

Games with Costly Endogenous Separation

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October 17, 2025

Abstract

Games with costly endogenous separation are repeated games where players have the option to leave their current partnership (with some cost) and keep on playing in a newly-formed partnership. Players can also be separated by exogenous factors. We study equilibria in these games. As a relevant case of application, cooperation in a repeated Prisoner's Dilemma is compromised in environments where individuals are free to leave their partners, since defectors can exploit cooperators and move on. Previous studies have explored diverse mechanisms to prevent defect-and-leave strategies from taking hold. We show that, for large enough separation costs, not only full cooperation, but actually any symmetric sequence of outcomes can be sustained as a (path-protecting) neutrally stable state, which is Lyapunov stable under the replicator dynamics. *JEL* classification numbers: C72, C73.

Keywords: Endogenous separation; conditional dissociation; separation costs; path-protecting; voluntarily repeated games.

1 Introduction

Games with endogenous separation¹ are repeated games where players have the option to leave their current partnership and continue playing in a newly-formed one. These games are motivated by a wide range of real-world settings where individuals can terminate relationships at will and seek new ones without necessarily transmitting their past reputation. Examples include labor and credit markets, marriages and friendships, on-line platforms, collaborative partnerships and many models of interaction in the animal world.

The study of games with endogenous separation has shed new light on the mechanisms that can sustain cooperation in dynamic strategic environments where players are free to leave their current partners. In contrast to the classical repeated games framework, where the duration of partnerships is fixed exogenously, the possibility of separation and re-matching fundamentally alters both the incentives faced by individuals

¹Games with endogenous separation (Rob and Yang, 2010; Deb et al., 2020) are also known as *voluntarily separable repeated games* (Fujiwara-Greve and Okuno-Fujiwara, 2009; Fujiwara-Greve et al., 2012, 2015), *voluntary partnership or voluntary continuation games* (Vesely and Yang, 2010, 2013) or *games with conditional dissociation* (Izquierdo et al., 2010, 2014).

and the resulting equilibria of the system. In the particular context of social dilemmas, partial cooperation in games with endogenous separation can be sustained through trust-building equilibria (where mutual cooperation is established only after sufficiently long periods of low payoffs) or through polymorphic equilibria involving endogenous positive assortment of cooperative and defective types (Fujiwara-Greve and Okuno-Fujiwara, 2009; Fujiwara-Greve et al., 2015; Izquierdo et al., 2010, 2014). In both cases, the threat of abandonment followed by re-matching under grimmer perspectives disciplines short-term incentives to defect and enables equilibrium paths that would not be possible with traditional trigger strategies.

A growing body of experimental research has further underscored the importance of endogenous separation. One prominent line of work examines how the ability to leave and re-match affects cooperative behavior in social dilemmas. Experiments consistently show that cooperation is significantly enhanced when players are given the option to terminate uncooperative relationships and rematch with others (Boone and Macy, 1999; Hauk, 2003; Wang et al., 2012; Zhang et al., 2016; Nosenzo and Tufano, 2017; Honhon and Hyndman, 2020; Lee, 2020).

Further support for this view comes from recent simulation studies. Graser et al. (2025) show that when players in a repeated prisoner’s dilemma are given the option to terminate a partnership and rematch, cooperation levels increase substantially compared to the standard repeated game without such an option, as partner switching generates endogenous assortment among cooperators. Similarly, Wubs et al. (2016) analyze the coevolution of positive reciprocity, punishment, and partner switching, and find that when interactions are sufficiently long, partner switching tends to dominate and yields high levels of cooperation, since cooperators can reliably avoid defectors and match with other cooperators. These results reinforce the experimental and theoretical evidence that the ability to abandon defectors and rematch with others constitutes a robust mechanism for sustaining partial cooperation in repeated interactions.

More generally, these theoretical, experimental and simulation findings highlight that endogenous separation is not just a behavioral side note but a fundamental feature with profound implications for the dynamics of repeated interactions.

Nevertheless, the literature on endogenous separation has so far focused primarily on the case of costless separation. In this setting, a player can leave a partnership at no additional expense, re-enter the matching pool, and immediately continue playing with a new partner. While analytically convenient, this assumption is arguably restrictive. In many real-world environments, leaving a partnership entails explicit or implicit costs: time spent searching for a new partner (Enquist and Leimar, 1993), foregone payoffs during rematching, or even direct switching costs. Recent experimental studies, such as Lee (2020), have begun to address these issues by explicitly comparing costless and costly separation. Lee (2020) shows that when separation is costless, defectors can take advantage of the rematching process by leaving uncooperative relationships without penalty and immediately seeking new partners. This undermines long-run cooperation. By contrast, when separation is made costly —whether through a direct penalty or by introducing a delay before rematching— the incentive to exploit rematching opportuni-

ties is reduced. Under these conditions, cooperative outcomes are more stable, as players must weigh the immediate benefits of defection against the costs of separating and re-matching. Yet, despite these insights, a systematic theoretical framework for games with costly endogenous separation has so far been lacking. This is precisely the gap our paper seeks to fill.

In this paper, we extend the framework of symmetric two-player games with endogenous separation (Fujiwara-Greve and Okuno-Fujiwara, 2009; Izquierdo and Izquierdo, 2024) to explicitly incorporate *costly* separation. Our analysis shows that separation costs crucially affect both the existence and stability of equilibria. In particular, we establish that sufficiently high separation costs turn a broad class of deviator-leaving strategies into *path-protecting strategies*. A path-protecting strategy (Izquierdo and Izquierdo, 2024) is such that, if a population of players adopts this strategy, any player who deviates from the equilibrium path obtains a strictly lower payoff than the incumbents (this is a stronger equilibrium condition than Nash and than neutral stability). This result highlights costly separation as an alternative mechanism for sustaining equilibrium outcomes, distinct from other approaches such as suspicion and gossip (Enquist and Leimar, 1993). Moreover, we show how equilibrium outcomes depend on the magnitude of separation costs and provide constructive procedures to identify path-protecting strategies in repeated games with costly separation.

We also consider an important case of separation costs: delayed re-matching (Enquist and Leimar, 1993). In many applications, individuals who terminate a partnership must wait before establishing a new one, effectively incurring an opportunity cost. We model this as a delay during which no payoffs are obtained and analyze its effect on equilibrium dynamics. Our results demonstrate that such delays reinforce the disciplining role of direct separation costs and expand the set of stable equilibria that support partial or full cooperation. This is consistent with the results by Enquist and Leimar (1993), who show that, with long search times, full cooperation can be sustained in equilibrium.

Overall, this paper makes three contributions. First, it introduces a general theoretical framework for repeated games with costly endogenous separation, extending earlier models of costless separation. Second, it characterizes strategies that can constitute conventions or stable equilibria under direct separation costs. Third, it considers delayed re-matching as an alternative source of separation costs and shows how this affects the existence and stability of equilibria.

The rest of the paper is structured as follows. In [section 2](#) we define games with costly endogenous separation derived from normal-form stage games, and we present their main elements: strategies, population states, pool states and payoff functions. In [section 3](#) we define and analyze the properties of two different notions of equilibrium in these games: Nash equilibrium and neutrally stable equilibrium. In [Section 4](#) we discuss the existence of *path-protecting* strategies, which ensure neutrally stable equilibria that are robust against unilateral deviations from the equilibrium path. We show how path-protecting strategies can be checked for, and found, stemming from *deviator-leaving* strategies: strategies which leave players who deviate from the equilibrium path and stay otherwise. Our main result in this section states that, for large enough separation costs,

every deviator-leaving strategy is path-protecting (and, consequently, neutrally stable). For a given separation cost, we also show how to find or check for path-protecting strategies, focusing on a family of deviator-leaving strategies whose path ends up (infinitely) repeating some finite pattern or sequence of action profiles (outcomes). [Section 5](#) considers separation costs caused by delayed re-matching. Here, players who separate are assumed to spend some average number of periods searching for a new partner, during which no payoff is obtained. Finally, in [section 6](#) we present some conclusions. Proofs are detailed in [appendix A](#).

2 Repeated games with costly endogenous separation

We consider a population of agents who are matched in couples or *partnerships* to play a symmetric two-player normal-form stage game. The stage game $G = \{A, U\}$ is defined by an action set $A = \{a_1, \dots, a_n\}$, and a payoff function $U: A^2 \rightarrow \mathbb{R}$, where $U(a_k, a_l)$ represents the payoff obtained by a player using action a_k whose opponent plays action a_l . Every stage game G has an associated repeated game with costly endogenous separation G^{CES} , characterized by a set of strategies² and a payoff function derived from the payoffs of the stage game.

After playing a stage game G at a given (discrete) time period, partnerships may remain together and play the stage game again the next time period. A partnership is broken if some exogenous factor breaks the partnership (exogenous separation) or if either one of the players, according to their strategy, decides to break it (endogenous separation). The probability that a partnership survives exogenous separation after each stage game is called the *continuation probability* δ . If a partnership survives exogenous separation, it may still break up endogenously. At the beginning of every time period, all single players are randomly (re-)matched in partnerships, and then all players play the stage game. There is no information flow between partnerships (Ghosh and Ray, 1996), so there are no reputation effects: single players (those who are rematched) are anonymous.³ Separation entails a cost for the separated players. We consider a cost c_0 after exogenous separation and a (possibly different) cost $c > 0$ after endogenous separation. [Figure 1](#) captures the structure of the model in a flowchart.

²We refer to choices in the stage game G as *actions*, reserving *strategy* for behavior in the repeated game (Mailath and Samuelson, 2006).

³Fujiwara-Greve et al. (2012) consider a model where players may voluntarily provide information across partnerships in the context of the Prisoner's Dilemma.

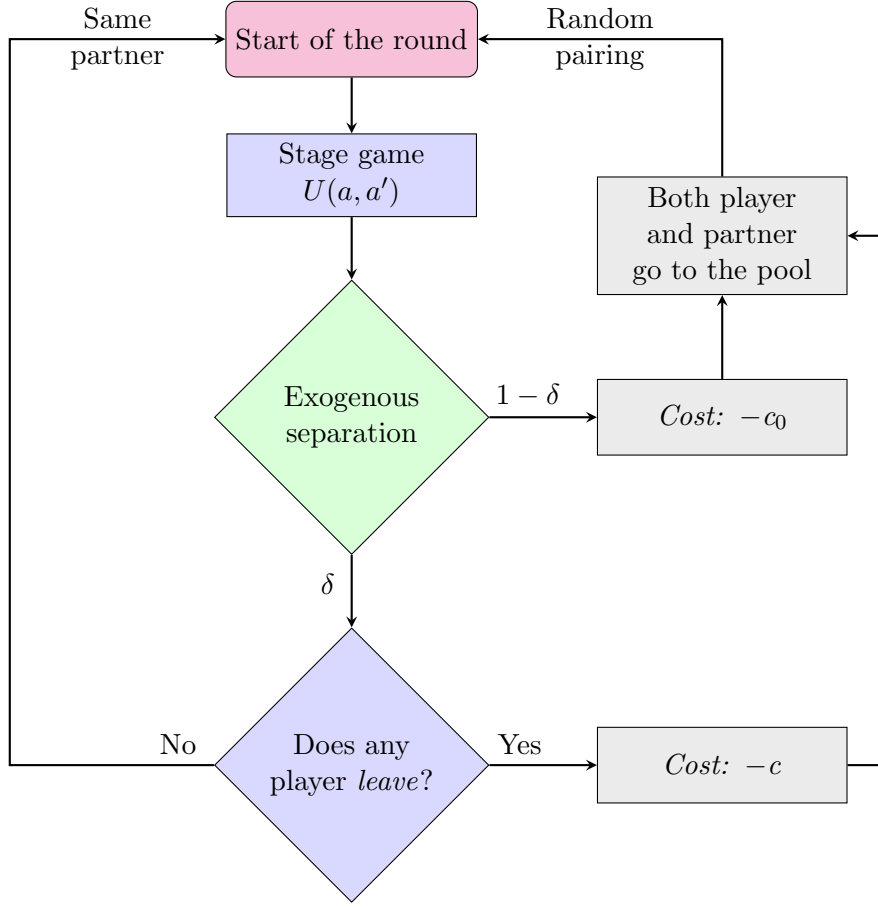


Figure 1: Structure of a repeated 2-players game with costly endogenous separation. Both players incur a cost $-c_0$ after exogenous separation and a cost $-c$ when endogenous separation occurs.

2.1 Strategies in G^{CES}

A strategy i for a player determines the choice that the player makes given any past history of play within a partnership.

For any given partnership, index $t \in \{1, 2, \dots\}$ is used to denote the t^{th} time that the stage game is played in that partnership, assuming the partnership is not broken before.

A *history of play of length* $t \geq 1$, $a^{[1,t]} = (a^{[1]}, a^{[2]}, \dots, a^{[t]}) \in (A^2)^t$, is a sequence of t action profiles. The set of all possible histories of any length (including also the empty history $a^{[1,0]} = \emptyset$, or history of length 0) is

$$\mathcal{H} \equiv \bigcup_{t=0}^{\infty} (A^2)^t,$$

where we have considered $(A^2)^0 \equiv \{\emptyset\}$.

Let $\tilde{A} \equiv A \cup \{break\}$ be the set of choices, where *break* represents the decision to break the current partnership. A strategy i for the repeated game is a mapping $i : \mathcal{H} \rightarrow \tilde{A}$, from the set of possible histories to the set of choices, that prescribes one choice $i(a^{[1,t]}) \in \tilde{A}$ for every possible history. As players in a new partnership are assumed to play at least once together before deciding whether to break their partnership, we require $i(\emptyset) \in A$. Let Ω be the set of strategies.

For any pair of strategies i and j , their endogenous breakup period $T_{ij} \geq 1$ is the number of stage games they are meant to play together before one of them decides to break up. If an i - j partnership never breaks up endogenously, then $T_{ij} = \infty$.

2.2 Population states, pool states and payoffs in G^{CES}

We consider populations where the number of different strategies being played is finite. Let x_i be the fraction of the population using strategy $i \in \Omega$. A (population) state $\mathbf{x}$ is a strategy distribution over Ω with finite support $\mathbb{S}(\mathbf{x}) \subset \Omega$, i.e., $\mathbf{x}$ is a function from Ω to $[0, 1]$ that:

- i) assigns a positive value $x_i > 0$ to each strategy i in a finite set $\mathbb{S}(\mathbf{x})$,
- ii) assigns the value 0 to strategies that are not in $\mathbb{S}(\mathbf{x})$, and
- iii) satisfies $\sum_{i \in \mathbb{S}(\mathbf{x})} x_i = 1$.

Let $\mathbb{D}$ be the set of distributions with finite support. Monomorphic states at which all players use the same strategy i are represented as $\mathbf{e}_i$. These are distributions such that $x_i = 1$ and $x_j = 0$ for every $j \in \Omega \setminus \{i\}$.

Before matching, the set of players who are single and are to be rematched is called the matching pool. In order to calculate the payoffs in G^{CES} we assume that the distribution of strategies in the matching pool has reached a stationary distribution $\mathbf{p} \in \mathbb{D}$ consistent with the distribution of strategies in the population $\mathbf{x}$.

Normalizing the mass of the matching pool to 1 and assuming stationarity, we have that, at every time period, the mass of i -players in partnerships with j -players and playing its t^{th} time together is $p_i p_j \delta^{t-1}$, for $t \in \{1, \dots, T_{ij}\}$. The total mass of i -players in the population is then

$$mass_i = p_i \sum_{j \in \mathbb{S}(\mathbf{x})} p_j \sum_{t=1}^{T_{ij}} \delta^{t-1} = \frac{p_i}{1-\delta} \sum_{j \in \mathbb{S}(\mathbf{x})} p_j (1 - \delta^{T_{ij}})$$

And the population distribution $\mathbf{x}$ corresponding to pool distribution $\mathbf{p}$ is

$$x_i = \frac{p_i \sum_{j \in \mathbb{S}(\mathbf{p})} p_j (1 - \delta^{T_{ij}})}{\sum_{k,j \in \mathbb{S}(\mathbf{p})} p_k p_j (1 - \delta^{T_{kj}})}. \quad (1)$$

Equation (1) defines an invertible⁴ function $f : \mathbb{D} \rightarrow \mathbb{D}$ such that $\mathbf{x} = f(\mathbf{p})$. Given a population distribution $\mathbf{x}$, there is a unique pool distribution $\mathbf{p} = f^{-1}(\mathbf{x})$ satisfying (1).

⁴See Izquierdo and Izquierdo (2024).

For $t \in \{1, \dots, T_{ij}\}$, let $U_{ij}^{[t]}$ denote the payoff obtained by an i -player in a partnership with a j -player the t^{th} time they play together. The (stationary) mass of such players is $p_i p_j \delta^{t-1}$. Adding the mass of all i -players with their payoff (including their cost if they separate), we find that the average payoff to strategy i given pool distribution $\mathbf{p}$ is:⁵

$$\begin{aligned}\hat{F}_i(\mathbf{p}) &\equiv \frac{p_i \sum_{j \in \mathbb{S}(\mathbf{p})} p_j \left(\sum_{t=1}^{T_{ij}} \delta^{t-1} U_{ij}^{[t]} - \sum_{t=1}^{T_{ij}} \delta^{t-1} (1-\delta) c_0 - \delta^{T_{ij}} c \right)}{\text{mass}_i} = \\ &= (1-\delta) \frac{\sum_{j \in \mathbb{S}(\mathbf{p})} p_j \left(\sum_{t=1}^{T_{ij}} \delta^{t-1} U_{ij}^{[t]} - \delta^{T_{ij}} c \right)}{\sum_{j \in \mathbb{S}(\mathbf{p})} p_j (1-\delta^{T_{ij}})} - (1-\delta) c_0\end{aligned}\quad (2)$$

Consequently, the payoff to strategy i at monomorphic population $\mathbf{e}_j$ is

$$F_{ij} \equiv \hat{F}_i(\mathbf{e}_j) = \frac{1-\delta}{1-\delta^{T_{ij}}} \left(\sum_{t=1}^{T_{ij}} \delta^{t-1} U_{ij}^{[t]} - \delta^{T_{ij}} c \right) - (1-\delta) c_0 \quad (3)$$

It follows from equations 2 and 3 that, for $\mathbf{p} = f^{-1}(\mathbf{x})$, the payoff to strategy i at population state $\mathbf{x}$ is

$$F_i(\mathbf{x}) = \hat{F}_i(\mathbf{p}) = \sum_{j \in \mathbb{S}(\mathbf{x})} \frac{p_j (1-\delta^{T_{ij}})}{\sum_{k \in \mathbb{S}(\mathbf{x})} p_k (1-\delta^{T_{ik}})} F_{ij}, \quad (4)$$

which shows that $F_i(\mathbf{x})$ is a convex combination of the payoffs F_{ij} for $j \in \mathbb{S}(\mathbf{x})$, with (strictly) positive coefficients for the convex combination.

Finally, for a group of players with strategy distribution $\mathbf{y} \in \mathbb{D}$ entering a population with strategy distribution $\mathbf{x}$, the average *payoff of $\mathbf{y}$ against $\mathbf{x}$* , $E(\mathbf{y}, \mathbf{x})$, is defined by:

$$E(\mathbf{y}, \mathbf{x}) \equiv \sum_{i \in \mathbb{S}(\mathbf{y})} y_i F_i(\mathbf{x}). \quad (5)$$

For fixed δ , the last term in (2), $(1-\delta)c_0$, is a constant that affects every strategy in the same way. This leads to the following observation, which implies that c_0 does not affect the strategic nature of the game.

Observation 1. *The cost of exogenous separation c_0 does not affect the payoff differences between strategies.*

Given that the cost of exogenous separation c_0 is merely a translation of the payoffs, for the purpose of analyzing equilibria and stability under various dynamics such as the replicator dynamics, we can assume $c_0 = 0$ without loss of generality. Taking $c_0 = 0$ leads to

$$F_{ij} = \frac{1-\delta}{1-\delta^{T_{ij}}} \left[\sum_{t=1}^{T_{ij}} \delta^{t-1} U_{ij}^{[t]} - \delta^{T_{ij}} c \right] \quad (6)$$

⁵Here we are assuming that exogenous separation happens before endogenous separation (see figure 1).

3 Equilibria in games with costly endogenous separation

3.1 Definitions: Nash and neutrally stable strategies in G^{CES}

A strategy distribution $\mathbf{y} \in \mathbb{D}$ is a best response to state $\mathbf{x}$ if (and only if) $E(\mathbf{y}, \mathbf{x}) \geq E(\mathbf{z}, \mathbf{x})$ for every $\mathbf{z} \in \mathbb{D}$. It follows from (5) that $\mathbf{y}$ is a best response to $\mathbf{x}$ if every strategy in its support $\mathbb{S}(\mathbf{y})$ is a best response to $\mathbf{x}$, i.e., if $F_j(\mathbf{x}) \geq F_k(\mathbf{x})$ for every $j \in \mathbb{S}(\mathbf{y})$ and $k \in \Omega$.

Definition 1 (Nash equilibrium state). *A state $x \in \mathbb{D}$ is Nash if it is a best response to itself.*

If a monomorphic state $\mathbf{e}_i$ is Nash, we say that strategy i is a Nash strategy. Consequently, a strategy i is Nash if and only if $F_{ii} \geq F_{ji}$ for every $j \in \Omega$.

Definition 2 (Neutrally stable state). *A state $\mathbf{x} \in \mathbb{D}$ is neutrally stable if*

$$\begin{aligned} E(\mathbf{x}, \mathbf{x}) &\geq E(\mathbf{y}, \mathbf{x}) && \text{for every } \mathbf{y} \in \mathbb{D}, \text{ i.e., } \mathbf{x} \text{ is Nash, and} \\ E(\mathbf{x}, \mathbf{y}) &\geq E(\mathbf{y}, \mathbf{y}) && \text{for every } \mathbf{y} \in \mathbb{D} \text{ such that } E(\mathbf{y}, \mathbf{x}) = E(\mathbf{x}, \mathbf{x}). \end{aligned}$$

A strategy i is said to be neutrally stable (NS) if and only if its associated monomorphic state $\mathbf{e}_i$ is neutrally stable.

There are other alternative definitions of neutral stability in the literature (Bomze and Weibull, 1995). The definition we adopt here guarantees Lyapunov stability in the replicator dynamics for games with endogenous separation (Izquierdo and Izquierdo, 2024). We could consider stronger equilibrium conditions, such as evolutionary stability, but by the standard argument for repeated games (Boyd and Lorberbaum, 1987), there are no evolutionarily stable states (at least with finite support) in games with costly endogenous separation: given any equilibrium and an incumbent strategy i , there is always some alternative strategy $j \neq i$ with the same behavior as i against the equilibrium strategies.

3.2 Nash strategies

The following three lemmas are adaptations of equivalent results for games with endogenous separation and no cost ($c = 0$) in Izquierdo and Izquierdo (2024). The first two lemmas are independent of the separation cost, while the third lemma opens the door to new equilibrium strategies in games with separation costs (vs. no costs).

Lemma 3.1. *If i is a Nash strategy with finite T_{ii} , then the action profile at the breakup stage T_{ii} of an i - i partnership is a Nash profile of the stage game G .*

The reason is that, if the last action profile is not Nash, then a deviation in action at that last stage (to a best-response action of the stage game, leaving afterwards) would be profitable.

Lemma 3.2. *If (a, a) is a Nash action profile of the stage game, any strategy i that (independently of its partner's behavior) chooses action a for some arbitrary number $T \geq 1$ of periods and breaks the partnership at period T is a Nash strategy of G^{CES} .*

The proof of lemma 3.2 considers that it is not possible to obtain a larger stage game payoff against i (than i itself) in any stage before the T^{th} , or to extend the interaction with i beyond that stage.

Lemma 3.3. *The first action a^0 played by a Nash strategy in G^{CES} must satisfy*

$$U^{BR}(a^0) \leq M + \delta c,$$

where $U^{BR}(a^0)$ is the best-response stage payoff to action a^0 and $M = \max_{a \in A} U(a, a)$ is the maximum symmetric stage-game payoff.

The proof of lemma 3.3 considers a strategy that plays a best response action to a^0 and then leaves. The payoff to such a strategy (against the Nash strategy which plays a^0 as initial action) is $U^{BR}(a^0) - \delta c$. This cannot be greater than the payoff to the Nash strategy, which is in turn bounded above by M .

To illustrate some applications of each result, we consider the Prisoner's Dilemma and the Hawk-Dove game, with actions C and D and payoffs U_{CC}, U_{CC}, U_{DC} and U_{DC} . (table 1). In the Prisoner's Dilemma, C stands for cooperate and D for defect; in the Hawk-Dove game, C corresponds to Dove and D to Hawk. In both cases, coordinating on C is more efficient than on D (the maximum symmetric stage payoff is $U_{CC} > U_{DD}$), and D is the minmax action, i.e., the action that minimizes the payoff obtained by a player who adopts a best-response action.

In the Prisoner's Dilemma ($U_{CD} < U_{DD} < U_{CC} < U_{DC}$), D is a dominant action and (D, D) is a Nash action profile. In the Hawk-Dove ($U_{DD} < U_{CD} < U_{CC} < U_{DC}$), the best-response to each action is the other action (this is an anti-coordination game) and there is no (pure) Nash action profile.

	C	D		C	D
C	$\begin{pmatrix} 3 & 1 \end{pmatrix}$		C	$\begin{pmatrix} 3 & 2 \end{pmatrix}$	
D	$\begin{pmatrix} 4 & 2 \end{pmatrix}$		D	$\begin{pmatrix} 4 & 1 \end{pmatrix}$	

Table 1: Left: A Prisoner's Dilemma game, with C for Cooperate and D for Defect. Right: A Hawk-Dove game, with C for Dove and D for Hawk.

For the Prisoner's Dilemma, our previous lemmas lead to these results:

- Strategies that always play D (regardless of their partner's behavior) for a fixed number of periods in a partnership and then leave are Nash (lemma 3.2).
- The condition $c \geq \frac{U_{DC} - U_{CC}}{\delta}$ is necessary for a nice strategy (a strategy that begins a partnership by playing C) to be Nash. This can be considered quite a strong requirement: in order to have a cooperative equilibrium, the separation

cost must be greater than the payoff advantage of unilateral defection over mutual cooperation ([lemma 3.3](#)).

For the Hawk-Dove we obtain these results:

- A strategy i with finite breakup period T_{ii} cannot constitute a Nash equilibrium ([lemma 3.1](#)).
- The condition $c \geq \frac{U_{DC} - U_{CC}}{\delta}$ is necessary in order to have a Nash strategy that begins a partnership by playing C , i.e. Dove ([lemma 3.3](#)).

3.3 Neutrally stable strategies

The following lemma will be useful to compare payoffs for different strategies. We need to introduce some notation first:

- Let an i - j path $a_{ij}^{[1, T_{ij}]} = (a_{ij}^{[1]}, a_{ij}^{[2]}, \dots, a_{ij}^{[T_{ij}]})$ be the sequence of T_{ij} action profiles played in an i - j partnership until their breakup period.⁶
- Let an i - j repeated path $h_{ij}^{[\infty]} = (a_{ij}^{[1, T_{ij}]}, a_{ij}^{[1, T_{ij}]}, \dots)$ be an infinite repetition of $a_{ij}^{[1, T_{ij}]}$. Note that if $T_{ij} = \infty$ then $h_{ij}^{[\infty]} = a_{ij}^{[1, \infty]}$.

Lemma 3.4. *If two strategies i_1 and i_2 are such that $h_{i_1 j}^{[\infty]} = h_{i_2 j}^{[\infty]}$, then*

- *If $T_{i_1 j} > T_{i_2 j}$ then $F_{i_1 j} > F_{i_2 j}$.*
- *If $T_{i_1 j} = T_{i_2 j}$ then $F_{i_1 j} = F_{i_2 j}$.*

[Lemma 3.4](#) states that if two strategies i_1 and i_2 generate the same repeated path when playing against j , then the strategy with the longest breakup period is “better” against j (because it suffers separation costs less often). This result implies a vulnerability for strategies which break up in the equilibrium path, and leads to the following proposition.

Proposition 1. *With costly endogenous separation, a neutrally stable strategy never breaks up in the equilibrium path.*

Note that [proposition 1](#) does not hold for games without separation costs ($c = 0$), which admit neutrally stable strategies with finite breakup period (Izquierdo and Izquierdo, 2024).

⁶ $a_{ij}^{[1, t]}$ represents some sequence of t action profiles, while $a_{ij}^{[1, t]}$ represents the first t action profiles generated by strategy i when playing against strategy j , assuming they do not break up before stage t .

4 Path-protecting strategies and deviator-leaving strategies

4.1 Introduction

In this section we develop the main contribution of the paper. Our goal is to study stable equilibria in games with costly endogenous separation, focusing on *path-protected* equilibria supported by some *path-protecting strategy*. These strategies ensure that any unilateral deviation from the equilibrium path yields a strictly lower payoff for the deviator. Path-protecting strategies are neutrally stable in games with costless separation (Izquierdo and Izquierdo, 2024); here we show that this result extends to games with costly separation. Furthermore, increasing separation costs expands the range of path-protecting strategies.

We also define *deviator-leaving strategies*, which are strategies that stay with their partner if and only if their partner mirrors their actions. We show that, for sufficiently large separation costs, every deviator-leaving strategy becomes path-protecting, which implies that any infinite series of symmetric outcomes can be sustained in equilibrium. Finally, we illustrate our results by applying them to the family of trust-building strategies, highlighting how the presence of costs affects their ability to sustain cooperation.

4.2 Properties

A strategy i is path-protecting (Izquierdo and Izquierdo, 2024) if, when playing against i (i.e., against a monomorphic population $\mathbf{e}_i$ of i -players), any strategy j that deviates from i 's choices obtains a strictly lower payoff than by not deviating. Formally:

Definition 3 (Path-protecting strategy). *A strategy $i \in \Omega$ is path-protecting if for all $j \in \Omega$:*

$$a_{jj}^{[1, T_{jj}]} \neq a_{ii}^{[1, T_{ii}]} \implies F_{ji} < F_{ii}.$$

For the case with no separation costs ($c = 0$), it has been shown (Izquierdo and Izquierdo, 2024) that a path-protecting strategy i must have an infinite i - i path and, also, that path-protecting strategies are neutrally stable. These results can be extended to G^{CES} , as we show next.

Lemma 4.1. *In games with and without separation costs, if i is path-protecting then $T_{ii} = \infty$.*

The proof of [lemma 4.1](#) considers that, if T_{ii} is finite, then one can find alternative strategies $j \neq i$ such that the outcomes in path $a_{jj}^{[1, T_{jj}]}$ are the same as the outcomes in path $a_{ii}^{[1, T_{ii}]}$ up to period T_{ii} , i.e. $a_{jj}^{[1, T_{ii}]} = a_{ii}^{[1, T_{ii}]}$, but $T_{jj} > T_{ii} = T_{ji}$, because i breaks an i - j partnership at period T_{ii} , but j does not. Such strategies satisfy $F_{ji} = F_{ii}$, so we have strategies j that generate a path $a_{jj}^{[1, T_{jj}]} \neq a_{ii}^{[1, T_{ii}]}$, but obtain the same payoff against i as i does, i.e. $F_{ji} = F_{ii}$.

Proposition 2. *In games with and without separation costs, path-protecting strategies are neutrally stable.*

The proof of [proposition 2](#) shows that, if j and k are alternative best responses to i , then they cannot obtain a larger payoff (than i against them) when playing against each other.

If the stage game has some strict Nash symmetric action profile, one can obtain path-protecting strategies from such actions: any strategy i whose path $a_{ii}^{[1,\infty]}$ is an infinite repetition of the same strict Nash profile of the stage game, and which breaks the partnership if its partner deviates from such path, is path-protecting.

Our next proposition shows that the property of being path-protecting is not lost when separation costs increase.

Proposition 3. *If a strategy is path-protecting for some separation cost $c_1 \geq 0$ then it is path-protecting for every separation cost $c \geq c_1$.*

It follows from [proposition 3](#) that if a strategy is path-protecting in a game with endogenous separation and no cost ($c = 0$), then it is path-protecting with costs. Consequently, the following result, which is adapted from Izquierdo and Izquierdo (2024) for $c = 0$, holds for G^{CES} with $c > 0$. The result considers *deviator-leaving* strategies, which are defined below.

Definition 4 (Deviator-leaving strategy). *A strategy $i \in \Omega$ is deviator-leaving if it has an infinite path $a_{ii}^{[1,\infty]}$ and breaks a partnership if its partner deviates from i 's own action.*

Proposition 4. *Let $\Phi = (\Phi^{[t]})_{t=1}^T$ be any finite sequence of T symmetric action profiles with average stage payoff $\bar{U}_\Phi = \frac{\sum_{t=1}^T U(\Phi^{[t]})}{T}$ strictly greater than the pure minmax payoff of the stage game. For large enough continuation probability $\delta < 1$, there are deviator-leaving path-protecting strategies whose equilibrium path, after a finite transient phase in which a minmax profile is played, is an infinite repetition of the sequence Φ .*

As an example, in a Prisoner's Dilemma, and in a Hawk-dove game, the minmax profile is DD ,⁷ and the pure minmax payoff is U_{DD} . The deviation-detering or minmax phase in [proposition 4](#) is a T_m -long series of DD action profiles. For the pattern-playing phase, the infinitely repeated finite pattern Φ_p can be any finite sequence of DD and CC action profiles with at least one CC in the sequence, which guarantees an average stage payoff $\bar{U}_{\Phi_p} > U_{DD}$.

Our next proposition shows that, for large enough separation costs, every deviator-leaving strategy is path-protecting and, consequently, any infinite series of symmetric outcomes can be sustained in equilibrium.

Proposition 5. *Given a fixed continuation probability $\delta \in (0, 1)$, for a sufficiently large separation cost c , any infinite sequence of (symmetric) outcomes is the equilibrium path of a path-protecting strategy.*

Specifically, for $c > \frac{\mathfrak{B}-\mathfrak{b}}{\delta(1-\delta)}$, where $\mathfrak{B}$ is the maximum stage game payoff and $\mathfrak{b}$ is the minimum symmetric stage game payoff, every deviator-leaving strategy is path-protecting (independently of the generated path $a_{ii}^{[1,\infty]}$).

⁷For compactness, here we represent action profiles (D, D) as DD .

There are several differences between the standard folk theorem and the results of [proposition 5](#). These differences show that a game with very large separation costs is not equivalent to a game in which separation is not allowed. The standard folk theorem states that, for a sufficiently large continuation probability, any individually rational sequence of outcomes (i.e., such that players obtain at least their stage-game minmax payoff) can be sustained in equilibrium. In contrast, [proposition 5](#), using a stronger notion of equilibrium, allows any continuation probability in $(0, 1)$ and any payoffs (corresponding to symmetric outcomes) even if they are below the stage-game minmax payoff. The threat of breaking up a partnership inflicting a large damage (even if also suffered by the leaver) can stabilize even series of outcomes that are not individually rational in the standard framework.

Example 1. *If the stage game is a Prisoner's Dilemma or a Hawk-Dove, for $c > \frac{U_{DC} - U_{DD}}{\delta(1-\delta)}$, every deviator-leaving strategy is path-protecting.*

To check whether a deviator-leaving strategy i is path-protecting (whether every deviation is harmful) one has to consider an infinite number of possible deviations along the i - i path $a_{ii}^{[1, \infty]}$. However, if the path $a_{ii}^{[1, \infty]}$ of the deviator-leaving strategy ends up repeating some finite sequence of outcomes, or pattern Φ_p , our next lemma allows to reduce considerably (in some cases to just one) the number of periods that must be checked to ensure that all possible deviations are harmful. Specifically, it is enough to consider deviations until the pattern is played once.

In preparation of the lemma, let Φ_0 , Φ_p and Φ_1 represent finite sequences of action profiles with respective length $T_0 \geq 0$, $T_p \geq 1$ and $T_1 \geq 1$. Let $(\Phi)^k$, where k is a non-negative integer, represent k repetitions of the action profiles in Φ , with $(\Phi)^0$ being empty, and let $(\Phi)^\infty$ represent an infinite repetition.

Lemma 4.2. *Let i and j be two (not necessarily different) strategies with infinite path $a_{ji}^{[1, \infty]} = (\Phi_0, (\Phi_p)^\infty)$, where Φ_0 may be empty. If j_1 and j_2 are strategies such that*

$$a_{j_1 i}^{[1, T_{j_1 i}]} = (\Phi_0, \Phi_1) \text{ and}$$

$$a_{j_2 i}^{[1, T_{j_2 i}]} = (\Phi_0, (\Phi_p)^k, \Phi_1) \text{ for some non-negative integer } k,$$

then

$$F_{j_1 i} < F_{ji} \iff F_{j_2 i} < F_{ji}$$

and

$$F_{j_1 i} > F_{ji} \iff F_{j_2 i} > F_{ji}.$$

As an example, if the stage game is the Prisoner's Dilemma or the Hawk-Dove, we saw in [section 3](#) that the condition $c \geq \frac{U_{DC} - U_{CC}}{\delta}$ is necessary for a nice strategy (a strategy that begins a partnership by playing C) to be Nash. Consider a deviator-leaving strategy i with fully cooperative path $a_{ii}^{[1, \infty]} = (CC)^\infty$. Its payoff against itself is $F_{ii} = U_{CC}$. To check whether i is Nash or path-protecting, applying [lemma 4.2](#), it is enough to consider a deviation at the first stage. The payoff (against i) to a strategy j that plays D at the first stage is $F_{ji} = U_{DC} - \delta c$. Consequently:

- For $c < \frac{U_{DC}-U_{CC}}{\delta}$ there are no fully cooperative equilibria (lemma 3.3).
- For $c = \frac{U_{DC}-U_{CC}}{\delta}$ there are fully cooperative Nash equilibria (lemma 4.2).
- For $c > \frac{U_{DC}-U_{CC}}{\delta}$ there are fully cooperative path-protected equilibria. For instance, the strategy that always plays C and breaks a partnership if its partner deviates from playing C is path-protecting (lemma 4.2).

4.3 Application: Trust-building strategies with costly separation

To gain insight into the effect of the cost c on the existence and stability of equilibria, let us consider the family of trust-building strategies (Fujiwara-Greve and Okuno-Fujiwara, 2009), which were initially defined for the Prisoner's Dilemma but can also be extended to the Hawk-Dove. A k -period trust-building strategy is a deviator-leaving strategy i with path $a_{ii}^{[1,\infty]} = (DD)^k(CC)^\infty$, for some non-negative integer k . The trust-building phase is $(DD)^k$ and the cooperation phase is $(CC)^\infty$. For $k = 0$ we actually have a fully cooperative strategy (when playing against itself), with no trust-building phase. The payoff to a k -period trust-building strategy i against itself is

$$F_{ii} = (1 - \delta^k)U_{DD} + \delta^k U_{CC} > U_{DD}$$

A strategy j that deviates during the trust-building phase obtains a payoff $F_{ji} < U_{DD} < F_{ii}$ (since j obtains payoffs U_{DD} and $U_{CD} < U_{DD}$ only). To ensure that deviations during the cooperation phase are harmful, given lemma 4.2, it is enough to study a strategy j that deviates to playing D the first time the trust-building strategy plays C . The payoff to such a strategy is

$$F_{ji} = \frac{(1 - \delta^k)U_{DD} + (1 - \delta)\delta^k U_{DC} - (1 - \delta)\delta^{k+1}c}{1 - \delta^{k+1}}$$

Consequently, a trust-building strategy is path-protecting if

$$(1 - \delta)\delta c + (1 - \delta^{k+1})U_{CC} > \delta(1 - \delta^k)U_{DD} + (1 - \delta)U_{DC} \quad (7)$$

On the other hand, any Nash strategy with path $(DD)^k(CC)^\infty$ must satisfy

$$(1 - \delta)\delta c + (1 - \delta^{k+1})U_{CC} \geq \delta(1 - \delta^k)U_{DD} + (1 - \delta)U_{DC} \quad (8)$$

because it needs to obtain at least as much as a strategy that at period $k+1$ deviates to playing D and leaves. So the strict inequality (7) is enough to have a path-protected equilibrium with path $(DD)^k(CC)^\infty$, and the non-strict inequality (8) is necessary to have a Nash equilibrium with that path.

We could also consider k -period trust-building grim strategies with path $a_{ii}^{[1,\infty]} = (DD)^k(CC)^\infty$ which punish deviations by perpetual defection without leaving. However, the following proposition shows that deviator-leaving strategies are more effective ensuring cooperation than grim strategies, since the former ones are path-protecting in cases where the latter ones are not.

Proposition 6. *Let i be a k -period trust-building grim strategy. Then, i is path-protecting if and only if both*

1. *Condition (7) holds, so the corresponding deviator-leaving strategy is path-protecting, and*
2. $\delta > \frac{U_{DC} - U_{CC}}{U_{DC} - U_{DD}}.$

Figure 2 shows the combinations of continuation probability δ and cost c needed for different k -period trust-building strategies to be path-protecting, with stage game payoffs $U_{CC} = 3, U_{DC} = 4, U_{DD} = 1$ and $U_{CD} < 3$. The stage game can be either a Prisoner's Dilemma (if $U_{CD} < 1$) or a Hawk-Dove (if $1 < U_{CD} < 3$). For values of c and δ below a k -line, the path generated by the corresponding k -period trust-building strategy cannot be achieved in equilibrium. For $k = 0$ (full cooperation), this line is $c = \frac{U_{DC} - U_{CC}}{\delta}$. In the grey region $\delta < \frac{U_{DC} - U_{CC}}{U_{DC} - U_{DD}}$ trust-building grim strategies cannot constitute an equilibrium.

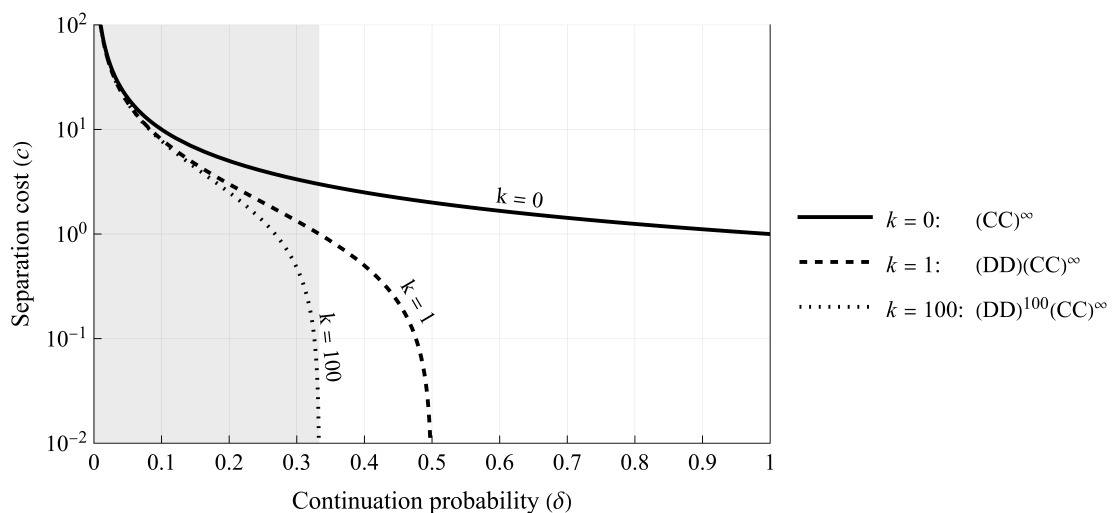


Figure 2: The k -lines separate the regions where path $(DD)^k(CC)^\infty$ can be a (path-protected) equilibrium path (above the line) from the region where it cannot be an equilibrium path (below the line). The gray zone $\delta < \frac{U_{DC} - U_{CC}}{U_{DC} - U_{DD}}$ represents a region where trust-building grim strategies cannot be Nash. Stage game payoffs $U_{CC} = 3, U_{DC} = 4, U_{DD} = 1, U_{CD} < 3$. The stage game can be a Prisoner's Dilemma or a Hawk-Dove.

5 Separation costs due to delayed re-matching

In some contexts it seems natural to assume that, after a breakup, partners remain unmatched looking for a partner, for some number of periods, until they find a new partner. If the stage game payoffs are non-negative (e.g., if payoffs represent Darwinian fitness) and players receive no payoff while they remain single, this delayed re-matching constitutes an alternative separation cost.

Let us assume that, after a breakup, partners remain single, without playing the stage game, for w periods during which they receive no payoff. Normalizing the mass of the matching pool to 1 and assuming stationarity, the mass of i -players who are matched (each period) to a j -player is $p_i p_j$, and the mass of single i -players who are to be matched at some point during the following w periods to a j -player is $p_i p_j w$. The total mass of i -players in a population $\mathbf{x}$ is $\sum_{j \in \mathbb{S}(\mathbf{x})} p_i p_j (w + 1 + \delta + \dots + \delta^{T_{kj}-1})$, and the payoff to an i -player in a population of j -players is then:

$$F_{ij}^w \equiv \frac{\sum_{t=1}^{T_{ij}} \delta^{t-1} U_{ij}^{[t]}}{w + \frac{1-\delta^{T_{ij}}}{1-\delta}} \quad (9)$$

A direct adaptation of [lemma 4.2](#) can be shown to hold for F_{ij}^w . Consequently, if we consider a deviator-leaving strategy i whose (infinite) path $a_{ii}^{[1,\infty]}$ ends up repeating some sequence of outcomes, then, in order to check that all deviations from the path are harmful, it is enough to check a small number of deviations.

Focusing on the Prisoner's Dilemma and the Hawk-Dove, we can study the conditions to have a fully cooperative path-protected equilibrium in this framework. Let i be a deviator-leaving strategy with path $a_{ii}^{[1,\infty]} = (CC)^\infty$. We have

$$F_{ii}^w = \frac{U_{CC}}{(1-\delta)w + 1}$$

To ensure that all deviations are harmful, given [lemma 4.2](#), it is enough to consider a strategy j that deviates to playing D on the first period of a partnership with i . The payoff to such a strategy is

$$F_{ji}^w = \frac{U_{DC}}{w + 1}$$

Consequently, strategy i is path-protecting if $F_{ji}^w < F_{ii}^w$, which is the case if and only if:

$$\delta > 1 - \frac{U_{CC}}{U_{DC}} \text{ and } w > \frac{U_{DC} - U_{CC}}{U_{CC} - (1-\delta)U_{DC}} \quad (10)$$

Otherwise, it is also easy to check that if $F_{ji}^w > F_{ii}^w$ then there are no fully cooperative equilibria (for this, consider a strategy that plays D and leaves).

Using the same method as in [section 4.3](#) for k -period trust-building strategies, we find that the region where they constitute equilibria is given by

$$\frac{(1-\delta^k)U_{DD} + (1-\delta)\delta^k U_{DC}}{(1-\delta)w + 1 - \delta^{k+1}} < \frac{(1-\delta^k)U_{DD} + \delta^k U_{CC}}{(1-\delta)w + 1}.$$

This region is shown in [Figure 3](#) for some specific payoff values of the stage game. For the case considered in [Figure 3](#), a searching period $w = 1$ is enough for full cooperation to be neutrally stable when $\delta > \frac{1}{2}$. In contrast, full cooperation cannot be sustained in equilibrium in the costless case ($w = 0$). This is consistent with the results by [Enquist](#)

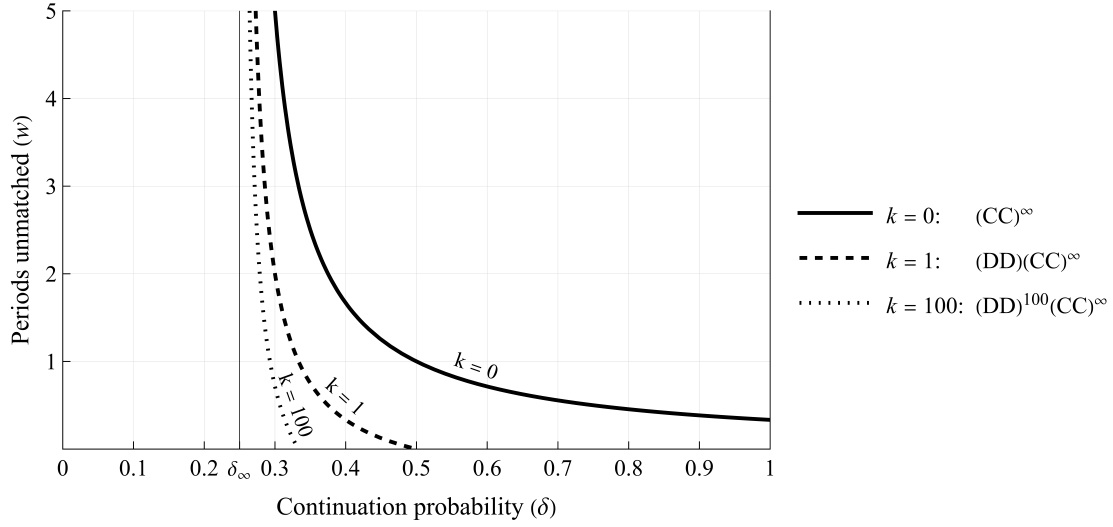


Figure 3: The k -lines separate the regions where path $(DD)^k(CC)^\infty$ can be a (path-protected) equilibrium path (above the line) from the region where it cannot be an equilibrium path (below the line). Stage game payoffs $U_{CC} = 3, U_{DC} = 4, U_{DD} = 1, U_{CD} < 3$. The stage game can be a Prisoner's Dilemma or a Hawk-Dove. The vertical line corresponding to $\delta_\infty = 1 - \frac{U_{CC}}{U_{DC}}$ represents the asymptote of every k -line.

and Leimar (1993) who, using a related model, show that increasing the search time for new partners helps to stabilize full cooperation.

Although delayed re-matching constitutes a cost for the separating players, it is inherently different from the direct separation cost c studied previously. Cooperation may be sustained for every δ if c is large enough, but only if δ is greater than a threshold δ_∞ under delayed re-matching. For values of $\delta < \delta_\infty$, no trust-building strategy can constitute an equilibrium regardless of the value of w (see figure 3).

A direct separation cost c can also be introduced in the delayed re-matching case. Figure 4 shows that full cooperation can be sustained in equilibrium for costs $c < U_{DC} - U_{CC}$ if searching period w is large enough. Therefore, the separation cost must no longer be greater than the payoff advantage of unilateral defection. In Figure 4, a small direct cost ($c = 0.625$) combined with a minimal searching period ($w = 1$) allows full cooperation to be sustained in scenarios with low continuation probability ($\delta > 0.4$). The inclusion of c lowers the threshold δ_∞ and the k -lines in Figure 4 with respect to figure 3.

6 Conclusions

This paper studies the effect of separation costs in games with endogenous separation. It focuses on the characterization of strategies that can support a path-protected equilibrium, i.e., an equilibrium such that any strategy which deviates from the equilibrium path obtains a strictly lower payoff than by not deviating. Path-protected equilibria are

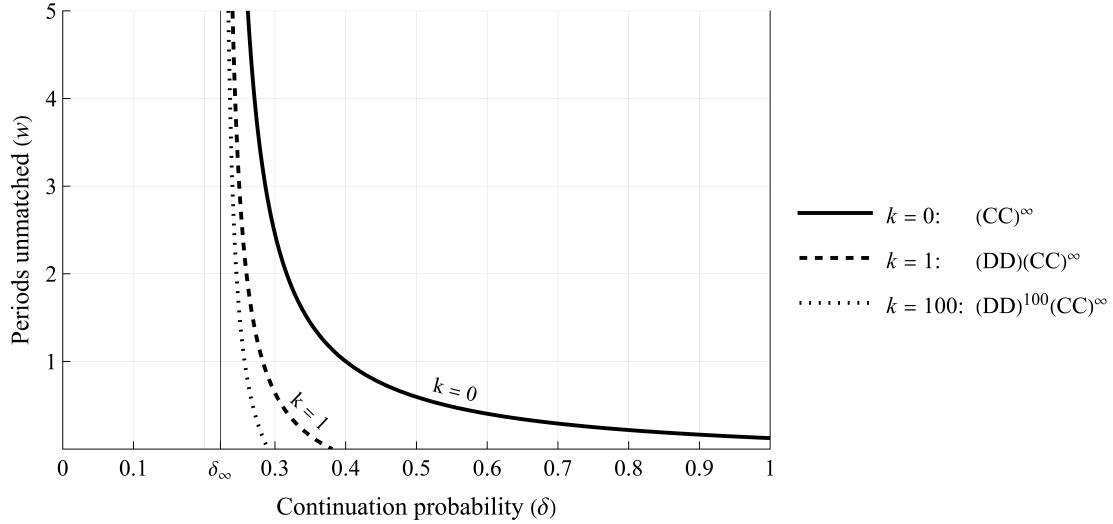


Figure 4: A separation cost $c = 0.625$ is considered. The k -lines separate the regions where path $(DD)^k(CC)^\infty$ can be a (path-protected) equilibrium path (above the line) from the region where it cannot be an equilibrium path (below the line). Stage game payoffs are $U_{CC} = 3$, $U_{DC} = 4$, $U_{DD} = 1$, and $U_{CD} < 3$. The stage game can be a Prisoner's Dilemma or a Hawk-Dove. The vertical line corresponding to $\delta_\infty = \frac{1}{2c} \left(c + U_{DC} - \sqrt{c^2 + 2c(2U_{CC} - U_{DC}) + U_{DC}^2} \right)$ represents the asymptote of every k -line.

neutrally stable.

We show that separation costs facilitate the existence of path-protected equilibria: if a strategy is path-protecting for a given separation cost, it is also path-protecting for any larger cost. A larger separation cost can also reduce the minimum continuation probability δ required for a strategy to be path-protecting.

We find that an important family of strategies is the set of deviator-leaving strategies: strategies which leave partners who deviate from the equilibrium path or convention, and stay otherwise. When considering the effects of separation costs, one might argue that deviator-leaving strategies should be harmed by large separation costs, as they suffer a large cost if separating. However, we show that, for large enough separation cost, every deviator-leaving strategy is actually path-protecting: the threat of separation acts as a strong deterrent against deviations from the path, but there is no actual separation in the equilibrium path.

For the special case of the Prisoner's Dilemma with endogenous separation, it is well known that full cooperation cannot be sustained in equilibrium with no separation cost, because a defector who leaves after its first interaction would exploit a population of initial cooperators. In contrast, with separation costs, we show that full cooperation can be sustained as a path-protected equilibrium. However, this requires an arguably large separation cost – greater than the stage payoff advantage of unilateral defection (temptation) over mutual cooperation (reward)–, and under this condition many other paths can also be sustained in equilibrium. The direct cost of separation needed to

support full cooperation can be eliminated, or considerably reduced, if combined with costs due to delayed re-matching.

Acknowledgements

Financial support from the Spanish State Research Agency (PID2024-159461NB-I00/MICIU and PID2020-118906GB-I00/MCIN, AEI/10.13039/501100011033/EU-FEDER) and from the Regional Government of Castilla y León with the EU-FEDER program (CLU-2019-04 - BIOECOUVA Unit of Excellence of the University of Valladolid) is gratefully acknowledged.

A Proofs

Proof of lemma 3.1. Consider a strategy j that, when playing against i , behaves like i up to period $T_{ii} - 1$ and at period T_{ii} chooses a best response action (of the stage game) to the action chosen by i and leaves. If the last action profile $a_{ii}^{[T_{ii}]}$ is not Nash, then $F_{ji} > F_{ii}$, so i is not Nash. $\square$

Proof of lemma 3.2. When playing against itself, a strategy i as in the lemma repeats the Nash profile (a, a) for T times before endogenous breakup. For every strategy j , we have $T_{ji} \leq T$. When playing against i , any strategy j that does not choose action a cannot obtain a larger stage game payoff $U_{ji}^{[t]}$ than by choosing a at any stage before T , and, considering (6), if $T_{ji} < T$, then $F_{ji} < F_{ii}$. And if $T_{ji} = T$, then $F_{ji} \leq F_{ii}$. $\square$

Proof of lemma 3.3. Let i be a strategy whose first action is $a^\emptyset$ and let j be a strategy whose first action is a (stage game) best response to $a^\emptyset$ and which leaves i after the first stage (so $T_{ji} = 1$). From (6), we have $F_{ji} = U^{BR}(a^\emptyset) - \delta c$ and $F_{ii} \leq M$. The Nash condition $F_{ji} \leq F_{ii}$ leads to $U^{BR}(a^\emptyset) - \delta c \leq M$. $\square$

Proof of lemma 3.4. For a sequence of T action profiles $a^{[1,T]} = (a^{[1]}, a^{[2]}, \dots, a^{[T]})$, where $a^{[t]} \in A^2$ is the t^{th} action profile in the sequence, let the *normalized discounted value* $V(a^{[1,T]})$ be

$$V(a^{[1,T]}) \equiv \frac{1 - \delta}{1 - \delta^T} \sum_{t=1}^T \delta^{t-1} U(a^{[t]}). \quad (11)$$

From (6), and considering that $V(a_{ij}^{[1,T_{ij}]}) = V(h_{ij}^{[\infty]})$, we have:

$$F_{ij} = V\left(h_{ij}^{[\infty]}\right) - \frac{1 - \delta}{1 - \delta^{T_{ij}}} \delta^{T_{ij}} c \quad (12)$$

The last term in (12) is negative and monotonic increasing with T_{ij} , which proves the result. $\square$

Proof of proposition 1. By contradiction. Suppose that i is neutrally stable and $T_{ii} < \infty$. Then there is a strategy j with $a_{jj}^{[1, T_{jj}]} = (a_{ii}^{[1, T_{ii}]}, a_{ii}^{[1, T_{ii}]})$ such that $h_{jj}^{[\infty]} = h_{ij}^{[\infty]} = h_{ii}^{[\infty]}$, $T_{ij} = T_{ii}$ and $T_{jj} = 2T_{ii}$, i.e., a j - j partnership behaves like an i - i partnership up until period T_{ii} and then restarts the same sequence of outcomes again without breaking the partnership until $T_{jj} = 2T_{ii}$. By lemma 3.4, we obtain $F_{ji} = F_{ii}$ and $F_{ij} < F_{jj}$, so i is not neutrally stable. $\square$

Proof of lemma 4.1. if T_{ii} is finite, then a strategy j that behaves like i (when playing with i) until period T_{ii} but does not choose to leave i at that period obtains $F_{ji} = F_{ii}$, with $T_{jj} > T_{ii}$, so i is not path-protecting. $\square$

Proof of proposition 2. If $\mathbf{y}$ is a best response to strategy i (equivalently, to state $\mathbf{e}_i$) then every strategy in the support of $\mathbf{y}$ is a best response to i . Suppose that i is path-protecting. Then any two strategies j and k that are best response to i generate the same history $a_{jk}^{[1, \infty]} = a_{ii}^{[1, \infty]}$ when playing in partnerships, because they are never the first to deviate from the path. So if i is path-protecting then i is Nash (this is immediate from the definition) and the following implication holds: $E(\mathbf{y}, \mathbf{e}_i) = F_{ii} \implies E(\mathbf{e}_i, \mathbf{y}) = F_{ii} = E(\mathbf{y}, \mathbf{y})$. $\square$

Proof of proposition 3. Let F_{ij}^c represent the payoff F_{ij} for separation cost c . If i is path-protecting for separation cost $c_1 < c_2$, then, from (12), $F_{ii}^{c_2} = F_{ii}^{c_1}$ (because $T_{ii} = \infty$) and $F_{ji}^{c_2} \leq F_{ji}^{c_1}$ for every $j \in \Omega$. $\square$

Proof of proposition 5. Let $a^{[1, \infty]}$ be an arbitrary path of symmetric outcomes and let i be a strategy with path $a_{ii}^{[1, \infty]} = a^{[1, \infty]}$ which breaks the partnership if its partner deviates from the path. Let j be a strategy that, when playing with i , deviates from i 's choice at period T (so $T_{ij} = T$). Let $\mathfrak{B}$ be the maximum stage game payoff and let $\mathfrak{b}$ be the minimum symmetric stage game payoff, $\mathfrak{b} = \min_{a \in A} U(a, a)$. From (6) we have

$$F_{ii} = (1 - \delta) \left[\sum_{t=1}^{T-1} \delta^{t-1} U_{ii}^{[t]} + \delta^{T-1} U_{ii}^{[T]} + \delta^T \sum_{t=1}^{\infty} \delta^{t-1} U_{ii}^{[t+T]} \right]$$

and

$$F_{ji} = (1 - \delta) \left[\sum_{t=1}^{T-1} \delta^{t-1} U_{ji}^{[t]} + \delta^{T-1} U_{ji}^{[T]} - \delta^T c \right] + \delta^T F_{ji}$$

so

$$F_{ii} \geq (1 - \delta) \left[\sum_{t=1}^{T-1} \delta^{t-1} U_{ii}^{[t]} + \delta^{T-1} \mathfrak{b} \right] + \delta^T \mathfrak{b}$$

$$F_{ji} \leq (1 - \delta) \left[\sum_{t=1}^{T-1} \delta^{t-1} U_{ji}^{[t]} + \delta^{T-1} \mathfrak{B} - \delta^T c \right] + \delta^T \mathfrak{B}$$

and

$$F_{ii} - F_{ji} \geq \delta^T \left[(1 - \delta)c - \frac{\mathfrak{B} - \mathfrak{b}}{\delta} \right]$$

Consequently, for $c > \frac{\mathfrak{B}-\mathfrak{b}}{\delta(1-\delta)}$ we have $F_{ii} > F_{ji}$, so i is path-protecting. $\square$

Proof of lemma 4.2. For any sequence Φ of length $T \geq 1$, let $V(\Phi)$ be defined as in (11):

$$V(\Phi) = \frac{1-\delta}{1-\delta^T} \sum_{t=1}^T \delta^{t-1} U(\Phi^{[t]}),$$

and define

$$V_c(\Phi) = \frac{1-\delta}{1-\delta^T} \left[\sum_{t=1}^T \delta^{t-1} U(\Phi^{[t]}) - \delta^T c \right].$$

If Φ is empty (if its length is $T = 0$) let $V(\Phi) = 0$. Then

$$F_{ji} = (1 - \delta^{T_0})V(\Phi_0) + \delta^{T_0}V(\Phi_p), \text{ and}$$

$$F_{j_2i} = \frac{(1 - \delta^{T_0})V(\Phi_0) + \delta^{T_0}(1 - \delta^{kT_p})V(\Phi_p) + \delta^{T_0+kT_p}(1 - \delta^{T_1})V_c(\Phi_1)}{1 - \delta^{T_0+kT_p+T_1}}, \text{ so}$$

$$F_{j_2i} - F_{ji} = \frac{\delta^{T_0+kT_p}}{1 - \delta^{T_0+kT_p+T_1}} \left[\delta^{T_1}(1 - \delta^{T_0})V(\Phi_0) - (1 - \delta^{T_0+T_1})V(\Phi_p) + (1 - \delta^{T_1})V_c(\Phi_1) \right].$$

It is clear from this last expression that the sign of $F_{j_2i} - F_{ji}$ does not depend on k , which proves the result. $\square$

Proof of proposition 6. Let i be a k -period trust-building grim strategy with path $a_{ii}^{[1,\infty]} = (DD)^k(CC)^\infty$. The payoff to any strategy that deviates against i during the trust-building phase (either by playing C or leaving) is clearly lower than F_{ii} . Idem for strategies that play C after the trust-building phase and leave at period $T > k$. Additionally, the payoff to a strategy that plays D at period $T_1 \geq k+1$ and leaves at period $T_2 \geq T_1$ is bounded by that of a strategy that keeps playing D after T_1 and leaves at T_2 . Let j be one of the latter strategies, then

$$\begin{aligned} F_{ji} &= \frac{1-\delta}{1-\delta^{T_2}} \left[\sum_{t=1}^k \delta^{t-1} U_{DD} + \sum_{t=k+1}^{T_1-1} \delta^{t-1} U_{CC} + \delta^{T_1-1} U_{DC} + \sum_{t=T_1+1}^{T_2} \delta^{t-1} U_{DD} - \delta^{T_2} c \right] \\ &= \frac{-\delta^{T_2} (U_{DD} + c(1-\delta)) + \delta^{T_1-1} (U_{DC} - U_{CC} + \delta(U_{DD} - U_{DC}))}{1 - \delta^{T_2}} \\ &\quad + \frac{\delta^k (U_{CC} - U_{DD}) + U_{DD}}{1 - \delta^{T_2}}. \end{aligned}$$

This payoff can be differentiated with respect to T_2 to obtain

$$F'_{ji} = \frac{\delta^{T_2}}{(1 - \delta^{T_2})^2} A,$$

where A is constant in T_2 . Therefore, F_{ji} is either monotonically increasing or monotonically decreasing in T_2 , which means that it is enough to compare i with strategies j that deviate at period T_1 playing D and leaving, or deviate playing D ad infinitum since period T_1 .

For the first kind of strategies, lemma 4.2 leads to condition (7). For the second kind,

$$F_{ji} = \delta^{T_1-1}(U_{DC} - U_{CC} + \delta(U_{DD} - U_{DC})) + \delta^k(U_{CC} - U_{DD}) + U_{DD},$$

and $F_{ii} > F_{ji}$ is satisfied as long as $\delta > \frac{U_{DC}-U_{CC}}{U_{DC}-U_{DD}}$, since $F_{ii} = \delta^k(U_{CC} - U_{DD}) + U_{DD}$. Up to this point we have shown that the conditions in the proposition are sufficient for i to be path-protecting. As in order to be path-protecting the path needs to be protected against any path-deviator, including the strategies we have considered before, the conditions are also necessary. $\square$

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