall's coefficient is proposed in a context of partial preorders. To this aim, this author models rankings through IF-sets but, previously, he motivates the need of dealing with missing information or hesitance. In this last case, the alternatives in the arrangement above would be assigned: 1 (2 or 3) (2 or 3) 4. However, it is pointed out that "if tied observations also appear then the most common practice for dealing with them, as in most other nonparametric procedures, is to assign equal ranks to indistinguishable observations", which is not true for the ordinal rank.

[4] Fine and Fine (1974) coined the terms *strict*, *weak* and *average* corresponding to the standard, modified and fractional ranks, respectively. Also, the fractional rank was called *mid-rank* by Kendall (1945, 1948).

[5] Note, however, that given a sequence S of numbers, Kammer et al. (2025) define: "The *competitive rank* of each $x \in S$ is the number of elements in S that are smaller than x. The *dense rank* of each $x \in S$ is the number of distinct elements in S that are smaller than x, i.e., competitive rank counts duplicate elements and dense rank does not".

[6] Some exceptions are https://documentation.sas.com/doc/en/imlug/15.2/imlug_langref_sect399.htm and https://rosettacode.org/wiki/Ranking_methods.

[7] In this paper it is called standard rank.

[8] See Edmunds (2003) for details.

[9] A binary relation R on X is complete if $x_i \, R \, x_j$ or $x_j \, R \, x_i$, for all $x_i, x_j \in X$.

transitive[10] binary relation on X. A *linear order* on X is an antisymmetric[11] weak order on X. With $\mathcal{W}(X)$ and $\mathcal{L}(X)$ we denote the sets of weak and linear orders on X, respectively. Given $R \in \mathcal{W}(X)$, with P and I we denote the asymmetric and symmetric parts of R, respectively: $x_i \, P \, x_j$ if not $x_j \, R \, x_i$; and $x_i \, I \, x_j$ if $(x_i \, R \, x_j$ and $x_j \, R \, x_i)$.

Given $R \in \mathcal{W}(X)$ and a permutation σ on $\{1, 2, \ldots, n\}$, we denote by R^σ the weak order obtained from R by relabeling the alternatives according to σ, i.e., $x_i \, R \, x_j \;\Leftrightarrow\; x_{\sigma(i)} \, R^\sigma \, x_{\sigma(j)}$, for all $i, j \in \{1, 2, \ldots, n\}$. In a similar way, we denote by R^{-1} the weak order obtained from R by inversion, i.e., $x_i \, R^{-1} \, x_j \;\Leftrightarrow\; x_j \, R \, x_i$, for all $i, j \in \{1, 2, \ldots, n\}$.

Given $R \in \mathcal{W}(X)$ and $Y \subseteq X$, the restriction of R to Y, $R|_Y$, is defined as $x_i \, R|_Y \, x_j$ if $x_i \, R \, x_j$, for all $x_i, x_j \in Y$. Note that $R|_Y \in \mathcal{W}(Y)$.

In turn, $\#Y$ is the cardinality of Y.

Given $R \in \mathcal{W}(X)$, we next consider the number of alternatives preferred and indifferent to $x_i \in X$. Following the convention of representing the alternatives from top (best) to bottom (worst), we denote by

$$A_i = \left\{ x_j \in X \mid x_j \, P \, x_i \right\}$$

the set of alternatives ***above*** x_i (or dominating x_i), and by a_i its cardinality, i.e., $a_i = \#A_i$.

Also, we denote by

$$B_i = \left\{ x_j \in X \mid x_i \, I \, x_j \right\}$$

the set of alternatives ***beside*** x_i, itself included, and by b_i its cardinality, i.e., $b_i = \#B_i$.

In this way, $a_i \in \{0, 1, \ldots, n-1\}$, $b_i \in \{1, 2, \ldots, n\}$ and $a_i + b_i \leqslant n$.

In the following proposition we show the basic relationships between the weak order and these values. Its proof appears in the Appendix.

Proposition 1. *Given* $R \in \mathcal{W}(X)$ *and* $x_i, x_j \in X$:

1. $x_i \, P \, x_j \;\Leftrightarrow\; a_i < a_j$.
2. $x_i \, I \, x_j \;\Leftrightarrow\; a_i = a_j$.
3. $x_i \, I \, x_j \;\Rightarrow\; b_i = b_j$.

Note that the introduced values faithfully reflect the original structure. Let us consider again the Wikipedia arrangement $\left[\begin{smallmatrix} \cdot \\ \cdot\;\cdot \\ \cdot \end{smallmatrix}\right]$ to show this. If we label the item on top as x_1, then $a_1 = 0$ (none above) and $b_1 = 1$ (itself counted). If x_2 and x_3 share the middle position, then $a_2 = a_3 = 1$ (one alternative above both of them) and $b_2 = b_3 = 2$ (two items compared equal). Finally, if x_4 is at the bottom, then $a_4 = 3$ (three items above) and $b_4 = 1$ (itself counted). Conversely, it is straightforward that the original structure can be totally recovered from these values a_i and b_i.

Definition 1. Let $D = \{(a, b) \in \{0, 1, \ldots, n-1\} \times \{1, 2, \ldots, n\} \mid a + b \leqslant n\}$. The mapping $C_R : X \longrightarrow D$ is defined as $C_R(x_i) = (a_i, b_i)$.

We say that the vector $\big(C_R(x_1), \ldots, C_R(x_n)\big) \in D^n$ codifies $R \in \mathcal{W}(X)$.

It is easy to check that $\#D = \dfrac{n \cdot (n+1)}{2}$.

Example 1. Consider the following weak order $R \in \mathcal{W}(\{x_1, \ldots, x_{10}\})$

$$\begin{array}{c} x_2 \;\; x_7 \\ x_1 \\ x_5 \;\; x_8 \;\; x_{10} \\ x_3 \;\; x_4 \;\; x_6 \;\; x_9 \end{array}$$

where alternatives in upper rows are preferred to those located in lower rows, while the ones in the same row are indifferent. It is codified through

$$C_R(x_2) = C_R(x_7) = (0, 2)$$
$$C_R(x_1) = (2, 1)$$
$$C_R(x_5) = C_R(x_8) = C_R(x_{10}) = (3, 3)$$
$$C_R(x_3) = C_R(x_4) = C_R(x_6) = C_R(x_9) = (6, 4).$$

Note that the vector

$$\big((2,1), (0,2), (6,4), (6,4), (3,3), (6,4), (0,2), (3,3), (6,4), (3,3)\big) \in D^{10}$$

codifies $R \in \mathcal{W}(X)$, but not all vectors of D^{10} correspond to the codification of a weak order. For example, among other relationships to appear in what follows, the second coordinate b_i for each component (a_i, b_i) must be the number of components that share the first coordinate a_i.

For small values of n, we can make a more accurate description:

- If $n = 2$, then $D = \{(0,1), (0,2), (1,1)\}$ and $\big((0,1), (1,1)\big)$, $\big((1,1), (0,1)\big)$ and $\big((0,2), (0,2)\big)$ are the vectors of D^2 that codify the three weak orders on $\{x_1, x_2\}$. However, the other 6 vectors of D^2 do not codify any weak order on $\{x_1, x_2\}$.
- If $n = 3$, then $D = \{(0,1), (0,2), (0,3), (1,1), (1,2), (2,1)\}$ and only 13 out of 216 triples of D^3 codify the 13 weak orders on $\{x_1, x_2, x_3\}$.
- If $n = 4$, then $D = \{(0,1), (0,2), (0,3), (0,4), (1,1), (1,2), (1,3), (2,1), (2,2), (3,1)\}$ and only 75 out of 1000 4-tuples of D^4 codify the 75 weak orders on $\{x_1, x_2, x_3, x_4\}$.

Exact values of the number of weak orders on X can be found in Santos-García and Alcantud (2025), focusing on the particular case $k = n$ (or, equivalently, $k = 0$) in Theorem 1, which provides the cardinality of preference approval structures on X.

In order to achieve a general description of the structure of those vectors in D^n codifying weak orders on X, we use another way of sorting alternatives already introduced in García-Lapresta and Martínez-Panero (2024).

Given $R \in \mathcal{W}(X)$, for each $a \in \{0, 1, \ldots, n-1\}$, with T^a we denote the set of all the alternatives that have a alternatives above (tier):

$$T^a = \{x_i \in X \mid a_i = a\}.$$

Each tier is an indifference class, because $T^a = B_i$ for every $x_i \in T^a$; and then, $\#T^a = b_i$ (see items 2 and 3 of Proposition 1). Note also that, unless $R \in \mathcal{L}(X)$, some T^a will be empty. This is the reason why we define

$$T = \left\{ a \in \{0, 1, \ldots, n-1\} \mid T^a \neq \emptyset \right\}.$$

Obviously, $T \neq \emptyset$, as $T^0 \neq \emptyset$. From now on, when we say tiers, we are referring to non-empty tiers, i.e., T^a with $a \in T$. Notice that a decomposition of X in tiers, $X = \bigcup\limits_{a \in T} T^a$, is associated with $R \in \mathcal{W}(X)$.

It is interesting to consider only the set of vectors codifying weak orders:

$$D^* = \Big\{ \big((a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\big) \in D^n \mid \exists R \in \mathcal{W}(X)$$
$$\big(C_R(x_1), C_R(x_2), \ldots, C_R(x_n)\big) = \big((a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\big) \Big\}.$$

In the following result we characterize D^* in a constructive recursive way.

Proposition 2. *A vector* $\big((a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\big) \in D^n$ *belongs to* D^* *if and only if*

$$\begin{array}{lcl} \#\big\{ i \in \{1, 2, \ldots, n\} \mid a_i = 0 \big\} = c_0 \geqslant 1 & \text{and} & b_i = c_0, \\ \#\big\{ i \in \{1, 2, \ldots, n\} \mid a_i = c_0 \big\} = c_1 & \text{and} & b_i = c_1, \\ \#\big\{ i \in \{1, 2, \ldots, n\} \mid a_i = c_0 + c_1 \big\} = c_2 & \text{and} & b_i = c_2, \\ \qquad\vdots & \vdots & \quad\vdots \\ \#\big\{ i \in \{1, 2 \ldots, n\} \mid a_i = c_0 + c_1 + \cdots + c_{t-1} \big\} = c_t & \text{and} & b_i = c_t, \end{array}$$

where the recursive process ends whenever $c_0 + c_1 + \cdots + c_t = n$.

The proof appears in the Appendix.

Remark 1. As there is a bijection between weak orders and their codifications, Santos-García and Alcantud (2025) also provide the value of $\#D^*$.

[10] A binary relation R on X is transitive if $(x_i \, R \, x_j$ and $x_j \, R \, x_k)$ implies $x_i \, R \, x_k$, for all $x_i, x_j, x_k \in X$.

[11] A binary relation R on X is antisymmetric if $(x_i \, R \, x_j$ and $x_j \, R \, x_i)$ implies $x_i = x_j$, for all $x_i, x_j \in X$.

Note that the decomposition of X into tiers associated with R can be more precisely described taking into account the structure of valid codifications according to Proposition 2: for example, x_0 belongs to the top tier T^0 if and only if $a_0 = 0$. And x_t belongs to the bottom tier T^t if and only if $a_t + b_t = n$. On the other hand, x_i is just above x_j if and only if $a_j = a_i + b_i$.

3. Assigning positions through aggregation functions

A key idea in what follows is to assign natural numbers to the objects in a sequential manner if there are no ties among them (linear orders); otherwise (weak orders), those objects involved in a tie should share the same position after an aggregation process from the linearized situation in any way (for example, at random). Eventually, in our analysis we may need to deal with some objects added or withdrawn throughout the process of assigning positions. Next, we formalize these considerations.

Definition 2. Given $R \in \mathcal{L}(X)$, the *natural sequential function* on R is the mapping $N_R : X \longrightarrow \{1, 2, \dots, n\}$ that assigns 1 to the alternative ranked first, 2 to the alternative ranked second, and so on:

$$N_R(x_i) = \#\{x_j \in X \mid x_j\, R\, x_i\} = \#\{x_j \in X \mid x_j\, P\, x_i\} + 1 = a_i + 1.$$

In this case, $N_R(x_i)$ will be called the *natural position* of x_i.

The notion of position operator, introduced in García-Lapresta and Martínez-Panero (2024), allows to deal with the alternatives in a dynamical way, similarly to what happens in voting theory when a variable electorate is considered (see Smith (1973)).

Definition 3. Given a universe of alternatives U and $X \subseteq U$ finite, a *position operator* O assigns to each $R \in \mathcal{W}(X)$ a function $O_R : X \longrightarrow \mathbb{R}$. We say that $O_R(x_i)$ is the *position* of the alternative $x_i \in X$ in the weak order R.

Note that we do not impose here any restrictions to the reached position values although, later on, we will include some conditions that position operators may fulfill. For example, it seems reasonable to consider stable position operators, i.e., those that assign the same position to all the alternatives in each indifference class[12].

Perhaps, the most immediate way to assign positions is to use tiers as entity units, and then apply the natural sequential function on them. This could be formalized through the concept of quotient set (see García-Lapresta & Martínez-Panero, 2024), but here we follow a more direct approach.

Definition 4. The *dense rank* is the position operator defined as

$$D_R(x_i) = \#T - \#\{a' \in T \mid a' > a\} = \#\{a' \in T \mid a' < a\} + 1,$$

where $x_i \in T^a$ in $R \in \mathcal{W}(X)$.

Example 2. Following with Example 1, we can obtain the dense rank as the number of tiers above plus one, as appearing in Table 2.

Another possible way to achieve stable position operators is that all the alternatives in each tier or indifference class share the same value as the result of applying a compensative aggregation function to their positions, if they were randomly linearized (ordinal rank). We note that, previously, a similar idea was successfully used in a voting context by García-Lapresta and Martínez-Panero (2017).

An *aggregation function* is a function $G : [0,1]^t \longrightarrow [0,1]$ that fulfills the boundary conditions: $G(0,0,\dots,0) = 0$ and $G(1,1,\dots,1) = 1$, and monotonicity: for all $(y_1, y_2, \dots, y_t), (z_1, z_2, \dots, z_t) \in [0,1]^t$ such that $y_1 \geqslant z_1, y_2 \geqslant z_2, \dots, y_t \geqslant z_t$, it holds $G(y_1, y_2, \dots, y_t) \geqslant G(z_1, z_2, \dots, z_t)$. Additionally, if $t = 1$, then $G(y) = y$ for every $y \in [0,1]$.

Table 2
Positions under dense rank in Example 1.

R	tier in	tiers above	$D_R(x_i)$
x_2 x_7	T^0	none	1
x_1	T^2	T^0	2
x_5 x_8 x_{10}	T^3	T^0, T^2	3
x_3 x_4 x_6 x_9	T^6	T^0, T^2, T^3	4

An aggregation function $G : [0,1]^t \longrightarrow [0,1]$ is *compensative* (or *averaging*) if $\min\{y_1, y_2, \dots, y_t\} \leqslant G(y_1, y_2, \dots, y_t) \leqslant \max\{y_1, y_2, \dots, y_t\}$, for every $(y_1, y_2, \dots, y_t) \in [0,1]^t$.

It is easy to see that, by monotonicity, compensativeness is equivalent to idempotency: $G(y, y, \dots, y) = y$ for every $y \in [0,1]$.

We can generalize compensative aggregation functions to intervals $[c,d]$ as functions $G : [c,d]^t \longrightarrow [c,d]$ that fulfill

1. Boundary conditions: $G(c, c, \dots, c) = c$ and $G(d, d, \dots, d) = d$.
2. Monotonicity: for all $(y_1, y_2, \dots, y_t), (z_1, z_2, \dots, z_t) \in [c,d]^t$ such that $y_1 \geqslant z_1, y_2 \geqslant z_2, \dots, y_t \geqslant z_t$, it holds
   $G(y_1, y_2, \dots, y_t) \geqslant G(z_1, z_2, \dots, z_t)$.
3. Compensativeness: for every $(y_1, y_2, \dots, y_t) \in [c,d]^t$, it holds
   $\min\{y_1, y_2, \dots, y_t\} \leqslant G(y_1, y_2, \dots, y_t) \leqslant \max\{y_1, y_2, \dots, y_t\}$,
   hence idempotency: $G(y, y, \dots, y) = y$ for every $y \in [c,d]$.
4. If $t = 1$, then $G(y) = y$ for every $y \in [c,d]$.

On aggregation functions, see Beliakov et al. (2007).

Definition 5. Given a compensative aggregation function $G : [1,n]^t \longrightarrow [1,n]$, the *position operator associated with* G is defined as

$$O_R(x_i) = G(a_i + 1, a_i + 2, \dots, a_i + b_i). \quad (1)$$

Note that in the previous definition $t = b_i$.

Example 3. Consider again Example 1. Since

$a_2 = a_7 = 0$ and $b_2 = b_7 = 2$

$a_1 = 2$ and $b_1 = 1$

$a_5 = a_8 = a_{10} = 3$ and $b_5 = b_8 = b_{10} = 3$

$a_3 = a_4 = a_6 = a_9 = 6$ and $b_3 = b_4 = b_6 = b_9 = 4$,

we have

$O_R(x_2) = O_R(x_7) = G(1,2)$

$O_R(x_1) = G(3) = 3$

$O_R(x_5) = O_R(x_8) = O_R(x_{10}) = G(4,5,6)$

$O_R(x_3) = O_R(x_4) = O_R(x_6) = O_R(x_9) = G(7,8,9,10)$.

The most prominent compensative aggregation functions are the minimum, the arithmetic mean, the median and the maximum, which lead by aggregation (according to Definition 5) to the competition ranks, as shown in what follows.

3.1. Minimum

If G is the minimum, Eq. (1) becomes

$$O_R(x_i) = \min\left(a_i + 1, a_i + 2, \dots, a_i + b_i\right) = a_i + 1,$$

i.e., x_i has the best position in the indifference class[13].

[12] We formalize this stability condition as *equality* in Definition 12. As commented before, this is the main reason why, although the ordinal rank will appear as a provisional stage in some forthcoming processes, we do not consider this method to have a similar status to other ranks (competition, dense, etc.) discussed in the paper. Even more, as shown below, all of the latter fit the (deterministic) definition of position operator, while the ordinal rank does not, due to the randomness or arbitrariness required to decide tiebreakers.

[13] In words of Kendall: "ties should all be ranked as if they were the highest member of the tie. This is subject to the obvious disadvantages that it gives different results if one ranks from the other end of the scale and that it destroys the useful property that the mean rank of the whole series shall be $\frac{n+1}{2}$" (see Kendall (1945) and references therein).

Definition 6. The *standard (competition) rank* is the position operator defined as

$$S_R(x_i) = a_i + 1.$$

It is the most common way to establish a ranking with only natural numbers. Just to mention an interesting example of its use in sport competitions, if we identify gold, silver and bronze medals in the Olympic Games with positions 1, 2 and 3, respectively, if there is a tie among laureate athletes and under some circumstances, these medals are awarded according to the standard rank (see García-Lapresta and Martínez-Panero (2024) and references therein for details).

It corresponds to the `RANK` or `RANK.EQ` functions in `Excel`. It also appears as `LOW` in `SPSS` and it is called `min` by the `R` programming language in their methods to handle ties.

It is interesting to mention that the standard rank has been defined here in a similar way to that of the restricted Borda function considered by Gärdenfors (1973) in his analysis of positional voting systems, where also "the alternatives in a tie are assigned the minimum they would have become in any straightening to a linear order". And the same idea is underlying in Bridges and Mehta (1995, Theorem 1.2.1), where a utility function is associated with a weak order by means of the cardinalities of their strict lower sections[14]. On the other hand, Alcantud et al. (2013) and González-Arteaga et al. (2016) have considered the standard rank as a suitable way of assigning positions in the design of consensus measures.

3.2. Arithmetic mean

If G is the arithmetic mean, Eq. (1) becomes (using the expression of the sum of terms in an arithmetic progression)

$$O_R(x_i) = \frac{(a_i+1)+(a_i+2)+\cdots+(a_i+b_i)}{b_i} = \frac{\dfrac{(a_i+1)+(a_i+b_i)}{2}\cdot b_i}{b_i} = a_i + \frac{b_i+1}{2}.$$

In this way, x_i has the average position in the indifference class[15].

Definition 7. The *fractional rank* is the position operator defined as

$$F_R(x_i) = a_i + \frac{b_i+1}{2}.$$

Remark 2. Since $a_i+1,\, a_i+2, \ldots, a_i+b_i$ are consecutive integer numbers, their median and arithmetic mean coincide. On the other hand, from

$$b_i \cdot F_R(x_i) = (a_i+1)+(a_i+2)+\cdots+(a_i+b_i)$$

in each tier T^{a_i} with $a_i \in T$, we obtain that all the positions assigned by means of the fractional rank add up $1+\cdots+n = \dfrac{n\cdot(n+1)}{2}$, even under ties.

Sometimes the fractional rank is used but not cited by this name. For example, in Cook (2006) several formats of representing ordinal data are surveyed, and that called "vector representation" uses this rank to obtain the corresponding coordinates. On the other hand, the fractional rank corresponds to the `RANK.AVG` function in `Excel` and `MEAN` in `SPSS`. Also, it is called `average` by the `R` programming language. It is used in some situations where positions need to be translated into scores, as happens with some positional voting systems known as scoring rules[16]; or in many sports, where the fractional rank is used (implicitly) in pairwise tournaments or matches where a victory computes 1 point, a tie 0.5 points and a defeat 0 points. In Remark 5 we will develop this last assertion.

3.3. Maximum

If G is the maximum, Eq. (1) becomes

$$O_R(x_i) = \max\left(a_i+1, a_i+2, \ldots, a_i+b_i\right) = a_i + b_i,$$

i.e., x_i has the greatest value in the indifference class.

Definition 8. The *modified (competition) rank* is the position operator defined as

$$M_R(x_i) = a_i + b_i.$$

Therefore, if some elements share the same rank, the worst position would be assigned (if all ties were broken at random). Consequently this rank guarantees that an alternative achieves the position k if and only if there are k alternatives at the same level or higher (what is not true for standard or fractional ranks).

The modified rank corresponds to `HIGH` in `SPSS` and it appears as `max` in the `R` programming language. However, it is not considered in `Excel`. And taking into account the reversal of positions/utilities pointed out in footnote 14, it can be related to the restricted Borda funcion considered by Gärdenfors (1973) rather than to the standard rank. Also, the modified rank has been used in some situations to apply the Ockham's razor principle, because its maximum possible penalization could prevent the duplication of information data (see Singer et al., 2014 and Walk et al., 2015). On the other hand, it is used in some competitions and sports (see later, Remark 5).

Remark 3. Note that $F_R(x_i) = \dfrac{S_R(x_i)+M_R(x_i)}{2}$. In Section 6, some more connections will appear that relate these ranks.

3.4. The Ω family of position operators

We now introduce a family of position operators that generalize the competition ranks. To this aim, notice that the expression of the fractional rank appearing immediately above in Remark 3 uses the arithmetic mean of the standard and modified ranks, but other weighted means can also be employed in the aggregation process. In other words, this can be practically done by allowing any convex combination of the worst and best possible positions in the indifference class of x_i, i.e.:

$$\lambda \cdot \overbrace{(a_i+b_i)}^{M_R(x_i)} + (1-\lambda)\cdot \overbrace{(a_i+1)}^{S_R(x_i)} = a_i + 1 + \lambda\cdot(b_i-1),$$

with $\lambda \in [0,1]$.

Equivalently, this family generalizes competition ranks in a continuous way through linear parametric interpolation, allowing all possible intermediate values between those corresponding to extreme treatments of ties.

[14] Note, however, that in the frameworks of utility and voting functions, the larger, the better; while in the framework of position operators, the smaller the value of the position, the better. Concerning these connections, see Remark 4 in García-Lapresta and Martínez-Panero (2024). Take also into account that if the position of an alternative improves, then its corresponding position number decreases; and, similarly, if the position of an alternative worsens, then its corresponding position number increases.

[15] Again, according to Kendall (1945), "the method of allocating ranking numbers to tied individuals in general use [in ranking correlation methods] is to average the ranks which they cover. This is known as the mid-rank method and is the only one I shall consider".

[16] The arithmetic mean (and, hence, the fractional rank) is related to the most relevant positional voting system, the Borda count, when it is adjusted to allow weak orders in agents' preferences (see, for instance, Gärdenfors (1973), Black (1976) and Cook (2006), among many others). It is worth mentioning that the Borda count (and indirectly the fractional rank) has been used recurrently to achieve rankings of sets of objects based on rankings of the single objects and, related with this approach, in fair division and allocation problems (see Darmann and Klamler (2019) and references therein).

Definition 9. Given $R \in \mathcal{W}(X)$ and $\lambda \in [0,1]$, the position operator Ω^λ is defined as

$$\Omega^\lambda_R(x_i) = a_i + 1 + \lambda \cdot (b_i - 1). \tag{2}$$

We denote by Ω the parameterized family of position operators Ω^λ for $\lambda \in [0,1]$.

Remark 4. Note that $\Omega^0_R(x_i)$, $\Omega^{0.5}_R(x_i)$ and $\Omega^1_R(x_i)$ are the three most relevant members of this family, corresponding to the best, the average (and also the median) and the worst positions in the indifference class of x_i, respectively, i.e., to the standard rank S_R, the fractional rank F_R and the modified rank M_R.

Remark 5. Other values of λ naturally arise or can be considered. For example, in football (soccer) matches before 1994, 2 points were given to the winner, 0 points to the loser, and 1 point to each team in case of a tie. After this date, FIFA changed the victory score from 2 to 3 (wins are worth more than two ties) aiming to encourage the competition (see Garicano & Palacios-Huerta, 2014, Chapter 8).

We can understand this last situation as follows:

- One team wins (position 1, 3 points) and the another team is defeated (position 2, 0 points)
- There is a tie (joint position to be determined, 1 point each).

If we translate scores into positions (usually the process is just the opposite) through a linear affine function (i.e., by linear interpolation), it is easy to check that the resultant joint position is 5/3, obtained by taking $\lambda = 2/3$ in the Ω family[17]. Also it is straightforward that before 1994 the resultant joint position was 1.5, obtained by taking $\lambda = 1/2$, which corresponds to the fractional rank.

Interestingly, in GO Battle League, a feature in the mobile game Pokémon GO[18], ties are a loss for both opponents and only wins are counted. This fact was expressly taken into account in Crane et al. (2021, Algorithm 2) and, following our scheme, it would correspond to consider the extreme value $\lambda = 1$ in the Ω family, corresponding to the modified competition rank[19].

3.5. Other position operators obtained through aggregation

Another relevant family of compensative aggregation functions is that of power means, defined as

$$G_r(y_1, y_2, \ldots, y_t) = \left(\frac{y_1^r + y_2^r + \cdots + y_t^r}{t} \right)^{1/r},$$

with $r \neq 0$.

This family includes some of the previous aggregation functions and introduces some other new[20] :

- The minimum (leading to the standard rank), the maximum (this, to the modified rank) and the geometric mean, by convergence when r tends to $-\infty$, ∞ and 0, respectively;
- The harmonic mean (whose inverse value is called *mean reciprocal rank)*, if $r = -1$;
- The arithmetic mean (leading to the fractional rank), if $r = 1$
- The quadratic mean (leading to the *DuBois rank*, if $r = 2$.

Due to its (theoretical) importance, next we develop the last one.

Table 3
Positions under competition and DuBois ranks in Example 1.

R	$N_R(x_i)$	$S_R(x_i)$	$M_R(x_i)$	$F_R(x_i)$	$B_R(x_i)$
x_2 x_7	1, 2	1	2	1.5	1.58
x_1	3	3	3	3	3
x_5 x_8 x_{10}	4, 5, 6	4	6	5	5.06
x_3 x_4 x_6 x_9	7, 8, 9, 10	7	10	8.5	8.57

3.6. The DuBois rank

As mentioned just above, if G is the quadratic mean, i.e.

$$G(y_1, y_2, \ldots, y_t) = \sqrt{\frac{y_1^2 + y_2^2 + \cdots + y_t^2}{t}},$$

then, Eq. (1) becomes

$$O_R(x_i) = G(a_i + 1, a_i + 2, \ldots, a_i + b_i) =$$

$$\sqrt{\frac{(a_i+1)^2 + (a_i+2)^2 + \cdots + (a_i+b_i)^2}{b_i}}. \tag{3}$$

While the arithmetic mean leads to position values so that they add up $1 + \cdots + n = \dfrac{n \cdot (n+1)}{2}$, ever under ties (see Remark 2), Dubois (1939) considered the use of the quadratic mean instead of the average. In words of Kendall (1945): "Dubois (1939) […] has suggested allotting the ties an equal rank but proposes to determine it so that the sum of squares of the ranks shall be that of an untied ranking, namely, of the first n integers, $1^2 + 2^2 + \cdots + n^2 = \dfrac{n \cdot (n+1) \cdot (2n+1)}{6}$".

After some computations and renaming O_R according to the previous comment, Eq. (3) appears as below.

Definition 10. The *DuBois rank* is the position operator defined as

$$B_R(x_i) = \sqrt{(a_i+1)^2 + \frac{(b_i - 1) \cdot (6a_i + 2b_i + 5)}{6}}.$$

Under linear orders, as $b_i = 1$, the DuBois rank replicates the natural position for x_i, i.e., $B_R(x_i) = a_i + 1$.

Example 4. Following with Examples 1 and 3, taking A as the minimum, maximum, arithmetic mean and quadratic mean of the natural positions in any linearization (ordinal rank), we obtain what appears in Table 3.

Note that all the expressions of the ranks appearing just above involve a_i and b_i through suitable functions. However, it will be shown in what follows that this feature is not shared in general.

3.7. Representable position operators

We now introduce a broad class of position operators that assign positions to the alternatives through a function of their codifying vectors. This means that alternatives with the same codification, even across different preference profiles, should be assigned identical positions.

[17] Equivalent scores leading to the same positions have also been used in chess tournaments. For example, for tie-breaking purposes, Kashdan's system adds four points for each game won, two points for each game drawn, and one point for each game lost (see *Wikipedia*: Kashdan, in https://en.wikipedia.org/wiki/Tie-breaking_Swiss-system_tournaments).

[18] More concretely, it is a matchmaking system where players compete against each other in online trainer battles around the world, earning rewards and improving their global ranking. Additional information can be found at https://pokemongo.fandom.com/wiki/GO_Battle_League.

[19] In this way, if the Olympic athletes in a tie had to decide between breaking the tie or be awarded the corresponding medal following the modified rank, they would surely choose the tiebreaker, which allows them to improve. The current use of the standard rank does not encourage the competition, because it assigns the best (maximum) possible position (see García-Lapresta and Martínez-Panero (2024)).

[20] Other compensative aggregation functions can be considered; for instance, other OWA operators (the maximum, the arithmetic mean, the median and the minimum are also specific cases of OWA operators) or quasiarithmetic means. On this, see Beliakov et al. (2007, Chapter 2).

Definition 11. A position operator O is *representable* if there exists a function $f : D \longrightarrow \mathbb{R}$ such that

$$O_R(x_i) = f\big(C_R(x_i)\big) = f(a_i, b_i) \tag{4}$$

for every $R \in \mathcal{W}(X)$.

Although the parameterized family Ω and, more generally, those position operators obtained through a tie-breaking aggregation process, as the DuBois rank, are representable (this can be seen from their defining expressions), next we prove that the dense rank is not.

Proposition 3. *The dense rank is not representable.*

Proof. Consider the two following orders:

R_1	R_2
$x_1\ x_2$	x_1
x_3	x_2
	x_3

Using the dense rank, x_3 occupies the second and third positions in R_1 and R_2, respectively, i.e., $D_{R_1}(x_3) = 2$ and $D_{R_2}(x_3) = 3$. However $C_{R_1}(x_3) = C_{R_2}(x_3) = (2, 1)$; thus, if the dense rank were representable, x_3 should have the same position: $f(2, 1)$ in both cases. □

Remark 6. Since the dense rank is not representable, it cannot be obtained by means of any tie-breaking process through any aggregating function A (all of which lead to representable ranks, as mentioned above). If this were possible, according to Eq. (1), in the situation appearing in the proof of Proposition 3, it should be $D_{R_1}(x_3) = D_{R_2}(x_3) = A(2 + 1) = 3$, which is not the case.

In summary, for non-representable position operators, the previous argument shows that the mere knowledge of a_i and b_i, for a particular $x_i \in X$, it might not be enough to establish its position through a function of its codification. A direct consequence is that the dense rank, as a non-representable position operator, is aside the competition ranks and cannot be found in the Ω family for any $\lambda \in [0, 1]$.

4. Characterizations of the parameterized family Ω

We now consider some basic properties that position operators on weak orders might (or should) verify. Note that we do not *a priori* impose compelling requirements in order to assign the same positions to indifferent alternatives, etc. Some of these properties had already appeared in García-Lapresta and Martínez-Panero (2024).

Definition 12. Let O be a position operator and $O_R : X \longrightarrow \mathbb{R}$ the function that assigns a position to each alternative of X in the weak order $R \in \mathcal{W}(X)$. We say that the position operator O satisfies the following conditions, when they are fulfilled for all $X \subseteq U$ finite and $R \in \mathcal{W}(X)$:

1. *Sequentiality*: if $R \in \mathcal{L}(X)$, then $O_R(x_i) = N_R(x_i)$ for every $x_i \in X$.
2. *Equality*: $x_i\, I\, x_j \ \Rightarrow\ O_R(x_i) = O_R(x_j)$, for all $x_i, x_j \in X$.
3. *Monotonicity*: $x_i\, R\, x_j \ \Leftrightarrow\ O_R(x_i) \leqslant O_R(x_j)$, for all $x_i, x_j \in X$.
4. *Neutrality*: $O_{R^\sigma}(x_{\sigma(i)}) = O_R(x_i)$ for every permutation σ on $\{1, 2, \dots, n\}$.
5. *Independence of dominated alternatives*[21]: $O_{R|_{X\setminus\{x_j\}}}(x_i) = O_R(x_i)$, for every $x_i \in X$ such that $x_i\, P\, x_j$.

Remark 7. Sequentiality is a compelling condition of extension to weak orders the particular case of linear orders and their natural positions, formalizing the convention of assigning unit-equidistant positions starting from one if there are no ties.

Equality entails that indifferent alternatives are indistinguishable from a positional point of view.

Monotonicity, a stronger condition than equality, means that the better the alternative, the less the position value, and vice-versa.

Neutrality guarantees an equal treatment of alternatives.

Independence of dominated alternatives requires that the deletion of alternatives below that the one to be assigned a position has no effect on it. It is important to note that this property entails that not only removing, but also adding alternatives below, will not change the positions of the alternatives above, since by successive withdrawals, the situation prior to the enlargement could be attained once more.

Remark 8. Standard, modified, fractional and dense ranks satisfy all the above properties (see García-Lapresta & Martínez-Panero, 2024 for details). The same stands for the DuBois rank[22].

Consequently, as we are interested in characterization results for the family Ω enclosing the standard, modified and fractional ranks (but not the dense rank, which stands aside of them (see Proposition 3 and, later on, Remark 13), those properties in Definition 12 will not be selective enough for our purposes; so we have to analyze further appropriate conditions capturing the very essence of the parameterized family.

We first introduce a set of properties concerning alternatives not affected by ties.

Definition 13. We say that a position operator O satisfies the following conditions, when they are fulfilled for all $X \subseteq U$ finite and $R \in \mathcal{W}(X)$:

1. *Natural subsequency*: if $R \in \mathcal{W}(X)$, then $O_R(x_i) = a_i + 1$ for every $x_i \in X$ such that $b_i = 1$.
2. *Primacy*: if $R \in \mathcal{W}(X)$ and $x_i\, P\, x_j$ for every $x_j \neq x_i$, then $O_R(x_i) = 1$.
3. *Ultimacy*: if $R \in \mathcal{W}(X)$ and $x_j\, P\, x_i$ for every $x_j \neq x_i$, then $O_R(x_i) = n$.

Remark 9. Natural subsequency means that if there are no ties at a tier, being this a singleton, then the position of its unique element, subsequent to those above, must be precisely the number of such previous alternatives, whatever their arrangement will be, plus one (itself counted). It is a stronger condition than sequentiality and it also implies primacy and ultimacy as particular cases: if an alternative stays alone on the top or at the bottom, its position will be 1 or n, respectively (these and other relations will be gathered in Proposition 4).

Also, in the following characterizations we will consider some other properties related to the positional behavior of alternatives when affected by the appearance of new others (or the withdrawal of existing ones).

Definition 14. We say that a position operator O satisfies the following conditions, when they are fulfilled for all $X \subseteq U$ finite and $R \in \mathcal{W}(X)$:

1. *Posteriority*: if $R' \in \mathcal{W}(X')$, with $X' = X \cup \{x_{n+1}\}$ such that $x_{n+1} \notin X$, $R'|_X = R$ and $x_{n+1}\, P'\, x_i$ for some $x_i \in X$, then $O_{R'}(x_i) = O_R(x_i) + 1$.
2. *Uniform variation*: if $R' \in \mathcal{W}(X')$, with $X' = X \cup \{x_{n+1}\}$ such that $x_{n+1} \notin X$, $R'|_X = R$ and $x_{n+1}\, I'\, x_i$ for some $x_i \in X$, then there exists $\lambda \in [0, 1]$ such that $O_{R'}(x_i) = O_R(x_i) + \lambda$.

Remark 10. Posteriority means that if a new alternative appears above another one staying below it, then the position of the latter worsens by exactly one unit (equivalently, the alternative above can also be removed and in such case the position of any alternative below improves by one unit).

Uniform variation can be explained in a dynamical manner as follows: if a considered alternative has a (provisional) position in a situation in progress and, finally, another appearing alternative reaches

[21] In fact, this condition is equivalent to that of *truncation* appearing in García-Lapresta and Martínez-Panero (2024) characterizing the dense rank.

[22] In fact, it is straightforward that all representable position operators (a class including the competition and DuBois ranks), due to their expression just involving a_i and b_i, verify neutrality (and hence equality) and independence of dominated alternatives.

the former one in a tie, then the previous position number can increase by at most a step not greater than one, being such step uniform, i.e., regular at any tier (again, the withdrawal of an alternative at the same level also means a decrease of the position number by the same step of those remaining at the same level). In particular, duplication, a property appearing in the characterization of the dense rank (García-Lapresta & Martínez-Panero, 2024), implies uniform variation when the step is null ($\lambda = 0$).

Notice that these properties establish what happens when new alternatives are considered above or at the same level than that already positioned. Their natural complement, concerning alternatives below, is independence of dominated alternatives, already defined (see Definition 12).

Proposition 4.

1. *Primacy and posteriority together imply natural subsequency.*
2. *Independence of dominated alternatives and ultimacy together imply natural subsequency.*
3. *Sequentiality and independence of dominated alternatives imply primacy.*
4. *Sequentiality and posteriority together imply ultimacy.*
5. *Natural subsequency implies sequentiality, primacy and ultimacy.*

Proof.

1. Suppose that $C_R(x_i) = (a_i, 1)$ and first, applying posteriority a_i times, delete all a_i alternatives dominating x_i, i.e, belonging to the subset $A_i \subseteq X$. Then, due to primacy, we have $O_{R|_{X \setminus A_i}}(x_i) = 1$. Restoring the deleted alternatives for recovering R, and again by posteriority, we finally obtain $O_R(x_i) = 1 + a_i$.
   Notice that the reciprocal does not hold. Although natural subsequency implies primacy (as commented in Remark 9 and will be shown in item 5 of this proposition), it does not imply posteriority: see the DuBois rank in Remark 11.
2. Suppose that $C_R(x_i) = (a_i, 1)$ and first, applying independence of dominated alternatives $n - a_i - 1$ times, delete all $n - a_i - 1$ alternatives dominated by x_i, i.e, belonging to the subset $X \setminus \left(A_i \cup \{x_i\}\right)$. Then, due to ultimacy, we have $O_{R|_{X \setminus (A_i \cup \{x_i\})}}(x_i) = a_i + 1$. Restoring the deleted alternatives to recover R, and again by independence of dominated alternatives, we finally obtain $O_R(x_i) = a_i + 1$.
   Also, the reciprocal is not true, because natural subsequency does imply ultimacy as a particular case (again, as commented in Remark 9 and will be shown in item 5 of this proposition), but not independence of dominated alternatives: it suffices to consider $O_R(x_i) = a_i + 1 + (b_i - 1) \cdot n$, which verifies natural subsequency, but not independence of dominated alternatives.
3. Suppose that x_i dominates all other alternatives. Then, by independence of dominated alternatives, $O_R(x_i) = O_{R|_{\{x_i\}}}(x_i) = 1$, this value coming from sequentiality applied to the singleton $\{x_i\}$.
4. If t x_i is dominated by all other alternatives, then $O_{R|_{\{x_i\}}}(x_i) = 1$, due to sequentiality applied to the singleton $\{x_i\}$. Restoring all other $n - 1$ alternatives above x_i to recover R and iterating posteriority, it should be $O_R(x_i) = 1 + (n - 1) = n$.
5. First, sequentiality trivially holds, just applying natural subsequency to linear orders. On the other hand, primacy and ultimacy are but particular cases of natural subsequency. Indeed, if $x_i \, P \, x_j$ for all $x_j \neq x_i$, then x_i is the only alternative on the top, and hence $C_R(x_i) = (0, 1)$. Then, by natural subsequency $O_R(x_i) = 0 + 1 = 1$. Similarly, if $x_j \, P \, x_i$ for all $x_j \neq x_i$, then x_i is the only alternative at the bottom, and hence $C_R(x_i) = (n - 1, 1)$. Again, by natural subsequency $O_R(x_i) = (n - 1) + 1 = n$.

□

Remark 11. In Table 4, it is shown the fulfillment of the forthcoming characterization conditions by the position operators appearing in this paper.

Table 4
Properties and their fulfillment.

	Ω family	DuBois	Dense rank
Sequentiality	yes	yes	yes
Indep. dom. alt.	yes	yes	yes
Nat. subseq.	yes	yes	no
Primacy	yes	yes	yes
Ultimacy	yes	yes	no
Posteriority	yes	no	no
Uniform var.	yes	no	yes

It is easy to check the total fulfillment of the properties by the Ω family. Taking into account items 1 and 5 of Proposition 4, it suffices to prove primacy, posteriority, independence of dominated alternatives, and uniform variation. These properties easily follow from the expression $\Omega^\lambda_R(x_i) = a_i + 1 + \lambda \cdot (b_i - 1)$:

- Primacy: if x_i dominates all other alternatives, $C_R(x_i) = (0, 1)$ and hence $\Omega^\lambda_R(x_i) = 0 + 1 + \lambda \cdot 0 = 1$.
- Posteriority: if a new alternative arises above x_i, then a_i increases one unit, so that its position also increases one unit.
- Independence of dominated alternatives: if a new alternative arises below x_i, as a_i and b_i do not change, neither does the position of x_i (as mentioned in footnote 22, this is also true for all representable position operators).
- Uniform variation: if a new alternative arises beside x_i, then b_i increases one unit, so that its position also increases one unit multiplied by λ.

The DuBois rank fulfills natural subsequency: due to its expression (see Definition 10), if $b_i = 1$, then $B_R(x_i) = a_i + 1$. Consequently, by item 5 in Proposition 4, sequentiality, primacy and ultimacy also hold. It also satisfies independence of dominated alternatives (again, because it is representable). However, it does not satisfy posteriority: if we extend R in Example 1 by adding a new alternative x_{11} indifferent to x_1, then all three alternatives just below drop from position 5.06 to 6.05, which is less than one unit. And it does not fulfill uniform variation neither: in this new situation, the joint position of both x_1 and x_{11} would be 3.53 instead of 3 for x_1 alone; but if any other new alternative still appears in the top tier, the joint position of the former ones in the top tier would drop from 1.58 to 2.16, with a non-uniform variation: $0.58 \neq 0.53$.

Finally, the dense rank satisfies sequentiality, what is trivial from Definition 4 taking into account that tiers are singletons in linear orders; and primacy, because all the alternatives in the top tier should have position 1. Also uniform variation and independence of dominated alternatives are fulfilled (see Remark 10 and footnote 21, respectively). However, ultimacy does not hold: in the proof of Proposition 3, according to R_1, the alternative x_3 appears alone at the bottom and reaches position 2, although $\#X = 3$. And, consequently, natural subsequency and posteriority do not hold neither (see items 4 and 5 of Proposition 4). Note that the unfulfillment of these last three properties is another argument for excluding the dense rank of the Ω family.

Next we show how some combinations of the aforementioned properties lead to different characterizations of the Ω family.

Theorem 1. *A position operator O satisfies natural subsequency and uniform variation if and only if $O = \Omega^\lambda$ for some $\lambda \in [0, 1]$.*

Proof. We have already pointed out that the parameterized family satisfies these properties (see Remark 11). Conversely, suppose that $C_R(x_i) = (a_i, b_i)$ and first delete all $b_i - 1$ alternatives indifferent to x_i, itself excluded. Then, we have $O_{R|_{(X \setminus T^{a_i}) \cup \{x_i\}}}(x_i) = a_i + 1$, due to natural subsequency. Now, restoring the previously deleted $b_i - 1$ alternatives to recover R, and adding λ each time by uniform variation, we finally obtain $O_R(x_i) = a_i + 1 + \lambda \cdot (b_i - 1) = \Omega^\lambda_R(x_i)$. □

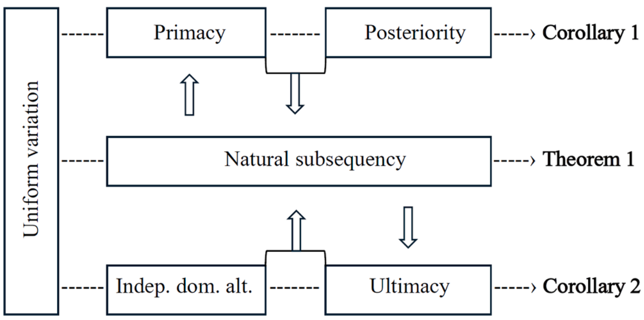

Fig. 1. Properties involved in the characterizations of the Ω family.

The following corollaries provide other characterizations with weaker conditions than natural subsequency, i.e., primacy or ultimacy, but then adding independence of dominated alternatives or posteriority, respectively. They easily follow from Theorem 1 and items 1 and 2 of Proposition 4.

Corollary 1. *A position operator O satisfies primacy, posteriority and uniform variation if and only if $O = \Omega^\lambda$ for some $\lambda \in [0, 1]$.*

Corollary 2. *A position operator O satisfies independence of dominated alternatives, ultimacy and uniform variation if and only if $O = \Omega^\lambda$ for some $\lambda \in [0, 1]$.*

Next we prove the independence of the above characterizing properties. Also, it will be immediately shown why we cannot use just sequentiality, as happened in the characterizations of the dense rank provided by García-Lapresta and Martínez-Panero (2024).

Proposition 5. *The conditions appearing in Theorem 1, Corollaries 1 and 2 are independent.*

Proof. See Remark 11 about the properties mentioned in this proof and their fulfillment by the dense and the DuBois ranks.

1. Concerning Theorem 1, the DuBois rank verifies natural subsequency but not uniform variation. The dense rank verifies uniform variation for $\lambda = 0$, but not natural subsequency.
2. Concerning Corollary 1, the position operator defined as $O_R(x_i) = a_i + 1 + (b_i - 1)^2$ verifies primacy and posteriority, but not uniform variation. The dense rank verifies primacy and uniform variation, but not posteriority. The position operator defined as $O_R(x_i) = a_i + 1 + b_i$ verifies posteriority and uniform variation, but not primacy.
3. Concerning Corollary 2, the DuBois rank verifies ultimacy and independence of dominated alternatives, but not uniform variation. The position operator defined as $O_R(x_i) = n$ verifies ultimacy and uniform variation, but not independence of dominated alternatives. The dense rank verifies independence of dominated alternatives and uniform variation, but not ultimacy.

□

Remark 12. Natural subsequency cannot be weakened by sequentiality in Theorem 1, because the dense rank satisfies sequentiality and uniform variation (see Remarks 8 and 10), but it does not belong to the parameterized family (see Remark 6). This also stands for primacy and posteriority in Corollary 1, as well as for independence of dominated alternatives and ultimacy in Corollary 2 (due to items 1 and 2 of Proposition 4).

Fig. 1 shows an overview of the characterizations obtained for the parameterized family of position operators. Some other relationships between the properties involved will appear in Section 6 (Fig. 2).

5. Particular characterizations of the competition ranks

Next we present specific characterizations of the standard, modified and fractional ranks attending to the size and place of the gaps between position values of the alternatives in connection to their ties.

Proposition 6. *Let $X = T^{l_1} \cup T^{l_2} \cup \cdots \cup T^{l_t}$ be the decomposition of X into tiers, from top to bottom, associated with the weak order $R \in \mathcal{W}(X)$, where $x_{i_k} \in T^{l_k}$. Then, for any position operator O it holds:*

1. *$O_R(x_{i_1}) = 1$ and $O_R(x_{i_{k+1}}) = O_R(x_{i_k}) + b_{i_k}$ if and only if $O_R = S_R$.*
2. *$O_R(x_{i_1}) = b_{i_1}$ and $O_R(x_{i_{k+1}}) = O_R(x_{i_k}) + b_{i_{k+1}}$ if and only if $O_R = M_R$.*
3. *$O_R(x_{i_1}) = \dfrac{b_{i_1} + 1}{2}$ and $O_R(x_{i_{k+1}}) = O_R(x_{i_k}) + \dfrac{b_{i_k} + b_{i_{k+1}}}{2}$ if and only if $O_R = F_R$.*

Proof.

1. First we show that S_R satisfies the above conditions. As $x_{i_1} \in T^{l_1} = T^0$, $a_{i_1} = 0$ and $S_R(x_{i_1}) = 0 + 1 = 1$. In addition, notice that x_{i_k} is just above $x_{i_{k+1}}$; hence, $a_{i_{k+1}} = a_{i_k} + b_{i_k}$. Then, $S_R(x_{i_{k+1}}) = a_{i_{k+1}} + 1 = (a_{i_k} + b_{i_k}) + 1 = (a_{i_k} + 1) + b_{i_k} = S_R(x_{i_k}) + b_{i_k}$.
   Conversely, if O_R satisfies both conditions, it must coincide with S_R. Indeed, the first one ensures $O_R(x_{i_1}) = 1 = 0 + 1 = a_{i_1} + 1 = S_R(x_{i_1})$. And by induction on k, again taking into account the contiguity of x_{i_k} and $x_{i_{k+1}}$, also the second condition is easily checked: $O_R(x_{i_{k+1}}) = O_R(x_{i_k}) + b_{i_k} = S_R(x_{i_k}) + b_{i_k} = (a_{i_k} + 1) + b_{i_k} = (a_{i_k} + b_{i_k}) + 1 = a_{i_{k+1}} + 1 = S_R(x_{i_{k+1}})$, as $O_R(x_{i_k}) = S_R(x_{i_k})$ by induction hypothesis.
2. In a similar way, as $x_{i_1} \in T^{l_1} = T^0$, $a_{i_1} = 0$ and $M_R(x_{i_1}) = 0 + b_{i_1} = b_{i_1}$. And using again the contiguity of x_{i_k} and $x_{i_{k+1}}$ we now obtain $M_R(x_{i_{k+1}}) = a_{i_{k+1}} + b_{i_{k+1}} = (a_{i_k} + b_{i_k}) + b_{i_{k+1}} = M_R(x_{i_k}) + b_{i_{k+1}}$.
   Conversely, if O_R satisfies both conditions, it must coincide with M_R, because the first one ensures $O_R(x_{i_1}) = b_{i_1} = 0 + b_{i_1} = M_R(x_{i_1})$, as $a_{i_1} = 0$. And by induction on k, again taking into account the contiguity of x_{i_k} and $x_{i_{k+1}}$, also we have $O_R(x_{i_{k+1}}) = O_R(x_{i_k}) + b_{i_{k+1}} = M_R(x_{i_k}) + b_{i_{k+1}} = (a_{i_k} + b_{i_k}) + b_{i_{k+1}} = a_{i_{k+1}} + b_{i_{k+1}} = O_R(x_{i_{k+1}})$, as $O_R(x_{i_k}) = M_R(x_{i_k})$ by induction hypothesis.
3. Immediate, taking into account Remark 3.

□

Remark 13. In each case, the values $O_R(x_{i_1})$ jointly with the other conditions relating contiguous positions provide alternative expressions of the competition ranks that are interesting for algorithmic or programming purposes, due to their recursive nature. It is also interesting to emphasize that only by knowing the position values reached under a competition rank and how many alternatives share them (i.e, repetitions in position numbers), it is possible to determine (up to permutations) the corresponding weak order. Next we show this in simple practical situations.

If we know that a competition rank has been used and the positions reached (with repetitions) have been $(1, 2, 2, 4)$, then the standard rank ought to be the one employed, because of the gap after 2, and the original structure would correspond to the Wikipedia example $\begin{bmatrix} \bullet \\ \bullet\,\bullet \\ \bullet \end{bmatrix}$. On the other hand, if the ranks were $(2, 2, 3, 4)$, this would necessarily be produced by the modified rank (due to the gap before 2), being $\begin{bmatrix} \bullet\,\bullet \\ \bullet \\ \bullet \end{bmatrix}$ the corresponding arrangement. Finally, with positions $(1, 2, 3.5, 3.5)$, the only possibility is $\begin{bmatrix} \bullet \\ \bullet \\ \bullet\,\bullet \end{bmatrix}$ under the fractional rank (gaps on both sides of 3.5). Note in each case how gaps in position numbers are directly related to the associated b_i values.

Even more, with competition ranks we would be able to recover the original weak order only by knowing the number of alternatives and the reached positions without repetitions. This does not hold for the

Table 5
Positions and their inverse values.

R	$S_R(x_i)$	$M_R(x_i)$	$F_R(x_i)$	$S_{R^{-1}}(x_i)$	$M_{R^{-1}}(x_i)$	$F_{R^{-1}}(x_i)$
x_2 x_7	1	2	1.5	9	10	9.5
x_1	3	3	3	8	8	8
x_5 x_8 x_{10}	4	6	5	5	7	6
x_3 x_4 x_6 x_9	7	10	8.5	1	4	2.5

dense rank[23], which depends on the global tier structure ignoring tie sizes, and not merely on the local codification of each alternative, as competition ranks do. Definitely, this fact sheds light on how the dense rank essentially differs from the standard, modified and fractional ranks.

Note also that if the position operator O already belongs to the Ω family (which has not been assumed in the statement of Proposition 6), the imposition of the initial values $O_R(x_{i_1})$ can be avoided, because they become forced, leaving only the corresponding recurrence conditions. For example, if $\Omega^\lambda_R(x_i) = a_i + 1 + \lambda \cdot (b_i - 1)$ verifies the recurrence $\Omega^\lambda_R(x_{i_{k+1}}) = \Omega^\lambda_R(x_{i_k}) + b_{i_k}$, it must hold

$$a_{i_{k+1}} + 1 + \lambda \cdot (b_{i_{k+1}} - 1) = a_{i_k} + 1 + \lambda \cdot (b_{i_k} - 1) + b_{i_k}.$$

As $a_{i_k+1} = a_{i_k} + b_{i_k}$,

$$a_{i_k} + b_{i_k} + 1 + \lambda \cdot (b_{i_{k+1}} - 1) = a_{i_k} + 1 + \lambda \cdot (b_{i_k} - 1) + b_{i_k}.$$

Simplifying, $\lambda \cdot (b_{i_{k+1}} - b_{i_k}) = 0$.

Now, to maintain the generality of the result, as $b_{i_{k+1}}$ may be different from b_{i_k}, it will be necessary that $\lambda = 0$, which corresponds to the standard rank (similar arguments can be presented for the other competition ranks).

6. Duality

Suppose a situation where there are no ties and, consequently, the positions are $1, 2, \ldots, n$. But there has been a mistake in the order scale and everything is upside down, so those positions should be reallocated. The problem is trivial in this case, because the new positions should be $n, n-1, \ldots, 2, 1$, where in both sequences the relationship between homologous terms adds up to $n+1$.

More generally, can we recover the positions in a weak order R from those of its inverse order R^{-1}? Table 5 shows what happens with the competition ranks when applied to the weak order corresponding to Example 1, as well as to the inverse.

Some relationships can be observed, and the following concept of dual position operator is helpful to understand and formalize them.

Definition 15. Given a position operator O, its *dual* O^d is the position operator defined as

$$O^d_R(x_i) = n + 1 - O_{R^{-1}}(x_i).$$

Proposition 7. *The standard and modified ranks are mutually dual and the fractional rank is self-dual, i.e.,* $M^d_R = S_R$, $S^d_R = M_R$, $F^d_R = F_R$ *and, in general,* $(\Omega^\lambda_R)^d = \Omega^{1-\lambda}_R$.

Proof. Taking into account Remark 4, it is sufficient to prove the last expression. To this aim, notice that if $C_R(x_i) = (a_i, b_i)$, it also holds that $C_{R^{-1}}(x_i) = (n - a_i - b_i, b_i)$. Then,

$$\Omega^{1-\lambda}_R(x_i) + \Omega^\lambda_{R^{-1}}(x_i) =$$

$$a_i + 1 + (1-\lambda) \cdot (b_i - 1) + n - a_i - b_i + 1 + \lambda \cdot (b_i - 1) =$$

$$1 + n - b_i + (1 - \lambda + \lambda) \cdot b_i = 1 + n - b_i + b_i = 1 + n.$$

Consequently, $\Omega^{1-\lambda}_R(x_i) = n + 1 - \Omega^\lambda_{R^{-1}}(x_i) = (\Omega^\lambda_R)^d(x_i)$. □

Remark 14. It is worth to emphasize the role of the fractional rank as the only self-dual position operator within the Ω family, which will be determinant in Theorem 2. This happens because such dual symmetry forces (and is forced by) the balanced treatment of ties. Indeed, this uniqueness result can be directly obtained taking into account that $\Omega^\lambda_R(x_i) = (\Omega^\lambda_R)^d(x_i)$ if and only if $a_i + 1 + \lambda \cdot (b_i - 1) = n + 1 - (n - a_i - b_i + 1 + \lambda \cdot (b_i - 1))$. Then, after easy computations, we obtain $\lambda \cdot (b_i - 1) = (1 - \lambda) \cdot (b_i - 1))$. As this may occur being $b_i \neq 1$, it follows that necessarily $\lambda = 1 - \lambda$, and hence $\lambda = 0.5$, which corresponds to the fractional rank.

Remark 15. As happens in different contexts where this notion arises, dualization is an involution: $\left(O^d\right)^d = O$.

On the other hand, the dense rank is also self-dual if we change $n+1$ for $\#T + 1$ in Definition 15.

Proposition 8.

1. *Sequentiality is self-dual; i.e., a position operator satisfies sequentiality if and only if its dual also does.*
2. *Primacy and ultimacy are mutually dual; i.e., a position operator satisfies primacy (respectively, ultimacy) if and only if its dual verifies ultimacy (respectively, primacy).*
3. *Posteriority and independence of dominated alternatives are mutually dual; i.e., a position operator satisfies posteriority (respectively, independence of dominated alternatives) if and only if its dual verifies independence of dominated alternatives (respectively, posteriority).*
4. *Natural subsequency is self-dual; i.e., a position operator satisfies natural subsequency if and only if its dual also does.*
5. *Uniform variation is self-dual; concretely, a position operator satisfies uniform variation for* $\lambda \in [0, 1]$ *if and only if its dual also does for* $1 - \lambda$.

Proof. It is enough to prove the direct statements, because the reciprocals are fulfilled by duality (see Remark 15).

1. If the position operator O satisfies sequentiality and $R \in \mathcal{L}(X)$, then $O_R(x_i) = a_i + 1$ for each $x_i \in X$ such that $C_R(x_i) = (a_i, 1)$. Consequently, also $R^{-1} \in \mathcal{L}(X)$ and $O^d_R(x_i) = n + 1 - O_{R-1}(x_i) = n + 1 - ((n - a_i - 1) + 1) = a_i + 1$, because $C_{R^{-1}}(x_i) = (n - a_i - 1, 1)$. Hence, O^d satisfies sequentiality too.
2. Suppose that O satisfies primacy. If x_i stays alone at the bottom in R, it will be on the top in R^{-1}. Then, $O_{R^{-1}}(x_i) = 1$, and hence $O^d_R(x_i) = n + 1 - 1 = n$. Consequently, O^d satisfies ultimacy.

   The statement interchanging the properties can be proven in a similar way.
3. Consider $R' \in \mathcal{W}(X')$, with $X' = X \cup \{x_{n+1}\}$ such that $x_{n+1} \notin X$, $R'|_X = R$ and $x_i \, P' x_{n+1}$ for some $x_i \in X$.

   Then, as we assume that O satisfies posteriority and x_{n+1} is above x_i in R^{-1}, we have $O_{(R')^{-1}}(x_i) = O_{R^{-1}}(x_i) + 1$. Now, taking into account the extra element,

   $(n+1) + 1 - O_{(R')^{-1}}(x_i) =$

   $(n+1) + 1 - (O_{R^{-1}}(x_i) + 1) =$

   $n + 1 - O_{R^{-1}}(x_i),$

   and hence $O^d_{R'}(x_i) = O^d_R(x_i)$, i.e., the dual operator verifies independence of dominated alternatives.

   Again, the statement interchanging the properties can be proven in a similar way.
4. Suppose that O satisfies natural subsequency and x_i stays alone at its tier in R, i.e., $C_R(x_i) = (a_i, 1)$. Then (similarly to what happens in item 1 of this proposition), as it is also alone in its tier in R^{-1}, where $C^{-1}_R(x_i)(n - a_i - 1, 1)$, we have $O_{R^{-1}}(x_i) = (n - a_i - 1) + 1 = n - a_i$. Now

   $n + 1 - O_{R^{-1}}(x_i) = n + 1 - (n - a_i) = a_i + 1.$

[23] As already argued in García-Lapresta and Martínez-Panero (2024) for a situation with three alternatives, if just positions 1 and 3 are reached by them under the standard rank, this necessarily should correspond to the following arrangement: any two of them on top and the other on the bottom. The same stands for positions 2 and 3 with the modified rank, as well as 1.5 and 3 with the fractional rank. However, if the dense rank is used, only knowing that positions 1 and 2 have been occupied, we cannot recover the weak order structure; this information only allows us to affirm that the alternatives are distributed in two tiers, but we cannot determine the cardinality of each tier.

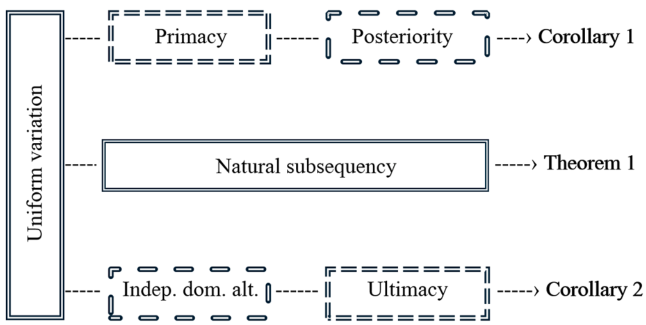

Fig. 2. Duality in characterizing properties.

Consequently, $O_R^d(x_i) = a_i + 1$ and the dual position operator satisfies natural subsequency.

5. Consider $R' \in \mathcal{W}(X')$, with $X' = X \cup \{x_{n+1}\}$ such that $x_{n+1} \notin X$, $R'|_X = R$ and $x_i \, I' \, x_{n+1}$ for some $x_i \in X$.

   Then, as we assume that O satisfies uniform variation for λ and x_{n+1} is also beside x_i in R^{-1}, we have $O_{(R')^{-1}}(x_i) = O_{R^{-1}}(x_i) + \lambda$. Now, taking into account the extra element,

   $$O^d_{R'}(x_i) = (n+1) + 1 - O_{(R')^{-1}}(x_i) =$$
   $$(n+1) + 1 - (O_{R^{-1}}(x_i) + \lambda) =$$
   $$n + 1 - O_{R^{-1}}(x_i) + 1 - \lambda = O^d_R(x_i) + 1 - \lambda,$$

   i.e., the dual operator verifies uniform variation for $1 - \lambda$.

□

Remark 16. Fig. 2 shows the above duality relationships between the properties already appeared in Fig. 1. Here, solid lines frame self-dual properties, while dashed lines with the same design frame mutually dual properties.

On the other hand, notice that duality also explains the different jumps in the recurrent expressions of the standard and modified ranks appearing in items 1 and 2 of Proposition 6, taking into account the following equivalent identities:

$$S_{R^{-1}}(x_{i_k}) = S_{R^{-1}}(x_{i_{k+1}}) + b_{i_{k+1}}$$
$$n + 1 - S_{R^{-1}}(x_{i_k}) = n + 1 - S_{R^{-1}}(x_{i_{k+1}}) - b_{i_{k+1}}$$
$$S^d_R(x_{i_k}) = S^d_R(x_{i_{k+1}}) - b_{i_{k+1}}$$
$$M_R(x_{i_k}) = M_R(x_{i_{k+1}}) - b_{i_{k+1}}$$
$$M_R(x_{i_{k+1}}) = M_R(x_{i_k}) + b_{i_{k+1}}.$$

Next we present another characterization theorem of the fractional rank within the class of representable ranks, that takes into account its symmetry[24].

Theorem 2. *A position operator O is representable and self-dual if and only if $O_R = F_R$ for every $R \in \mathcal{W}(X)$.*

Proof. The fractional rank is representable according to Definitions 7 and 11: its expression just involves a_i and b_i for assigning the position of x_i. It is also self-dual (see Proposition 7).

Conversely, let O be a representable and self-dual position operator. Consider $x_i \in X$ such that $C_R(x_i) = (a_i, b_i)$. Being O representable, there exists a function f such that $O_R(x_i) = f(a_i, b_i)$. Now, as O satisfies independence of dominated alternatives (see footnote 22), if necessary, we can add or remove alternatives below x_i to obtain a new weak order $R' \in \mathcal{W}(X')$ in such a way that x_i has as many alternatives above as below; i.e., $O_{R'}(x_i) = O_R(x_i)$ and $n' - a_i - b_i = a_i$, where $n' = \#X'$. Even more, $C_{R'}(x_i) = C_R(x_i) = (a_i, b_i)$ and also $C_{(R')^{-1}}(x_i) = (n' - a_i - b_i, b_i) = (a_i, b_i)$, so that $O_{R'}(x_i) = O_R(x_i) = O_{(R')^{-1}}(x_i) = f(a_i, b_i)$. Then, by self-duality of O, we have

$$O_{R'}(x_i) = O_{(R')^d}(x_i) = n' + 1 - O_{(R')^{-1}}(x_i) = n' + 1 - O_{R'}(a_i, b_i).$$

Consequently, $2 \cdot O_{R'}(x_i) = n' + 1$, and hence

$$O_R(x_i) = O_{R'}(x_i) = \frac{n' + 1}{2} = \frac{2a_i + b_i + 1}{2} = a_i + \frac{b_i + 1}{2} = F_R(x_i).$$

□

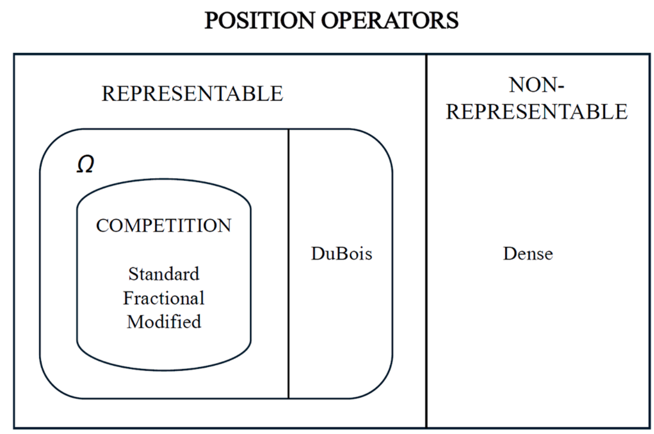

Fig. 3. Map of position operators.

7. Concluding remarks

Gärdenfors (1973, p. 2) asserted that "the positionalist concept is somewhat vague". In his relevant Social Choice analysis, the positionalist voting functions are not defined in a formal precise way, just pointing out that they "are those social choice functions where the positions of the alternatives in the voter's preference orders crucially influence the social ordering of the alternatives". In this way, implicitly, the Borda function takes into account the fractional rank, the restricted Borda function considers the modified rank, and the ranking level function operates with the dense rank. Although Gärdenfors is very careful with the properties that are fulfilled in each case, other authors have used these and other Borda-type procedures indiscriminately and without any precaution (see, for instance, Madani et al. (2014)).

To avoid vagueness or poor implementation in fields like Social Choice, Contest Theory, Bibliometrics, etc., it seems appropriate to establish the foundations of a theory of positions or ranks as a first step towards further developments. In this way, the present paper, as well as García-Lapresta and Martínez-Panero (2024), provide the basis on which positionalist approaches could be built. As a guide, Fig. 3 synoptically shows the place of the competition and the dense ranks with respect to other ranks and families or classes of position operators. The two papers mentioned provide a catalog of properties that characterize them.

As future research we can point out some possible lines. It would be interesting to make a positional analysis of n-tiles (introduced by Galton in 1885), where monotonicity fails (an alternative might be better than another one, and be allocated in the same n-tile). This lack of monotonicity also appears when establishing weights in scoring rules, the main positionalist voting functions (an alternative might be better than another, even sharing the same weights), and this is the reason why the connection between positions and weights should be properly determined. And an important task is to characterize the representable

[24] This symmetry, which was already considered in Remark 14, is the main reason for the extended use of the fractional rank (mid-rank) in correlation analysis (see footnotes 13 and 15). Similarly, note also that the symmetry of the Borda rule (whose relationship to the fractional rank was explained in footnote 16) gives this method a relevant role within the class of scoring rules in Social Choice.

position operators, a result which would clearly place the border between the competition ranks and the dense rank.

CRediT authorship contribution statement

Miguel Martínez-Panero: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization; **José Luis García-Lapresta:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Methodology, Investigation, Formal analysis.

Declaration of competing interest

The authors declare they have no conflict of interest.

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Appendix

Proof of Proposition 1

1. $\Rightarrow$) Suppose $x_i \, P \, x_j$. First we show that $A_i \subseteq A_j$. If $x_k \in A_i$, then $x_k \, P \, x_i$ and, by transitivity of P, we have $x_k \, P \, x_j$, hence $x_k \in A_j$. Since $x_i \in A_j \setminus A_i$, we obtain $A_i \subsetneq A_j$ and, consequently, $a_i < a_j$.
   $\Leftarrow$) By way of contradiction, suppose that $a_i < a_j$ and not $x_i \, P \, x_j$. There exist two cases:
   - $x_j \, P \, x_i$. Following the same reasoning as before, we have $A_j \subsetneq A_i$ and, consequently, $a_j < a_i$, that is a contradiction.
   - $x_j \, I \, x_i$. If $x_k \in A_j$, then $x_k \, P \, x_j$ and, by transitivity of R, we have $x_k \, P \, x_i$, hence $x_k \in A_i$; then, $A_j \subseteq A_i$ and $a_j \leqslant a_i$, that is a contradiction too.
2. $x_i \, I \, x_j$ if and only if neither $x_i \, P \, x_j$ nor $x_j \, P \, x_i$. Taken into account item 1 of this proposition, this is equivalent to both $a_i \geqslant a_j$ and $a_j \geqslant a_i$, i.e., $a_i = a_j$.
3. First we show that $B_i \subseteq B_j$. If $x_k \in B_i$, then $x_k \, I \, x_i$ and, by transitivity of I, we have $x_k \, I \, x_j$, hence $x_k \in B_j$. Analogously, we obtain $B_j \subseteq B_i$. Then, we have $B_i = B_j$, and hence $b_i = b_j$. $\square$

Proof of Proposition 2

In what follows, it is taken into account that $C_R(x_i) = (a_i, b_i)$ if and only if $x_i \in T^{a_i}$ and $\#T^{a_i} = b_i$ (see Proposition 1).

$\Rightarrow$) If $\big((a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)\big) \in D^*$ codifies $R \in \mathcal{W}(X)$, we will show that it has the above structure.

- Take $c_0 = \#T^0 = \#T^{l_1} \geqslant 1$ and every $x_i \in T^{l_1}$ will be represented by $C_R(x_i) = (0, c_0)$.
- Take $c_1 = \#T^{l_2}$ and every $x_i \in T^{l_2}$ will be represented by $C_R(x_i) = (c_0, c_1)$.
- Take $c_2 = \#T^{l_3}$ and every $x_i \in T^{l_3}$ will be represented by $C_R(x_i) = (c_0 + c_1, c_2)$.
- $\vdots$
- Take $c_t = \#T^{l_t}$ and every $x_i \in T^{l_t}$ will be represented by $C_R(x_i) = (c_0 + c_1 + \cdots + c_{t-1}, c_t)$.

And the process necessarily stops whenever $c_0 + c_1 + \cdots + c_t = n$.

$\Leftarrow$) Conversely, if $\big((a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)\big)$ has the structure of the proposition, we will find $R \in \mathcal{W}(X)$ codified by such vector, determining its associated decomposition into tiers, as follows:

- All c_0 couples with $a_i = 0$ will represent those $x_i \in T^{l_1} = T_0$.
- All c_1 couples with $a_i = c_0$ will represent those $x_i \in T^{l_2}$.
- All c_2 couples with $a_i = c_0 + c_1$ will represent those $x_i \in T^{l_3}$.
- $\vdots$
- All $c_t = n - (c_1 + c_2 + \cdots + c_{t-1})$ couples with $a_i = c_1 + c_2 + \cdots + c_{t-1}$ will represent those $x_i \in T^{l_t}$. $\square$

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