



# Ergodic Properties of Composition Semigroups on the Disc Algebra

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Received: 2 October 2020 / Accepted: 13 March 2021 / Published online: 1 April 2021  
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## Abstract

Every semigroup  $\{\varphi_t\}_{t \geq 0}$  of self-maps of the disc defines a semigroup  $\{C_{\varphi_t}\}_{t \geq 0}$  of composition operators on the space of holomorphic functions on the disc. We characterize the (uniform) mean ergodicity (in the sense of continuous means) and the asymptotic behaviour of these operators when they define a  $C_0$ -semigroup on the disc algebra, in terms of the Denjoy–Wolff point and the associated planar domain in the sense of Berkson and Porta. Finally we deal with the case of Hardy and Bergman spaces.

**Keywords** Semigroups · Composition operators · (Uniform) mean ergodicity · Disc algebra

**Mathematics Subject Classification** 47D06 · 47A35 · 47B33 · 30D05 · 30C45 · 30H05

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Communicated by Filippo Bracci.

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# 1 Introduction, Notation and Preliminaries

## 1.1 Introduction

The convergence in the strong operator topology of the means of the first iterates  $T^n$  of a linear and continuous operator  $T : X \rightarrow X$  on a Banach space  $X$  is called *mean ergodicity*. If this convergence occurs uniformly on bounded sets of  $X$ , then  $T$  is called *uniformly mean ergodic*. This discrete idea can be extended to “fractional iterates.” Given a  $C_0$ -semigroup  $\{T_t : X \rightarrow X\}_{t \geq 0}$  of linear and continuous operators (i.e.  $T_0x = x$ ,  $T_{t+s} = T_tT_s$  and  $\|T_t x - x\| \rightarrow 0$  as  $t \rightarrow 0^+$ ), we are concerned about the convergence of the Cesàro means

$$T_{[R]} = \frac{1}{R} \int_0^R T_t dt, \quad R > 0$$

in the strong operator topology (mean ergodicity) and uniformly on bounded sets (uniform mean ergodicity). This idea has been studied by many authors. See for example [1, 10, 12, 13].

Each semigroup  $\{\varphi_t : \mathbb{D} \rightarrow \mathbb{D}\}_{t \geq 0}$  of analytic functions (i.e.  $\varphi_0(z) = z$ ,  $\varphi_{t+s} = \varphi_t \circ \varphi_s$  and  $\varphi_t(z) \rightarrow z$  as  $t \rightarrow 0^+$ ) defines a semigroup of composition operators  $\{C_{\varphi_t} : H(\mathbb{D}) \rightarrow H(\mathbb{D}), f \mapsto f \circ \varphi_t\}_{t \geq 0}$ . The question whether  $\{C_{\varphi_t}\}_{t \geq 0}$  is a  $C_0$ -semigroup on different Banach spaces  $X \subseteq H(\mathbb{D})$  has been thoroughly studied, beginning with the seminal work of Berkson and Porta [6]. We refer the reader to [16] for a complete survey by Siskakis on these results and to the book [7]. The particular case in which we are interested is the disc algebra  $A(\mathbb{D})$ , which was studied in [8] by Contreras and Díaz-Madrigal. They show that the semigroup is a  $C_0$ -semigroup if, and only if, the semigroup of functions is contained in  $A(\mathbb{D})$ . This is actually equivalent to the existence of a  $t_0 > 0$  such that  $\varphi_{t_0} \in A(\mathbb{D})$  (see [7, Theorem 11.3.8]).

In Sect. 2 we study the (uniform) mean ergodicity and the asymptotic behaviour of the  $C_0$ -semigroup  $\mathcal{T} = \{C_{\varphi_t}\}_{t \geq 0}$  considered in  $A(\mathbb{D})$ . The case of discrete iterates was studied in [5] by Beltrán Meneu, Gómez-Collado, Jordá and Jornet, utilizing the Denjoy–Wolff point. We show that  $\mathcal{T}$  is mean ergodic if, and only if,  $\Phi = \{\varphi_t\}_{t \geq 0}$  has a unique common fixed point (its Denjoy–Wolff point). If  $\Phi$  has its Denjoy–Wolff point in  $\mathbb{D}$ , then mean ergodicity, uniform mean ergodicity, pointwise convergence of  $C_{\varphi_t}$  as  $t \rightarrow \infty$ , norm convergence of  $C_{\varphi_t}$  as  $t \rightarrow \infty$  and uniform convergence of  $\varphi_t$  on  $\overline{\mathbb{D}}$  to the Denjoy–Wolff point as  $t \rightarrow \infty$  are all equivalent. The situation differs when the Denjoy–Wolff point is on  $\partial\mathbb{D}$ . In this case pointwise convergence of  $C_{\varphi_t}$  as  $t \rightarrow \infty$  is equivalent to uniform convergence of  $\varphi_t$  on  $\overline{\mathbb{D}}$  to the Denjoy–Wolff point as  $t \rightarrow \infty$ . Moreover,  $\mathcal{T}$  is never uniformly mean ergodic in this case. We also characterize all these properties in terms of the planar domain  $\Omega$  associated to  $\Phi$  in the sense of Berkson and Porta [6].

In Sect. 3 we consider the semigroups on Hardy spaces  $H^p$  and Bergman spaces  $A^p$ , for  $1 \leq p < \infty$ . We show that  $\mathcal{T}$  is mean ergodic if, and only if,  $\mathcal{T}$  is equicontinuous if, and only if, the Denjoy–Wolff point of  $\Phi$  is in  $\mathbb{D}$ .

### 1.2 Notation and Preliminaries

A family of continuous and linear operators  $\mathcal{T} = \{T_t\}_{t \geq 0}$  on a Banach space  $X$  is called a *semigroup* if  $T_0 = I$  is the identity operator and  $T_{t+s} = T_t T_s$  for every  $t, s \geq 0$ . If in addition  $\mathcal{T}$  satisfies  $\lim_{t \rightarrow 0^+} T_t x = x$  for all  $x \in X$ , then  $\mathcal{T}$  is called a  $C_0$ -*semigroup* (or a strongly continuous semigroup).

If  $\mathcal{T}$  is a  $C_0$ -semigroup we define its Cesàro means as

$$T_{[R]}x = \frac{1}{R} \int_0^R T_t x dt, \quad x \in X, R > 0.$$

These means are continuous and linear operators. If the net  $\{T_{[R]}x\}_{R>0}$  converges as  $R \rightarrow \infty$  for every  $x \in X$ , we say  $\mathcal{T}$  is *mean ergodic*. If the net  $\{T_{[R]}\}_{R>0}$  converges in norm as  $R \rightarrow \infty$ , we say  $\mathcal{T}$  is *uniformly mean ergodic*, see [1,10,12,13].

As usual,  $H(\mathbb{D})$  denotes the space of holomorphic functions on the unit disc. A family  $\Phi = \{\varphi_t : \mathbb{D} \rightarrow \mathbb{D}\}_{t \geq 0} \subset H(\mathbb{D})$  is a *semigroup* if it satisfies  $\varphi_0(z) = z$ ,  $\varphi_{t+s}(z) = \varphi_t(\varphi_s(z))$  and  $\lim_{t \rightarrow 0^+} \varphi_t(z) = z$ . A semigroup of analytic self-maps of the disc always defines a semigroup of composition operators on  $H(\mathbb{D})$  by  $\{C_{\varphi_t} : H(\mathbb{D}) \rightarrow H(\mathbb{D}), f \mapsto f \circ \varphi_t\}_{t \geq 0}$ . This type of composition semigroups has been intensively studied by Berkson and Porta in [6]. They showed that every semigroup of analytic functions defines a  $C_0$ -semigroup of composition operators on the Hardy spaces  $H^p$ , for  $1 \leq p < \infty$ . Siskakis showed in [15] that this also holds for the Bergman spaces  $A^p$ , with  $1 \leq p < \infty$  and explains in [16, p.8] how there are no non-trivial semigroups  $\Phi$  defining  $C_0$ -semigroups on  $H^\infty$ . Contreras and Díaz-Madrigal in [8] studied the case of the disc algebra  $A(\mathbb{D})$  (see Theorem 1.3).

Before presenting the general theory of semigroups of analytic self-maps of  $\mathbb{D}$ , we consider two simple but typical examples.

- $\varphi_t(z) = e^{-ct}z$ , for  $\text{Re } c \geq 0$ . If  $\text{Re } c = 0$ , we have that each  $\varphi_t$  is a rotation. If  $\text{Re } c > 0$ , then the  $\varphi_t$  are contractions of the disc, and, for each  $z \in \mathbb{D}$ ,  $\varphi_t(z) \rightarrow 0$  as  $t \rightarrow \infty$ .

- $\varphi_t(z) = e^{-t}z + 1 - e^{-t}$ . Each symbol of this semigroup maps the unit disc into a disc of radius  $e^{-t}$  tangent to the unit disc at 1. It satisfies that  $\varphi_t(1) = 1$  and for each  $z \in \mathbb{D}$ ,  $\varphi_t(z) \rightarrow 1$  as  $t \rightarrow \infty$ .

The convergence of  $\varphi_t$  to 0 and 1 respectively in these examples is actually similar to the general case.

**Theorem 1.1** (Denjoy–Wolff) *If  $\Phi = \{\varphi_t : \mathbb{D} \rightarrow \mathbb{D}\}_{t \geq 0} \subset H(\mathbb{D})$  is a semigroup of analytic functions, which is not the trivial semigroup, then it is of one of the following three mutually exclusive types:*

- (00) *The  $\varphi_t$  have a unique common fixed point  $b \in \mathbb{D}$  and  $|\varphi'_t(b)| = 1$ .*
- (0) *The  $\varphi_t$  have a unique common fixed point  $b \in \mathbb{D}$  and  $\varphi_t \rightarrow b$  as  $t \rightarrow \infty$  uniformly on compact sets of  $\mathbb{D}$ .*
- (1) *There is  $b \in \partial\mathbb{D}$  such that  $\varphi_t \rightarrow b$  as  $t \rightarrow \infty$  uniformly on compact sets of  $\mathbb{D}$ .*

Following the theory of self-maps of the disc, we call the point  $b$  in (0) and (1) the *Denjoy–Wolff point* of  $\Phi$ .

In the cases (00) and (0) we can always consider  $b = 0$ , since the Möbius transformation

$$\phi_b(z) = \frac{b - z}{1 - \bar{b}z}, \quad z \in \mathbb{D}$$

interchanges  $b$  and  $0$ . Therefore  $\psi_t := \phi_b \circ \varphi_t \circ \phi_b$  fixes  $0$  and  $C_{\psi_t} = C_{\phi_b} C_{\varphi_t} C_{\phi_b}$  keeps the dynamical properties of  $C_{\varphi_t}$ . In particular if  $\Phi$  is of the first type, then each  $\varphi_t$  is an *elliptic automorphism* and  $\psi_t(z) = \alpha^t z$  for some  $\alpha \in \partial\mathbb{D}$ .

In the case (1), we can always consider  $b = 1$  by interchanging  $b$  with  $1$  by  $\psi_t(z) = b\varphi_t(\bar{b}z)$ . In this case we have that  $\varphi_t$  has a continuous extension to  $b (= 1)$ , with  $\varphi_t(b) = b$  (this is in fact not trivial, and it is not true in general for holomorphic self maps, see [11] and [7, Corollary 11.3.7]). In our work we mainly consider  $\varphi_t \in A(\mathbb{D})$ , then  $b$  is a fixed point in all the cases.

Before considering the geometric representation of semigroups, we give some more examples.

- Consider the domain  $\Omega := \{Re z \geq -1/2\} \subset \mathbb{C}$ . The map  $h : \mathbb{D} \rightarrow \Omega$ , given by

$$h(z) = \frac{z}{1 - z},$$

is holomorphic and injective. Since  $e^{-t}\Omega \subseteq \Omega$ , for  $t \geq 0$ , we can define  $\varphi_t(z) := h^{-1}(e^{-t}h(z))$  and we have

$$\varphi_t(z) = \frac{z}{e^t + z(1 - e^t)}.$$

Then  $\varphi_t \in A(\mathbb{D})$  and they satisfy  $\varphi_t(0) = 0$  and also  $\varphi_t(1) = 1$ . We have that for each  $z \in \mathbb{D} \setminus \{1\}$ ,  $\varphi_t(z) \rightarrow 0$  as  $t \rightarrow \infty$ .

- For the same  $\Omega$  and  $h$ , we have that  $\Omega + t \subseteq \Omega$ , then we can define  $\varphi_t(z) := h^{-1}(h(z) + t)$ . We get

$$\varphi_t(z) = \frac{z + t(1 - z)}{1 + t(1 - z)}.$$

Then also for all  $t \geq 0$ ,  $\varphi_t \in A(\mathbb{D})$  and  $\varphi_t(1) = 1$  is the only common fixed point. We have that, for each  $z \in \mathbb{D}$ ,  $\varphi_t(z) \rightarrow 1$  as  $t \rightarrow \infty$ .

It turns out that these examples are very typical.

**Theorem 1.2** (Berkson and Porta [6]) *Let  $\Phi = \{\varphi_t : \mathbb{D} \rightarrow \mathbb{D}\}_{t \geq 0} \subset H(\mathbb{D})$  be a semigroup of analytic functions. Then:*

(0) *If  $\Phi$  is of type (00) or (0), then there is a univalent  $h : \mathbb{D} \rightarrow \mathbb{C}$  such that  $h(b) = 0$ ,  $h'(b) = 1$  and, for  $\Omega := h(\mathbb{D})$  and  $t \geq 0$*

$$e^{-ct} \cdot \Omega \subset \Omega,$$

for some  $c \in \mathbb{C}$  with  $\operatorname{Re} c \geq 0$  and such that

$$\varphi_t(z) = h^{-1}(e^{-ct} \cdot h(z)).$$

Furthermore  $\Phi$  is of type (00) if, and only if,  $\operatorname{Re} c = 0$ .

(1) If  $\Phi$  is of type (1), then there is a univalent  $h : \mathbb{D} \rightarrow \mathbb{C}$  such that, for  $\Omega := h(\mathbb{D})$  and  $t \geq 0$ ,

$$\Omega + t \subset \Omega,$$

and such that

$$\varphi_t(z) = bh^{-1}(h(\bar{b}z) + t).$$

In particular if  $b = 1$ , we have  $\varphi_t(z) = h^{-1}(h(z) + t)$ .

In both cases, the domain  $\Omega$  is called the *associated planar domain* of  $\Phi$ . The existence of  $\Omega$  and  $h$  is unique, up to composition with affine transformations (see [7, Proposition 9.3.10] and also [4,9]), therefore we will consider  $h(0) = 0$  in the type (1) in the rest of this work.

From this description one gets that for semigroups of types (0) and (1) each  $\varphi_t$  is actually univalent.

This geometric description of the semigroup  $\Phi$  is the central point of our study of  $\{C_{\varphi_t}\}_{t \geq 0}$  which is inspired by results of Contreras and Díaz-Madrigal in [8].

To formulate one of their main results we denote by  $\partial A$  the boundary of the set  $A$  in  $\mathbb{C}$  and by  $\partial_\infty A$  the boundary of the set  $A$  in  $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ . A set  $A$  in a topological space is called *locally connected* if each point of  $A$  has a basis of neighbourhoods formed of connected open sets.

**Theorem 1.3** (Contreras and Díaz-Madrigal [8]) *Let  $\Phi = \{\varphi_t : \mathbb{D} \rightarrow \mathbb{D}\}_{t \geq 0}$  be a semigroup with associated planar domain  $\Omega$ . The following assertions are equivalent:*

- (i)  $\Phi$  defines a  $C_0$ -semigroup of operators on  $A(\mathbb{D})$ ,
- (ii)  $\varphi_t \in A(\mathbb{D})$ , for all  $t > 0$ ,
- (iii)  $\varphi_t \in A(\mathbb{D})$ , for some  $t > 0$ .

Furthermore, the following are also equivalent:

- (iv)  $\partial_\infty \Omega$  is locally connected,
- (v)  $h : \mathbb{D} \rightarrow \Omega$  has a continuous extension  $\hat{h} : \bar{\mathbb{D}} \rightarrow \hat{\mathbb{C}}$ .

In addition, (iv) implies (i) and if  $\Phi$  has an interior Denjoy–Wolff point, then also (i) implies (iv).

The relation between the first four assertions is originally the work of [8] and it is refined in [7, Theorem 11.3.8], while the equivalence of (iv) and (v) already appears in the theory of univalent functions of Carathéodory. See for example [14, Chapter 2].

We now mention some examples of domains with not locally connected boundary:

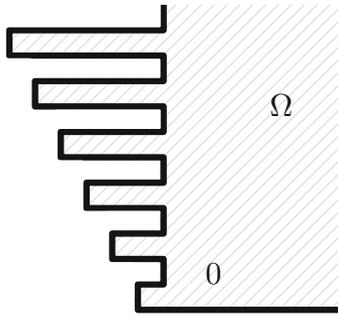


Fig. 1 The domain defined by the set  $A$

- The domain given in [8],  $\Omega = \mathbb{D} \setminus (\bigcup_{n \in \mathbb{N}} [0.4, 1] \cdot e^{i/n} \cup [0.4, 1])$ , defines a semigroup of type (0), but the boundary is not locally connected at  $[0.4, 1]$ .
- With the same ideas,  $\Omega = \{Im z > -1\} \setminus (\bigcup_{n \in \mathbb{N}} (-\infty, 0] - i(1 - \frac{1}{n}))$  defines a semigroup of type (1), but the boundary is not locally connected at  $(-\infty, 0) - i$ .
- Consider the set (see Fig. 1)

$$\begin{aligned}
 A = & [-1, +\infty) \times \{-1\} \cup \bigcup_{k \in \mathbb{N} \cup \{0\}} [-k - 2, -1] \times \{2k - 1, 2k\} \\
 & \cup \bigcup_{k \in \mathbb{N} \cup \{0\}} \{-k - 2\} \times [2k - 1, 2k] \\
 & \cup \bigcup_{k \in \mathbb{N} \cup \{0\}} \{-1\} \times [2k, 2k + 1].
 \end{aligned}$$

Define  $\Omega$  as the domain limited by  $A$  containing 0. Then  $\partial_\infty \Omega = A \cup \{\infty\}$ . This domain defines a semigroup of type (1), but the boundary is not locally connected at  $\infty$ .

## 2 Mean Ergodic Operators on the Disc Algebra

In this section we study the mean ergodicity and the asymptotic behaviour of composition  $C_0$ -semigroups on the disc algebra  $A(\mathbb{D}) = C(\mathbb{D}) \cap H(\mathbb{D})$  endowed with the supremum norm.

Throughout this section,  $\Phi = \{\varphi_t : \mathbb{D} \rightarrow \mathbb{D}\}_{t \geq 0} \subset A(\mathbb{D})$  is a semigroup and  $h$  are as in Theorem 1.2. We assume from this point that  $\partial_\infty \Omega$  is locally connected and then  $\mathcal{T} = \{T_t = C_{\varphi_t}\}_{t \geq 0}$  is the associated  $C_0$ -semigroup of composition operators defined in  $A(\mathbb{D})$ . Furthermore we denote the continuous extension of  $h$  by the same symbol.

We start by characterizing the common fixed points of  $\Phi$ , in terms of  $h$  and  $\Omega$ .

**Proposition 2.1** *The following assertions hold for a semigroup  $\Phi$  which is not formed by elliptic automorphisms:*

- (i) If  $z \in \overline{\mathbb{D}}$ , then  $h(z) = \infty$  if, and only if,  $z \in \partial\mathbb{D}$  and  $\varphi_t(z) = z$  for all  $t \geq 0$ .
- (ii) If  $h^{-1}(\infty)$  contains at least two points, then  $\partial\Omega$  is disconnected (Fig. 2).
- (iii) If  $b \in \mathbb{D}$ , then there are no more common fixed points if, and only if,  $\partial\Omega$  is bounded.
- (iv) If  $b \in \partial\mathbb{D}$ , then there are no more common fixed points if, and only if,  $\partial\Omega$  is connected.
- (v) If  $z \in \partial\mathbb{D}$  satisfies  $\{\varphi_t(z)\}_{t \geq 0} \subseteq \partial\mathbb{D}$ , then either  $z$  is a fixed point or  $b \in \partial\mathbb{D}$  and  $\varphi_t(z) \rightarrow b$ .

**Proof** • Assertion (i) is precisely the Case I in the proof of [8, Theorem 1.2], see also Proposition 13.6.1 in [7].

- (ii) follows from Proposition 4.3.5 in [7].
- We prove (iii). Clearly if there is any other common fixed point, it must be on  $\partial\mathbb{D}$ , else  $\Phi$  is the trivial semigroup. By (i), if there are more fixed points, then  $\infty \in \partial_\infty\Omega$  and therefore  $\partial\Omega = \partial_\infty\Omega \cap \mathbb{C}$  is not bounded. Conversely, if  $\infty \in \partial_\infty\Omega$ , then there is  $z \in \partial\mathbb{D}$  with  $h(z) = \infty$  and, by (i),  $z$  is a common fixed point.
- Assertion (iv) follows directly from (i) and (ii).
- It is a consequence of the Boundary Denjoy–Wolff Theorem, see [7, Theorem 14.1.1].

□

Now we characterize the mean ergodicity of  $\mathcal{T}$ . This result should be compared with the characterization of mean ergodicity in the case of discrete iterates given in [5, Theorems B1 and B2].

**Theorem 2.2**  $\mathcal{T}$  is mean ergodic if, and only if,  $\Phi$  has a unique common fixed point or  $\Phi = \{id\}$ .

**Proof** • First we consider the case in which  $\Phi$  is a group of elliptic automorphisms. In this case there is a unique common fixed point and we have to show that  $\mathcal{T}$  is mean ergodic. Without loss of generality, up to a conjugation with a Möbius transformations, we may assume  $\varphi_t(z) = e^{i\alpha t}z$  with  $\alpha \in \mathbb{R}$ , and since the case  $\alpha = 0$  is obvious we consider  $\alpha \neq 0$ .

We show that  $T_{[R]}$  converges to the evaluation at 0,  $C_0 : A(\mathbb{D}) \rightarrow A(\mathbb{D}), f \mapsto f(0)$ . This holds for the polynomials. Indeed, let  $p_n(z) = z^n$  for  $n \geq 0$ . Then, clearly  $(T_{[R]}p_0)(z) = 1 = (C_0(p_0))(z)$ . Now, for  $n > 0$ , we have

$$\begin{aligned} |(T_{[R]}p_n)(z) - (C_0p_n)(z)| &= \left| \frac{1}{R} \int_0^R e^{i\alpha t n} z^n dt - 0 \right| \\ &= \left| \frac{z^n (e^{i\alpha R n} - 1)}{R i \alpha n} \right| \leq \frac{2}{R|\alpha|n}, \end{aligned}$$

which converges to 0 as  $R \rightarrow \infty$ . By the density of the polynomials in  $A(\mathbb{D})$  and since  $\|T_t\|_\infty \leq 1$ , for  $t \geq 0$ , we deduce that  $T_{[R]} \rightarrow C_0$  in  $A(\mathbb{D})$ .

- Now we consider the case that  $\Phi$  has a Denjoy–Wolff point  $b \in \overline{\mathbb{D}}$ . First note that for all  $z \in \mathbb{D}$  and all  $f \in A(\mathbb{D})$  one has  $f(\varphi_t(z)) \rightarrow f(b)$  as  $t \rightarrow \infty$ , the only possible

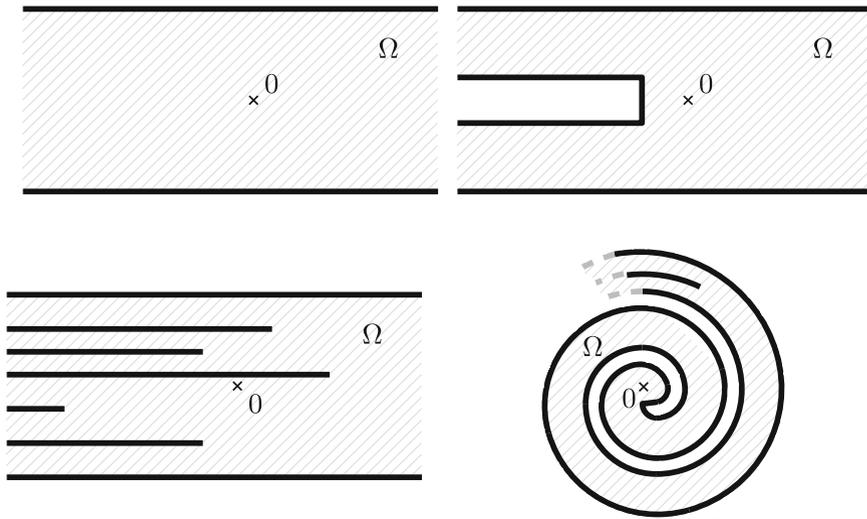


Fig. 2 Examples of disconnected boundaries of  $\Omega$

limit for  $\{T_t\}_{t \geq 0}$  and  $\{T_{[R]}\}_{R > 0}$  is thus the evaluation at  $b$ ,  $C_b : A(\mathbb{D}) \rightarrow A(\mathbb{D})$ ,  $f \mapsto f(b)$ .

Assume  $z_0 \in \partial\mathbb{D}$  is another common fixed point of  $\Phi$ . Let  $f \in A(\mathbb{D})$  be the identity, then for each  $R > 0$ ,

$$|((T_{[R]} - C_b)(f))(z_0)| = |f(z_0) - f(b)|,$$

and therefore  $\mathcal{T}$  is not mean ergodic.

Assume now that  $\Phi$  has a unique common fixed point. We show that  $\varphi_t(z) \rightarrow b$  as  $t \rightarrow \infty$ , for all  $z \in \overline{\mathbb{D}}$ . Indeed, this obviously holds for  $z \in \mathbb{D}$  and also for  $z \in \partial\mathbb{D}$  with  $\varphi_t(z) \in \mathbb{D}$  for some  $t \geq 0$ . Assume  $z_0 \in \partial\mathbb{D}$  satisfies  $\{\varphi_t(z_0)\}_{t \geq 0} \subseteq \partial\mathbb{D}$ . If  $b \in \mathbb{D}$  then this is a contradiction to  $\Phi$  having a unique common fixed point, by Proposition 2.1.(v). If  $b \in \partial\mathbb{D}$ , then  $\varphi_t(z_0)$  converges precisely to  $b$ , again by Proposition 2.1.(v).

Now let  $\varepsilon > 0$ ,  $f \in A(\mathbb{D})$  and  $z \in \overline{\mathbb{D}}$ . Then there is  $t_0 \geq 0$  such that  $|f(\varphi_t(z)) - f(b)| < \varepsilon/2$ , for all  $t \geq t_0$ . Then, for  $R \geq \max\left\{\frac{4t_0\|f\|_\infty}{\varepsilon}, t_0\right\}$ ,

$$\begin{aligned} |[(T_{[R]} - C_b)(f)](z)| &= \left| \frac{1}{R} \int_0^R (f(\varphi_t(z)) - f(b)) dt \right| \\ &\leq \left| \frac{1}{R} \int_0^{t_0} (f(\varphi_t(z)) - f(b)) dt \right| + \left| \frac{1}{R} \int_{t_0}^R (f(\varphi_t(z)) - f(b)) dt \right| \\ &\leq \frac{t_0 2\|f\|_\infty}{R} + \frac{R - t_0}{R} \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

This way we have shown that  $(T_{[R]}f)(z)$  converges to  $f(b)$  as  $R \rightarrow \infty$  for every  $f \in A(\mathbb{D})$  and every  $z \in \overline{\mathbb{D}}$ . Since  $A(\mathbb{D})$  is a closed subspace of the space of continuous

functions on the closed disc, pointwise convergence of bounded sequences in  $A(\mathbb{D})$  implies weak convergence. This is a standard argument using the Riesz representation theorem and Lebesgue dominated convergence theorem. But weak convergence is enough to ensure mean ergodicity by a classical theorem of Eberlein [10], see also [13, p.76] or [1, Proposition 3].  $\square$

The characterization for mean ergodicity is thus independent of the Denjoy–Wolff point. Other asymptotic properties however crucially depend on whether it is in the interior or in the boundary of the disc.

**Remark 2.3** In the proof of Theorem 2.2 we show that if  $\Phi$  has a unique common fixed point  $b$  which is a Denjoy–Wolff point, then  $\varphi_t(z) \rightarrow b$  as  $t \rightarrow \infty$  for every  $z \in \overline{\mathbb{D}}$  and from this fact we deduce the mean ergodicity of  $\mathcal{T}$ . Since the mean ergodicity already implies the uniqueness of the common fixed point, we have that these three assertions are actually equivalent when  $\mathcal{T}$  is not formed by elliptic automorphisms.

Note that if  $\varphi_t(z) \rightarrow b \in \mathbb{D}$  as  $t \rightarrow \infty$  for every  $z \in \overline{\mathbb{D}}$ , then actually  $\varphi_t \rightarrow b$  uniformly in  $\overline{\mathbb{D}}$ . This follows from a direct application of Dini’s theorem. Therefore pointwise convergence and uniform convergence are equivalent. This does not hold if  $b \in \partial\mathbb{D}$  in general, as can be seen in Example 2.7 below.

**Theorem 2.4** *If  $\Phi$  has a Denjoy–Wolff  $b \in \mathbb{D}$ , then the following assertions are equivalent:*

- (i)  $\mathcal{T}$  is mean ergodic,
- (ii)  $\Phi$  has a unique common fixed point,
- (iii)  $\partial\Omega$  is bounded,
- (iv)  $\varphi_t \rightarrow b$  as  $t \rightarrow \infty$  pointwise (or uniformly) in  $\overline{\mathbb{D}}$ ,
- (v)  $\|T_t - C_b\| \rightarrow 0$  as  $t \rightarrow \infty$ ,
- (vi)  $T_t f \rightarrow f(b)$  as  $t \rightarrow \infty$ , for all  $f \in A(\mathbb{D})$ ,
- (vii)  $\mathcal{T}$  is uniformly mean ergodic.

**Proof** The equivalence of (i), (ii) and (iv) follows from Theorem 2.2 and Remark 2.3. The equivalence between (ii) and (iii) is Proposition 2.1.(iii).

Now we consider (iv) $\Rightarrow$ (v). Again, consider  $b = 0$  and let  $f \in A(\mathbb{D})$  with  $\|f\|_\infty \leq 1$ . Define  $g(z) := \frac{1}{2}(f(z) - f(0))$ . Then  $g \in A(\mathbb{D})$ ,  $\|g\|_\infty \leq 1$  and  $g(0) = 0$ . Therefore, by the Schwarz lemma,  $|g(z)| \leq |z|$  for all  $z \in \overline{\mathbb{D}}$ . We have, for  $z \in \overline{\mathbb{D}}$

$$|((T_t - C_0)(f))(z)| = |f(\varphi_t(z)) - f(0)| \leq 2|\varphi_t(z)|.$$

Since by (iv) we have that for each  $\varepsilon > 0$  there is  $t_0 \geq 0$  such that  $|\varphi_t(z)| < \varepsilon/2$  for all  $t \geq t_0$  and  $z \in \overline{\mathbb{D}}$ , we deduce that if  $t \geq t_0$ ,  $\|T_t - C_0\| < \varepsilon$ .

Finally, we see that (v) clearly implies (vi) and (vii) and also that both (vi) and (vii) imply (i), and we conclude.  $\square$

**Theorem 2.5** *If  $\Phi$  has a Denjoy–Wolff  $b \in \partial\mathbb{D}$  the following assertions are equivalent:*

- (i)  $\mathcal{T}$  is mean ergodic,
- (ii)  $\Phi$  has a unique common fixed point,

- (iii)  $\partial\Omega$  is connected,
- (iv)  $\varphi_t \rightarrow b$  as  $t \rightarrow \infty$  pointwise in  $\overline{\mathbb{D}}$ .

**Proof** The equivalence follows from Theorem 2.2, Proposition 2.1.(iv) and Remark 2.3. □

**Theorem 2.6** *If  $\Phi$  has a Denjoy–Wolff  $b \in \partial\mathbb{D}$  the following assertions are equivalent:*

- (i)  $T_t f \rightarrow f(b)$  as  $t \rightarrow \infty$ , for all  $f \in A(\mathbb{D})$ ,
- (ii)  $\varphi_t \rightarrow b$  as  $t \rightarrow \infty$  uniformly in  $\overline{\mathbb{D}}$ ,
- (iii)  $\partial\Omega$  is connected and  $\overline{\Omega}$  does not contain horizontal lines.

**Proof** As usual we assume that  $b = 1$ . The equivalence of (i) and (ii) is clear by the continuity of the  $f$  on  $\overline{\mathbb{D}}$ . For the equivalence of (ii) and (iii) note that since (ii) implies (i), which implies (i) in Theorem 2.5, we may assume that  $\partial\Omega$  is connected and that  $\Phi$  has a unique common fixed point for the rest of the proof. We first show that (ii) is equivalent to

$$\forall K \subset \mathbb{C} \text{ compact } \exists t_0 \forall t \geq t_0, (\overline{\Omega} + t) \cap K = \emptyset. \tag{*}$$

Indeed, assume  $\varphi_t \rightarrow b$  as  $t \rightarrow \infty$  uniformly in  $\overline{\mathbb{D}}$  and let  $K \subset \mathbb{C}$  be a compact set. Then  $\hat{\mathbb{C}} \setminus K$  is a neighbourhood of  $\infty = h(1)$  in  $\hat{\mathbb{C}}$ . By the continuity of  $h : \overline{\mathbb{D}} \rightarrow \hat{\mathbb{C}}$ , we can find  $\varepsilon > 0$  such that

$$h(B(1, \varepsilon) \cap \overline{\mathbb{D}}) \subseteq \hat{\mathbb{C}} \setminus K.$$

For this  $\varepsilon > 0$ , there is  $t_0 \geq 0$  such that for  $t \geq t_0$ ,  $\varphi_t(\overline{\mathbb{D}}) \subseteq B(1, \varepsilon) \cap \overline{\mathbb{D}}$ . Therefore, we get for  $t \geq t_0$ , (with  $\infty + t = \infty$ )

$$\overline{\Omega} + t = h(\overline{\mathbb{D}}) + t = h(\varphi_t(\overline{\mathbb{D}})) \subseteq h(B(1, \varepsilon) \cap \overline{\mathbb{D}}) \subseteq \hat{\mathbb{C}} \setminus K.$$

For the converse, assume (\*) holds. Let  $\varepsilon > 0$  and define the compact set  $M := \overline{\mathbb{D}} \setminus B(1, \varepsilon)$ . Since  $h^{-1}(\{\infty\}) = \{1\}$ , we have that  $K := h(M)$  is a compact set in  $\mathbb{C}$ . Then, for  $t_0 \geq 0$  as in (\*), we have for  $t \geq t_0$

$$\varphi_t(\overline{\mathbb{D}}) = h^{-1}(\overline{\Omega} + t) \subseteq h^{-1}(\hat{\mathbb{C}} \setminus K) \subseteq B(1, \varepsilon) \cap \overline{\mathbb{D}}.$$

Now we show that (\*) is equivalent to (iii). First assume that (iii) does not hold. Then there is  $c \in \mathbb{R}$  such that  $ci + \mathbb{R} \subseteq \overline{\Omega}$ . Let  $K = \{ci\}$ . Since  $ci - t \in \overline{\Omega}$  for all  $t \geq 0$ , we have that for all  $t \geq 0$ ,  $(\overline{\Omega} + t) \cap K \neq \emptyset$ , which contradicts (\*).

For the converse, assume (\*) does not hold, then there are a compact set  $K \subset \mathbb{C}$  and a sequence  $t_n \nearrow \infty$ , with  $(\overline{\Omega} + t_n) \cap K \neq \emptyset$ . Thus there exists a sequence  $(z_n)_n \subset \overline{\Omega}$  with  $z_n + t_n \in K$ . By passing to a subsequence, we may assume that  $c_n = \text{Im}(z_n + t_n) = \text{Im} z_n$  converges to  $c \in \mathbb{R}$ . Then the line  $ci + \mathbb{R}$  is contained in  $\overline{\Omega}$ , contradicting (iii).

Indeed, if  $s + ic \notin \overline{\Omega}$  for some  $s \in \mathbb{R}$ , then, since  $\mathbb{C} \setminus \overline{\Omega}$  is open, there exists  $\varepsilon > 0$  such that  $s + i(c - \varepsilon, c + \varepsilon) \subseteq \mathbb{C} \setminus \overline{\Omega}$ . By the geometry of  $\overline{\Omega}$ , we have that

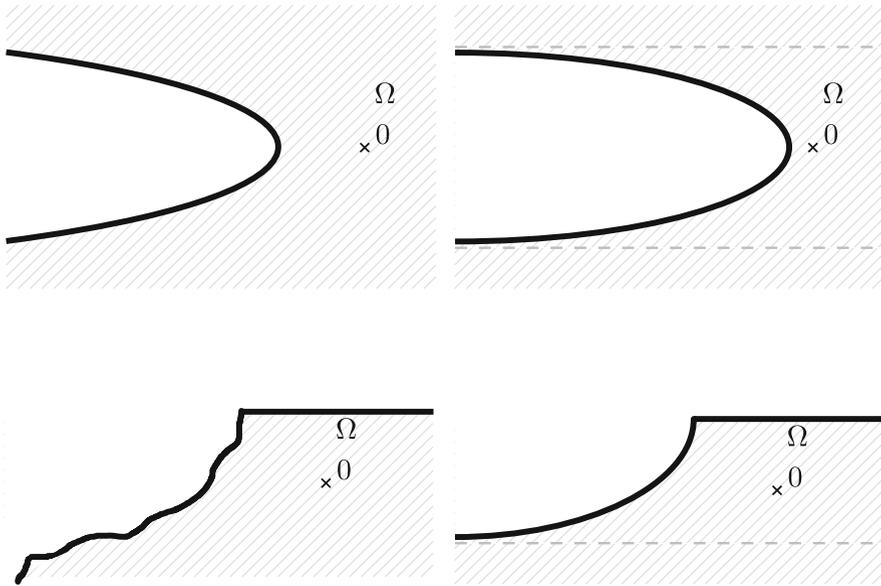


Fig. 3 Pairs of similar domains with different behaviour

$\{t + i(c - \varepsilon, c + \varepsilon) : t \leq s\} \subseteq \mathbb{C} \setminus \overline{\Omega}$ , which contradicts the convergence of  $c_n \rightarrow c$ .  
□

**Example 2.7** We show an example where  $\partial\Omega$  is connected and contains horizontal lines. Define the (not normalized) Koebe map by

$$h_K(z) := \left(\frac{1+z}{1-z}\right)^2 - 1.$$

Then  $\Omega = h_K(\mathbb{D}) = \mathbb{C} \setminus (-\infty, -1]$  has connected boundary and contains horizontal lines.

According to Theorems 2.5 and 2.6, the semigroup  $\mathcal{T}$  defined by  $\varphi_t(z) = h_K^{-1}(h_K(z) + t)$  is mean ergodic but  $C_{\varphi_t}$  is not pointwise convergent. One can actually see this by noting that for each  $t \geq 0$  there is  $\alpha_t \in \partial\mathbb{D}$  with  $\varphi_t(\alpha_t) = -1$ .

Some other examples showing the difference between Theorem 2.5 and Theorem 2.6 can be seen in Fig. 3.

**Theorem 2.8** *If  $\Phi$  has a Denjoy–Wolff  $b \in \partial\mathbb{D}$ , then  $\mathcal{T}$  is not uniformly mean ergodic. In particular,  $\mathcal{T}$  is not convergent in norm.*

**Proof** We may assume that  $\partial\Omega$  is connected since, if it were not, then by Theorem 2.5,  $\mathcal{T}$  would not even be mean ergodic.

Consider the Koebe map  $h_K : \mathbb{D} \rightarrow \Omega_K := \mathbb{C} \setminus (-\infty, -1]$ , defined by

$$h_K(z) := \left(\frac{1+z}{1-z}\right)^2 - 1.$$

This map is a biholomorphism, has a continuous extension to  $\overline{\mathbb{D}}$  and satisfies  $h_K(0) = 0$ ,  $h_K(1) = \infty$  and  $h_K(-1) = -1$ . Furthermore,  $h_K$  keeps the sign of the imaginary part, i.e.,  $\text{sgn}(\text{Im}(z)) = \text{sgn}(\text{Im}(h_K(z)))$ , for all  $z \in \mathbb{D} \cup \{-1\}$ .

Since  $\Omega \neq \mathbb{C}$ , there is  $\omega_0 \notin \Omega$ . Therefore, also  $\omega_0 - t \notin \Omega$ . Let  $s_{\omega_0} : \Omega \rightarrow \Omega_K$  be defined by the shift  $s_{\omega_0}(\omega) = \omega - \omega_0 - 1$ .

Now, for  $0 < \delta < 1$ , define  $g_\delta : \mathbb{D} \rightarrow \mathbb{D}$  by

$$g_\delta(z) = h_K^{-1}(\delta \cdot s_{\omega_0}(h(z))).$$

We have  $\delta \cdot \Omega_K \subset \Omega_K$  and therefore the functions  $g_\delta$  are well-defined. Since each of the functions forming  $g_\delta$  is holomorphic, also  $g_\delta$  is holomorphic in  $\mathbb{D}$ . We show that  $g_\delta \in A(\mathbb{D})$ . Let  $\alpha \in \partial\mathbb{D}$  and define  $\alpha_\omega := \delta \cdot s_{\omega_0}(h(\alpha))$ . If  $\alpha_\omega \in \Omega_K$ , then  $g_\delta(\alpha) \in \mathbb{D}$ , since  $h_K^{-1}$  is well-defined in  $\Omega_K$ . If  $\alpha_\omega = \infty$ , then  $g_\delta(\alpha) = 1$ . And if  $\alpha_\omega = -1$ , then  $g_\delta(\alpha) = -1$ . Now, consider  $\alpha_\omega \in (-\infty, -1)$ , then  $h_K^{-1}(\alpha_\omega)$  contains two points of  $\partial\mathbb{D}$ , whose imaginary parts have different signs. However, if  $(z_n)_n \subseteq \mathbb{D}$  is a sequence approaching  $\alpha$ , then  $(\delta \cdot s_{\omega_0}(h(z_n)))_n$  approaches  $\alpha_\omega$ . Hence, there is  $n_0 \in \mathbb{N}$  such that for  $n \geq n_0$ , we have  $\text{Im}(\delta \cdot s_{\omega_0}(h(z_n))) > 0$  (resp.  $< 0$ ). Thus  $\lim_{n \rightarrow \infty} g_\delta(z_n)$  is the point of  $h_K^{-1}(\alpha_\omega)$  with positive (resp. negative) imaginary part.

To conclude, let  $R > 0$  and  $0 < t \leq R$ . We have

$$\begin{aligned} g_\delta(\varphi_t(0)) &= h_K^{-1}(\delta \cdot s_{\omega_0}(h(\varphi_t(0)))) \\ &= h_K^{-1}(\delta \cdot s_{\omega_0}(h(0) + t)) \\ &= h_K^{-1}(\delta(-\omega_0 - 1 + t)). \end{aligned}$$

Hence, for every  $\varepsilon > 0$  and every  $R > 0$ , there is  $\delta > 0$  such that  $|g_\delta(\varphi_t(0))| < \varepsilon$ , for  $0 < t \leq R$ , since  $h_K^{-1}(0) = 0$ . Therefore, we have  $|(T_{[R]}(g_\delta))(0)| < \varepsilon$ . Now, for each  $R \geq 0$  and each  $0 < \varepsilon < 1$ , we have, for  $\delta$  small enough,

$$|((T_{[R]} - C_1)g_\delta)(0)| \geq |(C_1g_\delta)(0)| - |(T_{[R]}(g_\delta))(0)| > 1 - \varepsilon.$$

From this we deduce that  $\mathcal{T}$  is not uniformly mean ergodic. Thus it is also not norm convergent. □

**Open Question** At the beginning of the current section we assumed  $\partial_\infty\Omega$  to be locally connected. We do this to ensure the existence of the continuous extension of  $h$  to  $\overline{\mathbb{D}}$ . This is not necessary if the Denjoy–Wolff point is in the interior of the disc (see Theorem 1.3). Do the Theorems 2.5 and 2.6 still hold without this assumption?

### 3 Mean Ergodic Operators on Bergman and Hardy Spaces

In this section we deduce a characterization for mean ergodicity on the spaces  $A^p$  and  $H^p$ , for  $1 \leq p < \infty$ .

For  $1 \leq p < \infty$ , the Bergman space of order  $p$  is

$$A^p := \{f \in H(\mathbb{D}) : \|f\|_{A^p}^p := \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty\},$$

where  $dA$  is the normalized Lebesgue measure on  $\mathbb{D}$ .

For  $1 \leq p < \infty$ , the Hardy space of order  $p$  is

$$H^p := \{f \in H(\mathbb{D}) : \|f\|_{H^p}^p := \lim_{r \rightarrow 1^-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty\}.$$

In [6] and [15] it is proved that every semigroup  $\Phi$  defines a  $C_0$ -semigroup of composition operators in  $H^p$  and  $A^p$ , for  $1 \leq p < \infty$ . Arendt et al. already considered this type of  $C_0$ -semigroups in [2,3].

In the case of the discrete iterates  $\{T^n\}_n$  of an operator  $T$  the concept of *power boundedness* means that the set  $\{T^n\}_n$  is equicontinuous. In the case when  $T$  is defined on a Banach space, power boundedness happens when  $\sup \|T^n\| < \infty$ . In our continuous case we consider the property of  $\{C_{\varphi_t}\}_{t \geq 0}$  being equicontinuous, related to the mean ergodicity. We remark here that in the case of  $A(\mathbb{D})$ , we have  $\|C_{\varphi_t}\| \leq 1$ , for every  $t \geq 0$ , therefore the equicontinuity is trivial.

**Theorem 3.1** *Let  $\Phi = \{\varphi_t : \mathbb{D} \rightarrow \mathbb{D}\}_{t \geq 0}$  be a semigroup, with a Denjoy–Wolff point  $b \in \overline{\mathbb{D}}$ , generating the semigroup of operators  $\mathcal{T} = \{T_t = C_{\varphi_t}\}_{t \geq 0}$  on  $A^p$  or  $H^p$ , with  $1 \leq p < \infty$ . Then the following assertions are equivalent:*

- (i)  $b \in \mathbb{D}$ ,
- (ii)  $\mathcal{T}$  is equicontinuous,
- (iii)  $\mathcal{T}$  is mean ergodic.

**Proof** Assume (i), without loss of generality, we may assume that  $b = 0$ . By [17, Theorem 11.6],

$$\|C_{\varphi_t}\|_{A^p} \leq \left( \frac{1 + |\varphi_t(0)|}{1 - |\varphi_t(0)|} \right)^{\frac{2}{p}}$$

and

$$\|C_{\varphi_t}\|_{H^p} \leq \left( \frac{1 + |\varphi_t(0)|}{1 - |\varphi_t(0)|} \right)^{\frac{1}{p}}.$$

Since  $\varphi_t(0) = 0$  for all  $t \geq 0$ , we deduce (ii).

Now assume (ii). For  $p > 1$ , the spaces  $A^p$  and  $H^p$  are reflexive. Therefore, equicontinuity already implies mean ergodicity, by well-known results, see e.g. [1, Corollary 2].

For the case of  $p = 1$  note that the continuous inclusions  $A^2 \hookrightarrow A^1$  and  $H^2 \hookrightarrow H^1$  have dense range, therefore  $(C_{\varphi_t})_t$  is an equicontinuous net, which is convergent to  $C_b$  in the weak operator topology on a dense subspace. Thus  $(C_{\varphi_t})_t$  also converges in

the weak operator topology to  $C_b$  in  $\mathcal{L}(A^1)$  and  $\mathcal{L}(H^1)$ . Again, we deduce the mean ergodicity from the convergence in the weak operator topology, see e.g. [13, p.76].

To prove that (iii) implies (i), suppose  $b \in \partial\mathbb{D}$ . Then  $f(z) = \log(b - z)$  defines a function  $f \in A^p$  and  $f \in H^p$ . Let  $\delta_0$  be the evaluation at 0, then  $\delta_0 \in (A^p)'$  and  $\delta_0 \in (H^p)'$ . We have

$$\begin{aligned} \lim_{R \rightarrow \infty} |\delta_0(T_{[R]}f)| &\geq \lim_{R \rightarrow \infty} |\operatorname{Re} \delta_0(T_{[R]}f)| \\ &= \lim_{R \rightarrow \infty} \left| \operatorname{Re} \frac{1}{R} \int_0^R \log(b - \varphi_t(0)) dt \right| = \infty. \end{aligned}$$

Hence  $\{T_{[R]}f\}_{R>0}$  is not weakly bounded, it is therefore not bounded. Thus it is not convergent and  $\mathcal{T}$  is not mean ergodic.  $\square$

**Author Contributions** Not applicable.

**Funding** The second author was partially supported by the project MTM2016-76647-P and also partially supported by the grant PAID-01-16 of the Universitat Politècnica de València.

**Code availability** Not applicable.

## Declarations

**Conflict of interest** The authors declare that they have no conflicts of interest.

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