



Mean Ergodic Multiplication Operators on Weighted Spaces of Continuous Functions

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Abstract. Multiplication operators on weighted Banach spaces and locally convex spaces of continuous functions have been thoroughly studied. In this note, we characterize when continuous multiplication operators on a weighted Banach space and on a weighted inductive limit of Banach spaces of continuous functions are power bounded, mean ergodic or uniformly mean ergodic. The behaviour of the operator on weighted inductive limits depends on the properties of the defining sequence of weights and it differs from the Banach space case.

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1. Introduction, Notation, and Motivation

Multiplication operators on weighted spaces of (vector valued) continuous functions have been investigated by Manhas [13, 14], Singh and Manhas [15, 19, 21], Oubbi [17, 18] and Klilou and Oubbi [9] among others. See also the book by Singh and Manhas [20]. However, it seems that power boundedness and mean ergodicity of multiplication operators on weighted Banach spaces and weighted inductive limits of spaces of continuous functions had not been investigated. The case of multiplication operators on weighted spaces of analytic functions on the complex unit disc was studied by Bonet and Ricker [8]. Many tools used in that paper, like the maximum modulus principle and certain properties of the Banach spaces of analytic functions, are not available in the case of continuous functions. Thus a different approach is needed. Our aim is to characterize when continuous multiplication operators on a weighted Banach space or on a weighted inductive limit of Banach spaces of continuous functions are power bounded, mean ergodic or uniformly mean ergodic. Our main results are Theorems 2.6 and 2.7 for operators on Banach spaces and Theorem 3.7 and Corollary 3.9 for operators on weighted inductive limits. Several examples show that the characterization of uniformly mean ergodic

operators for weighted inductive limits does not coincide with the Banach space case.

We refer the reader to [22] for undefined terminology about set topology. We fix the notation for weighted spaces of continuous functions. In the sequel X denotes a Hausdorff, normal, locally compact topological space. For example X might be an open subset of \mathbb{R}^n or a discrete space, i.e. and index set. The space of continuous functions from X to \mathbb{C} is denoted by $C(X)$ and the locally convex topology on $C(X)$ of pointwise convergence is denoted by τ_p . We say $v \in C(X)$ is a weight if it is strictly positive. A function $f \in C(X)$ vanishes at infinity if for every $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $|f(x)| < \varepsilon$ if $x \in X \setminus K$. The weighted Banach spaces of continuous functions are defined by

$$C_v := C_v(X) = \left\{ f \in C(X) : \|f\|_v := \sup_{x \in X} v(x) |f(x)| < \infty \right\}$$

$$C_v^0 := C_v^0(X) = \{f \in C(X) : vf \text{ vanishes at infinity}\}$$

equipped with the norm $\|\cdot\|_v$. In case X is a discrete space, then we use the notation $\ell_\infty(v) = C_v$ and $c_0(v) = C_v^0$ [7]. Clearly $C_v^0 \subset C_v$ and every continuous function with compact support on X is contained in C_v^0 . Note that for $v \equiv 1$, C_v coincides with the Banach space $CB(X)$ of bounded continuous functions endowed with the supremum norm $\|\cdot\|_\infty$, and C_v^0 is the space $C_0(X)$ of continuous functions vanishing at infinity, also endowed with the supremum norm. Given $\varphi \in C(X)$, the multiplication operator $M_\varphi \in \mathcal{L}(C(X))$ is defined by $M_\varphi : f \mapsto \varphi f$.

Our notation for functional analysis, locally convex spaces and inductive limits is standard. We refer the reader to [1, 16, 23]. All the locally convex spaces are assumed to be Hausdorff. The weak topology of a locally convex Hausdorff space E is denoted by $\sigma(E, E')$, where E' is the topological dual space of E . The space of continuous linear operators from E into itself is denoted by $\mathcal{L}(E)$. We write $\mathcal{L}_s(E)$ and $\mathcal{L}_b(E)$ to denote $\mathcal{L}(E)$ when it is equipped with its strong operator topology and with the topology of uniform convergence on bounded sets of E , respectively. The Cesàro means of an operator $T \in \mathcal{L}(E)$ are defined by

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^n T^m, \quad n \in \mathbb{N}. \quad (1.1)$$

The operator T is mean ergodic (resp. uniformly mean ergodic) if the sequence $\{T_{[n]}\}_{n=1}^\infty$ converges in $\mathcal{L}_s(X)$ (resp. in $\mathcal{L}_b(X)$). We say T is power bounded if $\{T^n\}_{n=1}^\infty$ is an equicontinuous set of $\mathcal{L}(X)$. The standard text for mean ergodic operators in Banach spaces is [10]; see also [23].

2. Multiplication Operators on Weighted Banach Spaces of Continuous Functions

In this section, the multiplication operator M_φ on C_v and C_v^0 is denoted by $T_\varphi := M_\varphi|_{C_v} : C_v(X) \longrightarrow C(X)$ and $S_\varphi := M_\varphi|_{C_v^0} : C_v^0(X) \longrightarrow C(X)$,

respectively. The following result is well known. More general results can be seen in [17, Theorem 4], [19, Theorem 2.1] and [21, Theorem 3.1] for example.

Proposition 2.1. (i) *Let v, w be weights. Then $M_\varphi : C_v \longrightarrow C_w$ is continuous if and only if $\frac{w}{v}\varphi$ is bounded if and only if $M_\varphi : C_v^0 \longrightarrow C_w^0$ is continuous. Moreover, $\|M_\varphi\| = \sup_{x \in X} \frac{w(x)}{v(x)} |\varphi(x)|$ in both cases.*
(ii) *$T_\varphi \in \mathcal{L}(C_v)$ if and only if $S_\varphi \in \mathcal{L}(C_v^0)$ if and only if φ is bounded. Moreover,*

$$\|T_\varphi\|_v = \|S_\varphi\|_v = \|\varphi\|_\infty.$$

Lemma 2.2. *Let E be a locally convex space containing the functions with compact support on X such that E is continuously included in $(C(X), \tau_p)$. Assume that $M_\varphi \in \mathcal{L}(E)$. If $\lim_n (M_\varphi^n f)(x)/n = 0$ for every $f \in E$, then $|\varphi(x)| \leq 1$. In particular, if M_φ is either power bounded or mean ergodic, then $\|\varphi\|_\infty \leq 1$.*

Proof. If M_φ is power bounded or mean ergodic, then $\lim_n (M_\varphi^n f)(x)/n = 0$ for every $f \in E$ and every $x \in X$. Given $x \in X$, select a continuous function g with compact support such that $g(x) = 1$. We have $0 = \lim_n |(M_\varphi^n g)(x)|/n = \lim_n |\varphi(x)|^n/n$. This yields $|\varphi(x)| \leq 1$. \square

Corollary 2.3. *Let $\varphi \in CB(X)$. Then, $T_\varphi \in \mathcal{L}(C_v)$ is power bounded if and only if $S_\varphi \in \mathcal{L}(C_v^0)$ is power bounded if and only if $\|\varphi\|_\infty \leq 1$.*

Remark 2.4. Let E be a locally convex space containing the functions with compact support on X such that E is continuously included in $(C(X), \tau_p)$ and assume that $M_\varphi \in \mathcal{L}(E)$. For each $f \in E$ and $n \in \mathbb{N}$ we have

$$((M_\varphi)_{[n]}f)(x) = \frac{f(x)}{n} \sum_{m=1}^n (\varphi(x))^m, \quad x \in X, \quad (2.1)$$

and

$$((M_\varphi)_{[n]}f)(x) = \frac{f(x)\varphi(x)}{n} \cdot \frac{1 - (\varphi(x))^n}{1 - \varphi(x)}, \quad x \in X \setminus \varphi^{-1}(1). \quad (2.2)$$

In the case $\|\varphi\|_\infty \leq 1$, these formulas imply the convergence of $(M_\varphi)_{[n]}f$ uniformly on the compact subsets of X to the (not necessarily continuous) function h_f defined by

$$h_f(x) = \begin{cases} f(x) & \text{if } \varphi(x) = 1 \\ 0 & \text{if } \varphi(x) \neq 1. \end{cases} \quad (2.3)$$

Proposition 2.5. *Let E be a locally convex space containing the functions with compact support on X such that E is continuously included in $(C(X), \tau_p)$. If $M_\varphi \in \mathcal{L}(E)$ and it is mean ergodic, then $\varphi^{-1}(1)$ is open.*

Proof. By Lemma 2.2, $\|\varphi\|_\infty \leq 1$. Now suppose $\varphi^{-1}(1)$ is not open, then there exists $x \in X$ such that $\varphi(x) = 1$ but it is not in the interior of $\varphi^{-1}(1)$. Therefore, for each $U \subset X$ open with $x \in U$, there exists $x_U \in U$ with $\varphi(x_U) \neq 1$. These x_U form a net converging to x . Select a continuous function g with compact support such that $g(x) = 1$. By assumption $g \in E$. Since

M_φ is mean ergodic, the function h_g defined in Remark 2.4 is continuous. However, $h_g(x) = 1$ and $h_g(x_U) = 0$, for every U , which is a contradiction, and $\varphi^{-1}(1)$ is open. \square

Theorem 2.6. *Let $\varphi \in CB(X)$. Then, $S_\varphi \in \mathcal{L}(C_v^0)$ is mean ergodic if and only if $\|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open.*

Proof. If S_φ is mean ergodic, the conclusion follows from Lemma 2.2 and Proposition 2.5. To prove the converse, assume that $\|\varphi\|_\infty \leq 1$ and $Z := \varphi^{-1}(1)$ is open. Hence, Z and $X \setminus Z$ are both open and closed and thus $C_v^0(X) = C_v^0(Z) \oplus C_v^0(X \setminus Z)$. Since the restriction of S_φ to the first space is the identity operator, we may assume $\varphi \neq 1$ in X . From the formula (2.2) it follows that $(v(S_\varphi)_{[n]}f)_n$ is pointwise convergent to 0 for each $f \in C_v^0$ (even for the compact topology). If \widehat{X} is the Alexandroff compactification of X , the isometry $C_v^0(X) \hookrightarrow C(\widehat{X})$, $f \mapsto vf$, together with Riesz's representation theorem yield that actually $((S_\varphi)_{[n]}f)_n$ is weakly convergent to 0 for every $f \in C_v^0$. Now, since S_φ is power bounded, it follows from the classical mean ergodic theorem (see the equivalence of (ii) and (iii) of Theorem 1.1 in [10, page 72]) that S_φ is mean ergodic. \square

Theorem 2.7. *Let $\varphi \in CB(X)$. The following assertions are equivalent:*

1. $T_\varphi \in \mathcal{L}(C_v)$ is uniformly mean ergodic.
2. $S_\varphi \in \mathcal{L}(C_v^0)$ is uniformly mean ergodic.
3. $\|\varphi\|_\infty \leq 1$, $\inf\{|\varphi(x) - 1| : x \in X \setminus \varphi^{-1}(1)\} > 0$ and $\varphi^{-1}(1)$ is open.
4. $T_\varphi \in \mathcal{L}(C_v)$ is mean ergodic.

Proof. Clearly (1) implies (2) and (4).

To see that (2) implies (3) suppose that S_φ is uniformly mean ergodic. Then, S_φ is mean ergodic and by Theorem 2.6, $\|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open. Thus, as in the proof of Theorem 2.6, we may assume $\varphi \neq 1$ in X . In that case it is easy to see that $I - S_\varphi$ is injective. Since S_φ is uniformly mean ergodic, we can apply a theorem by Lin [11] (see also [10, Theorem 2.1]) to conclude that $I - S_\varphi$ is also surjective. The continuous operator $I - S_\varphi$ is bijective if and only if $\frac{1}{1-\varphi}$ is bounded. This implies $\inf\{|\varphi(x) - 1| : x \in X\} > 0$.

We show that (3) implies (1). Since $\varphi^{-1}(1)$ is open, it is enough to consider the case $\varphi \neq 1$ in X . The assumption (3) and the formula (2.2) imply that there is $C > 0$ such that for each $f \in C_v$ and each $n \in \mathbb{N}$ we have $\|(T_\varphi)_{[n]}f\|_v \leq C\|f\|_v/n$. Thus, $\|(T_\varphi)_{[n]}\| \rightarrow 0$ as $n \rightarrow \infty$, and T_φ is uniformly mean ergodic.

Finally we prove that (4) implies (3). Since T_φ is mean ergodic, also S_φ is mean ergodic and then $\|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open. Again we can restrict ourselves to the case $\varphi \neq 1$ in X . Let βX be the Stone-Ćech compactification of X and denote by \widehat{g} the unique extension of a function $g \in CB(X)$ to βX . Consider the isomorphisms $C_v(X) \longrightarrow CB(X)$, $f \mapsto vf$ and $CB(X) \longrightarrow C(\beta X)$, $h \mapsto \widehat{h}$. Also note that, since βX is compact, $C(\beta X)$ coincides with the space of functions vanishing at infinity $C_0(\beta X)$. Thus we have an isomorphism between $C_v(X)$ and $C_0(\beta X)$. Apply Theorem 2.6 to $S_\varphi \in \mathcal{L}(C_0(\beta X))$ to find that $(\widehat{\varphi})^{-1}(1)$ is open in βX . Since we have assumed

that $\varphi \neq 1$ in X , we must have $(\widehat{\varphi})^{-1}(1) \subset \beta X \setminus X$. But $\beta X \setminus X$ has empty interior and thus $(\widehat{\varphi})^{-1}(1) = \emptyset$. From this follows that $\inf_{x \in X} |1 - \varphi(x)| > 0$. \square

Remark 2.8. If the space X is discrete, then $C_v = l_\infty(v)$ is a Grothendieck Banach space with the Dunford–Pettis property (briefly a GDP space). In this case a theorem of Lotz [12] implies that T_φ is mean ergodic if and only if it is uniformly mean ergodic. This argument cannot be used to prove Theorem 2.7 in full generality, since not all $C_v(X)$ spaces have the GDP property. It is well known that if K is a compact space then $C(K)$ is a Grothendieck space if and only if K is extremely disconnected if and only if c_0 is not complemented in $C(K)$.

It is now easy to exhibit examples of multiplication operators that are mean ergodic but not uniformly mean ergodic: Define $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ as $\varphi(2n) = 1$, $\varphi(2n+1) = (1 - 1/n)$. For every weight v on \mathbb{N} we have that $S_\varphi : c_0(v) \rightarrow c_0(v)$, $(a_n) \mapsto (\varphi(n)a_n)$ is mean ergodic by Theorem 2.6 but not uniformly mean ergodic by Theorem 2.7 and $T_\varphi : l_\infty(v) \rightarrow l_\infty(v)$, $(a_n) \mapsto (\varphi(n)a_n)$ is not mean ergodic by Theorem 2.7.

3. Multiplication Operators on Weighted Inductive Limits of Spaces of Continuous Functions

Weighted inductive limits of spaces of continuous and holomorphic functions have been thoroughly investigated since the seminal work of Bierstedt et al. [6, 7]; see [1–3, 5] and the references therein. In this section, we characterize power bounded and (uniformly) mean ergodic multiplication operators defined on weighted inductive limits of spaces of continuous functions defined on a normal locally compact Hausdorff topological space X .

Throughout this section $V = (v_k)_k$ denotes a decreasing sequence of continuous strictly positive weights. The weighted inductive limits associated to V are $VC = VC(X) := \text{ind}_k C_{v_k}$ and $V_0C = V_0C(X) := \text{ind}_k C_{v_k}^0$. They are Hausdorff (LB)-spaces and $V_0C \subset VC$ is a topological subspace of VC by [6, Theorem 1.3]. In case X is discrete, these spaces are precisely Köthe co-echelon spaces of order infinite and zero [7].

The Nachbin family associated to V is

$$\overline{V} := \{\overline{v} : X \longrightarrow (0, \infty) : \overline{v} \text{ is upper semicontinuous and for each } k \in \mathbb{N}, \frac{\overline{v}}{v_k} \text{ is bounded on } X\}.$$

The weighted spaces of continuous functions associated with VC and V_0C are defined as follows:

$$C\overline{V} = \left\{ f \in C(X) : p_{\overline{v}}(f) := \sup_{x \in X} \overline{v}(x) |f(x)| < \infty, \forall \overline{v} \in \overline{V} \right\},$$

$$C\overline{V}_0 = \{f \in C(X) : \overline{v}f \text{ vanishes at infinity, } \forall \overline{v} \in \overline{V}\}.$$

They are endowed with the locally convex topology generated by the seminorms $p_{\overline{v}}, \overline{v} \in \overline{V}$. It is well known that $VC = C\overline{V}$ algebraically and the topology in $C\overline{V}$ is in general coarser, but they share bounded sets; in fact

every bounded subset of $C\bar{V}$ is contained and bounded in a step C_{v_n} . Moreover, $C\bar{V}_0$ is a closed subspace of $C\bar{V}$ and V_0C is a topological subspace of $C\bar{V}_0$, hence of VC . We refer the reader to [6] for these results.

The characterization of continuous multiplication operators follows at once from Grothendieck's factorization theorem [16, Theorem 24.33] and Proposition 2.1.

Proposition 3.1. *The following assertions are equivalent:*

1. $M_\varphi : VC(X) \longrightarrow VC(X)$ is continuous.
2. For every k there exists l such that $M_\varphi : C_{v_k}(X) \longrightarrow C_{v_l}(X)$ is continuous.
3. For every k there exists l such that $\frac{v_l}{v_k}\varphi$ is bounded on X ,
4. $M_\varphi : V_0C(X) \longrightarrow V_0C(X)$ is continuous,
5. for every k there exists l such that $M_\varphi : C_{v_k}^0(X) \longrightarrow C_{v_l}^0(X)$ is continuous.

Proposition 3.2. $M_\varphi \in \mathcal{L}(VC)$ is power bounded if, and only if, $M_\varphi \in \mathcal{L}(V_0C)$ is power bounded, if and only if $\|\varphi\|_\infty \leq 1$.

Proof. The necessity of $\|\varphi\|_\infty \leq 1$ for both VC and V_0C follows from Lemma 2.2. For the sufficiency assume $\|\varphi\|_\infty \leq 1$. Since both V_0C and VC are barrelled spaces, by Banach Steinhaus theorem it is enough to show that every orbit $((M_\varphi)^n f)_n$ is bounded for each f in the space. Fix $f \in V_0C$. There exist k and $\lambda > 0$ such that $f \in \lambda B_k$, where B_k is the closed unit ball of C_{v_k} . This implies $(M_\varphi)^n f \in \lambda B_k \cap V_0C$ for each n . Now it is enough to keep in mind that $(kB_k \cap V_0C)_k$ is a fundamental system of bounded sets of V_0C by [6, Theorem 1.3]. The case of VC is similar. \square

Proposition 3.3. $M_\varphi \in \mathcal{L}(V_0C)$ is mean ergodic if and only if $\|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open.

Proof. If $M_\varphi \in \mathcal{L}(V_0C)$ is mean ergodic, then by Lemma 2.2 and Proposition 2.5, we get the necessary conditions.

Conversely, if $\|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open, then $S_\varphi \in \mathcal{L}(C_{v_k}^0)$ is mean ergodic for every $k \in \mathbb{N}$ by Theorem 2.6. By the properties of inductive limits, it follows that $M_\varphi \in \mathcal{L}(V_0C)$ is mean ergodic. \square

The next technical lemma was proved in [4, Proposition 3] and it is very useful in our setting.

Lemma 3.4. *There is a fundamental system \mathcal{U} of neighbourhoods of 0 for VC such that if $U \in \mathcal{U}$ and $f \in U$, then, for every $g \in VC$ with $|g| \leq |f|$, one gets $g \in U$.*

Proposition 3.5. *The following assertions are equivalent:*

1. $M_\varphi \in \mathcal{L}(VC)$ is mean ergodic.
2. $M_\varphi \in \mathcal{L}(VC)$ is uniformly mean ergodic.

Furthermore, if these equivalent conditions hold, then $\|\varphi\|_\infty \leq 1$, $\varphi^{-1}(1)$ is open, h_f as in (2.3) belongs to VC for each $f \in VC$, and $(M_\varphi)_{[n]} \rightarrow P$ in $\mathcal{L}_b(VC)$, where $P \in \mathcal{L}(VC)$ with $P(f) = h_f$.

Proof. It suffices to show that M_φ is uniformly mean ergodic whenever it is mean ergodic. Let B_k denote the closed unit ball of C_{v_k} . By [6] every bounded set of VC is contained in a multiple of some B_k . Since M_φ is mean ergodic, h_g belongs to VC for each $g \in VC$. Fix k . We have to show that $(M_\varphi)_{[n]}g - h_g$ converges to 0 uniformly on $g \in B_k$. Set $f_k := 1/v_k$. If $U \in \mathcal{U}$, where \mathcal{U} is the basis of neighbourhoods of VC in Lemma 3.4, there exists n_0 such that for $n \geq n_0$, $(M_\varphi)_{[n]}f_k - h_{f_k} \in U$, since M_φ is mean ergodic. Every $g \in B_k$ satisfies $|g| \leq f_k$, then a simple computation and Lemma 3.4 yields $(M_\varphi)_{[n]}g - h_g \in U$, for $n \geq n_0$.

The consequences of conditions (1) and (2) in the statement now follow from arguments in Lemma 2.2, Remark 2.4 and Proposition 2.5. \square

Although M_φ is mean ergodic if and only if it is uniformly mean ergodic on VC , these conditions do not necessarily imply $\inf\{|\varphi(x) - 1| : x \in X \setminus \varphi^{-1}(1)\} > 0$. Compare with Theorem 2.7. The first examples can be obtained assuming a condition on the sequence V .

The sequence V satisfies condition (S) if for each n there exists $m > n$ such that v_m/v_n vanishes at infinity. It is easy to see that condition (S) holds if and only if $VC = V_0C$. For example, the sequence $V = (v_n)_n$ of weights $v_n(z) := (1 - |z|)^n, z \in \mathbb{D}, n \in \mathbb{N}$, on the complex unit disc \mathbb{D} satisfies condition (S). We refer the reader to [6] for the relevance and more examples of condition (S). In the context of Köthe echelon and co-echelon spaces, condition (S) characterizes the property of being Schwartz [7]. The next corollary is a consequence of Propositions 3.3 and 3.5.

Corollary 3.6. *Assume that the sequence V satisfies condition (S). The following conditions are equivalent for $M_\varphi \in \mathcal{L}(VC)$:*

1. $M_\varphi \in \mathcal{L}(VC)$ is uniformly mean ergodic.
2. $M_\varphi \in \mathcal{L}(VC)$ is mean ergodic.
3. $\|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open.

Now we characterize when M_φ is uniformly mean ergodic on V_0C .

Theorem 3.7. *The following assertions are equivalent:*

1. $M_\varphi \in \mathcal{L}(V_0C)$ is uniformly mean ergodic.
2. $\|\varphi\|_\infty \leq 1$, $\varphi^{-1}(1)$ is open in X and for every $k \in \mathbb{N}$ and every $\bar{v} \in \bar{V}$,

$$\lim_{n \rightarrow \infty} \sup_{x \in Y} \frac{1}{n} \frac{\bar{v}(x)}{v_k(x)} |\varphi(x)| \frac{|1 - \varphi(x)^n|}{|1 - \varphi(x)|} = 0,$$

where $Y = X \setminus \varphi^{-1}(1)$.

Proof. It is enough to prove the result when $\varphi(x) \neq 1$ for each $x \in X$. Thus we assume $X = Y$. We set, for each $k, n \in \mathbb{N}$, $\bar{v} \in \bar{V}$ and $x \in X$,

$$R_n^{k, \bar{v}}(x) := \frac{1}{n} \frac{\bar{v}(x)}{v_k(x)} |\varphi(x)| \frac{|1 - \varphi(x)^n|}{|1 - \varphi(x)|}.$$

Assume first that condition (2) holds. Fix $k \in \mathbb{N}$. For each $f \in V_0C$ such that $|f| \leq 1/v_k$ and each $\bar{v} \in \bar{V}$ we have

$$\bar{v}(x) |(M_\varphi)_{[n]}f(x)| \leq R_n^{k, \bar{v}}(x)$$

for each $n \in \mathbb{N}$ and each $x \in X$. Given $\varepsilon > 0$ we apply (2) to find $n(0) \in \mathbb{N}$ such that $R_n^{k,\bar{v}}(x) < \varepsilon$ for each $x \in X$ and each $n \geq n(0)$. Hence $\bar{v}(x) |(M_\varphi)_{[n]}f(x)| < \varepsilon$ for each $x \in X, n \geq n(0)$ and each $f \in V_0C$ such that $|f| \leq 1/v_k$. By [6, Theorem 1.3] V_0C is a topological subspace of $C\bar{V}_0$ and every bounded set of V_0C is contained in a multiple of $\{f \in V_0C : |f| \leq 1/v_k\}$ for some k . Therefore $((M_\varphi)_{[n]})_n$ converges to 0 in $\mathcal{L}_b(V_0C)$, and M_φ is uniformly mean ergodic.

To prove the converse, suppose that $M_\varphi \in \mathcal{L}(V_0C)$ is uniformly mean ergodic. Fix $k \in \mathbb{N}$ and $\bar{v} \in \bar{V}$. Given $\varepsilon > 0$ there is $n(0) \in \mathbb{N}$ such that $\bar{v}(x) |(M_\varphi)_{[n]}f(x)| < \varepsilon$ for each $x \in X, n \geq n(0)$ and each $f \in V_0C$ such that $|f| \leq 1/v_k$. Now, for an arbitrary $z \in X$, there is a continuous function with compact support $h \in C(X)$ such that $0 \leq h \leq 1$ and $h(z) = 1$. Then $g := h/v_k \in V_0C$ and $|g| \leq 1/v_k$ on X . This implies, for $n \geq n(0)$,

$$R_n^{k,\bar{v}}(z) = \frac{\bar{v}(z)}{k} |(M_\varphi)_{[n]}g(z)| < \varepsilon.$$

Since $z \in X$ is arbitrary, we have shown that $\lim_{n \rightarrow \infty} \sup_{z \in X} R_n^{k,\bar{v}}(z) = 0$. The other statements in condition (2) follow from Proposition 3.3. \square

Corollary 3.8. *If $\|\varphi\|_\infty \leq 1$, $\varphi^{-1}(1)$ is open in X and for every $k \in \mathbb{N}$ there exists $l \geq k$ such that*

$$\lim_{n \rightarrow \infty} \sup_{x \in Y} \frac{1}{n} \frac{v_l(x)}{v_k(x)} |\varphi(x)| \frac{|1 - \varphi(x)^n|}{|1 - \varphi(x)|} = 0,$$

then $M_\varphi \in \mathcal{L}(V_0C)$ is uniformly mean ergodic.

Proof. This is a consequence of Theorem 3.7 since, for every $\bar{v} \in \bar{V}$ and each $l \in \mathbb{N}$, there exists $\alpha_l > 0$ such that $\bar{v} \leq \alpha_l v_l$. \square

Corollary 3.9. *Assume that $VC = C\bar{V}$ holds topologically. Then $M_\varphi \in \mathcal{L}(VC)$ is uniformly mean ergodic if and only if condition (2) of Theorem 3.7 holds.*

Proof. If $M_\varphi \in \mathcal{L}(VC)$ is uniformly mean ergodic, then $M_\varphi \in \mathcal{L}(V_0C)$ is uniformly mean ergodic, since V_0C is a topological subspace of VC . We can apply Theorem 3.7 to conclude that condition (2) in this theorem holds. Conversely, as $VC = C\bar{V}$ algebraically and topologically, it is not hard to prove that condition (2) in Theorem 3.7 implies that the sequence $((M_\varphi)_{[n]})_n$ converges in $\mathcal{L}_b(VC)$ and $M_\varphi \in \mathcal{L}(VC)$ is uniformly mean ergodic. \square

The topological identity $VC = C\bar{V}$ was characterized by a condition (D) formulated in terms of the weights $V = (v_n)_n$ by Bastin, Bierstedt, Bonet and Vogt. See [2, 3, 5].

Corollary 3.10. *Assume that $\|\varphi\|_\infty \leq 1$ and $\varphi(x) \neq 1$ for each $x \in X$.*

(i) If for all k there is $l > k$ such that

$$\sup_{x \in X} \frac{v_l(x)}{v_k(x)} \frac{1}{|1 - \varphi(x)|} < \infty,$$

then $M_\varphi \in \mathcal{L}(V_0C)$ is uniformly mean ergodic.

- (ii) If $\inf\{|1 - \varphi(x)| : x \in X\} > 0$, then $M_\varphi \in \mathcal{L}(V_0C)$ is uniformly mean ergodic.

Proof. The hypothesis of part (i) implies the assumption in Corollary 3.8. Part (ii) follows from (i). \square

Examples 3.11. We mention two examples of sequences $V = (v_n)_n$ which do not satisfy condition (S) and uniformly mean ergodic multiplication operators $M_\varphi \in L(V_0C)$ such that $\inf\{|1 - \varphi(x)| : x \in X \setminus \varphi^{-1}(1)\} > 0$ is not satisfied. These examples show that the converse of Corollary 3.10 does not hold and they should be compared with Theorem 2.7 and Corollary 3.6.

1. Let $X = \mathbb{D}$ be the complex unit disc. Consider $V = (v_k)_k$ with $v_k(z) = \min(1, |1 - z|^k)$. The sequence V does not satisfy condition (S), since $v_k(-1 + a) = 1$ for each $a \in]0, 1[$. Take $\varphi(z) = z$. Clearly $\varphi(z) \neq 1$ for each $z \in \mathbb{D}$, but $\inf\{|1 - \varphi(x)| : x \in X\} = 0$.

On the other hand, for each $k \in \mathbb{N}$ we have, for $|1 - z| \leq 1$,

$$\frac{v_{k+1}(z)}{v_k(z)} \cdot \frac{1}{|1 - \varphi(z)|} = \frac{|1 - \varphi(z)|^{k+1}}{|1 - \varphi(z)|^{k+1}} = 1$$

and for $|1 - z| \geq 1$,

$$\frac{v_{k+1}(z)}{v_k(z)} \cdot \frac{1}{|1 - \varphi(z)|} = \frac{1}{|1 - z|} \leq 1.$$

We can apply Corollary 3.10 (i) to conclude that M_φ is uniformly mean ergodic in V_0C .

2. Let $X = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the upper half plane in \mathbb{C} . Define $v_k(z) := \exp(-k \operatorname{Im} z)$, $z \in X$, $k \in \mathbb{N}$, and $V = (v_k)_k$. It is easy to check that the sequence V does not satisfy condition (S). The function $\varphi(z) = 1 - e^{iz}$, $z \in X$, satisfies $\varphi(z) \neq 1$ for each $z \in X$ and $\inf\{|1 - \varphi(x)| : x \in X\} = 0$.

We have

$$\frac{v_{k+1}(z)}{v_k(z)} \cdot \frac{1}{|1 - \varphi(z)|} = 1.$$

Therefore M_φ is uniformly mean ergodic in VC by Corollary 3.10 (i).

3. It is not hard to show that both sequences V in the examples above are regularly decreasing in the sense of [6, Definition 2.1]. This implies that $VC = C\overline{V}$ holds topologically and the multiplication operators M_φ are also uniformly mean ergodic on VC .

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