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PHD PROGRAM IN MATHEMATICS

DOCTORAL THESIS:

TWO PROBLEMS IN THE LOCAL GEOMETRY OF REAL ANALYTIC DYNAMICAL SYSTEMS. PLANAR DIFFEOMORPHISMS AND THREE DIMENSIONAL HOPF VECTOR FIELDS

Submitted by María Martín Vega in fulfillment
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"Luck is when preparation meets opportunity"

Seneca

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Introduction

This PhD thesis can be framed in the Local Geometry of Real Analytic Dynamical Systems. We deal with two different (but related) problems concerning this topic. The first problem deals with the discrete dynamical systems generated by the iteration of the germ of a two dimensional real analytic diffeomorphism that fixes a point. The second problem deals with the dynamics generated by the flow of a germ of three dimensional real analytic singular vector field. Roughly, the two main results that we present in this text are the following ones:

> **Problem I. Sectorial decomposition of germs $F \in \mathbf{Diff}_1(\mathbb{R}^2, 0)$ of tangent to the identity real analytic plane diffeomorphisms**. Under the (necessary) hypothesis that F is of "non center-focus type", we prove that there exists a partition of a neighborhood U of the fixed point $0 \in \mathbb{R}^2$ into a finite number of topological submanifolds, such that the orbits of F on each submanifold have a uniform well-established asymptotic behavior. As a consequence, we obtain that the set of periodic points of F in U coincides with the set of fixed points. Under some non-degeneracy conditions (that hold in particular when 0 is an isolated fixed point of F), U can be assumed to be a semi-analytic open set and that each stratum is a real analytic submanifold.
>
> **Problem II. Description of the local cycle locus and Dulac's problem for germs of real analytic vector fields at** $(\mathbb{R}^3, 0)$. Here we consider such germs $\xi \in \mathfrak{X}^{\omega}(\mathbb{R}^3, 0)$ with a Hopf singularity at 0; i.e. whose linear part has two conjugated purely imaginary eigenvalues. For these vector fields, we prove that the union of all the cycles (periodic trajectories) in a sufficiently small neighborhood of $0 \in \mathbb{R}^3$ is empty, or equal to a finite number of subanalytic surfaces, or a dense open set (in fact the complement of the singular locus $\mathrm{Sing}(\xi)$). We also give a characterization of the last situation in terms of the analytic linearization of the foliation generated by ξ and in terms of complete integrability. As a consequence, we obtain that there cannot exist infinitely many isolated cycles of ξ accumulating and collapsing to 0 (Dulac's property).

The initial motivation of this PhD thesis was Problem II, that is, we were interested in generalizing Dulac's problem to a higher dimension, starting with the case of the Hopf singularities in dimension three. We obtained first the stated results in II for isolated Hopf singularities, which have already been published in [23]):

N. Corral, M. Martín-Vega, and F. Sanz Sánchez. Surfaces with Central Configuration and Dulac's Problem for a Three Dimensional Isolated Hopf Singularity. *Journal of Dynamics and Differential Equations*, 2024

One important argument in that reference concerned the study of some two dimensional analytic diffeomorphisms, coming from the Poincaré map defined on cross sections of cycles obtained by a process of reduction of singularities of a normal form of ξ. In particular, for such Poincaré maps, we found that the set of periodic points coincides locally with the set of fixed points, a property that permits to describe the cycle locus of ξ.

That study led us to pursue a more ambitious objective: to describe the asymptotic behavior of all orbits of a planar analytic diffeomorphism, coming or not from a Poincaré map of an analytic vector field. That is, we tackled the sectorial decomposition stated in Problem I.

We returned then to the study of vector fields with Hopf singularity. With the results of Problem I in hand, we obtained finally the description of the cycle-locus for all the cases, including those vector fields with a non-isolated singularity (Problem II). As a final result, we gave a characterization of *three dimensional Hopf centers*, namely, the case in which the cycle-locus is an open dense set.

In the rest of the Introduction, we provide more details about Problems I and II, and how are they are framed in the literature. In addition, we give precise statements of the results that we have obtained and outline their proofs.

Problem I: Sectorial decomposition of diffeomorphisms

We have already anticipated that our interest to study this problem relied first on the study of fixed and periodic points of the analytic diffeomorphisms given by Poincaré maps that we find in Problem II. However, the decomposition of the dynamics of $F \in \mathrm{Diff}_1(\mathbb{R}^2, 0)$ is a problem of great independent interest.

The decomposition of real analytic two dimensional vector fields dates back to Poincaré [66] and Bendixon [6] (see also a relatively modern proof by Andronov et al. [3]). It is valid for those $\xi \in \mathfrak{X}^\omega(\mathbb{R}^2, 0)$ with an isolated singularity which are not of *center-focus* type. We say that ξ is of center-focus type if there is no integral curve that accumulates at $0 \in \mathbb{R}^2$ with a defined tangent (also called a characteristic direction). For vector fields which are not of center-focus type, Poincaré and Bendixon obtained that a neighborhood of the singularity can be decomposed into finitely many curvilinear sectors invariant for ξ and where the dynamics of the vector field are uniformly described. Namely, for each sector, all trajectories accumulate at $0 \in \mathbb{R}^2$ in one direction and escape in the other direction (*parabolic sector*), or all trajectories accumulate at 0 in both directions (*elliptic sectors*), or all trajectories escape the sector in both directions (*hyperbolic sector*). The sectorial decomposition is not only useful for understanding the dynamics of $\xi \in \mathfrak{X}^\omega(\mathbb{R}^2, 0)$

around $0 \in \mathbb{R}^2$, but also for understanding the topological properties of the oriented foliation generated by ξ at the singularity. For instance, the Poincaré index $I(\xi, 0)$ can be computed from the sectorial decomposition by Bendixon formula

$$I(\xi, 0) = 1 + \frac{e-h}{2},$$

where e denotes the number of elliptic sectors and h denotes the number of hyperbolic ones. F. Dumortier in [27] also gave a sectorial decomposition of $\mathcal{C}^\infty$ vector fields fulfilling a Łojasiewicz inequality (as well as the non center-focus condition) and proved that it is finitely determined by some jet of the vector field. The Łojasiewicz inequality implies that the vector field has an isolated singularity and that it has a jet truncation which is different from zero, so that we can mirror in this case the proof for analytic vector fields.

The sectors are separated by characteristic curves, also invariant by ξ. Sectors, characteristic curves separating them and the singularity $0 \in \mathbb{R}^2$ form a stratification of a neighborhood by invariant topological manifolds (in fact analytic or $\mathcal{C}^\infty$ if ξ is so) with a parabolic, elliptic or hyperbolic behavior. In higher dimensions, very recently Alonso and Sanz in [1, 2] have generalized the sectorial decomposition for three dimensional vector fields under hypotheses that avoid center-focus behavior.

Our objective is to find the discrete counterpart of the decomposition of the dynamics of two dimensional vector fields, that is, a decomposition of the dynamics of analytic two dimensional diffeomorphisms. It is more difficult since the orbits of a diffeomorphism are discrete sets in $\mathbb{R}^2$, instead of continuous curves. This means that, in principle, wild behavior occurs, even for real analytic diffeomorphisms. As a kind of motivation, we want to recall one of the main results in holomorphic dynamics: Leau-Fatou's Flower Theorem, which provides the description of the dynamics of a holomorphic one dimensional tangent to the identity diffeomorphism $F \in \mathrm{Diff}_1(\mathbb{C}, 0)$. This result was originally stated by L. Leau [52] and P. Fatou [31, 32, 33]. Assuming that $F \neq Id$, if k is the multiplicity of the map $F - Id$, there exist k attracting directions v_i^+ and k repelling directions v_i^-. In addition, for each direction $v_i^\pm$, there is a sectorial region $V_i^\pm$ (a "petal"), bisected by $v_i^\pm$, invariant by $F^{\pm 1}$ where all the positive orbits of $F^{\pm 1}$ accumulate at 0. Moreover, the union $V_1^+ \cup \cdots \cup V_k^+ \cup V_1^- \cup \cdots \cup V_k^-$ is a punctured neighborhood of $0 \in \mathbb{C}$. We can always interpret F as a real analytic diffeomorphism at $(\mathbb{R}^2, 0)$. Hence, the intersections $V_i^+ \cap V_{i+1}^-$ and $V_i^- \cap V_{i+1}^+$ play the role of elliptic sectors and the complements of these elliptic sectors in the petals play the role of parabolic sectors so that we obtain a sectorial decomposition of this type of diffeomorphism F. Concerning also $\mathrm{Diff}_1(\mathbb{C}, 0)$, J. Écalle [30] and S. Voronin [74] obtained in fact a stronger result, the moduli of analytic classification.

On the other hand, if we take a holomorphic diffeomorphism $F \in \mathrm{Diff}(\mathbb{C}, 0)$ which is not tangent to the identity, we can have wild behavior for the orbits. Notably, R. Perez-Marco reveals in [64] certain chaotic behavior if $F'(0) \in \mathbb{S}^1$ and it is not a root of unity. In particular, under some conditions he shows the existence of a "hedgehog" K, which is a compact, connected, full subset of a neighborhood U of 0, invariant for F and such that $0 \notin K^\circ$. This subset contains periodic points of every period $n \in \mathbb{Z}$, i.e. fixed points of F^n. It turns out that nothing that resembles a sectorial decomposition can be found for this type of diffeomorphisms.

Even in the tangent to the identity case, a real analytic diffeomorphism may have infinitely many periodic points of different periods accumulating to $0 \in \mathbb{R}^2$ (see the example below).

Example. The time-1 flow of $\xi = (x^2+y^2)\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)$ is a center-focus diffeomorphism. This diffeomorphisms has expression

$$F(x,y) = (x + (x^2+y^2)(-y - \tfrac{1}{2}xy^2 - \tfrac{1}{2}x^3 + h.o.t), y + (x^2+y^2)(x - \tfrac{1}{2}x^2y - \tfrac{1}{2}y^3 + h.o.t.)).$$

We impose on the diffeomorphisms treated in this memory to be tangent to the identity and of non center-focus type. A real analytic diffeomorphism $F \in \mathrm{Diff}_1^\omega(\mathbb{R}^2, 0)$ is non center-focus if there is a formal invariant curve for F with defined tangent at 0. In the text, we see that this property is given in terms of a reduction of singularities of its infinitesimal generator $\mathrm{Log}(F) = \xi \in \widehat{\mathfrak{X}}(\mathbb{R}^2, 0)$, as it is the case for real analytic vector fields.

The main work aiming for a two dimensional sectorial decomposition has been done by F. Dumortier, P. Rodrigues and R. Roussarie in [29], nearly 40 years ago, and for $\mathcal{C}^\infty$ diffeomorphisms under the non center-focus condition and some extra hypotheses. They required a Łojasiewicz inequality for the diffeomorphisms, which means that the Taylor expansion of $F - Id$ has non-zero dominant terms (it is not flat) and it implies $0 \in \mathbb{R}^2$ is an isolated fixed point of F.

For our purposes, we need to study diffeomorphisms that may have curves of fixed points, then, not necessarily fulfilling the Łojasiewicz inequality. In the case of vector fields with a non-isolated singularity, a sectorial decomposition is obtained directly from a sectorial decomposition of the saturated foliation with isolated singularity. However, there is not an analogy of the saturation of a vector field for diffeomorphisms and curves of fixed points are unavoidable in our study. Introducing curves of fixed points leads to several new scenarios with respect to Dumortier-Rodrigues-Roussarie's work and entails new challenges (we humbly think that our main contribution to Problem I lies in addressing, and somehow solving, some of these difficulties):

- The condition of being $\mathcal{C}^\infty$ is too weak to have a reasonable behavior around the curves of fixed points. We assume analyticity of the diffeomorphism in order to overcome the possible uncontrolled behavior coming from the existence of flat functions.

- Arguments in Dumortier et al. [29] for establishing a sectorial decomposition of F used some technical results that they prove, concerning the possibility of replacing F in some cone around a characteristic invariant curve, by a conjugate diffeomorphism which is embedded in a flow of a $\mathcal{C}^\infty$ vector field. Such results seem to be quite difficult to generalize in order to cover cones of curves of fixed points of F, even with the assumption of analyticity. Without detracting from the conviction that pursuing this type of generalization has a high interest in the theory of dynamical systems, we adopt here more elementary arguments. Taking advantage of the total order of the field of real numbers, we analyze from the expression of F in convenient coordinates zones where F can exhibit certain monotonic behavior of the orbits. Although these arguments cannot be generalized to holomorphic dynamics, several ideas behind our proof could be useful for real analytic diffeomorphisms in higher dimension.

- Once the origin $0 \in \mathbb{R}^2$ is not an isolated fixed point, other fixed points may be accumulation points of nearby orbits. Thus, the three types of sectors, parabolic, elliptic and hyperbolic, are not sufficient to describe the sectorial decomposition theorem. On the other hand, it is conceivable that each point in a curve of fixed points is the center of a sectorial decomposition, and that these decompositions are not uniform while approaching 0, thus obstructing the description of a finite stratification of the local dynamics. We prove in this memory that such a wild behavior never happens.

We state now our main result concerning Problem I in more precise terms. Before stating it, we define some usual concepts. In general, if $F : W \to W'$ is a diffeomorphism between some open sets of $\mathbb{R}^n$ and $A \subset W$, we define the positive orbit $\mathrm{Orb}^+_{F,A}(p)$ issued from p in A to be the subset of A that contains exactly p and every $F^n(p) \in A$ with $n \in \mathbb{N}$ such that $F^l(p) \in A$ for $l \in \mathbb{N}$ with $l < n$. We define the negative orbit $\mathrm{Orb}^-_{F,A}(p)$ issued from p in A to be $\mathrm{Orb}^+_{F^{-1},A}(p)$. Thus, if some iterate $F^n(p)$ does not lie in A, we say that the orbit $\mathrm{Orb}^+_{F,A}(p)$ escapes A. We define also $\omega_{F,A}(p) = \bigcap_{n\in\mathbb{Z}_{\geq 0}} \mathrm{Orb}^+_{F,A}(F^n(p))$, pointing out that it is empty if $\mathrm{Orb}^+_{F,A}(p)$ escapes A, and we define similarly $\alpha_{F,A}(p) = \bigcap_{n\in\mathbb{Z}_{\leq 0}} \mathrm{Orb}^-_{F,A}(F^{-n}(p))$. When the diffeomorphism F or the subset A is clear from the context, we will drop the subindices F or A in the orbits and the α-and ω-limit sets. Let $q_0 \in \overline{A}$. We say that A is an attracting (or repelling) parabolic set of F at q_0 if for any $p \in A$, one has $\omega_A(p) = \{q_0\}$ and $\mathrm{Orb}^-_{F,A}(p)$ escapes A (or $\alpha_{F,A}(p) = \{q_0\}$ and $\mathrm{Orb}^+_{F,A}(p)$ escapes A). We say that A is an elliptic set of F at q_0 if, for any $p \in A$, one has $\omega_A(p) = \{q_0\}$ and $\alpha_A(p) = \{q_0\}$. Finally, we say that A is hyperbolic if for any $p \in A$, the sets $\mathrm{Orb}^-_{F,A}(p)$ and $\mathrm{Orb}^+_{F,A}(p)$ are finite.

The following is the main result of the first part of the thesis.

Theorem A. *Let $F \in \mathrm{Diff}^\omega_1(\mathbb{R}^2, 0)$ be a germ of a real analytic diffeomorphism with $F(0) = 0$, $F \neq Id$, tangent to the identity and of non center-focus type. Then, for any open neighborhood W of 0 where a representative of F and F^{-1} is defined, there exist a neighborhood $U \subset W$ of 0, and a finite partition $\mathcal{S}$*

of

$$U = \bigcup_{A \in \mathcal{S}} A,$$

into invariant $\mathcal{C}^0$ submanifolds of $\mathbb{R}^2$ such that, for any $A \in \mathcal{S}$, we have

0. $\dim A = 0$ *if and only if* $A = \{0\}$.

1. If $\dim A = 1$ *then,* $0 \in \overline{A} \setminus A$ *and either A is a connected component of $Fix(F)\setminus\{0\}$ or $A \cap Fix(F) = \emptyset$ and A is an attracting or repelling set (curve) at* 0.

2. If $\dim A = 2$, *then* $0 \in \overline{A} \setminus A$, $A \cap Fix(F) = \emptyset$ *and A is of one of the following six types.*

- *A is an attracting or repelling parabolic set at* 0.
- *A is an elliptic set at* 0.
- *A is a hyperbolic set.*
- *A is dicritical-parabolic (or D-parabolic): there exists $\Gamma_A \in \mathcal{S}$ with* $\dim \Gamma_A = 1$ *and $\Gamma_A \subset Fix(F)$ such that for each $p \in A$, either there is $q_p \in \Gamma_A$ with $\alpha_A(p) = \{q_p\}$ and* $\mathrm{Orb}^+_A(p)$ *escapes A, or there is $q_p \in \Gamma_A$ with $\omega_A(p) = \{q_p\}$ and* $\mathrm{Orb}^-_A(p)$ *escapes A.*
- *A is dicritical-elliptic (or D-elliptic): there exists $\Gamma_A \in \mathcal{S}$ with* $\dim \Gamma_A = 1$ *and $\Gamma_A \subset Fix(F)$ such that for each $p \in A$, either there is $q_p \in \Gamma_A$ with $\alpha(p) = \{q_p\}$ and $\omega_A(p) = \{0\}$, or there is $q_p \in \Gamma_A$ with $\omega_A(p) = \{q_p\}$ and $\alpha_A(p) = \{0\}$.*
- *A is dicritical-dicritical (or D-D): there exist $\Gamma_A, \Gamma'_A \in \mathcal{S}$ with* $\dim \Gamma_A = \dim \Gamma'_A = 1$, $\Gamma_A \neq \Gamma'_A$ *and $\Gamma_A \Gamma'_A \subset Fix(F)$ such that for each $p \in A$, there is $q_p \in \Gamma_A$ with $\alpha_A(p) = \{q_p\}$ and there is $q'_p \in \Gamma'_A$ with $\omega_A(p) = \{q'_p\}$.*

The pair $(U, \mathcal{S})$ fulfilling the properties of the previous theorem will be called a *sectorial decomposition*. The two dimensional sets in $\mathcal{S}$ will be called the *sectors of* $(U, \mathcal{S})$, and they are of *parabolic*, *elliptic*, *hyperbolic*, *D-parabolic*, *D-elliptic* or *D-D* type, accordingly to the properties presented in the second item of the Theorem. See Figure 1 for a schematic picture of the sectors and see Figure 2 for an example of a sectorial decomposition. It is worth highlighting the following remarks, discussed in more detail in the text.

- The curves Γ_A in the "new" types of sectors (D-parabolic, D-elliptic, D-D) have a dicritical behavior in both sides: each $q \in \Gamma_A$ sufficiently close to $0 \in \mathbb{R}^2$ is not only a limit point of a positive or negative orbit of F contained in A, but q is also a limit point of a positive or negative orbit of another sector $A' \neq A$ of type D-parabolic, D-elliptic, D-D. These components $\Gamma_A \in \mathcal{S}$ will be called *bidicritical curves*.

- Further to the point above, if Γ_A is a bidicritical curve on the boundary of the sector A, the map $\phi : A \to \Gamma_A$, $p \mapsto q_p$ (defining q_p as in the statement of Theorem A) is continuous

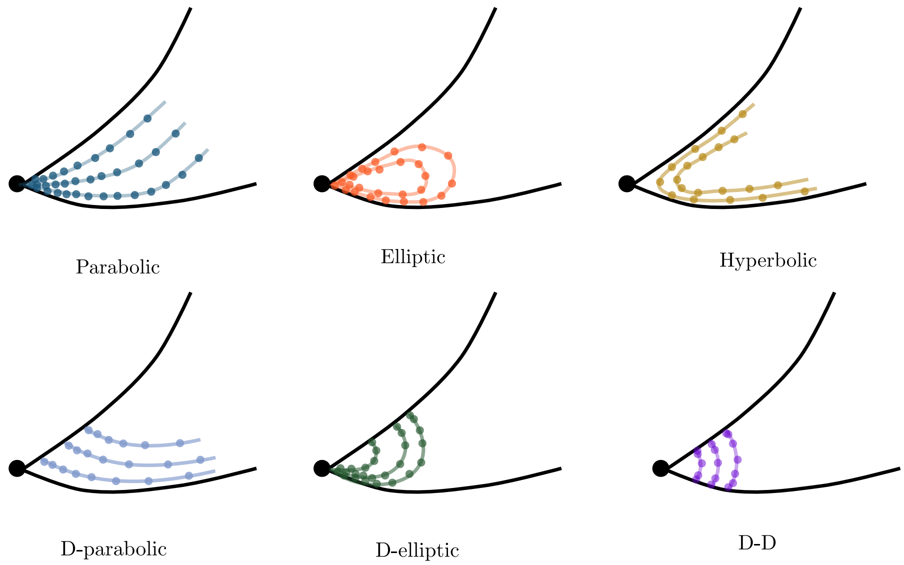

Figure 1: Types of sectors

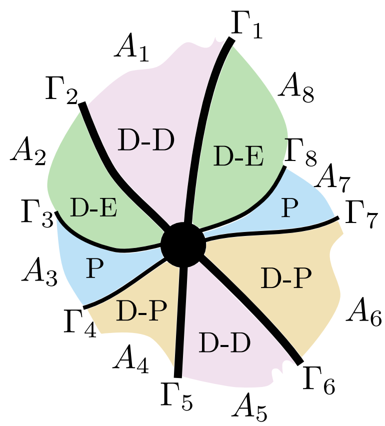

Figure 2: A example of a sectorial decomposition.

and its fibers $\{\phi^{-1}(q)\}_{q\in\Gamma_A}$ are parabolic curves of F attached to the different points of Γ_A (all attracting or repelling). The germ of $\phi^{-1}(q)$ is uniquely defined and coincides with the germ at q of the set of points p in the side of Γ_A that intersects A and whose orbit $\mathrm{Orb}_A^+(p)$ or $\mathrm{Orb}_A^-(p)$ accumulates at q.

- If A is a D-D sector, one has two such maps $\phi : A \to \Gamma_A$ and $\phi' : A \to \Gamma'_A$, one for each bidicritical curve which is in the boundary of A. It is natural to ask whether the fibers of ϕ and ϕ' coincide. We do not know the answer. Of course, one can construct examples for which the fibers of ϕ and ϕ' coincide: coming from the flow of an analytic vector field with arbitrary curves of singularities which are generically transverse to the saturated foliation (isolated singularity) that it generates. But we do not know if there are examples where the fibers do not coincide. Or wilder behavior, as depicted in Figure 3 where the fibers $\phi^{-1}(q)$, which by definition accumulate into singletons of Γ_A, may accumulate into a whole interval in the other bidicritical Γ'_A with non-empty interior. Such phenomenon, if it exists, would be proper to diffeomorphisms (with non-isolated fixed points) and would be an obstruction for embedding such diffeomorphisms into flows (even up to topological conjugation).

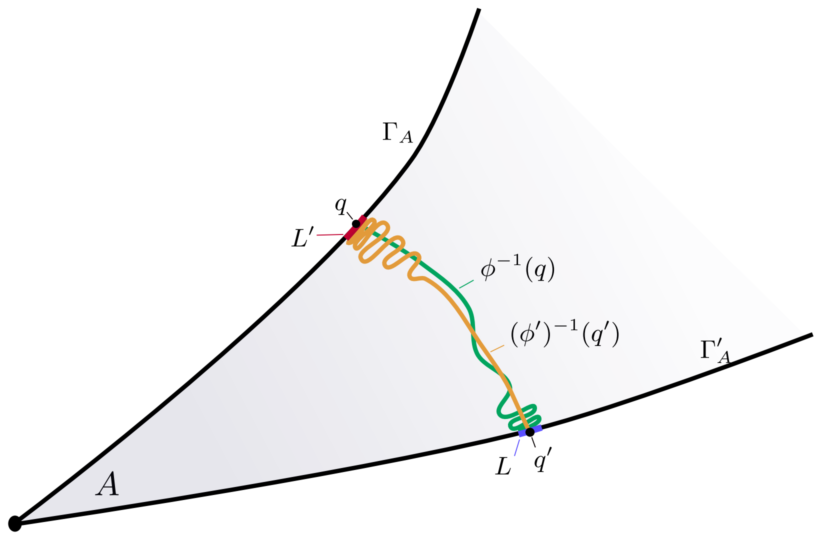

Figure 3: A fiber of ϕ and a fiber of ϕ'. The fiber $\phi^{-1}(q)$ accumulates in a compact subset L of Γ'_A, and the fiber $\phi^{-1}(q')$ accumulates in a compact subset L' of Γ_A.

- With regard to Theorem A, if there are no D-D sectors, then the statement can be improved in the sense that a sectorial decomposition $(U, \mathcal{S})$ can be taken so that U is an open neighborhood and $\mathcal{S}$ is an analytic stratification of U (the boundary of $A \in \mathcal{S}$ is a union of lower dimensional elements of $\mathcal{S}$ and the strata are analytic submanifolds of $\mathbb{R}^2$). Finally, if there are no bidicritical curves, then U can be chosen to be a semi-analytic subset of $\mathbb{R}^2$.

A basic consequence of the previous theorem is the following.

Corollary A. *Let $F \in \mathit{Diff}_1(\mathbb{R}^2, 0)$ be non center-focus. Then, there is a neighborhood U of 0 where a representative is taken, such that the only periodic points of F in U are fixed points. Hence, $Per(F) = Fix(F)$ as germs.*

Outline of the proof of Theorem A As we mentioned before, our proof is independent of the proof of the sectorial decomposition for $\mathcal{C}^\infty$ vector fields by Dumortier, Rodrigues and Roussarie in [29]. However, we follow some common steps, specially at the beginning of the proof. In particular, we make use of results in the literature about reduction of singularities of vector fields. To start, it is well known that any germ of a tangent to the identity diffeomorphism $F \in \mathrm{Diff}_1(\mathbb{R}^2, 0)$ has an infinitesimal generator, that is, a two dimensional formal vector field $\xi \in \widehat{\mathfrak{X}}(\mathbb{R}^2, 0)$. See for instance [13, 58] for the proof of the existence and uniqueness of such vector field. The formal infinitesimal generator ξ of F has a singularity of order two or greater than two at 0. Its formal singular locus $\mathrm{Sing}(\xi)$ coincides with the germ of fixed points $\mathrm{Fix}(F)$ of F and thus, this vector field is not necessarily saturated or reduced.

A classical result by Seidenberg [70] states that a two dimensional vector field with an isolated singularity admits a reduction of singularities: after a finite number of blowing-ups, the pull-back of the vector field only has *simple singularities* (in particular with at least one non-zero eigenvalue, see below). This result does not demand convergence of the vector field and it is equally applicable to a formal vector field with an isolated singularity. For vector fields with non-isolated singularity, we have not found a complete result on the reduction of singularities with the properties that we need. We propose a version of such a result on the reduction of singularities, specially adapted to the real formal setting in Chapter 1 (Section 1.4.3). Then, we apply this result to the infinitesimal generator ξ obtaining a sequence of blowing-ups $\pi : (M,E) \to (\mathbb{R}^2, 0)$ such that $E = \pi^{-1}(0)$ and $\pi|_{M\setminus E} : M \setminus E \to \mathbb{R}^2 \setminus \{0\}$ is an analytic isomorphism. We highlight that M is an analytic manifold with boundary and corners and E is its boundary. We use a result in [12] that ensures that F can be lifted to a diffeomorphism $\widetilde{F}$ in (M,E) that fixes each point of E. From this point, the object of work will be a diffeomorphism $\widetilde{F} : (M,E) \to (M,E)$ with $\widetilde{F}|_E = Id_E$. We look for a decomposition of $\widetilde{F}$ in some neighborhood $\widetilde{U}$ of E in (M,E) that will be projected to $\mathbb{R}^2$ in order to provide the desired sectorial decomposition $(U, \mathcal{S})$.

Let $\widetilde{\xi}'$ be the strict transform of ξ by π (locally defined by dividing the total transform $\widetilde{\xi}$ by an equation of the divisor E of maximal multiplicity). This strict transform permits to write $E = E_1 \cup \cdots \cup E_n$, where each E_i is either dicritical (transverse to the strict transform $\widetilde{\xi}'$ of ξ by π) or non-dicritical (tangent to $\widetilde{\xi}'$) for $\widetilde{\xi}'$. Of special interest is the set $\mathrm{Sing}(\widetilde{\xi}')$, composed of the corners between two non-dicritical components of E and the non-corner singularities of $\widetilde{\xi}'$. In any case, being simple singular points, there are exactly two formal mutually transverse separatrices of $\widetilde{\xi}'$ at each $q \in \mathrm{Sing}(\widetilde{\xi}')$. By construction of π, the components of E through $q \in \mathrm{Sing}(\widetilde{\xi}')$ and also the germ of the strict transform $(\mathrm{Fix}(F))'$ of $\mathrm{Fix}(F)$ at q (either empty or a simple non-singular analytic curve through q transverse to E) must coincide with the separatrices of $\widetilde{\xi}'$ at q. Extending E to $E \cup (\mathrm{Fix}(F))'$, we have a new normal crossing divisor where the points of $\mathrm{Sing}(\widetilde{\xi}') \cap (\mathrm{Fix}(F))'$ play the role of new corner points. We extend this normal crossing divisor $E \cup (\mathrm{Fix}(F))'$ at some non-corner points $q \in \mathrm{Sing}(\widetilde{\xi}')$. Concretely, at the points $q \in \mathrm{Sing}(\widetilde{\xi}') \setminus (\mathrm{Fix}(F))'$ where ξ' is not a saddle-node with the weak separatrix transverse to E and with a complete "node behavior" outside E (details below). At any q among such points, we will construct a parabolic curve γ_q of $\widetilde{F}$ that is asymptotic to the formal separatrix at q of E. We find these curves by using results in [5, 57, 56] concerning existence of (holomorphic) parabolic curves associated to formal invariant curves. We call $\widetilde{E}$ the extension of E by these parabolic curves and the strict transform of the fixed points. The set $\widetilde{E}$ is a normal crossing divisor. The curves in $\widetilde{E} \setminus E$ will be part of the one dimensional strata of the sectorial decomposition.

We prove in Section 2.3 some results on the local dynamics of $\widetilde{F}$. The results in this part of the thesis are technical, but the type of arguments that we use are very natural, based mainly on the use of monotonic functions. We start defining the set $\mathfrak{S}$ containing $\mathrm{Sing}(\widetilde{\xi}')$ and the corner points between a dicritical and a non-dicritical component of E. Given any compact connected subset of $E \setminus \mathfrak{S}$, we find a neighborhood of it where a monotonic function (on the orbits lying on this set) can be defined. We call these neighborhoods *monotonic domains* and we distinguish the dicritical and non-dicritical case. We also define the *quadrants* at each $q \in \mathfrak{S}$, which are basically connected components of the germ of $M \setminus \widetilde{E}$ at q. At each quadrant, we will find monotonic functions on the orbits of the diffeomorphism and three types of behaviors: saddle, node or dicritical. We also call these sets monotonic domains, because of the existence of monotonic functions on them. We conclude that E has a neighborhood defined by the union of monotonic domains and the curves of fixed points and parabolic curves in $\widetilde{E}$.

To construct the sectors in Section 2.4, we will proceed in two ways. We define the paths of quadrants as connected subsets of E that have two extreme points in $\mathfrak{S}$, and such that the intermediate points in $\mathfrak{S}$ are of a specific saddle type. The idea of the construction of the sectors consists in gluing the monotonic domains covering the path, and then, choosing a smaller open region, we ensure that orbits have a specific behavior inside the sector, so that the projection of this open set is of one of the types in Theorem A. We call *path sectors* to the sectors constructed from paths of quadrants. On the other hand, the union of these sectors is not necessarily a neighborhood of E. We complete this union with some parabolic sectors, obtaining finally a neighborhood $\widetilde{U}$ of E whose projection $U = \pi(\widetilde{U})$ is the required neighborhood of $0 \in \mathbb{R}^2$ in Theorem A.

In the last sections of Chapter 2, we make a refinement of the sectorial decomposition. We start the section by showing refinements on all the sectors. On the path sectors that are not adjacent to bidicritical curves, we consider what we call a "fundamental domain", which is a set that generates the sector by the saturation of it by $\widetilde{F}$ (the union of all the images by F and F^{-1}) and that contains a single element of each orbit in the sector. Taking a fundamental domain which is itself semi-analytic will serve us to prove that the sector has semi-analytic boundary outside 0. On the other hand, for a sector adjacent to a bidicritical curve (D-parabolic, D-elliptic or D-D), we are able to define parabolic curves of $\widetilde{F}$ at each point of the bidicritical curve and use it as the boundary of the sector. In the non-path sectors, we also make a refinement of the choice of the boundary. Using the refinements on the construction of the sectors, we prove the two main results of the section. In the absence of D-D sectors we can take U to be an open set in $\mathbb{R}^2$. In the absence of bidicritical curves, we can take U to be semi-analytic. We also make some comments on the reasons that lead us to think these restrictions are optimal.

Problem II: The structure of the cycle-locus of vector fields with Hopf singularity

As we mentioned before, our main motivation in this thesis was Dulac's problem in dimension three. Before stating the problem, recall that a cycle of a vector field is a periodic trajectory, thus homeomorphic to $\mathbb{S}^1$, and an *isolated cycle* is a cycle having a neighborhood where there are not other cycles. In $\mathbb{R}^2$ an isolated cycle is more frequently called a *limit cycle* because it is the accumulation set of some non-closed trajectories. In higher dimension, it is more proper to keep the name *isolated cycles*. One of the main tools in the study of cycles of vector fields are the Poincaré first return maps, which map points in a cross-section of a cycle to the first intersection of the trajectory again with the cross-section. Every cycle that intersects the cross-section, generates a periodic point of the Poincaré map. Therefore, studying the cycles near a given cycle is equivalent to studying the periodic points of its Poincaré map. The definition of this type of maps is generalized also to polycycles (closed union of trajectories and singularities of a vector field).

Dulac's problem claims that an analytic vector field cannot possess an infinite number of isolated cycles in a sufficiently small neighborhood of the singularity. The problem was originally stated in dimension two by H. Dulac [26] in 1923. It can be seen as a local version of (the second part of) the famous Hilbert's sixteenth problem, posed at [40] in 1902. Dulac himself provided a proof of it, but it turned out to have a mistake, discovered by Y. Ilyashenko in 1982, published in [44]. So far, there are two different and independent solutions to this problem, one by Y. Ilyashenko [43] and the other by J. Écalle [30]. Both are very intricate and not very well understood by the mathematical community. Recently, there are attempts revising Ilyashenko's proof by Yeung [77], who found a gap in Ilyashenko's proof, or approaching the problem with different tools, for instance o-minimal geometry [71, 45, 25, 35]. We highlight that the less degenerated case in dimension 2 is very easy to prove, and in fact it has been known since the work of Lyapunov [53]: it is the case where the vector field has a couple of conjugated non-zero purely imaginary eigenvalues (Hopf singularity). In this case, after a blowing-up of the origin we find a cycle (with no singularities), so that the analyticity of the first return map, being a diffeomorphism in one variable, gives the result.

Dulac's problem in higher dimension has not been very much treated so far, to our knowledge. Since the theorem in dimension two is already very intricate, it seems too optimistic to solve the problem in dimension three with full generality. It is very natural to attempt to generalize the easier case in dimension two to dimension three. Accordingly, from now on, we work with the family of vector fields with a *Hopf singularity*:

$$\mathcal{H}^3 := \{\xi \in \mathfrak{X}^\omega(\mathbb{R}^3, 0) : \operatorname{Spec}(D\xi(0)) = \{\pm bi, c\}, \text{ where } b, c \in \mathbb{R} \text{ and } b \neq 0\}.$$

We remark that for any $\xi \in \mathcal{H}^3$, there is a unique curve $\widehat{\Omega}$ which is non-singular and tangent to the eigendirection of the eigenvalue c. It is called the *formal rotational axis*. When $\xi \in \mathcal{H}^3$ has a non-zero real eigenvalue $c \neq 0$, we say that ξ has a *semi-hyperbolic Hopf singularity*. In this case, the Reduction to the Center Manifold Theorem applies and has important consequences: it implies that cycles sufficiently close to 0, if they exist, are contained in any center manifold, which is a surface. For some bibliography on center manifolds, see for instance [20, 47]. Then, the dimension of the problem is reduced to two, but the main difficulty is that the center manifolds may not be analytic, and then the restriction to the center manifold is not exactly the two dimensional Dulac's problem. Nevertheless, for Hopf-singularities, there are many authors that faced this problem and solved it successfully. We highlight the result of Aulbach [4] for a vector field with $n-2$ real non-zero eigenvalues and 2 imaginary ones. The conclusion is that either the center manifold is composed entirely of non-isolated cycles (and then, it is analytic) or it contains finitely many cycles. Other proofs that make use of the existence of a first integral can be found in [46, 48, 49, 68, 69, 76].

In the case where the eigenvalue c is zero, we say that ξ has a *Hopf-zero singularity*. We found fewer attempts to show Dulac's problem in this case. However, some authors have investigated the dynamics in generic families of Hopf-zero singularities. To mention the ones more related with this text:

- Dumortier and Bonckaert proved in [8] the existence of $\mathcal{C}^\infty$ realizations of the formal rotational axis $\widehat{\Omega}$.
- Dumortier in [28] considered Hopf-zero vector fields in the $\mathcal{C}^\infty$ class fulfilling two Łojasiewicz inequalities. One of them implies that the singularity is isolated. The other (stated for the infinitesimal generator of a Poincaré map of the central cycle obtained after blowing-up a realization of the rotational axis) implies that there are no local cycles. With these hypotheses, he obtains a complete description of the asymptotic behavior of the trajectories in a neighborhood of the origin.
- I. García has studied in [37] generic families of Hopf-zero vector fields, and he found that the number of isolated cycles generated in that family and making a finite number ν of turns is uniformly bounded in terms of ν. However, he does not give an answer on finiteness of limit cycles of individual fields making any number of turns.

We state now our main results. First, we introduce the following notation. Let $\xi \in \mathfrak{X}^\omega(\mathbb{R}^n, 0)$ be a germ of a real analytic vector field, and let U be an open neighborhood of 0 where ξ is defined. We denote by $\mathcal{C}_U(\xi)$ the union of all the cycles of $\xi|_U$ entirely contained in U. This set depends strongly on the neighborhood U and it is not ensured that it behaves as representatives of a germ of a set that we can associate to ξ. When the germs $\mathcal{C}_U(\xi)_0$, $\mathcal{C}_{U'}(\xi)_0$ of $\mathcal{C}_U(\xi)$ and $\mathcal{C}_{U'}(\xi)$ at $0 \in \mathbb{R}^n$

coincide for every pair of sufficiently small neighborhoods U, U' of 0, we say that ξ has a *local cycle-locus* equal to $\mathcal{C}(\xi) := \mathcal{C}_U(\xi)_0$.

Theorem B. *Let $\xi \in \mathcal{H}^3$. Then there is a neighborhood U of $0 \in \mathbb{R}^3$, where a representative of ξ is defined, for which exactly one of the following possibilities holds:*

(i) $\mathcal{C}_U(\xi) = \emptyset$.

(ii) There is a finite non-empty family $\mathcal{S} = \{S_1, \ldots, S_r\}$ of connected mutually disjoint smooth analytic two dimensional submanifolds of $U \setminus \{0\}$, invariant for ξ and subanalytic as sets satisfying $\overline{S_j} = S_j \cup \{0\}$ for any j, and there is a neighborhood basis $\mathcal{V}$ of the origin in U such that every $V \in \mathcal{V}$ satisfies

$$\mathcal{C}_V(\xi) = (S_1 \cup S_2 \cup \cdots \cup S_r) \cap V. \tag{1}$$

(iii) The singular locus $\mathrm{Sing}(\xi|_U)$ of ξ in U is a smooth analytic curve in U and there is a neighborhood basis $\mathcal{V}$ of the origin in U such that every $V \in \mathcal{V}$ satisfies

$$\mathcal{C}_V(\xi) = V \setminus (V \cap \mathrm{Sing}(\xi|_U)). \tag{2}$$

Consequently, the local cycle locus $\mathcal{C}(\xi)$ of ξ exists and it is equal to the empty germ, to the germ of $S_1 \cup \cdots \cup S_r$ or to the complement of the germ of $\mathrm{Sing}(\xi)$ in cases (i), (ii) or (iii), respectively.

As an immediate consequence, Dulac's problem has a positive answer for three dimensional Hopf singularities.

Corollary B (Dulac)**.** *If $\xi \in \mathcal{H}^3$, then there is a neighborhood of $0 \in \mathbb{R}^3$ where there are no isolated cycles of ξ.*

The surfaces in item (ii) are called the *central limit surfaces*. Each of them is filled with a one-parameter family of cycles or ξ. In the semi-hyperbolic singularity case, there is at most one limit central surface (either (i) or (ii) with $r = 1$), and if there is one, then it is a center manifold, non-singular and analytic at 0.

In the last result, we provide a characterization of the third situation of Theorem B in terms of linearizability and integrability of the vector field.

Theorem C. *Let $\xi \in \mathcal{H}^3$ be a Hopf-zero singularity, the following statements are equivalent.*

(1) ξ is formally orbitally linearizable (i.e. formally equivalent to $G(x,y,z)\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)$, where G is a unit in $\mathbb{R}[[x,y,z]]$).

(2) ξ is analytically orbitally linearizable.

(3) *There is a neighborhood U of* 0 *such that* $\mathcal{C}_U = U \setminus Sing(\xi)$.

(4) ξ *is analytically completely integrable (i.e. there exist two analytic first integrals f, g at* 0 *which are independent, i.e.* $df \wedge dg \not\equiv 0$*).*

We want to remark that some of the implications in Theorem C are not completely original:

- On the one hand, once we obtain that (1) implies that a formal normal form (discussed along this text) of ξ is proportional to the linear part $-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, the implication (1) $\Rightarrow$ (2) is a particular case of Brjuno's results on analytic normalization of analytic vector fields [10] (see also [59]).
- The implication (1) $\Rightarrow$ (4) can also be seen as a consequence of a result by Zhang [78] (see also [54]).
- The implication (3) $\Rightarrow$ (4) deserves a separate comment. It can be interpreted as a three dimensional version of the classical Poincaré-Lyapunov Center Theorem [66, 53], asserting that an analytic center at $(\mathbb{R}^2, 0)$ has an analytic first integral. It is already stated by I. García [36], but tacitly assuming an a priori stronger hypothesis than (3); namely, that all cycles in $U \setminus \mathrm{Sing}(\xi)$ perform a single turn around the curve $\mathrm{Sing}(\xi)$ before closing. In our proof, we surpass this difficulty, since we do not assume this extra condition: the scheme is more precisely the sequence of implications (3) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (4), the first being a consequence of (a part of) the proof of Theorem B. As a consequence of (2), we affirm the condition on the number of turns of all cycles near 0.

On the whole, we believe that our main contribution with Theorem C, apart from gathering several separate results about Hopf-zero singularities in a single statement, is the completion of the proof of the generalization of Poincaré-Lyapunov result commented above.

Outline of the proof of Theorem B We recall first that a proof of Theorem B for $\xi \in \mathcal{H}^3$ with an isolated singularity (and hence only (i) or (ii) can occur) has already been established in [23]. In this thesis, with the use of Theorem A, we have generalized Theorem B for any $\xi \in \mathcal{H}^3$ and we have also shortened the proof in [23] for the isolated singularity case. All in all, several steps on that proof remain the same. Let us summarize the main ideas. Fix any Hopf vector field $\xi \in \mathcal{H}^3$. We distinguish several cases.

1. The *semi-hyperbolic case*. This case is developed in Section 3.2. In this case, there is a one-dimensional invariant stable or unstable manifold W (tangent to the eigendirection of $c \neq 0$) and a two dimensional invariant center manifold W^c (tangent to the eigendirections of $\pm bi$). The manifold W is unique and analytic, but W^c can only be chosen of class $\mathcal{C}^k$. One

of the essential properties of the center manifolds is that they contain the cycles existing in a neighborhood of the singularity. In other words, $\mathcal{C}_U(\xi) \subset W^c$ for U sufficiently small. We perform a blowing-up $\pi : (M, E) \to (\mathbb{R}^3, W)$ centered at W and we find that the fiber $\pi^{-1}(0)$ is a cycle of the strict transform of ξ. We end using the Poincaré map on a cross-section Δ of this cycle (see for instance[61] for a definition of the Poincaré map): this map is analytic and its set of periodic points is contained in the curve $\Delta \cap W^c$. We conclude that either there is a single curve of fixed points, or that there are not periodic points except for $\Delta \cap \pi^{-1}(0)$. We obtain Theorem B with a single limit central surface equal to W^c (item (ii) with $r = 1$) in the first case, and item (i) in the second case.

2. ξ is a Hopf-zero vector field with an isolated singularity $\mathrm{Sing}(\xi) = \{0\}$. This case is developed in Sections 3.3-3.5. We use the theory of normal forms in this case (see [10, 59, 72]). We fix a (formal) normal form $\hat{\xi}$ for ξ; it is a formal vector field formally conjugated to ξ and written as
$$\hat{\xi} = A(x^2+y^2,z)\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + B(x^2+y^2,z)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + C(x^2+y^2,z)\frac{\partial}{\partial z},$$
where $A, B, C \in \mathbb{R}[[u,v]]$. We consider also a collection of analytic vector fields $\{\xi_\ell\}_{\ell \in \mathbb{N}^*}$, all analytically conjugated to ξ, so that the Taylor expansion of ξ_ℓ approximates the normal form $\hat{\xi}$ up to some order that grows with ℓ. From the rotational symmetry of the formal normal form $\hat{\xi}$, we guess that its cycles (if the surface were convergent) turn once around the z–axis and cut the transverse section $\{y = 0, x > 0\}$ at the singular points of the auxiliary vector field $\hat{\eta}$ defined as $\hat{\eta} = B(x^2,z)x\frac{\partial}{\partial x} + C(x^2,z)\frac{\partial}{\partial z}$. Its singular set being at most a curve, we "get" Theorem B (i) or (ii) for the formal vector field. It is quite natural to investigate to what extent this formal description is reflected on an analytic approximation ξ_ℓ for some ℓ sufficiently large (notice that it is enough to prove Theorem B for ξ_ℓ, being analytically conjugated to ξ).

We proceed as follows. First, since $\hat{\eta}|_{z-\mathrm{axis}} \neq 0$ (by the isolated singularity hypothesis), there is a cone of finite order around the z–axis where there are no "cycles" of $\hat{\xi}$. We show that the same happens for ξ_ℓ, that is, there is a cone around the z–axis free of cycles when ℓ is sufficiently large. In terms of blowing-ups, there is a sequence of blowing-ups $\pi : (M, E) \to (\mathbb{R}^3, 0)$ along the first k infinitely near points of the z–axis Z[1], and a neighborhood V_{k+1} of $\overline{\pi^{-1}(Z \setminus \{0\})} \cap E$ (the $k+1$-th infinitely near point) in M such that $\pi^*\xi_\ell$ has no cycles in $V_{k+1} \setminus E$. In fact, for our problem, we use real oriented blowing-ups along infinitely near points of both half-axes. Now, again $\hat{\xi}$ serves as a guiding vector field that points where to look for the cycles: the map π determines a finite family of compact curves in E, what we call *characteristic cycles*.

[1] The infinitely near points of Z are the intersection points of the strict transform of Z and the exceptional divisor of the blowing-up centered at $0 \in \mathbb{R}^3$ (the first infinitely near point), or the blowing-up centered at the previous infinitely near point (in the rest of the cases)

They are given by a family of actual cycles of the restriction of the strict transform $\hat{\xi}'$ of $\hat{\xi}$ to E (a true analytic vector field). These cycles are obtained also as the curves generated by rotation of the set of singularities of the strict transform $\hat{\eta}'$ of $\hat{\eta}$ along the one dimensional divisor $\pi^{-1}(\{y=0, x>0\}) \cap E$.

If ℓ is sufficiently large, the characteristic cycles are also cycles of the strict transform ξ_ℓ' of ξ_ℓ by π. Moreover, using flow-boxes along some compact subsets of different components of E, we show that cycles of ξ_ℓ', outside but sufficiently near to E, are localized on neighborhoods of the characteristic cycles. If γ is such a characteristic cycle, the nearby cycles of ξ_ℓ' are given by the periodic points of the Poincaré first return map $P_{\ell,\gamma} : \Delta_\gamma \to \Delta_\gamma$ of ξ_ℓ' defined in a local cross-section Δ_γ at some point of γ. The diffeomorphism $P_{\ell,\gamma}$ is not of type center-focus since $E \cap \Delta_\gamma$ is an invariant curve. Once we show $P_{\ell,\gamma}$ is not the identity, Corollary A of Theorem A permits to conclude Theorem B (i) or (ii), where the family of limit central surfaces corresponds to the family of connected components of $\mathrm{Fix}(P_{\ell,\gamma}) \setminus E$ for the different characteristic cycles γ.[2]

3. Suppose that ξ is a Hopf-zero vector field with a non-isolated singularity. Then, in a sufficiently small neighborhood of $0 \in \mathbb{R}^3$, the singular set $\mathrm{Sing}(\xi)$ is a non-singular analytic curve tangent to the eigenvalue $c = 0$ and coincides with the rotational axis. We distinguish two cases in terms of a normal form $\hat{\xi}$: either $\hat{\xi}$ is not proportional to the linear part $L\hat{\xi}(0)$ (that is, the auxiliary two dimensional vector field $\hat{\eta}$ is not identically 0) or it is proportional, that is, $\hat{\eta} \equiv 0$. It can be shown that this distinction does not depend on the chosen normal form $\hat{\xi}$. They are called respectively the *formally non-degenerated case* and the *formally degenerated case*.

   The first case is developed in Section 3.6. As in the isolated singularity case, only possibilities (i) or (ii) of Theorem B may occur. But the proof of the isolated singularity case does not necessarily apply in this case. Concretely, since $\hat{\xi}$ vanishes along the rotational axis, the arguments above for obtaining a cone around the axis free of cycles do not work. Instead of making point blowing-ups, we perform the blowing-up $\pi : (M, E) \to (\mathbb{R}^2, \mathrm{Sing}(\xi))$ centered at the analytic curve $\mathrm{Sing}(\xi)$. It turns out that $\pi^{-1}(0)$ is a cycle of the transform $\widetilde{\xi} = \pi^*(\xi)$. The Poincaré map of $\widetilde{\xi}$ in a transverse section of a point in this cycle satisfies the conditions of Theorem A and we finish by Corollary A as above.

   Finally, the formally degenerated case is developed in Section 3.7. We are in the situation (1) of Theorem C. By virtue of this result, this is equivalent to (3), which proves Theorem B (iii).

[2] In our article [23], we continue the sequence of blowing-ups centered at the different characteristic cyles (following a reduction of singularities of $\hat{\eta}$), obtaining more simple local situations around each characteristic cycle. Then, we prove Corollary A without having Theorem A for the corresponding Poincaré maps obtained after this reduction of singularities.

Chapter 1

Blowing-ups and reduction of singularities

In this chapter, we give a common context to the two problems treated in this thesis. We start studying *germs of real analytic singular vector fields*. We present a very powerful tool in the topological study of vector fields: *blowing-ups*. We give general definitions of the blowing-ups and then study more deeply the two-dimensional case. We state results on *reduction of singularities* of two-dimensional real vector fields in dimension 2 by the iteration of blowing-ups. We present some known results, adapting their statements so that we have a common language in this thesis, and we give only some of the proofs. Then, we introduce a graph obtained from the reduction of singularities, which does not have to be mistaken for the classical dual graph. We end the chapter by studying *germs of tangent to the identity diffeomorphisms*, their relation with germs of vector fields and the adaptation of the results presented in the previous section to diffeomorphisms.

1.1 Germs of real analytic vector fields with a singularity at 0

As we anticipated, in this section we work with germs of analytic vector fields and formal vector fields at a point of an analytic manifold. Without loss of generality, we can consider this point to be the origin of $\mathbb{R}^n$. That is, we study elements in $\mathfrak{X}^\omega(\mathbb{R}^n,0)$ and in $\widehat{\mathfrak{X}}(\mathbb{R}^n,0)$. We will denote $\mathfrak{X}^\omega(\mathbb{R}^n,0)$ simply by $\mathfrak{X}(\mathbb{R}^n,0)$. We start studying the analytic vector fields. Fixing coordinates $\mathbf{x}=(x_1,\ldots,x_n)$, a germ of a vector field $\xi\in\mathfrak{X}(\mathbb{R}^n,0)$ is written as

$$\xi = a_1(\mathbf{x})\frac{\partial}{\partial x_1}+\cdots+a_n(\mathbf{x})\frac{\partial}{\partial x_n},\quad a_i(\mathbf{x})=\sum_{\alpha\in\mathbb{N}^n} a_{i,\alpha}\mathbf{x}^\alpha\in\mathbb{R}\{\mathbf{x}\}, \tag{1.1}$$

where each a_i is a germ of analytic function in $\mathcal{O}_{n,0}$ expressed as a convergent series. Another way to express a vector field is the following

$$\dot{\mathbf{x}}=\xi(\mathbf{x}),$$

by the differential equation that it defines. We define the vector field

$$L\xi=\sum_{|\alpha|=1} a_{1,\alpha}\mathbf{x}^\alpha\frac{\partial}{\partial x_1}+\cdots+\sum_{|\alpha|=1} a_{n,\alpha}\mathbf{x}^\alpha\frac{\partial}{\partial x_n}. \tag{1.2}$$

Notice that the *linear part* $D\xi(0)=DL\xi(0)$ is an $n\times n$ matrix A. We can also write $\dot{\mathbf{x}}=A\mathbf{x}+F(\mathbf{x})$, where F is of order equal to or greater than 2. We denote the order or multiplicity of ξ at 0 as $\min_{i=1,\ldots,n}\{\nu(a_i)\}$, where $\nu(a_i)$ is the *order* or *multiplicity* of a_i at 0, i.e. $\nu(a_i)=\min_{j\in\mathbb{N}}\{j : j=|\alpha|,\ a_{i,\alpha}\neq 0\}$.

Now, a formal vector field $\xi\in\widehat{\mathfrak{X}}(\mathbb{R}^n,0)$ is a derivation in $\widehat{\mathcal{O}}_{n,0}\simeq\mathbb{R}[[\mathbf{x}]]$. Taking coordinates $\mathbf{x}=(x_1,\ldots,x_n)$ centered at 0 we can write ξ as

$$\xi = a_1(\mathbf{x})\frac{\partial}{\partial x_1}+\cdots+a_n(\mathbf{x})\frac{\partial}{\partial x_n},\quad a_i(\mathbf{x})=\sum_{\alpha\in\mathbb{N}^n} a_{i,\alpha}\mathbf{x}^\alpha\in\mathbb{R}[[\mathbf{x}]], \tag{1.3}$$

where each a_i is a formal series. As in the analytic case, we can define $L\xi$ by (1.2) and its linear part $D\xi(0)$. Notice that $\mathfrak{X}(\mathbb{R}^n,0)\subset\widehat{\mathfrak{X}}(\mathbb{R}^n,0)$, therefore, many of the definitions that we give in the rest of the section apply to both analytic and formal vector fields.

Concerning the eigenvalues of the linear part $D\xi(0)$ of ξ, there are two types of singularities: elementary and nilpotent ones.

Definition 1.1. *Let $\xi\in\widehat{\mathfrak{X}}(\mathbb{R}^n,0)$ be a vector field with a singularity at $0\in\mathbb{R}^n$. We say that* 0 *is an* elementary singularity *if its linear part $D\xi(0)$ has, at least, one non-zero eigenvalue. Otherwise, we say that* 0 *is a* nilpotent singularity.

In the study of real vector fields, as we will see, the real part of the eigenvalues of the linear part reveals useful information. We distinguish the following types of singularities concerning the number of eigenvalues whose real part is different from zero.

Definition 1.2. *Let $\xi \in \widehat{\mathfrak{X}}(\mathbb{R}^n, 0)$ be a vector field with a singularity at $0 \in \mathbb{R}^n$ and $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ the eigenvalues of $D\xi(0)$. We say that 0 is a* hyperbolic singularity *if $Re(\lambda_i) \neq 0$ for $1 \leq i \leq n$. We say that 0 is a* semi-hyperbolic singularity *if there is at least one λ_i with $Re(\lambda_i) \neq 0$. We say that 0 is a* completely non-hyperbolic singularity *if $Re(\lambda_i) = 0$ for $1 \leq i \leq n$.*

From this point to the end of the section, we introduce some useful concepts for two-dimensional real analytic and formal vector fields. Fix a two-dimensional vector field $\xi \in \widehat{\mathfrak{X}}(\mathbb{R}^2, 0)$ and coordinates (x, y) centered at $(0, 0)$, then

$$\xi = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}.$$

We suppose first that $a(0, 0) = b(0, 0) = 0$ and that $a(x, y)$ and $b(x, y)$ have no common (non-unit) factors. Such singularities are called *algebraically isolated* or, simply, isolated.

Definition 1.3. *Let ξ be a vector field with a singularity at $0 \in \mathbb{R}^2$ and λ, μ the eigenvalues of $D\xi(0)$. We say that ξ has a* (real) simple singularity *at 0 if $\lambda, \mu \in \mathbb{R}$, one of the eigenvalues, for example μ, is different from 0 and the ratio $\frac{\lambda}{\mu} \notin \mathbb{Q}_{>0}$.*

Remark 1.4. We want to remark that the concept of simple singularity exists for complex two-dimensional vector fields and the definition is the same allowing that $\lambda, \mu \in \mathbb{C}$. For instance the linear vector field $\eta = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2, 0) \subset \mathfrak{X}(\mathbb{C}^2, 0)$ has eigenvalues $i, -i$. The point 0 is not a real simple singularity. However, if we consider η as a complex vector field, it is a simple singularity. Along this text, as we will work with real vector fields, we will reserve the name simple singularity for the real ones.

We will highlight the importance of simple singularities in the following sections. In particular, as we will see, they are stable under blowing-ups and thus they will be considered final situations in the process of reduction of singularities, which will be discussed later. Another remarkable feature is that at simple singularities there always exist two unique formal invariant curves for ξ. Before stating the result, we give the definition of a separatrix and refer the reader to Appendix A.2.1 to recall the definition of formal curves.

Definition 1.5. *An analytic, C^k or formal curve Γ at $(\mathbb{R}^2, 0)$ is a* separatrix *of ξ if it is invariant for ξ and it has a defined tangent at 0.*

Theorem 1.6 (Existence of separatrices at simple singularities [9])**.** *Let $\xi \in \widehat{\mathfrak{X}}(\mathbb{R}^2, 0)$ be a formal vector field with a simple singularity at 0 having eigenvalues λ, μ, suppose, for instance, that $\mu \neq 0$.*

Then there exist two unique formal separatrices Γ_λ and Γ_μ, the first tangent to the eigendirection of λ and the second to the one of μ. In addition, if $\xi \in \mathfrak{X}^\omega(\mathbb{R}^2, 0)$, the curve Γ_μ is an analytic separatrix.

In the light of the previous theorem, we say that the eigendirection $v_\lambda \in \mathbb{P}^1_{\mathbb{R}}$ associated to an eigenvalue λ of ξ is a *strong* direction when $\lambda \neq 0$ and a *weak* direction when $\lambda = 0$.

The above theorem can be refined in the analytic case. In particular, a simple singularity of an analytic vector field has $\mathcal{C}^k$ or analytic separatrices associated to a formal one. That is, either a formal separatrix Γ of the previous theorem is convergent, or there is a $\mathcal{C}^k$ separatrix γ whose Taylor development is given by that of Γ.

The separatrices we have presented are real curves, and they have two real half-branches (see Appendix A.2.1). Sometimes, we will make an abuse of notation and name a real half-branch of a separatrix, directly a separatrix. Each half-branch is a different trajectory of the vector field, and in the literature, the trajectories accumulating to 0 in a concrete direction are called *characteristic trajectories*.

When $a(0,0) = b(0,0) = 0$ and the coefficients $a(x,y)$ and $b(x,y)$ have common factors, the singularity is said to be algebraically non-isolated. Suppose that $f_1(x,y), \ldots, f_r(x,y)$ are the irreducible common factors such that $f_i \neq f_j$ for $i \neq j$. We define the *singular locus of ξ* to be the ideal $\mathrm{Sing}(\xi) = (f_1 \cdots f_r)$. In the analytic case, this ideal provides a germ of analytic set and we will name this set as well the singular locus of ξ. Recall, however, that in real analytic geometry there is no one-to-one correspondence between prime ideals and real analytic sets. If any of these common factors provides a (formal) curve Γ, we have that Γ is invariant for ξ (the vector field restricted to the curve is 0), and hence we say that the curve is a *degenerate separatrix*.

1.2 Blowing-ups

In this section, we will present two types of blowings-ups: the classical blowing-ups and the real or oriented blowing-ups. We will work only with the second type of blowing-ups, but we present the first type for the sake of completeness and in order to understand better the second. The ambient space where we will work is $\mathbb{K}^n$ with $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, but the generalization to open subsets of $\mathbb{K}^n$ and to open sets of any analytic manifold M works well using the restriction in the first case and local charts in the second. We end this introduction indicating that we will present blowing-ups as transformations of the ambient space, and then see how manifolds, varieties and vector fields transform. In the rest of our work, we will mainly use the blowing-ups centered at points in $\mathbb{R}^2$ and $\mathbb{R}^3$. However, only in this section, we will define the concepts with more generality.

1.2.1 The classical blowing-up

Consider $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We define the *blowing-up of* $\mathbb{K}^n$ *centered at* 0 as the projection $\pi : M \to \mathbb{R}^n$ with

- M is the blown-up manifold and it is given by

$$M = \{(x_1, \dots, x_n, [y_1, \dots, y_n]) \in \mathbb{K}^n \times \mathbb{P}^{n-1}_{\mathbb{K}} : x_i y_j = x_j y_i,\ i \neq j\} \subset \mathbb{K}^n \times \mathbb{P}^{n-1}_{\mathbb{K}}.$$

- $\pi((x_1, \dots, x_n, [y_1, \dots, y_n])) = (x_1, \dots, x_n)$.

We define the *exceptional divisor* E *of* π by $E = \pi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1}_{\mathbb{K}}$. Notice that for a point $p \neq 0$ of $\mathbb{K}^n$, the set $\pi^{-1}(p)$ is a single point. In the contrary, $E = \pi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1}_{\mathbb{K}}$ contains infinite points. In particular, the map $\pi|_{M \setminus E} : M \setminus E \to \mathbb{K}^n \setminus \{0\}$ is an isomorphism.

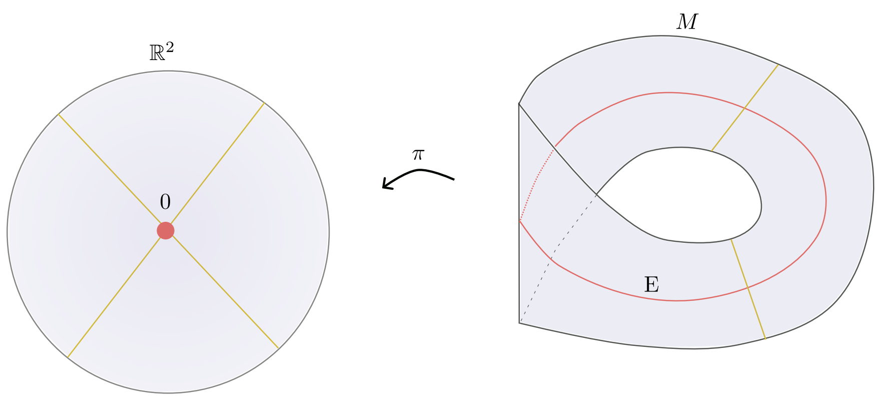

Figure 1.1: Classical blowing-up centered at $0 \in \mathbb{R}^2$

We want to hightlight that M, which is defined as an algebraic variety, is indeed an n−dimensional analytic manifold. We provide an altas and prove that the changes of coordinates are analytic. Let $U_j \subset M$ be the open set defined by $M \cap (\mathbb{K}^n \times (\mathbb{P}^{n-1}_{\mathbb{K}} \cap \{y_j \neq 0\}))$. We define the homeomorphism $\varphi_j : U_j \to \mathbb{K}^n$ for $j = 1, \dots, n$ as follows

$$\varphi_j((x_1, \dots, x_n, [y_1, \dots, y_n])) = (\frac{y_1}{y_j}, \cdots, \frac{y_{j-1}}{y_j}, x_j, \frac{y_{j+1}}{y_j}, \cdots, \frac{y_n}{y_j}).$$

We remark that if $p \in E$, then $p = (0, v)$ with $v \in \mathbb{P}^{n-1}_{\mathbb{K}} \cap \{y_j \neq 0\}$ and $\varphi_j(p) \in \{z_j = 0\}$. Hence, the divisor is mapped to the hyperplane given by $\{z_j = 0\}$. The expression of π in this chart is given by $\pi \circ \varphi_j^{-1} : \mathbb{K}^n \to \pi(U_j)$ as follows

$$\pi \circ \varphi_j^{-1}(z_1, \dots, z_n) = (z_j z_1, \dots, z_j z_{j-1}, z_j, z_j z_{j+1}, \dots, z_j z_n)$$

For the sake of simplicity, we will identify U_j and $\mathbb{K}^n$, and many times we will simply write that the expression of the blowing-up π in the chart U_j is given by $\pi(z_1,\dots,z_n) := \pi|_{U_j}(z_1,\dots,z_n) = \pi\circ\varphi_j^{-1}(z_1,\dots,z_n) = (z_jz_1,\dots,z_jz_{j-1},z_j,z_jz_{j+1},\dots,z_jz_n)$.

To end this section, we prove that the change of chart is an analytic morphism. Let $\varphi_i : U_i \to \mathbb{K}^n$ and $\varphi_j : U_j \to \mathbb{K}^n$ be two charts with $i \neq j$ and, for instance, $i < j$. Notice that $U_i \cap U_j = M \cap (\mathbb{K}^n \times (\mathbb{P}^{n-1}_{\mathbb{K}} \cap \{y_j \neq 0, y_i \neq 0\}))$. Notice that $U_i \cap U_j$ is mapped by φ_j to $\varphi_i(U_i \cap U_j) = \mathbb{K}^n \setminus \{z_i = 0\}$ and that $U_i \cap U_j$ is mapped by φ_i to $\varphi_j(U_i \cap U_j) = \mathbb{K}^n \setminus \{z_j = 0\}$. Let $z_1,\dots,z_n$ be the coordinates of the image of φ_j. Then, the morphism $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_j \cap U_i) \to \varphi_i(U_j \cap U_i)$ is given by

$$\varphi_i \circ \varphi_j^{-1}(z_1,\dots,z_n) = (z_1,\dots,z_{i-1},z_jz_i,z_{i+1},\dots,z_{j-1},\frac{1}{z_j},z_{j+1},\dots,z_n).$$

1.2.2 The real blowing-up

In this subsection, we will explain another type of blowing-ups, which will suit nicely our work. Some of the main references to study this type of blowing-up are [42, 24], the latter providing a more general construction of weighted blowing-ups. Another reference is [27] in the two-dimensional case, in which the author develops real blowing-ups from different perspectives. The main difference with the blowing-ups in the previous section is that the blown-up manifold has boundary (and maybe corners) but it is orientable. We explain the blowing-up centered at a point and then extend the construction to blowing-ups of smooth analytic submanifolds.

Suppose that $\mathbb{R}^n$ is equipped with the Euclidean norm $\|-\|$. We define *the real or oriented blowing-up of* $\mathbb{R}^n$ *with center at* 0 as the projection $\pi : M \to \mathbb{R}^n$ with

- M is the blown-up manifold of π and it is defined as the closure of $\{(p,q) \in \mathbb{R}^n \times \mathbb{S}^{n-1} : p \neq 0, \frac{p}{\|p\|} = q\}$ in $\mathbb{R}^n \times \mathbb{S}^{n-1}$. That is,

$$M = \left\{(p,q) \in \mathbb{R}^n \times \mathbb{S}^{n-1} : p \neq 0, \frac{p}{\|p\|} = q\right\} \cup \left(\{0\} \times \mathbb{S}^{n-1}\right).$$

- $\pi(x_1,\dots,x_n,y_1,\dots,y_n) = (x_1,\dots,x_n)$, where $y_1^2 + \cdots + y_n^2 = 1$.

We define the *exceptional divisor* E of π as $E = \pi^{-1}(0) = \{0\} \times \mathbb{S}^{n-1}$ and we will often denote $\pi : (M,E) \to (\mathbb{R}^n, 0)$. Notice that for a point $p \neq 0$ in $\mathbb{R}^n$, the set $\pi^{-1}(p)$ is a single point. On the other hand, $E = \pi^{-1}(0)$ contains infinitely many points. In particular, $\pi|_{M\setminus E} : M \setminus E \to \mathbb{R}^n \setminus \{0\}$ is an isomorphism.

The real blown up manifold M of π centered at 0 is not anymore an analytic manifold in the classical sense: it is an analytic manifold with boundary. In particular, $M \simeq \mathbb{R}_{\geq 0} \times \mathbb{S}^{n-1}$. We provide the bijection $\psi : M \to \mathbb{R}_{\geq 0} \times \mathbb{S}^{n-1}$. First, take a point in M. If $p = (0,\dots,0,y_1,\dots,y_n) \in M$, its image by ψ is $\psi(p) = (0,y_1,\dots,y_n)$. If $p = (x_1,\dots,x_n,y_1,\dots,y_n) \in M$, its image is $(\|(x_1,\dots,x_n)\|, y_1,\dots,y_n)$ (notice that $(y_1,\dots,y_n) = \frac{(x_1,\dots,x_n)}{\|(x_1,\dots,x_n)\|}$). On the other hand, if $q = (0,y_1,\dots,y_n)$, we have $\psi^{-1}(q) =$

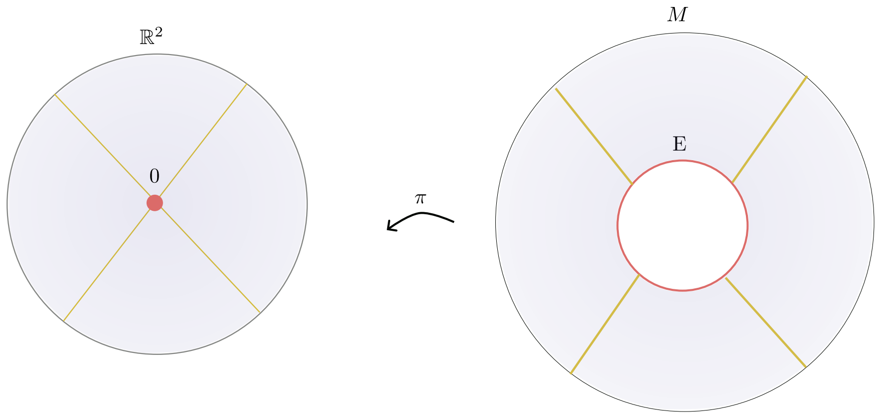

Figure 1.2: Real blowing-up centered at $0 \in \mathbb{R}^2$.

$(0,\dots,0,y_1,\dots,y_n)$. If $q = (r,y_1,\dots,y_n)$, we have $\psi^{-1}(q) = (ry_1,\dots,ry_n,y_1,\dots,y_n)$ We will consider M to be $M = \mathbb{R}_{\geq 0} \times \mathbb{S}^{n-1}$, by the previous identification.

We take the usual coordinates r in $\mathbb{R}_{\geq 0}$ and $(x_1,\dots,x_n)$ in $\mathbb{S}^{n-1}$, with $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. We can see how π is expressed in the given coordinates of $\mathbb{R}_{\geq 0} \times \mathbb{S}^{n-1}$,

$$\pi(r,x_1,\dots,x_n) = (rx_1,\dots,rx_n).$$

Now, as we announced, we prove that M is an analytic manifold with boundary, by defining an atlas that contains $2n$ charts through which we can identify open sets U_j^ϵ of M $(1 \leq j \leq n,\ \epsilon = +,-)$ with open sets of $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$ (with the usual subspace topology). Let $U_j^\epsilon \subset M$ be the open set defined by $\mathbb{R}_{\geq 0} \times (\mathbb{S}^{n-1} \cap \{x_j > 0\})$ if $\epsilon = +$ and $\mathbb{R}_{\geq 0} \times (\mathbb{S}^{n-1} \cap \{x_j < 0\})$ if $\epsilon = -$. In other words, the open set U_j^+ covers the positive direction of x_j and U_j^- the negative one. We define $\varphi_j^\epsilon : U_j^\epsilon \to \mathbb{R}^{j-1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-j}$ as

$$\varphi_j^\epsilon(r,x_1,\dots,x_n) =, \left(\frac{x_1}{x_j},\dots,\frac{x_{j-1}}{x_j},\epsilon r x_j,\frac{x_{j+1}}{x_j},\dots,\frac{x_n}{x_j}\right)$$

The blowing-up $\pi \circ (\varphi_j^\epsilon)^{-1} : \mathbb{R}^{j-1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-j} \to \pi(U_j^\epsilon)$ is given by

$$\pi|_{U_j}(z_1,\dots,z_n) := \pi \circ (\varphi_j^\epsilon)^{-1}(z_1,\dots,z_n) = (z_jz_1,\dots,z_jz_{j-1},\epsilon z_j,z_jz_{j+1},\dots,z_jz_n)$$

Notice that this expression is similar to the one obtained for the classical blowing-ups, but remark the presence of the sign ϵ and that neither the domain of definition of the map nor the image are the same. Thus, we can omit showing the change of chart expression, because it is similar to the change of chart in the classical blowing-up.

It is also possible to define blowing-ups with higher dimensional smooth centers. Let $N \subset \mathbb{R}^n$ be an analytic manifold of codimension m. We can take locally analytic coordinates $(z_1,\dots,z_n)$ so that $N = \{z_{n-m+1} = 0, \cdots, z_n = 0\}$. Therefore, we only give the idea of the blowing-ups with centers at $(n-m)$–planes, since deriving the coordinate expression is as in the blowing-up centered at a single point. The blowing-up of $\mathbb{R}^n$ centered at $\{z_{n-m} = 0, \cdots, z_n = 0\}$ is the projection $\pi : M \to \mathbb{R}^n$ where the blown-up manifold M is $\mathbb{R}^{n-m} \times \mathbb{R}_{\geq 0} \times \mathbb{S}^{m-1}$. The exceptional divisor is $E = \pi^{-1}(N)$. The expression of the blowing up $\pi : \mathbb{R}^{n-m} \times \mathbb{R}_{\geq 0} \times \mathbb{S}^{m-1} \to \mathbb{R}^n$ in coordinates $(y_1, \ldots, y_{n-m}, r, x_1, \cdots, x_m)$ with $(x_1, \ldots, x_m) \in \mathbb{S}^{m-1}$ is given by $\pi(y_1, \ldots, y_{n-m}, r, x_1, \ldots, x_m) = (y_1, \ldots, y_{n-m}, rx_1, \ldots, rx_m)$. The blown-up manifold can be provided with an atlas that contains $2m$ charts given at open sets U_j^ϵ with $n-m+1 \leq j \leq n$ and $\epsilon = +, -$, in the positive and negative directions of x_j, as before.

The last generalization we want to introduce, in order to be able to iterate the process of blowing-up, are the blowing-ups on $(\mathbb{R}_{\geq 0})^k \times \mathbb{R}^{n-k}$. We will only put the example of the blowing-up of the origin, but it can also be generalized to any other smooth center. The blowing-up of $(\mathbb{R}_{\geq 0})^k \times \mathbb{R}^{n-k}$ with center at 0 is the projection $\pi : M \to (\mathbb{R}_{\geq 0})^k \times \mathbb{R}^{n-k}$ of the analytic manifold with boundary and corners $M = \mathbb{R}_{\geq 0} \times (\mathbb{S}^{n-1} \cap ((\mathbb{R}_{\geq 0})^k \times \mathbb{R}^{n-k}))$ given by $\pi(r, x_1, \ldots, x_n) = (rx_1, \ldots, rx_n)$. Notice that it is simply the restriction of the blowing-up morphism of $\mathbb{R}^n$ with center at 0. In this case, since there are not negative directions for k variables, we can cover the blowing-up manifold with $k + 2(n-k)$ charts.

1.2.3 Sequences of blowing-ups

The process of blowing-up can be iterated. We will inductively define sequences of blowing-ups of length $i \in \mathbb{N}$. We start by blowing up $\mathbb{R}^n$ with center at N_0, obtaining a blown-up manifold M_1 with exceptional divisor $E_1 = \pi_1^{-1}(N_0)$ and blowing-up morphism $\pi_1 : (M_1, E_1) \to (\mathbb{R}^n, N_0)$. We say that π_1 is *a sequence of blowing-ups of length 1*. Then, we choose a new smooth center in $N_1 \subset E_1$ and repeat the process of blowing-up, obtaining a new manifold M_2 and exceptional divisor $\widetilde{E}_2 = \pi_2^{-1}(N_1)$. That is, $\pi_2 : (M_2, \widetilde{E}_2) \to (M_1, N_1)$. We define the *exceptional divisor of the sequence* $\pi_1 \circ \pi_2$ as $E_2 = \pi_2^{-1}(E_1) = (\pi_1 \circ \pi_2)^{-1}(N_0) = (E_1 \setminus N_1) \cup \widetilde{E}_2$.

Let $\pi_1 \circ \cdots \circ \pi_{i-1} : (M_{i-1}, E_{i-1}) \to (\mathbb{R}^n, N_0)$ be a sequence of blowing-ups of length $i-1$. A *sequence* $\pi_1 \circ \cdots \circ \pi_i : (M_i, E_i) \to (\mathbb{R}^n, N_0)$ *of blowing-ups of length i* is defined as the composition of a sequence of blowing-ups $\pi_1 \circ \cdots \pi_{i-1} : (M_{i-1}, E_{i-1}) \to (\mathbb{R}^n, N_0)$ of length $i-1$ and a blowing-up $\pi_i : (M_i, \widetilde{E}_i) \to (M_{i-1}, N_{i-1})$ with smooth center N_{i-1} inside the divisor E_{i-1} of the sequence of blowing-ups $\pi_1 \circ \cdots \pi_{i-1}$. The exceptional divisor of the sequence is $E_i = \pi_i^{-1}(E_{i-1}) = (\pi_1 \circ \cdots \circ \pi_i)^{-1}(N_0) = (E_{i-1} \setminus N_{i-1}) \cup \widetilde{E}_i$.

Notice that the blown-up manifolds M_i at each step of a sequence of blowing-ups are analytic manifolds with boundary and corners. Its boundary E_i is a subanalytic manifold of codimension 1. The names of the usual charts after a single blowing-up are fixed and well defined: $U_1^+, U_1^-, \ldots, U_n^+, U_n^-$. We also know well how to cover M_{i+1} with charts once we have fixed coordinates on N_i, since we can take the previous charts and add the usual charts of the blowing-up

π_{i+1}. However, giving a good name to the usual charts from the second blowing-up can be more tricky, because we need to fix coordinates centered at N_i and this highly depends on N_i. In general, we will just assume that the atlas of M_{i+1} is given by $\{U_I\}_{I\in\mathcal{I}}$, where $\mathcal{I}$ is a set of indices and each of the charts U_I are the ones obtained after blowing-up. When we need to give names to the charts that cover M_i, we will explain how we take them in the context. We show an example of a sequence of blowing-ups and a systematic way to choose the names of the charts.

Example 1.7. Suppose that we start blowing up $0 \in \mathbb{R}^2$. The blown-up manifold M_1 is covered by 4 charts $U_1^+, U_1^-, U_2^+, U_2^-$. Then, $\pi_2 : (M_2, \widetilde{E}_2) \to (M_1, p_{1+})$ is the blowing-up centered at the iterated tangent p_{1+} of the positive x–axis, i.e. the intersection of the positive half branch of the x–axis with the exceptional divisor. The blown-up manifold is then covered by the previous charts $\pi_2^{-1}(U_1^-), \pi_2^{-1}(U_2^+), \pi_2^{-1}(U_2^-)$, where π_2 is an isomorphism, and we identify them simply with U_1^-, U_2^+, U_2^-. It is also covered by the new charts U_{11}^{++}, U_{12}^{++}, U_{12}^{+-}. Then, we define $\pi_3 : (M_3, \widetilde{E}_3) \to (M_2, p_{2-})$. The blown-up manifold M_3 is covered by the previous charts $U_1^-, U_2^+, U_{11}^{++}, U_{12}^{++}, U_{12}^{+-}$ and new charts $U_{22}^{--}, U_{21}^{--}, U_{21}^{-+}$. Finally, we perform a blowing-up $\pi_4 : (M_4, \widetilde{E}_4) \to (M_3, p_{2-1+})$ centered at p_{2-1+}. Proceeding as in the previous steps, we have the sequence of blowing-ups $\pi = \pi_1 \circ \pi_2 \circ \pi_3 \circ \pi_4 : (M_4, E_4) \to (\mathbb{R}^2, 0)$ and the covering of M_4 is $U_1^-, U_2^+, U_{11}^{++}, U_{12}^{++}, U_{12}^{+-}, U_{22}^{--}, U_{21}^{--}, U_{211}^{-++}, U_{212}^{-+-}$.

1.2.4 Other coordinates to express blowing-ups

In the previous section, we gave a general definition of blowing-ups. Now, we will show other choices of coordinates in two particular examples, as we will find them useful in this text.

Polar coordinates for the blowing-up at the origin of $\mathbb{R}^2$. Let $\pi : (M, E) \to (\mathbb{R}^2, 0)$ be blowing-up centered at the origin of $\mathbb{R}^2$. As we have seen, the blown-up manifold M is identified with $\mathbb{R}_{\geq 0} \times \mathbb{S}^1$, where $E = \pi^{-1}(0) = \{0\} \times \mathbb{S}^1$. Hence, we will take polar coordinates as usually (r, θ) at $\mathbb{R}^2$, except that we do not identify the points having $r = 0$. That is, $r \in \mathbb{R}_{\geq 0}$ and $\theta \in \mathbb{S}^1$. The advantage of taking polar coordinates is that the blown-up manifold is covered by a single chart. The disadvantage is that the blowing-ups are not given by polynomial functions, they require the use of sines and cosines. The expression of π in this chart is $\pi(r, \theta) = (r\cos\theta, r\sin\theta)$.

Polar coordinates can also be used in higher dimensional ambient spaces. Consider a blowing-up of $\mathbb{R}^n$ with center at a $(n-2)$–plane. With polar coordinates it is possible to cover the blown-up manifold by a single chart. A particularly useful example is the blowing-up centered at the z-axis of $\mathbb{R}^3$ with coordinates (x, y, z). The blown-up manifold is given by $\mathbb{R}_{\geq 0} \times \mathbb{S}^1 \times \mathbb{R}$ with coordinates (ρ, θ, z') and the expression of the blowing-up is $\pi(\rho, \theta, z) = (\rho\cos\theta, \rho\sin\theta, z')$. The exceptional divisor of this blowing-up is $E = \pi^{-1}(\{x = 0, y = 0\}) = \{0\} \times \mathbb{S}^1 \times \mathbb{R}$.

Cylindrical coordinates at the blowing-up at the origin of $\mathbb{R}^3$. Now, we show a choice of coordinates for the real blowing-up of $\mathbb{R}^3$ centered at the origin. This time we will not cover the full blown-up manifold by a single chart, we will use three instead. Recall that M is identified with $\mathbb{R}_{\geq 0} \times \mathbb{S}^2$ and take coordinates (r, x_1, x_2, x_3) with $(x_1, x_2, x_3) \in \mathbb{S}^2$ and take coordinates (x, y, z) at $0 \in \mathbb{R}^3$.

The first chart of M is one of the usual ones: $U_\infty = M \cap \{x_3 > 0\}$, but tagged with a different subindex. It is identified by φ_∞ with the half-space $\mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ provided with coordinates $(x^{(\infty)}, y^{(\infty)}, z^{(\infty)})$. The homeomorphism φ_∞ is given by $\varphi_\infty(r, x_1, x_2, x_3) = (\frac{x_1}{x_3}, \frac{x_1}{x_3}, rx_3)$. We denote $\pi|_{U_\infty} = \pi \circ \varphi_\infty^{-1}$, which is given by $\pi|_{U_\infty}(x^{(\infty)}, y^{(\infty)}, z^{(\infty)}) = (z^{(\infty)}x^{(\infty)}, z^{(\infty)}y^{(\infty)}, z^{(\infty)})$. The second chart $U_{-\infty} = M \cap \{x_3 < 0\}$ is defined in a similar manner. The third chart is given by $U_0 = M \cap (\{x_1 \neq 0\} \cup \{x_2 \neq 0\})$. It is identified with $\mathbb{R}_{\geq 0} \times \mathbb{S}^1 \times \mathbb{R}$ with coordinates (ρ, θ, z'). The homeomorphism φ_0 is given by $\varphi_0(r, x_1, x_2, x_3) = (r, \arctan\left(\frac{x_2}{x_1}\right), \frac{x_3}{\sqrt{x_1^2 + x_2^2}})$. We denote $\pi|_{U_0} = \pi \circ \varphi_0^{-1}$, which is given by $\pi|_{U_0}(\rho, \theta, z') = (\rho\cos\theta, \rho\sin\theta, \rho z')$. See Figure 1.3 for an illustration of the covering of M.

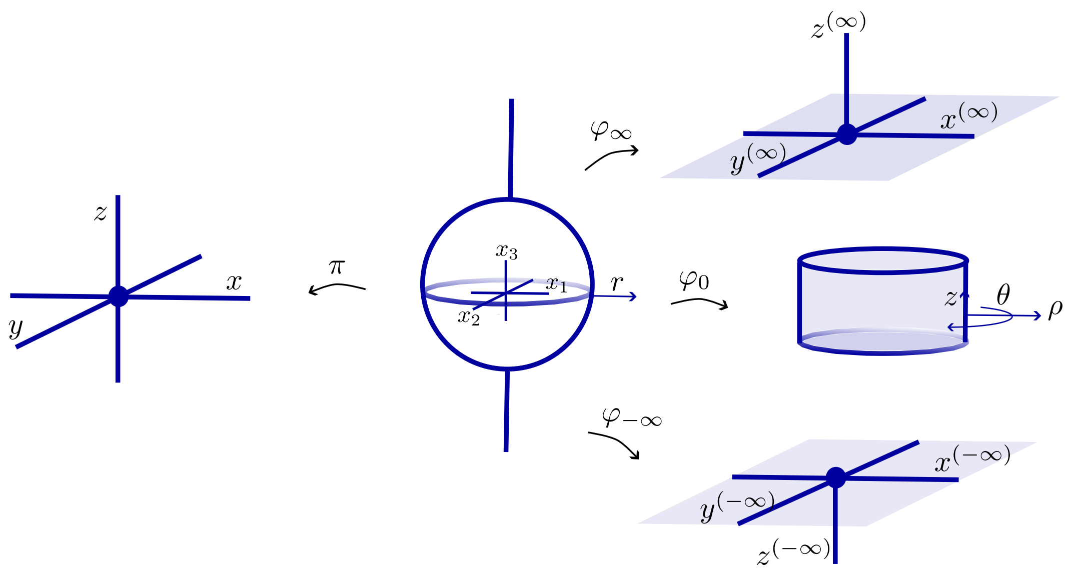

FIGURE 1.3: Blowing-up $\pi : (M, E) \to (\mathbb{R}^3, 0)$ and covering of M.

1.3 The total and strict transform of varieties and vector fields under blowing-ups

1.3.1 The transform of an analytic variety under a blowing-up

Let $X \subset \mathbb{R}^n$ be an analytic variety given by equations $f_1 = 0, \ldots, f_s = 0$ at 0. Suppose that $\pi : (M, E) \to (\mathbb{R}^n, N)$ is a blowing-up of $\mathbb{R}^n$ with center at N and that $N \subset X$ is smooth. We define the following.

Definition 1.8. *Let $X \subset \mathbb{R}^n$ be an analytic variety given by equations $f_1 = 0, \dots, f_s = 0$ at 0. Consider a blowing-up $\pi : (M, E) \to (\mathbb{R}^n, N)$ of $\mathbb{R}^n$ centered at a smooth subvariety $N \subset X$ at 0.*

- *The* total transform of X by π *is $\widetilde{X} = \pi^{-1}(X)$. Its defining ideal at each $p \in \pi^{-1}(0)$ is generated by $\pi^* f_1 = f_1 \circ \pi, \dots, \pi^* f_s = f_s \circ \pi$ at each $p \in \pi^{-1}(0)$.*
- *The* strict transform $\widetilde{X}'$ of X by π *is $\overline{\pi^{-1}(X \setminus N)}$. In a point p in $\pi^{-1}(0)$, the strict transform $\widetilde{X}'$ is given by the ideal generated by $\frac{1}{g_i}\pi^* f_i$ for $i = 1, \dots, s$, with $h_1 \cdots h_r = 0$ a reduced equation of E at p, and $g_i = h_1^{k_{i,1}} \cdots h_r^{k_{i,r}}$ where $k_{i,j} = \max\{k \in \mathbb{N} : h_j^k$ divides $\pi^* f_i\}$ for $j = 1 \cdots r$.*

Note that $\widetilde{X} = E \cup \widetilde{X}'$. If X has positive dimension and $N \neq X$, there is always some chart of the blown-up manifold where the strict transform is non-empty. This follows since π outside the exceptional divisor E is an isomorphism.

It is also possible to extend this definition to formal varieties, given by formal ideals at 0.

Definition 1.9. *Let $X \subset \mathbb{R}^n$ be a formal variety given by by equations $f_1 = 0, \dots, f_s = 0$ at 0. Consider a blowing-up $\pi : (M, E) \to (\mathbb{R}^n, N)$ of $\mathbb{R}^n$ with center at $N \subset X$.*

- *The total transform of X by π is the collection of ideals generated by $\pi^* f_1 = f_1 \circ \pi, \dots, \pi^* f_s = f_s \circ \pi$ at each $p \in \pi^{-1}(0)$.*
- *The strict transform $\widetilde{X}'$ of X by π is the collection of ideals generated by $\frac{1}{g_i}\pi^* f_i$ for $i = 1, \dots, s$ at the points $\pi^{-1}(0)$, with $h_1 \cdots h_r = 0$ a reduced equation of E at p, and $g_i = h_1^{k_{i,1}} \cdots h_r^{k_{i,r}}$ where $k_{i,j} = \max\{k \in \mathbb{N} : h_j^k$ divides $\pi^* f_i\}$ for $j = 1 \cdots r$.*

Now, we provide two related definitions dealing with the tangents of curves. The first one concerns parameterized curves and the second one concerns analytic and formal curves (see Appendix A.2.1). We take from [51] the following definitions. Recall that the ω–limit of a parameterized curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^n$ is defined by $\omega(\gamma) = \bigcap_{t_0 \in I} \overline{\gamma(I \cap \{t \geq t_0\})}$.

Definition 1.10. *Let γ be a parameterized curve with $\omega(\gamma) = \{0\}$ and let $\pi : (M_1, E_1) \to (\mathbb{R}^n, 0)$ be the blowing-up centered at 0 and set $p_0 = 0$. We say that γ has a (first)* tangent at 0 *if there exists a unique point $p_1 \in E_1$ such that strict transform $\gamma_1' = \widetilde{\gamma}'$ of γ by π_1 fulfills $\omega(\gamma_1') = \{p_1\}$. The point p_1 is named the tangent of γ. We define recursively the n–iterated tangents. We say that the curve γ has the* property of iterated tangents *if the process can be continued indefinitely and the family of points $\mathrm{IT}(\gamma) = \{p_i\}_{i \in \mathbb{N}}$ obtained in the process is the* sequence of iterated tangents.

The half-branches of analytic curves always have the property of iterated tangents in the above sense. In this sense, we can generalize the above definition to formal curves.

Definition 1.11. *Let Γ be an irreducible analytic or formal curve at 0 and recall that each irreducible formal curve has two half-branches Γ^+ and Γ^-. Let $\pi : (M_1, E_1) \to (\mathbb{R}^n, 0)$ be the blowing-up centered at 0. We define the tangent of Γ^ϵ at 0 as the only point $p_1 \in E_1$ where the ideal $(\Gamma_1^\epsilon)' = (\widetilde{\Gamma}^\epsilon)' \neq \mathcal{O}_{p_1}$ is not* *the total ideal. This process can be continued indefinitely and the* sequence of iterated tangents of Γ^ϵ *is the family of points* $IT^\epsilon(\Gamma) = \{p_i\}_{i\in\mathbb{N}}$ *obtained in the process.*

The previous definition applies for each irreducible component of an analytic or formal curve at 0. Suppose that $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_r$ is an analytic or formal curve such that each Γ_i is irreducible and $\Gamma_i \neq \Gamma_j$ for $i \neq j$. We will denote $\mathrm{IT}(\Gamma_i) = \mathrm{IT}^+(\Gamma_i) \cup \mathrm{IT}^-(\Gamma_i)$ and $\mathrm{IT}(\Gamma) = \bigcup_{i=1}^r \mathrm{IT}(\Gamma_i)$

1.3.2 The transform of a vector field by a blowing-up

Let $\xi \in \widehat{\mathfrak{X}}(\mathbb{R}^n, 0)$ be a formal vector field and $\pi : (M, E) \to (\mathbb{R}^n, N)$ be a blowing-up centered at an analytic manifold N. The following result is well known, see for instance [17]. It proves that is is possible to define a vector field at $\pi^{-1}(0) \subset E$ compatible with ξ.

Proposition 1.12. *Let $p \in \pi^{-1}(0)$ be a point in E. If N is invariant for ξ, then there exists a formal vector field $\widetilde{\xi}_p$ in p such that*

$$\widetilde{\xi}_p(f \circ \pi) = (\xi(f)) \circ \pi. \tag{1.4}$$

for any $f \in \widehat{\mathcal{O}}_0$

The vector field obtained in the previous proposition is called the *local transform of ξ at p by the blowing-up π.*

Definition 1.13. *We define the* total transform $\widetilde{\xi} = \pi^*\xi$ of ξ by π *as the collection of local transforms $\widetilde{\xi}_p$ under π, that is, $\widetilde{\xi} := \{\widetilde{\xi}_p\}_{p\in\pi^{-1}(0)}$.*

Remark 1.14. When the center of the blowing-up is not invariant for the vector field, it is still possible to obtain a vector field on the blowing-up manifold defined as in Proposition 1.12, however this vector field lies in the meromorphic class. Even though this vector field is meromoprhic, it is equivalent to a formal/analytic one up to multiplication by the equation of the divisor.

It often happens that the local transform at $p \in \pi^{-1}(0)$ can be divided by some equation of E. We remark that given an analytic vector field η, the trajectories of η and $f\eta$ are the same at any point outside $f = 0$, however the parameterization is different. Let $g_p = g_{1,p} \cdots g_{r,p}$ be a local reduced equation of the divisor E at p such that at $M \setminus E$ the sign of each $g_{i,p}$ is positive. We define the *real strict transform $\widetilde{\xi}'_p$ of ξ at p by π* as $\widetilde{\xi}'_p = \frac{1}{g_{1,p}^{k_1} \cdots g_{r,p}^{k_r}} \widetilde{\xi}_p$, where k_i is the maximum k such that $g_{i,p}^k$ divides $\widetilde{\xi}_p$. As this definition depends on the choice of the equation g_p of the divisor, we will consider the real strict transform modulo its product with any positive unit.

Definition 1.15. *We define the* (real) strict transform $\widetilde{\xi}'$ of ξ by π *as the collection of local (real) strict transforms $\widetilde{\xi}'_p$ under π, that is, $\widetilde{\xi}' := \{\widetilde{\xi}'_p\}_{p\in\pi^{-1}(0)}$.*

It is possible, and it will be more convenient, to work with vector fields on each of the usual charts of the blowing-up, transversely formal to $E \cap \pi^{-1}(0)$, which coincide with $\widetilde{\xi}_p$ at each $p \in \pi^{-1}(0)$ in the corresponding chart. We will show the expression of these vector fields showing only a single blowing-up and a single chart, but it works in the same manner iterating the process of blowing-up.

Take coordinates $(x_1, \dots, x_n)$ at 0 such that the analytic manifold N is a codimension m plane given by $x_{n-m+1} = 0, \dots, x_n = 0$. The vector field has expression

$$\xi = \sum_{i=1}^{n} a_i(\mathbf{x}) \frac{\partial}{\partial x_i},$$

with $a_i \in \mathbb{R}[[x_1, \dots, x_n]]$. Given the chart U_j^{ϵ} of M, we define a formal vector field $\pi|_{U_j^{\epsilon}}^{*}(\xi) = \xi^{(j\epsilon)}$ along the divisor $E \cap U_j^{\epsilon} \cap \pi^{-1}(0)$, as the only vector field such that

$$\xi^{(j\epsilon)}(f \circ \pi|_{U_j^{\epsilon}}) = \xi(f) \circ \pi|_{U_j^{\epsilon}}.$$

For the sake of simplicity, we give the expression of $\pi^{*}\xi$ only for the chart U_n^{+} with coordinates $\mathbf{x}' = (x_1', \dots, x_n')$, but the expression is similar on any other chart.

$$\xi^{(n+)} = \sum_{j=1}^{n-m} (a_j \circ \pi|_{U_n^{\epsilon}}) \frac{\partial}{\partial x_j'} + \sum_{j=n-m+1}^{n-1} \frac{1}{x_n'} ((a_j \circ \pi|_{U_n^{\epsilon}}) - x_j'(a_n \circ \pi|_{U_n^{\epsilon}})) \frac{\partial}{\partial x_j'} + (a_n \circ \pi|_{U_n^{\epsilon}}) \frac{\partial}{\partial x_n'} \tag{1.5}$$

where $a_j \circ \pi|_{U_n^{\epsilon}}(x_1', \dots, x_n') = a_j(x_1', \dots, x_{n-m}', x_n' x_{n-m+1}', \dots, x_n' x_{n-1}', x_n')$. The vector field $\xi^{(n+)}$ can be rewritten as

$$\xi^{(n+)} = \sum_{i=1}^{n} a_i^{(n+)}(\mathbf{x}') \frac{\partial}{\partial x_i'}$$

with coefficients, reordering variables, $a_i^{(n+)} \in \mathbb{R}[x_{n-m+1}, \dots, x_{n-1}][[x_1, \cdots, x_{n-m}, x_n]]$. We will prove this last fact with more detail in Lemma 1.50.

Since $a_i^{(n+)}$ are polynomial in the variables $x_{n-m+1}, \dots, x_{n-1}$, which can be considered as coordinates of $U_n^{+} \cap \pi^{-1}(0)$, the vector field $\xi^{(n+)}$ is well defined at each point of $U_n^{+} \cap \pi^{-1}(0)$. It can also be seen that the vector fields $\xi^{(j\epsilon)}$ coincide at the intersection of two charts.

It is also possible that $\xi^{(j\epsilon)}$ can be divided by some equation of the divisor. Consider U_j^{ϵ} and let $(g = (x_j')^{k_j} = 0)$ be an equation of the divisor with k_j maximum such that $(x_j')^{k_j}$ divides $\xi^{(j\epsilon)}$. We define the strict transform of ξ at U_j^{ϵ} to be the vector field $(\xi^{(j\epsilon)})' = \frac{1}{g}\xi^{(j\epsilon)}$. Notice that we have made a choice for the equation of E associated to the chart and that the coordinate function x_j is positive in $U_j^{\epsilon} \setminus E$.

Remark 1.16. The germ of the strict transform $(\xi^{(j\epsilon)})'$ at a point $p \in E$ on the chart U_j^{ϵ} coincides with the germ of $\widetilde{\xi}'_p$, taking into account that the strict transform at p is determined up to a multiplication by a positive unit.

The construction of the total and strict transform in the chart U_j^ϵ can be generalized to other usual charts after further blowing-ups. Given a sequence of blowing-ups $\pi = \pi_1 \circ \ldots \circ \pi_m$, recall that there is an atlas of usual charts $\{U_I\}_{I \in \mathcal{I}}$. We can inductively define the total and strict transform of ξ inside each chart U_I from the total transform of ξ at $\pi_m(U_I) \subset M_{m-1}$. We will give details for the concrete blowing-ups that we will use in this work.

1.3.3 Normal crossing divisors

We introduced before the exceptional divisor of a blowing-up as the counter image of the center of the blowing-up. We introduce now the concept of normal crossing divisor.

Definition 1.17 ([18])**.** *A* normal crossing divisor *E is a finite union of hyperplanes that fulfill the following property. At any point of $p \in E$ there exists analytic coordinates $(x_1, \ldots, x_n)$ such that the equation of E at p is $\bigcup_{i=1}^m \{x_i = 0\}$ for some $m \leq n$. We say that the set of coordinates $(x_1, \ldots, x_n)$ fulfilling this property is* adapted to *E. The components of E are called the* irreducible components of the divisor*. We also define $e_p(E) \in \mathbb{N}$ as the number of irreducible components of E that contain p. A* formal normal crossing divisor *I_E at p is an ideal in $\widehat{\mathcal{O}}_p$ given in formal coordinates $(x_1, \ldots, x_n)$ by the ideal generated by $\prod_{i=1}^m x_i$. An irreducible component of E is the ideal given by a single x_i with $i = 1, \ldots, m$.*

Notation 1.18. Given a formal normal crossing divisor I_E at p, we will often denote it by E, as it were a geometric object. This abuse of notation relies on the fact that we will often consider that E is the germ of an analytic normal crossing divisor in the sense of the first definition. In that case $I_E = I(E)$ and $E = V(I_E)$[1]. It will be clear in the context.

Consider a normal crossing divisor E at $0 \in \mathbb{R}^2$. We say that a formal curve Γ has *normal crossings with E at* 0 if $E \cup \Gamma$ is a formal normal crossing divisor at 0.

1.3.4 Saturated and non saturated vector fields

We start by defining the concept of a saturated vector field.

Definition 1.19. *Let $\xi \in \widehat{\mathfrak{X}}(\mathbb{R}^n, 0)$ be a formal vector field. We say that ξ is* saturated or reduced *if there is no other formal vector field η such that $\xi = f\eta$ with f a non-unit element in $\widehat{\mathcal{O}}_0$. Otherwise, we say that ξ is* non-saturated*. We say that η is a* saturation of *ξ if η is saturated and there is some non-unit f such that $\xi = f\eta$. We denote $S(\xi) = \eta$.*

[1]For an analytic set in $\mathbb{R}^n$, we denote $I(X)$ as the vanishing ideal of $\mathcal{O}_{n,0}$ of the subset X. For an ideal I in $\mathcal{O}_{n,0}$, we denote $V(I)$ the zero locus of the ideal.

Remark 1.20. The definition given for the saturation of a vector field is not uniquely determined. With this, we mean that it highly depends on the non-unit f that divides ξ. Suppose that we choose f so that $\xi = f\eta$ and η is saturated. We can take any unit u in $\mathcal{O}_0$ and define $\bar{f} = uf$. Now, the vector field $\bar{\eta} = u^{-1}\eta$ is another saturation of ξ, since $\bar{f}\bar{\eta} = ufu^{-1}\eta = f\eta = \xi$.

Now, we introduce some local vector fields.

Definition 1.21. *Let $\xi \in \widehat{\mathfrak{X}}(\mathbb{R}^n, 0)$ be a formal vector field, $S(\xi)$ a saturation of ξ with $\xi = hS(\xi)$, and E a normal crossing divisor. Let I_E be the formal ideal defining the divisor E. We define $\xi' := \frac{1}{g}\xi$ where $g \in I_E$ is a product $g = g_1^{k_1} \cdots g_s^{k_s}$ of the irreducible components g_i of E with maximal multiplicity k_i so that $g_i^{k_i}$ divides ξ.*

Remark 1.22. Notice that $S(\xi)$ is equal to $S(\xi')$ up to multiplication by a unit. As in Remark 1.20, the definition of ξ' and $S(\xi)$ is not canonically given. When adapted coordinates are given, that is, a system of coordinates $\mathbf{x} = (x_1, \ldots, x_n)$ in which the normal crossing divisor E is a union of coordinate hyperplanes, say $x_1 = 0, \ldots, x_s = 0$, we will take $g = x_1^{k_1} \cdots x_s^{k_s}$.

Notice that the definition of ξ' coincides with the definition of the strict transform of a vector field at a point of E when the normal crossing divisor E has been obtained as an exceptional divisor of a sequence of blowing-ups.

1.4 Reduction of singularities in dimension 2

In this section, we will work with vector fields in $\mathbb{R}^2$. We summarize the results concerning reduction of singularities in dimension 2 for reduced or saturated vector fields and non-saturated vector fields. We will basically present the results in the references [18, 27] without proof, adapted to real formal vector fields.

Consider a vector field $\xi \in \widehat{\mathfrak{X}}(\mathbb{R}^2, 0)$ and a sequence of blowing-ups $\pi : (M, E) \to (\mathbb{R}^2, 0)$. Let $\widetilde{\xi} = \{\widetilde{\xi}_p\}_{p \in E}$ and $\widetilde{\xi}' = \{\widetilde{\xi}'_p\}_{p \in E}$ be the total transform and strict transform defined in section 1.3.2. We make the abuse of notation $\widetilde{\xi}, \widetilde{\xi}' \in \widehat{\mathfrak{X}}(M, E)$ to indicate that it makes sense to consider them on all the points of the divisor. The objective of reduction of singularities is to obtain more simple singularities and normal crossings of the vector field with the exceptional divisor. Then, studying the total and strict transforms on (M, E) we are able to describe better the original vector field at $(\mathbb{R}^2, 0)$.

We define first the dicritical and non-dicritical components of the divisor for the vector field ξ.

Definition 1.23. *Let $\xi \in \widehat{\mathfrak{X}}(\mathbb{R}^2, 0)$ be a formal vector field, $\pi : (M, E) \to (\mathbb{R}^2, 0)$ be a sequence of blowing-ups and $E = E_1 \cup \cdots \cup E_s$ the exceptional divisor where each E_i is an irreducible component for $i = 1, \ldots, s$. We say that E_i is* non-dicritical *for ξ if E_i is invariant for every $\widetilde{\xi}_p$. Otherwise, we say that E_i is* dicritical *for ξ.*

We define the *singular locus* $\widetilde{\mathrm{Sing}}(\widetilde{\xi}, E)$ *of* ξ *relatively to* E as the set of points $p \in E$ such that either $p \in \mathrm{Sing}(\widetilde{\xi}'_p)$ or $p \notin \mathrm{Sing}(\widetilde{\xi}'_p)$ and the (formal) integral curve Γ of $\widetilde{\xi}'_p$ at p does not have normal crossings with E at p.

Notation 1.24. We can also denote $\widetilde{\mathrm{Sing}}(\widetilde{\xi}, E)$ by $\widetilde{\mathrm{Sing}}(\xi, E)$ or by $\widetilde{\mathrm{Sing}}(\widetilde{\xi}', E)$.

Notice that the above set contains all the singularities of $\widetilde{\xi}'$ in the divisor E, as well as all the points in which the formal integral curve of $\widetilde{\xi}'_p$ is tangent to E but different from E. One of the objectives of the reduction of singularities of a formal vector field is to remove points of $\widetilde{\mathrm{Sing}}(\xi, E)$ that are not the admissible final situations: adapted simple singularities.

Definition 1.25. *Let* $\xi \in \widehat{\mathfrak{X}}(\mathbb{R}^2, 0)$ *be a formal vector field and* E *a normal crossing divisor at* 0*. We say that* $0 \in E$ *is an* adapted simple singularity of ξ relatively to E *if*

TYPE I - *The origin* 0 *is a simple singularity of* $\xi = S(\xi)$ *and the irreducible components of* E *that contain* 0 *are invariant for* ξ*. In particular, the irreducible components of* E *at* 0 *are separatrices of* ξ*.*

TYPE II - *The origin* 0 *belongs to a curve of singularities* Γ *that has normal crossings with* E *and one of the following holds.*

a) The origin 0 *is a simple singularity of* $S(\xi)$ *and the curves* Γ *and* E *are invariant for* $S(\xi)$*.*

b) The origin 0 *is a regular point of* $S(\xi)$ *and the divisor* E *is invariant for* $S(\xi)$*.*

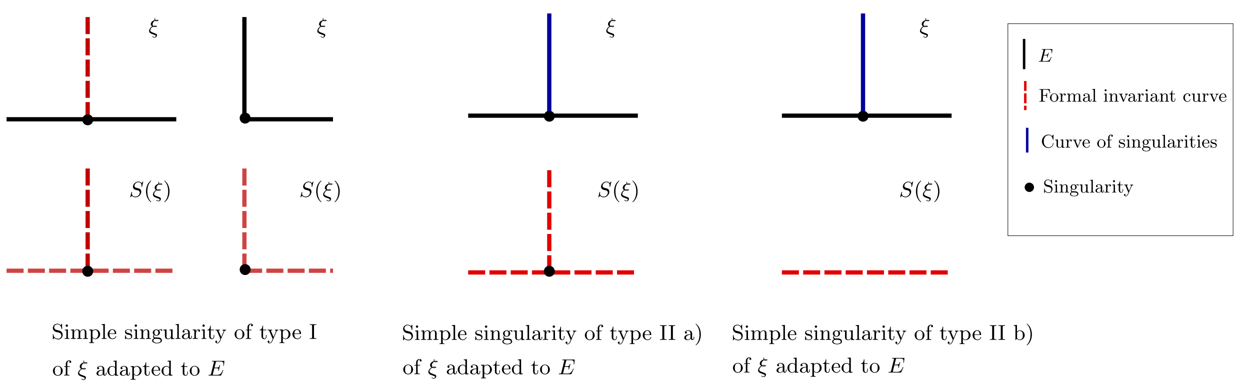

FIGURE 1.4: Adapted simple singularities.

We show possible types of adapted simple singularities of ξ relatively to a normal crossing divisor E in Figure 1.4. In the case of adapted simple singularities relatively to E of type I, the origin is a corner when $e_0(E) = 2$ or non-corner when $e_0(E) = 1$. In the case of simple singularities relatively to E of type II it is only possible that $e_0(E) = 1$. The second type of adapted simple singularities only appear when we work with non-saturated vector fields.

Notation 1.26. When it is clear from the context which is the normal crossing divisor E, we will simply use the name *adapted simple singularity*. For example, in the process of reduction of singularities, the normal crossing divisor E will be assumed to be the exceptional divisor.

Definition 1.27. *Consider a formal vector field $\xi \in \widehat{\mathfrak{X}}(\mathbb{R}^2, 0)$, E a normal crossing divisor at 0 and $S(\xi)$ the saturation of ξ. Suppose that 0 is an adapted simple singularity of ξ relatively to E.*

- *When the origin 0 is a singularity of type II b), the curve of singularities Γ is transverse to E at 0 and Γ is called a* bidicritical curve.
- *When 0 is of type I or II a) and λ, μ are the eigenvalues of the linear part of $S(\xi)$, we say that 0 is an* adapted simple singularity of saddle type *if $\mu\lambda < 0$, an* adapted simple singularity of node type *if $\mu\lambda > 0$ and an* adapted simple singularity of saddle-node type *if $\mu\lambda = 0$. For shortening the notation, we can simply name them:* adapted simple saddle, adapted simple node or adapted simple saddle-node.

In the case of the bidicritical curve, we find that the curve of singularities is transverse to the saturation $S(\xi)$. In the case of the node singularities, we have that the restriction of $S(\xi)$ to any formal (half) separatrix Γ induces a one dimensional vector field that is either formally attracting or repelling to 0. In the case of the saddle singularities, the restriction into two of the half separatrices (associated to the same eigenvalue) is attracting and the other two repelling (associated to the other eigenvalue). In the case of the saddle-node singularities, it is possible that ξ resembles a node, a saddle, or a node in one half space and a saddle in the other. Consequently, in the saddle-node singularity case, three of the separatrices are formally attracting and one is formally repelling or viceversa.

Example 1.28. We show an example of each type of adapted simple singularity in Definition 1.27. In all the examples, we suppose that $E = \{x = 0\}$.

- At the singularity at 0 of the vector field $\xi = y\frac{\partial}{\partial y}$, the curve $\{y = 0\}$ is a bidicritical curve of singularities. This curve is transverse to E and $S(\xi) = \frac{\partial}{\partial y}$ is regular and transverse to it.
- The singularity at 0 of the vector field $\xi = x\frac{\partial}{\partial x} + \sqrt{2}y\frac{\partial}{\partial y}$ is an adapted simple node. The restriction of $\xi = S(\xi)$ to $\{x = 0, y \geq 0\} \subset E$ is given by $\sqrt{2}y\frac{\partial}{\partial y}$, which is a repelling vector field. The same occurs in the rest of the separatrices.
- The singularity at 0 of the vector field $\xi = -x\frac{\partial}{\partial x} + \sqrt{2}y\frac{\partial}{\partial y}$ is an adapted simple saddle. The restriction of $\xi = S(\xi)$ to $\{x = 0, y \geq 0\} \subset E$ is given by $\sqrt{2}y\frac{\partial}{\partial y}$ (repelling), and the restriction of ξ to $\{y = 0, x > 0\}$ is given by $-x\frac{\partial}{\partial x}$ (attracting).
- The singularity at 0 of the vector field $\xi = x^2\frac{\partial}{\partial x} + \sqrt{2}y\frac{\partial}{\partial y}$ is a saddle-node. The restriction of $\xi = S(\xi)$ to $\{x = 0\} = E$ is given by $\sqrt{2}y\frac{\partial}{\partial y}$ (repelling), the restriction of ξ to $\{y = 0, x > 0\}$ is given by $x^2\frac{\partial}{\partial x}$ (repelling) and the restriction of ξ to $\{y = 0, x < 0\}$ is given by $x^2\frac{\partial}{\partial x}$ (attracting).

1.4.1 Stability of simple singularities adapted to a normal crossing divisor

Adapted simple singularities will be our final situations in after the reduction of singularities. It is well known that the blowing-up centered at a classical simple singularity, namely an adapted simple singularity of type I, provides simple singularities at each iterated tangent of the half separatrices of the vector field of the original singularity. This result is very classical and requires a simple computation. In this subsection, we prove the corresponding result for adapted simple singularities.

Lemma 1.29. *Let $\xi \in \widehat{\mathfrak{X}}(\mathbb{R}^2, 0)$ be a formal vector field, E a normal crossing divisor and suppose that 0 is a simple singularity adapted to E. Let $\sigma : (M, E, F) \to (\mathbb{R}^2, E_0, 0)$ be a blowing-up centered at 0 where the exceptional divisor is $F = \sigma^{-1}(0)$, the divisor E_0' is the strict transform of E_0, the divisor $E = \sigma^{-1}(E_0) = E_0' \cup F$ and $\widetilde{\xi} = \sigma^*(\xi) \in \widehat{\mathfrak{X}}(M, F)$. Let $\widetilde{\xi}'$ the strict transform $\widetilde{\xi}'$ of ξ, then $p \in \widetilde{\mathrm{Sing}}(\widetilde{\xi}', E) \cap F$ is an adapted simple singularity of $\widetilde{\xi}'$.*

Proof. In this section, we suppose that E_0 is a normal crossing divisor that has a single irreducible component at 0 and $0 \in \widetilde{\mathrm{Sing}}(\xi, E_0)$. The case in which E_0 has two irreducible components is left aside, because it is similar. We will only make some comments at the end of the proof. We work in formal coordinates in this proof.

- Suppose first that $\xi = S(\xi)$ has a type I simple singularity adapted to the divisor E_0 at 0. Take coordinates (x, y) so that E_0 is given by $x = 0$. Notice that E_0 must coincide with one of the separatrices of ξ by definition of adapted simple singularity. Then, the other formal separatrix Γ is transverse to E_0 and given by an equation $y - h(x)$. Making a formal change of coordinates $\tilde{x} = x$ and $\tilde{y} = y - h(x)$, we have that the equation of the other separatrix is exactly $\tilde{y} = 0$, and, up to renaming again the coordinates $(x, y) = (x, \tilde{y})$ the two separatrices are $x = 0$ and $y = 0$ and the vector field is written as

$$\xi = x(\lambda + \sum_{i+j>0} a_{ij} x^i y^j)\frac{\partial}{\partial x} + y(\mu + \sum_{i+j>0} b_{ij} x^i y^j)\frac{\partial}{\partial y}.$$

  Now, after a blowing-up $\sigma : (M, E) \to (\mathbb{R}^2, E_0)$ of 0, we can study the total and strict transforms of this vector field. We find that $F = \pi^{-1}(0)$ is a new component of the divisor and $E = E_0' \cup F$. Using the usual charts, we find that $\widetilde{\xi} = \sigma^* \xi$ is written in the chart U_1^+ as

$$\xi^{(1+)} = x(\lambda + \sum_{ij} a_{ij} x^{i+j} y^j)\frac{\partial}{\partial x} + y(\mu - \lambda + \sum_{ij} (b_{ij} - a_{ij}) x^{i+j} y^j)\frac{\partial}{\partial y},$$

  where we have renamed the coordinates $(x, y) = (x^{(1+)}, y^{(1+)})$ to simplify the notation. The strict transform in this chart coincides with the total transform, since the equation of the divisor

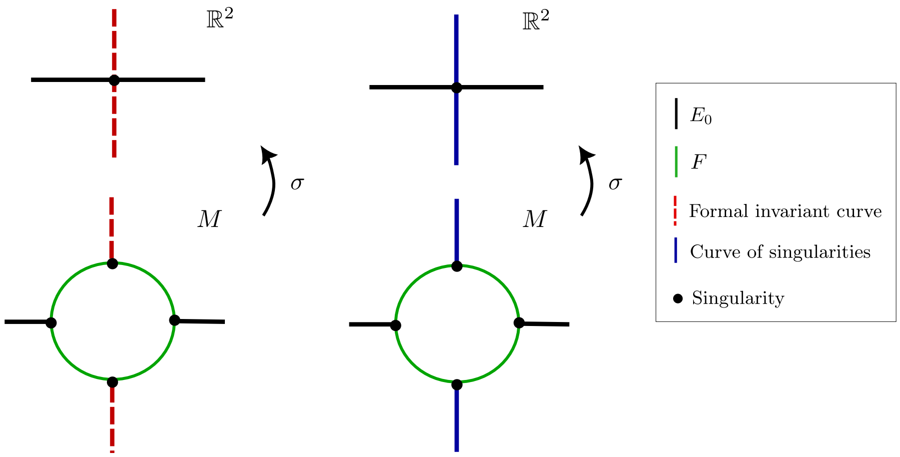

Figure 1.5: Effect of the blowing-up of the origin on the separatrices and curves of singularities of ξ.

cannot be factorized. This vector field has an isolated simple singularity at the origin of this chart. The two invariant curves at this simple singularity are F and the strict transform $\widetilde{\Gamma}'$ of Γ. The rest of the points of $F \cap U_1^+$ are regular.
At the chart U_2^+, the expression of $\widetilde{\xi}$ is

$$\xi^{(2+)} = x(\lambda - \mu + \sum_{ij}(a_{ij} - b_{ij})x^i y^{j+i})\frac{\partial}{\partial x} + y(\mu + \sum_{ij} b_{ij}x^i y^{j+i})\frac{\partial}{\partial y},$$

where we have renamed the coordinates $(x,y) = (x^{(2+)}, y^{(2+)})$ to simplify the notation. The origin is an adapted simple singularity of type I and the divisors E_0' and F are the separatrices of $\widetilde{\xi}$ and have normal crossings at the origin of U_2^+. The rest of the points of $F \cap U_2^+$ are regular. Finally, the study of the vector field $\widetilde{\xi}$ at the chart U_1^- is parallel to the one on U_1^+ and the study at U_2^- is parallel to the one of U_2^+. To clarify more the effect of the blowing-up on the half-separatrices, see the left diagram in Figure 1.5.

- Suppose that 0 is a simple singularity of type II a). There is a smooth curve of singularities Γ transverse to the divisor, 0 is a simple singularity of $S(\xi)$, the divisor E_0, given by $x = 0$, is a separatrix of $S(\xi)$ and the curve Γ is the other separatrix of $S(\xi)$ at 0. Making formal changes of coordinates as before, we have that

$$\xi = y^k \left(x(\lambda + \sum_{i+j>0} a_{ij}x^i y^j)\frac{\partial}{\partial x} + y(\mu + \sum_{i+j>0} b_{ij}x^i y^j)\frac{\partial}{\partial y} \right).$$

Now we make the blowing-up σ centered at the origin . Using the usual charts, we find in the chart U_1^+ that

$$\xi^{(1+)} = y^k x^{k+1}(\lambda + \sum_{ij} a_{ij} x^{i+j} y^j)\frac{\partial}{\partial x} + y^{k+1} x^k(\mu - \lambda + \sum_{ij}(b_{ij} - a_{ij}) x^{i+j} y^j)\frac{\partial}{\partial y},$$

where we have renamed the coordinates $(x,y) = (x^{(1+)}, y^{(1+)})$ to simplify the notation. The exceptional divisor at this chart is $F = \{x = 0\}$. Dividing by x^k, it can be noticed that the strict transform $(\xi^{(1+)})'$ has an adapted simple singularity of type II a) at the origin of U_1^+. The exceptional divisor F is invariant for $(\xi^{(1+)})'$, and hence F is non-dicritical.
In the chart U_2^+, we find that

$$\xi^{(2+)} = y^k x(\lambda - \mu + \sum_{ij}(a_{ij} - b_{ij}) x^i y^{j+i})\frac{\partial}{\partial x} + y^{k+1}(\mu + \sum_{ij} b_{ij} x^i y^{j+i})\frac{\partial}{\partial y},$$

where we have renamed the coordinates $(x,y) = (x^{(2+)}, y^{(2+)})$ to simplify the notation. The equation of F is $y = 0$ and the equation of E_0 is $x = 0$, at this chart. Hence, the origin of U_2^+ is a corner point. Dividing $\xi^{(2+)}$ by y^k, we obtain the strict transform $(\xi^{(2+)})'$ that has a type I simple singularity adapted to E at the origin. Studying the remaining two charts, we conclude that after this blowing-up there is a new non-dicritical component of the divisor, two type II a) adapted simple singularities and two type I adapted simple singularities. To clarify more the effect of the blowing-up on the curve of singularities, see the right diagram in Figure 1.5.

- Suppose that 0 is a simple singularity of type II b). There is a smooth curve of singularities Γ transverse to the divisor, 0 is regular for $S(\xi)$ and the divisor E_0 is invariant for ξ. In particular, we can write $\xi = f S(\xi)$. Using Theorem A.14, there is a change of coordinates such that $S(\xi)$ is $\frac{\partial}{\partial y}$. Since the divisor E_0 is invariant for $S(\xi)$ at 0 by definition, E_0 is given by $x = 0$. The curve of singularities is transverse to E_0, and hence, up to a formal change of coordinates, we can suppose that the curve of singularities Γ has equation $y = 0$ and that the vector field is written

$$\xi = y^k \frac{\partial}{\partial y}.$$

Now we make the blowing-up σ centered at the origin. Using the usual charts, we find in the chart U_1^+ that

$$\xi^{(1+)} = x^{k-1} y^k \frac{\partial}{\partial y},$$

where we have renamed the coordinates $(x,y) = (x^{(1+)}, y^{(1+)})$ to simplify the notation. The exceptional divisor at this chart is $F = \{x = 0\}$, which is invariant for $(\xi^{(1+)})' = y^k \frac{\partial}{\partial y}$, and hence F is non-dicritical. The origin of this chart is hence an adapted simple singularity of type II b).

In the chart U_2^+, we find that

$$\xi^{(2+)} = -y^{k-1}x\frac{\partial}{\partial x} + y^k\frac{\partial}{\partial y}.$$

The strict transform $(\xi^{(2+)})' = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ has a simple singularity at the origin of this chart. The origin is an adapted simple singularity of type I and the divisors E_0' and F are the separatrices of $\widetilde{\xi}'$ and have normal crossings at the origin of U_2^+. The situation in the charts U_1^-, U_2^- is parallel. Hence, $\widetilde{\mathrm{Sing}}(\widetilde{\xi}', E)$ contains 4 points, and all of them are simple singularities adapted to E, two of type I and two of type II b). To clarify more the effect of the blowing-up on the curve of singularities, see the right diagram in Figure 1.5.

In case that E_0 has two irreducible components at 0, there is only one case to study, the singularities of type I. We proceed in the same manner considering that the two irreducible components of E_0 are the two separatrices of ξ at 0. □

We finally point out that we have worked in $\mathbb{R}^2$, but this procedure applies to $\mathbb{R}_{\geq 0} \times \mathbb{R}$ and $(\mathbb{R}_{\geq 0})^2$.

1.4.2 Reduction of singularities of saturated vector fields

In this section, we will only state the result of reduction of singularities for saturated vector fields. Saturated vector fields have an isolated singularity. Reduction of singularities of holomorphic saturated vector fields dates back to [70]. We adapt the statement of the Theorem as it appears in [18] to the real formal case.

Theorem 1.30. *Let $\xi \in \widehat{\mathfrak{X}}(\mathbb{R}^2, 0)$ be a formal vector field with isolated singularity at 0 and E_0 a normal crossing divisor at 0. Then, there is a sequence of blowing-ups $\pi : (M, E) \to (\mathbb{R}^2, E_0)$ such that each point $p \in \widetilde{\mathrm{Sing}}(\tilde{\xi}, E) \cap \pi^{-1}(0)$ is a simple singularity of ξ' adapted to E of type I. In particular two different dicritical components of E do not intersect.*

We highlight some references of the reduction of singularities in the real case and say some words about the main differences with the complex case. F. Dumortier treats the real $\mathcal{C}^\infty$ case in [27] and J.-J. Risler the real analytic case in [67]. We remark that the proof in the real case follows from the reduction of singularities in the complexification of the real plane, following the singularities in the real trace. We remark also the following fact. Any complex simple singularity in the real plane with complex ratio of eigenvalues is considered a final situation in $\mathbb{C}$. A blowing-up centered at such point provides two other complex simple singularities which do not belong to the real trace. That is, it is possible that in the real case we need to do one extra blowing-up to finish the process of reduction of singularities.

We want to recall another important result due to C. Camacho and P. Sad [16] that concerns existence of analytic separatrices for analytic vector fields. The idea of the proof consists in using the above result of reduction of singularities for holomorphic vector fields, and showing that there is necessarily a simple singularity at the exceptional divisor whose strong direction is transverse to the divisor. Hence, there exists an analytic curve transverse to the exceptional divisor on the blowing-up manifold which is projected to an analytic curve at $0 \in \mathbb{C}^2$.

Theorem 1.31 (Camacho-Sad)**.** *Let $\xi \in \mathfrak{X}(\mathbb{C}^2, 0)$ be a holomorphic vector field with isolated singularity at* 0*. Then, it has an analytic separatrix at 0.*

Unfortunately, the above result is not always true in the real analytic case. The complexification of a real analytic vector field will always have analytic separatrix, but it may not intersect the real plane. There are authors that have studied the existence of real analytic separatrices. See for instance [15, 67].

We end this section by showing some situations in the real case. In some cases, there are not separatrices of any type at 0, in other occasions, these curves are only of class $\mathcal{C}^k$ for some $k \in \mathbb{N} \cup \{\infty\}$. Let $\xi \in \mathfrak{X}(\mathbb{R}^2, 0)$ be a real analytic vector field with isolated singularity and let $\pi : (M, E) \to (\mathbb{R}^2, 0)$ be a reduction of singularities.

1. Suppose that E has a dicritical component. Then $\widetilde{\xi}$ necessarily has a family of invariant curves transverse to E and, thus ξ has a family of invariant curves with defined tangent.

2. Suppose that $\widetilde{\xi}'$ has at least one non-corner simple singularity $p \in \widetilde{\mathrm{Sing}}(\widetilde{\xi}, E)$ at E. When a strong direction of ξ'_p is transverse to the divisor, there is an analytic invariant curve Γ at p by Briot-Bouquet's Theorem 1.6, which is projected to $(\mathbb{R}^2,0)$. When there is only a strong direction tangent to the divisor and a weak direction transverse to it, there is not an analytic curve at this point, but there is a formal one $\hat{\Gamma}$. In addition, by the center manifold theorem, this formal curve $\hat{\Gamma}$ has a $\mathcal{C}^k$ realization Γ, i.e. a $\mathcal{C}^k$ curve whose Taylor development coincides with the one of $\hat{\Gamma}$ that is as well invariant for the vector field ξ.

3. Suppose that all the singularities of $\widetilde{\xi}'$ at E are saddles (or saddle-nodes that behave as saddles in the sense showed after Definition 1.27) placed at corner points of E. Suppose also that there are not dicritical components on E. Hence, every $p \in \widetilde{\mathrm{Sing}}(\widetilde{\xi}, E)$ is a corner saddle singularity of ξ'_p. Then, there are not invariant curves transverse to the divisor, because at the rest of the points $q \in \widetilde{\mathrm{Sing}}(\widetilde{\xi}, E)$, the vector field ξ'_q is regular and tangent to E.

The vector fields $\xi \in \mathfrak{X}(\mathbb{R}^2, 0)$ that have a reduction of singularities $\pi : (M, E) \to (\mathbb{R}^2, 0)$ such that every $p \in \widetilde{\mathrm{Sing}}(\widetilde{\xi}', E)$ is a corner saddle singularity of ξ'_p and such that E does not contain dicritical components are named *center-focus* vector fields (situation 3 above). The definition extends to formal vector fields since it depends only on the type of singularities after blowing-up. Being center-focus is equivalent to not having any $\mathcal{C}^k$ or even formal invariant curve with a defined tangent at 0.

1.4.3 Reduction of singularities of non-saturated vector fields

In this section, we make a reduction of singularities of vector fields that are non-saturated. This reduction is, roughly speaking, a combination of a reduction of singularities of a saturated vector field and a reduction of singularities of formal curves, with some extra normal crossings conditions. Notice that the notion of simple singularity adapted to a divisor concerns not only simple singularities in the classical sense, but also normal crossings of the strict transform of the curve of singularities $\mathrm{Sing}(\xi)$ with the divisor. We start by recalling the classical result of reduction of singularities of formal curves, adapted to the real case.

Theorem 1.32 ([22]). *Let Γ be a formal curve at $0 \in \mathbb{R}^2$. Then there exists a sequence of blowing-ups $\pi : (M, E) \to (\mathbb{R}^2, 0)$ centered at points such that the strict transform $\widetilde{\Gamma}'$ is non-singular on M and has normal crossings with E. Moreover, there is a minimal sequence with the previous property which is composed only by blowing-ups centered at iterated tangents of the half-branches of Γ.*

Now we state the general theorem of reduction of singularities for non-saturated vector fields. We show an extended version of the proof that we gave in [23], in order to understand better the final situations after the reduction of singularities. We also remark that this result is a joint reduction of singularities of the saturation of a non-saturated vector field and its curve of singularities, requiring additional normal crossings conditions.

Theorem 1.33. *Let $\xi \in \widehat{\mathfrak{X}}(\mathbb{R}^2, 0)$ be a formal vector field and E_0 an analytic normal crossing divisor at 0 with $0 \in \widetilde{\mathrm{Sing}}(\xi, E_0)$. Then there is a sequence of blowing-ups $\pi : (M, E) \to (\mathbb{R}^2, E_0)$ with $F = \pi^{-1}(0)$ and $E = E_0' \cup F = \pi^{-1}(E_0)$ fulfilling the following conditions:*

a) For any point $q \in F$, let ξ_q' be the strict transform of ξ by π at q, that is, $\xi_q' = \frac{1}{u^k v^l}\xi_q$ where $uv^\epsilon = 0$ is a local reduced equation of E at p, with $\epsilon = 0, 1$. Then $q \in \widetilde{\mathrm{Sing}}(\xi', E)$ if and only if $q \in Sing(\xi_q')$.

b) If $q \in F$ is a singular point of ξ_q', then q is an adapted simple singularity of ξ_q' relatively to E.

In particular, two dicritical components of E do not intersect.

In order to shorten the proof, we will prove first a local result that will be useful for the proof of this theorem.

Lemma 1.34. *Let $\xi \in \widehat{\mathfrak{X}}(\mathbb{R}^2, 0)$ be a formal non-saturated vector field such that $\Gamma = Sing(\xi)$ is the formal curve of singularities. Consider a formal normal crossing divisor E_0 at 0 such that $0 \in \widetilde{\mathrm{Sing}}(\xi, E_0)$, and one of the following situations holds:*

(i) $S(\xi)$ is regular at 0 and $E_0 = E_0^1$ is a single irreducible invariant curve for $S(\xi)$.

(ii) *$S(\xi)$ is regular, E_0 has two irreducible components E_0^1 and E_0^2 such that E_0^1 is invariant for $S(\xi)$ and $E_0^2 \not\subset \Gamma$.*

(iii) *The origin is an adapted simple singularity of $S(\xi)$ relatively to E_0 and E_0 is composed by the two formal separatrices E_0^1 and E_0^2 of $S(\xi)$ at 0.*

Consider $\pi : (M,F) \to (\mathbb{R}^2, 0)$ the minimal reduction of singularities of the formal curve $\Gamma \cup E_0$ with $F = \pi^{-1}(0)$ and $E = \pi^{-1}(E_0) = E_0' \cup F$. For every $t \in F$ we have one of the following situations:

a) *If $t \in IT(\Gamma)$, then $t \in Sing(\widetilde{\xi}_t')$ and either $S(\widetilde{\xi}_t')$ is regular, or $S(\widetilde{\xi}_t')$ has a simple singularity at t such that F_t and Γ_t' are the two separatrices.*

b) *If (i) or (iii) hold and t is a corner point of E, it is an adapted simple singularity of type I of $\widetilde{\xi}'$ relatively to E. If (ii) holds and t is a corner between $(E_0^2)'$ and F, the vector field $S(\widetilde{\xi}_t)$ is regular. If t is any other corner point of E, it is an adapted simple singularity of type I of $\widetilde{\xi}'$ relatively to E.*

c) *The point t is a regular point for $\widetilde{\xi}'$.*

Proof. Since E_0 is a normal crossing divisor, the blowing-ups in the sequence π are centered at the iterated tangents of Γ, and $\Gamma = \mathrm{Sing}(\xi)$, all the blowing-ups are admissible for the vector field ξ. It is enough to study the final situations after this sequence for the situations *(i)*, *(ii)* and *(iii)* and verify the statements in the thesis of the Lemma.

We start studying the iterated tangents of Γ, recalling that Γ' is a smooth curve at any of these points. Consider $p \in IT(\Gamma)$. There are two possibilities:

- The point $p \notin E_0'$. In the three cases, we have that $S(\widetilde{\xi}_p)$ is regular and F is the only invariant curve at p.
- The point $p \in E_0'$, that is, necessarily $(E_0)_p' = \Gamma_p'$.
  - In the cases (i) and (ii), we have $(E_0)_p' = (E_0^1)_p' = \Gamma_p'$. Since $S(\xi)$ is regular and E_0^1 is invariant for it, we have that the saturation $S(\widetilde{\xi}_p')$ of the transform $\widetilde{\xi}_p$ has a simple singularity whose separatrices coincide with $(E_0)_p'$ and F_p.
  - In the case (iii), from Lemma 1.29, we have that p is a simple singularity of $S(\widetilde{\xi}_p')$ and F_p and $(E_0)_p' = \Gamma_p'$ are the two separatrices.

In both cases, we get a).

Now we study the corner points $(E_0' \cap F)$ which are either regular or adapted simple singularities of $\widetilde{\xi}'$ of type I relatively to E. Consider a point $p \in (E_0' \cap F) \setminus \Gamma'$.

- Suppose $(E_0')_p$ is invariant for $S(\widetilde{\xi}_p')$. Notice that $(E_0)_p' = (E_0^1)_p'$ is the strict transform of E_0^1 in the cases (i) and (ii), and either $(E_0)_p' = (E_0^1)_p'$ or $(E_0)_p' = (E_0^2)_p'$ in the case (iii). In the three cases, p is an adapted simple singularity of $\widetilde{\xi}_p' = S(\widetilde{\xi}_p')$.

- Suppose that $(E_0')_p$ is not invariant for $S(\widetilde{\xi}_p')$. Notice that this situation occurs when $(E_0)_p' = (E_0^2)_p'$ in the case (ii). As the original vector field $S(\xi)$ is regular, the point p is a regular point of $\widetilde{\xi}_p' = S(\widetilde{\xi}_p')$ and F is an invariant curve of $\widetilde{\xi}_p'$.

As before, in both cases we get b). The rest of the points are regular and we finish the proof. $\square$

Proof of Theorem 1.33 . If the vector field ξ is saturated, the result follows from Theorem 1.30. Then, we suppose that ξ is non-saturated. We denote by $\Gamma = \mathrm{Sing}(\xi)$ the singular locus of ξ which is a finite union of m real formal irreducible curves $\Gamma^{(1)},\ldots,\Gamma^{(m)}$. We will denote the two half-branches of each $\Gamma^{(j)}$ as $\Gamma^{(j,\epsilon)}$ for $\epsilon = +,-$. The vector field can be rewritten as $\xi = f\eta$, where $\eta = S(\xi)$ is the saturation of ξ and $f = 0$ is an equation of Γ.

1. The vector field η is a saturated one, and it admits a reduction of singularities $\pi_1 : (M_1, E_1, F_1) \to (\mathbb{R}^2, E_0, 0)$ as in Theorem 1.30, where $E_1 = \pi^{-1}(E_0)$ and $F_1 = \pi^{-1}(0)$. In order to avoid confusion when we make more blowing-ups, we change slightly the notations in section 1.3.2. We will denote the total transforms η_1 and ξ_1 and the strict transforms η_1' and ξ_1', corresponding to the transforms of η and ξ, respectively. Another simplification of the notation is the following: at each point $p \in F_1$, we will denote η_p' and ξ_p' instead of $\eta_{1,p}'$ and $\xi_{1,p}$, respectively, when it is clear that $p \in F_1$.

   After the reduction of singularities π_1, the vector field η_p only has an adapted simple singularity of type I or it is regular at $p \in F_1$. Furthermore, the dicritical components of E_1 for η_1' are isolated from others. Thus, at the points $p \in E_1$ where ξ_p' is equivalent to η_p' up to the multiplication by a unit, we directly have that ξ_p' is regular or has an adapted simple singularity of type I.

   Let Γ_1' and Γ_1 denote the strict and total transforms of the curve Γ (again, changing the notations with respect to section 1.3.1). We also simplify the notation Γ_p' for $\Gamma_{1,p}'$ when it is clear that $p \in E_1$. The strict transform Γ_1' of Γ by π_1 is non-trivial only in a finite number of points $p_1,\ldots,p_s$, the iterated tangents of Γ at E_1. Notice that we can write $\xi_{p_i}' = f_{p_i}\eta_{p_i}'$ for $i = 1,\ldots,s$ since the germ f_{p_i} of Γ_{p_i} is not a unit in $\widehat{\mathcal{O}}_p$ at these points. In particular, the strict transform ξ_{p_i}' has a singularity at each p_i, that is, $p_i \in \mathrm{Sing}(\xi_{p_i}')$ and then $p_i \in \widetilde{\mathrm{Sing}}(\xi_1, E_1)$.

   In this step we conclude the result for all points of E_1 except $p_1,\cdots,p_s$. We have that $q \in \widetilde{\mathrm{Sing}}(\xi_1, E_1)$ if and only if q is an adapted simple singularity of type I in Definition 1.25 for any $q \in E \setminus \{p_1,\ldots,p_s\}$. Then, a) follows at the points in $E \setminus \{p_1,\ldots,p_s\}$ and b) follows at $\mathrm{Sing}(\xi_1) \setminus \{p_1,\ldots,p_s\}$.

2. Now, we study the points $p_1,\ldots,p_s$. Take some $p \in \{p_1,\ldots,p_s\}$ and consider the strict transform Γ_{p_i}' of Γ. Recall also from the previous point that p is a simple singularity of $S(\xi_p) = \eta_p$ adapted to E_1 or it is a regular point for this vector field. We consider three cases:

- The point $p \in E_1$ is in a non-dicritical component $D^1 \subset E_1$ and p is a regular point for $S(\xi_p) = \eta_p$. We consider the normal crossing divisor $D = D^1$ at p.
- The point $p \in E_1$ is in a dicritical component $D^2 \subset E_1$. We consider the invariant curve D^1 for $S(\xi_p) = \eta_p$ and define the normal crossing divisor $D = D^1 \cup D^2$ at p. Notice that we do not exclude the case that $D^1 \subset E_1$.
- The point $p \in E_1$ is an adapted simple singularity of $S(\xi_p) = \eta_p$ relatively to E_1. The component $D^1 \subset E_1$ is a non-dicritical component of the divisor. We define D^2 as the separatrix of $S(\xi_p) = \eta_p$ transverse to D^1. When p is a corner, notice that $D^2 \subset E_1$ is another non-dicritical component of E_1. We consider the normal crossing divisor $D = D^1 \cup D^2$ at p.

The three previous cases are in the hypothesis of Lemma 1.34 with respect to the normal crossing divisor D. We apply this Lemma to every $p \in \{p_1, \ldots, p_s\}$, obtaining a new sequence of blowing-ups $\pi_2 : (M_2, E_2) \to (M_1, E_1)$ where $E_2 = \pi^{-1}(E_1)$. See Figure 1.6 for an illustration of intermediate blowing-ups in the construction of π_2.

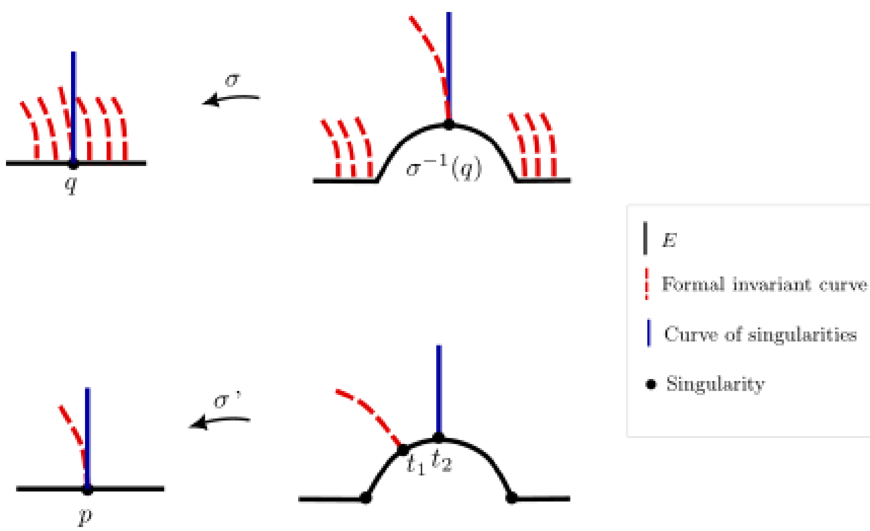

Figure 1.6: Illustration of intermediate steps in the sequence $\pi_2 : (M_2, E_2) \to (M_1, E_1)$.

The final situations of each of the points are the ones given in the thesis of the Lemma. We end by analyzing the final situations a), b) and c) with respect to the normal crossing divisor E_2. Let t be any point obtained above some $p \in \{p_1, \ldots, p_s\}$.

- Suppose t is in situation a), that is, it is an iterated tangent of Γ. If the point t is regular for the saturation $S(\xi_t')$ of the strict transform ξ_t' by π_2, we have that ξ_t' has an adapted simple singularity of type II b) at t relatively to the divisor E_2. If the point t is a simple singularity of $S(\xi_t')$, we have that Γ_t' is necessarily one of the separatrices and t is an adapted simple singularity of ξ_t' of type II a) relatively to E_2.

- Suppose that t is in situation b), that is, it is a corner point of the extended divisor $E_2 \cup D'$, where D' is the strict transform of the normal crossing divisor D defined at p. If one of the components D^1 or D^2 is not contained in E_1, then $E_2 \cup D'$ strictly contains E_2. Recall also that as these points are not iterated tangents of Γ, the strict transform ξ'_t is saturated, that is, $\xi'_t = S(\xi'_t)$. Suppose first that $(E_2)_t = (E_2)_t \cup (D')_t$, we have that t is an adapted simple singularity of ξ'_t of type I, placed at a corner of E_2. Suppose now that $(E_2)_t \subset (E_2)_t \cup (D')_t$, we also have that t is an adapted simple singularity of $\hat{\xi}'$ of type I, but in this case t is a non-corner point of E_2 and D' is the separatrix of ξ'_t transverse to E_2.
- The rest of the points t are regular for ξ'_t.

After the two steps, we conclude a) in the Theorem and that all the singularities are indeed adapted simple singularities (b) of the Theorem), as we wanted to prove. □

1.4.4 Adapted coordinates after reduction of singularities

Let $\widetilde{\xi} \in \widehat{\mathfrak{X}}(M,E)$ be the total transform of the formal vector field ξ. After a reduction of singularities of ξ, all the points of the divisor are associated either to regular points of $\widetilde{\xi}'$ or to adapted simple singularities of $\widetilde{\xi}'$. In this section, possibly performing more blowing-ups, we provide expressions of $\widetilde{\xi}$. The content of this section will be specially useful in Chapter 2. We follow [29] where the authors provide expressions of formal vector field after blowing-ups of a saturated vector field and we add the expressions in the presence of curves of singularities.

Before introducing these expressions, we remind the reader that after a reduction of singularities, at any adapted simple singularity there are exactly two formal transverse invariant curves, among which we necessarily find the components of the divisor. Suppose that p is an adapted simple singularity with $e(p) = 1$, that (x,y) are coordinates such that $y = 0$ is an equation of the divisor E and that the other formal invariant curve has equation $x = h(y)$. We can make a formal change of coordinates so that it becomes a coordinate hyperplane. However, in this section we will not be interested in making formal changes of coordinates, even if we work with formal vector fields, because when they are related to analytic objects, we want to preserve their analytic features. We use the adapted coordinates that are constructed working in the natural charts of the blowing-ups we explained in section 1.2.

- **Non-dicritical regular point.** At a non-dicritical regular point $p \in E_j \subset E$, we can choose adapted coordinates (x,y) so that $E = \{y = 0\}$. Recalling that E_j is a non-dicritical component, E_j is invariant for the strict transform $\widetilde{\xi}'$, i.e. $\widetilde{\xi}'((y)) \subset (y)$. Recall also that p is not singular for $\widetilde{\xi}'$, then we can write
$$\widetilde{\xi} = y^n\left(a(x,y)\frac{\partial}{\partial x} + yb(x,y)\frac{\partial}{\partial y}\right), \tag{1.6}$$

with $n \geq 1$, $a, b \in \mathbb{R}[x][[y]]$ and $a(x,y) = \sum_{j=0}^{\infty} a_j(x) y^j$ having $a_0(0) \neq 0$, since the point p is not a singularity for $\widetilde{\xi}'$.

- **Dicritical regular point.** At a dicritical regular point $p \in E_j \subset E$, we can again choose adapted coordinates (x,y) so that $E = \{y = 0\}$. Recalling that E_j is a dicritical component, E_j is not invariant for the strict transform $\widetilde{\xi}'$. Recalling also that p is not a singular point for $\widetilde{\xi}'$, we can write
$$\widetilde{\xi} = y^n (a(x,y) \frac{\partial}{\partial x} + b(x,y) \frac{\partial}{\partial y}), \tag{1.7}$$
with $n \geq 1$, $a, b \in \mathbb{R}[x][[y]]$ and $b(x,y) = \sum_{j=0}^{\infty} b_j(x) y^j$ having $b_0(0) \neq 0$, since $\widetilde{\xi}'$ is transverse to E_j at p.

- **Corner point between a dicritical curve and non-dicritical component of the divisor.** This situation applies to two cases: a normal crossing between a curve of singularities and the divisor (a singularity of type II-b) in Definition 1.25), when the first is of bidicritical type, and the normal crossing of a dicritical and a non-dicritical component of the divisor.

  First, let $p \in E$ be the intersection point of E_1 and E_2, a dicritical and a non-dicritical components of E at p. Take adapted coordinates (x,y) at p so that $E_1 = \{x = 0\}$ and $E_2 = \{y = 0\}$. Recall that p is a regular point for the strict transform, then, we can write
$$\widetilde{\xi} = x^n y^m \left(a(x,y) \frac{\partial}{\partial x} + y b(x,y) \frac{\partial}{\partial y} \right), \tag{1.8}$$
with $n, m \geq 1$, $a, b \in \mathbb{R}[x][[y]] \cap \mathbb{R}[y][[x]]$ and writing $a(x,y) = \sum_{j=0}^{\infty} a_j(x) y^j$ as a series in $\mathbb{R}[x][[y]]$, we have $a_0(0) \neq 0$.
Now, suppose that p is the intersection point of E_1 a non-dicritical component of the divisor and a formal invariant curve of bidicritical type. Then, we can take adapted coordinates (x,y) so that $E_1 = \{y = 0\}$ and the formal curve of singularities is given by an ideal generated by $x - h(y)$. In the analytic case, we can take the change of coordinates defined in the introduction of this section so that the vector field in adapted coordinates is reduced to equation (1.8). Otherwise, the vector field can be written
$$\widetilde{\xi} = (x - h(y))^n y^m (a(x,y) \frac{\partial}{\partial x} + y b(x,y) \frac{\partial}{\partial y}), \tag{1.9}$$
with $n, m \geq 1$, $a, b \in \mathbb{R}[x][[y]]$ and writing $a(x,y) = \sum_{j=0}^{\infty} a_j(x) y^j$ as a series in $\mathbb{R}[x][[y]]$, we have $a_0(0) \neq 0$.

- **Adapted simple singularity of type I a).** An adapted simple singularity $p \in E$ of type I is a point that lies in a non-dicritical component $E_1 \subset E$ of the divisor. We distinguish two cases based on whether p lies in the intersection of two components E_1 and E_2 of the divisor ($e_p = 2$) or not ($e_p = 1$). We have that $\widetilde{\xi}'$ at p has only two formal invariant curves. Since the divisor is invariant in both cases, we obtain that the two components of the divisor are the formal invariant curves in the first case, and that the only component of the divisor at p is one of the formal invariant curves. Then, in the first case, we can take adapted coordinates (x,y) at p so that $E_1 = \{y = 0\}$ and $E_2 = \{x = 0\}$. Suppose that the eigenvalues of $\widetilde{\xi}'$ at p are λ, μ, then, we can write

$$\widetilde{\xi} = x^n y^m \left(x a(x,y) \frac{\partial}{\partial x} + y b(x,y) \frac{\partial}{\partial y} \right),$$

with $n, m \geq 1$, $a, b \in \mathbb{R}[x][[y]] \cap \mathbb{R}[y][[x]]$ and writing $a(x,y) = \sum_{j=0}^{\infty} a_j(x) y^j$ and $b(x,y) = \sum_{j=0}^{\infty} b_j(x) y^j$ as a series in $\mathbb{R}[x][[y]]$, we have $a_0(0) = \lambda$ and $b_0(0) = \mu$. Furthermore, suppose that $\lambda = 0$. Then, performing a finite number of blowing-ups following the direction of x, we can ensure that there exists some $r \in \mathbb{N}$ such that we can write

$$\widetilde{\xi} = x^n y^m \left(x^r a(x,y) \frac{\partial}{\partial x} + y b(x,y) \frac{\partial}{\partial y} \right), \tag{1.10}$$

with $a(0,0) \neq 0$. This number is $r = \nu(S(\widetilde{\xi})_p|_{\{y=0\}})$. The intermediate points generated on this blowing-ups are regular ones and corner saddle points such that $DS(\widetilde{\xi})_q(q)$ has eigenvalues $\mu, -\mu \neq 0$ (observe the proof of Lemma 1.29). We enlarge the sequence of blowing-ups so that this situation is got.

In the non-corner case, we can take adapted coordinates (x,y) at p so that $E_1 = \{y = 0\}$. Suppose that the eigenvalues of $\widetilde{\xi}'$ at p are λ, μ, then, we can write

$$\widetilde{\xi} = y^m (a(x,y) \frac{\partial}{\partial x} + y b(x,y) \frac{\partial}{\partial y}), \tag{1.11}$$

with $m \geq 1$, $a, b \in \mathbb{R}[x][[y]]$ and writing $a(x,y) = \sum_{j=0}^{\infty} a_j(x) y^j$ and $b(x,y) = \sum_{j=0}^{\infty} b_j(x) y^j$ as a series in $\mathbb{R}[x][[y]]$, we have $a_0(x) = \lambda x + \cdots$ and $b_0(0) = \mu$. Notice that in this case, $a(x,y)$ is not necessarily divided by x. We remark again that there is a formal invariant curve Γ with equation $x - h(y) = 0$ that is transverse to the divisor. The restriction $\xi'|_\Gamma$ is a one dimensional vector field different from the 0 vector field, in particular, it has a non-vanishing r–jet. Suppose that $\mu = 0$. Proceeding similarly to the previous item, we can ensure that performing r blowing-ups following the iterated tangents of Γ, we can write

$$\widetilde{\xi} = y^m (a(x,y) \frac{\partial}{\partial x} + y^r b(x,y) \frac{\partial}{\partial y}). \tag{1.12}$$

- **Normal crossing of a smooth curve of singularities and the divisor at a simple singularity of**

$\widetilde{\xi}$. Notice that this point is an adapted simple singularity of type II$-a$: p is a simple singularity for $\widetilde{\xi}'$, the smooth curve of singularities Γ coincides with one of the two formal invariant curves of $\widetilde{\xi}'$ at p, and the component of the divisor E_1 at p is the other formal invariant curve. Then, choosing adapted coordinates (x,y) at p so that $E_1 = \{y = 0\}$, the curve Γ has equations $x - h(y)$ with $\nu(h) \geq 1$. We can write,

$$\widetilde{\xi} = (x - h(y))^n y^m (a(x,y)\frac{\partial}{\partial x} + yb(x,y)\frac{\partial}{\partial y}), \tag{1.13}$$

with $n, m \geq 1$, $a, b \in \mathbb{R}[x][[y]]$ and writing $a(x,y) = \sum_{j=0}^{\infty} a_j(x)y^j$ and $b(x,y) = \sum_{j=0}^{\infty} b_j(x)y^j$ as a series in $\mathbb{R}[x][[y]]$, we have $a_0(x) = \lambda x + \cdots$ and $b_0(0) = \mu$. As in the previous case, recall that there is a formal invariant curve Γ transverse to E. We have that the vector fields $S(\xi)|_\Gamma$ and $S(\xi)|_E$ are non-zero and each of them has a non-vanishing k–jet. Supposing that $\mu = 0$ and that r is the order of $S(\xi)|_\Gamma$, we can proceed as before, and perform r extra blowing-ups, so that we can write

$$\widetilde{\xi} = (x - h(y))^n y^m (a(x,y)\frac{\partial}{\partial x} + y^r b(x,y)\frac{\partial}{\partial y}), \tag{1.14}$$

for some $r \geq 1$.

1.4.5 The graph associated to the reduction of singularities in dimension 2

In this section, we present a combinatorial object constructed from the reduction of singularities of a formal vector field. It will be useful especially in Chapter 2. Vector fields in this section will be assumed to be different from 0. Let $\xi \in \widehat{\mathfrak{X}}(\mathbb{R}^2, 0)$ be a formal (non-saturated) vector field and $\pi : (M, E) \to (\mathbb{R}^2, 0)$ an oriented reduction of singularities.

Remark 1.35. Notice that π is by hypothesis an oriented reduction of singularities, then the divisor is homeomorphic to $\mathbb{S}^1$ and the number of components is finite. The number of adapted simple singularities is as well finite. At the non-corner adapted simple singularities there is a unique formal invariant curve. Finally, notice that the dicritical components of E are isolated as dicritical components, that is, two of them do not intersect.

There is a simple graph $G = G(\xi, \pi)$ associated to ξ and π that fulfills some properties. First, let us present the set of vertices $\mathcal{V} = \mathcal{V}(G)$ and edges $\mathcal{E} = \mathcal{E}(G)$ of the graph. We divide the vertices in two groups: $\mathcal{V} = \mathcal{V}_{div} \cup \mathcal{V}_{ndiv}$.

1. Every adapted simple singularity of $\widetilde{\xi}'$ and every corner provide a vertex to the set of vertices $\mathcal{V}_{div}$ of the graph, that is, for a point p (adapted simple singularity or, non-exclusively, corner point), there is $v_p \in \mathcal{V}_{div}$.

2. For each formal invariant curve Γ of $\widetilde{\xi}'$ transverse to the divisor at some adapted simple

singularity, we add a vertex v_Γ to $\mathcal{V}_{ndiv}$, in such a way that $v_\Gamma \neq v_{\Gamma'}$ if $\Gamma \neq \Gamma'$ and such that the set of vertices v_Γ is disjoint to the set of vertices defined in point 1 above.

The edges $\mathcal{E} = \mathcal{E}_{div} \cup \mathcal{E}_{ndiv}$ are given by:

3. Let p_v be the point from which $v \in \mathcal{V}_{div}$ has been obtained. Each connected component γ of $E \setminus \bigcup_{v \in \mathcal{V}_{div}} \{p_v\}$ provides a different edge e_γ that belongs to the set of edges $\mathcal{E}_{div}$.
4. For each formal invariant curve Γ of $\widetilde{\xi}'$ transverse to the divisor, there is also an edge e_Γ in $\mathcal{E}_{ndiv}$.
5. Adjacency is defined as follows. If $e \in \mathcal{E}_{div}$, then e joins the two vertices $v, v' \in \mathcal{V}_{div}$ that satisfy $\overline{\gamma_e} \setminus \gamma_e = \{p_v, p_{v'}\}$, where γ_e is the connected component of $E \setminus \bigcup_{v \in \mathcal{V}_{div}} \{p_v\}$ associated to e and $p_v, p_{v'}$ are the points associated to v, v', respectively. We denote $e = \{v, v'\}$. On the other hand, if Γ is a formal curve at a non corner adapted simple singularity p, the edge e_Γ associated to Γ joins $v_p \in \mathcal{V}_{div}$ and $v_\Gamma \in \mathcal{V}_{ndiv}$. We denote $e_\Gamma = \{v_p, v_\Gamma\}$.

See Figure 1.7 for an example of the construction of the graph. In the figure and to shorten the notation, we denote D component for a dicritical component of the divisor and N-D component for a non-dicritical one. The edges in $\mathcal{E}_{div}$ and vertices $\mathcal{V}_{div}$ form a subgraph G_{div}. In particular,

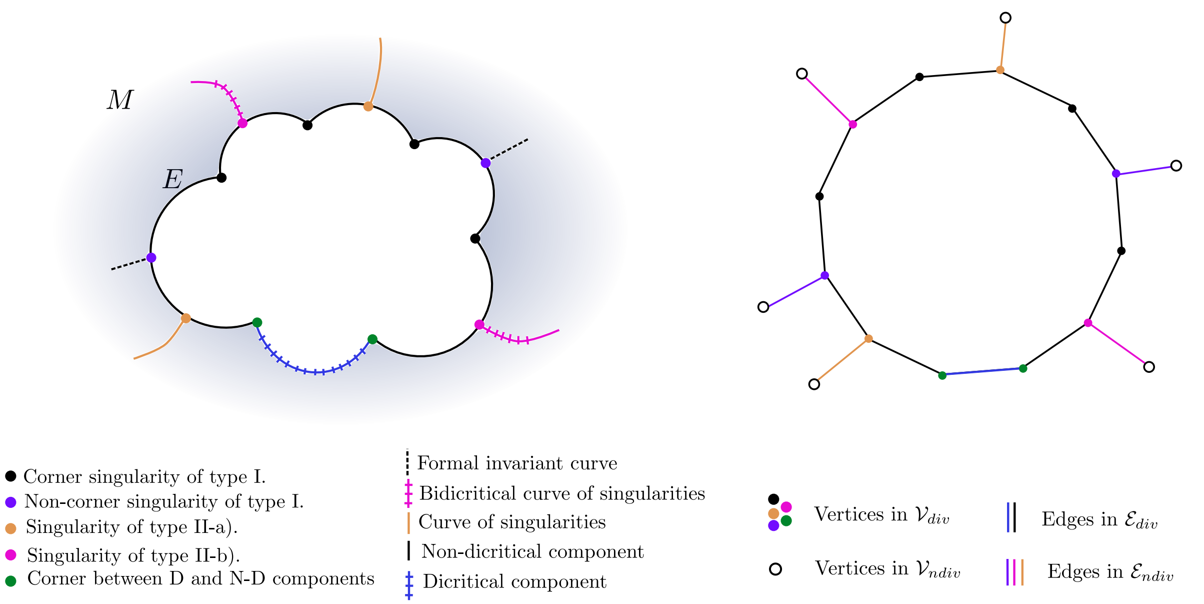

Figure 1.7: Manifold (M, E) and adapted simple singularities of $\pi^*(\xi)$ (left). Graph $G(\xi, \pi)$ (right).

without considering orientation, the graph G_{div} is a cycle (see remark 1.35). Now, we provide a partial orientation on the graph.

- Suppose $e = \{v, v'\} \in \mathcal{E}_{div}$. If e represents a connected component of $E \setminus \mathcal{V}$ contained in a dicritial component of E, then e is not oriented. Otherwise, we endow e with the orientation from v to v' or viceversa, depending on the orientation of restriction ξ' between p_v and $p_{v'}$.

- Suppose $e = \{v, v'\} \in \mathcal{E}_{ndiv}$ is associated to Γ where $v \in \mathcal{V}_{div}$ and $v' \in \mathcal{V}_{ndiv}$. There is a well defined orientation if Γ is not a curve of singularities, given by the restriction of ξ to Γ. Otherwise, if $\Gamma \subset \mathrm{Sing}(\xi)$, we have two possibilities. If $S(\xi)$ is regular at p_v with $v \in \mathcal{V}_{div}$, there is not orientation on e. Recall that in this case Γ is a bidicritical curve and it behaves as a dicritical component. If $S(\xi)$ has a simple singularity at v, we will provide later other type of orientation.

Definition 1.36. *The partially oriented graph $G = G(\xi, \pi) = (\mathcal{V}, \mathcal{E})$ constructed above is the* graph of ξ associated to the reduction of singularities π*.*

We will define finer combinatorial objects associated to ξ and to the graph $G(\xi, \pi)$. They are the quadrants of the graph. The graph together with the collection of quadrants will provide the combinatorial information we will need in further sections.

Definition 1.37. *Let $G = G(\xi, \pi)$ be the graph of ξ associated to π. A* quadrant of G *is a triplet $Q = (v, e, e')$ such that*

- *e, e' are adjacent to v.*
- *If v is a vertex of degree 3 whose adjacent edges are $e_1, e_2 \in \mathcal{E}_{div}$ and $e_3 \in \mathcal{E}_{ndiv}$, then, the triplet (v, e_1, e_2) is not a quadrant.*

We denote $\mathcal{Q}$ the collection of quadrants of G.

It will be convenient to work with quadrants because orientation of the non-divisor edges is well defined on them. We explain the orientation of the quadrants now. Let $Q = (v, e, e')$ be a quadrant. We define an orientation of e and e' inside the quadrant as follows.

- If e or e' are oriented in G, we take the same orientation inside Q.
- If e is not oriented in G and it does not correspond to a bidicritical curve or to a dicritical component of the divisor, we keep e without orientation.
- Suppose $e = e_\Gamma \in \mathcal{E}_{ndiv}$ is not oriented in G and it corresponds to a non-bidicritical curve $\Gamma \subset \mathrm{Sing}(\widetilde{\xi})$. Notice that e' must be oriented since it must correspond to a non-dicritical component γ of E. We define an orientation in the quadrant as follows. Since E and Γ have normal crossings, take formal coordinates (x_1, x_2) such that Q corresponds to $(\mathbb{R}_{\geq 0})^2$, the curve $\gamma \subset E$ (associated to e') corresponds to $x_1 = 0$, $x_2 \geq 0$ and $\Gamma = \{x_2 = 0,\ x_1 \geq 0\}$. Then, take a saturation $\eta = x_1^{-s} x_2^{-r} \widetilde{\xi}_{p_v}$ of ξ_{p_v}. The orientation of e inside the quadrant is given by the orientation of $\eta|_\Gamma$ (attracting or repelling to v).

We present some few remarks on the orientation of the quadrants. First, the orientation of the edges in $\mathcal{E}_{div}$ is determined by the orientation of the graph G. Secondly, notice that at least one of the edges is always oriented, since on the one hand two dicritical components do not intersect (see Remark 1.35) and on the other hand there are not singular points in a dicritical component. Finally, we want to remark that the same edge e can belong to two different quadrants Q, Q'. When it is a divisor edge, the orientation is the same in both of them. When e is not a divisor edge, this property does not longer hold. If $e = e_\Gamma$ with Γ a curve of singularities transverse to the divisor, the equation of Γ can have different sign at each side of Γ, and this implies that it may not be possible to endow e with a meaningful orientation in G.

We can already define the types of quadrants appearing in the graph, concerning their orientations.

Definition 1.38. *Let $Q = (v, e, e')$ be an oriented quadrant of G.*

- *We say that Q is a* dicritical quadrant *if one edge is oriented inside Q and the other is not. Furthermore, we say that Q is an* attracting dicritical quadrant *if the oriented edge is oriented towards v and a* repelling dicritical quadrant *otherwise.*
- *We say that Q is a* node quadrant *if both edges are oriented inside Q, either towards v or outwards v. In the first case, we say that it is an* attracting node quadrant *and in the second that it is a* repelling node quadrant.
- *We say that Q is a* saddle quadrant *if both edges are oriented inside Q, one towards v and the other one outwards v.*

We show an example of the orientation of the graph and its quadrants in Figure 1.8.

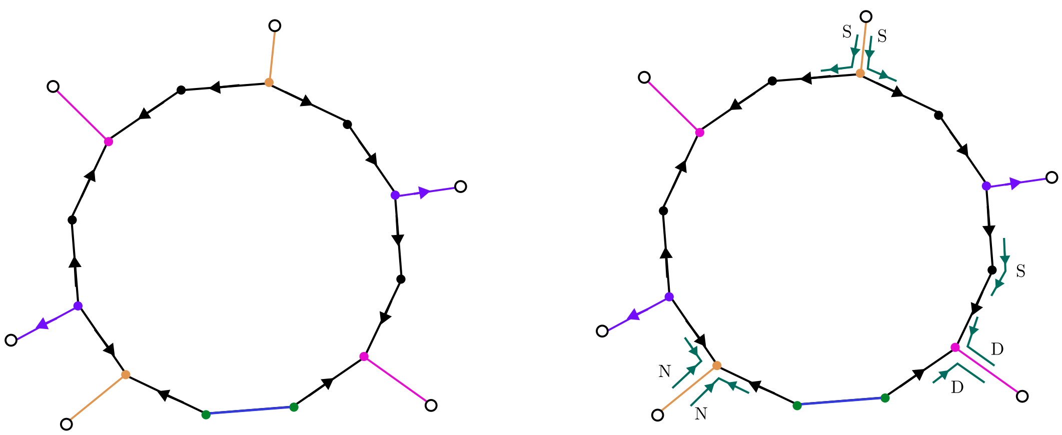

Figure 1.8: Orientation of the graph $G(\xi, \pi)$ given by the strict transform $\widetilde{\xi}'$ (left). Some example of quadrants of the graph where S denotes a saddle quadrant, N a node quadrant and D a dicritical quadrant (right).

We end the section by defining paths of quadrants.

Definition 1.39. *We say that a chain $(Q_1, \ldots, Q_s)$ with $s \geq 2$ is a* path of quadrants *if, for every $1 \leq i \leq s-1$, putting $Q_i = (v_i, e_i, e'_i)$, and $Q_{i+1} = (v_{i+1}, e_{i+1}, e'_{i+1})$, we have $\{e_i, e'_i\} \cap \{e_{i+1}, e'_{i+1}\} = \{e\} \subset \mathcal{E}_{div}$*

and e is oriented from v_i to v_{i+1}. A path of quadrants $(Q_1,\dots,Q_s)$ is maximal *if there is not other path of quadrants $(Q_{j_1},\dots,Q_{j_r})$ that contains $(Q_1,\dots,Q_s)$ as a subchain.*

Let $(Q_1,\dots,Q_s)$ be a path of quadrants, then the quadrants $Q_2,\dots,Q_{s-1}$ are necessarily saddle quadrants. Notice that a path of quadrants defines a path of edges of the subgraph G_{div} between the vertex v_1 and v_s of length $s-1$ in the classical sense. When Q_1 or Q_s are saddle quadrants (v,e,e') with $e,e'\in\mathcal{E}_{div}$, the path of quadrants can be extended.

1.5 Tangent to the identity diffeomorphisms

In this section, we will present the relation concerning tangent to the identity analytic diffeomorphisms and formal vector fields. Reduction of singularities of an associated vector field will help to study the dynamics of the former diffeomorphisms. We will also show how to lift tangent to the identity diffeomorphisms by the blowing-ups in the process of reduction of singularities. This section is based on [12], and it will be used in Chapter 2.

1.5.1 Infinitesimal generator of tangent to the identity diffeomorphisms

Let $\mathrm{Diff}_1(\mathbb{R}^n,0)$ denote the group of germs of analytic diffeomorphisms tangent to the identity. Recall that tangent to the identity diffeomorphisms fulfill that $DF(0)=Id$. Recall also that the *order* $\mathrm{ord}(F)$ of a diffeomorphism F is the minimum k such that $j_k(F-Id)\neq 0$. In this section, we associate a vector field to each element of $\mathrm{Diff}_1(\mathbb{R}^n,0)$. We denote by $\widehat{\mathfrak{X}}_2(\mathbb{R}^n,0)\subset\widehat{\mathfrak{X}}(\mathbb{R}^n,0)$ the submodule of formal vector fields at $0\in\mathbb{R}^n$ of order greater or equal to 2 and $\widehat{\mathrm{Diff}}_1(\mathbb{R}^n,0)$ the germs of formal diffeomorphisms tangent to the identity. We establish a bijection between $\widehat{\mathfrak{X}}_2(\mathbb{R}^n,0)$ and $\widehat{\mathrm{Diff}}_1(\mathbb{R}^n,0)$ via the exponential map. We recall that the exponential operator $\exp(t\xi)$: $\mathbb{R}[[x_1,\dots,x_n]]\to\mathbb{R}[[x_1,\dots,x_n,t]]$ of ξ is defined by

$$\exp(t\xi)(f)=\sum_{i=1}^{\infty}\frac{t^i}{i!}\xi^{(i)}(f). \tag{1.15}$$

where $\xi^{(0)}(f)=f$ and $\xi^{(i)}(f)=\xi(\xi^{(i-1)}(f))$ for $i\geq 1$. Notice that as $\nu(\xi)\geq 2$, then $\nu(\xi(f))\geq\nu(\xi)+\nu(f)-1\geq 1+\nu(f)$, and hence $\nu(\xi^{(i)}(f))\geq i+\nu(f)$. This implies that $\mathrm{Im}(\exp t\xi)\subset\mathbb{R}[t][[x_1,\cdots,x_n]]$, and substitution of any $t\in\mathbb{R}$ is well-defined. Hence, we will define the exponential map between formal vector fields of order 2 and tangent to the identity diffeomorphisms.

Definition 1.40. *The Exponential map* $\mathrm{Exp}:\widehat{\mathfrak{X}}_2(\mathbb{R}^n,0)\to\widehat{\mathit{Diff}}_1(\mathbb{R}^n,0)$ *is defined, by*

$$\mathrm{Exp}(\xi):(x_1,\cdots,x_n)\mapsto(\exp(\xi)(x_1),\dots,\exp(\xi)(x_n)).$$

This map is bijective. The fact that $\mathrm{Exp}(\xi)$ is indeed tangent to the identity is deduced form the fact that $\nu(\xi^{(i)}(x_j)) \geq i+1$ for every $1 \leq j \leq n$ and $i \geq 0$. Now, we prove that this map admits a global inverse, which implies that it is bijective. Let $F \in \widehat{\mathrm{Diff}}_1(\mathbb{R}^n, 0)$ be a tangent to the identity diffeomorphism given by $F = (F_1, \ldots, F_n)$ and $F_i(x_1, \ldots, x_n) = x_i + \sum_{j=2}^{\infty} f_{i,j}(x_1, \ldots, x_n)$ for $i = 1, \ldots, n$ where $f_{i,j}(x_1, \ldots, x_n)$ is an homogeneous polynomial of degree j. We write $x_i \circ F = x_i + \sum_{j=2}^{\infty} f_{i,j}(x_1, \ldots, x_n)$. We will see that there is a formal vector field $\xi = \sum_{j=2}^{\infty} \sum_{i=1,\ldots,n} a_{i,j}(x_1, \ldots, x_n) \frac{\partial}{\partial x_i}$, where $a_{i,j}(x_1, \ldots, x_n)$ is an homogeneous polynomial of degree j, such that $F = \mathrm{Exp}(\xi)$. It suffices to solve the triangular system given by

$$f_{i,j} = a_{i,j} + H_j\left(\sum_{r=2}^{j-1} (j_{j-1}(\xi))^{(r)}(x_i)\right),$$

where $H_j(g)$ denotes the homogeneous term of order j of g and $j_k(\xi)$ is the k–jet of ξ, as it can be seen in [12]. We denote by $\mathrm{Log} : \widehat{\mathrm{Diff}}_1(\mathbb{R}^n, 0) \to \widehat{\mathfrak{X}}_2(\mathbb{R}^n, 0)$ to the inverse of Exp.

Definition 1.41 (Infinitesimal generator)**.** *Given a formal diffeomorphism $F \in \widehat{Diff}_1(\mathbb{R}^n, 0)$, we say that $\xi = Log(F)$ is the* infinitesimal generator of *F.*

Notice that this bijection exists on the formal level due to formal convergence of Exp and Log. It is possible that the infinitesimal generator of an analytic diffeomorphism is only a formal vector field (there are results that prove that they belong to some Gevrey class that depends on the order of the diffeomorphism, see [13]). The other direction behaves better, in the sense that the exponential map of an analytic vector field is its time one flow, and hence, it is a germ of analytic diffeomorphism.

1.5.2 Reduction of singularities applied to tangent to the identity two-dimensional diffeomorphisms

In this section, we will restrict ourselves to the two-dimensional case, that is, $F \in \mathrm{Diff}_1(\mathbb{R}^2, 0)$ because we want to apply the reduction of singularities result. Let $\mathrm{Fix}(F)$ be defined by the ideal generated by $x \circ F - x$ and $y \circ F - y$ and let $\mathrm{Sing}(\xi)$ be the ideal generated by $\xi(x)$ and $\xi(y)$. The following result is also very classical, we found it in [12].

Proposition 1.42. *Given a formal diffeomorphism F and its infinitesimal generator ξ, we have $Fix(F) = Sing(\xi)$.*

As a consequence of the previous result, we have the following one, which means that even if the infinitesimal generator is only formal, at least its set of singularities is analytic.

Corollary 1.43. *Let F be an analytic diffeomorphism tangent to the identity and ξ its infinitesimal generator. Then $Fix(F)$ is analytic and so is $Sing(\xi)$.*

Before in this chapter, we proved in Theorem 1.33 that formal non-saturated vector fields admit reduction of singularities. Applying that result, we obtain a sequence of blowing-ups $\pi = \pi_1 \circ \cdots \circ \pi_s$ such that each blowing-up $\pi_i : (M_i, E_i) \to (M_{i-1}, E_{i-1})$ is centered at a single point in the divisor E_{i-1} of the chain $\pi_1 \circ \cdots \circ \pi_{i-1}$ of length $i-1$, and such that there is a finite number of simple singularities of the strict transform $\widetilde{\xi}'$ of ξ under π in E_s. The diffeomorphism can be lifted to $(M, E) = (M_s, E_s)$, using repeatedly the following proposition from [12].

Proposition 1.44. *Let $F \in Diff_1(\mathbb{R}^2, 0)$ be a germ of diffeomorphism tangent to the identity and ξ its infinitesimal generator. Let $\pi_1 : (M, E) \to (\mathbb{R}^2, 0)$ be the blowing-up centered at 0. Then,*

1. *There exists a unique germ $\widetilde{F}$ of diffeomorphism along E so that $\pi \circ \widetilde{F} = F \circ \pi$.*
2. *The infinitesimal generator of $\widetilde{F}$ at E is the total transform $\widetilde{\xi}$ at E.*
3. *For each $p \in E$, we have $ord_p(\widetilde{F}) = \nu_p(\widetilde{\xi})$ and if $p \in Sing(\widetilde{\xi}, E)$, then $ord_p(\widetilde{F}) \geq ord_0(F)$*

The total transform $\widetilde{\xi}$ in this proposition must be understood as the collection of germs of local transforms at each point of the divisor E. It is instructive presenting the proof from [12] in order to see the expression of the lifted diffeomorphism by the blowing-ups.

Proof. We prove this result using concrete charts. We start with the first statement. Suppose $\pi_1 : (M, E_1) \to (\mathbb{R}^2, 0)$ is the real blowing-up centered at 0 and let (x, y) be coordinates centered at 0. Suppose that F is given in these coordinates by

$$F(x, y) = (x + a(x, y), y + b(x, y)),$$

where a, b are series that written as a sum of homogeneous polynomials are given by $a(x, y) = \sum_{j=k}^{\infty} a_j(x, y)$ and $b(x, y) = \sum_{j=k}^{\infty} b_j(x, y)$ and $k \geq 2$ is the order of F.

We define the diffeomorphism $\widetilde{F}$ in each of the charts in the positive and negative directions of x and y as follows. Let U_1^+ be the chart in the positive direction of x, where the blowing-up is given by $\pi(x', y') = (x', x'y')$ and let U_2^+, U_1^-, U_2^- be the other charts defined as in section 1.2. We show the lifting only in U_1^+ and the rest is done similarly. Notice that $\widetilde{F}$ must fulfill $\pi \circ \widetilde{F} = F \circ \pi$, that is

$$(x' \circ \widetilde{F}, (x' \circ \widetilde{F})(y' \circ \widetilde{F})) = (x \circ F(x', x'y'), y \circ F(x', x'y').$$

Then, we find concrete expressions of $\widetilde{F}(x', y') = (x' + \tilde{a}(x', y'), y' + b(x', y'))$ as follows.

$$x' + \tilde{a}(x', y') = x' + a(x', x'y'),$$

$$y' + \tilde{b}(x', y') = (y' + \frac{1}{x'} b(x', x'y')) \cdot (1 + \frac{1}{x'} a(x', x'y'))^{-1}.$$

Notice that $\widetilde{F}$ fixes all the points of the divisor, which in this chart is given by $x' = 0$.

Now, we prove the second, that is, the infinitesimal generator of $\widetilde{F}$ at any $p \in E$ is the formal vector field $\widetilde{\xi}_p$. We again work only at U_1^+, since the procedure is the same in the four charts. By making a linear change of coordinates, we can assume that $p = (0,0)$. Then,

$$\begin{aligned}\widetilde{F}(x',y') =& \left(\exp \xi(x), \frac{\exp \xi(y)}{\exp \xi(x)}\right) \circ \pi = \left(\exp \widetilde{\xi}_p(x'), \frac{\exp \widetilde{\xi}_p(x'y')}{\exp \widetilde{\xi}_p(x')}\right) = \left(\exp \widetilde{\xi}_p(x'), \frac{\exp \widetilde{\xi}_p(x')\exp \widetilde{\xi}_p(y')}{\exp \widetilde{\xi}_p(x')}\right)\\ =& (\exp \widetilde{\xi}_p(x), \exp \widetilde{\xi}_p(y)),\end{aligned}$$

and thus $\widetilde{\xi}_p$ is the infinitesimal generator of $\widetilde{F}_p$.

The first part of the third item, $\mathrm{ord}_p(\widetilde{F}) = \nu_p(\widetilde{\xi})$, is a consequence of the previous work. For the second part consider that $p \in \mathrm{Sing}(\widetilde{\xi}', E)$ is in the chart U_1^+ and, under a linear change of coordinates, suppose $p = (0,0)$, as before. Suppose that $\xi = c(x,y)\frac{\partial}{\partial x} + d(x,y)\frac{\partial}{\partial y}$, with $\nu_0(\xi) = k$ then

$$\widetilde{\xi} = c(x',x'y')\frac{\partial}{\partial x'} + \frac{1}{x'}(d(x',x'y') - y'c(x',x'y'))\frac{\partial}{\partial y'}$$

We have in general that $(x')^k$ divides both $c(x',x'y')$ and $d(x',x'y')$, therefore $(x')^{k-1}$ divides $\widetilde{\xi}$. Then, if p is a singularity of $\widetilde{\xi}'$, we have that $\nu_p(\widetilde{\xi}') \geq 1$ since some power of x' or y' divide $\widetilde{\xi}'$. As a result, $\mathrm{ord}_p(\widetilde{F}) = \nu_p(\widetilde{\xi}) = k - 1 + \nu_p(\widetilde{\xi}') \geq k - 1 + 1 = k = \mathrm{ord}_0(F)$. □

A key consequence of the previous result is that, at any $p \in \mathrm{Sing}(\widetilde{\xi}', E)$, the infinitesimal generator $\widetilde{\xi}$ has order equal to or greater than 2 and the diffeomorphism is tangent to the identity. The blowing-ups can be iterated by choosing points in $\widetilde{\mathrm{Sing}}(\widetilde{\xi}', E)$ as centers. The obtainment of a infinitesimal generator after a sequence of blowing-ups and the corresponding lifted diffeomorphism is hence well-defined.

1.5.3 Strict and regular fixed points. Center-focus diffeomorphisms

As we have seen, after a reduction of singularities all the points in the divisor are fixed. We will distinguish two types of fixed points: regular and strict.

Definition 1.45. *Let $\widetilde{\xi}$ be the total transform of the infinitesimal generator of F and let $\widetilde{F}$ be the lifting of F by π. We say that $p \in E$ is a* strict fixed *point of F if it is an adapted simple singularity of $\widetilde{\xi}_p$ or a corner between a dicritical and a non dicritical component of E. We denote $\mathfrak{S}(F,\pi)$ the set of strict fixed points. Otherwise, we say that p is a* regular point.

We want to remark that the set of strict fixed points is exactly provided by $\mathcal{V}_{div} \subset \mathcal{V}(G(\xi,\pi))$. Among the strict fixed points we can make further distinctions.

Definition 1.46. *Let $\widetilde{\xi} \in \widehat{\mathfrak{X}}(M,E)$, $E = E_1 \cup \cdots \cup E_s$ and $p \in \mathfrak{S}(F,\pi)$ a strict fixed point. If $p \in E_i \cap E_{i+1}$ for some $1 \leq i \leq s$, we say that p is a* corner strict fixed point. *Otherwise we say that p is a* non-corner strict fixed point.

Among the regular points, we also make a further distinction.

Definition 1.47. *Let $\widetilde{\xi}$ be the total transform of the infinitesimal generator ξ of F, $E = E_1 \cup \cdots \cup E_s$ and $p \notin \mathfrak{S}(F,\pi)$. Suppose $p \in E_i$ for some $1 \leq i \leq s$. If E_i is dicritical for $\widetilde{\xi}$, we say that p is a* dicritical regular fixed point. *Otherwise we say that p is a* non-dicritical regular fixed point.

Definition of non center-focus diffeomorphisms

We give the definition of a center-focus diffeomorphism in terms of its reduction of singularities.

Definition 1.48. *Let $F \in Diff_1(\mathbb{R}^2, 0)$ be a germ of analytic diffeomorphism, ξ its infinitesimal generator and $\pi : (M,E) \to (\mathbb{R}^2, 0)$ a reduction of singularities of ξ. We say that F is* center-focus *if*

- *E does not have dicritical components for ξ.*
- *All the singularities of $\widetilde{\xi} = \pi^*(\xi)$ are corners of E of saddle type.*

We say that F is non center-focus *if it is not center-focus.*

Notice that this definition is parallel to the definition of center-focus vector fields (presented at the end of Section 1.4.2). We will see also that a non center-focus diffeomorphism has at least one invariant curve transverse to the divisor.

1.6 Technical results

1.6.1 Jet equalities

In this section, we will provide technical results in order to understand better the total transforms of vector fields in the usual sets of coordinates. Even if it is quite standard to work with jets of vector fields, we did not find explicitly these results in the literature.

Proposition 1.49. *Let $\pi = \pi_1 \circ \cdots \circ \pi_s$ be a sequence of admissible blowing-ups for $\xi \in \widehat{\mathfrak{X}}(\mathbb{R}^n, 0)$ centered at points. Then, the coefficients of the vector fields $\widetilde{\xi}^{(I)}$ and $(\widetilde{\xi}^{(I)})'$ are transversely formal to $\pi^{-1}(0)$ (formal in the variables transverse to $\pi^{-1}(0)$ and polynomial in the rest) for any of the usual charts $(U_I, (x_1^{(I)}, \ldots, x_n^{(I)}))$.*

Before proving it, we show the following lemma.

Lemma 1.50. *Let $\pi : \mathbb{R}^n \to \mathbb{R}^n$ be a quadratic morphism given by $\pi(x_1', \cdots, x_n') = (x_1', \ldots, x_{n-m}', x_j' x_{n-m+1}', \ldots x_j', \ldots, x_j' x_n')$. It induces an $\mathbb{R}$−algebra homomorphism $\pi^* : \mathbb{R}[[x_1, \ldots, x_n]] \to \mathbb{R}[[x_1', \ldots, x_n']]$ defined by $\pi^*(a) = a \circ \pi$ such that*

$$Im(\pi^*) \subset R[x_{n-m+1}', \ldots x_{j-1}', x_{j+1}', \ldots x_n'][[x_1', \ldots, x_{n-m}', x_j']].$$

Even more, we have

$$\pi^*(\mathbb{R}[x_{n-m+1},\dots,x_{n-1}][[x_1,\dots,x_{n-m},x_n]])\subset$$
$$\mathbb{R}[x'_{n-m+1},\dots x'_{j-1},x'_{j+1},\dots x'_n][[x'_1,\dots,x'_{n-m},x'_j]]\cap\mathbb{R}[x'_{n-m+1},\dots x'_{n-1}][[x'_1,\dots,x'_{n-m},x'_n]]$$

Proof. It suffices to take an element $a\in R[[x_1,\dots,x_n]]$ and study its image $\pi^*(a)=a\circ\pi$. Given

$$a(x_1,\dots,x_n)=\sum_{\alpha\in\mathbb{N}^n_{\geq 0}}a_\alpha\mathbf{x}^\alpha,$$

we obtain

$$\pi^*(a)=\sum_{\alpha\in\mathbb{N}^n_{\geq 0}}a_\alpha(x'_1)^{\alpha_1}\cdots(x'_{n-m})^{\alpha_m}(x'_{n-m+1})^{\alpha_{n-m+1}}\cdots(x'_{j-1})^{\alpha_{j-1}}(x'_{j+1})^{\alpha_{j+1}}(x'_n)^{\alpha_n}(x'_j)^{\alpha_{n-m+1}+\cdots+\alpha_n}.$$

We can reorder the terms in the above expression so that

$$\pi^*(a)=\sum_{\beta\in\mathbb{N}^{n-m+1}_{\geq 0}}b_\beta(x'_{n-m+1},\dots,x'_{j-1},x'_{j+1},x'_n)(x'_1)^{\beta_1}\cdots(x'_{n-m})^{\beta_{n-m}}(x'_j)^{\beta_{n-m+1}}$$

where b_β is a polynomial defined by

$$b_\beta=\sum_{\substack{\alpha_{n-m+1}+\cdots+\alpha_n=\beta_{n-m+1}\\ \alpha_1=\beta_1,\dots,\alpha_{n-m}=\beta_{n-m}}}a_\alpha(x'_{n-m+1})^{\alpha_{n-m+1}}\cdots(x'_{j-1})^{\alpha_{j-1}}(x'_{j+1})^{\alpha_{j-1}}\cdots(x'_n)^{\alpha_n},$$

which has degree equal or lower than β_{n-m+1}, and hence $\pi^*(a)\in\mathbb{R}[x'_{n-m+1},\dots x'_{j-1},x'_{j+1},\dots x'_n]$ $[[x'_1,\dots,x'_{n-m},x'_j]]$, as we wanted to prove.

Now, we prove the second part. Suppose that $j\neq n$. In that case, we can proceed as in the first part, finding $\pi^*(\mathbb{R}[x_{n-m+1},\dots,x_{n-1}][[x_1,\dots,x_{n-m},x_n]])\subset\mathbb{R}[x'_{n-m+1},\dots x'_{n-1}][[x'_1,\dots,x'_{n-m},x'_n]]$. Let $a\in\mathbb{R}[x_{n-m+1},\dots,x_{n-1}][[x_1,\dots,x_{n-m},x_n]]$ and consider, for shortening the notation, $\mathbf{y}=(x_{n-m+1},\dots,x_{n-1})$ and $\mathbf{z}=(x_1,\dots,x_{n-m},x_n)$. The series a is written as

$$a(x_1,\dots,x_n)=\sum_{\alpha\in\mathbb{N}^{n-m+1}_{\geq 0}}a_\alpha(\mathbf{y})\mathbf{z}^\alpha.$$

Applying π^*, we obtain

$$\pi^*(a)=\sum_{\alpha\in\mathbb{N}^n_{\geq 0}}a_\alpha(x'_{n-m+1}x'_j,\dots,x'_j,\dots,x'_{n-1})(x'_1)^{\alpha_1}\cdots(x'_{n-m})^{\alpha_{n-m}}(x'_n)^{\alpha_n}(x'_j)^{\alpha_n}.$$

Reordering the terms, we can check that

$$\pi^*(a) = \sum_{\beta \in \mathbb{N}_{\geq 0}^{n-m+1}} b_\beta(x'_{n-m+1}, \dots, x'_{j-1}, x'_{j+1}, x'_n)(x'_1)^{\beta_1} \cdots (x'_{n-m})^{\beta_{n-m}} (x'_j)^{\beta_{n-m+1}}$$

where b_β is a polynomial defined as follows. Writing the polynomials a_α in homogeneous terms $a_\alpha = \sum_{k_\alpha} a_{\alpha,k}(x_{n-m+1}, \dots, x_{n-1})$ we have $\pi^*(a_\alpha) = \sum_{k_\alpha} a_{\alpha,k_\alpha}(x'_{n-m+1}, \dots, x'_{j-1}, 1, x'_{j+1}, \dots, x'_{n-1})(x'_j)^{k_\alpha}$. Then,

$$b_\beta = \sum_{\substack{k_\alpha + \alpha_n = \beta_j \\ \alpha_1 = \beta_1, \dots, \alpha_{n-m} = \beta_{n-m}}} a_{\alpha,k}(x'_{n-m+1}, \dots, x'_{j-1}, 1, x'_{j+1}, \dots, x'_{n-1})(x'_n)^{\alpha_n}.$$

With a similar reordering, we can write as well

$$\pi^*(a) = \sum_{\gamma \in \mathbb{N}_{\geq 0}^{n-m+1}} c_\gamma(x'_{n-m+1}, \dots, x'_{j-1}, x'_{j+1}, x'_n)(x'_1)^{\gamma_1} \cdots (x'_{n-m})^{\gamma_{n-m}} (x'_n)^{\gamma_n}$$

where $c_\gamma = a_\alpha(x'_{n-m+1}x'_j, \dots, x'_j, \dots, x'_j x'_{n-1}) x_j^{\gamma_n}$. In conclusion, $\pi^*(a) \in \mathbb{R}[x'_{n-m+1}, \dots x'_{j-1}, x'_{j+1}, \dots x'_n][[x'_1, \dots, x'_{n-m}, x'_j]] \cap \mathbb{R}[x'_{n-m+1}, \dots x'_{j-1}, x'_{j+1}, \dots x'_n][[x'_1, \dots, x'_{n-m}, x'_n]]$, as we wanted to prove. ☐

Suppose that π_1 is a blowing-up centered at N, and take coordinates $(x_1, \dots, x_n)$ so that N is given by the ideal $(x_{n-m+1}, \cdots, x_n)$. Suppose that we study the chart of the blowing-up with coordinates $(x'_1, \dots, x'_n)$ such that E is given by $x'_n = 0$ (the procedure is similar in other charts). The expression of the vector field before the blowing-up is $\xi = \sum_{i=1}^n a_i(x_1, \dots, x_n) \frac{\partial}{\partial x_i}$ with $a_i \in \mathbb{R}[[x_1, \dots, x_n]]$. We recall the expression of the vector field after the blowing-up (1.5) in the chart U_n^+,

$$\begin{aligned} \xi^{(n)} = & \sum_{i \in \{1, \cdots, m\}, i = n} a_i(x'_1, \dots, x'_{n-m}, x'_n x'_{n-m+1}, \dots, x'_n x'_{n-1}, x'_n) \frac{\partial}{\partial x'_i} + \\ & \sum_{i \in \{n-m+1, \dots, n-1\}, i \neq j} \frac{1}{x'_n} (a_i(x'_1, \dots, x'_{n-m}, x'_n x'_{n-m+1}, \dots, x'_n x'_{n-1}, x'_n) \\ & - x'_i a_n(x'_1, \dots, x'_{n-m}, x'_n x'_{n-m+1}, \dots, x'_n x'_{n-1}, x'_n)) \frac{\partial}{\partial x'_i} \\ = & \sum_{i=1}^n \widetilde{a}_i(x'_1, \dots, x'_n) \frac{\partial}{\partial x'_i}, \end{aligned} \tag{1.16}$$

with $\widetilde{a}_i \in \mathbb{R}[x'_{n-m+1}, \dots x'_{n-1}][[x'_1, \dots, x'_{n-m}, x'_n]]$ by Lemma 1.50.

Notice that indeed the blowing-ups in Proposition 1.49 are centered at a single point. We can consider the expression $\xi^{(n)}$ for the codimension of the center of the being $m = n$.

Proof of Proposition 1.49. The proof for sequences of length 1 follows from the computation in (1.16).

Suppose by induction that $p \in E_{i-1}$ is the center of the blowing-up π_i with adapted coordinates $(x_1,\cdots,x_n)$ such that the divisor is given by equation $x_r\cdots x_n = 0$ for $1 \le r \le n$. Suppose also that the total transform of ξ after the first $i-1$ blowing-ups at p in the chart $(U,(x_1,\cdots,x_n))$ with coefficients in $\bigcap_{k=r}^{n}\mathbb{R}[x_1,\ldots x_{k-1},x_{k+1},\ldots x_n][[x_k]]$. Let $(U_j,(x'_1,\cdots,x'_n))$ be some chart of the blowing-up. By (1.16) and Lemma 1.50, we have that the coefficients of $\pi^*(\xi)$ in U_j belong to $\bigcap_{k=r,\ldots,n,\ k=j}\mathbb{R}[x_1,\ldots x_{k-1},x_{k+1},\ldots x_n][[x_k]]$.

□

We also prove a result that shows the jet dependence of a blown up vector field in terms of its jet before the blowing-up.

Proposition 1.51. *Let $\xi \in \widehat{\mathfrak{X}}(\mathbb{R}^n,0)$ be a formal vector field and $\pi:(M,E)\to(\mathbb{R}^n,N)$ be a blowing-up centered at $N\subset\mathbb{R}^n$. Suppose N is given by $x_{n-m}=0,\ldots,x_n=0$. Consider one of the usual charts $C_j=(U_j,(x'_1,\ldots,x'_n))$ and recall that the exceptional divisor of π in the chart C_j is given by $x'_j=0$. Then, the following jet equalities hold:*

$$j_k^{x'_j}(\xi^{(j)}) = j_k^{x_j}(\pi|_{C_j}^*(j_{k+1}^{x_{n-m},\ldots,x_n}(\xi))).$$

$$j_k^{x'_i}(\xi^{(j)}) = j_k^{x'_i}(\pi|_{C_j}^*(j_k^{x_i}(\xi))). \quad \textit{for } i\neq j$$

In the proof of Proposition 1.51 we use the following two lemmas.

Lemma 1.52. *Let $\pi:\mathbb{R}^n\to\mathbb{R}^n$ be a quadratic morphism given by $\pi(x'_1,\cdots,x'_n)=(x'_1,\ldots,x'_{n-m},x'_jx'_{n-m+1},\ldots x'_j,\ldots,x'_jx'_n)$. It induces an $\mathbb{R}$–algebra homomorphism $\pi^*:\mathbb{R}[[x_1,\ldots,x_n]]\to\mathbb{R}[[x'_1,\ldots,x'_n]]$ defined by $\pi^*(a)=a\circ\pi$ such that $j_k^{x'_j}(\pi^*(a))=\pi^*(j_k^{x_{n-m+1},\ldots,x_n}(a))$, $a\in\mathbb{R}[[x_1,\ldots,x_n]]$*

Proof. For the proof of this proposition, we write a in homogeneous components in $\mathbb{R}[[x_1,\ldots,x_{n-m}]]$ $[[x_{n-m+1},\ldots,x_n]]$, that is,

$$a=\sum_{i=0}^{\infty}a_i(x_1,\ldots,x_n)=\sum_{i=0}^{\infty}\sum_{i=i_{n-m+1}+\cdots+i_n}a_{i_{n-m+1},\cdots,i_n}(x_1,\ldots,x_{n-m})x_{n-m+1}^{i_{n-m+1}}\cdots x_n^{i_n},$$

where each $a_i\in\mathbb{R}[[x_1,\ldots,x_{n-m}]][x_{n-m+1},\ldots,x_n]_i$ and each $a_{i_{n-m+1},\cdots,i_n}(x_1,\ldots,x_{n-m})\in\mathbb{R}[[x_1,\ldots,x_{n-m}]]$.

Renaming $(x_1,\ldots,x_n)=(x'_1,\ldots,x'_n)$, we find the following expression,

$$\pi^*(a)=\sum_{i=0}^{\infty}\sum_{i=i_{n-m+1}+\cdots+i_n}a_{i_{n-m+1},\cdots,i_n}(x_1,\ldots,x_{n-m})x_{n-m+1}^{i_{n-m+1}}\cdots x_{j-1}^{i_{j-1}}x_j^{i_{n-m+1}+\cdots+i_n}x_{j+1}^{i_{j+1}}\cdots x_n^{i_n},$$

which implies the searched jet equality. In particular, the coefficient of $x_{n-m+1}^{i_{n-m+1}}\cdots x_{j-1}^{i_{j-1}}x_j^{i_j}x_{j+1}^{i_{j+1}}\cdots x_n^{i_n}$ is a polynomial in $\mathbb{R}[[x_1,\ldots,x_{n-m}]][x_{n-m+1},\ldots,x_{j-1},x_{j+1},\ldots,x_n]$ of degree equal or lower than i_j. □

Remark 1.53. When π is the expression of a point blowing-up in the chart U_j, we have the following.

$$j_k^{x'_j}(\pi^*(a)) = \pi^*(j_k(a))$$

Lemma 1.54. *Let $\pi : \mathbb{R}^n \to \mathbb{R}^n$ be a quadratic morphism given by $\pi(x'_1, \cdots, x'_n) = (x'_1, \ldots, x'_{n-m}, x'_j x'_{n-m+1}, \ldots x'_j, \ldots, x'_j x'_n)$. It induces an $\mathbb{R}$–algebra homomorphism $\pi^* : \mathbb{R}[[x_1, \ldots, x_n]] \to \mathbb{R}[[x'_1, \ldots, x'_n]]$ defined by $\pi^*(a) = a \circ \pi$ such that $j_k^{x'_i}(\pi^*(a)) = \pi^*(j_k^{x_i}(a))$, for any $i \neq j$.*

Proof. We write a as an element of $\mathbb{R}[[x_1, \ldots, x_n]][[x_i]]$,

$$a = \sum_{i=0}^{\infty} a_i(x_1, \ldots, x_n) x_i^i.$$

First suppose that $i \geq n-m+1$. Renaming $(x_1, \ldots, x_n) = (x'_1, \ldots, x'_n)$, we find the following expression

$$\pi^*(a) = \sum_{i=0}^{\infty} a_i(x_1, \ldots, x_{n-m}, x_{n-m+1}x_j, \ldots, x_n x_j) x_i^i x_j^i.$$

The jet equality then follows from this expression.

Now, suppose that $i \leq n-m$, then

$$\pi^*(a) = \sum_{i=0}^{\infty} a_i(x_1, \ldots, x_{n-m}, x_{n-m+1}x_j, \ldots, x_n x_j) x_i^i,$$

and the searched jet equality also follows from this expression. □

Proof of Proposition 1.51. It is enough to consider the expression of the coefficients $\widetilde{a_i}$ of the vector field (1.16) after a blowing-up and the previous Lemma 1.52. Notice that studying a $(k+1)$–jet in $x_1, \ldots, x_{n-m}, x_n$ is necessary since some coefficients $\tilde{a}_i$ are obtained from the coefficients a_i of ξ after dividing by x'_n. Observing (1.16), this means that the terms of degree r of $\xi^{(i)}$ in x'_n depend on terms of degree r and $r+1$ in $x'_{n-m+1}, \ldots, x'_n$, and hence the first jet equality follows.

For the second jet equality, it is enough to consider the expression of the vector field $\xi^{(j)}$ and use Lemma 1.54. □

Chapter 2

Sectorial decomposition of germs of diffeomorphisms

This chapter is devoted to the first problem of the thesis: the sectorial decomposition of germs of real analytic tangent to the identity plane diffeomorphisms. The aim of this chapter is to prove Theorem 2.6 (Theorem A in the Introduction). We start the chapter by giving the main definitions and statements of the results. Then, we fix a germ $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ of diffeomorphism and consider its infinitesimal generator ξ (cf. Definition 1.41). We apply Theorem 1.33 to the infinitesimal generator, obtaining a sequence of blowing-ups, and we study the lifting of the diffeomorphism by this sequence of blowing-ups. Then, we construct the sectors, concluding the existence of the sectorial decomposition. We finally refine this construction in order to obtain a sectorial decomposition having better topological and geometrical properties.

2.1 Formulation of the main results

In this section, we provide the main definitions and we state the main results.

2.1.1 Orbits and saturation

In this section, we introduce some technical definitions of sets that we need throughout this chapter, and we will also indicate some inclusions among these sets. In the whole section, denote by the same letter $F : W \to W'$ a representative of F, so that W is an open neighborhood of 0, $F|_W$ injective and $F(W) = W'$ (so W' is also an open neighborhood of 0). If $n \in \mathbb{N}$, denote by F^n the composition of F with itself n times defined on a maximal subdomain of W, denote also $F^{-n} = (F^{-1})^{|n|}$ when $-n \in \mathbb{Z}_{\leq 0}$, and $F^0 = Id$.

Definition 2.1. *Let $V \subset W$ be any set and let $p \in V$.*

- *The* positive orbit of p in V *is*

$$\mathrm{Orb}_V^+(p) = \{q \in V : \exists \ell_q \in \mathbb{Z}_{\geq 0} \text{ such that } F^n(p) \in V \text{ for } n \in \{0,1,\dots,\ell_q\} \text{ and } F^{\ell_q}(p) = q\}.$$

  If there exists $m \in \mathbb{N}$ such that $F^j(p) \in V$ for $j < m$ but $F^m(p) \notin V$, we say that the positive orbit of p in V escapes V.

- *The* negative orbit of p in V *is*

$$\mathrm{Orb}_V^-(p) = \{q \in V : \exists -\ell_q \in \mathbb{Z}_{\leq 0} \text{ such that } F^n(p) \in V \text{ for } n \in \{0,-1,\dots,-\ell_q\} \text{ and } F^{-\ell_q}(p) = q\}.$$

  If there exists $m \in \mathbb{N}$ such that $F^{-j}(p) \in V$ for $j < m$ but $F^{-m}(p) \notin V$, we say that the negative orbit of p in V escapes V.

- *The* orbit of p in V *is*

$$\mathrm{Orb}_V(p) = \mathrm{Orb}_V^+(p) \cup \mathrm{Orb}_V^-(p).$$

- *The* ω-limit of the point p with respect to V *is*

$$\omega_V(p) = \bigcap_{n \geq 0} \overline{\mathrm{Orb}_V^+(F^n(p))},$$

  where the closure is taken inside W.

- *The* α-limit set of the point p with respect to V *is*

$$\alpha_V(p) = \bigcap_{n \leq 0} \overline{\mathrm{Orb}_V^-(F^n(p))},$$

where the closure is taken inside W.

- *A point $p \in V$ is $V-$*periodic *if* $\mathrm{Orb}_V(p)$ *is finite and* $\omega_V(p) = \alpha_V(p) = \mathrm{Orb}_V(p)$.

Definition 2.2. *Let A, B be subsets of W with $A \subset B \subset W$.*

- *The* positive saturation of A in B *is*

$$\mathrm{Sat}^+_B(A) = \bigcup_{p \in A} \mathrm{Orb}^+_B(p).$$

- *The* negative saturation of A in B *is*

$$\mathrm{Sat}^-_B(A) = \bigcup_{p \in A} \mathrm{Orb}^-_B(p).$$

- *The* saturation of A in B *is*

$$\mathrm{Sat}_B(A) = \bigcup_{p \in A} \mathrm{Orb}_B(p).$$

We say that A is saturated *in B if* $\mathrm{Sat}_B(A) = A$.

Definition 2.3. *Let A, B be subsets of W with $A \subset B \subset W$. We say that A is a* fundamental domain in B *if* $\mathrm{Orb}_B(p) \cap A = \{p\}$ *for every $p \in A$.*

We obtain directly from the definitions:

- Suppose that $B' \subset B$ and $A \subset B' \subset B$. We have

$$\mathrm{Sat}_{B'}(A) \subset \mathrm{Sat}_B(A).$$

- Suppose $A' \subset A \subset B$. Then

$$\mathrm{Sat}_B(A') \subset \mathrm{Sat}_B(A).$$

- If $A', A \subset B$ satisfy that for every $p \in A'$ there is some $\ell_p \in \mathbb{Z}$ such that $F^{\ell_p}(p) \in A$, then $A' \subset \mathrm{Sat}_B(A)$.
- If $A' \subset \mathrm{Sat}_B(A)$, we have $\mathrm{Sat}_B(A') \subset \mathrm{Sat}_B(A)$ and equality holds if and only if for every $p \in A$ we have $\mathrm{Orb}_B(p) \cap A' \neq \emptyset$.

We end presenting three types of behaviors of subsets.

Definition 2.4. *Let A be a subset of W.*

- *A is* positively invariant *if $F(A) \subset A$. In other words, the positive iterates of every p remain forever in A.*

- *A is* negatively invariant *if $F^{-1}(A) \subset A$. In other words, the negative iterates of every p remain forever in A.*

- *A is an* attracting parabolic set at *$q_A \in \bar{A}$ if A is positively invariant and for every $p \in A$, $\alpha_A(p) = \{q_A\}$ and $\omega_A(p) = \emptyset$. A is a* repelling parabolic set at *$q_A \in \bar{A}$ if A is negatively invariant and for every $p \in A$, $\omega_A(p) = \{q_A\}$ and $\alpha_A(p) = \emptyset$. The point q_A is named the* attractor *or* repeller *of the parabolic set.*

- *A is an* elliptic set at *$q_A \in \bar{A}$ if for every $p \in A$, $\alpha_A(p) = \{q_A\}$ and $\omega_A(p) = \{q_A\}$. The point q_A is the attractor and repellor of the elliptic set.*

- *A is a* hyperbolic set, *if for every $p \in A$,* $\mathrm{Orb}_A(p)$ *is finite and p is not A–periodic, in particular, $\omega_A(p) = \emptyset$ and $\alpha_A(p) = \emptyset$.*

Now we define parabolic curves of diffeomorphisms.

Definition 2.5. *An* attracting analytic parabolic curve γ of F at 0 *is an injective analytic embedding $\gamma : (0, \epsilon) \to \mathbb{R}^2$ such that*

- *γ can be continuously extended to $\gamma(0) = 0$.*
- *$\gamma((0, \epsilon))$ is a parabolic set for F whose attractor is* 0.

A repelling analytic parabolic curve γ of F at 0 *is an attracting analytic parabolic curve γ of F^{-1} at* 0.

We have defined the analytic parabolic curve as a parameterized curve, but we will sometimes denote by parabolic curve its image. Notice that an analytic parabolic curve at 0 is a parabolic set at 0 with the additional property that it is an analytic submanifold.

We say that an analytic parabolic curve γ is *asymptotic to a formal curve* Γ at 0 if there is an irreducible formal parameterization $\beta \in \mathbb{R}[[s]]^2$ of Γ such that the asymptotic expansion of γ at 0 coincides with β, this means, for every $k \in \mathbb{N}$, there exists constants c_k, ϵ_k such that

$$\|\gamma(s) - j_k(\beta)(s)\| \leq c_k s^{k+1}, \quad s \leq \epsilon_k,$$

where $j_k(\beta)$ denotes the k-jet truncation of each component of β, which is a polynomial, and hence can be evaluated at s.

2.1.2 Main results

In this section, we will state the main results. We choose a first representative $F : W \to W'$ in which $(\mathrm{Fix}(F)\backslash\{0\}) \cap \partial_{\mathbb{R}^2} W$ has the same number of points as the number of connected components of $(\mathrm{Fix}(F) \setminus \{0\})$ in W, where $\mathrm{Fix}(F) = \{p \in W : F(p) = p\}$ (without further mention of the domain W

unless there is the risk of confusion). These connected components of the curve of fixed points will be called the half-branches of the curve of fixed points (compare with section A.2.1 in the appendix). Among the curves of fixed points, we will distinguish two types. Recall that, by Proposition 1.42, the germs of $\mathrm{Fix}(F)$ and $\mathrm{Sing}(\xi)$ at 0 coincide. We say that a half-branch Γ of $\mathrm{Fix}(F)$ is *bidicritical* when its germ at 0 is not invariant for $S(\xi)$, where $S(\xi)$ is any saturation of ξ.

Let us state the main result in this chapter.

Theorem 2.6. *Let $F \in Diff_1(\mathbb{R}^2, 0)$ be a germ of real analytic diffeomorphism with $F(0) = 0$, $F \neq Id$, tangent to the identity and of non center-focus type. For any open neighborhood W of 0 where a representative of F and F^{-1} are defined, there exists a neighborhood $U \subset W$ of 0, and a finite partition $\mathcal{S}$ of*

$$U = \bigcup_{A \in \mathcal{S}} A,$$

into $\mathcal{C}^0$ submanifolds of $\mathbb{R}^2$, such that, for any $A \in \mathcal{S}$, we have that A is saturated in U and

0. $\dim A = 0$ *if and only if* $A = \{0\}$.

1. *If* $\dim A = 1$ *then,* $0 \in \overline{A} \setminus A$ *and either A is a connected component of $Fix(F) \setminus \{0\}$ or $A \cap Fix(F) = \emptyset$. In the second case, A is an attracting or repelling parabolic set (curve) at 0.*

2. *If* $\dim A = 2$*, then $0 \in \overline{A} \setminus A$, $A \cap Fix(F) = \emptyset$ and A is of one of the following six types.*

    - *A is an attracting or repelling parabolic set at 0.*
    - *A is an elliptic set at 0.*
    - *A is a hyperbolic set.*
    - *A is dicritical-parabolic (or D-parabolic): there exists $\Gamma_A \in \mathcal{S}$ with $\dim \Gamma_A = 1$ and $\Gamma_A \subset Fix(F)$ such that either, for each $p \in A$, there is $q_p \in \Gamma_A \cap \overline{A}$ with $\alpha_A(p) = \{q_p\}$ and $\mathrm{Orb}^+_A(p)$ escapes A, or, for each $p \in A$, there is $q_p \in \Gamma_A$ with $\omega_A(p) = \{q_p\}$ and $\mathrm{Orb}^-_A(p)$ escapes A.*
    - *A is dicritical-elliptic (or D-elliptic): there exists $\Gamma_A \in \mathcal{S}$ with $\dim \Gamma_A = 1$ and $\Gamma_A \subset Fix(F)$ such that either, for each $p \in A$, there is $q_p \in \Gamma_A \cap \overline{A}$ with $\alpha(p) = \{q_p\}$ and $\omega_A(p) = \{0\}$, or, for each $p \in A$, there is $q_p \in \Gamma_A \cap \overline{A}$ with $\omega_A(p) = \{q_p\}$ and $\alpha_A(p) = \{0\}$.*
    - *A is dicritical-dicritical (or D-D): there exists $\Gamma_A, \Gamma'_A \in \mathcal{S}$ with $\dim \Gamma_A = \dim \Gamma'_A = 1$, $\Gamma_A \neq \Gamma'_A$ and $\Gamma_A, \Gamma'_A \subset Fix(F)$ such that for each $p \in A$, there is $q_p \in \Gamma_A \cap \overline{A}$ with $\alpha_A(p) = \{q_p\}$ and there is $q'_p \in \Gamma'_A \cap \overline{A}$ with $\omega_A(p) = \{q'_p\}$.*

The pair $(U, \mathcal{S})$ fulfilling the properties of the previous theorem will be called a *sectorial decomposition*. Elements in the partition will be called strata, although it is not necessarily a stratification in the usual sense. The two-dimensional sets in $\mathcal{S}$ will be called the *sectors of* $(U, \mathcal{S})$, and they are of *parabolic*, *elliptic*, *hyperbolic*, *D-parabolic*, *D-elliptic* or *D-D* type, correspondingly to the properties presented in the second item of the Theorem, which we call the *weak topological properties of the sector*, according to the terminology in [29]. Notice that each stratum, except the hyperbolic sectors, is either positively or negatively invariant.

See Figure 2.1 and Table 2.1 for a schematic explanation of the sectors, and Figure 2.2 for an example of a sectorial decomposition. In the example, $\mathrm{Fix}(F) = \{0\} \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_5 \cup \Gamma_6$, and all the half-branches of $\mathrm{Fix}(F)$ are bidicritical curves.

Sector A	$\alpha-$limit of $p \in A$	$\omega-$limit of $p \in A$	Bidicritical curves (in the boundary)
Parabolic	$\{0\}$	$\emptyset$	None
Parabolic	$\emptyset$	$\{0\}$	None
Elliptic	$\{0\}$	$\{0\}$	None
Hyperbolic	$\emptyset$	$\emptyset$	None
D-parabolic	$q_p \in \Gamma_A$	$\emptyset$	Γ_A
D-parabolic	$\emptyset$	$q_p \in \Gamma_A$	Γ_A
D-elliptic	$\{0\}$	$q_p \in \Gamma_A$	Γ_A
D-elliptic	$q_p \in \Gamma_A$	$\{0\}$	Γ_A
D-D	$q_p \in \Gamma_A$	$q_p \in \Gamma'_A$	Γ_A, Γ'_A

Table 2.1: Summary of the type of sectors depending on the asymptotic dynamics.

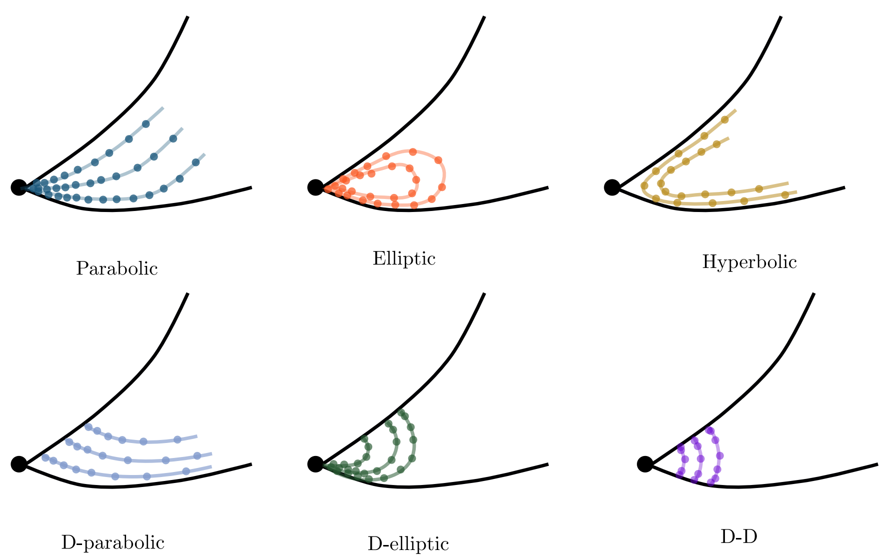

Figure 2.1: Types of sectors

As a direct consequence of Theorem 2.6, we find the following.

Corollary 2.7. *Given a non center-focus diffeomorphism $F \in Diff_1(\mathbb{R}^2, 0)$, there is a neighborhood U of 0, such that the only periodic points of F in U are fixed points, that is, $Per(F) = Fix(F)$ as germs.*

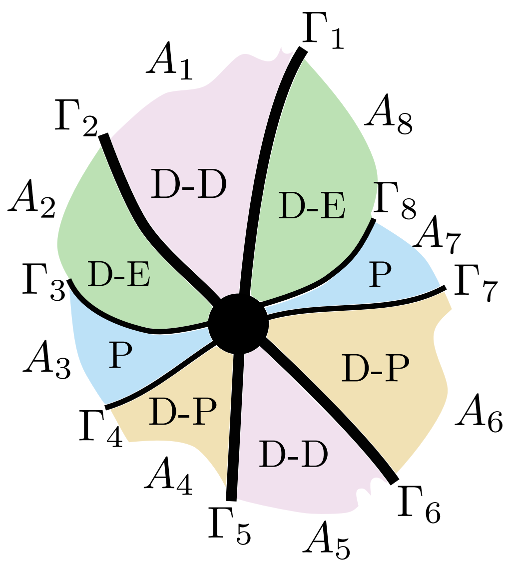

Figure 2.2: A sectorial decomposition.

The set U where the sectorial decomposition is defined is not necessarily open. In fact, in our construction, the set $U \setminus U^\circ$ is a union of segments non containing 0 of the bidicritical curves. We are interested in studying when the sectorial decompositions $(U,\mathcal{S})$ can be taken so that U is open. We also study when the set U can be chosen in the semi-analytic class. We prove the following results.

Proposition 2.8. *Let $F \in \mathit{Diff}_1(\mathbb{R}^2,0)$, $F \neq Id$ be non center-focus and $(U,\mathcal{S})$ a sectorial decomposition. If there are no $D-D$ sectors in $\mathcal{S}$, then, there is a sectorial decomposition $(U',\mathcal{S}')$ in which U' is an open set and $\mathcal{S}'$ is a stratification.*

Proposition 2.9. *Let $F \in \mathit{Diff}_1(\mathbb{R}^2,0)$, $F \neq Id$ be non center-focus. If there are no bidicritical curves, then there is a sectorial decomposition $(U,\mathcal{S})$ in which U is a semi-analyitic subset of $\mathbb{R}^2$ and $\mathcal{S}$ is a stratification.*

2.2 Realization of the graph $G(\xi,\pi)$ for F

Fix an element $F \in \mathrm{Diff}_1(\mathbb{R}^2,0)$ and take the infinitesimal generator $\xi = \mathrm{Log}(F)$ of F. Let $\pi : (M,E) \to (\mathbb{R}^2,0)$ be a reduction of singularities of ξ as in Section 1.4.3. In this section, we will define a geometric object: the realization $\mathrm{Supp}_F(G(\xi,\pi))$ of the graph $G(\xi,\pi)$ of ξ for the reduction of singularities π (cf. Section 1.4.5) and the diffeomorphism F, or for short, simply $\mathrm{Supp}(G(\xi,\pi))$. This object will be useful to construct a sectorial decomposition $(U,\mathcal{S})$ of F and will provide some of the one dimensional strata of $\mathcal{S}$.

2.2.1 Dynamical types of the strict fixed points

Consider the vertices in $\mathcal{V}_{div}$, each of them is associated to a strict fixed point, by definition. We classify the strict fixed points $\mathfrak{S}(F,\pi)$ of F on (M,E) by using the graph $G(\xi,\pi)$ of its infinitesimal generator. The types of fixed points are summarized in Figure 2.3.

- When p_v is a corner point, it can be of one of the following types.
  * d when it is a corner between a dicritical and a non-dicritical component of the divisor.
  * s when it is a corner between two non-dicritical components of the divisor and the only quadrant at v is a saddle quadrant.
  * n when it is a corner between two non-dicritical components of the divisor and the only quadrant at v is a node quadrant.
- When p_v is a non-corner point and it corresponds to a simple singularity of type I, that is, simple and isolated, it can be of one of the following types.
  * s-s when the two quadrants at v are saddle quadrants.
  * n-s when there is a node and a saddle quadrant at v.
  * n-n when the two quadrants at v are node quadrants.
- When p_v is a non-corner point and it corresponds to a simple singularity of type II, that is, it is a normal crossing between a curve of fixed points and the divisor, it can be of one of the following types.
  * f-s-s when the two quadrants at v are saddle quadrants.
  * f-n-s when there is a node and a saddle quadrant at v.
  * f-n-n when the two quadrants at v are node quadrants.
  * f-d-d when the two quadrants at v are dicritical quadrants.

We make a further distinction. Recall that the underlying singularity of a strict fixed point p of types s-s, n-s or n-n is a non-corner adapted simple singularity of type I. Let λ and μ be the eigenvalues of the strict transform of the infinitesimal generator of F at p, tangent, respectively to the eigendirection transverse to E and to the direction of E. Recall that the adapted simple singularity p can be a node, a saddle or a saddle-node, depending on the product $\lambda\mu$ being smaller, bigger or equal to 0. Recall also that when p is an adapted simple saddle singularity of the infinitesimal generator ξ, it is of type s-s as a strict fixed point of F. When p is an adapted simple node, we have that it is of type n-n as a strict fixed point. However, when it is an adapted simple saddle-node, it can be of type s-s, n-s or n-n as a strict fixed point. For technical reasons, we say that a strict fixed point of type n-n is of subtype n-n-1, when $|\lambda| > 0$, and that it is of subtype n-n-2 when $\lambda = 0$.

We summarize in the following diagram (Figure 2.4) the types of fixed points in terms of: its position (corner or non-corner), underlying singularity (regular point, type of adapted simple singularity) and quadrant or quadrants of $G(\xi,\pi)$ at the vertex v_p.

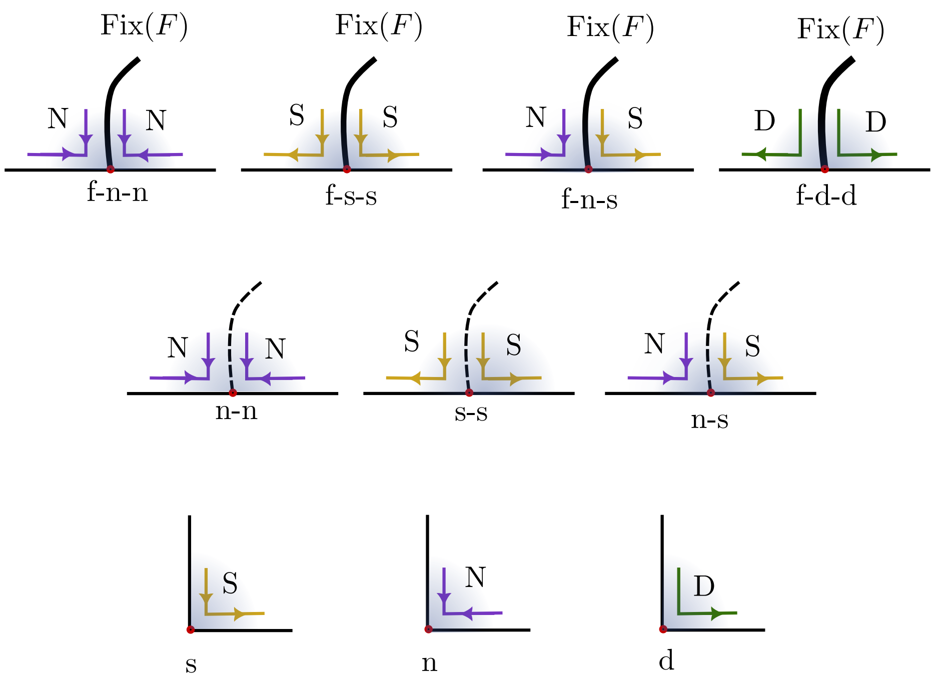

Figure 2.3: Types of strict fixed points.

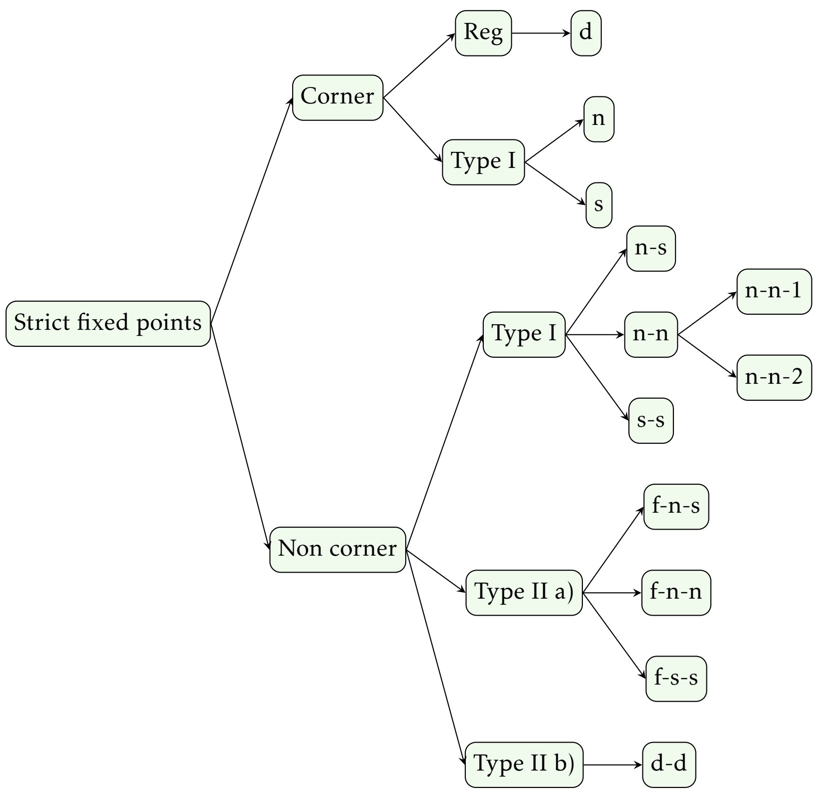

Figure 2.4: Diagram summarizing the strict fixed points in terms of the type of the underlying singularity or regular point.

2.2.2 Existence of parabolic curves and parabolic domains

Before explaining the realization of the graph, we recall a result that exists in the literature and which ensures the existence of parabolic curves at some type of strict fixed points. We also recall a result on existence of attracting domains that are finitely tangent to a formal invariant curve. The results given in this section apply only to strict fixed points of type s-s, n-s, n-n-1 and n-n-2.

The first result concerns existence of parabolic curves. We adapt the statement of the theorem in [56, Theorem 5.1] to our setting.

Theorem 2.10 (Existence of parabolic curves [5, 57, 56]). *Let $p \in \mathfrak{S}(F,\pi)$ be a non-corner strict fixed point of type s-s, n-s or n-n-1. Let Γ be the formal invariant curve of F that is transverse to the divisor. Then, there exists a unique analytic parabolic curve γ of F at p asymptotic to Γ.*

We emphasize that the curve γ obtained in the theorem may not be analytically extended to 0, recall Definition 2.5 of analytic parabolic curves. This result and similar ones have been proved by several authors. To mention some, Baldomá et al. in [5] proved the existence of parabolic curves for real analytic diffeomorphisms at fixed points in the cases in the hypotheses of the Theorem. On the other hand, Lopez et al. [57, 56] proved the existence of parabolic curves asymptotic to formal invariant curve for holomorphic diffeomorphisms. The hypothesis that they require is that the eigendirection of the invariant curve Γ belongs to the "saddle domain". In our setting in the real case, this assumption is satisfied if p is of type s-s, n-s or n-n-1. Let us remark that Dumortier et al. in [29] proved it in the context of $\mathcal{C}^\infty$ diffeomorphisms obtaining $\mathcal{C}^\infty$ parabolic curves.

As Theorem 2.10 does not provide existence of parabolic curves in all the cases, we will look for two dimensional parabolic sets at these points, as they are obtained in the work [56, Theorem 6.1]. Before stating the theorem, let us introduce a new concept, the reduced form of F with respect to a formal invariant curve. It is explained in [56, Section 4].

Definition 2.11. *Consider that the order of contact of the diffeomorphism and the identity is n, and the order of contact of the restriction $F|_\Gamma$ and the identity is $n+s$. We say that F is in* reduced form with respect to Γ *if there exist some coordinates (x,y), called reduced, for which Γ is tangent to $\{x=0\}$ and such that we can write*

$$F(x,y) = (x + y^n A(y)x + O(xy^{n+s+1}) + y^k b(y), y - y^{n+s+1} + O(y^{2n+2s+1})), \tag{2.1}$$

where $y=0$ is a line of fixed points, $k \geq 2n+2s+1$ and $A(y)$ is a polynomial of degree at most s. Notice that $\{y=0\} \subset Fix(F)$.

In [56], it is shown that if $F \in \mathrm{Diff}_1(\mathbb{C}^2,0)$ and Γ is an invariant curve such that $F|\Gamma \neq Id$, then after a finite number of blowing-ups centered at the iterated tangents of Γ and after changes of coordinates, the transformed diffeomorphism can be expressed in reduced form with respect to the strict transform of Γ. In our case, starting with a n-n-2 strict fixed point associated to an adapted simple singularity of type I, the real blowing-up provides two corner strict fixed points of type s and a non-corner strict fixed point of type n-n-2 (cf. Section 1.4.1). Recall also that the formal invariant curve Γ intersects the new component of the divisor at this last point. Then, we will suppose that π also encloses the sequence of blowing-ups centered at the strict fixed points of type n-n-2, so that at the n-n-2 points of $\mathfrak{S}(F,\pi)$, reduced coordinates can be taken.

Theorem 2.12 (Existence of parabolic domains [56])**.** *Let $p \in \mathfrak{S}(F,\pi)$ be a non-corner strict fixed point at the divisor E whose underlying singularity is isolated, simple and of type n-n-2. Let Γ be the formal invariant curve of F that is transverse to the divisor. Suppose also that F is in reduced form with respect to Γ with reduced coordinates (x,y) such that E is given by $y=0$ and Γ has a formal parameterization given by $(\gamma(t),t)$ with $j_{2n+2s+2}\gamma(s)=0$. Then, there exists a parabolic set Ω of F at p of the form*

$$\Omega = \{(x,y) : y \le \delta,\ |x| < y\}, \tag{2.2}$$

where $\delta > 0$. In addition, if for instance Ω is attracting parabolic, for every point $p \in \Omega$, the positive orbit $\mathrm{Orb}^+(p)$ is asymptotic to Γ.

The Theorem 2.12 on the existence of the set Ω is a direct consequence of [56, Theorem 6.1] for holomorphic diffeomorphisms, and its proof can be found therein. The proof adapts almost word by word to the real analytic diffeomorphism as stated. We do not enter into details but we want to make a comment on the last sentence of Theorem 2.12, based on the proofs presented in the article [56]. In short, the property that the orbits are asymptotic to the curve Γ means by definition that for every $p \in \Omega$ and for every N, there is some $k_{p,N}$ such that $F^k(p) \in \{|x - j_{n+s+N-1}(\gamma(y))| < y^N\}$ for $k \ge k_{p,N}$. That is, the orbits reach cones of arbitrarily large order of contact with the invariant curve. Alternatively, $\mathrm{Orb}^+(p)$ has iterated tangents[1] and the sequence of iterated tangents coincides with that of Γ.

We make a second refinement on the reduction of singularities that is valid after performing one additional blowing-up σ centered at a n-n-2 point. This blowing-up, as we mentioned before, generates two strict fixed points of type s and one new strict fixed point of type n-n-2. Moreover, after performing this blowing-up, we have that $\sigma^{-1}(\Omega)$ is an open parabolic set at the new strict fixed point of type n-n-2, that forms, adding the divisor, a neighborhood of the n-n-2 point. The expression of the transform of F is obtained as in Proposition 1.44, and it has the same form as (2.1). Summarizing, we will make the following assumption on the reduction of singularities π.

[1] We say that an orbit has a tangent if its α or ω limit is a single point in the exceptional divisor and iterated tangents are defined recursively by blowing up the subsequent tangents of the orbit. Compare with Definition 1.10 of iterated tangents of parameterized curves.

Assertion $*$. *Let $p \in \mathfrak{S}(F,\pi)$ be of type n-n-2. Then, there is a positively invariant neighborhood U_p of p such that for every $q \in U_p \setminus E$, $\omega(q) = \{p\}$.*

2.2.3 Support of the graph $G(\xi,\pi)$ for F in M and realization of quadrants

In this section, we need to recall the construction of graph $G = G(\xi,\pi) = (\mathcal{V}(G), \mathcal{E}(G))$ and the quadrants $\mathcal{Q}(G)$, together with the orientations (cf. section 1.4.5). We will also consider the reduction of singularities is refined so that we have Assertion $*$. We need also to recall that $\widetilde{F}$ is defined on a neighborhood $\pi^{-1}(W)$ of $E = \pi^{-1}(0)$. In short, we will provide a geometric support to this graph that will be a $\mathcal{C}^\infty$ normal crossing divisor, so that the dynamics of F will be reflected by the orientations given by G. More precisely:

- For any $v \in \mathcal{V}_{div}$, we define $\mathrm{Supp}(v) = p_v$, where p_v is the strict fixed point associated to v. Then, $\mathrm{Supp}(\mathcal{V}_{div})$ is the union of the strict fixed points of F.

- The edges in $\mathcal{E}_{div}$ are associated to the components of $E \setminus \mathrm{Supp}(\mathcal{V}_{div})$, by definition (recall the definition of $\mathcal{E}_{div}$ in section 1.4.5). In particular, if $e = \{v, v'\}$, we define $\mathrm{Supp}(e)$ as the connected component γ_e of $E \setminus \mathrm{Supp}(\mathcal{V}_{div})$ with extremities p_v and $p_{v'}$. Then, $\mathrm{Supp}(\mathcal{E}_{div})$ is the union of these connected components.

- Each edge in $\mathcal{E}_{ndiv}$ is associated either to curve of singularities of $\pi^*(\xi)$ or to a formal invariant curve Γ of $\pi^*(\xi)$, transverse to E. Fix some $e \in \mathcal{E}_{ndiv}$ that is adjacent to $v \in \mathcal{V}_{div}$ associated to a strict fixed point p_v. Recall that when the strict fixed point is of type f-n-n, f-n-s, f-s-s or f-d-d, the edge e is associated to a curve of fixed points. In this case, we directly consider the curve Γ (which indeed is analytic) and we define $\mathrm{Supp}(e) = \Gamma$, where this curve must be understood as a representative defined in some open set $U_e \subset \pi^{-1}(W)$. On the other hand, when Γ is not of fixed points, the strict fixed point p_v can be of type n-n-2, n-n-1, n-s or s-s. In the first case, we will not define a geometric support to e, that is, we will consider $\mathrm{Supp}(e) = \emptyset$. In the remaining cases, we use Theorem 2.10 in order to find a parabolic curve $\gamma \subset \pi^{-1}(W)$ of F transverse to the divisor at p_v. We define $\mathrm{Supp}(e) = \gamma$. Finally, $\mathrm{Supp}(\mathcal{E}_{ndiv})$ is by definition the union of $\mathrm{Fix}(F) \setminus E$ and parabolic curves transverse to E.

- We do not define a geometric support to the vertices $v \in \mathcal{V}_{ndiv}$.

Definition 2.13. *The* support $\mathrm{Supp}(G(\xi,\pi))$ of the graph $G(\xi,\pi)$ for F *is the union Supp*$(\mathcal{V}_{div}) \cup$ *Supp*$(\mathcal{E}_{div}) \cup$ *Supp*$(\mathcal{E}_{ndiv})$ *in a neighborhood of $E \subset M$.*

In order to lighten the notation, name $\widetilde{E} = \mathrm{Supp}(G(\xi,\pi))$. Notice that $\widetilde{E}$ is a $\mathcal{C}^\infty$ normal crossing divisor that extends the analytic one $E \cup \mathrm{Fix}(F)$. We show an example of the construction of the support of the graph $G(\xi,\pi)$ for the diffeomorphism F in Figure 2.5

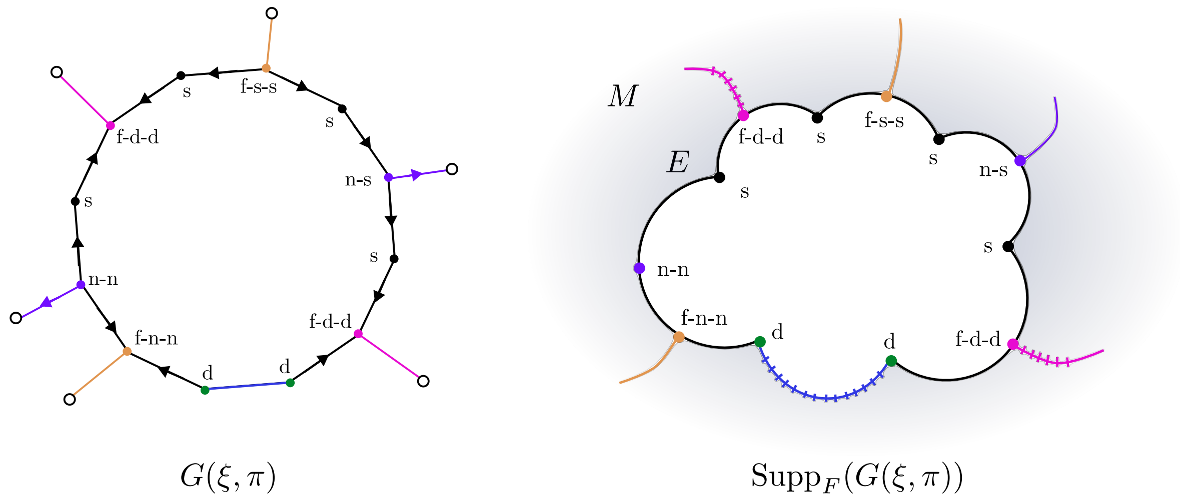

Figure 2.5: Graph $G(\xi,\pi)$ of the infinitesimal generator ξ of F and its realization for F.

Now, we define the realization of the quadrants of the graph as (half) neighborhoods of strict fixed points. This is an intermediate step in order to define suitable sets of coordinates in the following section.

Let $p \in E$ be a strict fixed point of F such that $\{p\} = \mathrm{Supp}(v)$ and such that p is not of type n-n-2. Recall that either there is a single quadrant $Q_1 = (v, e_1, e_2)$ of $G(\xi,\pi)$ that contains v, or there are two quadrants $Q_1 = (v, e_1, e_2), Q_2 = (v, e_1, e_3)$ of $G(\xi,\pi)$ that contain v. Take $\gamma_i = \mathrm{Supp}(e_i)$. Consider now the germ W_p of $\pi^{-1}(W)$ at p and the germ $\widetilde{E}_p$ of $\widetilde{E}$ at p. Then, $W_p \setminus \widetilde{E}_p$ has at most two connected components. That is, $W_p \setminus \widetilde{E}_p = U_{1,p}$ or $W_p \setminus \widetilde{E}_p = U_{1,p} \cup U_{2,p}$. We consider $U_{1,p}$ to be the connected component that fulfills $\overline{U_{1,p}} = U_{1,p} \cup \{p\} \cup \gamma_{1,p} \cup \gamma_{2,p}$ and $U_{2,p}$ to be the connected component that fulfills $\overline{U_{2,p}} = U_{2,p} \cup \{p\} \cup \gamma_{1,p} \cup \gamma_{3,p}$. A *realization of* Q_i, $i = 1, 2$, is a couple (Q_i, U_i), where U_i is a representative of $U_{i,p}$ such that $\overline{U}_i \cap \mathrm{Supp}(e_1)$ and $\overline{U} \cap \mathrm{Supp}(e_{j_i})$ are connected with $j_1 = 2$ and $j_2 = 3$. By an abuse of notation, we also call quadrants the realizations of the quadrants.

Now, let $p \in E$ be a strict fixed point of F such that $\{p\} = \mathrm{Supp}(v)$ and such that p is of type n-n-2. Notice that there are necessarily two quadrants of the graph, say $Q_1 = (v, e_1, e_2)$ and $Q_2 = (v, e_1, e_3)$, which contain v. However, as we have already pointed out, there is not a realization of the edge e_1. For this reason, a single quadrant cannot be realized in the above sense. We define a *joint realization of* Q_1 *and* Q_2 a triple (Q_1, Q_2, U), where U is a convex representative of the germ $\overline{W_p}$ such that $\overline{U} \cap \mathrm{Supp}(e_2)$ and $\overline{U} \cap \mathrm{Supp}(e_3)$ are connected.

2.2.4 Monotonic (coordinate) domains

In this section, we provide convenient expressions of $\widetilde{F}$ at regular and at strict fixed points of $\widetilde{E}$. This section is based on [29], allowing also curve of fixed points. At the strict fixed points, we work in a realization of a single quadrant, or in a joint realization of the two quadrants adjacent to a point n-n-2. We also provide changes of coordinates valid on a small enough realization of the quadrant(s).

Before studying each case, as a reminder from Section 1.5.1, we recall that, if $p \in E$, then the germ of $\widetilde{F}$ at p is given by $\widetilde{F}_p = \operatorname{Exp}(\widetilde{\xi}_p)$ where $\widetilde{\xi}_p$ is the total transform of ξ by π at p. We will simply denote $\widetilde{F}$ at p again as F, because throughout the text we will always indicate in which point we work, and we will also denote $\widetilde{\xi}_p$ by ξ. The expression of each F_i in $F(x,y) = (F_1(x,y), F_2(x,y))$ in terms of the expression of $\xi = c(x,y)\frac{\partial}{\partial x} + d(x,y)\frac{\partial}{\partial y}$ is given by the exponential map as follows.

$$\begin{aligned} F_1(x,y) &= \exp(\widetilde{\xi}_p)(x) = x + c(x,y) + \frac{1}{2}\left(c(x,y)\frac{\partial c(x,y)}{\partial x} + d(x,y)\frac{\partial c(x,y)}{\partial y}\right) + \cdots \\ F_2(x,y) &= \exp(\widetilde{\xi}_p)(y) = y + d(x,y) + \frac{1}{2}\left(c(x,y)\frac{\partial d(x,y)}{\partial x} + d(x,y)\frac{\partial d(x,y)}{\partial y}\right) + \cdots \end{aligned} \tag{2.3}$$

We also recall that (x,y) are analytic coordinates at p. Even if the vector field is only formal and its coefficients lie in $\mathbb{R}[[x,y]]$ (or sometimes in some other subalgebra), the diffeomorphism is analytic, and the components lie in $\mathbb{R}\{x,y\}$.

Now, we present the monotonic domains, namely, chart domains in which it is possible to define monotonic functions on the orbits of the diffeomorphism.

- **Non-dicritical regular point**. Let $p \in \mathcal{E} \setminus \mathfrak{S}(\mathsf{F},\pi)$ and assume that p is a non-dicritical component of E. As in (1.6), there are analytic coordinates (x,y) centered at p with $E = \{y = 0\}$ such that

  $$\xi = y^n\left(a(x,y)\frac{\partial}{\partial x} + yb(x,y)\frac{\partial}{\partial y}\right),$$

  with $n \in \mathbb{N}^*$, $a, b \in \mathbb{R}[[x,y]]$ and $a(0,0) \neq 0$.

  Then, the expression of F at p is

  $$F(x,y) = (x + y^n A(x,y), y + y^{n+k}B(x,y)), \tag{2.4}$$

  where $A, B \in \mathbb{R}\{x,y\}$, $n, k \in \mathbb{N}^*$, $y \geq 0$ and $A(0,0) \neq 0$. By considering a small enough domain U, we can suppose that $A(x,y) < 0$ or that $A(x,y) > 0$. In this case, we say that $(U,(x,y))$ is a *regular monotonic domain*.

  We have that this expression is valid in a greater domain. Recall, as we are assuming that we use the usual charts of the blowing-up (cf. Section 1.2), that the vector field $\widetilde{\xi}_p$ is $(T_{qp})_*\widetilde{\xi}_q$, where $\widetilde{\xi}_q := \xi^{(I)}$ is defined at the origin of the chart U_I and T_{qp} is the affine translation of q to the point p.

Remark 2.14. In terms of exponential maps, since T_{qp} is an affine map (a diffeomorphism), we have formally $\exp((T_{qp})_*\xi_q)(f) = \exp(\xi_q)(f \circ T_{qp}) \circ (T_{qp})^{-1}$.

We deduce the following from this remark.

Claim. We can consider a local expression of F, as the one in (2.4) that is valid on a neighborhood of a connected compact subset of $E \setminus \mathfrak{S}(\mathfrak{F},\pi)$ inside a non-dicritical component of E and in the domain of a single usual chart U_I.

- **Dicritical regular point**. Let $p \in \mathcal{E} \setminus \mathfrak{S}(\mathfrak{F},\pi)$ and assume that p is a dicritical component of E. We use (2.3) and the expression (1.7) given in analytic coordinates (x,y) at p where $E = \{y = 0\}$ by
$$\xi = y^n(a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}),$$
with $n \in \mathbb{N}^*$, $a,b \in \mathbb{R}[[x,y]]$ and $b(0,0) \neq 0$. Then,
$$F(x,y) = (x + y^n A(x,y), y + y^n B(x,y)), \tag{2.5}$$
with $A,B \in \mathbb{R}\{x,y\}$, $n,k \in \mathbb{N}^*$, $y \geq 0$ and $B(0,0) \neq 0$.

  As before, there is a small enough domain U such that $B(x,y) > 0$ or that $B(x,y) < 0$ at every point in U, we say that $(U,(x,y))$ is a *regular monotonic domain*. This expression can also be extended to a greater domain, a neighborhood of a connected compact smooth subset of $E \setminus \mathfrak{S}(F,p)$ and the Claim applies.

- **Strict fixed points of type d and f-d-d**: Let $p \in \mathfrak{S}(F,\pi)$ be an adapted simple singularity of type II b) (strict fixed point of type f-d-d) with E_1 a non-dicritical component and E_2 a bidicritical curve, or an intersection point of a non-dicritical component E_1 and a dicritical component E_2 of the divisor (strict fixed point of type d). The infinitesimal generator has expression as in (1.8) or (1.9) in analytic coordinates (x,y) such that $E_1 = \{y = 0\}$ and $E_2 = \{x - h(y) = 0\}$
$$\xi = (x - h(y))^m y^n \left(a(x,y)\frac{\partial}{\partial x} + y b(x,y)\frac{\partial}{\partial y}\right)$$
with $a,b \in \mathbb{R}[x][[y]]$, $n,m \in \mathbb{N}^*$ and $a(0,0) \neq 0$. As in (1.8), when E_2 is a dicritical component of the divisor, we consider $h \equiv 0$ and the germ of the space M at p is identified with $\{x \geq 0, y \geq 0\}$. When it is a bidicritical curve, h is a convergent series since $\mathrm{Fix}(F)$ is analytic, and the germ of the space is identified with $\{y \geq 0\}$. The expression of the diffeomorphism is
$$F(x,y) = (x + (x - h(y))^n y^m A(x,y), y + (x - h(y))^n y^{m+1} B(x,y)),$$
with $A,B \in \mathbb{R}\{x,y\}$, $n,m \in \mathbb{N}^*$, $y \geq 0$ and $A(0,0) \neq 0$. In a small enough domain U, we can

suppose that $A(x,y) < 0$ or $A(x,y) > 0$ for all points of U. Up to performing an analytic change of coordinates $z = x - h(y)$, $w = y$, we can always suppose that the diffeomorphism has the following expression

$$F(x,y) = (x + x^n y^m A(x,y), y + x^n y^{m+1} B(x,y)), \tag{2.6}$$

with $A, B \in \mathbb{R}\{x,y\}$, $n, m \in \mathbb{N}^*$, $y \geq 0$ and $A(0,0) \neq 0$, and again that $A(x,y) < 0$ or $A(x,y) > 0$ on a small enough domain U.

In the rest of the chapter we will work in some realization of the quadrants at p (one in the dicritical component case, or two in the bidicritical curve case). We will take as realizations of the quadrants at p the domains $U_1 = U \cap \{x \geq 0\}$ and $U_2 = U \cap \{x \leq 0\}$ (this last case only if there is a bidicritical curve). We call the charts $(U_i, (x,y))$ *dicirtical domains*, since for any point $p \in U_i$ either $x(F(p)) \geq x(p)$ or $x(F(p)) \leq x(p)$ (monotony) and the boundary curve $\{x = 0\}$ is dicritical. For simplicity, we will always suppose that we work in the positive quadrant of $\mathbb{R}^2$, that is, in $\{x \geq 0, y \geq 0\}$, this assumption can be made by performing the change of coordinates $z = -x$, $w = y$.

- **Strict fixed point of types f-s-s, f-n-s, f-n-n, n and s**. Let $p \in \mathfrak{S}(F,\pi)$ be a strict fixed point placed at an adapted simple singularity of type II a) (f-s-s, f-n-s and f-n-n) or at a corner adapted simple singularity (n and s). Recall the expressions of the infinitesimal generators in adapted coordinates (x,y) in (1.10) and (1.14) such that $E_1 = \{y = 0\}$ is a component of E and E_2 is another component of the divisor (p of type n,s) or a curve of fixed points (f-s-s, f-n-s and f-n-n). That is,

$$\xi = (x - h(y))^m y^n \left(a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} \right) \tag{2.7}$$

with $a, b \in \mathbb{R}[x][[y]]$, $h \in \mathbb{R}\{y\}$, $n, m \in \mathbb{N}^*$ and the germ of M at p is identified with $y \geq 0$ when p is of type f-s-s, f-n-s, f-n-n, or with $x \geq 0, y \geq 0$ (having $h \equiv 0$) when it is a corner. In addition, since the singularity is simple adapted to E, we assume that the vector field $S(\xi) = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$ is saturated and has a simple singularity (in the classical sense) such that $y = 0$ is one of the separatrices and that the other is given by the analytic equation $x - h(y) = 0$. Performing the analytic change of coordinates $z = x - h(y)$, $w = y$ as in the previous item, we obtain, rewriting again $(x,y) = (z,w)$, that the vector field is now

$$\xi = x^m y^n \left(x^s \tilde{a}(x,y)\frac{\partial}{\partial x} + y^r \tilde{b}(x,y)\frac{\partial}{\partial y} \right) \tag{2.8}$$

with $a, b \in \mathbb{R}[x][[y]]$, $n, m, r, s \in \mathbb{N}^*$, $1 \in \{s, r\}$, $\tilde{a}(0,0) \neq 0$, $\tilde{b}(0,0) \neq 0$. Notice that the fact that $\tilde{a}(0,0) \neq 0$, $\tilde{b}(0,0) \neq 0$ follows from the properties of the refined reduction of singularities π, highlighted in section 1.4.4. After the change of coordinates, notice that the two separatrices of the simple singularity of $S(\xi) = x^s\tilde{a}(x,y)\frac{\partial}{\partial x} + y^r\tilde{b}(x,y)$ are the two coordinate axis. One of the numbers r,s must necessarily be equal to 1, since at least one of the eigenvalues of $S(\xi)$ is different from 0. In these coordinates, the diffeomorphism F has expression

$$F(x,y) = (x + x^{n+s}y^m A(x,y), y + x^n y^{m+r} B(x,y)), \tag{2.9}$$

with $A,B \in \mathbb{R}\{x,y\}$, $n,m,r,s \in \mathbb{N}^*$, $1 \in \{s,r\}$ and $A(0,0) \neq 0$, $B(0,0) \neq 0$. The domain is $y \geq 0$ or $y \geq 0, x \geq 0$ when p is of type f-s-s, f-n-s, f-n-n, or of type n, s, respectively. Choosing a small enough domain U, we have $A(x,y) \neq 0, B(x,y) \neq 0$ at every point in U. As in the previous case, we will work separately on the realizations of the quadrants containing p. When p is of type f-s-s, f-n-s, f-n-n, there are two quadrants and when it is of type s,n there is only one. The realization of the quadrants are $U_1 = U \cap \{x \geq 0\}$ and $U_2 = U \cap \{x \leq 0\}$ (only for the f-s-s, f-n-s and f-n-n cases). For simplicity and after applying the change of coordinates $z = -x,\ w = y$, we will always suppose that we work on the positive quadrant of $\mathbb{R}^2$. If A and B have the same sign on the domain U_i, we say that $(U_i,(x,y))$ is a *node monotonic domain* since both coordinate functions x,y are monotonically increasing or decreasing on the orbits outside the fixed points. Otherwise, if A,B have opposite sign in U_i, we say that $(U_i,(x,y))$ is a *saddle monotonic domain*, since in this case, one of the functions x,y is monotonically increasing and the other is monotonically decreasing.

- **Strict fixed point of type s-s, n-n-1 and n-s**. Let $p \in \mathfrak{S}(F,\pi)$be a strict fixed point placed at a non-corner adapted simple singularity of type I. We suppose that the infinitesimal generator has expression in analytic coordinates (x,y) for which $E = \{y = 0\}$

$$\xi = y^n\left(a(x,y)\frac{\partial}{\partial x} + yb(x,y)\frac{\partial}{\partial y}\right)$$

with $a,b \in \mathbb{R}[x][[y]]$, $n,m \in \mathbb{N}^*$, and the germ of M at p is identified with $y \geq 0$. In addition, since the singularity is simple adapted to E, the vector field $S(\xi) = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$ is saturated and has a simple singularity (in the classical sense) such that $y = 0$ is one of the separatrices and that the other formal separatrix Γ is given by the formal equation $x - \hat{h}(y) = 0$. The expression of the diffeomorphism is then

$$F(x,y) = (x + y^n A(x,y), y + y^{n+r} B(x,y)),$$

with $A,B \in \mathbb{R}\{x,y\}$, $n,r \in \mathbb{N}^*$. Recall also that from Theorem 2.10, there exists a germ of parabolic curve of F transverse to $y = 0$ and asymptotic to the formal invariant curve Γ. The graph of this curve fulfills a $\mathcal{C}^\infty$ equation $x - h(y) = 0$, so that the Taylor expansion at $y = 0$ is $\hat{h}(y)$. Performing the $\mathcal{C}^\infty$ change of coordinates $z = x - h(y)$, $w = y$, and renaming $(x,y) = (z,w)$, we get from the above equation

$$F(x,y) = (x + y^n x^s A(x,y), y + y^{n+r} B(x,y)), \tag{2.10}$$

with $n,s,r \in \mathbb{N}^*$, $1 \in \{s,r\}$ and $A, B \in \mathcal{C}_p^\infty$ with $A(0,0) \neq 0$, $B(0,0) \neq 0$. We remark that y is analytic in any case, although x is C^∞. Notice that both curves $x = 0$ and $y = 0$ are invariant curves, and that s,r are taken to be maximal so that we have $A(0,0) \neq 0$, $B(0,0) \neq 0$ (they exist since both components of $F - Id$ are not flat in (x,y) components). The fact that $1 \in \{r,s\}$ follows since the saturation of the infinitesimal generator has at least one non-zero eigenvalue. We take a small enough neighborhood U such that $A(x,y) \neq 0, B(x,y) \neq 0$ for any $(x,y) \in U$.

Then, we consider realizations of the two quadrants at p by taking $U_1 = U \cap \{x \leq 0\}$ and $U_2 = U \cap \{x \geq 0\}$. In a realization of a node quadrant Q_i, the functions A and B have the same sign on U_i, and we say that $(U_i,(x,y))$ is a *node monotonic domain*. In the realization of a saddle quadrant Q_i, the functions A and B have different sign, and we say that $(U_i,(x,y))$ is a *saddle monotonic domain*. Notice that both coordinate functions x,y are monotonic on the orbits on the domain.

- **Strict fixed point of type n-n-2**. Let $p \in \mathfrak{S}(\xi,\pi)$ be a strict fixed point placed at a non-corner adapted simple singularity of type I. We will treat this case in a different manner, since we do not have a parabolic curve (although we use results from Section 2.2.2). We recall that the diffeomorphism is expressed as (2.1) and recall the assumption $*$, which follows after Theorem 2.12. That is, there is a positively invariant neighborhood of p in which the orbits converge to p. We will call a chart $(U,(x,y))$ in which (2.1) and the assumption (*) applies a *node monotonic domain*. For the sake of completeness of the section, we write again this expression.

$$F(x + y^n A(y)x + O(xy^{n+s+1}) + y^{2n+2s+1} b(y), y + \epsilon y^{n+s+1} + O(y^{2n+2s+1})),$$

where $A \in \mathbb{R}[y]_{\leq s}$ is a polynomial of degree lower than s and $\epsilon = +,-$. Notice that the coordinate y is monotonically increasing ($\epsilon = +$) or decreasing ($\epsilon = -$) in the orbits and convergence to p is ensured by (*).

2.3 Local dynamics after reduction of singularities

In this section, we study the blown-up diffeomorphism $\widetilde{F}$ of F after the reduction of singularities $\pi : (M,E) \to (\mathbb{R}^2, 0)$ of its infinitesimal generator ξ. Suppose that we are given any representative $F : R \to R'$, then, we will study $\widetilde{F}$ in a neighborhood $W_E \subset \pi^{-1}(R)$ of E as in Definition 2.13. We will study $\widetilde{F}$ locally at small compact subsets of the divisor E: strict fixed points and smooth connected compact subsets of $E \setminus \mathfrak{S}(\mathfrak{F},\pi)$, called regular arcs. Throughout this section, we will work in (M,E) and we will simplify the notation simply using F,ξ instead of $\widetilde{F},\widetilde{\xi}$.

2.3.1 Dynamics over regular non-dicritical arcs

We define a regular non-dicritical arc as follows.

Definition 2.15. *Let $\mathfrak{S}(F,\pi)$ be the set of strict fixed points of F after the sequence of blowing-ups π. A* regular non-dicritial arc *Γ (for F) is a connected compact subset of $E \setminus \mathfrak{S}(F,\pi)$ contained in a non-dicritical component of the divisor.*

Let Γ be a regular non-dicritical arc contained in a non-dicritical component of E. There is a chart of (M,E) where $\widetilde{\xi}'|_\Gamma$ is a polynomial vector field that does not have singularities, cf. section 2.2.4. The constant sign $\widetilde{\xi}'|_\Gamma$ induces an orientation in Γ, from one of the endpoints to the other one. We will consider the total ordering on Γ on the points $p \in \Gamma$ induced by this orientation. Choose a monotonic domain $(U,(x,y))$ centered at some point of Γ with $E = \{y = 0\}$. From section2.2.4, we may assume that U contains a neighborhood V_Γ of Γ in which there are constants $C_A, c_A, C_B > 0$ such that, for $(x,y) \in V_\Gamma$,

$$F(x,y) = (x + y^n A(x,y), y + y^{n+k} B(x,y)), \quad 0 < c_A \leq A(x,y) \leq C_A \text{ and } -C_B \leq B(x,y) \leq C_B, \tag{2.11}$$

with $n,k \geq 1$. The integer n coincides with that of equation

$$\xi = y^n (a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y})$$

of the infinitesimal generator. On the other hand, the integer k can be chosen differently after a change of coordinates that is valid in a neighborhood of the dicritical arc, as the following result shows.

Lemma 2.16. *Let Γ be a regular non-dicritical arc and let $(U,(x,y))$ be a monotonic domain at some $p \in \Gamma$ with $\Gamma \subset U$ so that F has the expression* (2.11) *with $k = 1$. Then, there exist another monotonic domain $(U'(z,w))$ with (z,w) centered at the same $p \in \Gamma$ such that the expression of F (renaming $(x,y) = (z,w)$ in* (2.11)*) has $k \geq 2$.*

Proof. First suppose that $\Gamma = (w_1,w_2) \times \{0\}$ in coordinates (x,y). Notice that the infinitesimal generator of F can be written as $\xi = y^n(a(x,y)\frac{\partial}{\partial x} + yb(x,y)\frac{\partial}{\partial y})$, where $a,b \in \mathbb{R}[x][[y]]$ and thus the coefficients $a(x,y) = a_0(x) + a_1(x)y + \cdots$ with $a_0(x) > 0$ at any point of Γ and $b(x,y) = b_1(x) + b_2(x)y + \cdots$. Let f be an analytic function that fulfills the ordinary differential equation $a_0(x)f'(x) - b_1(x)f(x) = 0$, which has solution defined in (w_1,w_2) since $a_0 \neq 0$ at any point. Now, we define the change of coordinates $x = z,\ y = wf(z)$, obtaining that the infinitesimal generator in these coordinates is

$$\widetilde{\xi} = w^n (f(z))^n a(z, wf(z)) \frac{\partial}{\partial z} + w^{n+1} (f(z))^n (b_1(z) w f(z) - a_0(z) w f'(z) + \cdots) \frac{\partial}{\partial w}$$

Recalling the definition of f, we get $a_0(z) f'(z) - b_1(z) f(z) = 0$, and hence after the change of coordinates there is not a term of degree $k+1$ in w in the coefficient $\frac{\partial}{\partial w}$ and neither will be at the second component of $\widetilde{F}$. □

We name a neighborhood $(V_\Gamma, (x,y))$ of Γ in which F is as (2.11) a *monotonic domain along* Γ. We prove the following result.

Proposition 2.17. *Let F be a diffeomorphism, Γ a non-dicritical regular arc and V_Γ a monotonic domain along Γ. Denote by $p_1 < p_2$ the endpoints of Γ. For every neighborhood $U_2 \subset V_\Gamma$ of p_2 there is some neighborhood $\widetilde{V_\Gamma}$ of Γ such that for every $p \in \tilde{V}_\Gamma$ we have* $\mathrm{Orb}^+_{V_\Gamma}(p) \cap U_2 \neq \emptyset$.

Proof. We assume that $k > 1$ thanks to Lemma 2.16. Put $p_1 = (w_1, 0)$ and $p_2 = (w_2, 0)$. Assume also that V_Γ contains an open box $V(\varepsilon, \delta) = (w_1 - \varepsilon, w_2 + \varepsilon) \times [0, \delta)$, for some $\varepsilon > 0, 1 > \delta > 0$ are small enough. We choose the constants δ, ε such that they satisfy several hypotheses, namely:

(1-δ) The open set U_2 contains a box $(w_2 - \varepsilon, w_2 + \varepsilon) \times [0, \delta)$.

(2-δ) δ fulfills

$$\delta < \left(\frac{2\varepsilon}{C_A} \right)^{\frac{1}{n}}.$$

We require an additional condition, that we explain now. First, we bound the growth of the monomials y^n on the orbits inside a box $V(\varepsilon, \delta)$ as follows: if $p = (x_0, y_0) \in V_\Gamma$ and $F(p) = (x_1, y_1)$, from

$$\frac{1}{y_1^n} - \frac{1}{y_0^n} = \frac{1}{y_0^n} \left(\frac{1}{(1 + y_0^{n+k-1} B(x_0, y_0))^n} - 1 \right),$$

we have

$$\frac{1}{y_1^n} - \frac{1}{y_0^n} = \frac{1}{y_0^n} \left(\frac{1}{(1 + y_0^{n+1} \widetilde{B}(x_0, y_0))^n} - 1 \right) = \frac{1}{y_0^n} \frac{y_0^{n+1}(n\widetilde{B} + O(y_0^{n+1}))}{1 + y_0^{n+1}(n\widetilde{B} + O(y_0^{n+1}))} = y_0 n \widetilde{B} + O(y_0^{n+1})), \quad \widetilde{B} = y_0^{k-2} B.$$

Then, there is a constant $K_0 \geq 0$ such that if both $p = (x_0, y_0)$, $F(p) = (x_1, y_1) \in V(\varepsilon, \delta)$, then

$$\left| \frac{1}{y_1^n} - \frac{1}{y_0^n} \right| \leq K_0 y_0 \leq K_0 \delta = K_\delta, \tag{2.12}$$

where $K_\delta := K_0 \delta$. Now we give the third condition of δ in terms of the previous equation.

(3-δ) δ fulfills

$$1 \leq \exp\left(\frac{K_\delta}{c_A}(w_2 - w_1)\right) < 2.$$

We take another constant $\widetilde{\delta}$ satisfying the following hypotheses.

(1-$\widetilde{\delta}$) $\widetilde{\delta}$ fulfills

$$\widetilde{\delta}^n < \frac{2 - \exp\left(\frac{K_\delta}{c_A}(w_2 - w_1)\right)}{K_\delta}.$$

(2-$\widetilde{\delta}$) $\widetilde{\delta}$ fulfills

$$\widetilde{\delta}^n \leq \frac{\delta^n\left(2 - \exp\left(\frac{K_\delta}{c_A}(w_2 - w_1)\right)\right)}{1 + \delta^n K_\delta}.$$

Finally we define the following function on y.

$$L(y) = \frac{\exp\left(\frac{K_\delta}{c_A}(w_2 - w_1)\right) - 1}{K_\delta y^n} + 1 \tag{2.13}$$

We prove the following claim.

Claim. For every $p = (x_0, y_0) \in V(\varepsilon, \widetilde{\delta})$ there is some $\ell_p \in \mathbb{N}$ such that

- $\ell_p \leq L(y_0)$
- $(x_i, y_i) = F^i(p) \in V(\varepsilon, \delta)$ for every $0 \leq i \leq \ell_p$
- $w_2 - \varepsilon < x_{\ell_p} < w_2 + \varepsilon$ and $y_{\ell_p} < \delta$, that is, $(x_{\ell_p}, y_{\ell_p}) \in U_2$.

Setting $\widetilde{V_\Gamma} = V(\varepsilon, \widetilde{\delta})$ and reminding that $V(\varepsilon, \delta) \subset V_\Gamma$, we have the thesis of this result, that is, for any point $p \in \widetilde{V_\Gamma}$ there is an iterate $\ell = \ell_p$ with $F^{\ell_p}(p) \in U_2$ and $F^i(p) \in V_\Gamma$ for every $0 \leq i \leq \ell_p$.

Proof of the claim. From the bound in (2.12) and using the triangular inequality, we find the following bound for the coordinate y on the orbits on $V(\varepsilon, \delta)$ of points $p = (x_0, y_0)$ with $y_0 > 0$:

$$\frac{y_0^n}{1 + \ell K_\delta y_0^n} \leq y_\ell^n \leq \frac{y_0^n}{1 - \ell K_\delta y_0^n}, \tag{2.14}$$

valid as long as $(x_i, y_i) \in V(\varepsilon, \delta)$ for every $0 \leq i \leq \ell$ and $1 - \ell K_\delta y_0^n > 0$. On the other hand, we find a lower bound for the coordinate x on the orbits on $V(\varepsilon, \delta)$ with $y_0 > 0$.

$$x_\ell \geq x_0 + c_A \sum_{j=0}^{\ell-1} y_j^n \geq x_0 + c_A \sum_{j=0}^{\ell-1} \frac{y_0^n}{1 + jK_\delta y_0^n} \geq x_0 + \frac{c_A}{K_\delta}\log(1 + \ell K_\delta y_0^n), \tag{2.15}$$

valid as long as $(x_i, y_i) \in V(\varepsilon, \delta)$ for every $0 \le i \le j$. We have used the lower bound of $A(x,y)$ in V_Γ in the first inequality, the lower inequality (2.14) in the second, and the classical integral criteria for finding a lower bound of sums with decreasing terms in the third.

We fix a point $(x_0, y_0) \in V(\varepsilon, \widetilde{\delta}) \setminus \{y = 0\}$. Taking any $\ell \ge L(y_0) - 1$, if the points $(x_0, y_0), (x_1, y_1), \dots, (x_{\ell-1}, y_{\ell-1}), (x_\ell, y_\ell) \in V(\varepsilon, \delta)$, we have $x_\ell \ge w_2 - \varepsilon$. To see this it is enough to notice that the lower bound of x_ℓ in (2.15) increases with ℓ. Then, consider $\ell \ge L(y_0) - 1$ and plug $L(y_0) - 1$ in the right member of (2.15). This substitution leads to

$$x_\ell \ge x_0 + \frac{c_A}{K_\delta} \log\left(1 + \left(\frac{\exp\left(\frac{K_\delta}{c_A}(w_2 - w_1)\right) - 1}{K_\delta y_0^n}\right) K_\delta y_0^n\right) = x_0 + w_2 - w_1 \ge w_2 - \varepsilon,$$

where the last inequality follows from $x_0 \ge w_1 - \varepsilon$. On the other hand, we also have that $y_\ell \le \delta$, again under the condition that it is possible to define the ℓ iterate. To see this, it is enough to use that $y_0 \le \widetilde{\delta}$ and proceed as follows.

- Notice first that the upper bound in (2.14) stands since $1 - \ell K_\delta y_0^n \ge 1 - L(y_0) K_\delta y_0^n > 0$, which can be seen by plugging the value of $L(y_0)$ in the former expression and using the bound of $\widetilde{\delta}$ given in the hypothesis (1-$\widetilde{\delta}$). More precisely:

$$\begin{aligned} 1 - \ell K_\delta y_0^n \ge& 1 - \left(\frac{\exp\left(\frac{K_\delta}{c_A}(w_2 - w_1)\right) - 1}{K_\delta y_0^n} + 1\right) K_\delta y_0^n = 2 - \exp\left(\frac{K_\delta}{c_A}(w_2 - w_1)\right) - K_\delta y_0^n \ge \\ \ge& 2 - \exp\left(\frac{K_\delta}{c_A}(w_2 - w_1)\right) - K_\delta \widetilde{\delta}^n > 2 - \exp\left(\frac{K_\delta}{c_A}(w_2 - w_1)\right) - K_\delta \left(\frac{2 - \exp\left(\frac{K_\delta}{c_A}(w_2 - w_1)\right)}{K_\delta}\right) = 0. \end{aligned}$$

- Now, we use (2.14), which is valid from the previous item. We have $y_\ell^n \le \frac{y_0^n}{1 - \ell K_\delta y_0^n} \le \frac{y_0^n}{1 - L(y_0) K_\delta y_0^n} \le \delta^n$, which can be seen again by plugging the value of $L(y_0)$ and using now the bound of $\widetilde{\delta}$ given in the hypothesis (2-$\widetilde{\delta}$). We have

$$\begin{aligned} y_\ell^n &\le \frac{y_0^n}{1 - \left(\frac{\exp\left(\frac{K_\delta}{c_A}(w_2 - w_1)\right) - 1}{K_\delta y_0^n} + 1\right) K_\delta y_0^n} = \frac{y_0^n}{2 - \exp\left(\frac{K_\delta}{c_A}(w_2 - w_1)\right) - K_\delta y_0^n} \le \\ &\le \frac{\widetilde{\delta}^n}{2 - \exp\left(\frac{K_\delta}{c_A}(w_2 - w_1)\right) - K_\delta \widetilde{\delta}^n} \le \frac{\frac{\delta^n\left(2 - \exp\left(\frac{K_\delta}{c_A}(w_2 - w_1)\right)\right)}{1 + \delta^n K_\delta}}{2 - \exp\left(\frac{K_\delta}{c_A}(w_2 - w_1)\right) - \left(\frac{\delta^n\left(2 - \exp\left(\frac{K_\delta}{c_A}(w_2 - w_1)\right)\right)}{1 + \delta^n K_\delta}\right) K_\delta} = \delta^n \end{aligned}$$

Since $p = (x_0, y_0) \in V(\varepsilon, \widetilde{\delta}) \setminus \{y = 0\}$, either $x_0 > w_2 - \varepsilon$ or $x_0 \le w_2 - \varepsilon$. In the first case, we already have that $p \in U_2$, and the claim is proved in this case. Now, we end the proof in the latter. From the fact that $y_j \le \delta$ for $j \le L(y_0)$ and that $x_i \ge w_2 - \varepsilon$ for any $i \ge L(y_0)$, there exists some $1 \le \ell_p < L(y_0)$ such that $x_\ell \le w_2 - \varepsilon$ and $y_\ell \le \delta$ for $\ell < \ell_p$ and $x_{\ell_p} > w_2 - \varepsilon$, that is, a first iteration in which $x_\ell > w_2 - \varepsilon$. On the other hand, from the hypothesis (2-δ) and the fact that $x_{\ell_p - 1} \le w_2 - \varepsilon$, we have also that $x_{\ell_p} \le x_{\ell_p-1} + C_A y^n_{\ell_p-1} \le x_{\ell_p-1} + C_A\delta^n < x_{\ell_p-1} + 2\varepsilon \le w_2 - \varepsilon + 2\varepsilon = w_2 + \varepsilon$. ■

□

As a consequence of the above result and using the definition of the saturation of a set, notice that $\widetilde{V}_\Gamma \subset \mathrm{Sat}_{V_\Gamma}(U_2) \cup \Gamma$.

Corollary 2.18. *Let F be a diffeomorphism, Γ a non-dicritical regular arc and V_Γ a monotonic domain. Then for any $p \in \Gamma$ and any neighborhood U of p, the set $\mathrm{Sat}_{V_\Gamma}(U) \cup \Gamma$ is a neighborhood of Γ.*

Proof. Let p_1 and p_2 be the extreme points of Γ. We define two regular arcs: Γ_1 from p_1 to p and Γ_2 from p to p_2. We apply Proposition 2.17 to Γ_1, the neighborhood U of p and the diffeomorphism F, obtaining a neighborhood $\widetilde{V}_{\Gamma_1}$ of Γ_1 for which $\widetilde{V}_{\Gamma_1} \setminus \Gamma_1 \subset \mathrm{Sat}_{V_\Gamma}(U)$. On the other hand, we apply Proposition 2.17 to Γ_2 and F^{-1}, obtaining a neighborhood $\widetilde{V}_{\Gamma_2}$ of Γ_2 for which $\widetilde{V}_{\Gamma_1} \setminus \Gamma_2 \subset \mathrm{Sat}_{V_\Gamma}(U)$. Hence $\tilde{V}_{\Gamma_1} \cup \tilde{V}_{\Gamma_2} \subset \mathrm{Sat}_{V_\Gamma}(U) \cup \Gamma$ is a neighborhood of Γ. □

2.3.2 Dynamics over regular dicritical arcs

We start defining the regular dicritical arcs.

Definition 2.19. *A* regular dicritial arc Γ *(for F) is a connected compact subset of $E \setminus \mathfrak{S}(F,\pi)$ contained in a dicritical component of the divisor for ξ.*

Let $E_i \subset E$ be the dicritical component of E containing a regular dicritical arc Γ. Recall that there is a neighborhood V_Γ in which F has expression (2.5) in coordinates (x,y) centered at some point of the dicritical arc. We prove the following result.

Proposition 2.20. *Let F be a diffeomorphism, Γ a regular dicritical arc and V_Γ a monotonic domain for F. There is some neighborhood $\widetilde{V}_\Gamma$ of Γ such that*

(a) For any $p \in \widetilde{V}_\Gamma \setminus E$ we have that $\mathrm{Orb}_{V_\Gamma}(p)$ is an infinite set.

(b) For any $p \in \widetilde{V}_\Gamma \setminus E$ the set $\omega(p)$ is a single point of E.

Proof. We start recalling the expression of F in V_Γ

$$F(x,y) = (x + y^n A(x,y), y + y^n B(x,y)),$$

with $A,B \in \mathbb{R}\{x,y\}$, $n,k \ge 1$, $y \ge 0$ and $B(0,0) \ne 0$. Consider the extreme points of Γ with coordinates $p_1 = (w_1, 0)$ and $p_2 = (w_2, 0)$ with $w_1 \le w_2$. Assume also that V_Γ contains a box $V(\varepsilon,\delta) =$

$(w_1-\varepsilon, w_2+\varepsilon)\times[0,\delta)$ for small enough $1>\delta>0$, $\varepsilon>0$. Assume as in the previous paragraph that if $p_0=(x_0,y_0), F(p),\dots,F^j(p)\in V(\varepsilon,\delta)$, we denote $F^j(p)=(x_j,y_j)$.

We start finding a lower bound for $\frac{1}{y_1^\tau}-\frac{1}{y_0^\tau}$, for some τ satisfying $n-\frac{1}{2}>\tau\geq n-1$, $\tau>0$ and inside $V(\varepsilon,\delta)$ with δ sufficiently small. Using the bound $B(x_0,y_0)\leq -c_B$ inside V_Γ in (2.5), we find that

$$\frac{1}{y_1^\tau}-\frac{1}{y_0^\tau}=\frac{1}{y_0^\tau}\left(\frac{1}{(1+y_0^{n-1}B(x_0,y_0))^\tau}-1\right)\geq\frac{1}{y_0^\tau}\left(\frac{1}{(1-y_0^{n-1}c_B)^\tau}-1\right)\geq c_B\tau y_0^{n-1-\tau}+O(y_0^{2n-2-\tau})\geq\widetilde{C}y_0^{n-1-\tau}, \tag{2.16}$$

for some $\tilde{C}>0$ (considering again that δ is small enough). Notice that $n-1-\tau\leq 0$.

Now, we fix τ,μ and K satisfying:

- $n-\frac{1}{2}>\tau\geq n-1,\ \tau>0$.
- $\mu=\frac{n}{n-\frac{1}{2}}$.
- $K>\frac{c_A}{\widetilde{C}}$

From those constants, we define $\widetilde{\delta}$ fulfilling the following condition

$$\widetilde{\delta}^{\mu\tau-n+1}<\frac{\varepsilon}{2K}.$$

We prove the following claim.

Claim. For every $p=(x_0,y_0)\in V(\frac{\varepsilon}{2},\widetilde{\delta})$ we have

a) $F^\ell(p)\in V(\varepsilon,\delta)$ for every $\ell\in\mathbb{N}$.

b) There is some $q\in\{y=0\}\cap\overline{V(\varepsilon,\delta)}$ such that $q=\omega(p)$.

As a consequence of this claim, we can choose $\widetilde{V_\Gamma}=V(\frac{\varepsilon}{2},\widetilde{\delta})$ which fulfills the thesis of this Proposition.

Proof of the Claim. We start proving a). We bound $|x_\ell-x_0|$ inside V_Γ. We consider that $x_\ell-x_0=\sum_{j=0}^{\ell-1}y_j^nA(x_j,y_j)$ and also that $y^n\leq y^{\tau\mu}$ since $\tau\mu<n-\frac{1}{2}\left(\frac{n}{n-\frac{1}{2}}\right)=n$.

$$\begin{aligned}|x_\ell-x_0|\leq&\sum_{j=0}^{\ell-1}C_Ay_j^n\leq\sum_{j=0}^{\ell-1}C_A(y_j^\tau)^\mu\leq C_A\sum_{j=0}^{\ell-1}\frac{y_0^{\tau\mu}}{(1+j\widetilde{C}y_0^{n-1})^\mu}\leq C_A\int_0^\ell\frac{y_0^{\tau\mu}}{(1+s\widetilde{C}y_0^{n-1})^\mu}ds\\ \leq&C_A\int_0^\infty\frac{y_0^{\tau\mu}}{(1+s\widetilde{C}y_0^{n-1})^\mu}ds=C_A(y_0^{\mu\tau-n+1}\frac{1}{\widetilde{C}(\mu-1)}+y_0^{\mu\tau})\leq Ky_0^{\mu\tau-n+1},\end{aligned}$$

for K defined as above and where we have used the bounds for A given in (2.5) on the first inequality, the above consideration on the second, the inequality (2.16) on the third, the integral criteria of series with decreasing terms on the fourth inequality. The fifth inequality stands since the function is strictly positive and the last inequality stands for δ small enough. Consider also that $\mu\tau - n + 1 > 0$.

Taking any point, the existence of its first iterate in V_Γ is granted since y is decreasing on the orbits and $|x_1 - x_0| \leq K\widetilde{\delta}^{\mu\tau-n+1} < \frac{\varepsilon}{2}$. The existence of further iterates is also granted by induction, since given the existence of $(x_1, y_1), \ldots, (x_{\ell-1}, y_{\ell-1})$, we have $0 < y_\ell < y_0$ and $x_0 - \frac{\varepsilon}{2} \leq x_\ell \leq x_0 + \frac{\varepsilon}{2}$.

We prove now b). Notice that the positive orbit of any $p \in V(\frac{\varepsilon}{2}, \widetilde{\delta})$ is confined in $V(\varepsilon, \widetilde{\delta})$ and hence has accumulation points in $\overline{V(\varepsilon, \widetilde{\delta})}$. In particular, the accumulation points of $\mathrm{Orb}^+_{V(\varepsilon,\delta)}(p)$ must lie in the set of fixed points, namely $\{y = 0\}$. To see that it is a unique point, we prove that given $p_0 = (x_0, y_0)$, the sequence $\{x_\ell\}_{\ell\in\mathbb{N}}$ converges to some x_∞, by proving that it is a Cauchy sequence. To prove this, it is enough to use that the sequence $\{y_\ell\}_{\ell\in\mathbb{N}}$ converges to zero and that $|x_n - x_{n-1}| \leq K y_n^{\mu\tau-n+1}$. Then, $\omega(p) = (x_\infty, 0)$. ■

□

2.3.3 Dynamics on saddle quadrants

In this section, we deduce the dynamics on realizations of saddle quadrants in saddle monotonic domains. The main result resembles Proposition 2.17, in the sense that, given a point in the divisor and a neighborhood of it, we can find a whole open set that has the property in the statement of 2.17. Before stating the result, let (Q, W) be a realization of Q and (x, y) be adapted coordinates at the saddle point p_* defined on W. Suppose that the curve Γ' is oriented towards the vertex and Γ oriented outwards. We denote this by $\Gamma' \leq \Gamma$. Then, the diffeomorphism can be written as in (2.10) or (2.9). Both cases can be summarized as

$$F(x,y) = (x + x^{n+s}y^m A(x,y), y + x^n y^{m+r} B(x,y)), \quad x \geq 0,\ y \geq 0, \tag{2.17}$$

with $m \geq 1$, $n \geq 0$, $s, r \geq 1$, $1 \in \{s, r\}$ and $A(x,y), B(x,y) \in \mathcal{C}^\infty(W)$ or $A(x,y), B(x,y) \in \mathbb{R}\{x,y\}$. In addition, $\Gamma = \{y = 0\}$, $\Gamma' = \{x = 0\}$ and there exist bounds $0 < c_A \leq A(x,y) \leq C_A$, $-C_B \leq B(x,y) \leq -c_B < 0$ inside W.

Proposition 2.21. *Let (Q, W) be a realization of a saddle quadrant on a saddle monotonic domain W. Then, for every $p_2 \in \Gamma$ and every neighborhood U_2 of p_2 there is another saddle monotonic domain $\widetilde{W}$ of p_*, such that, for every $p \in \widetilde{W} \setminus (\Gamma \cup \Gamma')$ we have $\mathrm{Orb}^+_{\widetilde{W}}(p) \cap U_2 \neq \emptyset$. In addition, $\mathrm{Orb}^+_{\widetilde{W}}(p)$ is a finite set and there is a non-empty subset $W^-_{esc} \subset W$ in which $\mathrm{Orb}^-_W(q) = \{q\}$ for $q \in W^-_{esc}$, with the property that $W \setminus W^-_{esc}$ is a neighborhood of $E \cap W$.*

The set W^-_{esc} is called the *negative escaping region* of F in W and it contains the points whose inverse image cannot be defined in W. Similar, we call positive escaping region W^+_{esc} of F in W as the negative escaping region of F^{-1}.

Proof. For the proof, we can assume that F is written as (2.17) with $p_2 = (w_2, 0)$ and that U_2 contains a box neighborhood $(w_2-\varepsilon, w_2+\varepsilon)\times[0,\delta)$ of p_2 for some $\varepsilon, \delta > 0$. The coordinate functions y and x are monotonic in the orbits inside $W \setminus (\Gamma \cup \{p_*\} \cup \Gamma')$, being y decreasing and x increasing. That is, denoting $(x_j, y_j) = F((x_{j-1}, y_{j-1}))$ for some $(x_0, y_0) \in W \setminus (\Gamma \cup \{p_*\} \cup \Gamma')$ such that $F^i(x_0, y_0) \in W$ for every i such that $0 \le i \le j$, we have that $x_j > x_{j-1}$ and $y_j < y_{j-1}$. We look for a neighborhood $\widetilde{W} = [0, w_2+\varepsilon)\times[0,\widetilde{\delta}) \cup (w_2-\varepsilon, w_2+\varepsilon)\times[0,\delta)$ of $p_* = (0,0)$, for some $\widetilde{\delta}$ with $\delta > \widetilde{\delta} > 0$.

First, we find a bound for the amount $x_1 - x_0$ for any point $(x_0, y_0) \in W \cap \{x \le w_2 - \varepsilon\}$, using the upper bound of the function A.

$$x_1 - x_0 \le x_0^{n+s} y_0^m C_A \le (w_2-\varepsilon)^{n+s} y_0^m C_A. \tag{2.18}$$

Imposing $y_0 \le \widetilde{\delta}$, with $\widetilde{\delta}^m \le \frac{2\varepsilon}{C_A (w_2-\varepsilon)^{n+s}}$, we find that $x_1 - x_0 \le 2\varepsilon$. Together with the fact that the coordinate y is decreasing on the orbits on W, this proves that for any point $(x_0, y_0) \in ([0, w_2 + \varepsilon) \times [0, \widetilde{\delta})) \cap \{x \le w_2 - \varepsilon\}$ such that $(x_1, y_1) \notin ([0, w_2+\varepsilon)\times[0,\widetilde{\delta})) \cap \{x \le w_2 - \varepsilon\}$, we necessarily have $(x_1, y_1) \in \widetilde{W} = [0, w_2+\varepsilon)\times[0,\widetilde{\delta})$. In other words, any orbit starting at $\widetilde{W}$ cannot "jump horizontally" the set U_2.

Now, we check that indeed every point p in $\widetilde{W}\setminus(\Gamma\cup\{p_*\}\cup\Gamma')$ fulfills that there is some $\ell_p \in \mathbb{Z}_{\ge 0}$ such that $F^{\ell_p}(p) \in U_2$. Suppose that $x(p) \le w_2 - \varepsilon$, since otherwise the point p belongs already to U_2. Suppose by contradiction that there is not such an iterate. Then, since y is decreasing and the upper bound $x_1 - x_0 \le 2\varepsilon$ we found above, the positive orbit $\mathrm{Orb}^+_W(p)$ of p must have accumulation points in the closure $\overline{\widetilde{W} \cap \{x \le w_2 - \varepsilon\}}$. By the increasing x condition, we find that no point in $\Gamma' \cup \{p_*\}$ can be an accumulation point for $\mathrm{Orb}^+_W(p)$, since $0 < x_0 < x_j$ for every $j > 0$. On the other hand, we find that no point q in Γ can be an accumulation point since either Γ is part of a non dicritical component of the divisor, a curve of fixed points or it is part of a parabolic curve. In the first two cases there is some regular non-dicritical arc inside Γ that contains q and this regular arc fulfills Proposition 2.17, which implies that q is not an accumulation point. In the second case, q is not a fixed point, and hence it can neither be an accumulation point. The rest of the points can neither be accumulation points since they are not fixed. Hence, we find a contradiction and conclude that the orbit $\mathrm{Orb}^+_{\widetilde{W}\cap\{x\le w_2-\varepsilon\}}(p)$ is finite. Then, let $(x_{\ell-1}, y_{\ell-1})$ be the last iterate of $p = (x_0, y_0)$ in the region $\widetilde{W} \cap \{x \le w_2 - \varepsilon\}$. Its image in $\widetilde{W}$ lies in the region $\widetilde{W} \cap \{x > w_2 - \varepsilon, y < y_{\ell-1}\}$ because of the bound in (2.18) and the fact that y decreases on the orbits in $\widetilde{W}$. Finally, since this region belongs to $(w_2-\varepsilon, w_2+\varepsilon)\times[0,\delta) \subset U_2$, we conclude that p has an iterate in U_2, as we wanted to prove.

With the same arguments, we can also prove that $\mathrm{Orb}^+_W(p)$ is finite, since W is relatively compact in M and there cannot be accumulation points of $\mathrm{Orb}^+_{\widetilde{W}}(p)$ in the closure of W. This implies that for every point $p \in W$, there exists $-m_p \in \mathbb{Z}_{\le 0}$ such that $F^{-m_p}(p) \in W^-_{esc}$ defined in the statement, which implies that W^-_{esc} is non-empty. Finally, to see that $W \setminus W^-_{esc}$ is a neighborhood of $E \cap W$, it is enough to consider that $E \cap W \subset \mathrm{Fix}(F)$. By continuity, there is a neighborhood of $E \cap W$ whose image is contained in the open set W. □

Notice that the above result implies that $\widetilde{W} \subset \mathrm{Sat}_W(U_2)$, and hence $\mathrm{Sat}_W(U_2)$ is a neighborhood of p_*.

Corollary 2.22. *For any $p_1 \in W \cap E$ and any neighborhood $U_1 \subset W$ of p_1, the set $Sat_W(U_1) \cup (\Gamma \cup \{p_*\} \cup \Gamma')$ is a neighborhood of $W \cap E$ in W.*

Proof. It is enough to apply Proposition 2.21 to the point p_1 and U_1 either for F if $p_1 \in \Gamma$ or F^{-1} if $p_1 \in \Gamma'$. Hence, we obtain a neighborhood $\widetilde{W_1}$ of p_* that is contained in $\mathrm{Sat}_W(U_1)$ and that it is a neighborhood of $\tilde{W}_1 \cap E$. To see that the saturation $\mathrm{Sat}_W(U_1)$ is a neighborhood of the rest of the points of E, we take any point $q \in E \cap \widetilde{W_1} \subset \mathrm{Sat}_W(U_1) \cup (\Gamma \cup \{p_*\} \cup \Gamma')$ and any neighborhood $U_q \subset \widetilde{W_1}$. Suppose that $\Gamma \subset E$ and that $q \in \Gamma \subset E$. Then for any other point, $q' \in E$, we define the regular non-dicritical arc $\Gamma_{q,q'}$ and apply Corollary 2.18. We find $\mathrm{Sat}_W(U_q) \cup \Gamma \subset \mathrm{Sat}_W(U_1) \cup (\Gamma \cup \{p_*\} \cup \Gamma')$ is a neighborhood of q'. We do the same process when $\Gamma' \subset E$, choosing $q \in \Gamma'$ so that $\mathrm{Sat}_W(U_1) \cup (\Gamma \cup \{p_*\} \cup \Gamma')$ is also a neighborhood of any other point in Γ'.

□

2.3.4 Dynamics over paths of divisor saddle quadrants

Now, we prove another result on transition. Let $(Q_2, \dots, Q_{s-1})$ be a path of quadrants (cf. Definition 1.39) with $Q_2 = (v_2, e_1, e_2)$ and $Q_i = (v_i, e_{i-1}, e_i)$ for $i = 2, \dots, s$. Let $p_2, \dots, p_{s-1}$ be strict fixed points corresponding to the vertices $v_2, \dots, v_{s-1}$. We say that $\mathbf{Q} = (Q_2, \dots, Q_{s-1})$ is a *path of divisor saddle quadrants* if the following are fulfilled

- Each Q_i is a saddle quadrant.
- The edges $e_i \in \mathcal{E}_{div}$ for every $i = 1, \dots s-1$.

We denote $\gamma_e = \mathrm{Supp}(e)$ as in Section 2.2.3. In particular for the path of divisor saddle quadrants, we have $\gamma_{e_i} \subset E$ for each $i = 1, \dots, s-1$. Notice that the path of divisor saddle quadrants can always be extended to other path of quadrants. This is because the edges e_1 and e_{s-1} are divisor edges adjacent to some other vertex in $\mathcal{V}_{div}$ say v_0 for e_1 and v_s for e_{s-1}. We take the path $(Q_1, Q_2, \dots, Q_{s-1}, Q_s)$ with $Q_1 = (v_1, e_0, e_1)$ and $Q_s = (v_1, e_{s-1}, e_s)$, where $e_0, e_s \notin \{e_1, \dots, e_{s-1}\}$.

Proposition 2.23. *Let $(Q_2, \dots, Q_{s-1})$ be a path of divisor saddle quadrants, $E_{\mathbf{Q}} = \gamma_{e_1} \cup \{p_2\} \cup \dots \cup \{p_{s-1}\} \cup \gamma_{e_{s-1}} \subset E$ the corresponding components of the divisor associated to this path. For any neighborhood V of $E_{\mathbf{Q}}$, there is other neighborhood $U_{\mathbf{Q}}$ of $E_{\mathbf{Q}}$ with the following properties. Take any point $t_{s-1} \in \gamma_{e_{s-1}}$, and other point $t_1 \in \gamma_{e_1}$ and any neighborhood U'_{s-1} of t_{s-1}, we have*

1. *(Transition) There is a neighborhood U_1 of t_1 such that for every $p \in U_1 \setminus E$ there is some $q_p \in U'_{s-1}$ such that $q_p = F^{\ell_p}(p)$ and $F^i(p) \in U_{\mathbf{Q}}$ for $0 \le i \le \ell_p$.*

2. *For any neighborhood U_1 of t_1, the set $\mathrm{Sat}_{V_{\mathbf{Q}}}(U_1) \cup E_{\mathbf{Q}}$ is a neighborhood of $E_{\mathbf{Q}}$.*

Proof.

$$U_{\mathbf{Q}} = \left(\bigcup_{i=1}^{s} W_i\right) \cup \left(\bigcup_{i=1}^{s-1} V_{\gamma_i}\right) \cap V$$

where

- $W_1, \dots, W_s$ are mutually disjoint and defined by
  - $(Q_2, W_2), \dots, (Q_{s-1}, W_{s-1})$ are realizations of saddle quadrants on mutually disjoint convex saddle monotonic domains $W_2, \dots, W_{s-1}$ with the property that they are contained in V. We also impose that they are mutually disjoint
  - When p_1, or respectively p_s, are not of type n-n-2, we will as well take a realization $(Q_1, \widetilde{W_1})$ of the quadrant Q_1 (resp. a realization $(Q_s, \widetilde{W_s})$ of Q_s) in the corresponding saddle, node or dicritical monotonic domain. We will also suppose that each W_i is convex. In case the strict fixed points p_1 (resp. p_s) is of type n-n, we take a joint realization $(Q_1, Q, \widetilde{W_1})$ of Q_1 and the other quadrant Q having the vertex v_1 (resp. a joint realization $(Q_s, Q, \widetilde{W_s})$ of Q_s and the other quadrant Q having v_s), but in order to unify the notation on the collection of cases, we will simply denote these by $(Q_1, \widetilde{W_1})$ (resp. $(Q_s, \widetilde{W_s})$). Then, we intersect them with the domain of work, that is, $W_i = \widetilde{W_i} \cap V$ for $i = 1, s$.

- Let $\Gamma_i = \gamma_{e_i} \cap W_i$ and $\Gamma'_i = \gamma_{e_i} \cap W_{i+1}$ for each $i = 1, \dots, s-1$. We select points $q_1 \in \Gamma_1$, $q'_1 \in \Gamma'_1, \dots, q'_{s-1} \in \Gamma_{s-1}$. As we can take q_1 as close to p_1 as desired, we will take q_1 so that q_1 lies between p_1 and t_1. Symmetrically, we take q'_{s-1} so that q'_{s-1} lies between p_s and t_{s-1}. Notice that every couple of points q_i, q'_i lie on the same component γ_{e_i} of the divisor, for $i = 1, \dots, s-1$, but $q_i \in W_i$ and $q'_i \in W_{i+1}$. We define the non-dicritical arcs $\gamma_i \subset \gamma_{e_i}$ limited by $q_i \le q'_i$, and consider mutually disjoint monotonic domains $V_{\gamma_i} \subset V$, defined as in section 2.3.1.

The construction of $U_{\mathbf{Q}}$ is sketched in Figure 2.6 for an example in which the path of divisor saddles is given by (Q_2, Q_3) and an extension is given by (Q_1, Q_2, Q_3, Q_4), where Q_1 is a saddle quadrant, Q_s is a node quadrant and p_s is a n-n-2 point.

We prove the first part of this result by recursively applying Propositions 2.17 and 2.21. First, take the dicritical arc β_{s-1} between q'_{s-1} and t'_{s-1} and a monotonic domain $V_{\beta_{s-1}} \subset V_{\gamma_{s-1}}$. We apply Proposition 2.17 to this arc, obtaining $\widetilde{V}_{\beta_{s-1}}$. We choose $U_{s-1} \subset W_{s-1} \cap \widetilde{V}_{\beta_{s-1}}$. Now, we apply the following steps.

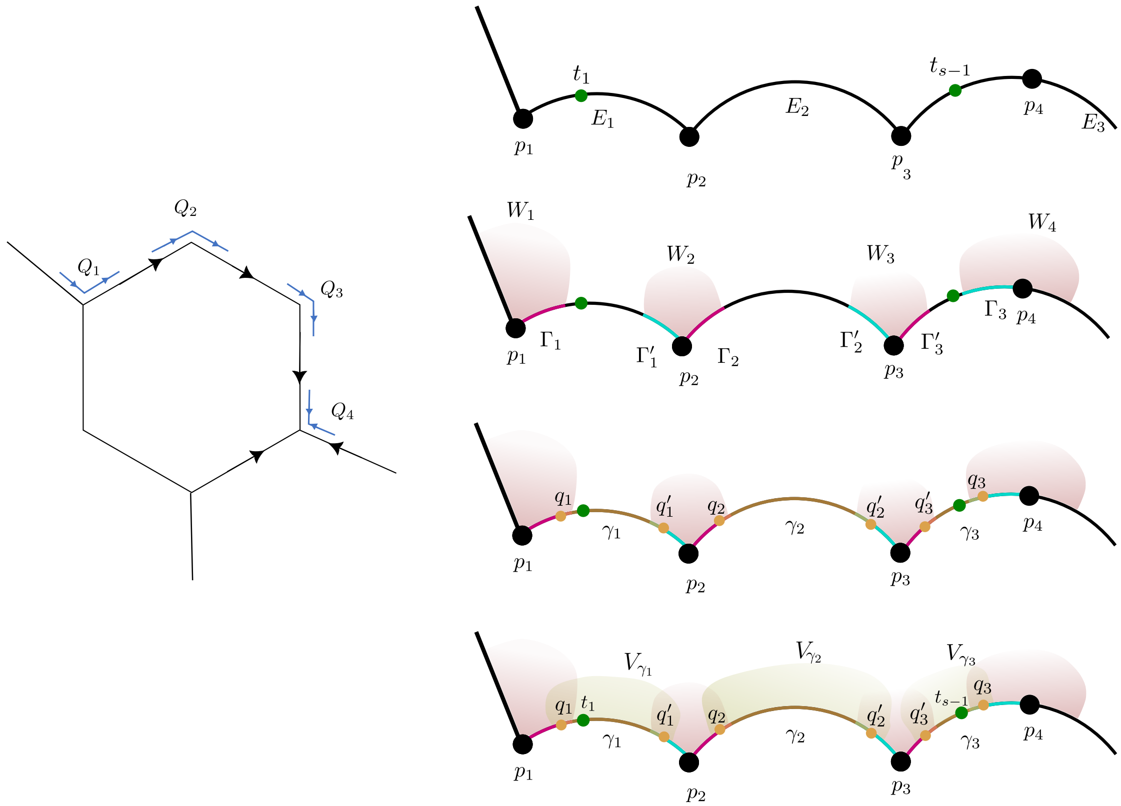

Figure 2.6: An example of the construction of the set $U_{\mathbf{Q}}$ for a path of quadrants $\mathbf{Q}$

Step i.b Consider the saddle monotonic domain W_i and take from the previous step $U_i \subset W_i \cap \widetilde{V}_{\gamma_i}$ a neighborhood of q_i, or U_{s-1} obtained as before it this is the first step. We apply Corollary 2.22 to these elements, obtaining $\widetilde{W_i} = \mathrm{Sat}_{W_i}(U_i)$ with the property that every point in $\widetilde{W_i}$ has an iterate in U_i and $\widetilde{W_i}$ is a neighborhood of $\Gamma_{i-1}' \cup \{p_*\} \cup \Gamma_i \subset E$. We take a neighborhood $U_{i-1}' \subset V_{\gamma_{i-1}} \cap \widetilde{W_i}$ of the point q_{i-1}'.

Step i.a Consider the non-dicritical arc γ_i and the neighborhood $U_i' \subset \widetilde{W}_{i+1} \cap V_{\gamma_i}$ of q_i' defined on the previous step. We apply Proposition 2.17 to these elements, obtaining $\widetilde{V}_{\gamma_i}$ fulfilling the property that every point in $\widetilde{V}_{\gamma_i}$ has an iterate in U_i'. In particular, $\widetilde{V}_{\gamma_i} \subset \mathrm{Sat}_{V_{\gamma_i}}(U_i') \cup V_{\gamma_i}$. We take a neighborhood $U_i \subset \widetilde{V}_{\gamma_i} \cap W_i$ of q_i.

We apply steps (s-1).b, (s-2).a, (s-2).b …, 2.b. We obtain a neighborhood U_1' of q_1'. Defining the regular non-dicritical arc β_1 between t_1 and q_1' and a monotonic domain $V_{\beta_1} \subset V_{\gamma_1}$, we apply Proposition 2.17 to $U_1' \cap V_{\beta_1}$, obtaining $\widetilde{V}_{\beta_1} \subset V_{\beta_1}$ with the property that $V_{\beta_1} \subset \mathrm{Sat}_{V_{\beta_1}}(U_1')$. Then, we claim that any neighborhood $U_1 \subset \widetilde{V}_{\beta_1}$ of t_1 fulfills the first statement. Let $p \in U_1 \setminus E$. Since $U_1 \subset \widetilde{V}_{\beta_1}$ and $\widetilde{V}_{\beta_1}$ is obtained from Proposition 2.17, there is $\ell_{1,a}$ such that $F^{\ell_{1,a}}(p) \in U_1'$ and the intermediate iterates remain in $V_{\beta_1} \subset V_{\gamma_1} \subset U_{\mathbf{Q}}$. Then, by Proposition 2.21, there is some $\ell_{2,b}$ such that $F^{\ell_{2,b}}(F^{\ell_{1,a}}(p)) \in U_2$ and the intermediate iterates lie in $W_2 \subset U_{\mathbf{Q}}$. Following this reasoning, we find that $\ell_p \in \mathbb{N}$ such that $F^{\ell_p}(p) \in U'_{s-1}$ and the intermediate iterates lie in $U_{\mathbf{Q}}$. It is given by $\ell_p = \ell_{1,a} + \ell_{2,b} + \ell_{2,a} + \cdots + \ell_{s-1,a}$. Applying Proposition 2.17, we have that the orbit of $F^{\ell_p}(p)$ either abandons $V_{\gamma_{s-1}}$ or reaches W_s.

We end proving that $\mathrm{Sat}_{U_{\mathbf{Q}}}(U_1) \cup (E_{\mathbf{Q}})$ is a neighborhood of $E_{\mathbf{Q}}$ for any neighborhood U_1 of t_1 (notice that the points p_1, p_s are excluded from $E_{\mathbf{Q}}$). It is enough to apply Corollaries 2.18 and 2.22 at each step. First, take any point q''_1 in γ_{e_1} and define the dicritical arc γ''_1 between t_1 and q''_1. Corollary 2.18 implies that $\mathrm{Sat}_{U_{\mathbf{Q}}}(U_1) \cup \gamma_{e_1}$ is a neighborhood of q''_1. Now take the neighborhood $\mathrm{Sat}_{U_{\mathbf{Q}}}(U_1) \cap U'_1$, its saturation $\mathrm{Sat}_{U_{\mathbf{Q}}}(\mathrm{Sat}_{U_{\mathbf{Q}}}(U_1) \cap U'_1) \subset \mathrm{Sat}_{U_{\mathbf{Q}}}(U_1)$ when adjoining $\Gamma'_1 \cup \Gamma_2$, it is a neighborhood of p_2 and the segments Γ'_1 and Γ_2 by Corollary 2.22, hence so is $\mathrm{Sat}_{U_{\mathbf{Q}}}(U_1)$. We apply Corollaries 2.18 and 2.22 to each $\mathrm{Sat}_{U_{\mathbf{Q}}}(U_1) \cap U_i$ and $\mathrm{Sat}_{U_{\mathbf{Q}}}(U_1) \cap U'_i$, concluding finally that $\mathrm{Sat}_{U_{\mathbf{Q}}}(U_1)$ is a neighborhood of $\gamma_{e_1} \cup \cdots \cup \gamma_{e_{s-2}} \cup \Gamma_{s-1} \cup \gamma_{s-1}$. We end by proving that $\mathrm{Sat}_{U_{\mathbf{Q}}}(U_1) \cup \gamma_{e_{s-1}}$ is a neighborhood of $\gamma_{e_{s-1}}$. It is enough to use, as before, Corollary 2.18 in the non-dicritical arc defined between t_{s-1} and any other point q''_{s-1} to prove that $\mathrm{Sat}_{U_{\mathbf{Q}}}(U_1) \cup \gamma_{e_{s-1}}$ is a neighborhood of q''_{s-1} and conclude that $\mathrm{Sat}_{U_{\mathbf{Q}}}(U_1)$ is a neighborhood of $E_{\mathbf{Q}}$.

□

2.3.5 Dynamics on node quadrants

In this section, we will introduce results on the dynamics on node monotonic domains. The node monotonic domains can be realization of a single quadrant or a joint realization of two node quadrants adjacent to the same vertex, depending on the type of point that p is. We will also prove that in general we cannot obtain a full neighborhood of p_* by saturating arbitrary neighborhoods of arbitrary points of the divisor, as we did before for saddles. We will distinguish two cases: strict fixed points of type f-n-n, f-n-s, n-s, n-n-1 and n, which have a node quadrant and for which realizations of the node quadrants can be defined (as quadrants in the geometric sense), and strict fixed points of type n-n-2, which have two node quadrants but only joint realizations exist (as half spaces in the geometric sense).

Quadrants

Suppose that p_* is a strict fixed point Let (Q, W) be a realization of Q with $(W, (x, y))$ a node monotonic domain at p_*. Let $\Gamma' = \{x = 0, y > 0\}, \Gamma = \{y = 0, x > 0\}$. Then, the diffeomorphism is expressed as in (2.9) or as in (2.10) in a node monotonic domain, in which both coordinate functions are decreasing on the orbits inside $W \setminus (\Gamma \cup \{p_*\} \cup \Gamma')$. Both expressions are summarized in

$$F(x, y) = (x + x^{n+s} y^m A(x, y), y + x^n y^{m+r} B(x, y)), \tag{2.19}$$

with $A,B \in \mathbb{R}\{x,y\}$ or $A,B \in \mathcal{C}^\infty(W)$, $m,r,s \geq 1, n \geq 0$, $1 \in \{s,r\}$ and $A(0,0) < 0$, $B(0,0) < 0$. Notice also that $\mathrm{Supp}(G(\xi,\pi)) \cap W = \Gamma \cup \{p_*\} \cup \Gamma'$.

Proposition 2.24. *Suppose for instance that $\Gamma \subset E$. For every $p \in \Gamma$ and any neighborhood U of p, the set $\mathrm{Sat}_W(U) \cup (\Gamma \setminus \{p_*\})$ is a neighborhood of $(\Gamma \setminus \{p_*\})$.*

Proof. Recall that the expression of F is given by (2.19), that $\Gamma = \{y = 0\}$ and $\Gamma' = \{x = 0\}$ and that we can assume there exist bounds $-C_A \leq A(x,y) \leq -c_A < 0$, $-C_B \leq B(x,y) \leq -c_B < 0$ inside W.

Take any $p \in \Gamma$, and any neighborhood U of p. To see that $\mathrm{Sat}_W(U) \cup (\Gamma)$ is a neighborhood of any $q \in \Gamma$, we directly apply Corollary 2.18 for the regular non-dicritical arc B between p,q. Noticing that $W \cap \{x > \mu\}$ with $\mu < x(p), x(q)$ is a monotonic domain for B, we apply the result for either F or F^{-1}, depending on $q \leq p$ or $p \geq p$ and the neighborhood U of p, obtaining $\tilde{V} \subset \mathrm{Sat}_W(U) \cup (\Gamma \setminus \{p_*\})$, which is a neighborhood of q. □

Using only arguments of monotony, we prove that any box type neighborhood of a point inside one component Γ or Γ' cannot fill in a full neighborhood of the corner p_*.

Proposition 2.25. *With the same notations as above, let $p = (w,0) \in \Gamma = \{y = 0\}$ and let $U = (w-\varepsilon, w+\varepsilon) \times [0,\delta)$ for some $0 < \varepsilon < w$ and some $\delta > 0$. Then, $\mathrm{Sat}_W(U) \cup \Gamma \cup \{p_*\} \cup \Gamma'$ is not a full neighborhood of p_*.*

Proof. Take any point $p' = (0,w') \in \Gamma'$ with $w' > \delta$ and a box neighborhood $U' = [0,\varepsilon') \times (w-\delta', w+\delta')$ with $0 < \delta' < w' - \delta$ and $\varepsilon' < w - \varepsilon$. Using Proposition 2.24, we get that $\mathrm{Sat}_W(U')$ is a neighborhood of Γ'. Suppose by contradiction that $\mathrm{Sat}_W(U) \cup \Gamma \cup \{p_*\} \cup \Gamma'$ is a neighborhood of p_*. Hence, there is a point $q \in \mathrm{Sat}_W(U') \cap \mathrm{Sat}_W(U)$. Then, there is some $\ell > 0$ such that $F^{-\ell}(q) \in U$ and there is some $\ell' > 0$ such that $F^{-\ell'}(q) \in U'$. Hence, $x(F^{-\ell}(q)) > w - \varepsilon > \varepsilon'$ and $y(F^{-\ell}(q)) < \delta < w - \delta'$ and $x(F^{-\ell'}(q)) < \varepsilon' < w - \varepsilon$ and $y(F^{-\ell'}(q)) > w' - \delta' > \delta$. Supposing that $\ell' > \ell$, we find a contradiction with the fact that x is decreasing on the orbits of F. Supposing that $\ell > \ell'$, we find a contradiction with the fact that y is decreasing in the orbits of F. □

In the following remark, we give some comments on the form that the saturations of box neighborhoods can take.

Remark 2.26 (On estimations on the saturation of box neighborhoods). Here, we just want to remark in which regions the saturation of box neighborhoods of points of the divisor lie. That is, sets of the form $U = (w-\varepsilon, w+\varepsilon) \times [0,\delta)$ as in the previous result. We can assume that F is written as in (2.9) or as in (2.10) with $\Gamma = \{y = 0\}$, $\Gamma' = \{x = 0\}$ and that $-C_A \leq A(x,y) \leq -c_A < 0$, $-C_B \leq B(x,y) \leq -c_B < 0$ inside W. We suppose first that $r = 1$ and $s \geq 1$. Let $p = (x_0, y_0) \in W$ and denote (x_j, y_j) the coordinates of $F^j(p)$. We estimate the growth of y_j/x_j^k for some $k \in \mathbb{N} \cup (1/\mathbb{N}^*)$.

Using the bounds for the functions A and B, we find the following

$$\frac{y_1}{x_1^k}-\frac{y_0}{x_0^k}\le\frac{y_0}{x_0^k}\left(\frac{1-C_Bx_0^ny_0^m}{(1-c_Ax_0^{n+s-1}y_0^m)^k}-1\right)\le\frac{y_0}{x_0^k}x_0^ny_0^m(-C_B+kc_Ax^{s-1}+O(x_0^ny_0^m)).$$

Since $s\ge 1$, we can choose $k\in\mathbb{N}\cup(1/\mathbb{N})$ for which y/x^k is decreasing along the orbits. Take $\kappa=\sup_{p\in U}\left\{\frac{y(p)}{x_(p)^k}\right\}$ and any $k\in\mathbb{N}\cup(1/\mathbb{N}^*)$ so that the function y/x^k is decreasing on the orbits. We find that for every $p\in U$ we have $\mathrm{Orb}_W(p)\subset\{(x,y)\in W\setminus\Gamma' : y/x^k\le\kappa\}$. This gives another proof of Proposition 2.25, that is , $\mathrm{Sat}_W(U)\cup\Gamma\cup\{p_*\}\cup\Gamma'$ cannot be a full neighborhood of p_*, since every neighborhood of p_* has non-empty intersection with the set $\{(x,y)\in W\setminus\Gamma' : y/x^k\ge\kappa\}$. For a more careful analysis, we distinguish two cases. When $s>1$, the function y/x^k is decreasing on the orbits for any choice of k. In short words, this means that the coordinate y of the orbits in the saturation of U decreases exponentially with x (the neighborhood where monotonicity holds depends on k). On the other hand, when $s=1$, we can just assume that there is a choice of $k\in\mathbb{N}\cup(1/\mathbb{N}^*)$ such that y/x^k is decreasing on the orbits.

When $r>1$ and $s=1$, we find a new estimation

$$\frac{x_1}{y_1^k}-\frac{x_0}{y_0^k}\le\frac{x_0}{y_0^k}x_0^ny_0^m(-C_A+c_Bky_0^{r-1}+O(x_0^ny_0^m)),$$

which implies that for every $k\in\mathbb{N}\cup(1/\mathbb{N})$ the function x/y^k is decreasing in the orbits if y_0 is small enough depending on k. Using this estimation, we find that the saturation of U may approach exponentially the curve Γ'.

Half-spaces

Now, the diffeomorphism on a realization (Q_1,Q_2,W) of two adjacent node quadrants at $p\in E$ in a node monotonic domain W is expressed as in (2.1). We summarize that expression.

$$F(x,y)=(x+y^nxA(x,y)+y^{2n+2s-1}b(y),y+y^{n+s-1}B(x,y)),\tag{2.20}$$

with $A,B\in\mathbb{R}\{x,y\}$, $n,s\ge 1$ and $B(x,y)<0$. Let $\Gamma=E\cap\{x>0\}\cap W$ and $\Gamma'=E\cap\{x<0\}\cap W$. Notice that now, the curves Γ and Γ' are not transverse. Recall also, that there is a formal invariant curve γ transverse to E at p_*, in which the diffeomorphism is formally attracting.

Proposition 2.27. *Let (Q_1,Q_2,W) be a realization of two adjacent node quadrants at p_* on a node monotonic domain W. Then, for every $p\in\Gamma$ (respectively $p'\in\Gamma'$) and neighborhood U of p (U' of p') the set* $\mathrm{Sat}_W(U)\cup\Gamma$ *is a neighborhood of* Γ ($\mathrm{Sat}_W(U')\cup\Gamma'$ *is a neighborhood of* Γ'*).*

Proof. The proof of this result follows from the application of Corollary 2.18. We apply this result to every non-dicritical arc defined between each $q \in \Gamma$ (or $q \in \Gamma'$, correspondingly) with $q \neq p$ and p. □

2.3.6 Dynamics on dicritical quadrants

In this section, we will introduce the results concerning the dynamics on realizations of dicritical quadrants. We will prove that there are dicritical monotonic domains which are positively or negatively invariant for F. We will prove that the orbits inside such dicritical monotonic domains accumulate either for positive or negative iterates into the dicritical curve, and that no orbit accumulates into the corner point. We will also prove existence of parabolic curves on every point of the dicritical curve. We will assume that Q is an attracting dicritical quadrant, but the results can be applied also for F^{-1} in the case of repelling dicritical quadrants.

Let (Q,W) be a realization of Q and (x,y) the coordinates at p_* defined on the dicritical monotonic domain W. Suppose that Γ is a non-dicritical component of the divisor given by $\{y=0\}$ and that Γ' is a dicritical curve given by $\{x=0\}$, which can be a component of the divisor or not. The diffeomorphism is thus given by (2.6).

Proposition 2.28. *Let (Q,W) be a realization of a dicritical quadrant such that W is a dicritical monotonic domain, Γ is a non-dicritical component of the divisor and Γ' is an (attracting) diciritcal component of the divisor or a bidicritical curve. There exists two other dicritical monotonic domains $\widetilde{W_1} \subset \widetilde{W_2} \subset W$ such that*

1. *For every $p \in \widetilde{W_1} \setminus (\Gamma \cup \{p_*\} \cup \Gamma')$, the orbit $\mathrm{Orb}^+_{\widetilde{W_2}}(p)$ is infinite.*
2. *For every $p \in \widetilde{W_1} \setminus (\Gamma \cup \{p_*\} \cup \Gamma')$, there exists a single $q = \omega(p)$ with $q \in \Gamma'$.*
3. *For every $p \in W \setminus (\Gamma \cup \{p_*\} \cup \Gamma')$, the orbit $\mathrm{Orb}^-_W(p)$ is finite.*
4. *For every point $p \in \Gamma$ and every open neighborhood U of p, $\mathrm{Sat}_W(U) \cup (\Gamma \cup \{p_*\} \cup \Gamma')$ is a neighborhood of $\Gamma \cup \{p_*\}$.*
5. *The function $\phi : \widetilde{W_1} \setminus (\Gamma \cup \{p_*\} \cup \Gamma') \to \Gamma'$ defined by $\phi(p) = \omega(p)$ is continuous.*

Proof. We recall the expression of the diffeomorphism given in (2.6) on a dicritical monotonic domain

$$F(x,y) = (x + x^n y^m A(x,y), y + x^n y^{m+1} B(x,y)), \quad x \geq 0,\ y \geq 0,$$

with $A(0,0) < 0$, $\Gamma = \{y=0\}$ and $\Gamma' = \{x=0\}$. Notice that the coordinate x decreases on the orbits of the points in $W \setminus (\Gamma \cup \{p_*\} \cup \Gamma')$, in particular there exist bounds $-C_A \leq A(x,y) \leq -c_A < 0$, $-C_B \leq B(x,y) \leq C_B$. We will show that the orbits of the diffeomorhism fulfill some bounds, which will be used along the proof. We will suppose first that W is sufficiently small so that the following bound stands for every point $p = (x_0, y_0)$ out of $\Gamma \cup \{p_*\} \cup \Gamma'$.

$$\frac{1}{x_1^\tau y_1^m} - \frac{1}{x_0^\tau y_0^m} \geq \frac{1}{x_0^\tau y_0^m}\left(\frac{1}{(1 - x^{n-1}y^m A(x_0, y_0))^\tau (1 + x^n y^m(x_0, y_0))^m} - 1\right) \geq \tau c_A x_0^{n-1-\tau} + O(x_0^{n-\tau}) \quad (2.21)$$

That is, up to taking a dicritical monotonic domain at p_* smaller than W, we can suppose that

$$\frac{1}{x_1^\tau y_1^m} - \frac{1}{x_0^\tau y_0^m} \geq \widetilde{C}, \tag{2.22}$$

for some $\widetilde{C} > 0$. If the iterates $(x_i, y_i) = F^i(x_0, y_0)$ exist and belong to W up to some $j \geq 0$, we have

$$x_j^\tau y_j^m \leq \frac{x_0^\tau y_0^m}{1 + j\widetilde{C}x_0^\tau y_0^m}, \tag{2.23}$$

which follows from the bound in (2.22). We want to bound $|y_\ell - y_0|$ for some ℓ such that $(x_i, y_i) = F^i(x_0, y_0)$ exist and are contained in W for i with $0 \leq i \leq \ell$. Following the same idea as in the proof of Proposition 2.20, we can bound $x^n y^{m+1} \leq (x^\tau y^m)^\mu$ for any $1 < \mu = \min\{\frac{n}{n-\frac{1}{2}}, \frac{m+1}{m}\}$.

$$\begin{aligned}|y_\ell - y_0| \leq & C_B \sum_{j=0}^{\ell-1} x_j^n y_j^{m+1} \leq C_B \sum_{j=0}^{\ell-1} (x_j^\tau y_j^m)^\mu \leq C_B \sum_{j=0}^{\ell-1} \frac{(x_0^\tau y_0^m)^\mu}{(1 + j\widetilde{C}x_0^\tau y_0^m)^\mu} \leq C_B \int_{j=0}^{\ell} \frac{(x_0^\tau y_0^m)^\mu}{(1 + t\widetilde{C}x_0^\tau y_0^m)^\mu} dt \leq \\ \leq & C_B \int_{j=0}^{\infty} \frac{(x_0^\tau y_0^m)^\mu}{(1 + t\widetilde{C}x_0^\tau y_0^m)^\mu} dt \leq C_B \left(x_0^{\tau\mu} y_0^m \mu + \frac{x_0^{\tau(\mu-1)} y_0^{m(\mu-1)}}{\widetilde{C}(\mu-1)}\right) \leq C_0 x_0^{\tau(\mu-1)} y_0^{m(\mu-1)},\end{aligned} \tag{2.24}$$

for some C_0. We have used first the bound of the function B, secondly the aforementioned bound of the monomials $x_j^n y_j^m$, later (2.23) and then the integral criteria for series with decreasing terms. Integrating on a larger domain, we have found a uniform bound, that is, a bound which does not depend on ℓ. Finally, up to reducing more the domain W, the last bound is valid.

We have the following claim, which we prove in the end of the proof of Proposition 2.28.

Claim (1). For every point $p = (0, w) \in \Gamma'$, let $\widetilde{D}_w = (\frac{3w}{4}, \frac{5w}{4}) \times [0, \varepsilon)$ and $D_w = (\frac{w}{2}, \frac{3w}{2}) \times [0, \varepsilon)$. Then, there is some $\varepsilon = \varepsilon(w)$ such that for each $q \in \widetilde{D}_w \setminus \Gamma'$ the orbit $\mathrm{Orb}^+_{D_w}(p)$ is infinite and there is some $p_q \in \Gamma' \cap \overline{D_w}$ such that $\omega(q) = p_q$. In addition, ε can be taken $\varepsilon = w^k$ for some $k \in \mathbb{N}$ with $k > \frac{1-m(\mu-1)}{\tau(\mu-1)}$.

- Proof of the first statement. We prove that there are two dicritical monotonic domains such that $\widetilde{W_1} \subset \widetilde{W_2}$ such that for every $p \in \widetilde{W_1} \setminus (\Gamma \cup \{p_*\} \cup \Gamma')$, the orbit $\mathrm{Orb}^+_{\widetilde{W_2}}(p)$ is infinite. It is enough to select a box $\widetilde{D}_w$ obtained by using Claim (1), and define $\widetilde{W_1} = [0, \varepsilon) \times [0, \frac{5}{4}w)$ and $\widetilde{W_2} = [0, \varepsilon) \times [0, \frac{3}{2}w)$. On the one hand, given any point $p = (x_0, y_0)$ in $\widetilde{W_1}$, its positive orbits lies in $\{x \leq x_0\}$ and $x_0 < \varepsilon$, which means that if the orbit is finite, there is an iterate with $y_\ell \geq \frac{3}{2}w$. However, the bound in (2.24) implies that $y_\ell - y_0 \leq \frac{1}{4}w$.

- Proof of the second statement. We need to prove that no orbit accumulates at $p_* = (0,0)$. Using the previous claim, we find that $\widetilde{D} = \bigcup_{w>0} \widetilde{D}_w$ is a region that fulfills the following: the orbit of any point $q \in D_w$ converges to $\Gamma' \setminus \{p_*\}$. Considering also that ε of $\widetilde{D}_w$ can be chosen to be $\varepsilon(w) = w^k$ for some $k \in \mathbb{N}$, we notice that the boundary curve of this region is $\gamma = \{x = C_3 y^k\}$ for some C_3.

  Now, we will make k new blowing-ups in order to separate the curves Γ' and $\gamma = \{x = C_3 y^k\}$, which intersect at p_*. We start blowing up p_*. We obtain a new non-dicritical component E_1 of the divisor free of inner strict fixed points and two points: a saddle point p_1 at the intersection of E_1 and the strict transform of Γ, and a dicritical corner p_{1*} at the intersection of E_1 and the strict transform of Γ'. In the usual chart of the blowing-up in the direction of y, we take coordinates (x', y') and we find that the strict transform of γ has equation $x' = C_2(y')^{k-1}$. We recursively blow up the point p_{i*} obtaining a new component of the divisor E_{i+1} and two points: a saddle point p_{i+1} at the intersection of E_i and E_{i+1} and a dicritical corner p_{i+1*} at the intersection of E_{i+1} and the strict transform of Γ'. After k blowing-ups, we find that the strict transform of γ has equation $x^{(k)} = C_3$ (in the usual coordinates $(x^{(k)}, y^{(k)})$ at p_{k*} after blowing-up). We make an example with $k = 2$ on Figure 2.7

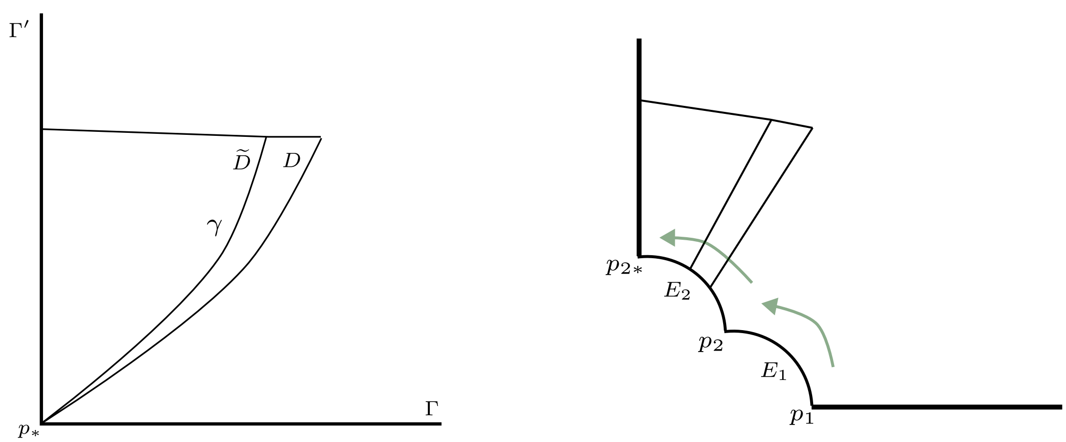

Figure 2.7: Example of sets D and $\widetilde{D}$ with boundary γ.

If any orbit accumulated into p_*, it should accumulate into $E_1 \cup \cdots \cup E_k$ after performing these blowing-ups. However, we find that every point at E_i is regular for every i such that $1 \leq i \leq k$, except its corners, and the points $p_1, \ldots, p_k$ are saddle points. We have already proved in the previous sections that any of these points cannot be an accumulation point of an orbit. The only point left to study is p_{k*}. Nevertheless, we have that p_{k*} is in the interior of the strict transform of $\widetilde{D}$, where all the points accumulate to $\Gamma' \setminus \{p_*\}$. We conclude then the second statement.

- Proof of the third statement,. We suppose that the negative orbit of some $p \in \widetilde{W_1} \setminus (\Gamma \cup \{p_*\} \cup \Gamma')$ is not finite. This implies that there exists an accumulation point of $\mathrm{Orb}^-_{\widetilde{\widetilde{W_2}}}(p)$ in $\overline{\widetilde{W_2}}$. Since $x(p) > 0$ and x is increasing for F^{-1}, the points in $\Gamma' \cup \{p_*\}$ cannot be accumulation points of $\mathrm{Orb}^-_{\widetilde{W_2}}(p)$. On the other hand the points on Γ are regular fixed points, and each of them belongs to a non-dicritical arc. Hence, they are neither accumulation points. Finally, the rest of the points of $\overline{\widetilde{W_2}}$ are regular, and they can neither be accumulation points.
- Proof the fourth statement. We see first the following claim, which is a weaker version of the statement. Before stating the claim, and for technical reasons, notice that the inverse F^{-1} of F has the expression

$$F^{-1}(x,y) = (x + x^n y^m \tilde{A}(x,y), y + x^n y^{m+1} \tilde{B}(x,y)), \quad x \geq 0,\ y \geq 0,$$

with $A(0,0) > 0$, and in $F(W) \cap W$ it has bounds $0 < \tilde{c}_A \leq \tilde{A}(x,y) \leq \widetilde{C}_A$ and $-\widetilde{C}_B \leq B(x,y) \leq \widetilde{C}_B$. For technical reasons as well, we will suppose that we work on some dicritical monotonic domain $W_1 \subset \widetilde{W_1}$ that is relatively compact in $\widetilde{W_1}$.

Claim (2). For any $\widetilde{U} = (a_1, a_2) \times [0, b) \subset F(W_1)$ where $0 < a_1 < a_2$ are chosen freely and $b > 0$ fulfills $a_2 - a_1 \geq \widetilde{C}_A a_1^n b^m$, we have $\mathrm{Sat}_W(\widetilde{U})$ is a neighborhood of p_*.

It is enough to prove this claim since given any neighborhood U of $p \in \Gamma$, and given any other point $r \in \Gamma \cap F(W_1)$, there is a box neighborhood as $\widetilde{U}$ of the statement contained in $\mathrm{Sat}_{\widetilde{W}}(U)$. This is achieved by selecting $a_2 > x(r)$ and $a_1 < x(r)$ and then defining the non-dicritical arc between $(a_1, 0)$ and p. There is a small enough b that fulfills simultaneously the hypothesis of the claim and that $\widetilde{U}$ is contained in the neighborhood of the non-dicritical arc obtained in Proposition 2.17.
- Proof of the last statement. Let $p \in W_1$, from the second statement, we have that $\{q\} = \omega(p)$. Then $\phi(p) = q$. Take any neighborhood I_q of $q \in \Gamma'$ and take any U_q of q such that $(U_q \cap \Gamma') \subset I_q$. Necessarily the set U_q is a neighborhood of a small dicritical arc γ_q that contains q. By Proposition 2.20, we have that this neighborhood contains one with the properties of Proposition 2.20, that is, all the points in the small V_q converge to a single point in $U_q \cap \Gamma'$. On the other hand, by the definition of the ω–limit $\omega(p)$, there exists some n_{V_q} such that $F^{n_{V_q}}(p) \in V_q$. We have that $F^{-n_{V_q}}(V_q)$ is an open neighborhood of p. Choosing any neighborhood B_p of p with $B_p \subset F^{-n_{V_q}}(V_q)$ we have that $\omega(p') = \{q'\}$ for some $q' \in U_q \cap \Gamma' \subset I_q$, that is $p' \in \phi^{-1}(I_q)$. Then, we conclude then that ϕ is continuous.

Proof of Claim (1). In a given box D_w of the statement, we have

$$|y_\ell - y_0| \leq C_0 \varepsilon^{\tau(\mu-1)} \left(\frac{3}{2} w\right)^{m(\mu-1)}, \tag{2.25}$$

since $x_0 < \varepsilon$ and $y_0 < \frac{3}{2}w$. We point out that this bound is also valid on the box $[0,\varepsilon)\times[0,\frac{3}{2}w)$. Now, we choose ε so that we ensure that any iterate of a point $p = (x_0, y_0) \in \widetilde{D}_w$ lies in D_w. It is enough to impose $C_0 \varepsilon^{\tau(\mu-1)} \left(\frac{3}{2}w\right)^{m(\mu-1)} \leq \frac{1}{4}w$, and take

$$\varepsilon \leq C_1 w^{\frac{1-m(\mu-1)}{\tau(\mu-1)}}. \tag{2.26}$$

Then, we take k so that $\varepsilon(w) = w^k$ with $k > \frac{1-m(\mu-1)}{\tau(\mu-1)}$, since $w^k \leq C_1 w^{\frac{1-m(\mu-1)}{\tau(\mu-1)}}$. We conclude that for any $(x_0, y_0) \in \widetilde{D}_w$, its iterate $(x_1, y_1) = F(x_0, y_0)$ exists in D_w since the coordinate x is decreasing and $x_0 - \frac{1}{4}w \leq y_1 \leq x_0 + \frac{1}{4}w$. By induction, we prove that any iterate $F^\ell(x_0, y_0)$ exists again since x is decreasing and since $|y_\ell - y_0| \leq \frac{1}{4}w$.

It remains to prove that $\omega(p)$ is a single point in $\overline{D_w} \cap \Gamma'$ for every $p = (x_0, y_0) \in \widetilde{D}_w$. We already know that the sequence $\{(x_\ell, y_\ell)\}_{\ell\in\mathbb{N}}$ exists. The only fixed points in $\overline{D_w} \cap \Gamma'$ belong to $\{x = 0\}$, hence the limit of the sequence of coordinates x, that is, $\{x_\ell\}_{\ell\in\mathbb{N}}$ is 0. On the other hand, from the bound (2.24), we have that $|y_{\ell+1} - y_\ell| \leq C_0 x_\ell^{\tau(\mu-1)} \left(\frac{3}{2}w\right)^{m(\mu-1)}$ and the convergence to 0 of x_ℓ, we get that $\{y_\ell\}_\ell$ is a Cauchy sequence, and hence it has a limit y_∞. Then, $\omega(p) = q_p = (0, y_\infty)$, as we wanted to prove. ■

Proof of Claim (2). We proceed by contradiction. Suppose that the statement is not true. Then, for every neighborhood of p_*, there exists a point t with $\mathrm{Orb}^-(t) \cap \widetilde{U} = \emptyset$. Because of the third item an iterate escaping the region $[0,a_1)\times[0,b)$ exists. In particular, the first negative iterate, say $F^{-n_t}(t)$, to abandon the region $[0,a_1)\times[0,b) \subset F(W_1)$ must do it with coordinate y greater than b. This is because by hypothesis this point is not intersecting $\widetilde{U}$ and because of the bound $a_2 - a_1 \geq C_A a_1^n b^m$. More precisely, because we have $x(F^{-n_t+1})(t) - x(F^{-n_t}(t)) \leq C_A a_1^n b^m \leq a_2 - a_1$. Then, we have that the point t fulfills that there exists some $F^{-n_t}(t) \in W_1 \cap \{y \geq b\}$, the iterate cannot scape the band $\{a_1 \leq x \leq a_2\}$.

Now, consider a decreasing sequence of neighborhoods $\{V_i\}_{i\in\mathbb{N}}$ of p_* such that $V_i \subset [0,a_2)\times[0,b)$, $V_i \subset V_{i-1}$ for all $i \in \mathbb{N}$ and $\bigcap_{i\in\mathbb{N}} V_i = \{p_*\}$, by the assumption, there exists a point t_i at each V_i and a first iterate $-n_i \in \mathbb{Z}_{\leq 0}$ such that $q_i = F^{-n_i}(t_i) \in W_1 \cap \{y \geq b\}$. The sequence of points $\{q_i\}_{i\in\mathbb{N}}$ must have an accumulation point q in the compact set $\overline{W_1 \cap \{y \geq b\}} \subset \widetilde{W}_1$. We find the contradiction as follows. To clarify more the ideas of the proof and the various sets that appear therein, see Figure 2.8.

- Suppose that $q \in \Gamma'$. Then, take a small enough regular arc Γ_q with $q \in \Gamma'_q$ and a small enough neighborhood $V_{\Gamma'_q}$ of Γ'_q such that $V_{\Gamma'_q} \cap V_i = \emptyset$ for all $i \geq i_0$. Applying Proposition 2.20, there exists a second neighborhood $\widetilde{V}_{\Gamma'_q} \subset V_{\Gamma'_q}$ with the property that the positive iterates of any $p \in \widetilde{V}_{\Gamma'_q}$ lie inside $V_{\Gamma'_q}$. We find a contradiction with the fact that $t_i = F^{n_i}(q_i) \in V_i$ and $V_i \cap V_{\Gamma'_q} \neq \emptyset$.

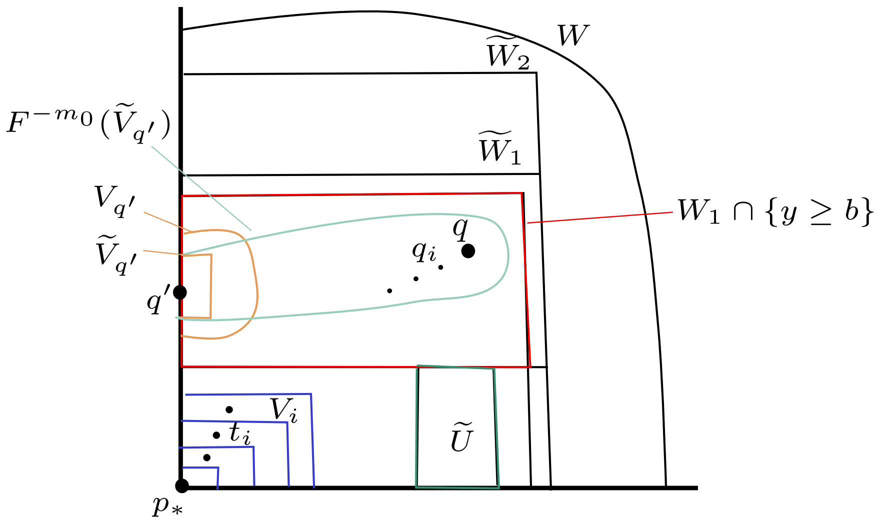

FIGURE 2.8: Sketch of the ideas of the proof of the fourth item.

- Suppose $q \in \Gamma'$. Since $q \in \widetilde{W}_1$, its orbit lies in $\widetilde{W}_2$ and there is some $q' \in \Gamma'$ such that $q' = \omega(q)$. As in the previous item, consider a small enough neighborhood $V_{q'}$ (of a dicritical arc that contains q') such that $V_{q'} \cap V_i = \emptyset$ for all $i \geq i_0$ for some large enough i_0. Applying Proposition 2.20, there exists a second neighborhood $\widetilde{V}_{q'} \subset V_{q'}$ with the property that the positive iterates of any $p \in \widetilde{V}_{q'}$ lie inside $V_{q'}$. Now, take some finite m_0 such that $q \in F^{-m_0}(V_q')$, which exists from the fact that $\omega(q) = q'$. Since q is an accumulation point of the points q_i, there exists a subsequence $\{q_{i_k}\}_{k\in\mathbb{N}} \subset F^{-m_0}(V_q')$. We claim that there exists a point q_{i_k} such that $n_{i_k} \geq m_0$ and $i_k \geq i_0$. This is because since the sequence of points $\{t_i\}_{i\in\mathbb{N}}$ accumulates into p_*, the number of iterations n_i of F^{-1} needed in order to surpass $\{y \geq b\}$ is not bounded. Then, we conclude that $F^{n_i}(p_i) \in V_i$ and $F^{n_i}(p_i) \in \widetilde{V}_{q'} \subset V_{q'}$. This is a contradiction since $V_{q'} \cap V_i = \emptyset$.

Then, there is a neighborhood V of p_* contained in the set $\mathrm{Sat}^+_W(\widetilde{U})$. On the other hand, by Corollary 2.18, we have that $\mathrm{Sat}^+_W(\widetilde{U})$ is neighborhood of any non-dicritical arc in Γ, and hence, of any point in Γ. ∎

□

Proposition 2.29. *Let (Q, W) be a realization of a dicritical quadrant such that W is a dicritical monotonic domain, Γ is a non-dicritical component and Γ' is a diciritcal component of the divisor or a bidicritical curve. Then, at every $q \in \Gamma'$ there is a parabolic curve γ_q transverse to Γ' and its germ is unique in the following sense: there exists a neighborhood basis $\mathcal{U}$ such that for any $U \in \mathcal{U}_q$, the curve $\gamma_q \cap U$ is characterized as*

$$\gamma_q \cap U = \{p \in U \setminus \Gamma' : \mathrm{Orb}^+_U(p) \text{ is infinite and } \omega(p) = q\}. \tag{2.27}$$

Proof. For the proof of this result, we distinguish two cases in terms of the integer n in the expression of the diffeomorphism.

$$F(x,y) = (x + x^n y^m A(x,y), y + x^n y^{m+1} B(x,y)), \quad x \geq 0,\ y \geq 0,$$

We make the affine change of coordinates $z = x$, $w = y - a$ and $a > 0$, so that we study the diffeomorphism at the point $p = (0,a) \in \Gamma'$.

$$F(z,w) = (z + z^n (w-a)^m A(z,w-a), w + z^n (w-a)^{m+1} B(z,w-a)), \quad z \geq 0,\ w \geq -a,$$

With $A(z,w-a) < 0$ and A, B have the same bounds as in the original coordinates.

- When $n = 1$, the diffeomorphism is not tangent to the identity at the point p, it is parabolic with eigenvalues $1 + (-a)^m, 1$. Then, the stable manifold theorem can be applied, finding an analytic curve that directly fulfills the statement of this result. See for instance the result in [56].

- When $n > 1$, the diffeomorphism is tangent to the identity. Then, we make a new blowing-up $\sigma : (M', E') \to (M, E)$ centered at p. Blowing up a dicritical point generates two dicritical corners p_{1*} and p_{2*}, and a single saddle point q' at the new component of the divisor. Using Theorem 2.10, we find that there exists indeed a parabolic curve $\gamma_{q'}$ at p' transverse to E. The germ of $\gamma_{q'}$ is unique in the sense of the statement and it can be blown down to $\gamma_q = \sigma(\gamma_{q'})$. The required curve is γ_p. □

The curves that have been obtained in the previous result are only local, in the sense that they are defined in neighborhoods of each point that can be arbitrarily small.

2.4 Construction of sectors

In this section, we will construct sectors as required in Theorem 2.6. We distinguish two situations; one of them coming from the maximal paths of quadrants of $G(\xi,\pi)$ and the other from dicritical components and node quadrants. Let $\mathrm{Supp}(G(\xi,\pi))$ be a realization of $G(\xi,\pi)$ for F in an open set W as described in Section 2.2 (we adopt the notations therein). We will fix this open set throughout all the section.

2.4.1 Sectors arising from maximal paths

The first type of sectors that we construct are based on a maximal path of quadrants. For this reason, we will name them *path sectors*. Let $\mathbf{Q} = (Q_1, \ldots, Q_s)$ be a maximal path with $Q_i = (v_i, e_{i-1}, e_i)$ for $i = 2, \ldots, s$. Let $p_1, \ldots, p_s$ be strict fixed points corresponding to the vertices $v_1, \ldots, v_s$. We have that the subchain $\mathbf{Q}' = (Q_2, \ldots, Q_{s-1})$ is a path of divisor saddle quadrants, in particular, the points $p_2, \ldots, p_{s-1}$ are strict fixed points of type s. Meanwhile, the points p_1 and p_s can be of any other type. We define $\gamma_{e_1}, \ldots, \gamma_{e_{s-1}}$ induced by the divisor path $(e_1, \ldots, e_{s-1})$, as we did in Section 2.3.4 for the path of saddle quadrants $(Q_2, \ldots, Q_{s-1})$, that is, as the union $E_{\mathbf{Q}} := \gamma_{e_1} \cup \{p_2\} \cup \gamma_{e_2} \cup \ldots \cup \gamma_{e_{s-2}} \cup \{p_{s-2}\} \cup \gamma_{e_{s-1}}$ contained in $\mathrm{Supp}(G(\xi, \pi))$ for F.

Proposition 2.30. *With the notations above, let V be a neighborhood of $\{p_1\} \cup E_{\mathbf{Q}} \cup \{p_s\}$. Then, there exists an open neighborhood $V_{\mathbf{Q}}$ of $E_{\mathbf{Q}}$ with $\overline{V}_{\mathbf{Q}} \subset V$ and satisfying the following properties:*

- *If p_1 is not of type n-n-2, there is a connected representative Γ_0 of $\mathrm{Supp}(e_0)$ contained in $\overline{V}_{\mathbf{Q}} \subset V$. If p_s is not of type n-n-2, there is a connected representative Γ_s of $\mathrm{Supp}(e_s)$ contained in $\overline{V}_{\mathbf{Q}} \subset V$. Both Γ_0 and Γ_s are part of the boundary $\partial V_{\mathbf{Q}}$ in V when they exist.*
- *There is a neighborhood $\widetilde{V}_{\mathbf{Q}} \subset V_{\mathbf{Q}}$ of $E_{\mathbf{Q}}$, saturated in $V_{\mathbf{Q}}$, and in which exactly one of the following situations holds for every $p \in \widetilde{V}_{\mathbf{Q}} \setminus E$.*
  1. *$\alpha(p) = p_1$ and $\omega(p) = \emptyset$.*
  2. *$\alpha(p) = \emptyset$ and $\omega(p) = p_s$.*
  3. *$\alpha(p) = p_1$ and $\omega(p) = p_s$.*
  4. *$\alpha(p) = \emptyset$ and $\omega(p) = \emptyset$.*
  5. *Γ_0 is a dicritical curve and there exists $q_p \in \Gamma_0$ such that $\alpha(p) = q_p$ and $\omega(p) = \emptyset$.*
  6. *Γ_s is a dicritical curve and there exists $q_p \in \Gamma_s$ such that $\alpha(p) = \emptyset$ and $\omega(p) = q_p$.*
  7. *Γ_0 is a dicritical curve and there exists $q_p \in \Gamma_0$ such that $\alpha(p) = q_p$ and $\omega(p) = p_s$.*
  8. *Γ_s is a dicritical curve and there exists $q'_p \in \Gamma_s$ such that $\alpha(p) = p_1$ and $\omega(p) = q'_p$.*
  9. *Γ_0 and Γ_s are dicritical curves and there exists $q_p \in \Gamma_0$ and $q'_p \in \Gamma_s$ such that $\alpha(p) = q_p$ and $\omega(p) = q'_p$.*

  In addition, the type only depends on the type of quadrants of Q_1 and Q_s.

In the statement of the Proposition, we obtain two neighborhoods $V_{\mathbf{Q}}$ and $\widetilde{V}_{\mathbf{Q}}$ of $E_{\mathbf{Q}}$. The first fulfills a specific property concerning its boundary. This will serve us to have more control in later proofs. The second fulfills two important properties; it is saturated in the first and that the positive and negative iterates are well controlled. This set $\widetilde{V}_{\mathbf{Q}}$ fulfills the conditions of the sectors in Theorem 2.6. We remark that, depending on the case, they can also be neighborhoods of p_1 and p_s (in the subspace topology in $V_{\mathbf{Q}}$). On the other hand, we remark that $\widetilde{V}_{\mathbf{Q}}$ and $V_{\mathbf{Q}}$ of the above result highly depend on the neighborhood V of $E_{\mathbf{Q}}$. In addition, there is not uniqueness on the choice of these sets.

Notation 2.31. Suppose that we obtain the neighborhood $\widetilde{V}_{\mathbf{Q}}$ of $E_{\mathbf{Q}}$ from a maximal path of quadrants $\mathbf{Q} = (Q_1, \ldots, Q_s)$ inside some open set U. We stress that $\widetilde{V}_{\mathbf{Q}}$ has been obtained inside V by denoting $\widetilde{V}_{\mathbf{Q}} = \widetilde{V}_{\mathbf{Q}}(V)$.

In the following table, we show the type of domains obtained in the previous result depending on Q_1 and Q_s.

Q_1	Q_s	F	F^{-1}
Node	Saddle	1	2
Saddle	Node	2	1
Node	Node	3	3
Saddle	Saddle	4	4
Dicritical	Saddle	5	6
Saddle	Dicritical	6	5
Dicritical	Node	7	8
Node	Dicritical	8	7
Dicritical	Dicritical	9	9

Table 2.2: Type of invariant sets (in terms of 1-9 of Proposition 2.30) for F and F^{-1} depending on the first and last quadrant of $\mathbf{Q}$.

Proof of Proposition 2.30. Consider a saddle, node or dicritical monotonic domain $W_i \subset V$ at p_i with $i = 1$ or s depending on the type of quadrant of Q_i.

- When the strict fixed point p_1 is of type n-n-2, W_1 is a joint realization of the quadrant Q_1 and the other quadrant Q with vertex v_1. In any other case, W_1 is a realization of a single quadrant. We call $\Gamma_1 := \operatorname{Supp}(e_1) \cap W_1$ and $\Gamma_0 := \operatorname{Supp}(e_0)$ (Γ_0 defined when p_1 is not of type n-n-2).
- When the strict fixed point p_s is of type n-n-2, W_s is a joint realization of the quadrant Q_s and the other quadrant Q' with vertex v_s. In any other case, W_s is a realization of a single quadrant. We call $\Gamma_{s-1} := \operatorname{Supp}(e_{s-1}) \cap W_s$, and $\Gamma_s := \operatorname{Supp}(e_s) \cap W_s$ (Γ_s defined when p_s is not of type n-n-2).

We need a technical requirement. We take a neighborhood $V' \subset V$ of $E_{\mathbf{Q}}$ (excluding the endpoints p_1, p_s) such that

- $\Gamma_0 \cap V' = \emptyset$ and $\Gamma_s \cap V' = \emptyset$, when Γ_0 or Γ_s exists.
- If Q_1 is a saddle quadrant, $V' \cap W^-_{1,esc} = \emptyset$ and $F(W^-_{1,esc}) \cap V' = \emptyset$ (recall that $W^-_{1,esc}$ is the region where the negative orbit escapes W_1), and if Q_s is a saddle quadrant, $V' \cap W^+_{s,esc} = \emptyset$ and $F^{-1}(W^+_{s,esc}) \cap V' = \emptyset$ (recall that $W^+_{s,esc}$ is the region where the positive orbit escapes W_1).

Notice that it is possible to obtain such a neighborhood V' of $E_{\mathbf{Q}} = \gamma_{e_1} \cup \{p_2\} \cup \ldots \cup \{p_{s-1}\} \cup \gamma_{e_{s-1}}$ fulfilling these conditions. The first is obvious since Γ_0, Γ_s does not intersect $\gamma_{e_1} \cup \{p_2\} \cup \ldots \cup \{p_{s-1}\} \cup \gamma_{e_{s-1}}$. The second is a consequence of Proposition 2.21: $W_1 \setminus W^-_{1,esc}$ is a neighborhood of $E \cap W_1$. Choosing V' even smaller we also have the condition of $F^{-1}(W^-_{1,esc})$ (since $W_1 \setminus F^{-1}(W^-_{1,esc})$ is not a neighborhood of $E \cap W_1$). We proceed in the same way to ensure the second statement in W_s. We apply Proposition 2.23 to $\mathbf{Q}'$ and V', so that we obtain a neighborhood $U_{\mathbf{Q}'} \subset V'$ of $E_{\mathbf{Q}}$ with the properties of that proposition. Let us recall that for any points $t_1 \in \gamma_{e_1}$, $t_{s-1} \in \gamma_{e_{s-1}}$ and neighborhood U'_{s-1} of t_{s-1}, there is a neighborhood $\widetilde{U}_1$ of t_1 such that for every $p \in \widetilde{U}_1 \setminus E$ there is some $q_p \in \operatorname{Orb}^+_{U_{\mathbf{Q}'}}(p) \cap U'_{s-1}$ (notice we have changed slightly the notation of the neighborhood $\widetilde{U}_1$

with respect to Proposition 2.23). Define the set $V_{\mathbf{Q}} \subset V$

$$V_{\mathbf{Q}} = W_1 \cup U_{\mathbf{Q}'} \cup W_s.$$

We distinguish the following cases.

I. When Q_s is a saddle quadrant, we choose $t'_{s-1} \in \Gamma'_{s-1}$ and a neighborhood $U'_{s-1} \subset U_{\mathbf{Q}'} \cap W_s$ and $U'_{s-1} \cap (\Gamma_s \cup \{p_s\}) = \emptyset$.

II. When Q_s is a node quadrant, let $t'_{s-1} \in \Gamma'_{s-1}$ and $U'_{s-1} \subset U_{\mathbf{Q}'} \cap W_s$ with $U'_{s-1} \cap (\Gamma \cup \{p_*\}) = \emptyset$.

III. When Q_s is a dicritical quadrant, we apply Proposition 2.28 to F^{-1}, obtaining another dicritical monotonic domain $\widetilde{W}_s \subset W_s$ (with the properties of $\widetilde{W}_1$ in the statement of that result). We choose $t'_{s-1} \in \Gamma'_{s-1} \cap \widetilde{W}_s$ and $U'_{s-1} \subset \widetilde{W}_s \cap U_{\mathbf{Q}'}$.

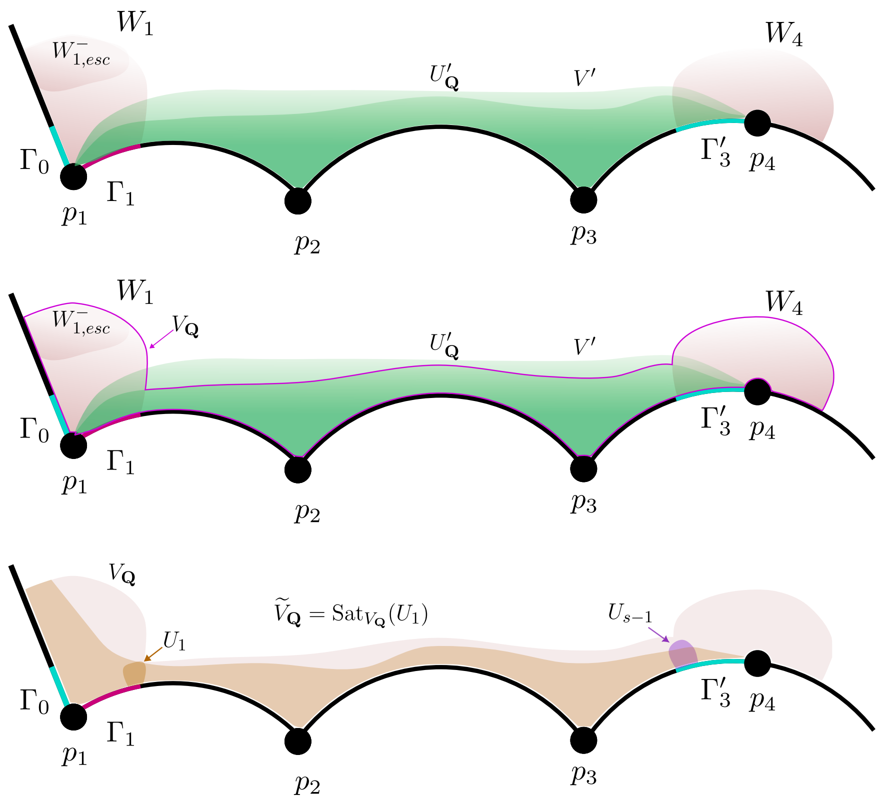

Figure 2.9: Construction of a path sector.

Notice that the positive iterates of any point p in U'_{s-1} can have different behaviors:

(+) By the technical requirement on the choice of V' and since $U'_{s-1} \subset W_s$, we have that the first iterate $F^n(p)$ to abandon $U_{\mathbf{Q}'}$ lies in W_s. Hence it is enough to study the positive iterate in W_s. In the case I, $\mathrm{Orb}^+_{V_{\mathbf{Q}}}(p)$ escapes W_s from Proposition 2.21. Notice that $\mathrm{Orb}^+_{V_{\mathbf{Q}}}(p)$ eventually reaches $W^+_{s,esc}$ and abandons W_s and also $V_{\mathbf{Q}}$ since $W^+_{s,esc} \cap V' = \emptyset$. In the case II, the orbit remains in W_s and converge to p_s from either Proposition 2.24 or 2.27 (depending on whether

p_s is of type n-n-2 or not). In the case III, $\mathrm{Orb}^+_{V_{\mathbf{Q}}}(p)$ remains in W_s and converge to some point in Γ_s, which depends on p by Proposition 2.28 (notice that $U'_{s-1} \subset \widetilde{W_s}$).

Now, we choose a point $t_1 \in W_1$.

I. When Q_1 is a saddle quadrant we choose any point in $t_1 \in \Gamma_1$.

II. When Q_1 is a node quadrant, we take any $t_1 \in \Gamma_1$.

III. When Q_1 is a dicritical quadrant, we apply Proposition 2.28 to F, obtaining $\widetilde{W_1}$ (with the properties of $\widetilde{W_1}$ in the statement of that result). We choose $t_1 \in \Gamma_1 \cap \widetilde{W_1}$.

As we have said, by Proposition 2.23, we obtain a neighborhood $\widetilde{U}_1$ of t_1 so that U'_{s-1} captures the positive orbits of the points $p \in \widetilde{U}_1$. In the dicritical case, we will take $U_1 = \widetilde{U}_1 \cap \widetilde{W_1}$. In the rest of the cases, we simply choose $U_1 = \widetilde{U}_1$. The negative iterates of any point $p \in U_1$ behave in a different manner depending on the type of quadrant that Q_1 is:

(-) By the technical requirement on the choice of V' and since $U_1 \subset W_1$, we have that the first iterate $F^{-n}(p)$ to abandon $U_{\mathbf{Q}'}$ lies in W_1. Hence it is enough to study the negative iterates in W_1. In the case I, $\mathrm{Orb}^-_{V_{\mathbf{Q}}}(p)$ escapes W_1 from Proposition 2.21. Notice that $\mathrm{Orb}^-_{V_{\mathbf{Q}}}(p)$ eventually reaches $W^-_{1,esc}$, and then abandons W_1 and also $V_{\mathbf{Q}}$ since $W^-_{1,esc} \cap V' = \emptyset$. In the case II, $\mathrm{Orb}^-_{V_{\mathbf{Q}}}(p)$ remains in W_1 and converge to p_1 from either Proposition 2.24 or 2.27 (depending on whether p_1 is of type n-n-2 or not). In the case III, it remains in W_1 and converge to some point in Γ_1, which depends on p by Proposition 2.28 (notice that $U_1 \subset \widetilde{W_1}$).

Then, we have control in the negative orbits of any point in U_1. By construction, we also have that the positive iterates of any point $p \in \widetilde{U}_1 \setminus E$ lie on $V' \subset V_{\mathbf{Q}}$ up to some ℓ_p with $F^{\ell_p}(p) \in U'_{s-1}$. The iterates $F^n(p)$ with $n \geq \ell_p$ have already been described in (+), since $F^{\ell_p}(p) \in U'_{s-1}$.

Then, we define the set

$$\widetilde{V}_{\mathbf{Q}} = \mathrm{Sat}_{V_{\mathbf{Q}}}(U_1) \cup (E_{\mathbf{Q}}).$$

By the construction above, we check that $\widetilde{V}_{\mathbf{Q}}$ fulfills one of the nine situations stated in the Proposition. It remains to prove that $\widetilde{V}_{\mathbf{Q}}$ is a neighborhood of $E_{\mathbf{Q}}$. It follows from Corollary 2.22, Proposition 2.24, Proposition 2.27 or the fourth item of Proposition 2.28 in the points in $E_{\mathbf{Q}} \cap W_1$ and $E_{\mathbf{Q}} \cap W_s$; as well as from Proposition 2.23 in the rest of the points.

□

Remark 2.32. We want to emphasize something to be proved later in this text; we can choose $\widetilde{V}_{\mathbf{Q}}$ such that the boundary $\partial\widetilde{V}_{\mathbf{Q}}$ is a $\mathcal{C}^0$ curve, as in section 2.5.1, by defining a fundamental domain in $V_{\mathbf{Q}}$ with certain properties and taking its saturation, instead of taking the saturation of a neighborhood U_1 of a point t_1 in the component γ_{e_1}, as in the proof of this result.

The following Proposition proves that the sets $\widetilde{V}_{\mathbf{Q}} \setminus \mathrm{Supp}(G(\pi,\xi))$ can be projected by the sequence of blowing-ups π to $A_{\mathbf{Q}} = \pi(\widetilde{V}_{\mathbf{Q}} \setminus \mathrm{Supp}(G(\pi,\xi)))$ and that they fulfill the asymptotic properties of the sectors. The proof is straightforward considering that the sequence of blowing-ups π is a diffeomorphisms outside of E and that E is mapped to 0. However, we remark that there is something of global nature yet to be proved. Namely, that the $A_{\mathbf{Q}}$ is self-saturated in a full open neighborhood of 0, roughly, $F(A_{\mathbf{Q}}) \subset A_{\mathbf{Q}}$. We will come later to this.

Proposition 2.33. *Let $\widetilde{V}_{\mathbf{Q}}$ be a saturated domain obtained from $Q = (Q_1, \ldots, Q_n)$ as in Proposition 2.30. Then, $A_{\mathbf{Q}} = \pi(\widetilde{V}_{\mathbf{Q}} \setminus \mathrm{Supp}(G(\pi,\xi)))$ is a sector with the properties of a two dimensional stratum in Theorem 2.6.*

Proof. As we anticipated, the behavior of the orbits inside $\widetilde{V}_{\mathbf{Q}}$ and hence inside $A_{\mathbf{Q}}$ is well described by the previous result. We only need to consider that E is mapped to 0 and that the parabolic curves and curves of fixed points out of the divisor are mapped to parabolic curves and curves of fixed points at 0. For instance, notice that it makes a difference having a dicritical quadrant with both edges in the divisor or a dicritical quadrant with one edge out of the divisor. This is because in the first case, the points will accumulate into a single point $0 \in \mathbb{R}^2$. Meanwhile in the second case, the projection of the curve is a curve at 0 instead of a single point. □

As before, we summarize the type of sectors that we can obtain in Table 2.3. At this stage, to determine the type of sector, we do not pay attention to whether we work with F or F^{-1}. To understand it better, let Γ_0 be the support of the edge e_0 of Q_1 and Γ_s the support of the edge e_s of Q_s. We denote Div to indicate that this curve belongs to E and NDiv when it is not. We denote $\emptyset$ to indicate that there does not exist a realization of e_0 for F.

Pairs $Q_1 - Q_s$	Pairs $\Gamma_0 - \Gamma_s$	Sector
Node - Saddle	Div, NDiv or $\emptyset$ - NDiv	Parabolic
Node - Node	Div, NDiv or $\emptyset$ - Div, NDiv or $\emptyset$	Elliptic
Saddle - Saddle	NDiv - NDiv	Hyperbolic
Dicritical - Saddle	NDiv - NDiv	D-parabolic
Dicritical - Saddle	Div - NDiv	Parabolic
Dicritical - Node	NDiv - Div, NDiv or $\emptyset$	D-elliptic
Dicritical - Node	Div - Div, NDiv or $\emptyset$	Elliptic
Dicritical - Dicritical	NDiv - NDiv	D-D
Dicritical - Dicritical	Div - NDiv	D-Elliptic
Dicritical - Dicritical	Div - Div	Elliptic

Table 2.3: Type of invariant sectors arising from saturated sets of paths.

By Proposition 2.30, we can construct the saturated domains $V_{\mathbf{Q}}$ in a common open neighborhood W_E of E in M. Let us put

$$V_{path}(W_E) := \bigcup_{\mathbf{Q}\ \max} \widetilde{V}_{\mathbf{Q}}(W_E) \cup \mathrm{Supp}(G(\xi,\pi)).$$

We highlight that each $\widetilde{V}_{\mathbf{Q}}$ is saturated in V_{path}. We see now that V_{path} is a neighborhood of all the points in E except a finite number of strict fixed points of type n, n-n, n-s, f-n-n and f-n-s, as well as the dicritical components of E for ξ.

Lemma 2.34. *The set $V_{path}(W_E)$ is a neighborhood of every regular point in the non-dicritical components of E for ξ, as well as the ones in $\mathfrak{S}(F,\pi)$ of type d, s-s, f-s-s or f-d-d.*

Proof. Suppose that $p \in \mathfrak{S}(F,\pi)$ of type d, s-s, f-s-s or f-d-d. In any of these cases, there is a curve $\gamma \subset \mathrm{Supp}(G(\xi,\pi))$ corresponding to a segment of parabolic curve, curve of fixed points or a component of the divisor. Let $\mathbf{Q}_+$ and $\mathbf{Q}_-$, be the two paths of quadrants with p as extreme point. By the first part of Proposition 2.30 applied on each path, the union $V_{\mathbf{Q}_+} \cup \gamma \cup V_{\mathbf{Q}_-}$ is a neighborhood of p since a small enough representative of the germ of the curve γ at p must belong to the boundary of $V_{\mathbf{Q}_\epsilon}$ from Proposition 2.30 for $\epsilon = +,-$.

Then, take saddle or dicritical monotonic domains $W_+, W_- \subset W_E$, correspondingly, realizations of the quadrants at p so that $W_+ \subset V_{\mathbf{Q}_+}$, $W_- \subset V_{\mathbf{Q}_-}$. Applying Corollary 2.22 (saddle) or the fourth item of Proposition 2.28 (dicritical) and the property that $\widetilde{V}_{\mathbf{Q}_\epsilon}$ is saturated in $V_{\mathbf{Q}_\epsilon}$ for $\epsilon = +,-$, we get that $V_{\mathbf{Q}_+} \cup \gamma \cup V_{\mathbf{Q}_-}$ is a neighborhood of p . More precisely, take any neighborhood $U_\epsilon \subset W_\epsilon \cap \widetilde{V}_{\mathbf{Q}}$ of any $q_\epsilon \in W_\epsilon$, for $\epsilon = +,-$. The saturation of U_ϵ(that lies in $W_\epsilon \subset V_{\mathbf{Q}_\epsilon}$) is a neighborhood of p in the subspace topology in W_ϵ in both the saddle and dicritical cases (cf. Corollary 2.22 and Proposition 2.28). Since $U_\epsilon \subset \widetilde{V}_{\mathbf{Q}_\epsilon}$, $U_\epsilon \subset W_\epsilon \subset V_{\mathbf{Q}_\epsilon}$ and $\widetilde{V}_{\mathbf{Q}_\epsilon}$ is saturated in $V_{\mathbf{Q}_\epsilon}$, we have that $\widetilde{V}_{\mathbf{Q}_\epsilon}$ is a neighborhood of $p \in V_{\mathbf{Q}_\epsilon}$ for $\epsilon = +,-$.

The open set V_{path} being a neighborhood of the rest of the regular non-dicritical points is a consequence of the construction of each $V_{\mathbf{Q}}$ in Proposition 2.30. □

2.4.2 Parabolic sectors at nodes and dicritical components of the divisor

Notice that to obtain a sectorial decomposition of F we need to obtain a partition of a neighborhood of $0 \in \mathbb{R}^2$. Notice that the projection $\pi(V_{path} \setminus E)$ once we remove $\mathrm{Supp}(G(\xi,\pi))$ is partitioned into sectors, the subsets $A_{\mathbf{Q}}$ obtained in Proposition 2.33. However $\pi(V_{path} \setminus E) \cup \{0\}$ is not a neighborhood of 0. In this section we construct new sectors to fill in a neighborhood of 0 with the properties of Theorem 2.6.

Proposition 2.35. *Let $V_{path}(W_E)$ be defined as above. Then, there exist a finite number of subsets*

$\{V_i\}_{i=1}^k$ *and a finite number of connected curves* $\{\gamma_j\}_{j=1}^s$ *such that*

$$V_{path} \cup \mathrm{Supp}(\mathcal{E}_{ndiv}) \cup V_1 \cup \dots \cup V_k \cup \gamma_1 \cup \dots, \gamma_r$$

is a neighborhood of E *and each* V_i *fulfills one of the following*

- *There is* $q_i \in \mathfrak{S}(F,\pi)$ *of type n, n-n, n-s, f-n-n or f-n-s such that for every* $p \in V_i$*, either* $\alpha(p) = q_i$ *and* $\omega(p) = \emptyset$ *or* $\alpha(p) = \emptyset$ *and* $\omega(p) = q_i$*.*
- *There is a regular dicritical arc* B *such that for every* $p \in V_i$*, there is some* $q_p \in B$ *such that either* $\alpha(p) = q_p$ *and* $\omega(p) = \emptyset$ *or* $\alpha(p) = \emptyset$ *and* $\omega(p) = q_p$*.*

In addition, the new curves γ_j *are contained in the common boundary of some* V_{i_j} *and some connected component of* V_{path}*.*

Proof. We will work in different points and dicritical arcs of the divisor.

- Suppose that p is a strict fixed point of type n, n-n, n-s, f-n-n or f-n-s.
  - Suppose that p is of type n or n-n-2. Suppose without loss of generality that it is attracting. There are two path quadrants, say $\mathbf{Q},\mathbf{Q}'$ sharing the endpoint at p, and there are not components of the germ of $\mathrm{Supp}(G(\xi,\pi))$ at p outside the divisor. Consider the open sets $V_{\mathbf{Q}}, \widetilde{V}_{\mathbf{Q}}, V_{\mathbf{Q}'}, \widetilde{V}_{\mathbf{Q}'}$ associated to the path quadrants $\mathbf{Q}$, $\mathbf{Q}'$. We take the corresponding node monotonic domain W at p as the one given in section 2.3.5, and small enough so that $W \subset V_{\mathbf{Q}} \cap V_{\mathbf{Q}'}$. Then the open set $V = W \setminus (E \cup \overline{\widetilde{V}_{\mathbf{Q}}} \cup \overline{\widetilde{V}_{\mathbf{Q}'}})$ is positively invariant (by F) and it is a parabolic set. Let us prove this. Recall that the whole node monotonic domain W is positively invariant and that $W \setminus E$ is a parabolic set at p. It is enough to see that V is positively invariant. Take $z \in W \setminus (E \cup \overline{\widetilde{V}_{\mathbf{Q}}} \cup \overline{\widetilde{V}_{\mathbf{Q}'}})$ and let $z' = F(z)$. We have $z' \notin E$, and since $W \cap \widetilde{V}_{\mathbf{Q}}$ and $W \cap \widetilde{V}_{\mathbf{Q}'}$ are saturated, so are $W \cap \overline{\widetilde{V}_{\mathbf{Q}}}$ and $W \cap \overline{\widetilde{V}_{\mathbf{Q}'}}$. Hence $z' \notin \widetilde{V}_{\mathbf{Q}} \cup \widetilde{V}_{\mathbf{Q}'}$. The set $V = W \setminus (E \cup \overline{\widetilde{V}_{\mathbf{Q}}} \cup \overline{\widetilde{V}_{\mathbf{Q}'}})$ is one of the V_i required in the statement of the result. By means of Remark 2.32, we also have that each curve $\gamma = \overline{V_i} \cap \overline{\widetilde{V}_{\mathbf{Q}}} \setminus E$ is a parabolic curve for F. This curve is not necessarily analytic, but it is of class $\mathcal{C}^0$. It divides W into two regions.
  - Suppose that p is of type f-n-n, f-n-s, n-s or n-n-1. There are two path quadrants, say $\mathbf{Q},\mathbf{Q}'$ sharing the endpoint at p, and there is one component Γ of the germ of $\mathrm{Supp}(G(\xi,\pi))$ at p outside the divisor. This is a curve of fixed points or an analytic parabolic curve. Let Q be one of the node quadrants at p and suppose without loss of generality that it is of attracting type and belongs to $\mathbf{Q}$. Let W be a small enough node monotonic domain corresponding to Q, so that it is limited by E and a segment of the curve Γ. We define the open set $V = W \setminus (E \cup \overline{\widetilde{V}_{\mathbf{Q}}})$. Proceeding as in the previous item, it is positively invariant

and of parabolic type. Its boundary $\gamma = \overline{V} \cap \overline{\widetilde{V}_{\mathbf{Q}}} \setminus E$, again by Remark 2.32 is a $\mathcal{C}^0$ parabolic curve, one of the collection $\{\gamma_j\}_{j=1}^s$ of the statement.

Summarizing, in this item, we have found as many sets V_i on the statement of the result as node quadrants there are at p, that is, one or two. We also obtain the corresponding parabolic curves in the boundaries of V with V_{path}.

- Suppose that E_i is a dicritical component. There are two dicritical quadrants that contain the edge e associated to E_i. There are also two paths of quadrants $\mathbf{Q}_-$ and $\mathbf{Q}_+$ meeting this edge, with endpoints p_- and p_+, the two dicritical corners of E_i. Let $\widetilde{V}_{\mathbf{Q}_\epsilon} \subset V_{\mathbf{Q}_\epsilon}$ be the domains obtained in Proposition 2.30 for the path $\mathbf{Q}_\epsilon$, with $\epsilon = +,-$. By construction and using 2.28, we have that the corresponding sets $\widetilde{V}_{\mathbf{Q}_\epsilon}$ is a neighborhood of p_ϵ. We choose two points q_- and q_+ such that the regular dicritical arc Γ_{q_-,q_+} contains $E_i \cap \partial \widetilde{V}_{\mathbf{Q}_-}$ and $E_i \cap \partial \widetilde{V}_{\mathbf{Q}_+}$. We apply Proposition 2.20 on a sufficiently small monotonic domain $V_{\Gamma_{q-q+}}$, obtaining $\widetilde{V}_{\Gamma_{q-q+}}$. Now, the searched set is $\widetilde{V}_{\Gamma_{q-q+}} \setminus (E \cup \overline{\widetilde{V}_{\mathbf{Q}_+}} \cup \overline{\widetilde{V}_{\mathbf{Q}_-}})$. By arguments similar to the ones in the previous item, we have that this set is positively invariant and has the property of one of the V_i in the statement of the result. Again by Remark 2.32, we also obtain the corresponding $\mathcal{C}^0$ curves in the family $\{\gamma_j\}_{j=1}^s$ in the boundaries of V with V_{path}.

□

We illustrate the output of Proposition 2.35 in Figure 2.10. In this figure, we put an example of three sectors generated over the paths of quadrants $\mathbf{Q_1}, \mathbf{Q_2}, \mathbf{Q_3}$. Then we define a parabolic sector V_1 at the dicritical component E_i and the $\mathcal{C}^0$ curves γ_1 and γ_2 on the common boundary of the new sector V_1 and $\widetilde{V}_{\mathbf{Q_1}}$ and $\widetilde{V}_{\mathbf{Q_2}}$, respectively. We construct the parabolic sectors on the ends of $\mathbf{Q_2}, \mathbf{Q_3}$, obtaining V_2, V_3 and the $\mathcal{C}^0$ parabolic curves γ_3, γ_4 defined, respectively, in the common boundary of $\widetilde{V}_{\mathbf{Q_2}}$ and V_2, and in the common boundary of $\widetilde{V}_{\mathbf{Q_3}}$ and V_3.

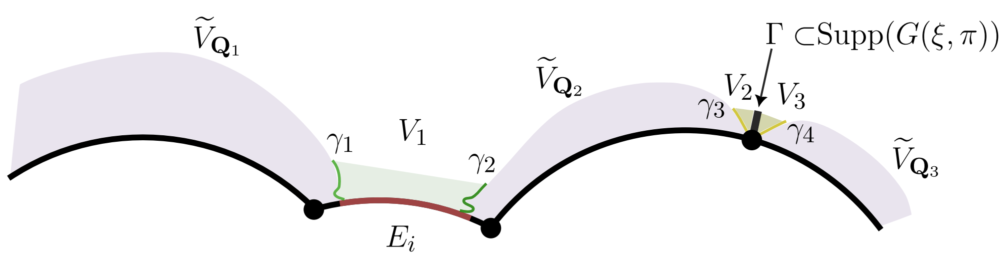

Figure 2.10: Example on the construction of parabolic sectors. The sector V_1 is constructed over a dicritical component of the divisor. The sectors V_2, V_3 are constructed over a node.

We name $V_{nopath} = V_1 \cup \cdots \cup V_s$. Just noticing that $\pi(E) = 0 \in \mathbb{R}^2$ and recalling the definition of the sets V_i in the above proposition, we have that $\pi(V_i)$ is a parabolic sector.

Proposition 2.36. *Let V_i be one of the saturated domains obtained in the previous result. Then $A_i = \pi(V_i)$ fulfills the weak topological properties of a parabolic sector in the sense of Theorem 2.6.*

2.4.3 Existence of a sectorial decomposition

In this section, we put together the results in the previous sections in order to prove the existence of a sectorial decomposition.

Proposition 2.37. *Let $F \in \mathrm{Diff}_1^{\omega}(\mathbb{R}^2, 0)$, $F \neq Id$ and not of center-focus type. There is a representative in an open neighborhood W of 0 and a second neighborhood $\widetilde{W}$ of 0 and a partition $\mathcal{S}$ of $\widetilde{W}$ in the conditions of Theorem 2.6.*

Proof. We have already made most of the work in the previous sections. First, we take a reduction of singularities $\pi : (M, E) \to (\mathbb{R}^2, 0)$ of the infinitesimal generator of F. We consider its graph $G(\xi, \pi)$ and define its maximal paths. Take a realization $\mathrm{Supp}(G(\xi, \pi))$ for F defined on some neighborhood $W_E \subset \pi^{-1}(W)$ of E. Applying Proposition 2.30 to each maximal path $\mathbf{Q}$, we obtain $\widetilde{V}_{\mathbf{Q}}$ saturated in open sets $V_{\mathbf{Q}}$. Then, we define $V_{path}(W_E)$ as the union of the sets $\widetilde{V}_{\mathbf{Q}}$ for each maximal path $\mathbf{Q}$.

Then, we apply Proposition 2.35 and we obtain a finite number of new sets V_i for $i = 1, \dots, k$ whose union is V_{nopath}. Then, by this result, the union $\widetilde{V}$ of V_{nopath}, V_{path}, $\mathrm{Supp}(G(\xi, \pi))$ and the one dimensional curves $\gamma_1, \dots, \gamma_r$ contained in $(\overline{V_{nopath}} \cap \overline{V_{path}}) \setminus E$ is a neighborhood of E in W_E.

We conclude that $\widetilde{W} = \pi(\widetilde{V})$ is the claimed neighborhood that admits the partition of Theorem 2.6. We have already proved in Proposition 2.33 and 2.36 that, respectively, the sets $A_{\mathbf{Q}} = \pi(\widetilde{V}_{\mathbf{Q}})$ for a maximal path $\mathbf{Q}$ and the sets $A_i = \pi(V_i)$ are sectors in the sense of the theorem. Since the curves $\{\gamma_j\}_{j=1}^r$ accumulate only at E, their projections $\{\pi(\gamma_j)\}_{j=1}^r$ are parabolic curves at $0 \in \mathbb{R}^2$. We also know that the curves of fixed points and parabolic curves on $\mathrm{Supp}(\mathcal{E}_{n-div})$ are one dimensional strata. Then, adding the point 0, we have found the partition of $\widetilde{W}$ we claimed. □

Remark 2.38. The partitioned set $\widetilde{W}$ obtained in the previous Proposition may not be open even if each two-dimensional stratum is. Later in this text, we will talk more about this topic. We anticipate that we can make a refinement on the choice of the strata so that, locally, the boundary $\partial\widetilde{W}$ and $\widetilde{W}$ do not intersect except on some points of the curves of fixed points of F. And in the absence of D-D sectors, we can achieve an open $\widetilde{W}$.

2.5 Refinements of the sectorial decomposition

In this section, we investigate two problems related to the topology and geometry of the sectorial decomposition: in which conditions we can find a sectorial decomposition $(U, \mathcal{S})$ such that U is open and in which conditions we can find a sectorial decomposition $(U, \mathcal{S})$ such that U is semi-analytic. The first section is more technical and we provide some results to be used in the following sections. We present some refinements on the construction of the sectorial decomposition so that each sector individually fulfills specific geometric properties. In the second section, we will see that in the absence of D-D sectors, the set U can be chosen to be open, as Proposition 2.8 claims. We also explain the reasons why we think that the existence of D-D sectors may be an obstruction for U being open. In the last section, we prove also that in the absence of bidicritical curves, U can be chosen indeed semi-analytic, as Proposition 2.9 claims, putting together the results seen in the first subsection.

2.5.1 Sector-wise refinements

In this section, we choose first a concrete sectorial decomposition, defined inside a neighborhood $\widetilde{W}$ of $0 \in \mathbb{R}^2$ that fulfills some monotonic properties. Then, we present refinements on the individual sectors of the diffeomorphism defined on a neighborhood of $0 \in \mathbb{R}^2$ that fulfills specific properties. We also show that it is possible to choose the boundary $\partial\widetilde{V}_{\mathbf{Q}}$ to be a curve in the $\mathcal{C}^0$ class, as we anticipated in Remark 2.32.

We present first some additional hypotheses to be imposed in the saddle and node domains.

H-S Let (Q,W) be a realization of a saddle quadrant at $p \in \mathfrak{S}(F,\pi)$ on a saddle domain W with coordinates (x,y). Recall that x is monotonically increasing and that y is monotonically decreasing on the orbits. Recall also that when p is of type s, f-s-s or f-n-s, the coordinate functions are analytic. On the other hand, when p is of type s-s or n-s, the coordinate x may only be $\mathcal{C}^\infty$ while y is analytic. We suppose that W is of box type in these coordinates, bounded by $\{y = C_1\}$ and $\{x = C_2\}$.

H-N-I Let (Q,W) be a realization of a node quadrant at $p \in \mathfrak{S}(F,\pi)$ on a node domain W with coordinates (x,y). Recall that both x,y are monotonically increasing or decreasing on the orbits. Recall also that when p is of type n, f-n-n or f-n-s, the coordinate functions are analytic. On the other hand, when p is of type n-n-1 or n-s, the coordinate x may only be $\mathcal{C}^\infty$ while y is analytic. We suppose that W is of box type in these coordinates, bounded by $\{y = C_1\}$ and $\{x = C_2\}$.

H-N-II Let (Q,Q',W) be a joint realization of two quadrants at $p \in \mathfrak{S}(F,\pi)$ of type n-n-2 on a node domain W with coordinates (x,y). Recall that y is an analytic function and monotonically increasing or decreasing on the orbits and that the orbits of points out of E converge to p. We suppose that W is bounded by $\{y = C_1\}$.

We fix an open neighborhood $\widetilde{W_E}$ of the divisor $E \subset M$ given as

$$\widetilde{W_E} = \left(\bigcup_{p\in\mathfrak{S}(F,\pi)} W_p^+ \cup W_p^- \right) \cup \bigcup_{e\in\mathcal{E}_{div}} V_{\Gamma_e}$$

where

- If p is of type f-n-n, f-n-s, f-s-s, s-s, n-s or n-n-1, the set W_p^- is a realization in the quadrant Q^- on the left as a saddle or node domain, correspondingly. The set W_p^+ is defined in the same manner. We choose the saddle and node domains fulfilling H-S and H-N-I, respectively. Notice that $W_p^- \cap W_p^+$ is a curve of fixed points or a parabolic curve.
- If p is of type f-d-d, the sets W_p^- and W_p^+ are realizations of the dicritical quadrants Q^- and Q^+ on dicritical domains.
- If p is n-n-2, we have that W_p^+ is a joint realization of both node quadrants at p in a node domain of type half-space fulfilling H-N-II, and $W_p^- = \emptyset$.
- If p is a corner of type s, n, d, there is a single quadrant at p. Let W_p^+ be a realization of it in a saddle, node or dicritical domain, correspondingly, and $W_p^- = \emptyset$. In the saddle and node cases, we suppose H-S or H-N-I, respectively.
- For each divisor edge e, let γ_e be its realization and p, p' its adjacent points. We take a regular arc (dicritical or non-dicritical) Γ_e joining two points $q, q' \in \gamma_e$ such that $q \in (W_p^{\epsilon})^{\circ} \cap \gamma_e$ and $q' \in (W_{p'}^{\epsilon'})^{\circ} \cap \gamma_e$, where ϵ and ϵ' correspond to the quadrants intersecting γ_e. Then, V_{Γ_e} is a monotonic domain.
- We can always take $W_p^{\epsilon} \cap W_{p'}^{\epsilon'} = \emptyset$ when $p \neq p'$ for any $\epsilon, \epsilon' \in \{+,-\}$. In addition, we can take $V_{\Gamma_e} \cap V_{\Gamma_{e'}} = \emptyset$ when $e \neq e'$.

Then, applying Proposition 2.30 and Proposition 2.35, we obtain a decomposition $(W_E, \mathcal{S})$ of a neighborhood $W_E \subset \widetilde{W}_E$. We have the decomposition of the neighborhood W_E given by

$$E \cup \left(\bigcup_{e \in \mathcal{E}_{ndiv}} \operatorname{Supp}(e) \right) \cup \left(\bigcup_{i=1}^{r} \gamma_i \right) \cup \left(\bigcup_{\mathbf{Q} max} \widetilde{V}_{\mathbf{Q}} \right) \cup \left(\bigcup_{i=1}^{k} V_i \right),$$

where $\operatorname{Supp}(e)$ is a parabolic curve of fixed points of the realization of $G(\xi, \pi)$ for F in W_E, the sets $\widetilde{V}_{\mathbf{Q}}$ are the path sectors, the V_i are parabolic sectors at nodes and dicritical components and the curves γ_i are the parabolic curves in the boundaries between a parabolic sector and a path sector. Using the sectorial decomposition we have just presented, we will introduce some refinements, so that the configuration of sectors will be the same. However, it is possible that the path sectors are strictly contained in the original ones, and that the parabolic sectors and the curves lying on their boundaries of can change.

Refinements of the path sectors

Let $\widetilde{V}_{\mathbf{Q}}$ be one of the path sectors associated to the maximal path $\mathbf{Q} = (Q_1, \ldots, Q_s)$. In this section, we propose a refinement of the path sectors. We will do it in two different ways, depending on the quadrants Q_1 and Q_s of the maximal path $\mathbf{Q}$.

- Neither Q_1 nor Q_s are dicritical quadrants.
- Either Q_1 or Q_s is a dicritical quadrant.

Neither Q_1 nor Q_s are dicritical quadrants. Consider any regular point p in some non-dicritical path Γ_e and the monotonic domain $V_{\Gamma_e} \cap \widetilde{V}_{\mathbf{Q}}$ with coordinates (x, y) such that x is monotonically increasing on the orbtis of the points $V_{\mathbf{Q}} \setminus E$. Then, we take

- $\gamma_0 = \{x = 0, y > 0\}$.
- $F(\gamma_0)$
- Fix $p_\beta \in \gamma_0$ and take any smooth semi-analytic curve that joins p_β and $F(p_\beta)$ fulfilling
  - $\beta \cap \gamma_0 = p_\beta$ and $\beta \cap F(\gamma_0) = F(p_\beta)$.
  - The curve β is the image $\tilde{\beta}|_{[0,1]}$ where $\tilde{\beta}$ is a parameterized curve $\tilde{\beta} : (-\epsilon, 1+\epsilon) \to U_p$ with $DF_{p_\beta}(\tilde{\beta}'(0)) = \tilde{\beta}'(1)$.

Consider the open region U' bounded by the above three curves. We finally define $U = U' \cup (\gamma_0 \cap \overline{U'}) \setminus \{p_\beta, p\}$. We claim that it is a fundamental domain in $\widetilde{V}_{\mathbf{Q}}$. Consider the equivalence relation in $\widetilde{V}_{\mathbf{Q}} \setminus E$ given by $p \sim q$ if and only if $p \in \mathrm{Orb}_{\widetilde{V}_{\mathbf{Q}}}(q)$. Saying that U is a fundamental domain in $\widetilde{V}_{\mathbf{Q}}$ is the same as saying that the map $c : \widetilde{V}_{\mathbf{Q}} \to \widetilde{V}_{\mathbf{Q}}/\sim$ restricted to U is injective.

Lemma 2.39. *The set U defined above is a fundamental domain in $\widetilde{V}_{\mathbf{Q}}$.*

Proof. We need to prove that $F^n(p) \notin U$ for every $p \in U$ and every $n \in \mathbb{Z} \setminus \{0\}$. First, by the fact that $\widetilde{V}_{\mathbf{Q}} \setminus V_{\Gamma_e}$ has two connected components, the transition in Proposition 2.21 and the fact that $\widetilde{V}_{\mathbf{Q}}$ is saturated, we have that the orbits transit from one connected component of $\widetilde{V}_{\mathbf{Q}} \setminus V_{\Gamma_e}$ to the other. Then, if one orbit abandons V_{Γ_e}, it does not return to it. Hence we can suppose that we work in the single monotonic domain V_{Γ_e}.

Suppose by contradiction that this is not true, that is, there exists some $p \in U$ and some $n \in \mathbb{Z} \setminus \{0\}$ such that $q = F^n(p) \in U$. We directly dismiss the possibility that $p \in \gamma_0$, since x is monotonically increasing on the orbits. Suppose without loss of generality that $n > 0$ and $x(q) \geq x(p) > 0$. Let γ_p be the curve $\{x = x(p)\}$. It is possible that a finite number of closed connected subsets of $F^i(\gamma_p)$ may escape V_{Γ_e}, and that $F^i(\gamma_p) \cap V_{\Gamma_e}$ has a finite number of connected open components. Each connected component has one point in the closure necessarily on the boundary with W_t^ϵ, where t is adjacent to $\gamma_e = \mathrm{Supp}(e)$. As $q \in U^\circ$, we have that the connected component of $F^n(\gamma_p)$ that contains q intersects $F(\gamma_0)$ in at least one point, say q', since the boundary of W_t^ϵ in V_{Γ_e} is in the exterior of U. The contradiction that we find is that, on the one hand, $x(F^{-1}(q')) > x(p) > 0$

since $q' \in F^{n-1}(\gamma_p)$. On the other hand, $x(F^{-1}(q')) = 0$ since $q' \in F(\gamma_0)$. See Figure 2.11 for an illustration of this proof. $\square$

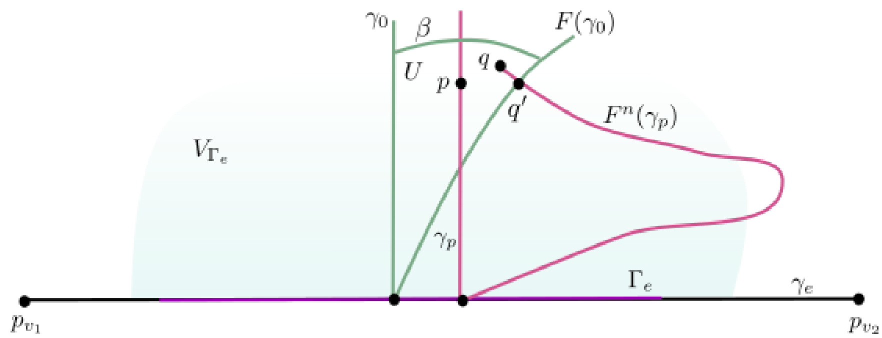

Figure 2.11: Illustration of the proof of Lemma 2.39

We want to remark that the image $c(U)$ of the fundamental domain by the quotient map is topologically a cylinder, since the curve $\gamma_0 \subset U$ is identified with $F(\gamma_0) \subset \overline{U}$. By the Whitney inmersion theorem, $c(U)$ can be embedded as a submanifold in some $\mathbb{R}^n$ with $n \geq 3$. Hence it will make sense to consider the tangent bundle of $c(U)$, curves, foliations and any other geometrical object. We prove the following property.

Lemma 2.40. *$B_{\mathbf{Q}} = \mathrm{Sat}_{\widetilde{V}_{\mathbf{Q}}}(U)$ is a neighborhood of $\gamma_{e_1} \cup \{p_2\} \cup \ldots \cup \{p_{s-1}\} \cup \gamma_{e_{s-1}}$ (excluding the endpoints p_1, p_s), where $\gamma_{e_1} \cup \{p_2\} \cup \ldots \cup \{p_{s-1}\} \cup \gamma_{e_{s-1}}$ are the inner components of the divisor for the path* **Q**. *Moreover, $B_{\mathbf{Q}}$ fulfills the properties in Proposition 2.30.*

Proof. It is enough to prove that $\mathrm{Sat}_{\widetilde{V}_{\mathbf{Q}}}(U)$ is a neighborhood of its basepoint $p_0 = \overline{U} \cap E$, since proving this and applying Proposition 2.23 we have that the saturation of a neighborhood of any point in $\gamma_{e_1} \cup \{p_2\} \cup \ldots \cup \{p_{s-1}\} \cup \gamma_{e_{s-1}}$ is a neighborhood of $\gamma_{e_1} \cup \{p_2\} \cup \ldots \cup \{p_{s-1}\} \cup \gamma_{e_{s-1}}$. Then, we will only work in a small monotonic domain U_{p_0}.

Consider the family of curves $\gamma_C = \{x = C\}$. Notice that the endpoints of these curves are fixed points placed in the divisor. Notice also that these curves can be parameterized by its y coordinate, having $y \geq 0$. There are two possibilities, either $F^{-1}(\gamma_C) \cap \gamma_0 = \emptyset$ or

$$(*) \quad F^{-1}(\gamma_C) \cap \gamma_0 \neq \emptyset$$

Choosing any γ_C with $\gamma_C \cap U \neq \emptyset$, which lies in $\{x > 0\}$ and considering $F^{-1}(\gamma_C \cap U)$ lying in $\{x \leq 0\}$ and the connectedness of the curve $F^{-1}(\gamma_0)$, the intersection $F^{-1}(\gamma_C) \cap \gamma_0$ is not empty. Then, we have the existence of curves γ_C with such property. Observe that provided one γ_C, any other $\gamma_{C'}$ with $0 \leq C' \leq C$ also fulfills $F^{-1}(\gamma_{C'}) \neq \emptyset$.

Take any curve γ_C with the property that $F^{-1}(\gamma_C) \cap \gamma_0 \neq \emptyset$ and select the point q_C such that $F^{-1}(q_C) \in \gamma_0$ with minimum coordinate y. We claim that for any point $q \in \gamma_C$ with $y(q) \leq y(q_C)$

there is $-n_q \in \mathbb{Z}_{\leq 0}$ such that $F^{-n_q}(q) \in U$. To prove this, recall that there are not strict fixed points in V_{Γ_e} and recall that x is monotonic out of E. Then, there exists $-m_q = \max\{-m \in \mathbb{Z} : x(F^{-m}(q)) < 0\}$. We prove that $-n_q = -m_q+1$. Among the points in γ_C with coordinate $y \leq y(q)$, choose the point q' such that $F^{-m_q}(q') \in \gamma_0$ and such that $y(q')$ is maximal. Existence of such a point is ensured by connectedness of the curve $F^{-1}(\gamma_C)$. Now, we indicate the segment of γ_C that lies between q and q' by $\gamma_C|_{q,q'}$. By the definition of q' having maximum y with the property that $F^{-m_q}(q') \in \gamma_0$, we have $F^{-m_q}(\gamma_C|_{q,q'}) \subset \{x \leq 0\}$ and that $F^{-m_q+1}(\gamma_C|_{q,q'}) \subset \{x \geq 0\}$.

On the one hand, we have that $F^{-m_q}(\gamma_C) \cap F^{-1}(\beta) = \emptyset$ for $m_q \geq 2$ and $F^{-m_q}(\gamma_C) \cap F^{-1}(\beta) \neq \emptyset$ for $m_q = 1$. However, the point $t \in F^{-1}(\gamma_C) \cap F^{-1}(\beta)$ with least coordinate y fulfills $y(t) > y(q_C)$, by the definition of q_C and the continuity of F. Since $y(q') < y(q) < y(q_C) < y(t)$ for $m_q = 1$ and $F^{-m_q}(\gamma_C) \cap F^{-1}(\beta) = \emptyset$ in the rest of the cases, we conclude that $F^{-m_q+1}(\gamma_C|_{q,q'}) \cap \beta = \emptyset$. On the other hand, by the maximality condition of q', there are not points between the extreme points $F^{-m_q}(q)$ and $F^{-m_q}(q')$ intersecting γ_0. Then, the segment $F^{-m_q+1}(\gamma_C|_{q,q'}) \setminus \{F^{-m_q+1}(q')\}$ lies inside U and, in particular, the point $F^{-m_q+1}(q) \in U$. See Figure 2.12 for the illustration of these arguments.

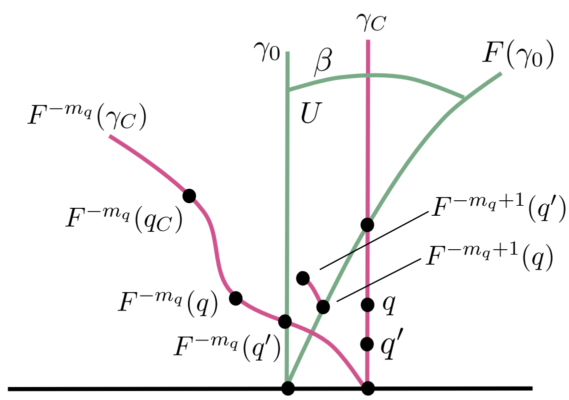

Figure 2.12: Illustration of the proof of Lemma 2.40.

Therefore, we have that the collection of points $B_+ = \{q = (C, y(q)) \in U_p : 0 < y(q) < y(q_C), C \leq C_m\} \cup U$ is contained in $\mathrm{Sat}_{V_{\Gamma_e}}(U) \subset \mathrm{Sat}_{\widetilde{V}_{\mathbf{Q}}}(U)$, where C_m is any $C_m > 0$ fulfilling the property (*). Proceeding similarly, we find another open set B_- having the points with $x \leq 0$ with an iterate in $F^{-1}(U)$ (and consequently in U), we find therefore, that there is an open neighborhood of p_0 contained in $\mathrm{Sat}_{\widetilde{V}_{\mathbf{Q}}}(U)$, as we claimed.

The fact that $B_{\mathbf{Q}}$ fulfills one of the properties in Proposition 2.30 follows from the construction of $B_{\mathbf{Q}}$ inside $\widetilde{V}_{\mathbf{Q}}$ and the fact that it is saturated. □

Notice that the set $B_{\mathbf{Q}} = \mathrm{Sat}_{\widetilde{V}_{\mathbf{Q}}}(U)$ is saturated in $\widetilde{V}_{\mathbf{Q}}$ which is also saturated in W_E. We conclude that $B_{\mathbf{Q}}$ is saturated in W_E.

We define the following curve

$$\begin{array}{rccl} \alpha: & I_{\mathbf{Q}} & \to & V_{\mathbf{Q}} \\ & t & \mapsto & \alpha(t) = F^n(\tilde{\beta}(t-n)) \text{ with } n = \lfloor t \rfloor \end{array}$$

where $I_{\mathbf{Q}}$ is the maximum open set in which the curve can be defined. We will set $I_{\mathbf{Q}}^- = I_{\mathbf{Q}} \cap \mathbb{R}_{\leq 0}$ and $I_{\mathbf{Q}}^+ = I_{\mathbf{Q}} \cap \mathbb{R}_{\geq 0}$. We claim that $I_{\mathbf{Q}}^- = \mathbb{R}_{\leq 0}$ when the quadrant Q_1 is of node type. It follows since given any point p in $V_{\mathbf{Q}}$, we have that $\mathrm{Orb}_{V_{\mathbf{Q}}}(p)$ is infinite and accumulates in a strict fixed point (in the node case). When the quadrant Q_1 is of saddle type, for every p in $V_{\mathbf{Q}}$, we have that $\mathrm{Orb}^-_{\widetilde{V}_{\mathbf{Q}}}(p)$ is finite. That is, for each $p \in \beta$ there is some $-n_p$ such that $F^{-n_p}(p) \in \widetilde{V}_{\mathbf{Q}}$ and $F^{-n_p-1}(p) \notin \widetilde{V}_{\mathbf{Q}}$. By the compactness of β, the n_p are bounded and $I_{\mathbf{Q}}^-$ is the union of an interval of the form $(a, 0]$ and a finite number of open intervals (a_i, b_i) with $a_i < b_i < a$. We study the shape of $I_{\mathbf{Q}}^+$ in the same manner. We define the curve γ to be union of the segments of α and the segments of $\partial\widetilde{V}_{\mathbf{Q}}$ without $\mathrm{Supp}(G(\xi, \pi))$.

See Figure 2.13 for an illustration of the construction of $\widetilde{B}_{\mathbf{Q}}$ in two examples: a hyperbolic sector and a parabolic sector.

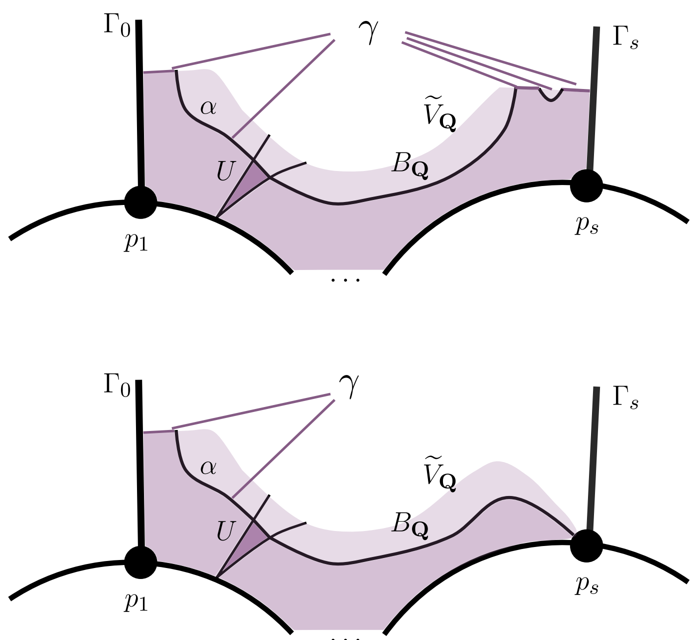

FIGURE 2.13: Hyperbolic and parabolic sectors obtained from a path of quadrants $\mathbf{Q}$ as the saturation of a fundamental domain U. The curve α is semi-analytic in a neighborhood of E except of the points in $\mathrm{Fix}(F)$ in both cases. In the second picture, it is also a parabolic curve at p_s.

Lemma 2.41. *The curve $\gamma \subset \partial B_{\mathbf{Q}}$ is semi-analytic in $W_E \setminus E$.*

Proof. The semi-analyticity of the segments α follows since β is semi-analytic and smooth. We remark that α is the union of β and $F^n(\beta)$ for arbitrary β. In addition this union is locally finite in $W_E \setminus E$, accumulation of segments $F^n(\beta)$, if it exists, occurs in strict fixed points or dicritical curves, but we are excluding the dicritical curves in this section. Then, given any point q of $W_E \setminus E$, we have that α is locally $F^n(\beta)$, and semi-analyticity follows.

Other segments that can be part of γ are segments of the boundary of W_1 and W_s, when they are saddle domains. By hypothesis H-S, semi-analyticity follows.

Finally, γ is the finite union of the segments presented above, then it is semi-analytic in $W_E \setminus E$. □

Either Q_1 or Q_s is a dicritical quadrant

We start by showing that the local parabolic curves that we found in the dicritical components (Proposition 2.29) can be globalized on the sector.

Lemma 2.42. *Let $\mathbf{Q}$ be such that Q_1 is a dicritical quadrant. Then, for every $q \in \Gamma_0$ sufficiently close to p_1, there exists a connected analytic parabolic curve γ at q saturated in $\widetilde{V}_{\mathbf{Q}}$.*

Proof. Taking the intersection $W_1 \cap \overline{\widetilde{V}_{\mathbf{Q}}}$, take an open neighborhood N_{p_1} of p_1 in the subspace topology, and take any point $q \in N_{p_1} \cap \Gamma_0$. Notice that such set N_{p_1} exists as a consequence of the fourth statement of Proposition 2.28. As a consequence of Proposition 2.29, there is a local analytic parabolic curve γ_q at q defined in U_q, asymptotic to a formal invariant curve (cf. Definition 2.5). Since $U_q \subset N_{p_1}$ and N_{p_1} is open and contained in $W_1 \cap \overline{\widetilde{V}_{\mathbf{Q}}}$, and the fact that $\widetilde{V}_{\mathbf{Q}}$ is saturated, we have that the parabolic curve can be extended to γ in $\widetilde{V}_{\mathbf{Q}}$, by taking $\mathrm{Sat}_{V_{\mathbf{Q}}}(\gamma_q)$. □

Now, we claim that such a curve γ in $\widetilde{V}_{\mathbf{Q}}$ encloses a neighborhood of $\gamma_{e_1} \cup \{p_2\} \cup \ldots \cup \{p_{s-1}\} \cup \gamma_{e_{s-1}}$ that has the properties of a sector. Recall that γ is invariant and saturated, then, it is a simple curve γ that admits a parameterization $\tilde{\gamma} : I \to \widetilde{V}_{\mathbf{Q}}$ such that I is open and $F(\tilde{\gamma}(t)) = \tilde{\gamma}(t+1)$ (when the image of $\tilde{\gamma}(t)$ is defined in $\widetilde{V}_{\mathbf{Q}}$) and such that $\alpha(\tilde{\gamma}) = q$ in the dicritical curve Γ_0. Indeed, as in the previous section, it is enough to study the saturation in $\widetilde{V}_{\mathbf{Q}}$ of the compact set $\gamma([r_0, r_0+1])$ for some $r_0 \in I$ such that the segment is contained in $\widetilde{V}_{\mathbf{Q}}$.

Lemma 2.43. *The parabolic curve γ encloses a neighborhood $B_{\mathbf{Q}}$ of $\gamma_{e_1} \cup \{p_2\} \cup \ldots \cup \{p_{s-1}\} \cup \gamma_{e_{s-1}}$. Moreover, $B_{\mathbf{Q}}$ fulfills one of the properties in Proposition 2.30.*

By Proposition 2.33, the set $B_{\mathbf{Q}}$ is projected to $\mathbb{R}^2$ into a sector. See Figure 2.14 for an illustration on the construction of $B_{\mathbf{Q}}$.

Proof. We construct first the region $B_{\mathbf{Q}}$, and then we prove that it fulfills one of the properties of Proposition 2.30. It is enough to distinguish the three possible cases: Q_s is a saddle, a node or a dicritical quadrant. In the three cases we study the saturation of the compact segment of γ given by $\tilde{\gamma}([r_0, r_0+1])$ for some r_0 as above. In the saddle case, recall that the saddle domain W_s

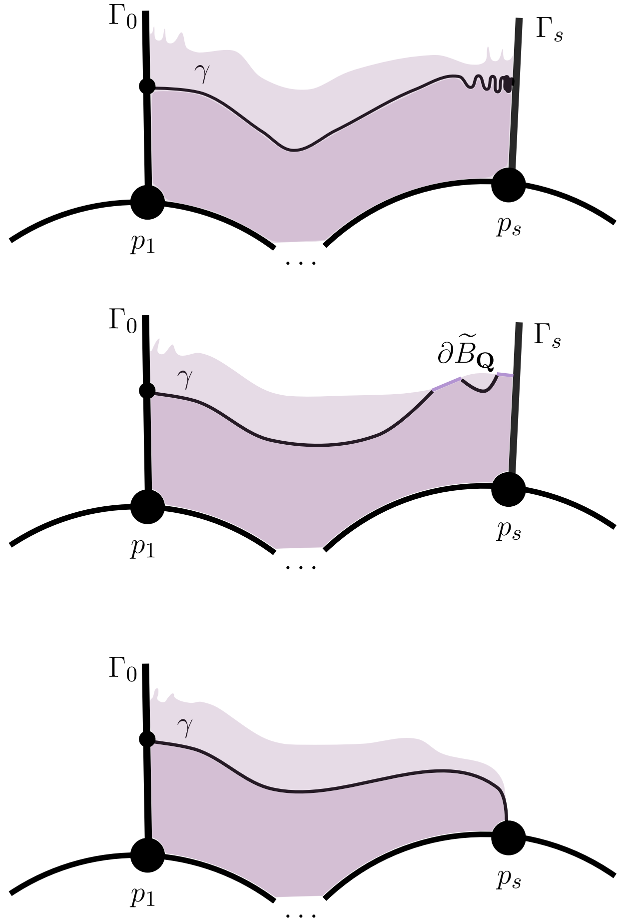

Figure 2.14: Construction of $B_{\mathbf{Q}}$ when $\mathbf{Q}$ has a dicritical quadrant. In the upper case, when the other extreme is also dicritical. In the second case, when the other extreme is saddle. In the third case, when it is a node (in particular, when p_s is of type n-n-2 and the edge e_s does not have a realization).

is in the conditions of H-S. We have that the curve γ intersects the semi-analytic component of the boundary $\{y = C_1\}$ a finite number of times. Then, we conclude the region $B_{\mathbf{Q}}$ enclosed by a finite union of segments of γ and a finite union of segments of $\{y = C_1\} \subset W_s$. In the node case, we have that $\omega(\tilde{\gamma}) = p_s$. Then, $B_{\mathbf{Q}}$ is the region enclosed by this connected curve. In the dicritical case, suppose that $\tilde{\gamma}([r_0, r_0 + 1]) \subset W_s$. By the continuity of the ω–limit in the dicritical domains shown in the fifth stament of Proposition 2.28, we have that $\omega(\tilde{\gamma})$ is a compact connected set that does not contain p_s. Then, we have γ encloses the region $B_{\mathbf{Q}}$ and it has semi-analytic boundary in $W_E \setminus \mathrm{Fix}(F)$.

Now, we prove that $B_{\mathbf{Q}}$ is indeed a sector by showing that it is saturated in $V_{\mathbf{Q}}$. To see this, take any point $p \in B_Q \setminus \mathrm{Fix}(F)$. We have that $\alpha(p)$ is a single point p' in $\Gamma_0 \cap \overline{\widetilde{V}_{\mathbf{Q}}}$. Take the saturated parabolic curve γ' of p'. We claim that γ' does not intersect with γ. This is because otherwise we find a contradiction. That is, the intersection point p'' would have $\alpha(p'') = \{q\}$ since $p'' \in \gamma$

and $\alpha(p'') = \{p'\}$. Then the curve γ', which contains $\mathrm{Orb}_{\widetilde{V}_{\mathbf{Q}}}(p)$, lies in $B_{\mathbf{Q}}$. From the property that $\mathrm{Orb}_{\widetilde{V}_{\mathbf{Q}}}(p) \subset B_{\mathbf{Q}}$, we have that $B_{\mathbf{Q}}$ is saturated. □

Refinements of the parabolic sectors

In the previous section we have obtained a new collection of path sectors, we call its union B_{path}. We apply Proposition 2.35 in W_E for the set B_{path}, obtaining a neighborhood of E given by

$$\widetilde{W} = E \cup B_{path} \cup \mathrm{Supp}(\mathcal{E}_{ndiv}) \cup \tilde{B}_1 \cup \cdots \cup \tilde{B}_k \cup \tilde{\gamma}_1 \cup \ldots \cup \tilde{\gamma}_r$$

Lemma 2.44. *There exists parabolic sectors $B_i \subset \tilde{B}_i$ and parabolic curves $\gamma_j \subset \tilde{\gamma}_j$ such that B_i has semi-analytic boundary on $W_E \setminus E$ and such that*

$$E \cup B_{path} \cup \mathrm{Supp}(\mathcal{E}_{ndiv}) \cup B_1 \cup \cdots \cup B_k \cup \gamma_1 \cup \ldots \cup \gamma_r$$

is a neighborhood of E.

Proof. It is enough to reduce the sector in any of the possible cases: $\tilde{B}_i$ is constructed on a node domain W_i at a point p_i fulfilling H-N-I, H-N-II, or on a monotonic domain V_{Γ_i} at the regular dicritical arc Γ_i. In any of the cases, taking the corresponding coordinates (x, y), we have that y is an analytic function monotonically increasing or decreasing on the orbits. In the case f-n-n There is always a path sector $B_{\mathbf{Q}}$ ending on p_i or on Γ_i. With this, we mean that $\overline{B_{\mathbf{Q}}} \cap \{p_i\} \neq \emptyset$ or $\overline{B_{\mathbf{Q}}} \cap \Gamma_i \neq \emptyset$. We define $B_i = \tilde{B}_i \cap \{y < C\}$ for some small enough $C > 0$ so that one of the following is fulfilled.

- If p_i is of type n or n-n-2, there are two paths $\mathbf{Q}$ and $\mathbf{Q}'$ fulfilling the above. We take $\{y = C\}$ joining two points in the boundary of $B_{\mathbf{Q}}$ in W_i and $B_{\mathbf{Q}'}$ in $\tilde{B}_i$.
- If p_i is of type f-n-n, f-n-s, n-n-1 or n-s. There is a single path $\mathbf{Q}$ fulfilling the above. We take $\{y = C\}$ joining a point in the boundary of $B_{\mathbf{Q}}$ in $\tilde{B}_i$ and the curve of fixed points (f-n-n or f-n-s) or parabolic curve (n-n-1 or n-s), given by $\{x = 0\}$.
- If Γ_i is a regular dicritical arc, there are two paths $\mathbf{Q}$ and $\mathbf{Q}'$ fulfilling the above. We take $\{y = C\}$ joining two points in the boundary of $B_{\mathbf{Q}}$ in V_{Γ_i} and $B_{\mathbf{Q}'}$ in $\tilde{B}_i$.

We have then, that the boundary of B_i is semi-analytic in $W_E \setminus E$. The curves $\gamma_i \subset \tilde{\gamma}_i$ are also refined so that its extreme is given by the corresponding interesection of the adjacent ∂B_j and $\partial B_{\mathbf{Q}}$. □

2.5.2 On the openness of the sectorial decomposition

In the absence of D-D sectors, we can construct a sectorial decomposition $(U,\mathcal{S})$ taking U open and ensuring that $\mathcal{S}$ is a stratification. We give the final refinements on the construction of $(U,\mathcal{S})$ in this section, concluding Proposition 2.8. Finally, we present the possible obstructions in order to obtain these properties when D-D sectors exists.

Let $e \in \mathcal{E}_{ndiv}$ be an edge out of the divisor and $\mathrm{Supp}(e)$ its support in some neighborhood of E in M. It is clear that in the points of type s-s, n-s, n-n-1, f-s-s, f-n-s and f-n-n, we can take the two realization of quadrants W_-, W_+ such that $W_- \cap \mathrm{Supp}(e) = W_+ \cap \mathrm{Supp}(e)$. Then, we choose the curve $\Gamma = W_+ \cap \mathrm{Supp}(e)$ to be a one dimensional stratum of $\mathcal{S}$. See Figure 2.15 for an illustration of this case.

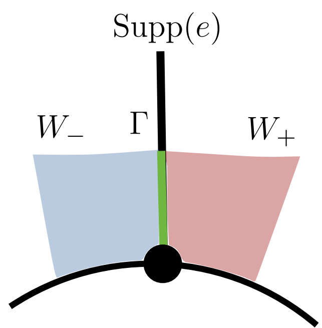

Figure 2.15: Choice of W_-, W_+ and one dimensional stratum Γ.

In this setting, it is possible to construct sectors whose union is open in $W_- \cup \Gamma \cup W_+$. In particular, in a saddle quadrant, we can construct a sector ending at W_ϵ and bounded by Γ, an invariant curve and non-invariant segments of the boundary of W_ϵ, as shown in the previous section. In a node quadrant, we can find a path sector $B_{\mathbf{Q}}$ and a parabolic sector B intersecting W_ϵ. The parabolic sector can be constructed so that its boundary contains Γ, an invariant curve in the boundary of $B_{\mathbf{Q}}$ and another non-invariant curve such that one of its endpoints is the endpoint of Γ, as we did in the previous section. We conclude then that in these types of points (s-s, n-s, n-n-1, f-s-s, f-n-s and f-n-n), the union of the sectors intersecting $W_- \cup \Gamma \cup W_+$ is open therein.

However, when $\mathrm{Supp}(e)$ is a bidicritical curve, we know that the two sectors adjacent to this curve will be of type D-parabolic, D-elliptic or D-D. We can take the realizations W_-, W_+ of the two dicritical quadrants such that $W_- \cap \mathrm{Supp}(e) = W_+ \cap \mathrm{Supp}(e)$, and choose $\Gamma = W_+ \cap \mathrm{Supp}(e)$ to be the one dimensional strata. In addition, at the time of constructing the sector on each of the paths ending at this curve, we can choose the curves γ_-, γ_+ in Lemma 2.42 such that both γ_- and γ_+ have the same basepoint $q \in \Gamma$. See Figure 2.16 for an illustration of this. Avoiding the presence of D-D sectors, we can also ensure that $\mathcal{S}$ is a stratification because, on the one hand, we can choose one dimensional strata $\Gamma = W_- \cap \mathrm{Supp}(e) = W_+ \cap \mathrm{Supp}(e)$. On the other hand, the rest of one dimensional strata are parabolic curves lying between path sectors and parabolic ones, and the curves γ in between are chosen fulfilling the boundary condition $\gamma = \partial V_i \cap \partial B_{\mathbf{Q}}$.

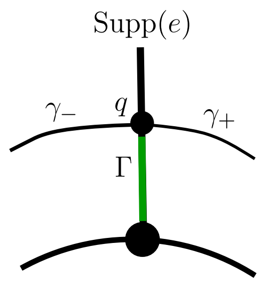

Figure 2.16: Choice of the boundary curves of two sectors ending on a bidicritical curve.

Now, we illustrate a possible chaotic behavior in the presence of D-D sectors. In a D-D sector, we have two dicritical curves Γ, Γ'. The parabolic curve γ at any basepoint $q \in \Gamma$ accumulates in a compact set $L_\gamma \subset \Gamma'$, as we have already proved in Proposition 2.28. If L_γ were a single point ($L_\gamma = \{q'\}$), we would have that γ is also a parabolic curve in the basepoint q', because of the characterization given in Proposition 2.29. Moreover, in the other sector adjacent to Γ' we would choose again a parabolic curve with basepoint q'. The problem is that we cannot ensure that L_γ is a single point. The points in L_γ must belong to $\pi^{-1}(U)$, since they are α or ω–limits of points in the sector. However, none of the points in L_γ can belong to the interior of U (since any neighborhood of any point in L_γ intersects the curve γ that necessarily belongs to the boundary of $B_{\mathbf{Q}}$). This fact is independent of the choice of the parabolic curve in the following sector adjacent to Γ'. See Figure 2.17 for an illustration.

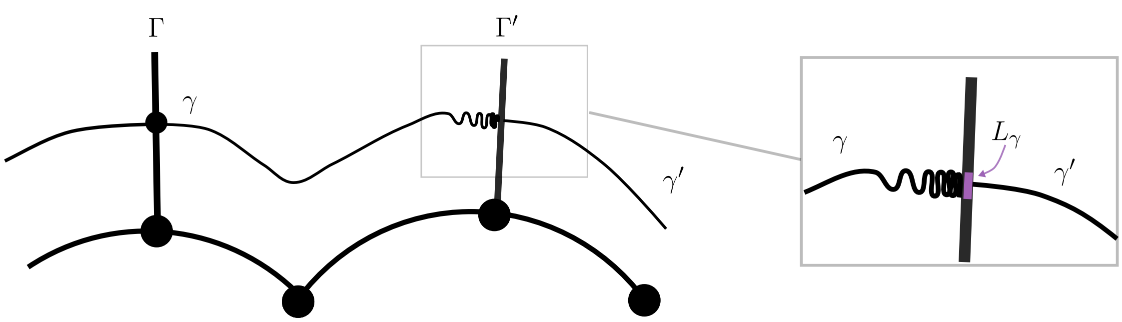

Figure 2.17: The problem of D-D sectors.

To see that $\mathcal{S}$ is not necessarily a stratification it is enough to consider the following example, where all the sectors are of type D-D. We illustrate this case in the Figure 2.18. We see that it is possible that $\overline{A}_2 = \Gamma_1 \cup A_2 \cup \tilde{\Gamma}_2$, where $\tilde{\Gamma}_2$ is strictly contained in Γ_2.

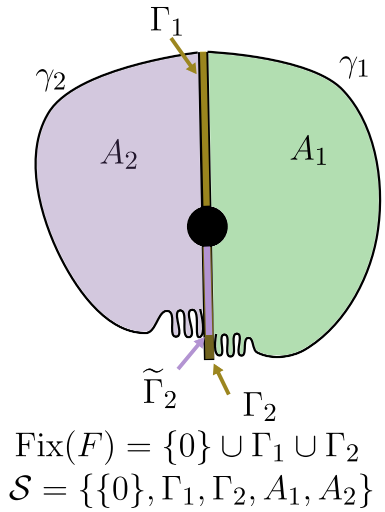

FIGURE 2.18: An example in which $U = \{0\} \cup \Gamma_1 \cup \Gamma_2 \cup A_1 \cup A_2$ is not open and $\mathcal{S}$ is not a stratification.

2.5.3 On the semi-analyticity of the sectorial decomposition

In this section, we use the results in Section 2.5.1 to prove that the set U is semi-analytic, under the hypothesis that there are not bidicritical curves, as Proposition 2.9 claims.

It is enough to consider an initial sectorial decomposition $(W_E, \mathcal{S})$ (before projection to $\mathbb{R}^2$) as in Section 2.5.1, and then make the refinements proposed in Lemma 2.40 and 2.44. We obtain a new sectorial decomposition $(\widetilde{U}, \mathcal{S})$ (before projection). On each of the sectors, we have semi-analytic (in $W_E \setminus E$) curves γ provided by the Lemmas. On the one hand, we have that $\partial\widetilde{U}$ is strictly contained in the closure of the union of the curves γ. On the other hand, $\overline{\partial\widetilde{U}} \cap E = \emptyset$, since $\widetilde{U}$ is by construction a neighborhood of E. Then, recalling that the semi-analyticity of the boundary curves γ could only drop in E, we conclude that $\partial\widetilde{U}$ is a semi-analytic curve and thus, the set $\widetilde{U}$ is semi-analytic. Since the sequence π is an analytic isomorphism out of E, we conclude that U is semi-analytic.

We just make some comments on why the semi-analytic property is not achieved in the presence of bidicritical curves of fixed points. At any point q in a bidicritical curve Γ, we can define an analytic parabolic curve asymptotic to a formal one. However, the definition of this curve is not necessarily analytically extended to the basepoint q in Γ, sometimes this extension is only $\mathcal{C}^\infty$. For this reason, we think that U cannot be chosen in the semi-analytic class, as in the absence of bidicritical curves. However, we think that semi-analyticity of $U \setminus \mathrm{Fix}(F)$ on $\mathbb{R}^2 \setminus \mathrm{Fix}(F)$ be obtained in the presence of bidicritical curves.

Chapter 3

Dulac's problem for vector fields with a Hopf singularity

This chapter is devoted to the second problem of the thesis, the study of the cycle locus of germs of analytic vector fields at $(\mathbb{R}^3, 0)$ having a Hopf singularity. The main objective is to give an answer to Dulac's problem for these vector fields. This problem states that there are no infinitely many isolated cycles accumulating and collapsing to a singular point. We resolve it by proving Theorem 3.1 (Theorem B in the Introduction). Here, we collect the result of the published article [23], in which we worked with isolated Hopf singularities, and extend that result to the non-isolated Hopf singularities. The proof in the isolated singularity case is simplified with respect to that in [23], by using Theorem 2.6 of Chapter 2. The proof in the non-isolated singularity case again uses that Theorem and also the other main result proved in this chapter, Theorem 3.3 (Theorem C in the Introduction).

3.1 Hopf singularities

Let us introduce the notation and necessary definitions in order to state the main result. We denote by $\mathfrak{X}^{\omega}(\mathbb{R}^3,0)$, or simply by $\mathfrak{X}(\mathbb{R}^3,0)$, the space of germs of analytic vector fields at the origin of $\mathbb{R}^3$ which are singular at 0. We say that an element ξ has a *Hopf singularity* if the linear part $D\xi(0)$ has eigenvalues $\pm bi, c$, with $b \in \mathbb{R}_{\neq 0}$ and $c \in \mathbb{R}$. If, moreover, the real eigenvalue c is equal to 0, we say that ξ has *Hopf-zero singularity*. Otherwise, we say that ξ has a *semi-hyperbolic Hopf singularity*. Denote by

$$\mathcal{H}^3 := \{\xi \in \mathfrak{X}^{\omega}(\mathbb{R}^3,0) : \operatorname{Spec}(D\xi(0)) = \{\pm bi, c\}, \text{ where } b,c \in \mathbb{R} \text{ and } b \neq 0\}.$$

the family of germs with a Hopf singularity.

Fix any $\xi \in \mathfrak{X}(\mathbb{R}^3,0)$. Consider an open neighborhood U of 0 where (a representative of) ξ is defined. We denote by $\mathcal{C}_U = \mathcal{C}_U(\xi)$ the union of all cycles of $\xi|_U$ (that is, cycles entirely contained in U). It is called the *cycle-locus of ξ in U*. Notice that this cycle-locus depends strongly on the neighborhood U and that it does not behave as a germ of a set that we can associate to the germ ξ (i.e., if $U' \subset U$ we can only assert that $\mathcal{C}_{U'} \subset \mathcal{C}_U$, but not $\mathcal{C}_{U'} = U' \cap \mathcal{C}_U$). When the germs $\mathcal{C}_U(\xi)_0$, $\mathcal{C}_{U'}(\xi)_0$ of $\mathcal{C}_U(\xi)$ and $\mathcal{C}_{U'}(\xi)$ coincide at 0 for every pair of sufficiently small neighborhoods U, U' of 0, we define the *local cycle-locus in U*

$$\mathcal{C}(\xi) = \mathcal{C}_U(\xi)_0 \text{ for a small enough } U.$$

It is not difficult to prove (see [8]) that a Hopf singularity $\xi \in \mathcal{H}^3$ has a unique formal invariant curve $\widehat{\Omega} = \widehat{\Omega}_\xi$ at 0. Such an invariant curve is non-singular and tangent to the eigenspace corresponding to the eigenvalue c. It is called the *(formal) rotational axis* of ξ. When $c \neq 0$ (the *semi-hyperbolic case*), the rotational axis is convergent and provides an analytic invariant curve, since in this case $\widehat{\Omega}$ coincides with the stable or unstable manifold of ξ (see for instance [21] for a proof of the analyticity of these invariant manifolds in general). On the contrary, when $c = 0$ (the *completely non-hyperbolic case* or *Hopf-zero singularity*), the rotational axis $\widehat{\Omega}$ may be convergent or not, although there is always an invariant $\mathcal{C}^\infty$-curve whose Taylor expansion at 0 coincides with $\widehat{\Omega}$. This is a result by Bonckaert and Dumortier in [8] in the case where ξ has an isolated singularity, and trivially true otherwise since, in this case, $\widehat{\Omega}$ coincides with the singular locus $\operatorname{Sing}(\xi)$, which is an analytic curve.

3.1.1 Statement of the main results

In this section we state the main results. We start with a result that provides the description of the cycle locus for Hopf singularities. It is Theorem B in the Introduction.

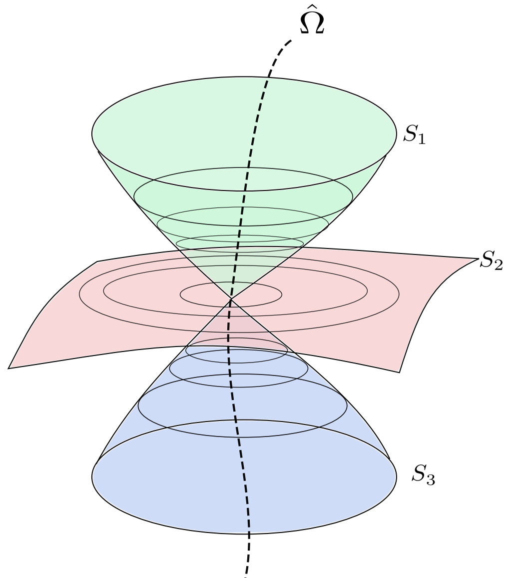

Figure 3.1: Limit central surfaces.

Theorem 3.1 (Structure of the local cycle locus)**.** *Let $\xi \in \mathcal{H}^3$. Then there is a neighborhood U of $0 \in \mathbb{R}^3$ where a representative of ξ is defined for which exactly one of the following possibilities holds:*

(i) $\mathcal{C}_U(\xi) = \emptyset$.

(ii) There is a finite non-empty family $\mathcal{S} = \{S_1, ..., S_r\}$ of connected smooth analytic two-dimensional submanifolds of $U \setminus \{0\}$, mutually disjoint, invariant for ξ, subanalytic sets satisfying $\overline{S_j} = S_j \cup \{0\}$ for any j, and there is a neighborhood basis $\mathcal{V}$ of the origin in U such that every $V \in \mathcal{V}$ satisfies

$$\mathcal{C}_V(\xi) = (S_1 \cup S_2 \cup \cdots \cup S_r) \cap V. \tag{3.1}$$

(iii) The singular locus $\mathrm{Sing}(\xi|_U)$ *of ξ in U is a smooth analytic curve in U and there is a neighborhood basis $\mathcal{V}$ of the origin in U such that every $V \in \mathcal{V}$ satisfies*

$$\mathcal{C}_V(\xi) = V \setminus (V \cap \mathrm{Sing}(\xi|_U)). \tag{3.2}$$

Consequently, the local cycle locus $\mathcal{C}(\xi)$ of ξ exists and it is equal to the empty germ, to the germ of $S_1 \cup \cdots \cup S_r$ or the complement of the germ of $\mathrm{Sing}(\xi)$ *in cases (i), (ii) or (iii), respectively.*

The surfaces in item (ii) will be called the *limit central surfaces* of ξ (see Figure 3.1 for an example). Notice that possibility (i) of the theorem can be included on the second fixing $r = 0$ when there are no limit central surfaces.

The main consequence of Theorem 3.1 is the one concerning Dulac's problem, solving it for this type of vector fields. It is Corollary B in the Introduction.

Corollary 3.2 (Dulac's Property). *If $\xi \in \mathcal{H}^3$, there is a neighborhood of $0 \in \mathbb{R}^3$ which is free of isolated cycles of ξ.*

Some comments about Theorem 3.1 are in order. In the particular semi-hyperbolic case ($c \neq 0$), we obtain that only possibilities (i) or (ii) with $r = 1$ of the theorem can occur. Moreover, in the last case, the unique limit central surface is a smooth analytic center manifold of ξ at 0. This situation has been treated by many authors, namely [4, 46, 49], but we will present a proof in Section 3.2 using the theory of center manifolds.

In the Hopf-zero case, any situation of the theorem can occur. The possibility (iii) means that all non-trivial trajectories are periodic. In the literature, this situation is known by saying that ξ is a *three-dimensional center*. It can be noticed that when ξ is a three-dimensional center, its singularity is not isolated, as a consequence of a Brunella's result [14]: if ξ has an isolated singularity at $0 \in \mathbb{R}^3$, then there is a non-trivial trajectory γ of ξ such that $\omega(\gamma) = \{0\}$ or $\alpha(\gamma) = \{0\}$.

We also provide a characterization of the three-dimensional centers of Hopf type, as follows. This Theorem corresponds to Theorem C in the Introduction.

Theorem 3.3. *Let $\xi \in \mathcal{H}^3$, the following statements are equivalent.*

(1) ξ is formally degenerate (its normal form has the form $G(x^2+y^2,z)(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y})$ where $G \in \mathbb{R}[[u,v]]$ is a unit).

(2) ξ is analytically orbitally linearizable (i.e. it is orbitally equivalent to its linear part up to multiplication by an analytic unit)

(3) There is a neighborhood U of 0 such that $\mathcal{C}_U = U \setminus Sing(\xi)$.

(4) ξ is analytically completely integrable (i.e. there are two analytic first integrals f, g at 0 satisfying $df \wedge dg \neq 0$).

We highlight that some of the equivalences and implications provided in this Theorem have also been obtained by other authors, as we presented in the Introduction. We will also show that (1) is equivalent to a weaker statement: ξ is formally orbitally linearizable. In section 3.1.3, we will associate a two dimensional formal vector field to each normal form. With that tool in hand, we define the formally degenerated Hopf-zero vector fields as those that have a zero vector field associated to them.

We finish the section by showing that any cycle makes a single turn around a rotational axis, which is a direct consequence of Theorem 3.1 in the situations (i) and (ii), and the fact that ξ is analytically orbitally linearizable provided in Theorem 3.3 in the situation (iii).

Corollary 3.4. *Let $\xi \in \mathcal{H}^3$ and suppose that its local cycle locus is non-empty. Let Ω_∞ be a C^∞ realization of the formal rotational axis. Then, the neighborhood basis $\mathcal{V}$ in Theorem 3.1, (ii) or (iii)* *can be chosen so that $V \setminus \Omega_\infty$ is homotopically equivalent*[1] *to $\mathbb{S}^1$ and any cycle $\gamma \subset C_V(\xi)$ is a generator of $\pi_1(V \setminus \Omega_\infty)$.*

We show now an example of a vector field that defines two limit central surfaces and then an example of a three dimensional center.

Example 3.5. Consider the following vector field in $\mathcal{H}^3$.

$$\xi = (-y - xz^2 + x(x^2+y^2))\frac{\partial}{\partial x} + (x - yz^2 + y(x^2+y^2))\frac{\partial}{\partial y} + (z^3 + z(x^2+y^2))\frac{\partial}{\partial z}.$$

It has isolated singularity. The two half-cones $S_1 = \{(x,y,z) : x^2+y^2-z^2 = 0, z > 0\}$ and $S_2 = \{(x,y,z) : x^2+y^2-z^2 = 0, z < 0\}$ are invariant. The restriction of ξ to any of the surfaces S_i is $\xi|_{S_i} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, which proves that ξ defines a central configuration in S_i, for $i = 1,2$.

In this example, one can see that there are no cycles outside $S_1 \cup S_2$ in a neighborhood of 0. Thus, we obtain that ξ is of type (ii) of Theorem 3.1 with two limit central surfaces ($r = 2$), given by S_1 and S_2.

Example 3.6. Consider the following vector field in $\mathcal{H}^3$.

$$\xi = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

The z–axis is a smooth curve of singularities of ξ. The rest of the trajectories are cycles, given by the intersection of the level surfaces of its two first integrals: the coordinate function z and x^2+y^2. Then, the vector field ξ is a three dimensional center (type (iii) in Theorem 3.1)

3.1.2 Normal form theorems applied to Hopf-zero singularity

Let $\xi \in \mathcal{H}^3$ be a Hopf-zero singularity. Using the normal forms theorems stated in section **??** (Theorems A.26 and A.29) for ξ, we have the following: There exists a formal automorphism at 0, expressed in terms of the chosen coordinates as

$$\hat{\varphi}(x,y,z) = (x + \hat{\varphi}_1(x,y,z), y + \hat{\varphi}_2(x,y,z), z + \hat{\varphi}_3(x,y,z)) \in \mathbb{R}[[x,y,z]]^3,$$

with $j_1(\hat{\varphi}_j) = 0$ for $j = 1,2,3$, such that the formal vector field $\hat{\xi} = \hat{\varphi}^*(\xi)$ is written in the form

$$\hat{\xi} = A(x^2+y^2,z)\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + B(x^2+y^2,z)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + C(x^2+y^2,z)\frac{\partial}{\partial z}, \tag{3.3}$$

[1]We recall that being homotopically equivalent implies that the two topological spaces have the same homotopy group. For a definition of this equivalence, see [38].

where $A,B,C \in \mathbb{R}[[u,v]]$ and $A(0,0)=1$. Note that $A(u,v), C(u,v)$ belong to the ideal (u,v). Neither the automorphism $\hat{\varphi}$ need to be convergent, nor the components of $\hat{\xi}$ need to belong to $\mathbb{R}\{x,y,z\}$. Any formal vector field $\hat{\xi}$ as in (3.3) obtained as above is called a formal normal form of ξ. We remark that $\hat{\xi}$ is not uniquely determined by ξ.

Remark 3.7. The z–axis is sent to the rotational axis $\widehat{\Omega}$ of ξ by $\hat{\varphi}$, that is, $\widehat{\Omega} = \hat{\varphi}(0,0,z)$. On the other hand, since $\hat{\varphi}$ must preserve the (formal) singular locus, the hypothesis that ξ has isolated singularity implies that $C(0,v) \neq 0$ for $v \neq 0$, in particular, we have $C(0,v)$ belongs to the ideal (v^2). Meanwhile, when ξ has a curve of singularities, any formal normal form $\hat{\xi}$ satisfies that $\mathrm{Sing}(\hat{\xi})$ is the z–axis.

Once we fix a formal normal form $\hat{\xi}$ of ξ given by $\hat{\xi} = \hat{\varphi}^*\xi$, we can consider normal forms of ξ up to some jet in the following way. For any $\ell \in \mathbb{N}_{\geq 2}$, let φ_ℓ be the polynomial tangent to the identity diffeomorphism of $(\mathbb{R}^3,0)$ given by

$$\varphi_\ell(x,y,z) = (j_{\ell+1}\hat{\varphi})(x,y,z) = (j_{\ell+1}(x\circ\hat{\varphi}), j_{\ell+1}(y\circ\hat{\varphi}), j_{\ell+1}(z\circ\hat{\varphi})).$$

The vector field $\xi_\ell = (\varphi_\ell)^*(\xi)$ has the same ℓ–jet as the formal one $\hat{\xi}$ in coordinates (x,y,z). That is, $j_\ell(\xi_\ell) = j_\ell(\hat{\xi})$. Notice that the vector field ξ_ℓ is analytically conjugated to ξ and formally conjugated to $\hat{\xi}$ for any ℓ. More precisely, we have the following formal equation:

$$\hat{\xi} = \psi_\ell^*\xi_\ell, \text{ where } \psi_\ell := \varphi_\ell^{-1}\circ\hat{\varphi}. \tag{3.4}$$

Assertion †. *It is sufficient to prove Theorem 3.1 for ξ_ℓ for some $\ell \geq 2$ when ξ has a Hopf-zero singularity.*

3.1.3 Two dimensional vector field associated to a normal form. Degenerated and Non-Degenerated Hopf-zero singularities.

Now, fix a normal form $\hat{\xi}$ of ξ, written as (3.3). First, consider that the coefficient A of $-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ is a unit, then we obtain an equivalent vector field ζ.

$$\zeta = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{B(x^2+y^2,z)}{A((x^2+y^2,z))}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + \frac{C(x^2+y^2,z)}{A(x^2+y^2,z)}\frac{\partial}{\partial z}, \tag{3.5}$$

given by the product with $\frac{1}{A}$. We write this vector field in cylindrical coordinates (θ,z,ρ), where $x = \rho\cos\theta$ and $y = \rho\sin\theta$. We obtain $\zeta(\theta) = 1$, so that, using θ as the time parameter, ζ is described by the following autonomous system of ODEs

$$\zeta : \begin{cases} \frac{dz}{d\theta} &= A(\rho^2, z)^{-1} B(\rho^2, z) &= \rho^n A_z(z, \rho) \\ \frac{d\rho}{d\theta} &= A(\rho^2, z)^{-1} C(\rho^2, z) &= \rho^n A_\rho(z, \rho). \end{cases} \tag{3.6}$$

where $A_z, A_\rho \in \mathbb{R}[[z, \rho]]$. We define the *associated two dimensional vector field* as the two dimensional vector field obtained from the previous system

$$\hat{\eta} = \frac{B(\rho^2, z)}{A(\rho^2, z)} \frac{\partial}{\partial \rho} + \frac{C(\rho^2, z)}{A(\rho^2, z)} \frac{\partial}{\partial z}.$$

The vector field $\hat{\eta}$ encloses the dynamical properties of $\hat{\xi}$, and will be of use in the following sections. We remark that this vector field may not be a saturated vector field.

Degenerated and Non-degenerated Hopf singularities

We will make this distinction in terms of $\hat{\eta}$. It is possible that $\hat{\eta}$ is exactly equal to 0. We will see that the fact of $\hat{\eta}$ being 0 is independent of the choice of formal normal form $\hat{\xi}$. In this case, we say that ξ has a *degenerated Hopf singularity.* Notice that an isolated Hopf singularity can never be degenerated, since, as we discussed before, the coefficient C must contain a non-zero term that only depends on z.

When $\hat{\eta} \neq 0$, we say that ξ has a *non-degenerated Hopf singularity*. As in the previous case, $\hat{\eta}$ depends on the choice of normal form, but the fact of being different from zero will not depend on this choice.

We will prove Theorem 3.1 distinguishing the following four cases:

1. Semi-hyperbolic case,
2. Isolated Hopf-zero singularity case. It is always a non-degenerated Hopf singularity,
3. Non-degenerated non-isolated Hopf-zero singularity case,
4. Degenerated Hopf-zero singularity case. The singularity in this case is always non-isolated.

3.2 The semi-hyperbolic case

We prove the result first for semi-hyperbolic vector fields of Hopf type. Assume that the eigenvalues of $D\xi(0)$ are i, $-i$ and c with $c \neq 0$. Then, applying the Center Manifold Theorem (Theorem A.22 in the Appendix) there is some neighborhood V_k where

- There exists a unique non-singular invariant one dimensional analytic manifold $W = W^s$ or W^u, tangent to the eigenspace of c. This is the stable (when $c < 0$) or unstable (when $c > 0$) manifold.

- There exists a non-singular invariant two dimensional $\mathcal{C}^k$ manifold W^c, tangent to the eigenspaces of i and $-i$. It contains every cycle of ξ in V_k. Moreover, for any sufficiently small neighborhood U of 0, we have $\mathcal{C}_U \subset W^c$.

We work in some neighborhood $U \subset V_k$ of 0 in which the above manifolds are well defined and the stated properties for them hold. Let $\pi : (M, E) \to (\mathbb{R}^3, W)$ be the blowing-up centered at the stable manifold. The divisor $\pi^{-1}(W)$ is topologically a cylinder and the fiber $\gamma_0 = \pi^{-1}(0)$ is a cycle of the total transform $\xi_1 = \pi^*\xi$ contained in E. Notice also that the strict transform $(\widetilde{W^c})' = \overline{\pi^{-1}(W^c \setminus \{0\})}$ is a surface of class $\mathcal{C}^{k-1}$, invariant for ξ_1 and transverse to E at γ_0.

Now, consider the point $p_0 \in \gamma_0$ given by the intersection of $\overline{\pi^{-1}(\{y=0\})}$ and γ_0, and two nested discs $\Delta' \subset \Delta \subset \overline{\pi^{-1}(\{y=0\})}$ in which ξ_1 is transverse to both Δ' and Δ and where the Poincaré map $P : \Delta' \to \Delta$ is defined. Notice that any cycle of ξ_1 transverse to Δ' provides a periodic point $\gamma \cap \Delta'$ of P. By the fact that $\mathcal{C}_U \subset W^c$, we have that any cycle γ is contained in $\pi^{-1}(W^c)$, which is a surface. Then, by arguments based on Jordan Curve Theorem (see [62]), we find that any cycle $\gamma \subset W^c$ intersects Δ' in a single point, which is necessarily a fixed point of P. In particular, in some open neighborhood of γ_0, the family of cycles of ξ_1 is in bijection with the set of fixed points of P. In addition, the set of fixed points must be contained in the curve $H = (\widetilde{W^c})' \cap \Delta'$.

With these tools we prove Theorem 3.1 in the semi-hyperbolic case. First, suppose that (i) does not hold, that is, $\mathcal{C}_V \neq 0$ for any open neighborhood V of 0. Then, there are infinitely many cycles of ξ that accumulate and collapse to 0. By the above reasoning, there are infinitely many fixed points accumulating to the point p_0 along the curve H. Since the map P is analytic and so is the set $\mathrm{Fix}(P)$, the set $\mathrm{Fix}(P)$ must be curve of positive dimension. Since $\mathrm{Fix}(P) \subset H$ and H is a curve of class $\mathcal{C}^{k-1}$, we conclude that the sets $\mathrm{Fix}(P)$ and H coincide (as germs at p_0). Suppose that $\tilde{V}$ is a neighborhood of 0 that fulfills

- $\widetilde{V} \cap \Delta = \Delta'$.
- $\widetilde{U} \cap (\widetilde{W^c})'$ is the saturation of a segment of H by the flow of ξ_1.
- $V = \pi(\widetilde{V}) \subset V_k$.

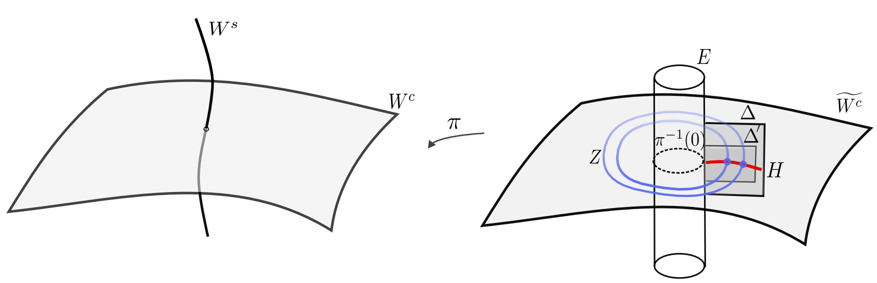

Figure 3.2: Arguments of the proof of Theorem 3.1 in the semi-hyperbolic case.

We get that V is a neighborhood of 0 and that $\mathcal{C}_V = (W^c \cap V) \setminus \{0\}$. Notice also that $(\widetilde{W^c})' \cap \widetilde{V}$ is an analytic set since H is an analytic curve. Being π proper and analytic, we conclude that $W^c \cap V$ is subanalytic, which proves Theorem 3.1 in this case. An illustration of the arguments in this proof can be found in Figure 3.2.

Remark 3.8. The proof above shows that, in the semi-hyperbolic case, there is at most one limit central surface S_1. Moreover, if S_1 exists, then $\overline{S_1} = W^c$ is a center manifold which is unique and analytic (using Tamm's Theorem [73], because W^c is of class $\mathcal{C}^k$ for every k and subanalytic in this case).

3.3 The isolated Hopf-zero singularity case: Admissible blowing-ups and reduction of singularities

In this section, we will apply a sequence of blowing-ups in order to simplify its dynamics. We will explain first the process for the normal form, and then adapt it to its jet approximations, choosing a sufficiently large jet approximation ξ_ℓ. In the first subsection, we will explain the admissible blowing-ups and introduce the notation. Later, we will give the result on reduction of singularities. We will finish by applying this result to the jet approximations. To compare this section with [23], we will restrict here the admissible blowing-ups to one of the two types presented therein, in particular, here we only make blowing-ups centered at points, instead of curves. In addition, the reduction of singularities result we need is weaker, and we will state it with the generality needed for this proof. This is one of the simplifications with respect to [23] we anticipated in the introduction of this chapter.

3.3.1 Admissible blowing-ups

The first blowing-up

The first blowing-up to be done is the real blowing-up $\sigma_0 : (M_0, E_0) \longrightarrow (\mathbb{R}^3, 0)$ with center at the origin. The blown-up space M_0 is a manifold having the divisor $E_0 = \sigma_0^{-1}(0)$ as its boundary. This divisor is homeomorphic to a sphere and represents the space of all the half-lines through 0. The morphism σ_0 defines an analytic isomorphism from $M_0 \setminus E_0$ to $\mathbb{R}^3 \setminus \{0\}$. We consider M_0 as in Section 1.2.4 changing slightly the notation. The manifold M_0 is covered by three charts $(C_0, (\theta, z^{(0)}, \rho^{(0)}))$, $(C_\infty, (x^{(\infty)}, y^{(\infty)}, z^{(\infty)}))$ and $(C_{-\infty}, (x^{(-\infty)}, y^{(-\infty)}, z^{(-\infty)}))$ where $C_0 \simeq \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}_{\geq 0}$ and $C_{\pm\infty} \simeq \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$. In these charts, the expression of σ_0 is given by:

$$\text{In } C_0 : \begin{cases} x &= \rho^{(0)} \cos\theta \\ y &= \rho^{(0)} \sin\theta \\ z &= \rho^{(0)} z^{(0)} \end{cases} \qquad (\cos\theta, \sin\theta) \in \mathbb{S}^1, z^{(0)} \in \mathbb{R}, \rho^{(0)} \geq 0 \tag{3.7}$$

$$\text{In } C_\infty : \begin{cases} x &= x^{(\infty)} z^{(\infty)} \\ y &= y^{(\infty)} z^{(\infty)} \qquad x^{(\infty)}, y^{(\infty)} \in \mathbb{R}, z^{(\infty)} \geq 0 \\ z &= z^{(\infty)} \end{cases} \tag{3.8}$$

$$\text{In } C_{-\infty} : \begin{cases} x &= x^{(-\infty)} z^{(-\infty)} \\ y &= y^{(-\infty)} z^{(-\infty)} \qquad x^{(-\infty)}, y^{(-\infty)} \in \mathbb{R}, z^{(-\infty)} \geq 0. \\ z &= -z^{(-\infty)} \end{cases} \tag{3.9}$$

See Figure 3.2 to understand better this blowing-up. The origins of the charts C_∞ and $C_{-\infty}$ will be denoted by γ_∞ and $\gamma_{-\infty}$, respectively. They are the points of the divisor E_0 corresponding to the half-lines contained in the z–axis and they are the only points of E_0 not covered by C_0. More explicitly, $\sigma_0(C_0) = \mathbb{R}^3 \setminus \{x = y = 0\}$.

We define the *(total) transform of $\hat{\xi}$ by σ_0 in the chart* C_0 as the pull-back

$$\hat{\xi}^{(0)} := (\sigma_0|_{C_0})^* \hat{\xi}.$$

Using simplified notation $(z, \rho) := (\rho^{(0)}, z^{(0)})$ and equations (3.3) and (3.7), the vector field $\hat{\xi}^{(0)}$ is given by

$$\hat{\xi}^{(0)} = A^{(0)}(z,\rho)\frac{\partial}{\partial \theta} + C^{(0)}(z,\rho)\frac{\partial}{\partial z} + B^{(0)}(z,\rho)\frac{\partial}{\partial \rho}, \tag{3.10}$$

where $A^{(0)}(z,\rho) = A(\rho^2, \rho z)$, $C^{(0)}(z,\rho) = \frac{1}{\rho}C(\rho^2,\rho z) - zB(\rho^2,\rho z)$ and $B^{(0)}(z,\rho) = \rho A(\rho^2,\rho z)$. Notice that $A^{(0)}, B^{(0)}, C^{(0)} \in \mathbb{R}[z][[\rho]]$, $A^{(0)}(0,0) = 1$ and that $(B^{(0)}, C^{(0)}) \neq (0,0)$ by the hypothesis that $\hat{\xi}$ is non-degenerated. Notice also that ρ divides $B^{(0)}, C^{(0)}$.

As in section 3.1.3, we can define a two dimensional vector field that describes the three dimensional one in this chart. Consider the system of ODEs

$$\hat{\eta}_0 : \begin{cases} \frac{dz}{d\theta} &= A^{(0)}(z,\rho)^{-1} B(z,\rho) &= \rho^{n^{(0)}} A_z^{(0)}(z,\rho) \\ \frac{d\rho}{d\theta} &= A^{(0)}(z,\rho)^{-1} C^{(0)}(z,\rho) &= \rho^{n^{(0)}} A_\rho^{(0)}(z,\rho), \end{cases} \tag{3.11}$$

and define the associated two-dimensional vector field $\hat{\eta}_0$ and $\hat{\eta}_0' = \frac{1}{\rho^{n^{(0)}}}\hat{\eta}_0$. Consider the blowing-up of 0 in $\mathbb{R} \times \mathbb{R}_{\geq 0}$ and the vector field $\hat{\eta}$.

Remark 3.9. Notice that, up to multiplication by a unit, the vector field $\hat{\eta}_0$ coincides with the total transform $\hat{\eta}^{(2+)}$ of $\hat{\eta}$ in the usual chart U_2^+ in the direction of ρ.

Consider the curve curve $F_0 := E_0 \cap \{\theta = 0\}$, and the set $\widetilde{\mathrm{Sing}}(\hat{\eta}_0, F_0)$, which is given by the singularities of $\hat{\eta}_0'$ and also points where $\hat{\eta}_0'$ is tangent to F_0 (in the dicritical case). Suppose that

$$\widetilde{\mathrm{Sing}}(\hat{\eta}_0, F_0) = \{(\omega_i^{(0)}, 0) : i = 1, \ldots, m_0\}, \quad \text{with } \omega_i^{(0)} < \omega_j^{(0)} \text{ if } i < j.$$

Definition 3.10. *The* characteristic cycles *of $\hat{\xi}$ in M_0 are the connected components of the set* $\mathbb{S}^1 \times \widetilde{Sing}(\eta_0'|F_0) \subset C_0$*, that is, the circles in the divisor E_0 given by $\gamma_i := \{z^{(0)} = \omega_i^{(0)}, \rho^{(0)} = 0\}$ for $i = 1, 2, \ldots, m_0$.*

We define also the transforms $\hat{\xi}^{(\infty)} := (\sigma_0|_{C_\infty})^* \hat{\xi}$ and $\hat{\xi}^{(-\infty)} := (\sigma_0|_{C_{-\infty}})^* \hat{\xi}$ of $\hat{\xi}$ in the charts $C_\infty, C_{-\infty}$, respectively. The expressions for $\hat{\xi}^{(\infty)}$, using simplified notation $(x,y,z) := (x^{(\infty)}, y^{(\infty)}, z^{(\infty)})$ is the following:

$$\begin{aligned}\hat{\xi}^{(\infty)} =& B^{(\infty)}(x^2+y^2,z)\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)+A^{(\infty)}(x^2+y^2,z)\left(-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\right)\\ &+C^{(\infty)}(x^2+y^2,z)\frac{\partial}{\partial z},\end{aligned} \tag{3.12}$$

where $B^{(\infty)}, A^{(\infty)}, C^{(\infty)} \in \mathbb{R}[x^2+y^2][[z]]$ are given by:

$$B^{(\infty)}(x^2+y^2,z) = B((x^2+y^2)z^2,z) - \frac{1}{z}C((x^2+y^2)z^2,z),$$

$$A^{(\infty)}(x^2+y^2,z) = A((x^2+y^2)z^2,z) \text{ and } C^{(\infty)}(x^2+y^2,z) = C((x^2+y^2)z^2,z).$$

In a similar way, we obtain expressions for $\hat{\xi}^{(-\infty)}$. Notice that the origin of these charts are singularities of $\hat{\xi}^{(\infty)}$.

Definition 3.11. *The origins $\gamma_\infty, \gamma_{-\infty}$ of the charts C_∞ and $C_{-\infty}$ (cf. equations* (3.8) *and* (3.9)*) are called the* characteristic singularities of $\hat{\xi}$ in M_0. *We use the term* characteristic elements *to refer either to the characteristic cycles or to characteristic singularities.*

Further blowing-ups

In the rest of this section, we define sequences of blowing-ups attached to $\hat{\xi}$ starting from the data defined above for the first blowing-up σ_0. More precisely, consider the tuple $\mathcal{M}_0 := (M_0, \sigma_0, \mathcal{A}_0, \mathcal{D}_0)$, where:

- $\mathcal{A}_0$ is the atlas of M_0 composed by the charts $C_{-\infty}, C_0, C_\infty$,
- $\mathcal{D}_0$ is the family of *characteristic elements* of $\hat{\xi}$ in M_0, that is, $\mathcal{D}_0 := \{\gamma_{-\infty}, \gamma_1, \ldots, \gamma_{m_0}, \gamma_\infty\}$.

By definition, we say that $\mathcal{M}_0$ a *sequence of admissible blowing-ups of length $l = 0$ for $\hat{\xi}$*. In further steps, we will only admit blowing-ups centered at one of the two characteristic points: the iterated tangents of the semi-branches of the formal rotational axis. Recall that this axis is simply the z–axis. This means that we will blow up the origins γ_∞ and $\gamma_{-\infty}$ of the charts C_∞ and $C_{-\infty}$ on the previous section.

For instance, suppose that we start with the blowing-up $\sigma_\infty : (M_1, E_1) \to (M_0, E_0)$ centered at the point γ_∞. It is expressed in two charts: $C_{\infty,0}$ and $C_{\infty,\infty}$ defined in the same way as C_0 and C_∞ in Equations 3.7 and (3.8). See Figure 3.3 for an illustration of these blowing-ups.

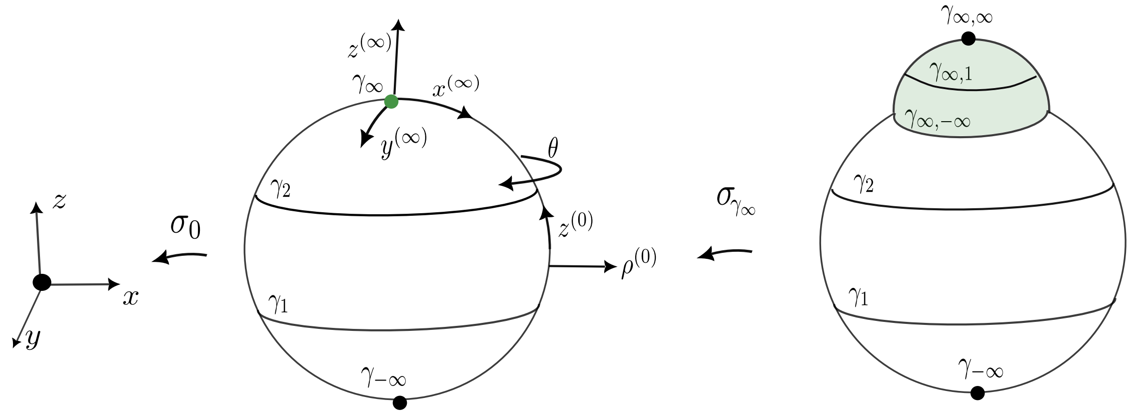

Figure 3.3: Admissible blowing-ups.

In the chart $(C_{\infty,0}, (x^{(\infty,0)}, y^{(\infty,0)}, z^{(\infty,0)}))$, we can study the characteristic cycles of $\hat{\xi}^{(\infty,0)} = (\sigma_\infty|_{C_{\infty,0}})^* \hat{\xi}^{(\infty)}$ whose expression is similar to (3.10). We define the associated two-dimensional vector field $\hat{\eta}_{\infty,0}$ as in (3.11) and study its singularities to determine the characteristic cycles in this chart. The origin of $C_{\infty,\infty}$ is again a singularity of $\hat{\xi}^{(\infty,\infty)} = (\sigma_\infty|_{C_{\infty,\infty}})^* \hat{\xi}^{(\infty)}$- The expression of $\hat{\xi}^{(\infty,\infty)}$ is similar to that of (3.12).

We denote by $\mathcal{A}_1$ the atlas of M_1 given by $(\mathcal{A}_0 \setminus \{C_\infty\}) \cup \{C_{\infty,0}, C_{\infty,\infty}\}$. We denote by $\mathcal{D}_1$ the set of characteristic cycles and characteristic points defined on each chart $C \in \mathcal{A}_1$.

The rest of the admissible blowing-ups $\mathcal{M}_i = (M_i, \pi_i, \mathcal{A}_i, \mathcal{D}_i)$ are defined similarly, by the composition of blowing-ups $\pi_i = \sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_{i-1}$ centered at the iterated tangents of the z−axis, and with $\mathcal{A}_i = \{C_J\}_{J \in \mathcal{J}}$ where each C_J has coordinates either of the form $(x^{(J)}, y^{(J)}, z^{(J)})$ or $(\theta^{(J)}, z^{(J)}, \rho^{(J)})$, and $\mathcal{D}_i = \{\gamma_I\}_{I \in I_i}$ where each γ_I is a characteristic cycle or a characteristic point.

3.3.2 Reduction of singularities

We saw in Section 1.4.3 the reduction of singularities of formal non-saturated vector fields. The vector field $\hat{\eta}$ suits in this class of vector fields. In order to find a suitable reduction of singularities of $\hat{\xi}$, we will make admissible blowing-ups centered at the iterated tangents of the rotational axis (the z−axis). The main difference with respect to our article [23] is that we do not pursue a reduction of singularities of the associated two dimensional vector fields $\hat{\eta}_J$ for $J \in \mathcal{J}$, instead, we will only resolve the points in the iterated tangents of the rotational axis of $\hat{\xi}$.

Just as a comment to prepare the following result and its proof, and according to the construction of sequences of admissible blowing-ups in the previous section, any $\mathcal{A} = \{C_J\}_{J \in \mathcal{J}}$ is composed of the two charts $C_{I^{\mathcal{M}}_{-\infty}}$ and $C_{I^{\mathcal{M}}_{\infty}}$ where $I^{\mathcal{M}}_{-\infty}, I^{\mathcal{M}}_{\infty}$ are, respectively, $(-\infty, \ldots, -\infty)$ and $(\infty, \ldots, \infty)$, and a finite number of charts:

$$\{C_J\}_{J\in\widetilde{\mathcal{J}}_0},\ \text{where } \widetilde{\mathcal{J}}_0 = \{0,(\infty,0),\ldots,(\infty,\ldots,\infty,0),(-\infty,0),\ldots,(-\infty,\ldots,-\infty,0)\},$$

the second group of charts having coordinates of the form $(\theta,z,\rho)\in\mathbb{R}\times(\mathbb{R}_{\geq 0})^2$, except for C_0 with z taking values in $\mathbb{R}$.

Proposition 3.12 (Adapted reduction of singularities)**.** *There exists an admissible sequence of blowing-ups $\mathcal{M}=(M,\pi,\mathcal{A},\mathcal{D})$ for $\hat{\xi}$ with $\mathcal{A}=\{C_J\}_{J\in\mathcal{J}}$, $\mathcal{D}=\{\gamma_I\}_{I\in\mathcal{I}}$ and total divisor $E=\pi^{-1}(0)$ such that the following holds. For $J\in\{I_\infty^{\mathcal{M}},I_{-\infty}^{\mathcal{M}}\}$, the transformed vector field $\hat{\xi}^{(J)}=(\pi|_{C_J})^*\hat{\xi}$ satisfies $\hat{\xi}^{(J)}(z^{(J)})=(z^{(J)})^t\cdot G$ where $t\geq 1$ and G is a unit in $\mathbb{R}[[x^{(J)},y^{(J)},z^{(J)}]]$.*

Proof. From Remark 3.7 there exists a term c_jz^j in the coefficient $\hat{\xi}(z)$ with $c_j\neq 0$. Assume, without loss of generality, that j is the minimal exponent with this condition. Notice that $j>0$. Write $\hat{\xi}(z)\in\mathbb{R}[[x,y,z]]$ as

$$\hat{\xi}(z)=Z(x^2+y^2,z)=z^{t_0}G(x,y,z)=z^{t_0}\sum_{k=\nu(G)}^{\infty}G_k(x,y,z),$$

where G_k is an homogeneous polynomial of degree k for each k, $t_0\geq 0$ is defined as the maximum integer such that z^{t_0} divides $\hat{\xi}(z)$. Then $G_{j-t_0}(x,y,z)$ contains the monomial $c_jz^{j-t_0}$ (notice that $j\geq t_0$ and the equality holds if and only if $\nu(G)=0$). Consider the first blowing-up σ_0 and study $\hat{\xi}^{(\infty)}(z^{(\infty)})$, where $\xi^{(\infty)}=(\sigma_0|_{C_\infty})^*\hat{\xi}$. Omitting super-indices for the coordinates $(x^{(\infty)},y^{(\infty)},z^{(\infty)})$, we have:

$$\hat{\xi}^{(\infty)}(z)=z^{t_0}\sum_{k=\nu(G)}^{\infty}G_k(x,y,1)z^k=z^{t_1}\sum_{k=\nu(G)}^{\infty}G_k(x,y,1)z^{k-\nu(G)},$$

where $t_1=t_0+\nu(G)\geq t_0$. Rewrite the series $G^{(1)}:=\sum_{k=\nu(G)}^{\infty}G_k(x,y,1)z^{k-\nu(G)}$ in homogeneous components:

$$\hat{\xi}^{(\infty)}(z)=z^{t_1}G^{(1)}(x,y,z)=z^{t_1}\sum_{k=\nu(G^{(1)})}^{\infty}G_k^{(1)}(x,y,z).$$

If $j=t_1$, we see that $G_0^{(1)}=c_j$ and thus, $G^{(1)}$ is a unit, which gives statement *(1)* of the proposition for $t=t_1$. Otherwise, if $t_1<j$, we see that $G_{j-t_0}^{(1)}(x,y,z)$ contains the term $c_jz^{j-t_1}$. Notice that in this case we have $t_1\geq t_0$ since, otherwise, if $t_1=t_0$ then $\nu(G)=0$ and $j=t_0=t_1$. Thus, $j-t_0>j-t_1\geq 0$. By recurrence over $j-t_0$, there exists an admissible sequence of blowing-ups $\widetilde{\mathcal{M}}=(\widetilde{M},\widetilde{\pi},\widetilde{\mathcal{A}},\widetilde{\mathcal{D}})$ with $\widetilde{\pi}$ a composition of s blowing-ups at the corresponding characteristic singularities $\gamma_{I_\infty^{\mathcal{M}_i}}$ such that, defining $t_0,t_1,\ldots,t_s$ as above, we have $j=t_s$. We conclude the statement for $\widetilde{\pi}^*\hat{\xi}$ at the characteristic singularity $\gamma_{I_\infty^{\widetilde{\mathcal{M}}}}$ with $t=t_s$. Analogously, up to blowing-up repeatedly the characteristic singularity $\gamma_{-\infty}^{\tilde{\mathcal{M}}}$, we may assume that the statement holds at $\gamma_{I_{-\infty}^{\widetilde{\mathcal{M}}}}$. ∎

3.3.3 Jet approximations under blowing-ups

In this section, we study the effect of sequences of admissible blowing-ups to the jet approximations ξ_ℓ of the formal normal form $\hat{\xi}$, for convenient values of ℓ. We follow the techniques introduced in Section 1.6.1. The main difficulty in comparison to the results in section 1.6.1 is that we are using also cylindrical coordinates, and thus, the coefficients of the vector field in those coordinates lie in a ring that we did not treat in section 1.6.1. First, we establish the jet dependence of the transform of $\hat{\xi}$ on such blowing-ups in the different charts.

Proposition 3.13. *Let $\hat{\xi}$ be a formal normal form of $\xi \in \mathcal{H}^3$. Consider an admissible sequence of blowing-ups $\mathcal{M} = (M, \pi, \mathcal{A}, \mathcal{D})$ for $\hat{\xi}$ of length $l > 0$, with $\mathcal{A} = \{C_J\}_{J \in \mathcal{J}}$. For every $J \in \mathcal{J}$ and for every $k \geq 1$, if u is a coordinate of the chart C_J such that $\{u = 0\} \subset E = \pi^{-1}(0)$, then we have*

$$j_k^u(\hat{\xi}^{(J)}) = j_k^u((\pi|_{C_J})^* j_{k+l+1}(\hat{\xi})). \tag{3.13}$$

Proof. The proof uses Proposition 1.51, which we summarize now. Let η be a vector field with coefficients in $A[[x_1, \ldots, x_n]]$ and let π be a quadratic morphism of the form $\pi(x_1, \ldots, x_n) = (x_1 x_i, \ldots, x_{i-1} x_i, x_i, x_{i+1} x_i, \ldots, x_n x_i)$. Then,

$$\begin{aligned} &j_k^{x_i}(\pi^* \eta) = j_k^{x_i}(\pi^* j_{k+1}(\eta)) \\ &j_k^{x_j}(\pi^* \eta) = j_k^{x_j}(\pi^* j_k^{x_j}(\eta)) = j_k^{x_j}(\pi^* j_{k+1}^{x_j}(\eta)), \ j \neq i. \end{aligned} \tag{3.14}$$

We proceed by induction on the length l of $\mathcal{M}$. If $l = 0$, that is, $\pi = \sigma_0$ is the blowing-up of the origin $0 \in \mathbb{R}^3$ described in section 3.3.1. We have (with simplified notation in the respective equalities $\rho := \rho^{(0)}$ in the first, $z := z^{(\infty)}$ in the second and $z = z^{(-\infty)}$ in the third)

$$\begin{aligned} j_k^\rho((\sigma_0|_{C_0})^* \hat{\xi}) &= j_k^\rho((\sigma_0|_{C_0})^* j_{k+1}(\hat{\xi})), \\ j_k^z((\sigma_0|_{C_\infty})^* \hat{\xi}) &= j_k^z((\sigma_0|_{C_\infty})^* j_{k+1}(\hat{\xi})), \\ j_k^z((\sigma_0|_{C_{-\infty}})^* \hat{\xi}) &= j_k^z((\sigma_0|_{C_{-\infty}})^* j_{k+1}(\hat{\xi})). \end{aligned} \tag{3.15}$$

The first item is developed as in the proof of 1.51. The other two follow from direct application of that result. Suppose $l > 0$ and that $\pi = \widetilde{\pi} \circ \sigma_{\gamma_I}$, where σ_{γ_I} is the blowing-up centered at some characteristic singularity γ_I of a sequence of admissible blowing-ups $\widetilde{\mathcal{M}} = (\widetilde{M}, \widetilde{\pi}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{D}})$ of length $l-1$. It is enough to study the transform $\hat{\xi}^{(J)}$ in the charts C_J when $\sigma_{\gamma_I}^{-1}(\gamma_I) \cap C_J \neq \emptyset$, since the map σ_{γ_I} is an isomorphism out of $\sigma_{\gamma_I}^{-1}(\gamma_I)$. According to the construction of $\mathcal{M}$ from $\widetilde{\mathcal{M}}$ and using the same notations as in section 3.3.1, we have several cases:

1. γ_I is a characteristic singularity (for instance $I = I_\infty^{\widetilde{\mathcal{M}}}$) and $J = I_\infty^{\mathcal{M}} = (\infty, \overset{s}{\ldots}, \infty)$. In this case $z^{(J)}$, is the only coordinate of the chart C_J in the conditions of the statement. Notice that $s \leq l + 1$.

Applying s times the first formula in (3.14), we get, for $u = z^{(J)}$, that

$$j_k^u(\hat{\xi}^{(J)}) = j_k^u((\pi|_{C_J})^* j_{k+s}(\hat{\xi})) = j_k^u((\pi|_{C_J})^* j_{k+s}(j_{k+l+1}(\hat{\xi}))) = j_k^u((\pi|_{C_J})^* j_{k+l+1}(\hat{\xi})).$$

2. γ_I is a characteristic singularity (for instance $I = I_\infty^{\widetilde{\mathcal{M}}}$) and J is not $I_\infty^{\mathcal{M}} = (\infty, \overset{s}{\ldots}, \infty)$. Notice that γ_I is the origin of a chart $(C_{J_I}, (x^{(J_I)}, y^{(J_I)}, z^{(J_I)}))$ of $\widetilde{\mathcal{A}}$ where $z^{(J_I)} = 0$ is the equation of the divisor $\widetilde{E} \cap C_{J_I}$ and $\sigma_{\gamma_I}|_{C_J} \colon C_J \to C_{J_I}$ has the same expression as (3.7) for σ_0, considering coordinates $(\theta, z^{(J)}, \rho^{(J)})$ for C_J and with the obvious change of notation. Notice that in C_J the two coordinates $u = \rho^{(J)}$ and $u = z^{(J)}$ are in the conditions of the statement. By the induction hypothesis, renaming $z = z^{(J_I)}$ for simplicity, we have, for any $k \geq 1$, that $j_k^z(\hat{\xi}^{(J_I)}) = j_k^z((\tilde{\pi}|_{C_{J_I}})^* j_{k+l}(\hat{\xi}))$. By the fact that $j_k(\chi) = j_k(j_k^z(\chi))$ for any vector field χ, we also have $j_k(\hat{\xi}^{(J_I)}) = j_k((\tilde{\pi}|_{C_{J_I}})^* j_{k+l}(\hat{\xi}))$. The result follows for $u = \rho^{(J)}$ from this similarly to the case of the first blowing-up σ_0. For $u = z^{(J)}$, it is a consequence of the second equation of (3.14).

□

Now, let us discuss the validity of Proposition 3.13 for the jets approximated normal forms ξ_ℓ.

Consider the first blowing-up σ_0 at $0 \in \mathbb{R}^3$, a singular point of ξ_ℓ for any ℓ. Being ξ_ℓ analytic, the total transform $\sigma_0^* \xi_\ell$ exists and is analytic in a neighborhood of the divisor $E_0 = \sigma_0^{-1}(0)$. Moreover, in terms of coordinates of the charts $C_{-\infty}$, C_0, C_∞ (c.f. section 3.3.1), we can prove (see for instance the computations in [4, sec. 3]):

a) For $(C_\infty, (x^{(\infty)}, y^{(\infty)}, z^{(\infty)}))$ (and analogously for $C_{-\infty}$) the coefficients of $\xi_\ell^{(\infty)} := (\sigma_0|_{C_\infty})^* \xi_\ell$ belong to $\mathbb{R}[x^{(\infty)}, y^{(\infty)}][[z^{(\infty)}]] \cap \mathbb{R}\{x^{(\infty)}, y^{(\infty)}, z^{(\infty)}\}$. In fact, they belong to the algebra $\mathbb{R}[x^{(\infty)}, y^{(\infty)}]\{z^{(\infty)}\}$ of convergent series with polynomial coefficients (c.f. Chapter 1).

b) For $(C_0, (\theta, z, \rho))$, the coefficients of $\xi_\ell^{(0)} := (\sigma_0|_{C_0})^* \xi_\ell$ belong to $\mathbb{R}[\cos\theta, \sin\theta, z][[\rho]] \cap \mathbb{R}[\cos\theta, \sin\theta]\{z, \rho\}$. In fact, they belong to $\mathbb{R}[\cos\theta, \sin\theta, z]\{\rho\}$.

Finally, taking into account that $\ell \geq 1$ (i.e. ξ_ℓ has the same linear part as ξ or $\hat{\xi}$), we may observe that $\xi_\ell^{(J)}|_{E_0 \cap C_J} = \hat{\xi}^{(J)}|_{E_0 \cap C_J}$ for any $J \in \{-\infty, 0, \infty\}$. In particular, the characteristic elements of $\hat{\xi}$ in E_0 are invariant for $\sigma_0^* \xi_\ell$, which admits then a transform which is analytic if we blow up again one of those characteristic elements. Using recursively the same kind of arguments, and with a similar proof, we obtain the following version of Proposition 3.13 for the jets approximations of the normal form.

Proposition 3.14. *Let $\mathcal{M} = (M, \pi, \mathcal{A}, \mathcal{D})$ be an admissible sequence of blowing-ups of length l with $\mathcal{A} = \{C_J\}_{J \in \mathcal{J}}$. Then, for $\ell \geq l+1$ and $J \in \mathcal{J}$ the transform $\xi_\ell^{(J)} := (\pi|_{C_J})^* \xi_\ell$ is analytic. Moreover, if*

$k \in \mathbb{N}$, u is a coordinate of C_J such that $\{u = 0\} \subset E = \pi^{-1}(0)$ and $\ell \geq k + l + 1$, then, we have

$$j_k^u(\xi_\ell^{(J)}) = j_k^u(\hat{\xi}^{(J)}).$$

Remark 3.15. As a part of the proof, we verify that the restriction of $\xi_\ell^{(J)}$ and $\hat{\xi}^{(J)}$ to the divisor coincide. Hence, the characteristic elements $\gamma_I \in \mathcal{D}$ are invariant for the total transform $\pi^*\xi_\ell$. They are called *characteristic singularities or characteristic cycles*, accordingly, of $\xi_\ell^{(J)}$. Moreover, we observe that the coefficients of $\xi_\ell^{(J)}$ are convergent series in the coordinates of the chart C_J; i.e. they satisfy the corresponding property a), respectively b), above when $J \in \{I_{-\infty}^{\mathcal{M}}, I_\infty^{\mathcal{M}}\}$ (resp. $J \notin \{I_{-\infty}^{\mathcal{M}}, I_\infty^{\mathcal{M}}\}$). In the last case, we can also interchange the roles of the coordinates z and ρ if C_J is a corner chart.

Recall from section 3.3.1 the definition of the associated two dimensional vector fields $\hat{\eta}_J$ to $\hat{\xi}^{(J)}$ for $J \in \mathcal{J} \setminus \{I_\infty^{\mathcal{M}}, I_{-\infty}^{\mathcal{M}}\}$ and the corresponding reduced vector fields $\hat{\eta}_J' = (\rho^{n_1^{(J)}} z^{n_2^{(J)}})^{-1}\hat{\eta}_J$, where (θ, z, ρ) are the coordinates in C_J. Write the transform $\xi_\ell^{(J)}$ as

$$\xi_\ell^{(J)} = B_\ell^{(J)}(\theta,z,\rho)\frac{\partial}{\partial\rho} + A_\ell^{(J)}(\theta,z,\rho)\frac{\partial}{\partial\theta} + C_\ell^{(J)}(\theta,z,\rho)\mathbf{z} \tag{3.16}$$

The *associated (to $\xi_\ell^{(J)}$) system of ODEs* $\eta_{\ell,J}$ is defined as:

$$\left\{ \begin{array}{lcl} \frac{dz}{d\theta} & = & C_\ell^{(J)}(\theta,z,\rho)\cdot(A_\ell^{(J)}(\theta,z,\rho))^{-1} \\ \frac{d\rho}{d\theta} & = & B_\ell^{(J)}(\theta,z,\rho)\cdot(A_\ell^{(J)}(\theta,z,\rho))^{-1} \end{array} \right. \tag{3.17}$$

Recall also that if $J \in \{I_\infty^{\mathcal{M}}, I_{-\infty}^{\mathcal{M}}\}$ and we use simplified notation $(x,y,z) := (x^{(J)}, y^{(J)}, z^{(J)})$, we have defined $n^{(J)}$ as the maximum $n \in \mathbb{N}$ such that $\hat{\xi}^{(J)}(z)$ is divisible by z^n. As well, if $J \in \mathcal{J} \setminus \{I_\infty^{\mathcal{M}}, I_{-\infty}^{\mathcal{M}}\}$, we have defined $n^{(J)} := \max\{n_1^{(J)}, n_2^{(J)}\}$. With those notations, we have the following Corollary of Proposition 3.14.

Corollary 3.16. *Let $\mathcal{M} = (M, \pi, \mathcal{A}, \mathcal{D})$ be an admissible sequence of blowing-ups of length $l > 0$ with $\mathcal{A} = \{C_J\}_{J\in\mathcal{J}}$. Define $\ell_{\mathcal{M}} := \max\{n^{(J)} : J \in \mathcal{J}\} + l + 1$. Fix $k \in \mathbb{N}_{\geq 0}$.*

1. *Let $J \in \mathcal{J} \setminus \{I_{-\infty}^{\mathcal{M}}, I_\infty^{\mathcal{M}}\}$ and put $(z,\rho) := (z^{(J)}, \rho^{(J)})$. For every $\ell \geq \ell_{\mathcal{M}} + k$, the monomial $(\rho)^{n_1^{(J)}}(z)^{n_2^{(J)}}$ divides the system $\eta_{\ell,J}$. Moreover, putting $\eta_{\ell,J}' := (\rho^{n_1^{(J)}} z^{n_2^{(J)}})^{-1}\eta_{\ell,J}$, if u is a coordinate with $\{u = 0\} \subset E \cap C_J$, then*
$$j_k^u(\eta_{\ell,J}') = j_k^u(\hat{\eta}_J').$$

2. *Let $J \in \{I_{-\infty}^{\mathcal{M}}, I_\infty^{\mathcal{M}}\}$ and put $(x,y,z) := (x^{(J)}, y^{(J)}, z^{(J)})$. For every $\ell \geq \ell_{\mathcal{M}} + k$, the series $\xi_\ell^{(J)}(z)$ is divisible by $z^{n^{(J)}}$, and*
$$j_k^z\Big(z^{-n^{(J)}}\xi_\ell^{(J)}(z)\Big) = j_k^z\Big(z^{-n^{(J)}}\hat{\xi}^{(J)}(z)\Big).$$

Proof. Both statements are direct consequence of the jet equality stated in Proposition 3.14. Since $k+\ell_{\mathcal{M}} \geq n_i^{(J)}+l+1$ for $i=1,2$ and for very $J \in \mathcal{J}\setminus\{I_{-\infty}^{\mathcal{M}}, I_{\infty}^{\mathcal{M}}\}$ and $k+\ell_{\mathcal{M}} \geq n^{(J)}+l+1$ when $J \in \{I_{-\infty}^{\mathcal{M}}, I_{\infty}^{\mathcal{M}}\}$, we have that the monomials of type $(\rho^{(J)})^{n_1^{(J)}}(z^{(J)})^{n_2^{(J)}}$ divide the system $\eta_{\ell,J}$ when $J \in \mathcal{J}\setminus\{I_{-\infty}^{\mathcal{M}}, I_{\infty}^{\mathcal{M}}\}$, or $(z^{(J)})^{n^{(J)}}$ divides $\xi_\ell^{(J)}(z^{(J)})$ when $J \in \{I_{-\infty}^{\mathcal{M}}, I_{\infty}^{\mathcal{M}}\}$. □

3.4 The isolated singularity case: Dynamics after the reduction of singularities

In this section, we study the dynamics of the vector fields ξ_ℓ for ℓ large enough after the reduction of singularities.

3.4.1 Characteristic cycles as limit sets

The objective of this section is to prove that the characteristic elements of ξ_ℓ after a sequence of admissible blowing-ups $\mathcal{M}$ are the only possible limit sets of the family of local cycles.

Along this section, we fix a sequence of admissible blowing-ups $\mathcal{M} = (M, \pi, \mathcal{A}, \mathcal{D})$ for $\hat{\xi}$, with $\mathcal{A} = \{C_J\}_{J\in\mathcal{J}}$ and $\mathcal{D} = \{\gamma_I\}_{I\in\mathcal{I}}$. Denote by $E = \pi^{-1}(0)$ the total divisor of π. We define also the *support of* $\mathcal{D}$ as $\mathrm{Supp}\mathcal{D} = \bigcup_{I\in\mathcal{I}} \gamma_I$. Recall the definition of $\ell_{\mathcal{M}}$ in Corollary 3.16.

Proposition 3.17. *Let $\ell \geq \ell_{\mathcal{M}}+1$ and W be a neighborhood of $\mathrm{Supp}\mathcal{D} = \bigcup_{I\in\mathcal{I}} \gamma_I$. There is some neighborhood $U = U(W)$ of $0 \in \mathbb{R}^3$ such that $\pi^{-1}(\mathcal{C}_U(\xi_\ell)) \subseteq W$.*

To prove this result, we need to introduce new notation and a technical lemma. Consider the set

$$\dot{E} := E \setminus \left(\left(\bigcup_{I:\gamma_I \in \mathcal{D} \text{ corner}} \{\gamma_I\} \right) \cup \{\gamma_{I_\infty^{\mathcal{M}}}\} \cup \{\gamma_{I_{-\infty}^{\mathcal{M}}}\} \right).$$

It has a finite family of connected components denoted by $\mathcal{E}_{\mathcal{M}} = \{L_0, L_1, \dots, L_{k_{\mathcal{M}}}\}$. Each $L_i \in \mathcal{E}_{\mathcal{M}}$ is open in E and contained in a chart C_{J_i} for $i = 0, 1, \dots, k_{\mathcal{M}}$. Therefore, we will call them simply *open components (of E)*. In addition, in case L_i is contained in two different charts, we choose C_{J_i} such that $L_i \subseteq \{\rho^{(J_i)} = 0\}$, which is always possible by the construction of sequences of admissible blowing-ups. Then, each open component $L_i = \mathbb{S}^1 \times (\lambda_i^-, \lambda_i^+) \times \{0\}$ in the coordinates of C_{J_i} where $\lambda_i^- \in \mathbb{R}\cup\{-\infty\}$ and $\lambda_i^+ \in \mathbb{R}\cup\{\infty\}$. An element $L_i \in \mathcal{E}_{\mathcal{M}}$ is said to be dicritical (respectively, non-dicritical) if the component of E that contains L_i is dicritical (respectively, non-dicritical).

Fix $L = L_i \in \mathcal{E}_{\mathcal{M}}$ and the corresponding chart C_J with $J = J_i$. Consider the formal vector field $\hat{\eta}_J$ associated to $\hat{\xi}^{(J)} = (\pi|_{C_J})^*\hat{\xi}$ as in equation (3.11). For the purpose of this section, we write, removing super-indices in (z,ρ):

$$\hat{\eta}_J = \rho^{n_1^{(J)}}(A_z^{(J)}(z,\rho)\frac{\partial}{\partial z} + A_\rho^{(J)}(z,\rho)\frac{\partial}{\partial \rho}). \tag{3.18}$$

We consider the vector field $\hat{\eta}_J'' := \rho^{-n_1^{(J)}}\hat{\eta}_J$ which has a finite number of adapted singularities along $\{\rho = 0\}$. The singularities determine the characteristic cycles contained in L_i. The z–coordinates of the characteristic cycles in L are denoted by $\omega_1^L,\ldots,\omega_{m_L}^L$ and the associated characteristic cycles by $\gamma_1^L,\ldots,\gamma_{m_L}^L$.

Define the collection of sets $\mathcal{V}(L,\varepsilon,\delta) := \{V_0, V_1,\ldots,V_{m_L-1},V_{m_L}\}$ depending on two parameters $\varepsilon,\delta > 0$ by:

$$\begin{array}{lcll} V_0 & = & \mathbb{S}^1\times\Omega_0(\varepsilon)\times(0,\delta], & \Omega_0(\varepsilon) = [\mu_-,\omega_1^L-\varepsilon], \\ V_j & = & \mathbb{S}^1\times\Omega_j(\varepsilon)\times(0,\delta], & \Omega_j(\varepsilon) = [\omega_j^L+\varepsilon,\omega_{j+1}^L-\varepsilon], \quad j = 1,\ldots,m_{L_i}-1, \\ V_{m_L} & = & \mathbb{S}^1\times\Omega_{m_L}(\varepsilon)\times(0,\delta], & \Omega_{m_L}(\varepsilon) = [\omega_{m_L}^L+\varepsilon,\mu_+], \end{array} \tag{3.19}$$

where $\mu_\pm = \lambda^\pm \mp \varepsilon$ when $|\lambda^\pm| < \infty$, $\mu_- = \omega_1^L - \frac{1}{\varepsilon}$ when $\lambda^- = -\infty$, and $\mu_+ = \omega_{m_L} + \frac{1}{\varepsilon}$ when $\lambda^+ = \infty$. Define the surfaces $\partial_{min}V_j$ and $\partial_{max}V_j$ as follows:

- $\partial_{min}V_0 = \mathbb{S}^1\times\{\mu_-\}\times(0,\delta]$ and $\partial_{min}V_j = \mathbb{S}^1\times\{\omega_j^L+\varepsilon\}\times(0,\delta]$ for $j = 1,2,\ldots,m_L$.
- $\partial_{max}V_j = \mathbb{S}^1\times\{\omega_{j+1}^L-\varepsilon\}\times(0,\delta]$ for $j = 0,1,\ldots,m_L-1$ and $\partial_{max}V_{m_L} = \mathbb{S}^1\times\{\mu_+\}\times(0,\delta]$.

Lemma 3.18. *Assume $\ell \geq \ell_{\mathcal{M}}+1$ and denote by $\xi_\ell^{(J)} = (\pi|_{C_J})^*\xi_\ell$. There exists $\varepsilon_0 > 0$ such that for every small ε with $\varepsilon_0 > \varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that the collection $\mathcal{V}(L,\varepsilon,\delta) = \{V_j\}_{j=1}^{m_L}$ satisfies:*

1. *In case L is non-dicritical, the function z is monotonic along the trajectories of $\xi_\ell^{(J)}$ in each V_j for any j. Otherwise, if L is dicritical, the function ρ is monotonic along the trajectories of $\xi_\ell^{(J)} = (\pi|_{C_J})^*\xi_\ell$ in each V_j, for any j.*
2. *If L is dicritical and $\rho^{n_1^{(J)}+1}$ does not divide $\xi_\ell^{(J)}(z)$, then $\xi_\ell^{(J)}(z)$ has constant sign along the surfaces $\partial_{min}V_j$ and $\partial_{max}V_j$, for any j.*
3. *Suppose that L is dicritical and $\rho^{n_1^{(J)}+1}$ divides $\xi_\ell^{(J)}(z)$. Denote $\mathcal{V}(L,\frac{\varepsilon}{2},\delta) = \{V_0', V_1',\ldots,V_{m_L}'\}$. Then, each element $V_j' \in \mathcal{V}(L,\frac{\varepsilon}{2},\delta)$ fulfills (1) and, moreover, any trajectory of $\pi^*\xi_\ell$ containing a point in V_j remains inside V_j' either for any positive time $t \geq 0$ or for any negative time $t \leq 0$.*

Proof. Taking into account Corollary 3.16 and since $\ell \geq \ell_{\mathcal{M}}$, the vector field $\xi_\ell^{(J)}$ is described by a non-autonomous two dimensional system of ODEs (see equation 3.17)

$$\left\{ \begin{array}{lcl} \frac{dz}{d\theta} & = & \rho^{n_1^{(J)}} A_z^{\ell,(J)}(\theta,z,\rho), \\ \frac{d\rho}{d\theta} & = & \rho^{n_1^{(J)}} A_\rho^{\ell,(J)}(\theta,z,\rho) \end{array} \right. \tag{3.20}$$

where $A_u^{\ell,(I)}(\theta,z,0) = A_u^{(J)}(z,0)$ for $u = \rho, z$. (As for the formal system of ODEs (3.18), we include the factor $z^{n_2^{(J)}}$ in $A_u^{\ell,(I)}$.)

We choose ε_0 satisfying the following conditions:

- In any case, we require $\varepsilon_0 < \frac{1}{2}\min_{i\neq j}\{|\omega_i^L - \omega_j^L|\}$.
- If L is dicritical and $A_z^{\ell,(I)}(\theta,z,0) \not\equiv 0$, being $\{t_1,\dots,t_s\}$ its set of zeroes, we require also

$$\varepsilon_0 < \frac{1}{2}\min\{|\omega_j^L - t_k| \;:\; 1 \le j \le m_L,\ 1 \le k \le s,\ \omega_j^L \neq t_k\}.$$

In the non-dicritical case, the function $A_z^{\ell,(J)}(\theta,z,0) \equiv A_z^{(J)}(z,0)$ is not identically zero and only depends on z. Being its zeroes $\omega_1^L,\dots,\omega_{m_L}^L$ by definition, it has constant sign when z belongs to the interval of $\Omega_j(\varepsilon)$ for $j \in \{0,\dots,m_L\}$ for any $0 < \varepsilon < \varepsilon_0$. By continuity and periodicity in θ, $A_z^{\ell,(J)}(\theta,z,\rho)$ has constant sign for (θ,z,ρ) in a set of the form $\mathbb{S}^1\times\Omega_j(\varepsilon)\times(0,\delta_j]$ for some $\delta_j = \delta_j(\varepsilon)$. Take δ fulfilling $\delta \le \min_{i=0,\dots,m_{L_i}}\{\delta_i\}$ and $B_{\ell,\theta}^{(J)} = \xi_\ell^{(J)}(\theta)$ has positive sign in $\mathbb{S}^1\times\Omega_j(\varepsilon)\times(0,\delta]$ for every $j = 0,\dots,m_L$. This is possible since $B_{\ell,\theta}^{(J)}(\theta,0,0) = 1$. Then, we define $V_j := \mathbb{S}^1\times\Omega_j(\varepsilon)\times(0,\delta]$. Taking into account that $\xi_\ell^{(J)}(z) = \rho^{n_1^{(J)}} A_z^{\ell,(J)}(\theta,z,\rho)\cdot B_{\ell,\theta}^{(J)}(\theta,z,\rho)$, we obtain the property *(1)* for the non-dicritical case.

In the dicritical case we proceed in the same way. Notice that $A_\rho^{\ell,(J)}(\theta,z,0) = \hat{A}_\rho^{(J)}(z,0)$ only depends on z and its set of zeros is by definition $\omega_1^L,\dots,\omega_{m_L}^L$. We get that $\xi_\ell^{(J)}(\rho)$ has constant sign in each V_j and statement *(1)* holds.

Let us show *(2)*, assuming that $\rho^{n_1^{(J)}+1}$ does not divide $\xi_\ell^{(J)}(z)$. By the choice of ε_0, we have that $A_z^{\ell,(J)}(\theta,z,0) = \hat{A}_{\mathbf{z}}^{(J)}$ does not vanish at any of the extreme values of $\Omega_j(\varepsilon)$. Since $\xi_\ell^{(J)}(z) = \rho^{n_1^{(J)}} A_z^{(J)}(\theta,z,\rho)\cdot B_{\ell,\theta}^{(J)}(\theta,z,\rho)$, we obtain *(2)*, up to taking a smaller δ.

Finally, we show *(3)*. Assume that L is dicritical and that $A_z^{(J)}(\theta,z,0) \equiv 0$. Then, the associated to $\xi_\ell^{(J)}$ system (3.20) can be written as

$$\left\{\begin{array}{lcl} \frac{dz}{d\theta} &=& \rho^{n_1^{(J)}+1}\widetilde{A}_z^{\ell,(J)}(\theta,z,\rho) \\ \frac{d\rho}{d\theta} &=& \rho^{n_1^{(J)}} A_\rho^{\ell,(J)}(\theta,z,\rho) \end{array}\right. , \tag{3.21}$$

where $A_\rho^{\ell,(J)}(\theta,z,0)$ does not depend on θ (by Corollary 3.16), vanishes exactly for $z \in \{\omega_1^L,\dots,\omega_{m_L}^L\}$, and $\widetilde{A}_z^{\ell,(J)}(\theta,z,0) \in \mathbb{R}[\cos\theta,\sin\theta,z]$. Proceeding as in the beginning of the proof, we take a constant $\delta > 0$ such that the collection $\mathcal{V}(L,\frac{\varepsilon}{2},\delta) = \{V_0', V_1',\dots,V_{m_L}'\}$ fulfills *(1)*, so that ρ is monotonic in every V_j'. Being $\overline{V}_j$ relatively compact, there are constants $a, K > 0$ such that for any $V_j' \in \mathcal{V}(L,\frac{\varepsilon}{2},\delta)$, we have

$$\inf_{p\in V_j'}\{|A_\rho^{\ell,(J)}(p)|\} \ge a, \quad \sup_{p\in V_j'}\{|\widetilde{A}_z^{\ell,(J)}(p)|\} \le K. \tag{3.22}$$

Fix V_j' and suppose, for instance, that $A_\rho^{\ell,(J)}|_{V_j'} < 0$. Then, if $\sigma : \mathbb{R} \longrightarrow M$ is a trajectory of $\xi_\ell^{(J)}$ parameterized as a solution $\sigma(\theta) = (\theta, z(\theta), \rho(\theta))$ of system (3.21), as long as it remains in $V_j' \setminus L$, the function $\rho(\theta)$ is strictly decreasing. Hence, σ can be parameterized by ρ instead of θ and we obtain from (3.21) and (3.22)

$$\left|\frac{dz}{d\rho}\right| \leq C\rho, \text{ where } C = \frac{K}{a}.$$

Now, consider the collection $\mathcal{V}(L, \varepsilon, \delta) = \{V_0, V_1, \ldots, V_{m_L}\}$ whose elements fulfill $V_j \subset V_j'$ for $j = 0, 1, \ldots, m_L$. If the trajectory σ starts at a point $p_0 = (\theta_0, z_0, \rho_0) \in V_j \subset V_j'$ with $\rho_0 > 0$, it satisfies, for $\theta > \theta_0$:

$$|z(\theta) - z_0| \leq \frac{C}{2}|\rho(\theta) - \rho_0|^2 \leq \frac{C}{2}\rho_0^2 \leq \frac{C}{2}\delta^2$$

as long as $\mathrm{Im}(\sigma|_{[\theta_0, \theta]}) \subset V_j'$. We obtain similar bounds for $|z(\theta) - z_0|$ when $A_\rho^{\ell,(J)}|_{V_j'} > 0$. Imposing $\delta < \sqrt{\frac{\varepsilon}{C}}$, we can conclude that $|z_0 - z(\theta)| < \frac{\varepsilon}{2}$ and guarantee, for any j and for any $p_0 \in V_j \in \mathcal{V}(L, \varepsilon, \delta)$, that the trajectory σ starting at p_0 satisfies $\mathrm{Im}(\sigma|_{[\theta_0, \infty)}) \subset V_j'$ (or $\mathrm{Im}(\sigma|_{(-\infty, \theta_0]}) \subset V_j'$ in case $A_\rho^{\ell,(J)}|_{V_j'} > 0$). □

From the proof above, we may observe that $\mathcal{V}(L, \varepsilon, \delta')$ also fulfills (1-3) of the lemma for any $\delta' < \delta$.

Notation 3.19. Given an open component $L \in \mathcal{E}_\mathcal{M}$ as above, in the notations of section 3.3.1 for $j \in \{0, \ldots, m_L\}$, let $I_j \in \mathcal{I}$ be the index of the corresponding characteristic cycle $\gamma_{I_j} = \{z = \omega_j, \rho = 0\}$. Let I_0, I_{m_L+1} be also the indices of either the corner characteristic cycles or characteristic singularities in the component $\bar{L}$. We say that the box $V_j \in \mathcal{V}(L, \varepsilon, \delta)$ with $j = 1, \ldots, m_L - 1$ is *adjacent* to γ_{I_j} and to $\gamma_{I_{j+1}}$ and we denote $\partial_{I_j} V_j = \partial_{min} V_j$ and $\partial_{I_{j+1}} V_j = \partial_{max} V_j$.

Proof of Proposition 3.17. Let W be a neighborhood of $\mathrm{Supp}\mathcal{D}$. For every $I \in \mathcal{I}$ we consider an open neighborhood $W_I \subset W$ of γ_I such that $W_I \cap W_{I'} = \emptyset$ if $I \neq I'$. Consider the collection $\mathcal{E}_\mathcal{M}$, and apply Lemma 3.18 to each $L_i \in \mathcal{E}_\mathcal{M}$, taking ε and δ small enough so that each family $\mathcal{V}(L_i, \varepsilon, \delta)$ also satisfies:

- For any $V \in \mathcal{V}(L_i, \varepsilon, \delta)$, we impose $V \cap W_I \neq \emptyset$ if and only if γ_I is adjacent to V.
- For any $V \in \mathcal{V}(L_i, \varepsilon, \delta)$, the boundaries $\partial_{min} V$ and $\partial_{max} V$ are contained in the corresponding neighborhoods W_I and $W_{I'}$, where γ_I and $\gamma_{I'}$ are adjacent to V.
- The set
  $$\bigcup_{I \in \mathcal{I}} W_I \cup \bigcup_{L \in \mathcal{E}_\mathcal{M}} \bigcup_{V \in \mathcal{V}(L, \varepsilon, \delta)} V$$
  is a neighborhood of the divisor $E = \pi^{-1}(0)$ in M.

Now, we define a closed neighborhood $\widetilde{W_I} \subset W_I$ of γ_I for each $I \in \mathcal{I}$ in such a way that (see Figure 3.4):

(i) The set

$$\widetilde{U} = \operatorname{int}\left(\bigcup_{I\in\mathcal{I}} \widetilde{W_I} \cup \bigcup_{L\in\mathcal{E}_{\mathcal{M}}} \bigcup_{V\in\mathcal{V}(L,\varepsilon,\delta)} V\right)$$

is a neighborhood of the divisor E in M.

(ii) For any $I \in \mathcal{I}$, $L \in \mathcal{E}_{\mathcal{M}}$ and $V \in \mathcal{V}(L,\varepsilon,\delta)$, $\widetilde{W_I} \cap \bar{V}$ is empty, in case V is not adjacent to γ_I, or, otherwise, it is of the form $\widetilde{W_I} \cap V = \mathbb{S}^1 \times \{c_I\} \times (0,\mu]$, where $0 < \mu \le \delta$ and c_I satisfies $V = \mathbb{S}^1 \times \{c_I\} \times (0,\delta]$.

Now, the set $U := \pi(\tilde{U})$ is an open neighborhood of 0 satisfying the requirements of the proposition. More precisely, we claim that $\pi^{-1}(\mathcal{C}_U) \subset \bigcup_{I\in\mathcal{I}} \widetilde{W_I}$.

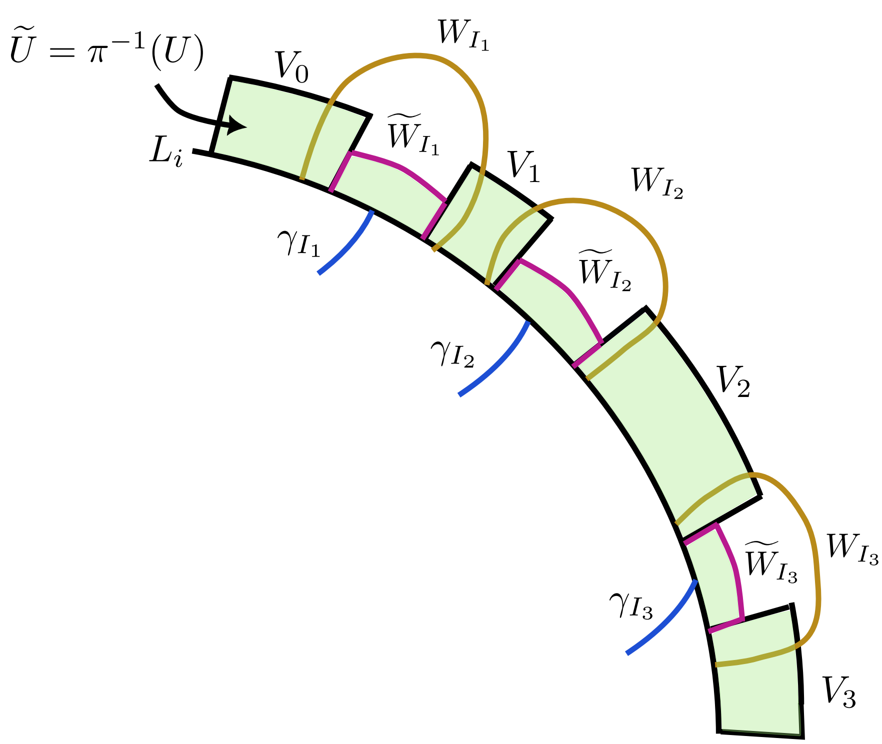

Figure 3.4: Construction of $\widetilde{U}$.

To prove this, suppose that there is a cycle Z of ξ_ℓ contained in U and such that $\widetilde{Z} := \pi^{-1}(Z)$ intersects some $V \in \mathcal{V}(L,\varepsilon,\delta)$ for some L. Consider a parametrization $\sigma : \mathbb{R} \longrightarrow \tilde{U}$ of $\widetilde{Z}$ as a trajectory of $\pi^*\xi_\ell$ such that $\sigma(0) \in V$. By the property *(1)* of Lemma 3.18, one of the coordinates z or ρ is monotonic along σ inside V, so it cannot be completely contained in V. As a consequence, σ leaves V so that for some $t_0 \ge 0$ we have $\sigma(t_0) \in \mathrm{Fr}(V) \cap \widetilde{W_I}$, where $I \in \mathcal{I}$ and γ_I is adjacent to

V. By construction (cf. item (ii) above), $\sigma(t_0)$ belongs to the boundary $\partial_I V$. We have two cases to consider (notations as in Lemma 3.18).

- $A_z^{\ell,(J)}(\theta,z,0) \neq 0$. By statement (2) of Lemma 3.18, the vector field $\pi^*\xi_\ell$ is transverse to $\partial_I V$, so that, for instance, we have $\sigma((t_0-c,t_0)) \subset \mathrm{int}(V)$ and $\sigma((t_0,t_0+c)) \subset \mathrm{ext}(V)$ for some $c>0$. Since σ is periodic, we must have that σ crosses $\mathrm{Fr}(V)$ at a first time $t_1 > t_0$ necessarily along one of the boundaries $\partial_{min}V, \partial_{max}V$ where $\pi^*\xi_\ell$ points towards $\mathrm{int}(V)$. If we denote $\{\partial_I V, \partial_{I'}V\} = \{\partial_{min}V, \partial_{max}V\}$, we must have $\sigma(t_1) \in \partial_{I'}V$ and $\sigma((t_0,t_1)) \subset \mathrm{ext}(V)$, but this contradicts the fact that, by construction, $\tilde{U} \setminus V = \widetilde{U}_1 \cup \widetilde{U}_2$ where $U_1, \tilde{U}_2$ are non-empty open sets such that $\tilde{U}_1 \cap \tilde{U}_2 = \emptyset$ and the closure of each $\tilde{U}_i$ cuts V only along one of the sets $\{\partial_{I'}V, \partial_I V\}$.
- $A_z^{\ell,(J)}(\theta,z,0) \equiv 0$. Using statement (3) of Lemma 3.18, we know that either $\sigma((t_0,\infty))$ or $\sigma((-\infty,t_0))$ is contained in the corresponding element V' of the collection $\mathcal{V}(L,\frac{\varepsilon}{2},\delta)$ and $\rho\circ\sigma$ is monotonic along that interval. This is also a contradiction with σ being periodic.

Consequently, we have proved that $\tilde{Z} \subset \bigcup_{I\in\mathcal{I}} \widetilde{W_I}$ (in fact, included in a single $\widetilde{W_I}$ by connectedness). Therefore, we have that:

$$\pi^{-1}(\mathcal{C}_U(\xi_\ell)) \subset \bigcup_{I\in\mathcal{I}} \widetilde{W_I} \subseteq \bigcup_{I\in\mathcal{I}} W_I \subseteq W,$$

as we wanted to prove.

□

3.4.2 Analysis of Infinitely Near Points of the Rotational Axis

We see first that we can find a neighborhood of the two characteristic singular points that does not contain cycles of a jet approximation ξ_ℓ of $\hat{\xi}$.

Proposition 3.20. *Given $\ell \geq \ell_{\mathcal{M}}+1$, there exist neighborhoods W_∞ of $\gamma_{I_{-\infty}^{\mathcal{M}}}$ and $W_{-\infty}$ of $\gamma_{I_\infty^{\mathcal{M}}}$ in M such that neither $W_\infty \setminus E$ nor $W_{-\infty}\setminus E$ contains cycles of $\pi^*\xi_\ell$.*

Proof. According to the construction in section 3.3.1, the point $\gamma_{I_\infty^{\mathcal{M}}}$ is the origin of the chart $(C_J,(x^{(J)},y^{(J)},z^{(J)}))$ with $J = I_\infty^{\mathcal{M}}$ and $E\cap C_J = \{z^{(J)}=0\}$. By means of Proposition 3.14, we have that $\xi_\ell^{(J)}(z^{(J)}) = (z^{(J)})^{n^{(J)}}\cdot F(x^{(J)},y^{(J)},z^{(J)})$ in a neighborhood of $\gamma_{I_\infty^{\mathcal{M}}}$ where $\xi_\ell^{(J)} = (\pi|_{C_J})^*\xi_\ell$, $n^{(J)} \in \mathbb{N}_{\geq 1}$ and $F(x^{(J)},y^{(J)},z^{(J)}) \in \mathbb{R}\{x^{(J)},y^{(J)},z^{(J)}\}$ satisfies $F(0,0,0)\neq 0$. Take a neighborhood W_∞ of $I_\infty^{\mathcal{M}}$ in M where F has a constant sign, positive or negative. We have that the trajectories of $\pi^*\xi_\ell$ in $W_\infty\setminus E$ can be parameterized by $z^{(J)}$, which contradicts the existence of cycles of $\pi^*\xi_\ell$ in $W_\infty \setminus E$. The proof for $\gamma_{I_{-\infty}}^{\mathcal{M}}$ is analogous.

□

3.4.3 Analysis of Characteristic Cycles

This section is new with respect to [23]. The main difference is that we do not require that the characteristic cycles correspond to adapted simple singularities. We are able to solve the problem for any characteristic cycle. The main tool is the definition of the Poincaré maps. Applying Theorem 2.6, we have that there are not periodic points, other than fixed points, for such Poincaré maps, in a sufficiently small neighborhood of the origin. In addition, if there are fixed points, they are contained in a finite number of curves of fixed points. Each of these curves implies the existence of a surface with center configuration. These arguments will serve us to conclude that there are not isolated cycles in a sufficiently small neighborhood of a characteristic cycle of ξ_ℓ for ℓ large enough.

In order to apply Theorem 2.6, we need to ensure that the Poincaré maps at the characteristic cycles are tangent to the identity but different from it.

Definition of Poincaré maps

Throughout this subsection, we suppose that γ_I is a characteristic cycle of $\hat{\xi}$ in $\mathcal{M}$ contained in a chart C_J for which $\{\rho^{(J)} = 0\}$ (or $\{z^{(J)}\rho^{(J)} = 0\}$) is the equation of $E \cap C_J$ and $\gamma_I = \{\rho^{(J)} = 0, z^{(J)} = w_I\}$ for some $\omega_I \in \mathbb{R}$. Consider the transform $\hat{\xi}^{(J)} = (\pi|_{C_J})^*\hat{\xi}$ in the translated coordinates $(z := z^{(J)} + w_I, \rho := \rho^{(J)})$. Its associated two-dimensional vector field is

$$\hat{\eta}_J := \frac{\hat{\xi}^{(J)}(\rho)}{\hat{\xi}^{(J)}(\theta)}\frac{\partial}{\partial \rho} + \frac{\hat{\xi}^{(J)}(z)}{\hat{\xi}^{(J)}(\theta)}\frac{\partial}{\partial z}.$$

More precisely, we write $\hat{\eta}_J = \rho^a \hat{\eta}_J'$ where $a \geq 1$ (or $\hat{\eta}_J = \rho^a z^b \hat{\eta}_J'$ where $a, b \geq 0$ and $a + b \geq 2$) and $\hat{\eta}_J'$ is a formal vector field in coordinates (z, ρ) with a singularity at the origin.

Remark 3.21. Because we are in the non-degenerate case, we have that the vector field $\hat{\eta}_J$ is not identically zero. This implies that this formal vector field has a non-vanishing jet $j_{k_I}(\hat{\eta}_J') \neq 0$ with $k_I \geq a + b + 1 \geq 2$, since on the one hand $(\omega_I, 0)$ is a singularity of $\hat{\eta}_J'$ and on the other hand $a \geq 1$ or $a + b \geq 2$. We will take $\ell \geq l + 1 + k_I$.

By Remark 3.15, γ_I is a trajectory of the vector field $\xi_\ell^{(J)} = (\pi|_{C_J})^*\xi_\ell$ for $\ell \geq \max\{\ell_{\mathcal{M}}+1, l+1+k_I\}$. This vector field is described by the system of ODEs $\eta_{\ell,J}$. We can define the Poincaré map $P = P_{\ell,I} : \Delta \to \{\theta = 0\}$ as the first-return map of $\xi_\ell^{(J)}$ relatively to γ_I, where Δ is a sufficiently small neighborhood of $(z, \rho) = (0, 0)$ in $\{\theta = 0\}$ in which P is analytic.

Notice that the Poincaré map does not depend on the parameterization of the trajectories of the vector field, and hence, we can define it using any equivalent vector field. In particular, we are going to consider the vector field $\tilde{\xi}_\ell^{(J)}$ equivalent to $\xi_\ell^{(J)}$ obtained by the multiplication by the inverse of $\xi_\ell^{(J)}(\theta)$. That is, we put

$$\tilde{\xi}_\ell^{(J)} = \frac{\partial}{\partial\theta} + \chi, \quad \text{where } \chi = \frac{\xi_\ell^{(J)}(z)}{\xi_\ell^{(J)}(\theta)}\frac{\partial}{\partial z} + \frac{\xi_\ell^{(J)}(\rho)}{\xi_\ell^{(J)}(\theta)}\frac{\partial}{\partial\rho}. \tag{3.23}$$

Notice that the components of χ are the right members of the system of ODEs $\eta_{\ell,J}$ introduced in section 3.3.3. They belong to the $\mathbb{R}$–algebra $\mathbb{R}[\cos\theta,\sin\theta]\{z,\rho\}$ (by Remark 3.15). Thus, we consider $\widetilde{\xi}_\ell^{(J)}$ as an analytic vector field on the domain $(\theta,z,\rho)\in\mathbb{R}\times(-\delta,\delta)^2$ for some $\delta>0$ and 2π–periodic in the variable θ. Notice, moreover, from Corollary 3.16, that ρ divides χ and hence $\widetilde{\xi}_\ell^{(J)}|_E = \frac{\partial}{\partial\theta}$.

Denote by $\Phi_t := \Phi_t^{\tilde{\xi}_\ell^{(J)}}$ the flow map of $\tilde{\xi}_\ell^{(J)}$. It is defined and analytic for $(t,(\theta,z,\rho))\in(-\varepsilon,2\pi+\varepsilon)\times((-\varepsilon,2\pi+\varepsilon)\times V)$ where V is a neighborhood of $0\in\mathbb{R}^2$. Using that $\widetilde{\xi}_\ell^{(J)}(\theta)=1$, we obtain

$$\Phi_t(\theta,z,\rho) = (\theta+t,\Psi_t^z(\theta,z,\rho),\Psi_t^\rho(\theta,z,\rho)), \tag{3.24}$$

that is, the angle θ is the natural time for $\tilde{\xi}_\ell^{(J)}$. By definition, the Poincaré map is given by

$$P(z,\rho) = (\Psi_{2\pi}^z(0,z,\rho),\Psi_{2\pi}^\rho(0,z,\rho)). \tag{3.25}$$

We are going to express the flow via the exponential map. To be precise, given any $G\in\mathbb{R}[\cos\theta,\sin\theta]$ $[[z,\rho]]$, we define:

$$\exp(t\tilde{\xi}_\ell^{(J)})(G) := \sum_{i=0}^{\infty}\frac{t^i}{i!}(\tilde{\xi}_\ell^{(J)})^{(i)}(G),$$

where, for any vector field ζ, $\zeta^{(0)}(G)=G$ and $\zeta^{(i)}(G)=\zeta(\zeta^{(i-1)}(G))$, if $i\geq 1$. Taking into account the above properties of the components of $\tilde{\xi}_\ell^{(J)}$, it is immediate to check that $\exp(t\tilde{\xi}_\ell^{(J)})(G)\in\mathbb{R}[\cos\theta,\sin\theta][[t,z,\rho]]$. In the following result, we get some useful properties of this exponential map and its relation with the flow map. Notice first that, if $G\in\mathbb{R}[\cos\theta,\sin\theta][[z,\rho]]$, then the composition $G\circ\Phi_t$, due to the analyticity of Φ_t, has a formal Taylor expansion at $t=0$, denoted by $T_0(G\circ\Phi_t)$, a formal power series in variables (t,z,ρ), with analytic functions of $\theta\in(-\varepsilon,2\pi+\varepsilon)$ as coefficients.

Proposition 3.22. *Let $G\in\mathbb{R}[\cos\theta,\sin\theta][[z,\rho]]$. We have:*

1. $T_0(G\circ\Phi_t) = exp(t\tilde{\xi}_\ell^{(J)})(G)\in\mathbb{R}[\cos\theta,\sin\theta][[t,z,\rho]]$
2. *For any $t_0\in[0,2\pi]$, the expression $exp(t_0\tilde{\xi}_\ell^{(J)})(G) = \sum_{i=0}^{\infty}\frac{t_0^i}{i!}(\tilde{\xi}_\ell^{(J)})^{(i)}(G)$ has a sense as a series in $\mathbb{R}[\cos\theta,\sin\theta][[z,\rho]]$ and we have*

$$G\circ\Phi_{t_0} = exp(t_0\tilde{\xi}_\ell^{(J)})(G) \tag{3.26}$$

Proof. We prove *(1)* with the same arguments as the case in [58, pag. 15] for holomorphic vector fields: expand $G\circ\Phi_t$ as a Taylor series in t at $t=0$, so that we get

$$T_0(G\circ\Phi_t)=\sum_{i=0}^{\infty}\frac{t^i}{i!}\left.\frac{\partial^i(G\circ\Phi_t)}{\partial t^i}\right|_{t=0},$$

and check that, for any $i\geq 1$, $\frac{\partial^i(G\circ\Phi_t)}{\partial t^i}=(\tilde{\xi}_\ell^{(J)})^{(i)}(G)\circ\Phi_t$.

Let us prove item *(2)*. First, we show that there exists $\alpha>0$ such that *(2)* is true for any $t_0\in[0,\alpha]$. For that, consider the particular case where G is either the coordinate z or ρ (with the notations of (3.29), $z\circ\Phi_{t_0}=\Psi^z_{t_0}$ and $\rho\circ\Phi_{t_0}=\Psi^\rho_{t_0}$). By analyticity of these functions and by item *(1)*. We get that $\exp(\tilde{\xi}_\ell^{(J)})(z),\exp(\tilde{\xi}_\ell^{(J)})(\rho)\in\mathbb{R}[\cos\theta,\sin\theta]\{t,z,\rho\}$. More precisely, they belong to $\mathbb{R}[\cos\theta,\sin\theta]\{t\}_\beta[[z,\rho]]$ for some $\beta>0$ (all coefficients in $\mathbb{R}[\cos\theta,\sin\theta]\{t\}$ have a common radius of convergence). We conclude that $\Psi^z_{t_0}=z\circ\Phi_{t_0}=\exp(t_0\tilde{\xi}_\ell^{(J)})(z)$ and $\Psi^\rho_{t_0}=\rho\circ\Phi_{t_0}=\exp(t_0\tilde{\xi}_\ell^{(J)})(\rho)$ for any $t_0\in[0,\alpha]$ with $0<\alpha<\beta$.

Let $G\in\mathbb{R}[\cos\theta,\sin\theta][[z,\rho]]$ be any formal series and write

$$G=\sum_{u,v}G_{uv}(\theta)z^u\rho^v,\ \text{with } G_{uv}(\theta)\in\mathbb{R}[\cos\theta,\sin\theta].$$

Consider the series

$$\bar{G}=\sum_{u,v}G_{uv}(\theta+t)z^u\rho^v$$

which belongs to $\mathbb{R}[\cos\theta,\sin\theta]\{t\}_\beta[[z,\rho]]$ since each $G_{uv}(\theta)$ is a trigonometric polynomial. Taking into account the expression of the flow Φ_t, we have that $G\circ\Phi_t$ is the result of substituting in the series $\bar{G}$ the variables z,ρ by Ψ^z_t,Ψ^ρ_t, respectively. Since the series Ψ^z_t,Ψ^ρ_t belong to $\mathbb{R}[\cos\theta,\sin\theta]\{t\}_\delta[[z,\rho]]$ and have positive order with respect the variables z,ρ, substitution has perfect sense and provides an element in $\mathbb{R}[\cos\theta,\sin\theta]\{t\}_\beta[[z,\rho]]$. Since, by item *(1)*, $T_0(G\circ\Phi_t)$ coincides with $\exp(t\widetilde{\xi}_\ell^{(J)})(G)$ as a series in $\mathbb{R}[\cos\theta,\sin\theta][[t,z,p]]$, we conclude item *(2)* and (3.26) for $t_0\in[0,\alpha]$. Notice that we can choose $\alpha>0$ which does not depend on G. Let us show that we can extend the property(3.26) to any $t_0\in[0,2\alpha]$ (and hence similar extensions will prove *(2)*). Let $t_0\in[\alpha,2\alpha]$ and write $t_0=s_0+\alpha$, where $s_0\in[0,\alpha]$. We have $G\circ\Phi_{t_0}=(G\circ\Phi_{s_0})\circ\Phi_\alpha$. Applying (3.26) for the values s_0 and α, and for G and $G\circ\Phi_{s_0}$, respectively, we get

$$\begin{aligned}G\circ\Phi_{t_0}&=\sum_i\frac{\alpha^i}{i!}(\tilde{\xi}_\ell^{(J)})^{(i)}(G\circ\Phi_{s_0})=\sum_i\frac{\alpha^i}{i!}(\tilde{\xi}_\ell^{(J)})^{(i)}\left(\sum_j\frac{s_0^j}{j!}(\tilde{\xi}_\ell^{(J)})^{(j)}(G)\right)\\&=\sum_k\left(\sum_{i+j=k}\frac{\alpha^i}{i!}\frac{s_0^j}{j!}(\tilde{\xi}_\ell^{(J)})^{(k)}(G)\right)=\sum_k\frac{(\alpha+s_0)^k}{k!}(\tilde{\xi}_\ell^{(J)})^{(k)}(G)=\exp(t_0\tilde{\xi}_\ell^{(J)})(G),\end{aligned}$$

which was to be proved. □

We can now prove an important feature of the Poincaré map.

Lemma 3.23. *There exists ℓ_I such that for any $\ell \geq \ell_I$, the Poincaré map $P = P_{\ell,I}$ is tangent to the identity but $P \neq Id$ as a germ of diffeomorphisms at $(0,0) \in \Delta$.*

Proof. This result is a consequence of the fact that the vector field $\hat{\eta}_J$ has a non-vanishing jet (c.f. Remark 3.21). Suppose that $\rho = 0$ is the equation of a component of the divisor and write $\hat{\eta}_J'$ in homogeneous components in ρ. We have

$$\hat{\eta}_J = \rho^a z^b \left(F(z,\rho) \frac{\partial}{\partial z} + G(z,\rho) \frac{\partial}{\partial \rho} \right) = \rho^a z^b \sum_{k \geq 1} \left(F_k(z) \rho^k \frac{\partial}{\partial z} + G_k(z) \rho^k \frac{\partial}{\partial \rho} \right),$$

and consider the minimum ν such that $F_\nu \neq 0$ or $G_\nu \neq 0$. We take $\ell_I = \nu + a + l + 1$, where l is the length of the sequence of blowing-ups $\mathcal{M}$ that is being considered.

By Corollary 3.14, choosing $\ell \geq \max\{\ell_I, \ell_{\mathcal{M}}\}$, we have that the system $\eta_{\ell,J}$ has the same $(\nu + a)$–jet in ρ as $\hat{\eta}_J$, namely $j^\rho_{\nu+a}(\eta_{\ell,J}) = j^\rho_{\nu+a}(\hat{\eta}_J)$. Recall also that the coefficients of χ in $\zeta_\ell^{(J)} = \frac{\partial}{\partial \theta} + \chi$ are given by the system $\eta_{\ell,J}$, and hence its $(\nu + a)$–jet in ρ is known.

Suppose first that $F_\nu \neq 0$. Then, we find that

$$\begin{aligned} \exp(t\zeta_\ell^{(J)})(\rho) =& \rho + tQ_1 + tQ_2 + \cdots \\ =& \rho + t(\rho^{a+\nu} z^b F_\nu(z) + O(\rho^{\nu+a+1})) + t^2(\rho^{2(a+\nu)-1} z^b (a+\nu) F_\nu^2 + O(\rho^{2(a+\nu)})) + O(t^3 \rho^{3(a+\nu)-1}). \end{aligned}$$

Notice that the terms summarized in $O(\rho^{\nu+a+1})$ can depend also on $\cos\theta, \sin\theta$ and z. We have to take care about the case $\nu = 0$ and $a = 1$. Since $(0,0)$ is a singularity of $\eta_{\ell,J}$, we have that $F\nu(0) = 0$, in particular, z divides F_ν. Then, the sum $tF_\nu + t^2 F_\nu^2 + \cdots$, which gives the lower order jet in ρ, is different from 0, for any value of $t \neq 0$ for which the flow is defined. In any other case, we have that independently of the value of ν, the order in ρ increases on each Q_i. And again, since $F_\nu \not\equiv 0$, we have that $\Psi^\rho_{2\pi} \neq \rho$.

A similar reasoning applies if $G_\nu \neq 0$, obtaining that $\Psi^z_{2\pi} \neq z$. We conclude that either $\Psi^\rho_{2\pi} \neq \rho$ or $\Psi^z_{2\pi} \neq z$, and hence $P_{\ell,I} \neq Id$. □

Existence of limit central surfaces

In the previous subsection, we showed that the Poincaré maps defined at the characteristic cycles $\gamma_I \in \mathcal{D}$ are tangent to the identity diffeomorphisms and different from the identity. Hence, Theorem 2.6 can be applied to these Poincaré maps. We will use this result to prove the following one.

Proposition 3.24. *Let $\gamma_I \in \mathcal{D}$ be a characteristic cycle of ξ_ℓ and $\ell \geq \ell_I$ in Lemma 3.23. Then, there exists a neighborhood W_I of γ_I such that exactly one of the following holds.*

- *There is a finite number of analytic surfaces $S_{I,1},\ldots,S_{I,s_I}$ that have center configuration. There are not other cycles of ξ_ℓ in W_I.*
- *There are no cycles of ξ_ℓ in W_I.*

Proof. For the proof of this result, take the Poincaré map $P_{\ell,I}$ defined on a cross section Δ of γ_I. We define W_I as $\Delta \times \mathbb{S}^1$, which is a neighborhood of γ_I. By Lemma 3.23, it is tangent to the identity. In addition, since the intersection of the plane Δ with E is invariant for $P_{\ell,I}$, we have that it is not center-focus. We are on the hypotheses of Theorem 2.6 and we can apply this result and its consequence, Corollary 2.7. We find that the only periodic points of $P_{\ell,I}$ that can appear are analytic curves $\Gamma_{I,1},\ldots,\Gamma_{I,s_I}$ of fixed points. Hence, the only cycles of ξ_ℓ in W_I that intersect this cross section do it in the curves of fixed points. Saturating these curves with the flow of ξ_ℓ, we find that $S_{I,1}\ldots,S_{I,s_I}$ are analytic surfaces with center configuration and no other cycles are contained in W_I. □

3.5 The isolated singularity case: End of the proof

In this section, we will collect all the results presented in sections 3.3 and 3.4 and prove Theorem 3.1 in the isolated singularity case.

Proposition 3.25. *There is a vector field ξ_ℓ with ℓ sufficiently large that fulfills Theorem 3.1.*

Proof. First, we will choose the ξ_ℓ with ℓ sufficiently large. Consider the reduction of singularities $\mathcal{M} = (M,\pi,\mathcal{A},\mathcal{D})$ of $\hat{\xi}$, with $\mathcal{A} = \{C_J\}_{J\in\mathcal{J}}$ and $\mathcal{D} = \{\gamma_I\}_{I\in\mathcal{I}}$ given in Proposition 3.12. Take $\ell \geq \max\{\ell_\mathcal{M},\ell_I : I \in \mathcal{I}\setminus\{I^{\infty}_{\mathcal{M}}, I^{-\infty}_{\mathcal{M}}\}\}$, where ℓ_I are defined as in Lemma 3.23. Applying Proposition 3.20, we obtain neighborhoods W_∞ and $W_{-\infty}$ of $\gamma_{I^{\infty}_{\mathcal{M}}}$ and $\gamma_{I^{-\infty}_{\mathcal{M}}}$ respectively, which are free of cycles of ξ_ℓ. Secondly, we apply Proposition 3.24 to each γ_I obtaining neighborhoods W_I of γ_I that are either free of cycles or in which the cycles of ξ_ℓ are organized in surfaces with center configuration. If necessary, we reduce these neighborhoods so that they are not overlapping. Taking the union

$$W = W_\infty \cup W_{-\infty} \cup \bigcup_{I\in\mathcal{I}\setminus\{I^{\infty}_{\mathcal{M}}, I^{-\infty}_{\mathcal{M}}\}} W_I,$$

we obtain a neighborhood W of $\operatorname{Supp}(\mathcal{D})$. We are then in the conditions of Proposition 3.17. Applying this result, we obtain a neighborhood $U(W)$ of $0 \in \mathbb{R}^3$ such that $\pi^{-1}(\mathcal{C}_U) \subset W$. From Proposition 3.20, we have that there are not cycles in W_∞ nor in $W_{-\infty}$. From Proposition 3.24, we have that either there are not cycles in W_I for $I \in \mathcal{I}\setminus\{I^{\infty}_{\mathcal{M}}, I^{-\infty}_{\mathcal{M}}\}$, or they belong to analytic surfaces in W_I.

To finish the proof, we project the surfaces $S_{I,k}$ for each $I \in \mathcal{I} \setminus \{I_{\mathcal{M}}^{\infty}, I_{\mathcal{M}}^{-\infty}$ and $1 \le k \le s_I$ under π to $\mathbb{R}^3$. They provide a collection of subanalytic limit central surfaces $S_1, \dots, S_r$ at $0 \in \mathbb{R}^3$. $\square$

Notice that any vector field ξ_ℓ is analytically conjugated to the original vector field ξ by construction and that any other $\xi_{\ell'}$ is as well analytically conjugated to ξ_ℓ. Hence, proving the theorem for one ξ_ℓ implies proving it for any other conjugated vector field.

3.6 The non-isolated singularity case for formally non-degenerated vector fields

Now we will prove Theorem 3.1 for non-degenerated non-isolated Hopf singularities. The proof will follow the lines of the semi-hyperbolic case, in the sense that we will be able to blow up an analytic curve and to define a Poincaré map from a cycle that arises after this blowing-up.

We have mentioned before that at the time of the publication of [23], we did not have a good description of the Poincaré maps, nor a strategy to prove non-accumulation of cycles at a degenerated rotational axis, the curve of singularities. Thanks to the contribution of Theorem 2.6, we overcome those difficulties and we are able to give a rather simple proof of the result in this case.

3.6.1 Blowing-up of the rotational axis and choice of a jet approximation

First, we make the assumption that the curve of singularities at 0 is the z–axis, for convenience. Fix a formal normal form $\hat{\xi}$ of ξ and consider its associated two dimensional vector field $\hat{\eta}$ as in section 3.1.2. We recall the expression of $\hat{\xi}$ in equation (3.3)

$$\hat{\xi} = A(x^2+y^2,z)\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + B(x^2+y^2,z)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + C(x^2+y^2,z)\frac{\partial}{\partial z},$$

where A is a unit in $\mathbb{R}[[u,v]]$ and C is divided by a power of x^2+y^2 because of the non-isolated singularity hypothesis. The expression of $\hat{\eta}$ is

$$\hat{\eta} = B(\rho^2,z)A(\rho^2,z)^{-1}\rho\frac{\partial}{\partial \rho} + C(\rho^2,z)A(\rho^2,z)^{-1}\frac{\partial}{\partial z}.$$

Since we are in the non-degenerated case, we know that the vector field $\hat{\eta}$ is a non-zero formal vector field of order equal or greater than 1 and it has a non-vanishing jet. Because of the isolated singularity condition, we also have that ρ divides $\hat{\eta}$.

Now, we blow up the z-axis, and present the transform vector field in the global chart C_0, where we take polar coordinates (θ, z, ρ).

$$\hat{\xi}^{(0)} = A(\rho^2,z)\frac{\partial}{\partial \theta} + B(\rho^2,z)\frac{\partial}{\partial \rho} + C(\rho^2,z)\frac{\partial}{\partial z} = A^{(0)}(z,\rho)\frac{\partial}{\partial \theta} + B^{(0)}(z,\rho)\frac{\partial}{\partial \rho} + C^{(0)}(z,\rho)\frac{\partial}{\partial z},$$

with $A^{(0)}, B^{(0)}, C^{(0)} \in \mathbb{R}[[z,\rho]]$. The associated two dimensional vector field $\hat{\eta}_0$ coincides with $\hat{\eta}$ and has a non-vanishing k–jet. Notice also that $\sigma_0^{-1}(0)$ is a cycle of $\hat{\xi}^{(0)}$ in $E = \sigma_0^{-1}(\Omega)$, where the restriction of $\hat{\xi}^{(0)}$ is equivalent to $\frac{\partial}{\partial\theta}$

Now, take any ξ_ℓ with $\ell \geq k$. We can apply the previous blowing-up σ_0 to ξ_ℓ, obtaining a vector field

$$\begin{aligned}\xi_\ell^{(0)} &= A_\ell(\rho\cos\theta,\rho\sin\theta,z)\frac{\partial}{\partial\theta} + B(\rho\cos\theta,\rho\sin\theta,z)\frac{\partial}{\partial\rho} + C(\rho\cos\theta,\rho\sin\theta,z)\frac{\partial}{\partial z}\\ &= A_\ell^{(0)}(\theta,z,\rho)\frac{\partial}{\partial\theta}B_\ell^{(0)}(\theta,z,\rho)\frac{\partial}{\partial\rho}C_\ell^{(0)}(\theta,z,\rho)\frac{\partial}{\partial z},\end{aligned}$$

with $A_\ell^{(0)}, B_\ell^{(0)}, C_\ell^{(0)} \in \mathbb{R}[\cos\theta,\sin\theta]\{z,\rho\}$. We recall also that, since the singularity is non-isolated, ρ divides both C_ℓ and B_ℓ. Using that $A_\ell^{(0)}$ is a unit in $\mathbb{R}[\cos\theta,\sin\theta]\{z,\rho\}$, we can define a system of ODEs $\eta_{\ell,0}$, as we did similarly in the isolated singularity case:

$$\left\{\begin{array}{lcl}\frac{dz}{d\theta} &=& A_\ell^{(0)}(\theta,z,\rho)^{-1}C_\ell^{(0)}(\theta,z,\rho),\\ \frac{d\rho}{d\theta} &=& A_\ell^{(0)}(\theta,z,\rho)^{-1}B_\ell^{(0)}(\theta,z,\rho).\end{array}\right. \tag{3.27}$$

By hypothesis, the ℓ–jet in ρ, z of $\xi_\ell^{(0)}$ is the same as that of $\hat{\xi}^{(0)}$. We will fix $\ell \geq k$ where k is a non-vanishing jet of $\hat{\xi}^{(0)}$. By this jet equality, we find that ξ_ℓ has a cycle γ at $\sigma_0^{-1}(0)$ as well. Following the ideas of section 3.4.3, we define a Poincaré map at an analytic cross section of this cycle.

Recall that for the definition of the Poincaré map, the parameterization of the trajectories is not important. Then, to define it, we will work with the vector field $\zeta_\ell^{(0)}$ which is equivalent to $\xi_\ell^{(0)}$ by multiplying by the unit $A_\ell^{(0)}(\theta,z,\rho)$. The expression of this vector field $\zeta_\ell^{(0)}$ in coordinates (θ,z,ρ) is given by

$$\zeta_\ell^{(0)} = \frac{\partial}{\partial\theta} + \chi,$$

where the coefficients of χ coincide with the right members of the system η_ℓ in (3.27). Since they are analytic, suppose that they are convergent on a domain $\mathbb{R}\times(-\delta,\delta)^2$ for some $\delta > 0$. Fix the plane $\{\theta = 0\}$. Following the same reasoning as in section 3.4.3, we will define the Poincaré map in a neighborhood Δ of $(0,0)$ in $\{\theta = 0\}$. First, we find that the coordinate θ acts like the time variable, since

$$\Phi_t(\theta,z,\rho) = (\theta + t, \Psi_t^z(\theta,z,\rho), \Psi_t^\rho(\theta,z,\rho)). \tag{3.28}$$

By definition, the Poincaré map $P = P_\ell$ is given by

$$P(z,\rho) = (\Psi_{2\pi}^z(0,z,\rho), \Psi_{2\pi}^\rho(0,z,\rho)). \tag{3.29}$$

A more precise expression of the Poincaré map is found using the expression of the flow given by the exponential map and using Proposition 3.22 that ensures that the formal expression of the flow makes sense for $t = 2\pi$. It remains to prove that the Poincaré map is different from the identity.

Lemma 3.26. *There exists $\ell_{\mathcal{M}}$ such that the Poincaré map $P = P_\ell$ for $\ell \geq \ell_{\mathcal{M}}$ at γ is tangent to the identity but $P \neq Id$ as a germ of diffeomorphism at $(0,0) \in \Delta$.*

Proof. This result is a consequence of the fact that $\hat{\eta}$ has a non-vanishing jet. We write this vector field in homogeneous components in ρ, z

$$\hat{\eta} = \rho\left(F(z,\rho)\frac{\partial}{\partial z} + G(z,\rho)\frac{\partial}{\partial \rho}\right) = \rho\sum_{k\geq 1}\left(F_k(z)\rho^k\frac{\partial}{\partial z} + G_k(z)\rho^k\frac{\partial}{\partial \rho}\right),$$

and consider the minimum ν such that $F_\nu \neq 0$ or $G_\nu \neq 0$. We take $\ell \geq \nu + 1$. Since $\hat{\eta}$ and η_ℓ share the same ℓ–jet in ρ, z, as we pointed out in the beginning of the section, we have $j_{\nu+1}(\eta_\ell) = j_{\nu+1}(\hat{\eta})$. Suppose that $F_\nu \neq 0$, then

$$\begin{aligned}\exp(t\zeta_\ell^{(J)})(\rho) =& \rho + tQ_1 + tQ_2 + \cdots \\ =& \rho + t(\rho F_\nu(z,\rho) + O(|(\rho,z)|^{\nu+2}) + t^2(\rho F_\nu(z,\rho)^2 + O(|(\rho,z)|^{\nu+3})) + O(t^3, |(\rho,z)|^{\nu+2}).\end{aligned}$$

Notice that the terms in $O(|(\rho,z)|^{\nu+k})$ can also depend on $\cos\theta$ and $\sin\theta$ and are divided by ρ. Since $\nu \geq 1$, we observe that the lower order term $t\rho F_\nu$ is different from 0, for any value of $t \neq 0$. Thus, we find that $\Psi_{2\pi}^{\rho} \neq \rho$, which implies that $P \neq Id$.

Following the same reasoning, we find that $\Psi_{2\pi}^{z} \neq z$ when $G_\nu \neq 0$, which also implies that $P \neq Id$. □

We finish the proof of Theorem 3.1 for non-degenerated vector fields with non-isolated singularity.

3.6.2 End of the proof

We prove the Theorem for some ξ_ℓ, which is analytically conjugated to ξ. Recall that by Assertion †, this suffices to prove Theorem 3.1.

Proposition 3.27. *There is some $\ell \geq 1$ such that ξ_ℓ satisfies Theorem 3.1.*

Proof. We choose ℓ as in Lemma 3.26, obtaining a Poincaré map P defined in $\Delta \subset \{\theta = 0\}$ that is different from the identity. On the other hand, this Poincaré map has an invariant curve given by $\Delta \cap E$, so it is not center-focus. We can apply Theorem 2.6 to P in $(0,0)$, obtaining a sectorial decomposition of P in $\tilde{\Delta} \subset \Delta$. In particular, all the periodic points of P in $\tilde{\Delta}$ lie on curves $\Gamma_1, \ldots, \Gamma_r$

of fixed points. Saturating these curves in $\tilde{\Delta}$ by the flow of ξ_ℓ, we obtain analytic surfaces $S_1, \ldots, S_r$ contained in $\tilde{\Delta} \times \mathbb{S}^1$ that have center configuration. Then, we find that $\sigma(S_1), \ldots, \sigma(S_r)$ are the only limit central surfaces at $0 \in \mathbb{R}^3$. By choosing a neighborhood of 0 in $\sigma(\tilde{\Delta} \times \mathbb{S}^1)$ conveniently, we find that the first item (when $r = 0$) or the second item (when $r \geq 1$) of the theorem are satisfied. □

Recall that the vector field ξ_ℓ is analytically conjugated to ξ by the diffeomorphism φ_ℓ (cf. section 3.1.2). Then, there is some neighborhood U of 0 in which ξ also fulfills one of the situations (i) or (ii) of Theorem 3.1.

3.7 The non-isolated singularity case for formally degenerated vector fields

In this case, we prove Theorem 3.1 by means of proving Theorem 3.3.

3.7.1 Characterization of three dimensional centers

One of the equivalent statements of Theorem 3.3 concerns complete integrability. We say that a three dimensional vector field is completely integrable if there exists two analytic first integrals f and g such that $df \wedge dg \neq 0$ in a dense neighborhood of 0.

Theorem 3.3. *Let $\xi \in \mathcal{H}^3$, the following statements are equivalent.*

(1) ξ is formally degenerated.

(2) ξ is analytically orbitally linearizable (i.e. it is orbitally equivalent to its linear part)

(3) There is a neighborhood U of 0 such that $\mathcal{C}_U = U \setminus Sing(\xi)$.

(4) ξ is analytically completely integrable.

Remark 3.28. In addition, (1) in the statement can be replaced by an a priori weaker statement:

(1') The vector field ξ is formally orbitally linearizable.

In fact, assume that ξ is formally conjugated to a vector field of the form $U(x,y,z)(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y})$ where U is a unit in $\mathbb{R}[[x,y,z]]$. Then following the normal form algorithm, we get that $U(x,y,z)(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y})$ has a degenerated normal form, and so does ξ, that is, (1') $\Rightarrow$ (1). The converse follows by definition.

Proof. The proof of this result is organized as follows. We start by proving the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). The equivalence with the latter is proved by showing (2) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Suppose that ξ is formally degenerated, i.e. there is a formal normal form $\hat{\xi}$ such that

$$\hat{\xi} = \varphi^*\xi = (1 + A(x^2 + y^2, z))\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right).$$

The idea of this proof is to use Brjuno's result on existence of an analytic normal form [10] that we state in Theorem A.28. We follow the statement of the result in the text of Martinet [59] that states that under certain hypotheses, a vector field has an analytic normal form. To check the hypotheses of this result, we will work in the complex case, since the result is stated for holomorphic vector fields. We will end the proof by obtaining a real analytic normal form that is analytically equivalent to a linear vector field, applying another result of Brjuno in [11].

The complex Jordan normal form of ξ the linear part is obtained by applying the linear automorphism $\psi(x,y,z) = (\frac{1}{\sqrt{2}}(x+iy), \frac{1}{\sqrt{2}}(x-iy), z) = (u,v,z)$. Its inverse is given by $\psi^{-1}(u,v,z) = (\frac{1}{\sqrt{2}}(u+v), \frac{1}{\sqrt{2}}(-iu+iv), z)$. We obtain then that

$$\psi_*\hat{\xi} = (1 + A(uv, z))\left(iu\frac{\partial}{\partial u} - iv\frac{\partial}{\partial v}\right).$$

The vector of eigenvalues is denoted by $\Lambda = (i, -i, 0)$. In order to see if Theorem A.28 (also of [59, Theorem 5]) is fulfilled, we need to verify the following two hypotheses.

- (Arithmetic condition) The series
$$\sum_k \frac{\log \omega_k}{2^k}$$
is convergent, where $\omega_k = \min\{|\alpha_Q| : |Q| \le 2^{k+1}, \alpha_Q \neq 0\}$, $Q = (q_1, q_2, q_3)$ with $q_1 + q_2 + q_3 \ge -1$ in $\{q_1 \in \mathbb{Z}_{\ge -1}, q_2 \in \mathbb{Z}_{\ge 0}, q_3 \mathbb{Z}_{\ge 0}\} \cup \{q_1 \in \mathbb{Z}_{\ge 0}, q_2 \in \mathbb{Z}_{\ge -1}, q_3 \mathbb{Z}_{\ge 0}\} \cup \{q_1 \in \mathbb{Z}_{\ge 0}, q_2 \in \mathbb{Z}_{\ge 0}, q_3 \mathbb{Z}_{\ge -1}\}$ and $\alpha_Q = \langle \Lambda, Q \rangle$.
- (Geometric condition) The formal vector field $\psi * \hat{\xi}$ is tangent to the foliations given by $u^{r_1} v^{r_2} z^{r_3} = c$, where c is a constant and $R = (r_1, r_2, r_3)$ such that $\langle \Lambda, R \rangle = 0$.

We also remark that the values α_Q that appear in the arithmetic condition are the eigenvalues of $[iu\frac{\partial}{\partial u} - iv\frac{\partial}{\partial v}, -] : \hat{\mathfrak{X}}_2(\mathbb{R}^3, 0) \to \hat{\mathfrak{X}}_2(\mathbb{R}^3, 0)$, which is used to construct the normal form. The eigenvectors associated to the eigenvalue $\langle \Lambda, Q \rangle = i(q_1 - q_2)$ are $u^{q_1} v^{q_2} z^{q_3} u\frac{\partial}{\partial u}$, $q_2 \ge -1, q_2, q_3 \ge 0, u^{q_1} v^{q_2} z^{q_3} v\frac{\partial}{\partial v}$, $q_1 \ge -1, q_1, q_3 \ge 0, u^{q_1} v^{q_2} z^{q_3} z\frac{\partial}{\partial z}$, $q_3 \ge -1, q_1, q_2 \ge 0$. For the proof of the arithmetic condition, we show that $\omega_k \ge 1$ for every $k \ge 0$, since $\alpha_Q = \langle \Lambda, Q \rangle = i(q_1 - q_2)$ has $|\alpha_Q| = 0$ if $\alpha_Q = 0$ or $|\alpha_Q| \ge 1$ otherwise. Hence, the term of the series decrease as 2^{-k} and it is convergent.

Now, we prove the geometric condition. The vectors R that we consider fulfill $r_1 = r_2$. To see that the vector is tangent to the foliations given by $u^{r_1} v^{r_1} z^{r_3} = c$ it is enough to see that $\psi_*\xi(u^{r_1} v^{r_1} z^{r_3} - c) = 0$.

We conclude that the arithmetic and geometric conditions are fulfilled and by Theorem A.28 there exists a holomophic normalization ψ_h and a holomorphic normal form ξ_h.

Existence of a real analytic normal form $\zeta = (1 + A(x^2 + y^2, z))\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)$ obtained by an analytic automorphism tangent to the identity $\widetilde{\psi}$ is guaranteed by Theorem 3 in [11], see also Theorem A.29. The vector field ξ is analytically conjugated to $\widetilde{\xi}$, which is orbitally equivalent to $-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. Hence, the vector field ξ is as well orbitally equivalent to $-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.

(2) $\Rightarrow$ (3). It is straightforward since by hypothesis ξ is orbitally equivalent to its linear part. The trajectories of this vector field are either cycles around the z-axis or singularities in this curve. Taking an appropriate U, that is, choosing U whose boundary is entirely composed by a union of cycles, item (3) follows.

(3) $\Rightarrow$ (1). To prove this item, we will suppose by contradiction that (1) does not hold, i.e. that ξ is formally non-degenerate. We have already proved Theorem 3.1 for every non-degenerated vector fields: in Section 3.2 for semi-hyperbolic Hopf vector fields, sections 3.3-3.5 for isolated singularity Hopf vector fields and in section 3.6 for non degenerated Hopf vector fields with non-isolated singularity. In all of these cases, only situations (i) and (ii) of Theorem 3.1 are permitted. This contradicts (3), and we conclude that existence of a continuum of cycles implies that ξ is formally degenerated.

(2) $\Rightarrow$ (4). This implication is also direct, since the linear vector field $-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$ has two independent first integrals, namely $x^2 + y^2$ and z. By (2), consider that the linear vector field $-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ and ξ are analytically orbitally equivalent by the analytic diffeomorphism ϕ. Then $\phi^*(x^2 + y^2)$ and $\phi^*(z)$ are two analytic first integrals of ξ.

(4) $\Rightarrow$ (1). Suppose that ξ has two analytic first integrals. To simplify the proof, we will work with the complex formal normal form $\psi_*\hat{\xi}$, as we did in the proof of (1) $\Rightarrow$ (2). The two analytic first integrals provide two formal first integrals f, g of the vector field $\psi_*\hat{\xi} = \tilde{\varphi}^*\xi$.

$$\psi_*\hat{\xi} = (1 + A(uv, z))\left(iu\frac{\partial}{\partial u} - iv\frac{\partial}{\partial v}\right) + B(uv, z)\left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right) + C(uv, z)\frac{\partial}{\partial z}.$$

Observing the that the series A, B, C belong to $\mathbb{R}[[uv, z]]$, we prove first the following claim: $f, g \in \mathbb{R}[[uv, z]] \subset \mathbb{R}[[u, v, z]]$. To do this, we will write both f, g and A, B, C in homogeneous components as elements of $\mathbb{R}[[u, v, z]]$.

$$f = \sum_{k \geq \nu_f} f_k,\ g = \sum_{k \geq \nu_g} g_k$$

$$A = \sum_{k\geq 1} A_k,\ B = \sum_{k\geq 1} B_k,\ C = \sum_{k\geq 2} C_k,$$

where $f_k, g_k, A_k, B_k, C_k \in \mathbb{R}[u,v,z]_k$.

Since f is a first integral of $\psi_*\hat{\xi}$, we have that $h := \psi_*\hat{\xi}(f) = 0$. We have that necessarily $h = \sum_{k\geq \nu_f} h_k = 0$, that is, each $h_k = 0$. Suppose that $\nu_f = n \geq 1$, we prove that $f_n \in \mathbb{R}[uv,z]$. First we write $f_n = \sum_{k_1+k_2+k_3=n} f_{k_1k_2k_3} u^{k_1}v^{k_2}z^{k_3}$. Imposing $h_n = \left(iu\frac{\partial}{\partial u} - iv\frac{\partial}{\partial v}\right)(f_\nu) = 0$, we get

$$h_n = \sum_{k_1+k_2+k_3=n} i f_{k_1k_2k_3}(k_1-k_2)u^{k_1}v^{k_2}z^{k_3} = 0.$$

Then, we have that the only possible non-zero $f_{k_1k_2k_3}$ fulfill $k_1 = k_2$. This implies that $f_n \in \mathbb{R}[uv,z]$. Now suppose by induction that $f_k \in \mathbb{R}[uv,z]$ for all $k \leq l$, we show that $f_{l+1} \in \mathbb{R}[uv,z]$. As before, we have that

$$\begin{aligned}
h_l &= h_{l,1} + h_{l,2} = 0,\\
h_{l,1} &= \left(iu\frac{\partial}{\partial u} - iv\frac{\partial}{\partial v}\right)(f_{l+1}),\\
h_{l,2} &= \sum_{k=1}^{l} A_k\left(iu\frac{\partial}{\partial u} - iv\frac{\partial}{\partial v}\right)(f_{l+1-k}) + B_k\left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)(f_{l+1-k}) + C_{k+1}\frac{\partial}{\partial z}(f_{l-k})
\end{aligned}$$

Using that $\left(iu\frac{\partial}{\partial u} - iv\frac{\partial}{\partial v}\right)(u^{k_1}v^{k_2}z^{k_3}) = i(k_1-k_2)u^{k_1}v^{k_2}z^{k_3}$, $\left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)(u^{k_1}v^{k_2}z^{k_3}) = (k_1+k_2)u^{k_1}v^{k_2}z^{k_3}$ and $\frac{\partial}{\partial z}(u^{k_1}v^{k_2}z^{k_3}) = k_3u^{k_1}v^{k_2}z^{k_3-1}$ and the induction hypothesis, we have that $h_{l,2} \in \mathbb{R}[uv,z]$. Then, $h_{l,1}$ must also belong to $\mathbb{R}[uv,z]$, which implies that $f_{l+1} \in \mathbb{R}[uv,z]$ as we wanted to prove. Proceeding in the same manner for g, we also have that $g \in \mathbb{R}[[uv,z]]$.

Since the formal first integrals of $\psi_*\hat{\xi}$ belong to $\mathbb{R}[[uv,z]]$, we have that the two formal first integrals of $\hat{\xi}$ belong to $\mathbb{R}[[x^2+y^2,z]]$, that is, they are $f(x^2+y^2,z)$ and $g(x^2+y^2,z)$. Then, consider the associated two dimensional vector field $\hat{\eta}$ of $\hat{\xi}$, defined in equation (3.1.3) section 3.1.3,

$$\hat{\eta} = \frac{B(\rho^2,z)}{A(\rho^2,z)}\frac{\partial}{\partial\rho} + \frac{C(\rho^2,z)}{A(\rho^2,z)}\frac{\partial}{\partial z}.$$

We have that $\hat{\eta}$ has two formal first integrals $\tilde{f}(z,\rho) = f(\rho^2,z)$ and $\tilde{g}(z,\rho) = g(\rho^2,z)$. These two first integrals are elements of $\mathbb{R}[[z,\rho]]$, and they fulfill $d\tilde{f}\wedge d\tilde{g} \neq 0$, since the original formal series fulfill $df\wedge dg \neq 0$. We conclude by pointing out that any two dimensional vector field that has two independent first integrals is neccessarily 0. Then $\hat{\eta} = 0$ and ξ is formally degenerated, as we wanted to prove.

$\square$

3.7.2 End of the proof in the formally degenerated case

We remark that Theorem 3.3 proves something stronger than Theorem 3.1 for formally degenerated vector fields. Theorem 3.1 corresponds only to the implication (1) $\Rightarrow$ (3). With this last step we have finished studying all possible Hopf vector fields (semi-hyperbolic, formally non-degenerated and formally degenerated), and we conclude the proof of Theorem 3.1.

3.8 Consequences of Theorem 3.1

In this last section of the chapter, we provide Corollary 3.4 and Corollary 3.2. We give the proof of both of them, which are direct consequences of Theorem 3.1.

Corollary 3.4. *Let $\xi \in \mathcal{H}^3$ and suppose that its local cycle locus is non-empty. Let Ω_∞ be a C^∞ realization of the formal rotational axis. Then, the neighborhood basis $\mathcal{V}$ in Theorem 3.1, (ii) or (iii) can be chosen so that $V \setminus \Omega_\infty$ is homotopically equivalent to $\mathbb{S}^1$ and any cycle $\gamma \subset C_V(\xi)$ is a generator of $\pi_1(V \setminus \Omega_\infty)$.*

Proof of Corollary 3.4. We are assuming that either (ii) or (iii) of Theorem 3.1 are fulfilled. In (ii) we have that $\mathcal{C}_U = S_1 \cup \cdots \cup S_r$ and after [8] that there is a non-singular realization Ω_∞ of the rotational axis $\widehat{\Omega}$. Since we are working with simply connected neighborhoods of 0, the fundamental group of $V \setminus \Omega_\infty$ is $\mathbb{Z}$. We want to prove that any cycle of ξ in U generates this fundamental group. With this objective, we prove that any cycle γ is a deformation retract of $V \setminus \Omega_\infty$. Notice that any cycle is contained in one of the surfaces. First, considering that $V \setminus \Omega_\infty$ is homeomorphic to a filled boounded cylinder and any of the surfaces S_i is homeomorphic to a disk without a point, it is well known that S_i is a deformation retract of $V \setminus \infty$. Secondly, any cycle γ in S_i, which gives a single turn after Jordan's curve theorem, is as well a deformation retract of S_i. Notice that this relation is associative, and then, γ is a deformation retract of $V \setminus \Omega_\infty$.

In (iii) the rotational axis $\Omega_\infty = \widehat{\Omega}$ and it is analytic and $V \setminus \Omega_\infty$ is filled with cycles. By (3) of Theorem 3.3, we have that ξ is orbitally linearizable. This implies that any cycle γ of ξ makes a single turn around Ω_∞, and hence, it generates the fundamental group of $V \setminus \Omega_\infty$. $\square$

The last result in this chapter concerns Dulac's problem on non-existence of accumulation of isolated cycles, which is an open problem in dimension 3. We give a positive answer for Hopf vector fields. It is a straightforward consequence of Theorem 3.1, since none of the possible structures of the cycle locus allows an infinite number of isolated cycles.

Corollary 3.2. *If $\xi \in \mathcal{H}^3$, there are no infinitely many isolated cycles of ξ collapsing to $0 \in \mathbb{R}^3$.*

Conclusions

In this PhD thesis we have studied two different problems.

I. Sectorial decomposition of germs of real analytic plane diffeomorphisms tangent to the identity.

II. Structure of the cycle-locus and Dulac's problem for germs of three dimensional vector fields with a Hopf singularity.

We present the conclusions dealing with each of them in the following paragraphs, as well as lines of future work.

Sectorial decomposition of germs of real analytic plane diffeomorphisms tangent to the identity. Concerning the first problem,we provide a sectorial decomposition $(U,\mathcal{S})$ of a diffeomorphism F under the condition that F is not of type center-focus (Theorem 2.6 or Theorem A in the Introduction). The sectorial decomposition is a partition $\mathcal{S}$ of a neighborhood U of $0 \in \mathbb{R}^2$ in submanifolds so that on each submanifold $A \in \mathcal{S}$ the asymptotics of the diffeomorphism F are uniformly described.

The non center-focus condition is our unique hypothesis, and it is also imposed on the sectorial decomposition of germs of real analytic plane vector fields. Comparing our result for diffeomorphisms with the analogous for vector fields, we find some differences.

- On the one hand, the dynamical types of the sectors of diffeomorphisms are essentially the same as for vector fields (allowing curves of singularities). For every point p in a single stratum $A \in \mathcal{S}$, we find that the orbit of p accumulates positively at the point $0 \in \mathbb{R}^2$, or at some point $q = q(p)$ in (a half-branch of) the curve of fixed points in the closure of A, or the orbits escape the sector. The negative orbits are also uniformly described on the sectors. We find six types of sectors concerning their dynamical types in terms of the asymptotic behavior of F or F^{-1}.
- On the other hand, we do not find "good" topological properties on the neighborhood U, in contrast with vector fields. The initial objective was to obtain a sectorial decomposition $(U,\mathcal{S})$ not only with the dynamical properties in Theorem 2.6 but also fulfilling that U is open and $\mathcal{S}$ is a stratification. For vector fields, we have that the boundary of U is given by the union of pieces of trajectories and curves transverse to the vector field, then the set U can always be chosen open and $\mathcal{S}$ is a stratification. In contrast, for diffeomorphisms we find an intrinsic difficulty since the orbits are discrete sets. Invariant curves can be constructed,

but they will not always have the same geometrical properties as the trajectories of the vector fields.

As we have seen in Chapter 2, only under some conditions we can ensure that U is open and $\mathcal{S}$ is a stratification. For instance, in a D-D sector, we have an invariant curve in its boundary, which may accumulate in a compact set of the curve of fixed points. If this compact set is not a single point, we have that U is not open since for any of the points of accumulation of the curve of the boundary, there is not a neighborhood of it completely contained in U. Indeed, to ensure that the invariant curve of the boundary accumulates in a single point (in both positive and negative directions once fixing a parameterization of the curve), we need that two uniquely given parabolic curves coincide (see Figure 3). We think that the generic case is that these curves do not coincide.

- In the case of vector fields, we can choose the open set U semi-analytic. However, for diffeomorphisms this is not always the case. For instance, in the presence of bidicritical curves, we can choose a parabolic curve that might not be analytically extended to its extreme. Choosing any other curve for the boundary, we find that it accumulates in a compact subset of the bidicritical curve, hence it is far from being a semi-analytic set.

From the second item, we think that our objective of choosing U open and $\mathcal{S}$ a stratification was very demanding for diffeomorphisms.

Future work:

- Following the lines in the work of Dumortier, Rodrigues and Roussarie in [29], we would like to treat the problem of the configuration of sectors being a weak topological invariant. They succeeded to prove this for $\mathcal{C}^\infty$ diffeomorphism fulfilling a Łojasiewicz inequality. However, because of this condition, D-parabolic, D-elliptic and D-D sectors do not arise and this simplifies the problem. We think that the presence of D-D sectors might be an obstruction to achieve a similar result.

- We would like to illustrate the phenomena of the boundary curve of D-D sectors (Section 2.5.2) with an example.

- We think it is possible to define foliations that are invariant by the diffeomorphism. We already know that a formal vector field generates diffeomorphisms tangent to the identity, by defining the time-1 flow. Our question is if there is an analytic foliation that might be preserved by the action of the diffeomorphism. This would imply that the diffeomorphism is given by the flow at the time given by some function of the point. As this is too strong to require, we are currently working in the construction of foliations on each sector, in order to determine if they can be continuously extended to other sectors. Once again, we think that the presence of D-D sectors may be an obstruction on the construction of such foliation.

- Finally, we would like to extend this work to higher dimensional diffeomorphism. In the case of three dimensional vector fields, we have the generalization of C. Alonso-González and F. Sanz Sánchez of the sectorial decomposition to dimension 3 in [1, 2], where they also generalize the non center-focus condition. We think that the techniques used in the proofs of Theorem 2.6 can be extended to germs of three dimensional diffeomorphisms imposing similar hypotheses that generalize the non center-focus one.

Structure of the cycle-locus and Dulac's problem for germs of three dimensional vector fields with a Hopf singularity. Considering the second problem of the thesis, we give a complete description of the cycle-locus of three dimensional vector fields with Hopf singularity (Theorem 3.1, Theorem B in the Introduction) and answering Dulac's problem for these vector fields (Corollary 3.2, Corollary B in the Introduction). Related to the first result, we find that, in sufficiently small neighborhoods of $0 \in \mathbb{R}^3$, the cycles of a vector field with Hopf singularity belong to a finite number of limit central surfaces (pairwise disjoint surfaces with center configuration) or every trajectory is a non-trivial cycle except a curve of singularities (three dimensional center). The second result gives a satisfactory answer to Dulac's problem for vector fields with Hopf singularity, that is, there is not an infinite number of isolated cycles accumulating to $0 \in \mathbb{R}^3$. We think this is a first step and a novel contribution to Dulac's problem in higher dimension.

Other result that we obtained related to this topic is a characterization of the three dimensional centers having a Hopf singularity. We collect some partial results in the literature and prove Theorem 3.3 (Theorem C in the Introduction) with our own methods. We remark that among the original results, we provide a generalization of Poincaré-Lyapunov Center Theorem: being a three dimensional center with Hopf singularity implies complete integrability in dimension 3, as Poincaré and Lyapunov proved in dimension 2. The last result that we obtain, which follows as a consequence of the previous ones, is that all the cycles in a small neighborhood of 0 make a single turn around a rotational axis (Corollary 3.4).

Our contributions help to understand better vector fields with a Hopf-zero singularity. In particular, we see that some chaotic phenomena is avoided, such as the possibility of having accumulation of cycles making arbitrary large number of turns around a rotational axis.

Future work:

- Notice that we have studied Dulac's problem for germs of three dimensional vector fields with at least two non-zero eigenvalues. The following natural step is to study the structure of the cycle-locus for vector fields with only one non-zero eigenvalue. A study of these vector fields in the case that the center manifold has nilpotent (but non-zero) linear part can be found in [65], where the authors give conditions on the center manifold to study if it has a central configuration. Our objective will be to give a complete description of the cycle-locus in all the cases.

- Other natural objective is to generalize our result to higher dimensional vector fields that have a singularity with eigenvalues $a_1, \ldots, a_{n-3}, bi, -bi, 0$, with $a_i, b \in \mathbb{R} \setminus \{0\}$ for each $i = 1, \ldots, n-3$. This generalization is not straightforward since the center manifold of such system is three dimensional but not necessarily analytic. Nevertheless, we think our result can be generalized to that context.

Appendix A

Basic notions on analytic manifolds and vector fields

In this appendix, we will provide general definitions and basic results, since along the rest of the text we have to make use of some classical results. The purpose of this appendix is to fix some notations and state some classical theorems.

A.1 Formal power series and formal maps

Let $\mathbb{K}$ be a field of characteristic 0, typically $\mathbb{R}$ or $\mathbb{C}$, A be a $\mathbb{K}-$algebra and $\mathbf{x} = (x_1, \ldots, x_n)$ variables.

A.1.1 Formal power series

In this section, we introduce the power series as $\mathbb{K}-$albegras. The $\mathbb{K}-$*algebra of formal power series in* $\mathbf{x}$ *with coefficients in* A is denoted by $A[[\mathbf{x}]]$. Elements $f \in A[[\mathbf{x}]]$ are written as

$$f = \sum_{\alpha \in \mathbb{N}^n_{\geq 0}} f_\alpha \mathbf{x}^\alpha, \text{where } f_\alpha \in A,\ \alpha = (\alpha_1, \ldots, \alpha_n) \text{ and } \mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Given $f, g \in A[[\mathbf{x}]]$, consider the operations of the $\mathbb{K}-$algebra $A[[\mathbf{x}]]$ induced by the operations of A

$$f + g = \sum_{\alpha \in \mathbb{N}^n_{\geq 0}} (f_\alpha + g_\alpha)\mathbf{x}^\alpha$$
$$f \cdot g = \sum_{\alpha \in \mathbb{N}^n_{\geq 0}} \left(\sum_{\beta + \gamma = \alpha} f_\beta g_\gamma \right) \mathbf{x}^\alpha,$$

The units of $A[[\mathbf{x}]]$ are those having $f_0 \in U(A)$. Now, we introduce a concept that will be used specially in Chapter 3: the jets of power series. Consider the ideal $\mathfrak{m} = (x_1, \ldots, x_n)$. We define the *k–jet of the power series f* as its image under the canonical projection $j_k : A[[\mathbf{x}]] \to A[[\mathbf{x}]]/\mathfrak{m}^{k+1}$, namely $j_k(f) = f + \mathfrak{m}^{k+1}$. In particular,

$$j_k(f) = j_k^{\mathbf{x}}(f) := \sum_{\alpha:|\alpha|\le k} f_\alpha \mathbf{x}^\alpha + \mathfrak{m}^{k+1}$$

where $|\alpha| := \alpha_1 + \cdots + \alpha_n$. The *order or multiplicity at* 0 *of* f, denoted by $\nu(f)$, is the first $k \geq 0$ (or $+\infty$ if it does not exist) such that $j_k(f) \neq 0$.

Remark A.1. Considering only the vector space nature of $A[[\mathbf{x}]]$ and $A[[\mathbf{x}]]/\mathfrak{m}^{k+1}$, notice that j_k sends elements of an infinite dimensional $\mathbb{K}$–vector space to a finite dimensional $\mathbb{K}$–vector space. Notice that the quotient $A[[\mathbf{x}]]/\mathfrak{m}^{k+1}$ is identified, only as a vector space, with the polynomials $A[\mathbf{x}]_{\le k}$ of degree equal or lower than k. We will tacitly make this abuse of notation, that is, we simply consider j_k as a truncation and $j_k(f) = j_k^{\mathbf{x}}(f) := \sum_{\alpha:|\alpha|\le k} f_\alpha \mathbf{x}^\alpha$.

Two important features of the jets of formal power series are deduced from the commutation of the two operations of the algebra and the jet projection. Let $f, g \in A[[\mathbf{x}]]$, then

- $j_k(f+g) = j_k(f) + j_k(g)$.
- $j_k(f \cdot g) = j_k(f) \cdot j_k(g)$. In fact, this property can be refined: if $k \geq \max\{\nu(f), \nu(g)\}$, then $j_k(f \cdot g) = j_{k-\nu(g)}(f) \cdot j_{k-\nu(f)}(g)$, considering the elements $j_{k-l}(h)$ as elements in $A[[\mathbf{x}]]/\mathfrak{m}^{k+1}$.
- $j_k(f^{-1}) = (j_k(f))^{-1}$, when $f \in A[[\mathbf{x}]]$ is a multiplicative unit.

Notice that, when $n > 1$, the variables of $\mathbf{x}$ can be separated into two groups $\mathbf{x} = (\mathbf{y}, \mathbf{z})$ where $\mathbf{y} = (y_1, \ldots, y_r)$ and $\mathbf{z} = (z_1, \ldots, z_t)$ with $n = r + t$. There is a natural identification between $A[[\mathbf{x}]]$ and $A[[\mathbf{y}]][[\mathbf{z}]]$

$$A[[\mathbf{x}]] \xrightarrow{\sim} A[[\mathbf{y}]][[\mathbf{z}]],\ f = \sum_{\alpha\in\mathbb{N}^n_{\ge 0}} f_\alpha \mathbf{x}^\alpha \mapsto \sum_{\beta\in\mathbb{N}^t_{\ge 0}} \left(\sum_{\gamma\in\mathbb{N}^r_{\ge 0}} f_{\gamma,\beta} \mathbf{y}^\gamma \right) \mathbf{z}^\beta$$

In these terms, we define the *k–jet of f with respect to the variables* $\mathbf{z}$ as the k–jet of f as an element of $B[[\mathbf{z}]]$ where $B = A[[\mathbf{y}]]$. We denote the k–jet of $f \in A[[\mathbf{y}]][[\mathbf{z}]]$ with respect to $\mathbf{z}$ as $j_k^{\mathbf{z}}(f)$. Finally, we make the following remarks.

- Notice that $j_k^{\mathbf{x}}(j_k^{\mathbf{z}}(f)) = j_k(f)$. Writing $j_k(f)$ and $j_k^{\mathbf{z}}(f)$ as power series in $A[[\mathbf{x}]]$, notice that an infinite number of terms can appear in the development of $j_k^{z}(f)$, while only a finite number of terms do in the development of $j_k(f)$. In particular, $j_k^{\mathbf{x}}(f) \in \mathbb{R}[\mathbf{x}]_{\le k}$ and $j_k^{\mathbf{z}}(f) \in A[\mathbf{y}][\mathbf{z}]_{\le k}$.
- We also remark that given a series $f \in A[\mathbf{y}][[\mathbf{z}]]$, we can obtain another via the automorphism $\chi_i : A[\mathbf{y}][[\mathbf{z}]] \to A[\mathbf{y}][[\mathbf{z}]]$ that sends each $y_j \mapsto y_j$ for $1 \le j \le r$ and $j \neq i$, $y_i \mapsto y_i + a$ and $z_k \mapsto z_k$ for $1 \le k \le s$. Then, $\chi_i(j_k^{\mathbf{z}}(f)) = j_k^{\mathbf{z}}(\chi_i(f))$.

We also recall the notion of convergence of power series, whenever A can be provided with a norm. This notion is important in this document, since we work with analytic functions. The *algebra of convergent* *series with coefficients in A* is the subalgebra of $A[[\mathbf{x}]]$ defined by

$$A\{\mathbf{x}\} := \bigcup_{\delta>0} A\{\mathbf{x}\}_\delta$$

where, by definition, a series $f = \sum_{\alpha\in\mathbb{N}^n_{\geq 0}} f_\alpha \mathbf{x}^\alpha A[[\mathbf{x}]]$ belongs to $A\{\mathbf{x}\}_\delta$ if there exists $C > 0$ such that $\|f_\alpha\| < C\delta^{|\alpha|}$ for any α. It is important to notice that there is an strict inclusion $A\{\mathbf{x}\} \subset A[[\mathbf{x}]]$.

Example A.2. We present two examples that appear in this document.

- $A = \mathbb{R}[\cos\theta, \sin\theta]$, the algebra of trigonometric polynomials, whose elements are considered indistinctively as a function on $\mathbb{R}$ or on $\mathbb{S}^1$, via the covering $\tau : \theta \to (\cos\theta, \sin\theta)$. It will be endowed with the supremum norm $\|f\| := \sup_{\theta\in\mathbb{R}} f(\theta)$. Notice that a given a convergent series $F \in \mathbb{R}[\cos\theta, \sin\theta]\{\mathbf{x}\}_\delta$, its partial sums converge absolutely and uniformly in the compact sets of the neighborhood $V = \mathbb{S}^1 \times (-\delta, \delta)^n$ of $\mathbb{S}^1 \times \{0\}$ (or the neighborhood $V = \mathbb{R} \times (-\delta, \delta)^n$ of $\mathbb{R} \times \{0\}$), thus providing an analytic function that we denote again f.
- In the case of $A = \mathbb{R}[\mathbf{z}]$ (respectively $\mathbb{R}[\cos\theta, \sin\theta, \mathbf{z}]$), where $\mathbf{z} = (z_1, \ldots, z_r)$, there is no unique natural norm on A. We will consider a norm for each compact set K of $\mathbb{R}^r$ (resp. $\mathbb{S}^1 \times \mathbb{R}^r$) with non-empty interior, defined by

$$\|f\|_K := \sup_{a\in K}\{|f(a)|\}.$$

  Denoting $A_K = (A, \|\cdot\|_K)$ such a normed space, we have the corresponding algebra of convergent series $A_K\{\mathbf{x}\}$. We define the algebra of convergent series with coefficients in A as the intersection of algebras $A_K\{\mathbf{x}\}$ where K runs all compact sets of such form. With an abuse of notation, we name this algebra $A\{\mathbf{x}\}$ for convenience. Each element $f \in A\{\mathbf{x}\}$ defines an analytic function on a neighborhood of $\mathbb{R}^r \times \{0\}$ (resp. $\mathbb{S}^1 \times \mathbb{R}^r \times \{0\}$) in $\mathbb{R}^r \times \mathbb{R}^n$ (resp. in $\mathbb{S}^1 \times \mathbb{R}^r \times \mathbb{R}^n$).

We associate convergent power series with coefficients in $\mathbb{R}$ in n variables to germs of analytic functions in points of $\mathbb{R}^n$. We extend this relation to formal power series and the formal completion of the germs of analytic functions. Recall that a *germ of analytic function at* $p \in \mathbb{R}^n$ is an equivalence class of the equivalence relation on the analytic functions that have p on their domain. That is, in $f : U \to \mathbb{K} \sim g : V \to \mathbb{K}$ if there is $W \subset U \cap V$ with $p \in W$ such that $f|_W = g|_W$. We will denote the germs of analytic functions as $\mathcal{O}_{n,p}$ or simply by $\mathcal{O}_p$, when the dimension is clear. Choosing a set of coordinates in a neighborhood of p, the correspondence between the germ of analytic function and a convergent power series is given by the Taylor series of the analytic function.

On the other hand, let $\mathfrak{m}_{\mathcal{O}_p}$ be the ideal of $\mathcal{O}_p$ given by $\{f \in \mathcal{O}_p : f(p) = 0\}$, and define the $\mathfrak{m}_{\mathcal{O}_p}$–adic topology generated by the basis of neighborhoods $\{f + \mathfrak{m}^k_{\mathcal{O}_p}\}_{k\in\mathcal{N}}$. We define the formal completion of the germs of analytic functions as the limit

$$\widehat{\mathcal{O}}_{n,0} = \varprojlim \mathcal{O}_{n,0}/\mathfrak{m}^k_{\mathfrak{m}_{\mathcal{O}_p}}.$$

The ring $\widehat{\mathcal{O}}_{n,0}$ is isomorphic to $\mathbb{R}[[\mathbf{x}]]$. See [60], for further details.

We also recall the derivations in the ring of formal power series since we work with analytic and formal vector fields. A *formal derivation* $\partial \in \mathrm{Der}(A[[\mathbf{x}]])$ on the ring of formal power series is a morphism $\partial : A[[\mathbf{x}]] \to A[[\mathbf{x}]]$ satisfying, for any $f, g \in A[[\mathbf{x}]]$ and $a, b \in A$

- $\partial(1) = 0$.
- (A–linearity) $\partial(af + bg) = a\partial(f) + b\partial(g)$.
- (Leibniz rule) $\partial(fg) = f\partial(g) + \partial(f)g$.

Defining $(f\partial)(g) = f\partial(g)$ and $(\partial+\partial')(f) = \partial(f)+\partial'(f)$, the formal derivations have the structure of a finitely generated $A[[\mathbf{x}]]$–module. The standard basis of this module is given by $\left\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right\}$. An element of this basis acts on a power series $f = \sum_\alpha f_\alpha \mathbf{x}^\alpha$ as follows.

$$\frac{\partial}{\partial x_i}(f) = \sum_{\alpha:\alpha_i>0} \alpha_i f_\alpha x_1^{\alpha_1} \cdots x_{i-1}^{\alpha_{i-1}} x_i^{\alpha_i - 1} x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n}$$

A.1.2 Formal maps and diffeomorphisms

A *formal map* is a tuple of formal power series in $A[[\mathbf{x}]]$, that is, $F = (F_1, \cdots, F_m) \in A[[\mathbf{x}]]^m$ such that $j_0(F_i) = 0$. Fixing m, the formal maps form a vector space with the sum defined component wise, and a $\mathbb{K}$–algebra considering the product component wise as well. In the same way as in the previous section, define the *k–jet of $F \in A[[\mathbf{x}]]^m$* as $j_k(F) := (j_k(F_1), \cdots, j_k(F_m))$.

The *group of formal maps $(A[[\mathbf{x}]]^n)^\circ$ with the composition* operation is the subset of elements of $A[[\mathbf{x}]]^n$ such that the determinant $\det(DF(0)) = \det(\frac{\partial}{\partial x_j}(F_i)(0)) \neq 0$ for each $F = (F_1, \dots, F_n) \in (A[[\mathbf{x}]]^n)^\circ$ and the following operation. For $F, G \in (A[[\mathbf{x}]]^n)^\circ$, the *composition operation* is defined by $(F \circ G)(\mathbf{x}) := F(G(\mathbf{x}))$ and the unit of this group is $Id = (x_1, \cdots, x_n)$. An important subgroup of formal maps is the group of *formal maps tangent to the identity* denoted by $A[[\mathbf{x}]]_1^n \subset A[[\mathbf{x}]]^n$ and such that $j_0(F) = 0$ and $j_1(F) = Id = (x_1, \cdots, x_n)$ with the composition $\circ$ in $A[[\mathbf{x}]]_1^n$.

As we did in the formal series case, convergence can be considered, studying the radius of convergence of the series on each component. Then, we can associate the convergent power series to the germs of *analytic diffeomorphisms*, denoted by $\mathrm{Diff}(\mathbb{R}^n, 0)$, via the Taylor series expansions. There is also a one to one correspondence between the convergent maps tangent to the identity and the germs of analytic diffeomorphisms tangent to the identity, denoted by $\mathrm{Diff}_1(\mathbb{R}^n, 0)$. Finally, this association can be extended to the formal completion of the analytic diffeomorphisms, denoted by $\widehat{\mathrm{Diff}}(\mathbb{R}^n, 0)$ (or by $\widehat{\mathrm{Diff}}_1(\mathbb{R}^n, 0)$ in the tangent to the identity case).

A.2 Analytic geometry

In this section, we introduce some definitions concerning real analytic curves and we will also recall the definition of semi-analytic and subanalytic sets.

A subset $X \subset \mathbb{R}^n$ is an *analytic set* if at each $p \in \mathbb{R}^n$ there is an open set U_p and a finite number of real analytic functions $f_1, \dots, f_s$ in $\mathcal{O}(U_p)$, such that $X \cap U_p = \{q \in U_p : f_1(q) = 0, \dots, f_s(q) = 0\}$. We can also define the germs of analytic sets at any $p \in \mathbb{R}^n$. They are the equivalence classes of the relation $X \sim Y$ if and only if there is U with $p \in U$ such that $(X \cap U) = (Y \cap U)$. Conversely, given an analytic function $f \in \mathcal{O}(U)$, we define the set generated by this function. Let $f \in \mathcal{O}(U)$ be an analytic function. We define the *zero set of f* as

$$V(f) := \{p \in U : f(p) = 0\}.$$

Given a germ of analytic function $f_p \in \mathcal{O}_{n,p}$ at p, it is possible to define a germ of analytic set, by taking a representative f and the germ of set at p that $V(f)$ defines. Under a translation, we can always suppose that $p = 0 \in \mathbb{R}^n$. The other way around also defines a germ of analytic set; given a germ or analytic set X_0, there is and ideal $I(X) \subset \mathcal{O}_{n,0}$ with the property $V(I)_0 = X_0$.

We remark that the theory of germs of analytic sets has been widely studied in the last century [39, 50]. In the complex case, there is a correspondence between prime ideals and irreducible components (those that cannot be decomposed into properly smaller analytic subsets) of an analytic subset. However, this fact does not hold in the real analytic case. We show now an example that appears in this text.

Example A.3. We work in $\mathcal{O}_{2,0}$ and coordinates (x,y) at 0. Let $I = (x^2+y^2) \subset \mathcal{O}_{2,0}$. We have that $V(I) = \{0\}$ and that $I(V(I)) = (x,y)$, the maximal ideal. Notice that $(x,y) \neq \sqrt{(x^2+y^2)} = (x^2+y^2)$.

Remark A.4. The concept of analytic set must not be mistaken for analytic manifold, even in the irreducible case. Under some conditions, an analytic set has the structure of an analytic manifold. Suppose that X is a germ of an analytic set given locally by $f_1,\dots,f_s$. In the presence of singularities, i.e. points in which $\operatorname{rank}(Df_1,\dots,Df_s) < n-s$, the analytic sets do not have the structure of an analytic manifold. Analytic manifolds can be immerse in $\mathbb{R}^m$ for some $m \in \mathbb{N}$ so that they are locally the set of zeroes of a finite number of smooth analytic functions.

A.2.1 Real analytic and formal curves

As we anticipated, we will only give more details for germs of analytic curves, that is, germs of one-dimensional analytic subsets of $\mathbb{R}^n$. Given a germ of analytic curve Γ, we denote $I_\Gamma \subset \mathcal{O}$ its corresponding generating ideal. We consider also that a real analytic curve has a finite number of *branches* or irreducible components, that is, $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_s$ where each Γ_i is an irreducible component. We take from [19, 51], the following definitions.

An *analytic parameterization of an irreducible branch* is an element $\gamma \in (t\mathbb{R}\{t\}^n) \setminus \{(0,\dots,0)\}$ such that $f(\gamma) = 0$ for any $f \in I_\Gamma$. Existence of analytic parameterizations of analytic curves is ensured by the classical Puiseux theorem. A parameterization γ is irreducible if there is not other parameterization $\tilde{\gamma}$ such that $\gamma = \tilde{\gamma}(s^k)$ for some $k \in \mathbb{N}_{>1}$. In the rest of the section, the considered parameterizations are irreducible. Given two different parameterizations γ_1 and γ_2, we say that they define the same branch if there is $\sigma \in \mathbb{R}\{t\}$ such that $\gamma_1 = \gamma_2 \circ \sigma$ and $\sigma(t) = at + \cdots \in \mathbb{R}\{t\}$ for some $a \in \mathbb{R} \setminus \{0\}$. In this way, there is a correspondence between the branches of real analytic curves and the classes of parameterizations.

We can also define the *half-branches* of a curve as the connected components of $\Gamma \setminus \{0\}$. Each branch Γ produces two half branches Γ^+, Γ^-, and given a parameterization γ of Γ, the half-branches can be parameterized by the restrictions $\gamma|_{\mathbb{R}_{>0}}, \gamma|_{\mathbb{R}_{<0}}$ of γ to $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$, respectively. Up to a change on the sign of the parameterization, we can always suppose that a half branch has an oriented parameterization γ, that is, a parameterization such that $\gamma|_{\mathbb{R}_{>0}}$ provides the half branch. In terms of oriented parameterizations, two oriented parameterizations γ_1 and γ_2 define the same half-branch if there is $\sigma \in \mathbb{R}\{t\}$ such that $\gamma_1 = \gamma_2 \circ \sigma$ and $\sigma(t) = at + \cdots \in \mathbb{R}\{t\}$ for some $a > 0$. Once again, there is a correspondence between half branches of real analytic curves and oriented parameterizations.

The definition of real analytic branches of curves as equivalence classes of parameterizations can be easily generalized to formal curves. A *formal curve* can be defined as a class on the equivalence relation of parameterizations in $(t\mathbb{R}[[t]]^n) \setminus \{(0,\ldots,0)\}$. Each formal curve is associated to a prime ideal $I_\Gamma \subset \mathbb{R}[[\mathbf{x}]]$. However, as in the real analytic world, not every prime ideal produces a formal curve. See for instance the following examples.

Example A.5. The curve $(t, \sum_{k\in N^*} k!t^k) \in t\mathbb{R}[[t]]^2$ is a formal curve and its generating ideal is $(y - \sum_{k\in N^*} k!x^k) \subset \mathbb{R}[[x,y]]$. Notice that it is not a real analytic curve.

Example A.6. The prime ideal $(x^2+y^2) \subset \mathbb{R}[[x,y]]$, whose unique generator is indeed convergent, does not provide a formal curve, because it does not admit a parameterization.

We end by pointing out that in Section 1.3.1, we define an important object related to real analytic and formal curves, the iterated tangents.

A.2.2 Definitions of semi-analytic and subanalytic sets

In this section, we simply introduce the definitions of semi-analytic and subanalytic sets, as we need them at some points in this thesis. These types of sets are relatively modern and they have been first defined in [34, 41, 55]. Other good reference is [7]. Before defining the semi-analytic sets, suppose that X is a real analytic set and let $p \in \mathbb{R}^n$ so that X_p is given by $f_1 = 0, \ldots, f_s = 0$. The equations $f_1 = 0, \ldots, f_s = 0$ can be summarized in $f_1^2 + \cdots + f_s^2 = 0$.

Definition A.7. *A set $S \subset \mathbb{R}^n$ is* semi-analytic *if for each $p \in \mathbb{R}^n$ there is a neighborhood U of p and analytic functions $f, g_1, \ldots, g_s \in \mathcal{O}(U)$ such that*

$$S \cap U = \{q \in U : f(q) = 0, g_1(q) > 0, \ldots, g_s(q) > 0\}.$$

Example A.8.

$$S = \{(x,y) \in \mathbb{R}^2 : y = \sin 2\pi x, -x^2 - y^2 > -1, y > 0\}$$

is a semi-analytic set. Notice also that it is relatively compact, since its closure is a compact set.

$$S' = \{(x,y) \in \mathbb{R}^2 : y = e^{\frac{-1}{x^2}}, x > 0\}$$

is not a semi-analytic set, since the function $e^{\frac{-1}{x^2}}$ cannot be analytically extended to 0.

The family of semi-analytic sets is closed under finite unions, finite intersections and complement, but it is not closed under projections.

Definition A.9. *A set $S \subset \mathbb{R}^n$ is* subanalytic *if at each point $p \in \mathbb{R}^n$ there is a neighborhood $U \subset \mathbb{R}^n$, some $m \in \mathbb{N}$ and a semi-analytic relatively compact set $\tilde{S} \subset \mathbb{R}^n \times \mathbb{R}^m$ so that $S \cap U = pr_1(\tilde{S})$, where $pr_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the first factor projection.*

The family of subanalytic sets is closed under finite unions and finite intersections, and even local projections and complements. Then subanalytic sets form an structure of sets with relatively good behavior.

A.3 Vector fields on analytic manifolds

First, we recall some notions on differentiable or analytic manifolds. We denote by M, or more descriptively by $(M,\mathcal{A})$, a differentiable, real analytic or complex analytic manifold of dimension n, provided with an atlas $\mathcal{A} = \{(U_i,\varphi_i)\}_{i\in\mathcal{I}}$ which endows M with a differentiable structure, in the $\mathcal{C}^r$,$\mathcal{C}^\infty$,$\mathcal{C}^\omega$ or holomorphic class.

From now on, we only work with real analytic manifolds, but the definitions also apply to other classes of manifolds. We denote by $\mathcal{O}(U)$ the real analytic functions on some open set $U \subset M$, and by $\mathcal{O}_p$ the germs of analytic functions. Lastly, we denote by $\mathcal{O}(M,N)$ the analytic maps from M to N. Of special interest are the diffeomorphisms of a manifold M with itself, denoted by Diff(M) and the germs of local diffeomorphisms of M that fix a point $p \in M$, denoted by Diff(M,p). On the other hand, the *tangent space* T_pM of an analytic manifold M at a point $p \in M$ is the vector space composed by the linear derivations of the ring of functions at the point. The *tangent bundle* of M is the vector bundle $\sigma : TM = \sqcup_{p\in M} T_pM \to M$ given by $\sigma(v) = p$ for $v \in T_pM$. As a matter of fact, the tangent bundle can be provided a differential structure, so that it is itself an analytic manifold. The topology provided to TM is the initial topology associated to the trivializations of this bundle map σ.

We also need the notion of real analytic manifold with boundary and corners. We start introducing the real analytic functions in open subsets of $(\mathbb{R}_{\geq 0})^m$. A function $f : U \subset (\mathbb{R}_{\geq 0})^m \to \mathbb{R}$ is analytic at a point $p \in U$ if there is an analytic function $\tilde{f} : V \to \mathbb{R}$ defined on an open set $V \subset \mathbb{R}^m$ with $U \subset V$ such that $\tilde{f}|_U = f$. In the same manner, an analytic map $F : U \subset (\mathbb{R}_{\geq 0})^m \to V \subset (\mathbb{R}_{\geq 0})^n$ is analytic at $p \in U$ if each component is analytic. With this consideration, the objects introduced in the previous paragraph, namely analytic manifolds, functions, maps, and tangent spaces are generalized to this setting.

A.3.1 Some algebraic properties of vector fields

Vector fields are, roughly speaking, assignments of vectors of a vector space at each point. It is worth pointing out that there is a different vector space at each point $p \in M$, the tangent space T_pM of M at p.

Definition A.10. *A* vector field *in an analytic manifold in an open set $U \subset M$ is an analytic section of the tangent bundle $\sigma : TU \to U$, that is, a map $\xi : U \to TU$ such that $\sigma \circ \xi = Id_U$. We denote the vector fields in U as $\mathfrak{X}(U)$.*

Notice that at each U, the vector fields $\mathfrak{X}(U)$ have the structure of a $\mathbb{K}$–vector space induced by the vector space structure at the tangent spaces at each point, since $(k_1\xi_1 + k_2\xi_2)(p) = k_1\xi_1(p) + k_2\xi_2(p) \in T_pM$ for any two $k_1,k_2 \in \mathbb{K}$, vector fields $\xi_1,\xi_2 \in \mathfrak{X}(U)$ and point $p \in U$. Even more, it has an $\mathcal{O}(U)$–module structure by setting $(f\xi)(p) = f(p)\xi(p)$ for any function $f \in \mathcal{O}(U)$, vector field $\xi \in \mathfrak{X}(U)$ and $p \in U$. Recall that the tangent space is the set of derivations of germs of functions, which means that the vector fields define a derivation at each point. In addition, analytic vector fields are analytic as maps, that is, these linear derivations change analytically. Hence it is natural to see vector fields as operators (derivations) as follows. Let $\xi \in \mathfrak{X}(U)$ be a vector field in U. Then $\xi : \mathcal{O}(U) \to \mathcal{O}(U)$ is defined by $\xi(f)(p) = \xi(p)(f)$.

We also remark that using coordinate charts $\mathbf{x} : U \to \mathbb{R}^n$ it easily follows that the vector fields $\mathfrak{X}$ are locally finitely generated. Namely, the the set $\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\}$ is a local basis of $\mathfrak{X}(U)$, acting on $\mathcal{O}(U)$ as $\frac{\partial}{\partial x_i}(f) = \frac{\partial}{\partial t_i}(f \circ \mathbf{x})$ in $\mathbf{x}(U)$.

We end by recalling that two vector fields $\xi_1 \in \mathfrak{X}(U), \xi_2 \in \mathfrak{X}(V)$ have the same germ at p if there is $W \subset U \cap V$ such that $\xi_1|_W = \xi_2|_W$. A *germ of vector field* ξ_p is an equivalence class of $\mathfrak{X}$ under the previous equivalence relation. The set of germs of vector fields at a point is denoted by $\mathfrak{X}_p$ or $\mathfrak{X}(M,p)$.

At any point of M, recall that there is always a local coordinate chart $(U,\mathbf{x})$. Using this chart, we can define as before an equivalence between the germs of analytic functions and the convergent power series in $\mathbb{R}\{\mathbf{x}\}$. Secondly, there is an equivalence between the $\mathcal{O}_p$−module of analytic vector fields and the $\mathbb{R}\{\mathbf{x}\}$−module of derivations of convergent power series. Along this text, it is also important to consider formal completions of germs of vector fields, $\hat{\mathfrak{X}}(M,p)$, which are isomorphic to the derivations of $\widehat{\mathcal{O}_p}$, see [18].

The last algebraic property we want to recall is the Lie algebra structure of vector fields.We recall the definition of the Lie bracket. Let $\xi_1\xi_2 \in \mathfrak{X}(U)$ be two vector fields. We define the *Lie bracket* of ξ_1, ξ_2 as the vector field $[\xi_1,\xi_2] : \mathcal{O}(U) \to \mathcal{O}(U)$ such that $[\xi_1,\xi_2](f) = \xi_1(\xi_2(f)) - \xi_2(\xi_1(f))$.

A.3.2 Some geometric properties of vector fields

Associated to any analytic vector field there exists a family of integral curves.

Definition A.11. *Let ξ an analytic vector field on M and $p \in M$. An* integral curve of ξ at p on M *is a parameterized curve $\gamma : I \to M$ where $I \subset \mathbb{R}$ is an open interval, $\gamma'(t) = \xi(\gamma(t))$ for $t \in I$ and $\gamma(t_0) = p$. We say that γ is a* cycle *if there is some $T \in \mathbb{R}_{\neq 0}$ such that $\gamma(t+T) = \gamma(t)$ for each $t \in \mathbb{R}$.*

Existence and uniqueness of integral curves (up to reparameterizations and choice of maximal interval of definition) is ensured by the general theory of ordinary differential equations, because they are solutions of the following system of ODEs

$$\gamma_i'(t) = a_i(\gamma(t)),\ i = 1, \dots, n,$$

where $a_i(\gamma(t))$ is simply the i−th component of the vector field, given by

$$\xi(\gamma(t)) = a_1(\gamma(t))\left(\frac{\partial}{\partial x_1}\right)_{\gamma(t)} + \cdots + a_n(\gamma(t))\left(\frac{\partial}{\partial x_n}\right)_{\gamma(t)}.$$

When coordinates are fixed, we can also use the notation $\dot{\mathbf{x}} = \xi(\mathbf{x})$ to refer to the vector field ξ, via the ordinary differential equation that it defines.

Using the existence and uniqueness of the integral curves, it is possible to define the flow of a vector field as the map $\Phi : D \to M$ defined by $\Phi(t,p) = \gamma_p(t)$, where $D \subset \mathbb{R} \times M$ and γ_p denotes the integral curve of ξ such that $\gamma(0) = p$ (see [75] for a reference of the flows of vector fields). A local expression of the flow at each point p can be obtained using the exponential map. Formally, choosing a set of coordinates $\mathbf{x}$ centered at p, the exponential map is given by $\exp t\xi : \mathbb{R}[[x_1, \dots, x_n]] \to \mathbb{R}[[x_1, \dots, x_n, t]]$, which is defined by

$$\exp t\xi(f) = \sum_{i=1}^{\infty} \frac{t^i}{i!} \xi^{(i)}(f). \tag{A.1}$$

where $\xi^{(0)}(f) = f$ and $\xi^{(i)}(f) = \xi(\xi^{(i-1)}(f))$ for $i \geq 1$. Now, we will "forget" that trajectories are parameterized curves, in order to define the foliation that a vector field generated.

Definition A.12. *Let γ be a (maximal) integral curve of the vector field ξ, the set $|\gamma| = Im(\gamma)$ is a* trajectory *of ξ. When $|\gamma|$ is a single point $q \in U$, we say that q is a* singular point *or a* singularity *of ξ.*

Notice that q is a singular point if and only if $\xi(q) = 0$.

Definition A.13 (Foliation of U defined by ξ)**.** *The partition of U into topological manifolds $|\gamma| \subset U$ of dimensions* 1 *and* 0 *is the* one dimensional analytic foliation $\mathcal{F}$ generated by ξ. *The* set of singularities *$Sing(\mathcal{F}) = Sing(\xi)$ is the union of the trajectories that are a single point. If $Sing(\mathcal{F}) = \emptyset$ we say that the foliation is regular, otherwise that it is singular.*

In fact, given a one dimensional analytic foliation $\mathcal{F}$, there is not a single vector field generating it. This is because the parameterization of the trajectories is not taken into account. For instance, if $\mathcal{F}$ is generated by ξ, it is also generated by $\lambda\xi$ for any $\lambda \in \mathbb{R} \setminus \{0\}$.

Let $p \in M$. We say that two foliations $\mathcal{F}$ of $U \subset M$ with $p \in U$ and $\mathcal{F}'$ of $U' \subset M$ and $p \in U'$ have the same germ at p if there is an open subset $W \subset U \cap U'$ where $\mathcal{F}|_W = \mathcal{F}'|_W$. The class of equivalence of $\mathcal{F}$ under this relation is the *germ of foliation $\mathcal{F}_p$ at p* or the *local foliation $\mathcal{F}_p$ at p*. The local foliations are generated by germs or vector fields.

We can define other equivalences between foliations. Two foliations $\mathcal{F}$ and $\mathcal{F}'$ defined on U and U', respectively, are *equivalent or homeomorphic*, if there is a homeomorphism $F : U \to U'$ such that for any $L \in \mathcal{F}$ we have some $L' \in \mathcal{F}'$ such that $F(L) = L'$. In the same spirit, two germs of foliations $\mathcal{F}_p$ and $\mathcal{F}_p'$ are equivalent if there are representatives $\mathcal{F}$ at U and $\mathcal{F}'$ at U' such that $\mathcal{F}$ and $\mathcal{F}'$ are equivalent.

We will be specially interested in the local properties of the foliations defined by vector fields, that is, in local foliations. In particular, the points of special interest in our work are the points where the vector field vanishes, the singularities, as we motivate now.

In the non singular points, the foliation is locally trivial, i.e. equivalent to one generated by a constant vector field. This is a consequence of the following classical theorem.

Theorem A.14 (Rectification theorem)**.** *Let ξ be a vector field in M, $\mathcal{F}$ the foliation that ξ defines and $p \in M$ a non-singular point. Then, there is an open neighborhood U of p and a homeomorphism $F : U \to U'$ such that $F(\mathcal{F})$ is generated by $(\frac{\partial}{\partial x_1})$, where $(x_1, \cdots, x_n)$ are coordinates in U'.*

The problem of studying the topological properties of analytic foliations is (almost) totally solved in dimension 2. We highlight the result on sectorial decomposition of non center-focus vector fields by [6, 66, 3]. There are several problems yet to be solved, that we will discuss later. For instance, the center-focus problem which consists on determining if a vector field is topologically a radial foliation or a collection of 1-spheres is transcendent.

A.3.3 Conjugation and equivalence of vector fields

When it comes to study the topology of local foliations, it is often convenient to work with vector fields that are some sort of equivalent to the original one, but which may have a simpler expression. In this section, we will outline which kinds of equivalence preserve the local foliation of a vector field. We start with the strongest type of equivalence, the analytic and $\mathcal{C}^k$ conjugation, for $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$. We include the analytic case by denoting the analytic functions by $\mathcal{C}^\omega$.

Definition A.15. *Let M,N be two analytic manifolds, $U \subset M$ and $V \subset N$ be two open sets and $\xi \in \mathfrak{X}(U)$ and $\eta \in \mathfrak{X}(V)$ be two analytic vector fields. We say that* ξ and η are locally $\mathcal{C}^k$ conjugated at a point $p \in M$ *if there exist two open sets $\widetilde{U} \subseteq U$ and $\widetilde{V} \subseteq V$ with $p \in \widetilde{U}$ and an analytic diffeomorphism $\varphi : \widetilde{U} \to \widetilde{V}$ such that $D\varphi(q)(\xi(q)) = \eta(\varphi(q))$ for any $q \in \widetilde{U}$. It is the same as saying that $\varphi_*(\xi) = \eta$.*

We find, as a consequence, that the conjugation diffeomorphism preserves not only the local foliation, but also the parameterization of trajectories around p.

Proposition A.16. *Let M,N be two analytic manifolds, $U \subset M$ and $V \subset N$ be two open sets and $\xi \in \mathfrak{X}(U)$ and $\eta \in \mathfrak{X}(V)$ be two analytic vector fields. Let Φ^ξ be the local flow of ξ at $p \in M$. If ξ and η are locally $\mathcal{C}^k$ conjugated at p, then $\varphi(\Phi^\xi(t,q)) = \Phi^\eta(t,\varphi(q))$, where $\varphi : \widetilde{U} \to \widetilde{V}$, the map Φ^η is the local flow at $\varphi(p)$ and $q \in \widetilde{V}$.*

At the thesis of last proposition, we see that the differentiable properties of φ are not used. Thus, we will extend the concept of conjugation to the homeomorphic or $\mathcal{C}^0$ case by means of conjugation of flows. Let $k \in \mathbb{N} \cup \{\omega, \infty\}$.

Definition A.17. *Let M,N be two analytic manifolds, $U \subset M$ and $V \subset N$ be two open sets and $\xi \in \mathfrak{X}(U)$ and $\eta \in \mathfrak{X}(V)$ be two analytic vector fields. Let Φ_ξ be the local flow of ξ at $p \in M$. We say that ξ and η are* locally $\mathcal{C}^k-$flow conjugated *(or simply locally $\mathcal{C}^k-$ conjugated) if there are $\widetilde{U} \subset U$ and $\widetilde{V} \subset V$ and a $\mathcal{C}^k-$differentiable map $\varphi : \widetilde{U} \to \widetilde{V}$, such that $\varphi(\Phi^\xi(t,q)) = \Phi^\eta(t,\varphi(q))$, where Φ^η is the local flow at $\varphi(p)$ and $q \in \widetilde{V}$.*

Notice that these conjugations still respect the parameterization of the trajectories. We can still define a weaker equivalence for which the parameterization of the trajectories is not necessarily preserved.

Definition A.18. *Let M,N be two analytic manifolds, $U \subset M$ and $V \subset N$ be two open sets and $\xi \in \mathfrak{X}(U)$ and $\eta \in \mathfrak{X}(V)$ be two analytic vector fields. Let Φ^ξ be the local flow of ξ at $p \in M$. We say that ξ and η are* locally $\mathcal{C}^k-$equivalent *or* locally $\mathcal{C}^k-$ orbitally equivalent *if there are $\widetilde{U} \subset U$ and $\widetilde{V} \subset V$ and a $\mathcal{C}^k-$differentiable map $\varphi : \widetilde{U} \to \widetilde{V}$, such that $\varphi(\Phi^\xi(t,q)) = \Phi^\eta(s,\varphi(q))$, where Φ^η is the local flow at $\varphi(p)$ and $q \in \widetilde{V}$ and $s \in \mathbb{R}$.*

Notice that the previous definition, implies exactly that the foliations generated by ξ and η are equivalent as foliations, since one leaf is sent to another.

Example A.19. An easy example of two vector fields that are $\mathcal{C}^1-$equivalent but not $\mathcal{C}^1-$conjugated are the following, both defined in $(-\varepsilon,\varepsilon) \subset \mathbb{R}$ with $\varepsilon << 1$ and given by $\eta = x\frac{\partial}{\partial x}$ and $\eta = (1+x^2)x\frac{\partial}{\partial x} = (x+x^3)\frac{\partial}{\partial x}$. The homeomorphism considered to see the equivalence can simply be the identity, but there is not a diffeomorphism conjugating them.

A.3.4 Invariant manifold theorems

In this section, we will present the definition of (local) invariant manifolds, invariant analytic sets and theorems of great importance on existence of invariant manifolds for vector fields.

Definition A.20. *Let $\xi \in \mathfrak{X}(U)$ be a vector field in $U \subset \mathbb{R}^n$ and $S \subset U \subset \mathbb{R}^n$ be a subset. We say that S is an* invariant set for ξ *if $\Phi(t,p) \in S$ for every $p \in S$ and t in $D \cap \mathbb{R} \times \{p\}$. Let $\xi \in \mathfrak{X}(\mathbb{R}^n, 0)$ a germ of vector field and S_0 a germ of subset at 0. We say that S_0 is invariant for ξ if there is a representative S of S_0 that is invariant for a representative of ξ.*

The invariant sets can sometimes have the structure of a submanifold, or be algebraic, analytic, semi-analytic or subanalytic. Notice that the algebraic and analytic sets are defined, only locally in the second case, as sets of zeros of algebraic and analytic functions. The invariance property in this case is translated to the following. Let $p \in M \subset U \subset \mathbb{R}^n$ be a point at an algrebraic or analytic set M. Let U_p be a neighborhood of p in $\mathbb{R}^n$ such that $M \cap U_p = V(I)$, where I is an ideal (in $\mathcal{O}(U_p)$) generating $M \cap U_p$. Then, the invariance condition is equivalent to the invariance of I under the derivation defined by ξ at p, that is $\xi_p(I) \subset I$.

This idea of invariant ideals leads to a useful generalization.

Definition A.21. *Let $\xi \in \widehat{\mathfrak{X}}(M,p)$ be a germ of formal vector field at p. We say that an ideal I in $\hat{\mathcal{O}}_p$ is a* formal invariant set *if $\xi(I) \subset I$.*

Notice that germs of analytic vector fields are convergent elements in the set of germs of formal vector fields. For this reason, it makes sense to speak about germs of formal invariant sets also for germs of analytic vector fields.

Let ξ be a vector field with a singularity in 0 and let $L\xi$ be defined as $L\xi = j_1(\xi)$. In coordinates, if

$$\xi = a_1(\mathbf{x})\frac{\partial}{\partial x_1} + \cdots + a_n(\mathbf{x})\frac{\partial}{\partial x_n},$$

where each $a_i(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}^n} a_{i,\alpha}\mathbf{x}^\alpha$, then its linear part is

$$L\xi = \sum_{|\alpha|=1} a_{1,\alpha}\mathbf{x}^\alpha \frac{\partial}{\partial x_1} + \cdots + \sum_{|\alpha|=1} a_{n,\alpha}\mathbf{x}^\alpha \frac{\partial}{\partial x_n}.$$

For a semi-hyperbolic vector field, we can always define linear subspaces E^u, E^s and E^c, associated to the eigenvalues of $D\xi(0)$ with positive real part, negative real part and vanishing real part, respectively. These linear subspaces are invariant for the linear vector field $L\xi$. With the definition of these subspaces in hand, we can provide some two classical results on real dynamical systems based on the existence of invariant manifolds.

Theorem A.22 (Center manifold theorem [20, 47])**.** *Let $\xi \in \mathfrak{X}(U)$ be an analytic vector field with hyperbolic singularity at 0 and defined in a neighborhood $U \subset \mathbb{R}^n$. Then, for each $k \in \mathbb{N}$ ($k \leq r$) there is a neighborhood V_k of 0 and manifolds $W_k^u, W_k^{cu}, W_k^c, W_k^{cs}, W_k^s$ of class $\mathcal{C}^k$, named* unstable, center-unstable, center, center-stable and stable *manifolds, respectively. The unstable manifold W_k^u is tangent to E^u of $L\xi$ at 0, the center-unstable manifold W_k^{cu} is tangent to $E^u \oplus E^c$ of $L\xi$ at 0, the center manifold W_k^c is tangent to E^c of $L\xi$ at 0, the center-stable manifold W_k^{cs} is tangent to $E^s \oplus E^c$ of $L\xi$ at 0, and the stable manifold W_k^s is tangent to E^s of $L\xi$ at 0. In addition:*

1. *Let $p \in W^s$. Then, the integral curve at p converges to the point 0 as t tends to $+\infty$ and leave V in negative time. In addition, there are constants $C, B > 0$ such that $\|\Phi_t(v)\| \leq Ce^{-Bt}$ for $t \geq 0$.*

2. *Let $p \in W^u$. Then, the integral curve at p converges to the point 0 as t tends to $-\infty$ and leave V in positive time. In addition, there are constants $C, B > 0$ such that $\|\Phi_t(v)\| \leq Ce^{Bt}$ for $t \leq 0$.*

3. *The center-stable manifold contains every $p \in V_k$ such that the integral curve γ_p is defined for all $t \geq 0$ and remains in V_k.*

4. *The center manifold contains every $p \in V_k$ such that the integral curve γ_p is defined for all $t \in \mathbb{R}$ and remains in V_k.*

5. *The center-unstable manifold contains every $p \in V_k$ such that the integral curve γ_p is defined for all $t \in \mathbb{R}$ and remains in V_k.*

Remark A.23. In the light of the previous theorem, we remark that in the center-unstable, center and center-stable manifolds, there might be orbits that leave V_k for both positive and negative times. It is also possible that one of these manifolds behaves as a stable or unstable manifold (depending on the case) in terms of integral curves. The behavior of the vector field inside them depends on the higher order terms of the vector field, which are less dominating than the linear ones.

The stable and unstable manifolds are commonly called *strong manifolds*, since they are (germ-wise) unique, and analytic when ξ is analytic. However, the center-unstable, center and center-stable manifolds may not be unique and neither analytic nor ∞–differentiable. Because of these facts, the center-unstable, center and center-stable manifolds are called the *weak manifolds*. There is a large list of works studying the properties of the center manifolds. We summarize the properties of greatest interest for our work and refer the reader to a collection of interesting papers on the topic.

- All center manifolds have the same Taylor expansion at 0. We will call this series the *formal center manifold* $\widehat{W}^c$. The same can be said for the formal center-unstable manifold $\widehat{W}^{cu}$ and the formal center-stable manifold $\widehat{W}^{cs}$.
- As a consequence of the previous item, any two invariant manifolds differ from each other on the order of $e^{-C/\|\mathbf{x}_c\|}$ for some constant $C > 0$ at 0, and $\mathbf{x}_c$ coordinates of the center manifold.
- Under certain conditions, the center manifold is unique and even analytic or differentiable. For instance, a center manifold full of cycles is necessarily unique.
- There is a collection of examples of center manifolds that are not infinitely differentiable.

We end the section by stating a result on topological conjugation of a vector field with semi-hyperbolic singularity and a simpler one. We also highlight the idea that the integral curves of a vector field lying outside a center manifold topologically follow the behavior of a linear vector field transverse to the center manifold while they also follow the flow inside the center manifold. The following theorem is a particular case of the result of F. Takens and J. Palis [63].

Theorem A.24 (Reduction to the center manifold theorem)**.** *Let $\xi \in \mathfrak{X}(U)$ be an analytic vector field in a neighborhood of $0 \in \mathbb{R}^n$ with semi-hyperbolic singularity at 0, and let $L\xi$ be its linear part. Then, there is a topological conjugation between ξ and $\xi|_{W^c} + L\xi|_{E^u \oplus E^s}$.*

A.4 Normal forms

We end this appendix by introducing normal forms and we state the main results in this topic. The obtainment of normal forms and the invariant manifold theorems, presented in the previous section, are two strong tools that combined simplify the description of the dynamics of vector fields, in many cases.

In short words, a normal form of a vector field is another vector field conjugated to the original one (analytically, $\mathcal{C}^k$, $\mathcal{C}^\infty$ or only formally) that has a more simple expression. Many authors have studied this problem with different approaches. We start giving the definition of formal normal form in terms of the Jordan decomposition of formal vector fields in $\mathbb{C}^n$. Recall that formal vector fields are derivations of $\widehat{O}_0 = \widehat{O}(\mathbb{C}^n, 0)$ and that $\widehat{O}_0$ is provided with the jet truncation morphisms. Notice also that a k–jet of vector field $j_k(\xi)$ defines a derivation (endomorphism) in $j_k(\widehat{O}_0)$. It is hence possible to obtain its Jordan decomposition

$$j_k(\xi) = \xi_{S,k} + \xi_{N,k},$$

with $[\xi_{S,k}, \xi_{N,k}]$. In fact, because of the commutation of this decomposition with jet truncations, we can take the limit of the Jordan decompositions and define the Jordan normal form as

$$\xi = \xi_S + \xi_N,$$

with $j_k(\xi_S) = \xi_{S,k}$ and $j_k(\xi_N) = \xi_{N,k}$.

Definition A.25. *A formal vector field $\xi = \xi_S + \xi_N$ is a* formal normal form *if ξ_S is a semi-simple linear diagonal vector field, ξ_N is a nilpotent vector field and $[\xi_S, \xi_N] = 0$. When ξ is analytic, we say that ξ is an analtyic normal form.*

In the literature, the formal normal form is also known as the *Poincaré-Dulac normal form*. The following is a well known result, see for instance [10] and the references therein, we find that a formal normal form always exists for every $\xi \in \mathfrak{X}(\mathbb{C}^n, 0)$. However, the formal normal form is generally far from being unique, as we will comment thereafter.

Theorem A.26 (Formal Normal Form)**.** *Let $\xi \in \mathfrak{X}(\mathbb{C}^n, 0)$. Then, there is a formal diffeomorphism φ at $0 \in \mathbb{C}^n$ such that $\varphi^*(\xi)$ is in normal form.*

To prove this theorem, it is enough to linearize the semi-simple part of ξ into $S = \sum_{i=1}^n \lambda_i x_i \frac{\partial}{\partial x_i}$, where λ_i are the eigenvalues of ξ. Notice that after a linear change of coordinates, we can assume that the linear part of the initial vector field is in the Jordan normal form S. We give some definitions and then outline the inductive steps on the construction of a formal normal form.

We define the Lie derivative operator $L_S : \widehat{\mathfrak{X}}(\mathbb{C}^n, 0) \to \widehat{\mathfrak{X}}(\mathbb{C}^n, 0)$ given by $L_S(\eta) = [S, \eta]$. Writing conveniently $\eta = \sum_{i=1}^n a_i(\mathbf{x}) \frac{\partial}{\partial x_i} = \sum_{i=1}^n b_i(\mathbf{x}) x_i \frac{\partial}{\partial x_i}$, allowing that b_i is meromorphic (when x_i does not divide a_i), we obtain a simple expression for $L_S(\eta)$ as follows. Let $b_i(\mathbf{x}) = \sum_{q_i \geq -1, q_k \geq 0, k=1,2,\ldots,n, k \neq i} b_{i,Q} \mathbf{x}^Q$, we have that $L_S(\eta)$ is expressed as a sum of monomial vector fields, then

$$L_S(\eta) = \sum_{i=1}^{n} \sum_{q_i \geq -1, q_k \geq 0, k=1,2,\ldots,n, k \neq 0} b_{i,Q} \langle \Lambda, Q \rangle \mathbf{x}^Q x_i \frac{\partial}{\partial x_i},$$

where $\langle -, - \rangle$ is the inner product in $\mathbb{C}^n$. The values $\alpha_Q = \langle \Lambda, Q \rangle$ are the eigenvalues of L_S, and the vector fields $\mathbf{x}^Q x_i \frac{\partial}{\partial x_i}$ are the corresponding eigenvectors. The linear subspace of vector fields η such that $L_S(\eta) = 0$ are the ones commuting with the semi-simple part and the vector fields that cannot be eliminated. The monomial vector fields $\mathbf{x}^Q x_i \frac{\partial}{\partial x_i}$ for $Q \in \{(q_1, \ldots, q_n) : q_i \in \mathbb{Z}_{\geq -1}, q_j \in \mathbb{Z}_{\geq 0} j \neq i\}$, $|Q| > 1$ and $i \in \{1, \ldots, n\}$ such that $\langle \Lambda, Q \rangle = 0$ are called the *resonances of* S.

By defining linear systems of equations, one can induce an analytic change of coordinates φ_2 such that the non-resonant terms of $\xi_2 = (\varphi_2)_*(\xi)$, are equal to zero up to its 2–jet. The analytic change of coordinates is defined as the exponential map of a polynomial vector field η_2 that fulfills that $j_2(\xi) - L_S(\eta_2) = S + \chi_2$, where χ_2 is an homogeneous vector field of degree 2 that contains only resonant terms. We highlight that this expression provides the homogeneous components of degree 2 of $(\varphi_2)_*(\xi)$ and it also affects terms of higher degree. In the presence of resonances, we remark that there is not uniqueness on the choice of η_2 and χ_2 in the aforementioned expression. Hence there is not uniqueness on the construction of normal forms.

We do not show more details on the construction, but there is an inductive method to construct the formal normal form jet by jet (indeed, following Martinet's text [59], we can double the jet at each steps). At each step we obtain a vector field ξ_i, $i = 2^k$, such that:

- ξ_i is conjugated by some analytic diffeomorphism φ_i to the vector field $\xi_{i/2}$ obtained in the previous step and it is also conjugated to the original ξ.
- $j_i(\xi_i)$ is in normal form.

The formal normal form is the formal vector field obtained as the limit.

Remark A.27. Even if the original vector field ξ is analytic, it is possible that there is not a choice of normal form that is analytic. In general, most likely, it will not be. And even if there is a choice of formal normal form that is analytic, the normalization formal diffeomorphism does not need to be analytic.

Brjuno has a very complete study about the normalization of analytic vector fields, and in a wide range of cases he gives conditions to determine if there is a choice of normal form that is analytic or if it is generically divergent. However, in between the convergence and divergence conditions, there are still many cases in which convergence is not known. We provide a result of him treating convergence that will be useful in our work, as it appears in [59, Theorem 5]. The author also restate the conditions of Brjuno in a more geometrical way. We present these conditions now.

- (Arithmetic condition) The series
$$\sum_k \frac{\log \omega_k}{2^k}$$
is convergent, where $\omega_k = \min\{|\alpha_Q| : |Q| \le 2^{k+1}, \alpha_Q \ne 0\}$, $Q \in \bigcup_{i=1}^n \{(q_1, \dots, q_n) : q_i \in \mathbb{Z}_{\ge -1}, q_j \in \mathbb{Z}_{\ge 0} j \ne i\}$ and $\alpha_Q = \langle \Lambda, Q \rangle$.
- (Geometric condition) The formal normal form $\hat{\xi}$ of ξ is tangent to the foliations given by $\mathbf{x}^R = c$, where c is a constant and $R = (r_1, \dots, r_n)$ such that $\langle \Lambda, R \rangle = 0$.

Note that the arithmetic condition depends only on the vector of eigenvalues and the geometric condition depends on a particular normal form of ξ. We say that ξ satisfies the geometric condition if one normal form of ξ does.

Theorem A.28. *Under the arithmetic and geometric conditions, and one of the following additional hypotheses, there is an analytic normal form $\widetilde{\xi}$ of ξ.*

- $n \le 4$.
- $0 \in Conv(\{\lambda_1, \cdots, \lambda_n\}) \subset \mathbb{C}$.

Now, we state the results concerning real analytic and $\mathcal{C}^k$ vector fields, for $k \in \mathbb{N} \cup \{\infty\}$. It is important to remark that the real formal normal form may not have diagonalized semi-simple part as before, but in any case, it is possible to use the linear vector field in real Jordan form, and we will assume this consideration when we speak about real normal forms. We start by stating the existence of the real formal normal form for real systems. The result is stated more generally for subfamilies of holomorphic vector fields, but its main application is its use for real vector fields, see [11].

Theorem A.29. *Let $\xi \in \mathfrak{X}(\mathbb{R}^n, 0)$ be a germ of real analytic vector field. Then, there is a real formal diffeomorphism φ such that $\varphi^*(\xi)$ is in formal normal form.*

The proof is based on the obtainment of normal forms for complex vector fields. This is because the real Jordan form and the complex Jordan form are related by a linear change of coordinates. We need also to use the fact that coefficients of the normalizing transformation can be chosen in such a way that the normal form is mapped into a real one by the linear change of coordinates, that is, we work in the image of the real vector fields under the diagonalizing linear map.

For $\mathcal{C}^k$ vector fields with $k \in \mathbb{N} \cup \{\infty\}$, we state the result of Takens [72].

Theorem A.30. *Let $\xi \in \mathfrak{X}^\infty(\mathbb{R}^n, 0)$ be a $\mathcal{C}^k$ vector field with $k \in \mathbb{N} \cup \{\infty\}$. Then, there exists a $\mathcal{C}^k$ diffeomorphism such that $\varphi^*(\xi) = \hat{\xi} + R_k$, where $\hat{\xi}$ is a formal normal form of ξ and R_k fulfills $j_k(R_k) = 0$.*

Notice that the k–jets of $\mathcal{C}^\infty$ vector fields are well defined, being $j_\infty(\xi)$ the limit of the l–jet decompositions with $l \in \mathbb{N}$, that is a formal vector field. The vector field R_∞ is a $\mathcal{C}^\infty$ vector field with Taylor expansion equal to 0 (plane).

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Index

Notation

$\mathbb{N}$	The set of natural numbers including 0.
$\mathbb{N}^*$	The set of natural numbers without 0.
$\mathbb{Z}$	The set of integer numbers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{R}_{\geq 0}$	The set of non-negative real numbers
i	Imaginary unit
$0 \in \mathbb{R}^n$	$(0,\ldots,0) \in \mathbb{R}^n$
$\mathbf{x} = (x_1,\ldots,x_n)$	Tuple of variables
$\mathbb{R}[\mathbf{x}]$	$\mathbb{R}$–algebra of polynomials in $\mathbf{x}$.
$\mathbb{R}[\mathbf{x}]_s$	$\mathbb{R}$–algebra of polynomials of degree s in $\mathbf{x}$.
$\mathbb{R}[[\mathbf{x}]]$	$\mathbb{R}$–algebra of formal power series in $\mathbf{x}$.
$\mathbb{R}\{\mathbf{x}\}$	$\mathbb{R}$–algebra of convergent power series in $\mathbf{x}$.
$M, (M, \mathcal{A})$	Differentiable or analytic n-dimensional manifold (with boundary and corners) and atlas $\mathcal{A}$.
$\mathbf{x} : U \to \mathbb{R}^n$	Coordinate chart of M.
$\mathcal{O}(U), \mathcal{C}^\omega(U)$	Analytic functions in $U \subset M$
$\mathcal{C}^k(U)$	k–differentiable functions in $U \subset M$ with $k \in \mathbb{N} \cup \{\infty\}$
$\mathcal{O}_{n,p}, \mathcal{O}_p$	Germs of analytic functions at $p \in M$
$\widehat{\mathcal{O}}_{n,p}, \widehat{\mathcal{O}}_p$	Formal completion of germs of analytic functions at $p \in M$
T_pM	Tangent space of M at p.
$TM = \cup_{p\in M} T_pM$	Tangent bundle of M.
T_p^*M	Cotangent space of M at p.
$T^*M = \cup_{p\in M} T_pM$	Cotangent bundle of M.
$\mathfrak{X}(U)$	Vector fields on $U \subset M$
$\mathfrak{X}(M,p), \mathfrak{X}^\omega(M,p)$	Germs of analytic vector fields at $p \in M$.
$\mathfrak{X}^k(M,p)$	Germs of $\mathcal{C}^k$ vector fields at $p \in M$.
$\widehat{\mathfrak{X}}(M,p)$,	Formal vector fields at $p \in M$.
$\Omega(U)$	1-forms on $U \subset M$
$\Omega(M,p)$	Germs of 1–forms at $p \in M$.
$\Phi^\xi : D \subset \mathbb{R} \times U \to U$	Flow of the vector field ξ on $U \subset M$.
$\mathrm{Diff}(U,V)$, $\mathrm{Diff}^\omega(U,V)$	Real analytic diffeomorphism from $U \subset M$ to $V \subset N$.
$\mathrm{Diff}(M,p)$, $\mathrm{Diff}^\omega(M,p)$	Germs of real analytic diffeomorphisms fixing p.
$\mathrm{Diff}_1(M,p)$, $\mathrm{Diff}_1^\omega(M,p)$	Germs of real analytic diffeomorphisms tangent to the identity at p (i.e. $D_pF = Id$ for $F \in \mathrm{Diff}_1(M,p)$).
$\widehat{\mathrm{Diff}}_1(M,p)$	Formal diffeomorphisms tangent to the identity at p (i.e. $D_pF = Id$ for $F \in \mathrm{Diff}_1(M,p)$).
$\mathrm{Fix}(F)$	Fixed points of $F \in \mathrm{Diff}(M,p)$
$\mathrm{Per}(F)$	Periodic points of $F \in \mathrm{Diff}(M,p)$