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Some results about diagonal operators on Köthe echelon spaces

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Abstract
Several questions about diagonal operators between Köthe echelon spaces are investigated: (1) The spectrum is characterized in terms of the Köthe matrices defining the spaces, (2) It is characterized when these operators are power bounded, mean ergodic or uniformly mean ergodic, and (3) A description of the topology in the space of diagonal operators induced by the strong topology on the space of all operators is given.

Keywords Echelon spaces · Diagonal operators · Mean ergodic operators · Power bounded operators

Mathematics Subject Classification 47B37 · 47A10 · 46A45

1 Introduction and notation

Diagonal operators between sequence spaces are defined on the space of all scalar sequences $\mathbb{C}^{\mathbb{N}}$ by $M_\varphi : \mathbb{C}^{\mathbb{N}} \longrightarrow C^{\mathbb{N}}$, $M_\varphi : (x_i)_i \mapsto (\varphi_i x_i)_i$, with $\varphi = (\varphi_i)_i$ a given sequence in $\mathbb{C}^{\mathbb{N}}$. If M_φ acts continuously from a space E into a space F, we say that φ is a multiplier from E to F. These operators have been investigated by many authors. We only mention here [3,4] and [10]. In the context of Köthe echelon spaces, diagonal operators were investigated by Crofts [9]. In this note we treat three aspects of diagonal operators between Köthe echelon spaces which, apparently had not been considered before: In Sect. 2 we describe the spectrum and the Waelbroeck spectrum of diagonal operators. Characterizations of power bounded and (uniformly) mean ergodic diagonal operators are given in Sect. 3. In the final Sect. 4 we describe the topology induced by the strong topology on the space of all operators into the subspace of diagonal operators; see Theorem 4.

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Our notation for Köthe echelon sequence spaces is as in [7] and [12]. We recall here the terminology needed below. A sequence $v = (v(i))_i \in \mathbb{C}^{\mathbb{N}}$ is called a weight if it is strictly positive. The weighted Banach spaces of sequences are defined by

$$\ell_p(v) := \{x = (x_i)_i \in \mathbb{C}^{\mathbb{N}} \ : \ p_v(x) := \|(v(i)x_i)_i\|_p < \infty\},\ 1 \le p \le \infty,$$
$$c_0(v) := \{x = (x_i)_i \in \mathbb{C}^{\mathbb{N}} \ : \ \lim_{i\to\infty} v(i)x_i = 0\},$$

where $\|\cdot\|_p$ denotes the usual ℓ_p norm. These spaces are Banach spaces with the corresponding norm p_v, and $c_0(v)$ is a Banach space with the norm of $\ell_\infty(v)$.

Now, given $A = (a_n)_n$, a Köthe matrix (i.e. a_n is a weight and $a_n(i) \le a_{n+1}(i)$ for all $i, n \in \mathbb{N}$), the echelon space of order $1 \le p \le \infty$ is defined by

$$\lambda_p(A) = \bigcap_{n\in\mathbb{N}} \ell_p(a_n) \quad \text{and} \quad \lambda_0(A) = \bigcap_{n\in\mathbb{N}} c_0(a_n),$$

endowed with the projective topologies $\lambda_p(A) := \mathrm{proj}_{n\in\mathbb{N}}\, \ell_p(a_n)$ and $\lambda_0(A) := \mathrm{proj}_{n\in\mathbb{N}}\, c_0(a_n)$. These spaces are Fréchet spaces with the topology defined by the corresponding seminorms $p_n := p_{a_n}$, with $n = 1, 2, \ldots$ Observe that we only consider in this paper Köthe echelon spaces with a continuous norm. It is easy to extend our results to the general case, since every Köthe echelon space is the countable product of Köthe echelon spaces with a continuous norm; see [7].

Our notation for functional analysis, locally convex spaces and inductive limits is standard. We refer the reader to [5,12] and [15]. All the locally convex spaces are assumed to be Hausdorff. The weak topology of a locally convex Hausdorff space E is denoted by $\sigma(E, E')$, where E' is the topological dual space of E. The space of continuous linear operators from E into other locally convex Hausdorff space F is denoted by $\mathcal{L}(E, F)$ and it is denoted by $\mathcal{L}(E)$ when $E = F$. We write $\mathcal{L}_s(E, F)$ and $\mathcal{L}_b(E, F)$ to denote $\mathcal{L}(E, F)$ when it is equipped with its strong operator topology and with the topology of uniform convergence on bounded sets of E, respectively.

If $T \in \mathcal{L}(E)$ and I is the identity operator on E, the *point spectrum* of T is the set $\sigma_{pt}(T, E)$ of all $\mu \in \mathbb{C}$ such that $T - \mu I$ is not injective and the *spectrum* is the set $\sigma(T, E)$ of all $\mu \in \mathbb{C}$ such that $T - \mu I$ is not invertible. The *resolvent* $\rho(T, E)$ is the complement of the spectrum in $\mathbb{C}$. It is well known that whenever E is a Banach space, the spectrum is a compact subset of $\mathbb{C}$. This is not, in general, the case for Fréchet spaces and this fact leads to the definition of the Waelbroeck spectrum [13, Chapter III, Sect. 3] which is defined as follows. A point $\lambda \in \mathbb{C}$ is in $\rho^*(T, E)$ if there exists $\delta > 0$ such that if $|\lambda - \mu| < \delta$, then $\mu \in \rho(T, E)$ and such that the set $\{(T - \mu I)^{-1} \ : \ |\lambda - \mu| < \delta\}$ is equicontinuous. The set $\rho^*(T, E)$ is open and its complement in $\mathbb{C}$ is the Waelbroeck spectrum $\sigma^*(T, E)$.

2 Spectrum of diagonal operators

First we characterize the multipliers between Köthe echelon spaces. The following result is well known and can be easily deduced from [8,9].

Lemma 1 *Let $A = (a_m)$ and $B = (b_n)$ be Köthe matrices. For $1 \le p \le \infty$ and $p = 0$, $M_\varphi : \lambda_p(A) \longrightarrow \lambda_p(B)$ is continuous if, and only if, for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that*

$$\sup_{i\in\mathbb{N}} \frac{b_n(i)|\varphi_i|}{a_m(i)} < \infty.$$

Lemma 2 *Let X be $\lambda_p(A)$ or ℓ_p for $1 \leq p \leq \infty$ or $p = 0$. Then,*

$$\sigma_{pt}(M_\varphi, X) = \{\varphi_i \ : \ i \in \mathbb{N}\}.$$

Proposition 1 *Let $1 \leq p \leq \infty$ or $p = 0$. Then the following assertions are equivalent:*

(1) $\mu \in \rho(M_\varphi, \lambda_p(A))$,
(2) *for each $n \in \mathbb{N}$, there exists $m \geq n$ such that*

$$\sup_{i \in \mathbb{N}} \frac{a_n(i)}{a_m(i)} \frac{1}{|\varphi_i - \mu|} < \infty.$$

Proof Set $\phi = \left(\frac{1}{\varphi_i - \mu}\right)_i$. Since M_ϕ is the inverse of $M_\varphi - \mu I$, whenever it exists, the equivalence of (1) and (2) is a consequence of Lemma 1. □

Corollary 1 *Let $1 \leq p \leq \infty$ or $p = 0$. Then,*

$$\overline{\sigma(M_\varphi, \lambda_p(A))} = \overline{\{\varphi_i \ : \ i \in \mathbb{N}\}}.$$

Proof Since $\{\varphi_i \ : \ i \in \mathbb{N}\} = \sigma_{pt}(M_\varphi, \lambda_p(A))$, we have

$$\overline{\{\varphi_i \ : \ i \in \mathbb{N}\}} \subset \overline{\sigma(M_\varphi, \lambda_p(A))}.$$

For the other inclusion let $\mu \in \mathbb{C} \setminus \overline{\{\varphi_i \ : \ i \in \mathbb{N}\}}$. Then there is $\delta > 0$ such that $|\mu - \varphi_i| > 2\delta$, for $i \in \mathbb{N}$. Proposition 1 yields $\mu \in \rho(M_\varphi, \lambda_p(A))$. Assume $\mu \in \overline{\sigma(M_\varphi, \lambda_p(A))}$. Then we must actually have $\mu \in \partial\sigma(M_\varphi, \lambda_p(A))$. Thus, there exists $\lambda \in \sigma(M_\varphi, \lambda_p(A))$ such that $|\mu - \lambda| < \delta$ and therefore, for every $i \in \mathbb{N}$, we have

$$|\lambda - \varphi_i| \geq |\mu - \varphi_i| - |\mu - \lambda| > 2\delta - \delta = \delta.$$

From this we deduce, again by Proposition 1, that $\lambda \in \rho(M_\varphi, \lambda_p(A))$, which is a contradiction. Hence, $\mu \notin \overline{\sigma(M_\varphi, \lambda_p(A))}$. □

Example 1 It is easy to give examples showing $\sigma(M_\varphi, \lambda_p(A)) \neq \overline{\sigma(M_\varphi, \lambda_p(A))}$. Let $1 \leq p \leq \infty$ or $p = 0$. Consider $A = (a_n)_n$ with $a_n(i) = i^n$ and take $\varphi_i = 1 - \frac{1}{i}$. We show that $1 \in \overline{\sigma(M_\varphi, \lambda_p(A))} \cap \rho(M_\varphi, \lambda_p(A))$. Clearly, using Corollary 1, $1 \in \overline{\{\varphi_i \ : \ i \in \mathbb{N}\}} = \overline{\sigma(M_\varphi, \lambda_p(A))}$. Now take $n \in \mathbb{N}$ and let $m = n + 1$. Then we have

$$\sup_{i \in \mathbb{N}} \frac{a_n(i)}{a_m(i)} \frac{1}{|\varphi_i - 1|} = \sup_{i \in \mathbb{N}} \frac{1}{i} \frac{1}{\frac{1}{i}} = 1 < \infty,$$

and thus $1 \in \rho(M_\varphi, \lambda_p(A))$, by Proposition 1.

Theorem 1 *Let $1 \leq p \leq \infty$ or $p = 0$. Then*

$$\sigma^*(M_\varphi, \lambda_p(A)) = \overline{\sigma(M_\varphi, \lambda_p(A))} = \overline{\{\varphi_i \ : \ i \in \mathbb{N}\}}.$$

Proof Set $X := \lambda_p(A)$. By the very definition, we have $\overline{\sigma(M_\varphi, X)} \subset \sigma^*(M_\varphi, X)$. It remains to prove the other inclusion. Let $\lambda \in \mathbb{C} \setminus \overline{\{\varphi_i \ : \ i \in \mathbb{N}\}}$. There exists $\delta > 0$ such that if $|\mu - \lambda| \leq 2\delta$, then $\mu \in \rho(M_\varphi, X)$. We show that the set

$$\{(M_\varphi - \mu I)^{-1} \ : \ |\lambda - \mu| \leq \delta\}$$

is equicontinuous. By Lemma 2, we know that $\varphi_i \in \sigma(M_\varphi, X)$, and thus $|\varphi_i - \lambda| > 2\delta$. Therefore, if $\mu \in \mathbb{C}$ satisfies $|\lambda - \mu| \leq \delta$, then for every $i \in \mathbb{N}$ we have

$$|\varphi_i - \mu| \geq |\varphi_i - \lambda| - |\lambda - \mu| > 2\delta - \delta = \delta.$$

Now let $x \in X$ and let $y^\mu = (M_\varphi - \mu I)^{-1}x = M_{\left(\frac{1}{\varphi_i - \mu}\right)_i} x$, for each $\mu \in \mathbb{C}$ with $|\lambda - \mu| \leq \delta$. Then for every $i \in \mathbb{N}$ we have

$$|y_i^\mu| = \left|\frac{x_i}{\varphi_i - \mu}\right| \leq \frac{1}{\delta}|x_i|.$$

From this we deduce

$$p_n((M_\varphi - \mu I)^{-1}x) \leq \frac{1}{\delta} p_n(x),$$

for all $n \in \mathbb{N}$ and for all μ with $|\lambda - \mu| \leq \delta$. Therefore the set $\{(M_\varphi - \mu I)^{-1} : |\lambda - \mu| \leq \delta\}$ is equicontinuous, hence $\lambda \in \rho^*(M_\varphi, X)$. ◻

3 Mean ergodicity of diagonal operators

The aim of this section is to characterize the power boundedness, the mean ergodicity and the uniform mean ergodicity of the diagonal operators defined on Köthe echelon spaces. An operator $T \in \mathcal{L}(E)$, defined on a locally convex Hausdorff space E is called *power bounded* if the set $\{T^k : k \in \mathbb{N}\}$ is equicontinuous. Here T^k denotes the composition of T with itself k-times. The operator is called *mean ergodic* (resp. *uniformly mean ergodic*) if the sequence of Cesàro means $T_{[k]} := \frac{1}{k}\sum_{j=1}^{k} T^j$ converges on $\mathcal{L}_s(E)$ (resp. on $\mathcal{L}_b(E)$).

Lemma 3 *Let $1 \leq p \leq \infty$ or $p = 0$. If $T := M_\varphi \in \mathcal{L}(\lambda_p(A))$ satisfies that $\frac{T^k x}{k} \to 0$ as $k \to \infty$ for each $x \in \lambda_p(A)$, then $\|\varphi\|_\infty \leq 1$. This holds in particular if T is power bounded or mean ergodic.*

Proof For each $j \in \mathbb{N}$, let $e^j = (\delta_{ij})_i \in \lambda_p(A)$. Then

$$\lim_k \frac{|\varphi_j^k|}{k} = \lim_k \frac{|(T^k e^j)_j|}{k} = 0,$$

with $(T^k e^j)_j$ the j-th coordinate of $T^k e^j$. This implies $|\varphi_j| \leq 1$.

If T is power bounded, then $\left(\frac{T^k}{k}\right)_k$ clearly converges to 0 in $\mathcal{L}_s(\lambda_p(A))$. On the other hand, if T is mean ergodic, then $\frac{T^k x}{k} = T_{[k]}x - \frac{k-1}{k}T_{[k-1]}x$ converges to 0 as $k \to \infty$ for each $x \in \lambda_p(A)$. ◻

Proposition 2 *For $1 \leq p \leq \infty$ and $p = 0$, $M_\varphi \in \mathcal{L}(\lambda_p(A))$ is power bounded if, and only if, $\|\varphi\|_\infty \leq 1$.*

Proof Necessity follows from Lemma 3. Conversely, if $\|\varphi\|_\infty \leq 1$, then $p_n(M_\varphi^k x) \leq p_n(x)$ for every $k, n \in \mathbb{N}$ and $x \in \lambda_p(A)$. Thus M_φ is power bounded. ◻

Lemma 4 *Let $(y^k)_k \subset \lambda_0(A)$ be bounded with $\lim_k y_i^k = 0$ for each $i \in \mathbb{N}$. Then $(y^k)_k$ converges to 0 for the weak topology $\sigma(\lambda_0(A), (\lambda_0(A))')$.*

Proof Let $u = (u_i)_i \in \lambda_0(A)'$. There is $m \in \mathbb{N}$ such that $(u_i/a_m(i))_i \in \ell_1$. Since $(y^k)_k$ is bounded, there is $M > 0$ such that $a_m(i)|y_i^k| \leq M$ for each $i, k \in \mathbb{N}$. Given $\varepsilon > 0$, there is $i(0) \in \mathbb{N}$ such that

$$\sum_{i=i(0)+1}^{\infty} \frac{|u_i|}{a_m(i)} < \frac{\varepsilon}{2M}.$$

Now, there is $k_0 \in \mathbb{N}$ such that for all $k \geq k(0)$ and $i = 1, ..., i(0)$, we have $|y_i^k| < \varepsilon/(2i(0)(|u_i| + 1))$. We have, for $k \geq k(0)$,

$$|\langle y^k, u\rangle| = \left|\sum_{i=1}^{\infty} y_i^k u_i\right| \leq \sum_{i=1}^{i(0)} |y_i^k u_i| + \sum_{i=i(0)+1}^{\infty} |y_i^k| a_m(i) \left|\frac{u_i}{a_m(i)}\right| < \varepsilon.$$

Therefore $(y^k)_k$ converges to 0 for the weak topology $\sigma(\lambda_0(A), (\lambda_0(A))')$. □

Theorem 2 *For $1 \leq p < \infty$ or $p = 0$, $M_\varphi \in \mathcal{L}(\lambda_p(A))$ is mean ergodic if, and only if, $\|\varphi\|_\infty \leq 1$.*

Proof If M_φ is mean ergodic, by Lemma 3, $\|\varphi\|_\infty \leq 1$.

Now we assume that $\|\varphi\|_\infty \leq 1$ and distinguish three cases.

1. If $1 < p < \infty$, the space $\lambda_p(A)$ is reflexive. By [2, Corollary 2.7], every power bounded operator in $\lambda_p(A)$, $1 < p < \infty$ is mean ergodic. The conclusion follows from Proposition 2.

For the rest of the proof, we may assume without loss of generality that $\varphi_i \neq 1$ for all $i \in \mathbb{N}$. Otherwise we split the space into two sectional subspaces and observe that in the subspace in which $\varphi_i = 1$, the diagonal operator acts as the identity.

Under this assumption, the following expression of the i-th coordinate of the means $(M_\varphi)_{[k]}x$ of the iterates evaluated at $x \in \lambda_p(A)$ is useful.

$$((M_\varphi)_{[k]}x)_i = \frac{\varphi_i + \cdots + \varphi_i^k}{k} x_i = \frac{\varphi_i}{k} \frac{1 - \varphi_i^k}{1 - \varphi_i} x_i.$$

2. For the case $p = 1$. Fix $x \in \lambda_1(A)$. We want to show that $(M_\varphi)_{[k]}x$ converges to 0 in $\lambda_1(A)$. Let $\varepsilon > 0$ and fix $n \in \mathbb{N}$. Since $x \in \lambda_1(A)$, there exists $i_0 \in \mathbb{N}$ such that

$$\sum_{i=i_0+1}^{\infty} a_n(i)|x_i| < \frac{\varepsilon}{2}.$$

For each $i \in \mathbb{N}$ we have

$$\lim_{k\to\infty} |((M_\varphi)_{[k]}x)_i| = \lim_{k\to\infty} \frac{|\varphi_i|}{k} \frac{|1 - \varphi_i^k|}{|1 - \varphi_i|} |x_i| \leq \lim_{k\to\infty} \frac{2|x_i|}{k} \frac{1}{|1 - \varphi_i|} = 0.$$

Then, for $i = 1, \ldots, i_0$, select $k_i \in \mathbb{N}$ such that for $k \geq k_i$,

$$|((M_\varphi)_{[k]}x)_i| < \frac{\varepsilon}{2a_n(i)i_0}.$$

If $k \geq \max\{k_i \ : \ i = 1, \ldots, i_0\}$, then

$$p_n((M_\varphi)_{[k]}x) = \sum_{i=1}^{\infty} a_n(i) \frac{|\varphi_i|}{k} \frac{|1 - \varphi_i^k|}{|1 - \varphi_i|} |x_i|$$

$$\leq \sum_{i=1}^{i_0} a_n(i)\frac{|\varphi_i|}{k}\frac{|1-\varphi_i^k|}{|1-\varphi_i|}|x_i| + \sum_{i=i_0+1}^{\infty} a_n(i)\left|\frac{\varphi_i+\cdots+\varphi_i^k}{k}\right||x_i|$$
$$\leq \sum_{i=1}^{i_0} a_n(i)\frac{\varepsilon}{2a_n(i)i_0} + \sum_{i=i_0+1}^{\infty} a_n(i)|x_i| < \varepsilon.$$

3. We now consider the case $p = 0$. Fix $x \in \lambda_0(A)$ and set $y^k := (M_\varphi)_{[k]}x$. Then

$$y_i^k = \frac{\varphi_i x_i}{k}\frac{1-\varphi_i^k}{1-\varphi_i}.$$

By Proposition 2, M_φ is power bounded and, in particular, $(y^k)_k$ is bounded. Clearly, for each $i \in \mathbb{N}$, $\lim_k y_i^k = 0$. Apply Lemma 4 to deduce that (y^k) converges to 0 for the weak topology $\sigma(\lambda_0(A), (\lambda_0(A))')$. By Yosida's mean ergodic theorem [15, Chapter VIII, Sect. 3] (see also [1, Theorem 2.2] and [11]) we conclude that M_φ is mean ergodic. □

The following useful description of the bounded sets in a Köthe echelon space is due to Bierstedt, Meise and Summers [7].

Lemma 5 *Let $1 \leq p \leq \infty$ or $p = 0$, then $B \subset \lambda_p(A)$ is bounded if, and only if, there exists $\overline{v}_0 \in \overline{V} := \{\overline{v} \in \lambda_\infty(A) : \overline{v}(i) \geq 0, \forall i \in \mathbb{N}\}$, with $\overline{v}_0 > 0$, such that $B \subset B_{\overline{v}_0} := \left\{(x_i)_i \in \mathbb{C}^{\mathbb{N}} : \left\|\left(\frac{x_i}{\overline{v}_0(i)}\right)_i\right\|_p \leq 1\right\}$.*

Theorem 3 *The following assertions are equivalent:*

(1) *$M_\varphi \in \mathcal{L}(\lambda_\infty(A))$ is mean ergodic,*
(2) *$M_\varphi \in \mathcal{L}(\lambda_\infty(A))$ is uniformly mean ergodic,*
(3) *$M_\varphi \in \mathcal{L}(\lambda_0(A))$ is uniformly mean ergodic,*
(4) *for $1 \leq p < \infty$, $M_\varphi \in \mathcal{L}(\lambda_p(A))$ is uniformly mean ergodic,*
(5) *$\|\varphi\|_\infty \leq 1$ and for each $n \in \mathbb{N}$ and each $\overline{v} \in \overline{V}$,*

$$\lim_{k\to\infty} \sup_{i\in\mathbb{N}\setminus J} \frac{a_n(i)\overline{v}(i)|\varphi_i|}{k}\frac{|1-\varphi_i^k|}{|1-\varphi_i|} = 0,$$

where $J = \{i \in \mathbb{N} : \varphi_i = 1\}$.

Proof Clearly (2) implies (1) and (3).

We prove that (1) implies (5). Clearly $\|\varphi\|_\infty \leq 1$, by Lemma 3. For the rest assume that $\varphi_i \neq 1$ for all $i \in \mathbb{N}$, i.e. $J = \emptyset$, and let $n \in \mathbb{N}$ and $\overline{v} \in \overline{V} \subset \lambda_\infty(A)$. Then, by mean ergodicity,

$$0 = \lim_{k\to\infty} p_n((M_\varphi)_{[k]}\overline{v}) = \lim_{k\to\infty} \sup_{i\in\mathbb{N}} a_n(i)\frac{|\varphi_i|}{k}\left|\frac{1-\varphi_i^k}{1-\varphi_i}\right|\overline{v}(i),$$

and we conclude.

We assume (5) and show (2) and (4) simultaneously. Assume that $\varphi_i \neq 1$ for all $i \in \mathbb{N}$. Fix $1 \leq p \leq \infty$, let $B \subset \lambda_p(A)$ be bounded and let $\overline{v}_0 \in \overline{V}$ be as in Lemma 5. Fix $n, k \in \mathbb{N}$. Then we have

$$\sup_{x\in B} p_n((M_\varphi)_{[k]}x) = \sup_{x\in B}\left\|\left(a_n(i)\frac{\overline{v}_0(i)|\varphi_i|}{k}\frac{|1-\varphi_i^k|}{|1-\varphi_i|}\frac{|x_i|}{\overline{v}_0(i)}\right)_i\right\|_p$$

$$\leq \sup_{i\in\mathbb{N}} \frac{a_n(i)\overline{v}_0(i)|\varphi_i|}{k} \frac{|1-\varphi_i^k|}{|1-\varphi_i|} \sup_{x\in B} \left\| \left(\frac{x_i}{\overline{v}_0(i)} \right)_i \right\|_p$$
$$\leq \sup_{i\in\mathbb{N}} \frac{a_n(i)\overline{v}_0(i)|\varphi_i|}{k} \frac{|1-\varphi_i^k|}{|1-\varphi_i|},$$

which converges to 0 by assumption.

Now we show that both (3) and (4) imply (5). Assume that $\varphi_i \neq 1$ for all $i \in \mathbb{N}$. Let $1 \leq p < \infty$ or $p = 0$ and fix $n \in \mathbb{N}$ and $\overline{v} \in \overline{V}$. Let $B = \{x \in \lambda_p(A) \ : \ |x_i| \leq \overline{v}(i), \forall i \in \mathbb{N}\}$ and $\varepsilon > 0$. By uniform mean ergodicity, there exists $k_0 \in \mathbb{N}$ such that for $k \geq k_0$, $p_n((M_\varphi)_{[k]}x) < \varepsilon$ for every $x \in B$. For each $s \in \mathbb{N} \setminus J$ let $x^s = (\delta_{is}\overline{v}(s))_i$. Note that $x^s \in B$ for every $s \in \mathbb{N}$. We have, for $s \in \mathbb{N}$ and $k \geq k_0$,

$$\frac{a_n(s)\overline{v}(s)|\varphi_s|}{k} \frac{|1-\varphi_s^k|}{|1-\varphi_s|} = p_n((M_\varphi)_{[k]}x^s) < \varepsilon.$$

□

Proposition 3 *Let $1 \leq p < \infty$ or $p = 0$ and let $A = (a_n)_n$ be a Köthe matrix. The space $\lambda_p(A)$ is Montel if and only if every mean ergodic diagonal operator M_φ on $\lambda_p(A)$ is uniformly mean ergodic.*

Proof Every mean ergodic operator on a Fréchet Montel space is uniformly mean ergodic by [1, Proposition 2.8]. If $\lambda_p(A)$ is not Montel, then it contains a sectional subspace which is diagonally isomorphic to ℓ_p (or to c_0 if $p = 0$). The conclusion follows, since there are mean ergodic not uniformly mean ergodic diagonal operators on ℓ_p. Compare with the proof of Proposition 2.9 in [1]. □

4 The topology of the space of diagonal operators

Let $\mathcal{A} = (a_m)_m$ and $\mathcal{B} = (b_n)_n$ be Köthe matrices and $1 \leq p \leq \infty$ or $p = 0$. Set $E = \lambda_p(\mathcal{A})$ and $F = \lambda_p(\mathcal{B})$. By Lemma 1, M_φ is continuous from E to F, that is φ is a multiplier, if and only if for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$\sup_{i\in\mathbb{N}} \frac{b_n(i)|\varphi_i|}{a_m(i)} < \infty.$$

This is actually equivalent to

$$\varphi \in \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \ell_\infty\left(\frac{b_n}{a_m}\right) =: E_{\mathcal{AB}}.$$

Endow the set $E_{\mathcal{AB}}$ of all multipliers from E to F with the topology

$$E_{\mathcal{AB}} = \mathrm{proj}_n \, \mathrm{ind}_m \, \ell_\infty\left(\frac{b_n}{a_m}\right).$$

Then $E_{\mathcal{AB}}$ is a countable projective limit of countable inductive limits of Banach spaces. These spaces are called (PLB)-spaces. We refer the reader to Wengenroth lecture notes [14] for more information about these spaces.

Inductive limits of sequence spaces have been thoroughly studied by several authors; see for example Bierstedt [5]. Here we state the notation and results we need below. Given a decreasing family of weights $V = (v_n)_n$ we define the co-echelon space of order ∞ by

$$\kappa_\infty(V) := \bigcup_{n\in\mathbb{N}} \ell_\infty(v_n),$$

endowed with the inductive topology $\kappa_\infty(V) := \operatorname{ind}_{n\in\mathbb{N}} \ell_\infty(v_n)$.

In case the Köthe echelon space $\lambda_1(A)$, $A := (1/v_n)_n$ is distinguished, it is possible to describe the topology of $\kappa_\infty(V)$ using a family of weighted seminorms. Let

$$\overline{V} = \overline{V}(V) := \{\overline{v} = (\overline{v}_i)_i \in \lambda_\infty(A) \mid \overline{v}_i \geq 0 \ \ i \in \mathbb{N}\},$$

and

$$K_\infty(V) := \{x = (x_i)_i \in \mathbb{C}^{\mathbb{N}} \ : \ \sup_{i\in\mathbb{N}} \overline{v}(i)|x_i| < \infty, \ \forall \overline{v} \in \overline{V}\},$$

endowed with the projective limit topology, $\operatorname{proj}_{\overline{v}} \ell_\infty(\overline{v})$. If $\lambda_1(A)$ is distinguished, then $\kappa_\infty(V) = K_\infty(V)$ algebraically and topologically. This result can be seen in [7]. Distinguished $\lambda_1(A)$ were characterized in terms of a condition (D) on the sequence Köthe matrix A by Bierstedt and Bonet and Meise. We refer the reader to the survey article [6] for more details and references.

We can now write $E_{\mathcal{AB}}$ as a projective limit of co-echelon spaces as

$$E_{\mathcal{AB}} = \operatorname{proj}_n \kappa_\infty(\mathcal{V}_n),$$

where $\mathcal{V}_n = \left(\frac{b_n}{a_m}\right)_m$ is a decreasing family of weights for each $n \in \mathbb{N}$. Now let $V = (1/a_m)_m$, then we have $\kappa_\infty(\mathcal{V}_n) \simeq \kappa_\infty(V)$, for each $n \in \mathbb{N}$. Furthermore, assuming that $\lambda_1(\mathcal{A})$ is distinguished, we get $\kappa_\infty(\mathcal{V}_n) \simeq K_\infty(V)$, for each $n \in \mathbb{N}$. This way, we get that, for each $n \in \mathbb{N}$, the topology of $\kappa_\infty(\mathcal{V}_n)$ is given by the seminorms

$$p_{\overline{v}}(x) := \sup_{i\in\mathbb{N}} b_n(i)\overline{v}(i)|x_i|, \quad x \in \kappa_\infty(\mathcal{V}_n), \overline{v} \in \overline{V}.$$

Then the fundamental system of seminorms of $E_{\mathcal{AB}}$ is

$$q_{n,\overline{v}}(x) := \sup_{i\in\mathbb{N}} b_n(i)\overline{v}(i)|x_i|, \quad x \in E_{\mathcal{AB}}, n \in \mathbb{N}, \overline{v} \in \overline{V}.$$

On the other hand, the topology of the space $\mathcal{L}_b(\lambda_p(\mathcal{A}), \lambda_p(\mathcal{B}))$, can be described as follows. First let $1 \leq p \leq \infty$. Denote the seminorms of $\lambda_p(\mathcal{B})$ by

$$q_n(x) := \|(b_n(i)x_i)_i\|_p, \quad x \in \lambda_p(\mathcal{B}), n \in \mathbb{N},$$

and for the case $p = 0$, take $\|\cdot\|_\infty$, as usual. Then, the seminorms defining the topology of $\mathcal{L}_b(\lambda_p(\mathcal{A}), \lambda_p(\mathcal{B}))$ are

$$r_{n,B}(M) := \sup_{x\in B} q_n(Mx), \quad M \in \mathcal{L}(\lambda_p(\mathcal{A}), \lambda_p(\mathcal{B})), B \subset \lambda_p(\mathcal{A}) \text{ bounded}.$$

The following theorem is the main result and ensures that the topology of the space $\mathcal{L}_b(\lambda_p(\mathcal{A}), \lambda_p(\mathcal{B}))$ induces on the space of multipliers $E_{\mathcal{AB}}$ from $\lambda_p(\mathcal{A})$ into $\lambda_p(\mathcal{B})$ the natural (PLB)-topology defined above.

Theorem 4 *Let $1 \leq p \leq \infty$ or $p = 0$. Assume that $\lambda_1(\mathcal{A})$ is distinguished. Then the map $T : E_{\mathcal{AB}} \longrightarrow \mathcal{L}_b(\lambda_p(\mathcal{A}), \lambda_p(\mathcal{B}))$, $\varphi \mapsto M_\varphi$, is a topological isomorphism into.*

Proof The injectivity of T is easy.

1. Continuity of T: Fix $n \in \mathbb{N}$ and let $B \subset \lambda_p(\mathcal{A})$ be bounded. Apply Lemma 5 to find $\overline{v} \in \overline{V}$ such that $B \subset B_{\overline{v}}$. For $\varphi \in E_{\mathcal{AB}}$ we have

$$
\begin{aligned}
r_{n,B}(T(\varphi)) &= r_{n,B}(M_\varphi) \\
&= \sup_{x\in B} \left\| \left(b_n(i)\overline{v}(i)\varphi_i \frac{x_i}{\overline{v}(i)} \right)_i \right\|_p \\
&\leq \sup_{i\in\mathbb{N}} b_n(i)\overline{v}(i)|\varphi_i| \sup_{x\in B} \left\| \left(\frac{x_i}{\overline{v}(i)} \right)_i \right\|_p \\
&\leq \sup_{i\in\mathbb{N}} b_n(i)\overline{v}(i)|\varphi_i| = q_{n,\overline{v}}(\varphi).
\end{aligned}
$$

2. Continuity of T^{-1}: Fix $n \in \mathbb{N}$ and $\overline{v} \in \overline{V}$. For $j \in \mathbb{N}$ let $e^j = (\delta_{ij})_i$. Thus, $e^j\overline{v}(j) \in B_{\overline{v}}$. For $j \in \mathbb{N}$ we have

$$
\begin{aligned}
r_{n,B_{\overline{v}}}(T(\varphi)) &= r_{n,B_{\overline{v}}}(M_\varphi) \\
&= \sup_{x\in B_{\overline{v}}} q_n(M_\varphi x) \\
&\geq q_n(M_\varphi(e^j\overline{v}(j))) \\
&= q_n(\overline{v}(j)\varphi_j e^j) = b_n(j)\overline{v}(j)|\varphi(j)|
\end{aligned}
$$

Since $j \in \mathbb{N}$ is arbitrary, we have $q_{n,\overline{v}}(\varphi) \leq r_{n,B_{\overline{v}}}(M_\varphi)$. □

The following result about the locally convex properties of the space $E_{\mathcal{AB}}$ of multipliers is a direct consequence of [1]. We refer the reader to this article and to [14] for the relevance of the technical conditions (Q) and (wQ).

Proposition 4 *Consider the family* $(\mathcal{V}_N)_N = \left(\left(\frac{b_N}{a_m}\right)_m\right)_N$.

(1) Assume $(\mathcal{V}_N)_N$ *satisfies condition* (Q)*, i.e.*

$$
\begin{aligned}
&\forall N\ \exists M \geq N\ \exists n\ \forall K \geq M\ \forall m, \varepsilon > 0\ \exists S\ \exists k\ \forall x\ : \\
&\frac{a_m(x)}{b_M(x)} \leq \max\left(\varepsilon\frac{a_n(x)}{b_N(x)}, S\frac{a_k(x)}{b_K(x)}\right).
\end{aligned}
\tag{Q}
$$

Then $E_{\mathcal{AB}}$ *is barrelled.*

(2) If $E_{\mathcal{AB}}$ *is barrelled, then* $(\mathcal{V}_N)_N$ *satisfies condition* (wQ)*, i.e.*

$$
\begin{aligned}
&\forall N\ \exists M \geq N\ \exists n\ \forall K \geq M\ \forall m\ \exists k\ \exists S \geq 0\ \forall x\ : \\
&\frac{a_m(x)}{b_M(x)} \leq S\max\left(\frac{a_n(x)}{b_N(x)}, \frac{a_k(x)}{b_K(x)}\right).
\end{aligned}
\tag{wQ}
$$

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