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Positive definite functions as uniformly ergodic multipliers of the Fourier algebra

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Abstract

Let G be a locally compact group and let ϕ be a positive definite function on G with $\phi(e) = 1$. This function defines a multiplication operator M_ϕ on the Fourier algebra $A(G)$ of G. The aim of this paper is to classify the ergodic properties of the operators M_ϕ, focusing on several key factors, including the subgroup $H_\phi = \{x \in G\colon \phi(x) = 1\}$, the spectrum of M_ϕ, or how "spread-out" a power of M_ϕ can be. We show that the multiplication operator M_ϕ is uniformly mean ergodic if and only if H_ϕ is open and 1 is not an accumulation point of the spectrum of M_ϕ. Equivalently, this happens when some power of ϕ is not far, in the multiplier norm, from a function supported on finitely many cosets of H_ϕ. Additionally, we show that the powers of M_ϕ converge in norm if, and only if, the operator is uniformly mean ergodic and $H_\phi = \{x \in G\colon |\phi(x)| = 1\}$.

Keywords Ergodic measure · Uniformly ergodic measure · Random walk · Mean ergodic operator · Uniformly mean ergodic operator · Convolution operator · Locally compact group · Measure algebra

1 Introduction

The ergodicity of the random walk governed by a probability measure μ on a group G can be described through the behaviour of the $L^1(G)$-convolution operator, $\lambda_1(\mu)f = \mu * f$, restricted to the augmentation ideal $L_1^0(G) = \{f \in L^1(G)\colon \int f(x)\mathrm{d}m_G(x) = 0\}$. When G is commutative, the properties of $\lambda_1(\mu)$ can be recast, via the Fourier–Stieltjes transform, in

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terms of the multiplication operators $M_{\widehat{\mu}}\colon A(\widehat{G}) \to A(\widehat{G})$, where $A(\widehat{G})$ denotes the Fourier algebra of $\widehat{G}$, the algebra of functions on $\widehat{G}$ that can be obtained as the Fourier transform of some function in $L_1(G)$.

While the Fourier–Stieltjes transform can hardly be given sense beyond commutative or compact groups, the Fourier algebra can still be meaningfully defined as an algebra of functions on any locally compact group. Harmonic Analysis on commutative groups, which is so often developed with the aid of transforms, admits then two noncommutative generalizations, one through convolution operators on the noncommutative algebra $L_1(G)$ and another one leaning on multiplication operators on the Fourier algebra $A(G)$.

In our previous works [15, 16] we dealt with the analysis of the convolution operator $\lambda_1(\mu)$ for general locally compact groups. We now address the multiplication operator approach. That means that we look at the bounded linear operator $M_\phi\colon A(G) \to A(G)$ induced by a positive definite function ϕ on G, given by $M_\phi u = \phi \cdot u$. This operator is said to be *mean ergodic* when $\frac{1}{n}(M_\phi + M_\phi^2 + \cdots + M_\phi^n)$ is convergent in the strong operator topology. If the strong operator topology is replaced by the operator norm topology, we say that M_ϕ is *uniformly mean ergodic*.

In this work, we particularly focus on the uniform problem. While our results parallel the ones obtained in [16], different approaches are often needed, especially due to the lack of a viable transform for discrete groups (transforms were important in our treatment of compact groups in [16]) and the absence of group structure in the quotient G/H_ϕ, where H_ϕ denotes the closed subgroup $\phi^{-1}(\{1\})$. It is known that M_ϕ is mean ergodic precisely when H_ϕ is open. We show in this paper that uniform mean ergodicity of M_ϕ is characterized through the spectral properties of M_ϕ, through the proximity of ϕ to a function supported on finitely many translates of H_ϕ, and through its relation with operator quasi-compactness. Our approach is also used to characterize under which conditions the iterates M_ϕ^n of M_ϕ converge.

We next dualize the ergodicity properties of the random walk induced by a probability measure and model them through ϕ. This consists in analyzing the convergence to 0 of the means $\frac{1}{n}(M_\phi u + \cdots + M_\phi^n u)$, for every $u \in A(G)$ with $u(e) = 0$. We prove that ϕ is ergodic if and only if ϕ is *adapted*, i.e. if $H_\phi = \{e\}$, precisely as in the commutative case and unlike the scenario for general locally compact groups in the convolution case. This equivalence can be deduced from the work of Kaniuth, Lau and Ülger [24] and of Guex [19], but the scope of these proofs is somewhat blurred by nonessential hypotheses or misguided connections. We provide here a short direct proof. Replacing in this context the strong operator topology by the uniform norm, we see that ϕ is uniformly ergodic if and only if G is discrete and 1 is isolated in the spectrum of ϕ. When it comes to powers of ϕ we prove that the operators M_ϕ^n converge to 0 uniformly on $A_0(G)$ if and only if ϕ is uniformly ergodic and $|\phi(x)| = 1$ implies $x = 1$.

2 Preliminaries

We need to establish the notation and basic facts concerning both ergodicity and Fourier algebras.

2.1 Preliminaries on mean ergodicity

For a Banach space X, we denote by $\mathcal{L}(X)$ the space of linear and continuous operators from X to itself and for a Hilbert space $\mathbb{H}$, the space of its unitary operators is denoted $\mathcal{U}(\mathbb{H})$. The Cesàro means of $T \in \mathcal{L}(X)$ are

$$T_{[n]} = \frac{1}{n}\sum_{j=1}^{n} T^j,$$

where $T^j = T \circ \cdots \circ T$ denotes the j-th iterate of T. We say that T is *mean ergodic* when $(T_{[n]})_n$ converges in the strong operator topology to an operator $P \in \mathcal{L}(X)$ (and then P has to be the projection on the subspace of X consisting of the vectors fixed by T), that is, when $(T_{[n]}x)_n$ converges to Px for each $x \in X$. When $(T_{[n]})_n$ converges to P in the operator norm, we say that T is *uniformly mean ergodic*. When T is mean ergodic, the space decomposes as $X = \overline{(I-T)(X)} \oplus \ker(I-T)$. For further information on this topic, see the second chapter of [25].

If X is a Banach space and $T \in \mathcal{L}(X)$ we denote by $\sigma(T)$ its spectrum, i.e. $\sigma(T) := \{z \in \mathbb{C} : \; zI - T \text{ is not invertible}\}$. The resolvent mapping $R(\cdot, T) : \mathbb{C}\backslash\sigma(T) \to \mathcal{L}(X)$, $z \mapsto R(z,T) := (zI - T)^{-1}$ is holomorphic when $\mathcal{L}(X)$ is endowed with its norm topology (cf [8, Chapter VII]).

We state here a useful characterization of uniform mean ergodicity, due to the combined results of Dunford and Lin.

Theorem 2.1 *Let $T \in \mathcal{L}(X)$, where X is a Banach space. If $(\|T^n\|)_n$ is bounded, then the following assertions are equivalent:*

(1) *T is uniformly mean ergodic,*
(2) *$(I-T)(X)$ is closed,*
(3) *either $1 \notin \sigma(T)$, or 1 is a pole of order 1 of the resolvent.*

The equivalence of (1) and (3) was proved by Dunford in [10] and Lin proved the equivalence with (2) in [29].

A property closely related to uniform mean ergodicity is quasi-compactness. An operator $T \in \mathcal{L}(X)$ is called *quasi-compact* if there is a compact operator K such that $\|T^n - K\| < 1$ for some $n \geqslant 1$. Yosida and Kakutani found that quasi-compactness provides sufficient conditions for uniform mean ergodicity and asymptotic convergence of the iterates of an operator. We record this in the following theorem. Here, (and elsewhere in the paper) we will denote by $\sigma_p(T)$ the point-spectrum of T, the set of its eigenvalues, and by $\mathbb{T}$ the set of complex numbers of modulus 1.

Theorem 2.2 (Theorem 4 and its Corollary in page 205 of [37]) *Let X be a complex Banach space and let $T \in \mathcal{L}(X)$ be a quasi-compact operator. If $(\|T^n\|)_n$ is a bounded sequence, the following assertions hold:*

(i) *$(T_{[n]})_n$ converges in norm to a finite rank projection P.*
(ii) *$(T^n)_n$ converges in norm to a finite rank projection P if, and only if, $\sigma_p(T) \cap \mathbb{T} \subseteq \{1\}$.*
(iii) *$(T^n)_n$ converges in norm to 0 if, and only if, $\sigma_p(T) \cap \mathbb{T} = \varnothing$.*

2.2 Preliminaries on Fourier algebras

The notation of [22] will be generally adopted here.

Let G be a locally compact group with identity e and Haar measure m_G.

2.2.1 The group C^* -algebra

The group C*-algebra of G, denoted by $C^*(G)$, is defined as the completion of $L^1(G)$ with the norm given by

$$\|f\|_* = \sup_{\pi} \|\pi(f)\|,$$

where the supremum is taken over the set of all unitary representations $\pi : G \longrightarrow \mathcal{U}(\mathbb{H}_\pi)$, and $\pi(f) \in \mathcal{L}(\mathbb{H}_\pi)$, is defined by

$$\langle \pi(f)\xi, \eta \rangle = \int_G \langle \pi(t)\xi, \eta \rangle\, f(t)\mathrm{dm}_G(t), \quad \xi, \eta \in \mathbb{H}_\pi.$$

2.2.2 The Fourier and Fourier–Stieltjes algebras

The dual space of $C^*(G)$ can be identified with the Fourier–Stieltjes algebra $B(G)$ consisting of functions on G of the form π_ρ^η, where $\pi : G \longrightarrow \mathcal{U}(\mathbb{H}_\pi)$ is a continuous unitary representation of G and $\xi, \eta \in \mathbb{H}_\pi$, that are given by

$$\pi_\xi^\eta(x) = \langle \pi(x)\xi, \eta \rangle .$$

The duality between $C^*(G)$ and $B(G)$ is provided by

$$\left\langle f, \pi_\xi^\eta \right\rangle = \int \pi_\xi^\eta(t) f(t)\mathrm{dm}_G(t) = \langle \pi(f)\xi, \eta \rangle , \quad f \in L^1(G),$$

which also defines the norm of $B(G)$, by taking the supremum over all $f \in L^1(G)$, with $\|f\|_* \leqslant 1$. The functions π_ξ^η are known as *matrix coefficients* of the representation π of G. When $\xi = \eta$ they are called *diagonal* matrix coefficients. Diagonal matrix coefficients are positive definite functions, so that $B(G)$ is spanned by the set of positive definite functions on G. We denote the set of continuous positive definite functions by $P(G)$ and by $P^1(G)$ those $\phi \in P(G)$ with $\phi(e) = 1$. A useful summary of their properties can be found in Section 1.4 of [22]. We would like to display here one that will be especially useful for us, it is proved in [22, Theorem 3.7.7]:

$$\text{if } \ \phi(y) = 1, \ \text{ then } \ \phi(xy) = \phi(x) \ \text{ for every } x \in G. \tag{2.1}$$

Among the unitary representations of a locally compact group G, one is specially relevant for its capacity to carry the properties of the group. This is the left regular representation $\lambda_2 \colon G \to \mathcal{U}(L_2(G))$ defined by $\lambda_2(t)f(s) = (\delta_t * f)(s) = f(t^{-1}s)$, $s, t \in G$ and $f \in L_2(G)$. As happens with every unitary representation of G, λ_2 can be extended to a representation of $M(G)$. This extension is given by

$$\lambda_2(\mu)f(s) = (\mu * f)(s) = \int_G f(t^{-1}s)d\mu(t) \quad s \in G.$$

The Fourier algebra $A(G)$ can be described in several ways. We outline two of them here. For a complete description we refer to [11] and Chapter 2 of [22]. If $C_c(G)$ denotes the space of continuous functions with compact support on G, one can define

$$A(G) = \overline{\langle B(G) \cap C_c(G) \rangle},$$

where the closure is taken in the norm of $B(G)$. One can then prove that

$$A(G) = \{f * \tilde{g} \ : \ f, g \in L^2(G)\},$$

where, for a given function $g\colon G\to\mathbb{C}$, $\tilde{g}\colon G\to\mathbb{C}$ is defined by $\tilde{g}(t)=\overline{g(t^{-1})}$. The elements of $A(G)$ can also be seen as matrix coefficients of the left regular representation: if $u=f*\tilde{g}$, $f,g\in L_2(G)$, then $u=(\lambda_2)_{g}^{\overline{f}}$. The Fourier algebra so defined becomes a closed ideal of the Fourier–Stieltjes algebra.

The Lebesgue decomposition of $B(G)$, first introduced in [2, Remarque 3.20] and developed in [23, 31], identifies a closed linear subspace $B_s(G)$ of $B(G)$ such that every $\phi\in B(G)$ can be expressed as $\phi=\phi_a+\phi_s$ with $\phi_a\in A(G)$ and $\phi_s\in B_s(G)$, and $\|\phi\|=\|\phi_a\|+\|\phi_s\|$.

At a few points, we will be using two other norms on $B(G)$, the uniform norm $\|\phi\|_\infty=\sup\{|\phi(x)|\colon x\in G\}$ and the *multiplier norm*. The multiplier norm is defined by

$$\|\phi\|_{MA(G)}=\|M_\phi\|.$$

With this norm, the algebra $B(G)$ is a subspace of the *multiplier algebra* $M(A(G))$ of $A(G)$, made of those operators $S\in\mathcal{L}(A(G))$ with $S(uv)=S(u)v$ for every $u,v\in A(G)$. It is well-known that such an S is always a multiplication operator by some bounded continuous function on G (see, for instance, Theorem 1.2.2 of [26]).

The norms $\|\cdot\|_{M(A(G))}$ and $\|\cdot\|_{B(G)}$ are equivalent (and even coincide) if and only if G is amenable. Amenability of $A(G)$ is also a necessary and sufficient condition for $M(A(G))=B(G)$, see [22, Chapter 5] for further information on $MA(G)$. It will be useful to record here that, for any locally compact group and $\phi\in B(G)$,

$$\|\phi\|_\infty\leqslant\|\phi\|_{MA(G)}\leqslant\|\phi\|_{B(G)}. \tag{2.2}$$

2.2.3 The dual spaces $A(G)^*$ and $B(G)^*$

The Banach space dual of $A(G)$ can be identified with the von Neumann algebra of G, denoted by $\mathrm{VN}(G)$. This algebra is defined as the closure, in the weak operator topology, of $\lambda_2(L^1(G))$ and its identification with the dual space of $A(G)$ is realized through the duality

$$\left\langle \lambda_2(f),(f_1*\tilde{f_2})^\vee\right\rangle=\langle\lambda_2(f)f_1,f_2\rangle_{L_2(G)},\quad f\in L_1(G),\ (f_1*\tilde{f_2})^\vee\in A(G).$$

When G is commutative and $\widehat{G}$ is the group of characters of G, the Fourier–Stieltjes transform establishes a linear isometry between $B(G)$ and $M(\widehat{G})$ and between $A(G)$ and $L^1(\widehat{G})$. The same identifications makes the algebra $C^*(G)$ isometrically isomorphic to $C_0(\widehat{G})$ and $VN(G)$ to $L_\infty(\widehat{G})$.

The dual space of $B(G)^*$ can be canonically identified with the universal enveloping von Neumann algebra of $C^*(G)$. We give some details here on that construction, see, e.g., [22, Remark 2.1.6], or [36, Section III.2], for full details and proofs. The GNS construction associates to each $\phi\in P^1(G)$, a specific representation, $\pi_\phi\colon G\to\mathcal{U}(\mathbb{H}_\phi)$ and a vector $v_\phi\in\mathbb{H}_\phi$ in such a way that $\phi=(\pi_\phi)_{v_\phi}^{v_\phi}$. The *universal representation* is the representation of G, $\omega\colon G\to\mathcal{U}\big(\mathbb{H}_{\mathrm{uni}}\big)$ by unitary operators on the Hilbert space $\mathbb{H}_{\mathrm{uni}}:=\bigoplus_{\phi\in P^1(G)}\mathbb{H}_\phi$ given by

$$\omega(g)\Bigg(\sum_{\phi\in P^1(G)}w_\phi\Bigg)=\sum_{\phi\in P^1(G)}\pi_\phi(g)w_\phi,\ \text{where } w_\phi\in\mathbb{H}_\phi.$$

The double commutant of $\omega(G)$ in $\mathcal{L}\big(\mathbb{H}_{\mathrm{uni}}\big)$ is then a von Neumann algebra, the *universal enveloping von Neumann algebra* of $C^*(G)$, denoted as $W^*(G)$. From the identification of $B(G)$ with $(C^*(G))^*$, one can obtain a natural identification

$$\langle\omega(g),\psi\rangle=\psi(g),\qquad\text{for any }\psi\in B(G).$$

The elements of $W^*(G)$ can be ultraweakly approximated by linear combinations of operators in $\omega(G)$. It follows that every unitary representation of $\pi : G \to \mathcal{U}(\mathbb{H})$, can be extended to a representation $\pi'' : W^*(G) \to \mathcal{L}(\mathbb{H})$. The duality between $W^*(G)$ and $B(G)$ is then given by the relation

$$\langle L, \phi \rangle = \left\langle \pi''(L)\xi, \eta \right\rangle \quad \text{for each } L \in W^*(G) \text{ and } \phi = \pi_\xi^\eta \in B(G). \tag{2.3}$$

The preceding construction furnishes $W^*(G)$ with a naturally defined multiplication, the multiplication of operators. There is a different way to address the multiplication of elements in $W^*(G)$, one that can be used in the second dual of any Banach algebra. It was introduced, in a more general form, in [1] and has since been known as the Arens-construction, or the Arens multiplication. It is well known, see [7, Theorem 7.1], that for $\mathcal{A} = C^*(G)$, this multiplication coincides with the operator multiplication that $W^*(G)$ acquires as the enveloping algebra of $C^*(G)$.

The Arens multiplication on the bidual $\mathcal{A}^{**}$ of a Banach algebra $\mathcal{A}$ is defined after introducing two module actions on $\mathcal{A}^*$, one by elements of $\mathcal{A}$ and the other by elements of $\mathcal{A}^{**}$. These actions, along with the Arens multiplication, are presented below, for $a, b \in \mathcal{A}, \phi \in \mathcal{A}^*$ and $T, T_1, T_2 \in \mathcal{A}^{**}$.

$$\begin{aligned} \phi \cdot a \in \mathcal{A}^* &\text{ is defined by } \langle \phi \cdot a, b \rangle = \langle \phi, ab \rangle, \\ T \cdot \phi \in \mathcal{A}^* &\text{ is defined by } \langle T \cdot \phi, a \rangle = \langle T, \phi \cdot a \rangle, \\ T_1 \cdot T_2 \in \mathcal{A}^{**} &\text{ is defined by } \langle T_1 \cdot T_2, \phi \rangle = \langle T_1, T_2 \cdot \phi \rangle. \end{aligned}$$

It is clear from the above definitions that

$$\begin{aligned} \|\phi \cdot a\| &\leqslant \|\phi\| \cdot \|a\|, \\ \|T \cdot \phi\| &\leqslant \|T\| \|\phi\|, \\ \|T_1 \cdot T_2\| &\leqslant \|T_1\| \|T_2\|. \end{aligned}$$

If $\mathcal{A}$ is a Banach algebra, $\Delta(\mathcal{A})$ will always denote its spectrum, i.e., the set of multiplicative bounded functionals of $\mathcal{A}$.

2.3 Preliminaries on ergodicity in Fourier algebras

We outline here the concepts that will let us dualize the ergodic theory of random walks on groups.

Every function $\phi \in B(G)$ defines a bounded linear operator $M_\phi : A(G) \to A(G)$ given by $M_\phi u(s) = u(s)\phi(s)$, $u \in A(G)$, $s \in G$. If we define the augmentation ideal in $A(G)$,

$$A_0(G) = \{u \in A(G) \,:\, u(e) = 0\},$$

then $A_0(G)$ is stable under the action of M_ϕ. We denote by $M_\phi^0 : A_0(G) \to A_0(G)$ the restriction of M_ϕ to $A_0(G)$.

We will also write, for $\phi \in B(G)$, $\phi_{[n]} = \frac{1}{n}(\phi + \cdots + \phi^n)$ and will use interchangeably the expressions $(M_\phi)_{[n]}$ and $M_{\phi_{[n]}}$, as well as the expressions $(M_\phi)^n$ and M_{ϕ^n}.

Definition 2.3 Let G be a locally compact group. We say that $\phi \in P^1(G)$ is:

- ergodic if $\lim_n M^0_{\phi_{[n]}} u = 0$, for every $u \in A_0(G)$,
- uniformly ergodic if $\lim_n \|M^0_{\phi_{[n]}}\| = 0$,
- completely mixing if $\lim_n M^0_{\phi^n} u = 0$, for every $u \in A_0(G)$,

- uniformly completely mixing if $\lim_n \|M^0_{\phi^n}\| = 0$.

A longstanding objective in the study of the random walk induced by a probability measure μ has been to classify its ergodic behaviour through algebraic properties of the support of μ. Among these properties two stand out: adaptedness, the support of μ is not contained in any proper closed subgroup of G, and strict aperiodicity, the support of μ is not contained in any translate of a proper closed normal subgroup of G. When G is abelian, these properties can be characterized by properties of the Fourier–Stieltjes transform $\widehat{\mu}$. We take these characterizations as definitions on positive definite functions:

Definition 2.4 Let G be a locally compact group and let $\phi \in P^1(G)$. We define the sets $H_\phi = \{x \in G : \phi(x) = 1\}$ and $E_\phi = \{x \in G : |\phi(x)| = 1\}$. We say ϕ is:

- *adapted,* if $H_\phi = \{e\}$,
- *strictly aperiodic,* if $E_\phi = \{e\}$.

These properties characterize the convergence of the means and powers of M_ϕ in the strong operator topology, see [32, Theorems 2.2 and 2.8]. By well-known results, e.g. [22, Theorem 3.7], this is also equivalent to the convergence of the means and powers of ϕ in the compact-open topology.

Proposition 2.5 *Let G be a locally compact group and let $\phi \in P^1(G)$. The following are equivalent.*

(i) *M_ϕ is mean ergodic.*
(ii) *H_ϕ is an open set.*
(iii) *$(\phi_{[n]})$ is convergent to $\mathbf{1}_{H_\phi}$ in the compact open topology.*

Proposition 2.6 *Let G be a locally compact group and let $\phi \in P^1(G)$. The following assertions are equivalent.*

(i) *$(M_{\phi^n})_n$ is convergent in $\mathcal{L}(A(G))$ endowed with the strong operator topology.*
(ii) *$H_\phi = E_\phi$ is an open set*
(iii) *$(\phi^n)_n$ is convergent to $\mathbf{1}_{H_\phi}$ in the compact open topology.*

3 Uniform mean ergodicity of M_ϕ

In our paper [16], the uniform mean ergodicity of the convolution operator $\lambda_1(\mu)$ was characterized in terms of the position of 1 in its spectrum, the nonsingularity of convolution powers of μ and quasicompactness of $\lambda_1(\mu)$. Here, we return to these perspectives in the context of multiplication operators.

3.1 Uniform mean ergodicity and the spectrum

By the Dunford–Lin Theorem, Theorem 2.1, 1 is isolated in $\sigma(T)$ whenever T is a uniformly mean ergodic operator. The converse may fail for general operators (the operator $T = I - V$ with V being the Volterra operator is such an example, see [16, Remark 5.26 (a)]) but it does hold in the case of convolution operators, see [16, Theorem 5.5] and, as we prove in this section, in the case of the multiplication operators M_ϕ discussed here.

Our main tools will be the two following general results.

Theorem 3.1 ([18],Theorem 1; [3],Corollary 10, Remark 11) *Let X be a Banach space, $H \subseteq X^*$ a separating subspace, $\Omega \subseteq \mathbb{C}$ a domain and $a \in \Omega$. If $f : \Omega\backslash\{a\} \longrightarrow X$ is a holomorphic function such that $x^* \circ f$ admits a holomorphic extension to Ω for each $x^* \in H$, then f admits a holomorphic extension to Ω.*

We apply this Theorem to subalgebras $\mathcal{A} \subseteq C_b(Y)$ of the space of bounded continuous functions of a topological space Y. Under the conditions of the proposition below, we use the family $\{\delta_{y,f} \ : \ y \in Y, f \in \mathcal{A}\} \subseteq \mathcal{L}(\mathcal{A})^*$, given by $\left\langle \delta_{y,f}, T \right\rangle = Tf(y)$, as a separating subspace of $\mathcal{L}(\mathcal{A})$ (i.e., for different $T_1, T_2 \in \mathcal{L}(\mathcal{A})$ there are y, f such that $T_1 f(y) \neq T_2 f(y)$).

Proposition 3.2 *Let Y be a Hausdorff, normal locally compact space and let $\mathcal{A}$ be a subalgebra of $C_b(Y)$ equipped with a Banach algebra norm $\|\cdot\|$ satisfying $\|\cdot\| \geqslant \|\cdot\|_\infty$ and*

$$\text{for each } y \in Y \text{ there is } f \in \mathcal{A} \text{ with } f(y) \neq 0. \tag{3.1}$$

If $\psi \in C_b(Y)$ is such that $M_\psi : \mathcal{A} \to \mathcal{A}$, $f \mapsto \psi f$ is a bounded linear operator with $\|M_\psi\| \leqslant 1$, then M_ψ is uniformly mean ergodic if and only if 1 is not an accumulation point in $\sigma(M_\psi)$.

Proof We only have to show that $M_\psi : \mathcal{A} \to \mathcal{A}$ is uniformly mean ergodic when 1 is not an accumulation point in $\sigma(M_\psi)$. In this proof we will need to work with two resolvent operators: for a given $z \in \mathbb{C}$, we will denote by $R^{C_b(Y)}(z, M_\psi)$ the resolvent operator with M_ψ seen as an operator in $\mathcal{L}(C_b(Y))$, and by $R^{\mathcal{A}}(z, M_\psi)$ the resolvent operator with values in $\mathcal{L}(\mathcal{A})$. We use analogous notation for the spectra. We observe that (3.1) implies that $\overline{\psi(Y)} \subseteq \sigma^{\mathcal{A}}(M_\psi)$. It is also well known that $\sigma^{C_b(Y)}(M_\psi) = \overline{\psi(Y)}$. As $\sigma^{C_b(Y)}(M_\psi) = \overline{\psi(Y)} \subseteq \sigma^{\mathcal{A}}(M_\psi)$, for each $f \in \mathcal{A} \subseteq C_b(Y)$ and $y \in Y$ we get that

$$R^{C_b(Y)}(z, M_\psi)f(y) = \frac{f(y)}{z - \psi(y)} \in C_b(Y), \text{ for each } z \in \mathbb{C} \setminus \overline{\psi(Y)},$$
$$R^{\mathcal{A}}(z, M_\psi)f(y) = \frac{f(y)}{z - \psi(y)} \in \mathcal{A}, \text{ for each } z \in \mathbb{C} \setminus \sigma^{\mathcal{A}}(M_\psi).$$

By hypothesis, there is $R > 0$ such that $B(1, R) \cap \sigma^{\mathcal{A}}(M_\psi) = \{1\}$. This yields that $|1 - \psi(y)| \geqslant R > 0$, for each $y \in Y$ with $\psi(y) \neq 1$. Furthermore, the set $\psi^{-1}(1) = \psi^{-1}(B(1, R))$ is then open in Y and the multiplication operator $M_\psi : C_b(Y) \to C_b(Y)$ is uniformly mean ergodic by [4, Theorem 2.7].

Then, by using (1) implies (3) in Theorem 2.1, we get that $(\cdot - 1)R^{C_b(Y)}(\cdot, M_\psi) : B(1, R)\backslash\{1\} \to \mathcal{L}(C_b(Y))$ admits holomorphic extension in $z = 1$.

We have that, for each $z \in B(1, R)$,

$$\left\langle \delta_{y,f}, (z-1)R^{\mathcal{A}}(z, M_\psi) \right\rangle = \left\langle \delta_{y,f}, (z-1)R^{C_b(Y)}(z, M_\psi) \right\rangle,$$

for every $y \in Y$ and every $f \in \mathcal{A}$.

Having seen that $\delta_{y,f} \circ (\cdot - 1)R^{\mathcal{A}}(\cdot, M_\psi)$ is holomorphic in $B(1, R)$ and, taking into account that $\{\delta_{y,f} \ : \ y \in Y, f \in \mathcal{A}\} \subseteq \mathcal{L}(\mathcal{A})^*$ is separating, we deduce from Theorem 3.1 that 1 is a pole of order 1 of $R^{\mathcal{A}}(\cdot, M_\Psi)$. We conclude by applying (3) implies (1) in Theorem 2.1. □

We can now characterize uniform mean ergodicity of the operators M_ϕ. The same approach can be used to characterize the convergence of the means in the norm of $B(G)$.

Theorem 3.3 *Let G be a locally compact group and let $\phi \in P^1(G)$.*

(i) $M_\phi \in \mathcal{L}(A(G))$ *is uniformly mean ergodic if and only if* 1 *is isolated in* $\sigma(M_\phi)$.
(ii) $(M_{\phi^n})_n$ *is convergent in the norm topology if and only if* $\sigma(M_\phi) \cap \mathbb{T} = \{1\}$ *and* 1 *is isolated in* $\sigma(M_\phi)$.
(iii) $(\phi_{[n]})$ *is convergent to* 1_{H_ϕ} *in* $B(G)$ *if and only if* 1 *is isolated in* $\sigma(\phi)$.
(iv) (ϕ^n) *is convergent to* 1_{H_ϕ} *in* $B(G)$ *if and only if* $\sigma(M_\phi) \cap \mathbb{T} = \{1\}$ *and* 1 *is isolated in* $\sigma(\phi)$

Proof All the statements follow from Proposition 3.2 with $A(G) \subseteq B(G) \subset C_b(G)$. For Statement (i), we apply Proposition 3.2 applied to $M_\phi : A(G) \to A(G)$ and for Statement (iii), we apply Proposition 3.2 applied to $\widetilde{M}_\phi : B(G) \to B(G)$, $f \mapsto \phi \cdot f$. Note that $\|\widetilde{M}_\phi\| = \|\phi\|$ (since $1 \in B(G)$) and $\sigma(\widetilde{M}_\phi) = \sigma(\phi)$. We obtain statements (ii) and (iv) after combining statements (i) and (iii), respectively, with [30, Corollaire 3]. □

Remark 3.4 The hypothesis $\phi \in P_1(G)$ in Theorem 3.3 can be relaxed. For instance, in (i) and (ii) we only need $\phi \in MA(G)$ and $(\|M_\phi^n\|)_n$ power bounded, and in (iii) and (iv) $\phi \in B(G)$ and $(\|\phi^n\|)_n$ bounded.

3.2 Uniform mean ergodicity and spread-out functions

If G is amenable and ϕ is adapted (or even if H_ϕ is compact), uniform mean ergodicity of M_ϕ implies that the means of ϕ converge to an element of $A(G)$ (namely, the characteristic function of H_ϕ). Since the singular part in the Lebesgue decomposition $B(G) = A(G) \bigoplus B_s(G)$ is a closed vector subspace, it follows immediately that some power of ϕ is not going to be singular. This is the original argument for convolution operators. In this latter case, the converse is proved using the connection between adapted spread-out measures (measures with some non-singular convolution power) and quasi-compact operators (see next subsection for this theme). If the measure is not adapted, one may consider the same measure restricted to its support group H_μ (the smallest closed subgroup of G that contains the support of μ) and show that $\lambda_1(\mu)$ is uniformly mean ergodic if and only if some power of μ is not singular with respect to the Haar measure of H_μ (see Remark 4.8 of [16]).

This approach faces some difficulties in the case of the multiplication operators M_ϕ on general locally compact groups. The reason for this is twofold.

Firstly, the norms of M_ϕ and ϕ do not necessarily coincide in nonamenable groups, making the Lebesgue decomposition less useful in that case. We will instead lean on the characters of $MA(G)$ to deduce that, given an adapted $\phi \in B(G)$, for the multiplier M_ϕ to be uniformly mean ergodic, it is necessary and sufficient that, for some $n \in \mathbb{N}$, ϕ^n is *not far,* in the multiplier norm, from $A(G)$.

Secondly, the reduction to the adapted case is not as straightforward as for convolution operators. The natural path here would be to work with the quotient G/H_ϕ, but this may fail to be a group. One way of remedying this could be to replace $A(G)$ and $B(G)$ by $A_{\lambda_{G/H}}$ and $B_{\lambda_{G/H}}$ as defined by Arsac, [2], with $\lambda_{G/H}$ denoting the quasi-regular representation of G/H. But this option brings some pathologies, such as $A_{\lambda_{G/H}}$ not being an ideal (nor a subalgebra) in $B_{\lambda_{G/H}}$. The *right* approach turns out to be recurring to the algebras $A(G/H)$ and $B(G/H)$ introduced in [13].

Definition 3.5 (*Forrest* [13]) Let H be an open subgroup of the locally compact group G and let $p\colon G \to G/H$ denote the quotient map.

- The *Fourier–Stieltjes algebra* of G/H is defined as

$$B(G/H) = \{\psi \in B(G) \,:\, \psi(xh) = \psi(x),\ x \in G, h \in H\}.$$

- The *Fourier algebra* of G/H is defined as

$$A(G/H) = \overline{\langle \psi \in B(G/H) \ : \ p(\operatorname{supp}\psi) \text{ is finite}\rangle}.$$

Note that

$$A(G/H) = \overline{\langle \mathbf{1}_{xH}, x \in G\rangle},$$

and that this definition of $A(G/H)$ brings back $A(G)$ when $H = \{e\}$.

We can now give sense to the notion of spread-out functions, in the nonadapted case.

Definition 3.6 Let G be a locally compact group and let $\phi \in P^1(G)$. Consider the quotient map $p\colon G \to G/H_\phi$ and let $\tilde{\phi} : G/H \to \mathbb{C}$ be defined so that $\phi = \tilde{\phi} \circ p$. We say that $\tilde{\phi}$ is *spread-out* if there exist $k \in \mathbb{N}$ and $v \in A(G/H_\phi)$ such that $\|M_{\tilde{\phi}}^k - M_v\| < 1$. We say that ϕ is spread-out when v can be taken in $A(G)$.

It is clear from the preceding definition that, when G is amenable, ϕ is spread-out if and only if $\phi^k \notin B_s(G)$ for some $k \in \mathbb{N}$.

The following fact, a straightforward adaptation of [22, Theorem 2.3.8], will be useful in the proof of Theorem 3.11 below.

Lemma 3.7 *Let G be a locally compact group and let H be an open subgroup of G. If $0 \neq \chi \in \Delta(A(G/H))$, then there is $x \in G$ with $\chi(u) = u(x)$ for every $u \in A(G/H)$.*

Proof Let $\chi \in \Delta(A(G/H))$ and assume $\chi = T_x$ for no $x \in G$, where $T_x \in \Delta(A(G/H)$ is the point evaluation $T_x u = u(x)$ for $u \in A(G/H)$. Then (see the proof of [21, Theorem 2.1.8]), $\ker\chi \neq \ker T_x$ for all $x \in G$. So, for each $x \in G$ there exists $f_x \in A(G/H)$ such that $\chi(f_x) = 1$ and $f_x(x) = 0$.

By density of the functions with finite support in G/H, we may assume that there is $f_0 = \sum_{j=1}^N \alpha_j \mathbf{1}_{x_j H}$ with $\chi(f_0) = 1$.

Now define $f = f_0 \cdot f_{x_1} \cdots f_{x_N} \in A(G/H)$. The definition of f, forces $f = 0$, since $f_{x_j}(x_j h) = 0$, for all $h \in H$, and $f_0(x) = 0$ whenever $x \notin \bigcup_{j=1}^N x_j H$. However, $\chi(f) = \chi(f_0)\cdot\chi(f_{x_1})\cdots\chi(f_{x_N}) = 1$, a contradiction. □

We now study the stability under multiplication of those elements of $W^*(G)$ that are bounded for the multiplier norm. We first give them a name.

Definition 3.8 Let G be a locally compact group. We define

$$W_M^*(G) = \left\{T \in W^*(G)\colon \ |\langle T, \psi\rangle| \leqslant C\|M_\psi\| \text{ for every } \psi \in B(G)\right\}.$$

If $T \in W_M^*(G)$, $\|T\|_{W_M^*(G)}$, is defined as

$$\sup\left\{|\langle T, \psi\rangle| : \psi \in B(G) \text{ and } \|M_\psi\| \leqslant 1\right\}.$$

Lemma 3.9 *$W_M^*(G)$ is a $*$-subalgebra of $W^*(G)$.*

Proof The set $W_M^*(G)$ is clearly a vector subspace of $W^*(G)$.

We now show that $T_1, T_2 \in W_M^*(G)$ implies that $T_1 \cdot T_2 \in W_M^*(G)$.

If $\psi \in B(G)$, then

$$\begin{aligned}\left|\langle T_1 \cdot T_2, \psi \rangle\right| &= \left|\langle T_1, T_2 \cdot \psi \rangle\right| \\ &\leqslant \|T_1\|_{W^*_M(G)} \cdot \|M_{T_2 \cdot \psi}\| \\ &= \sup_{\substack{u \in A(G) \\ \|u\| \leqslant 1}} \|T_1\|_{W^*_M(G)} \cdot \|T_2 \cdot \psi \cdot u\|_{B(G)} \\ &\leqslant \sup_{\substack{u \in A(G) \\ \|u\| \leqslant 1}} \|T_1\|_{W^*_M(G)} \cdot \|T_2\|_{W^*(G)} \cdot \|\psi \cdot u\|_{B(G)} \\ &\leqslant \|T_1\|_{W^*_M(G)} \cdot \|T_2\|_{W^*(G)} \cdot \|M_\psi\|.\end{aligned}$$

And $T_1 \cdot T_2 \in W^*_M(G)$.

That $T^* \in W^*_M(G)$ whenever $T \in W^*_M(G)$ is a consequence of $\langle T^*, \phi \rangle = \left\langle \pi''(T^*)\xi, \xi \right\rangle = \overline{\langle T, \phi \rangle}$ for every $\phi = \pi_\xi^\xi \in \mathcal{P}(G)$, and hence for every $\phi \in B(G)$. □

For our next proof, we need a description of the quasi-regular representation $\lambda_{G/H}$. If H is an open subgroup of a locally compact group, $\lambda_{G/H} \colon G \to \mathcal{U}\big(\ell_2(G/H)\big)$ is given by

$$\lambda_{G/H}(x) f(tH) = f(x^{-1}tH), \quad f \in \ell_2(G/H),\ x \in G,\ tH \in G/H.$$

If $\delta_{xH}, x \in G$, denotes the element of $\ell_2(G/H)$ that takes the value 1 on xH and 0 elsewhere, then

$$\big(\lambda_{G/H}\big)_{\delta_{xH}}^{\delta_{yH}} = \mathbf{1}_{yHx^{-1}}. \tag{3.2}$$

We have therefore that coefficients of $\lambda_{G/H}$ need not belong to $B(G/H)$. They do, however, belong when $xH = H$.

Lemma 3.10 *Let G be a locally compact group with an open subgroup H and let $T_1, T_2 \in W^*(G)$. If $\langle T_2, u \rangle = 0$ for every $u \in A(G/H)$, then $\langle T_1 \cdot T_2, \mathbf{1}_H \rangle = 0$.*

Proof We first observe that, given $f = \sum_{i=1}^N \alpha_i \delta_{x_i H} \in \ell_2(G/H)$ with finite support (in G/H), taking into account that $\sum_{i=1}^N \overline{\alpha_i}\, \mathbf{1}_{x_i H} \in A(G/H)$, we use (3.2) and the identification in (2.3) to get

$$\begin{aligned}\left\langle \lambda''_{G/H}(T_2)\delta_H, f \right\rangle &= \left\langle T_2, \big(\lambda_{G/H}\big)_{\delta_H}^{f} \right\rangle \\ &= \left\langle T_2, \sum_{i=1}^N \overline{\alpha_i}\, \mathbf{1}_{x_i H} \right\rangle = 0.\end{aligned}$$

Since functions with finite support are dense in $\ell_2(G/H)$, we deduce that $\lambda''_{G/H}(T_2)\delta_H = 0$. Hence,

$$\begin{aligned}\langle T_1 \cdot T_2, \mathbf{1}_H \rangle &= \left\langle \lambda''_{G/H}(T_1 \cdot T_2)\delta_H, \delta_H \right\rangle \\ &= \left\langle \lambda''_{G/H}(T_1)\big(\lambda''_{G/H}(T_2)\delta_H\big), \delta_H \right\rangle = 0.\end{aligned}$$

□

Theorem 3.11 *Let G be a locally compact group and let $\phi \in P^1(G)$. The following assertions are equivalent:*

(1) *The operator M_ϕ is uniformly mean ergodic.*
(2) *The subgroup H_ϕ is open and the function $\widetilde{\phi}$ is spread-out.*

Proof Assume that M_ϕ is uniformly mean ergodic. The subgroup H_ϕ must be open by Proposition 2.5. Suppose, towards a contradiction, that $\tilde{\phi}$ is not spread-out.

We consider then the Banach algebra $M(A(G/H_\phi))$ consisting of the multipliers $M_f \in M(A(G))$ with f constant on the left cosets of H_ϕ. The algebra $A(G/H_\phi)$ is then a closed ideal of $M(A(G/H_\phi))$. Let M/A and B/A denote, respectively, the quotients of $M(A(G/H_\phi))$ and $B(G/H_\phi)$ by $A(G/H_\phi)$. A consequence of $\tilde{\phi}$ not being spread-out is that, for every $j \in \mathbb{N}$

$$\|\phi^j\|_{\frac{M}{A}} = 1.$$

The spectral radius of ϕ in the quotient algebra M/A must therefore be 1 and, by Gelfand duality, there must be a character $T \in \Delta(M/A)$ with $\left|\left\langle T, \phi^j \right\rangle\right| = 1$, for every $j \in \mathbb{N}$. This character can be extended by the Hahn-Banach theorem to a functional in the unit ball of $MA(G)^*$ which, when restricted to $B(G)$, yields a functional on $B(G)$. We keep the name T for this functional. It is clear from the construction of T that $T \in W_M^*(G)$. It is also clear that

$$\left|\left\langle T, \phi^j \right\rangle\right| = 1, \text{ for every } j \in \mathbb{N} \text{ and}$$
$$\langle T, u \rangle = 0, \text{ for every } u \in A(G/H_\phi).$$

Choose now, for each $j \in \mathbb{N}$, a unitary representation $\pi_j : G \to \mathcal{U}(\mathbb{H}_j)$ and $\xi_j \in \mathbb{H}_j$, $\|\xi_j\| = 1$, so that $\phi^j = (\pi_j)_{\xi_j}^{\xi_j}$. We see then that, for each $j \in \mathbb{N}$,

$$\left|\left\langle \pi_j''(T)\xi_j, \xi_j \right\rangle\right| = \left|\left\langle T, \phi^j \right\rangle\right| = 1.$$

Since $1 = \|\xi_j\| \geqslant \|\pi''(T)\xi_j\|$, we deduce that $\pi''(T)\xi_j = \left\langle T, \phi^j \right\rangle \xi_j$.

We next consider the operator T^*T.

$$\begin{aligned}\left\langle T^*T, \phi^j \right\rangle &= \left\langle \pi''(T^*T)\xi_j, \xi_j \right\rangle \\ &= \left\langle \pi''(T)\xi_j, \pi''(T)\xi_j \right\rangle \\ &= |\left\langle T, \phi^j \right\rangle|^2 = 1.\end{aligned}$$

Since, by Lemma 3.9, $T^*T \in W_M^*(G)$ and, by Lemma 3.10, $\left\langle T^*T, \mathbf{1}_{H_\phi} \right\rangle = 0$, we obtain the following contradiction with the uniform mean ergodicity of M_ϕ.

$$\begin{aligned}1 &= \left\langle T^*T, \phi_{[n]} \right\rangle \\ &= \left\langle T^*T, \phi_{[n]} - \mathbf{1}_{H_\phi} \right\rangle \\ &\leqslant \|T^*T\|_{W_M^*(G)} \cdot \|M_{\phi_{[n]}} - \mathbf{1}_{H_\phi}\|.\end{aligned}$$

We have thus proved that Assertion (1) implies Assertion (2).

Assume next that $\widetilde{\phi}$ is spread-out. This means that there exist $\phi_a \in A(G/H_\phi)$ and $\phi_s \in B(G/H_\phi)$ with $\|M_{\phi_s}\| < 1$ such that $\phi^k = \phi_a + \phi_s$, for some $k \in \mathbb{N}$.

Suppose that $(z_n)_n \subseteq \sigma(M_\phi)$ is a sequence with $\lim_n z_n = 1$. There will be then $\chi_n \in \Delta M(A(G))$ such that $\chi_n(\phi) = z_n$. Since, for every $n \in \mathbb{N}$, $\chi_n|_{B(G)} \in W_M^*(G)$

$$|\chi_n(\phi_a)| = |z_n^k - \chi_n(\phi_s)| \geqslant |z_n|^k - \|\phi_s\|_{M(A(G))}.$$

We can therefore find $\alpha > 0$ and n_0 big enough so that $|\chi_n(\phi_a)| > \alpha > 0$, for all $n \geqslant n_0$.

Now, by Lemma 3.7, for each $n \in \mathbb{N}$ there is $x_n \in G$ such that $\chi_n(\phi_a) = \phi_a(x_n)$. As $|\phi_a(x_n)| > \alpha$, for all $n \geqslant n_0$, and functions in $A(G/H)$ must vanish at infinity in G/H ($A(G/H)$ is spanned by functions of finite support in a norm stronger that the uniform norm), we see that $\{x_n H_\phi \ : \ n \in \mathbb{N}\}$ is a finite set. The sets $\{\phi_a(x_n) : n \in \mathbb{N}\}$ and $\{\phi_s(x_n) : n \in \mathbb{N}\}$ will have to be finite as well, for both ϕ_a and ϕ_s are constant on cosets of H_ϕ.

Since $A(G/H_\phi)$ is an ideal of $B(G/H_\phi)$, $\phi_a\phi_s \in A(G/H_\phi)$ and we have that

$$\phi_a(x_n)\chi_n(\phi_s) = \chi_n(\phi_a)\chi_n(\phi_s) = \chi_n(\phi_a\phi_s) = \phi_a(x_n)\phi_s(x_n),$$

so that $\chi_n(\phi_s) = \phi_s(x_n)$, for any $n \geqslant n_0$.

Recalling that $z_n^k = \chi_n(\phi^k) = \phi_a(x_n) + \phi_s(x_n)$, we find that the set $\{z_n : n \in \mathbb{N}\}$ has to be finite. As $(z_n)_n$ was an arbitrary sequence in $\sigma(M_\phi)$ that approximates 1, we conclude that 1 is isolated in $\sigma(M_\phi)$. Theorem 3.3 shows that M_ϕ is uniformly mean ergodic. □

Corollary 3.12 *Let G be a locally compact group and let $\phi \in P^1(G)$. Then the sequence of means $(\phi_{[n]})_n$ converges to $\mathbf{1}_{H_\phi}$ if and only if H_ϕ is open and, for some $k \in \mathbb{N}$, there is $u \in A(G/H_\phi)$ such that $\|\phi^k - u\| < 1$.*

Proof The proof of Theorem 3.11 can be used here replacing $M(A(G))$ by $B(G)$, and using the right version of Theorem 3.3 throughout. □

Example 3.13 The operator M_ϕ can be uniformly mean ergodic, even if the sequence $(\phi_{[n]})_n$ is not convergent to $\mathbf{1}_{H_\phi}$.

Proof Let $G = F(X)$ denote the free group on countably many generators. For a given word $w \in F$, let $|w|$ denote its word-length.

We now consider the function $\phi : G \to \mathbb{C}$ given by $\phi(w) = 9^{-|w|}$ for every $w \in G$, where we are assuming, as usual, that $|e| = 0$.

This is a so-called *Haagerup* function and, as any such, is positive definite, [9, Theorem 1].

By [9, Theorem 2], applied to ϕ^k for each $k \in \mathbb{N}$, $\|\phi^k - u\| > 1$ for every $u \in A(G)$. Hence $(\phi_{[n]})_n$ is not convergent by Corollary 3.12.

On the other hand, if we define $\psi = \phi - \mathbf{1}_{\{e\}}$, then

$$\sup_{w\in G} \left|\psi(w)(1+|w|)^2\right| = \sup_{n\geqslant 1} 9^{-n}(n+1)^2 < \frac{1}{2}.$$

The estimate obtained in [5, Corollary 2.4], a consequence of Haagerup's inequality [20, Lemma 1.4], shows then that $\|M_\psi\| < 1$. This implies that $\lim (M_\psi)^n = 0$. Noting that, for each $n \in \mathbb{N}$, $\psi^n = \phi^n - \mathbf{1}_e$, we conclude that $\lim_n \|M_{\phi^n} - M_{\mathbf{1}_e}\| = 0$. □

Corollary 3.14 *Let G be a locally compact group and let $\phi \in P^1(G)$ be adapted. Then the following are equivalent*

(i) *$M_\phi : A(G) \to A(G)$ is uniformly mean ergodic.*
(ii) *G is discrete and ϕ is spread-out.*
(iii) *1 is isolated in $\sigma(M_\phi)$.*

All the preceding results can be easily adapted to analyze the convergence of the iterates M_ϕ^n. We state here the adapted case.

Corollary 3.15 *Let G be a locally compact group and let $\phi \in P^1(G)$ be adapted. Then the following are equivalent*

(i) $(M_\phi^n)_n$ *is norm convergent.*
(ii) 1 *is isolated in* $\sigma(M_\phi)$ *and* $\sigma(M_\phi)\cap\mathbb{T}=\{1\}$.
(iii) G *is discrete and* ϕ *is both spread-out and strictly aperiodic.*
(iv) $M_\phi : A(G) \to A(G)$ *is uniformly mean ergodic and* $(M_{\phi^n})_n$ *converges in the strong operator topology.*

Proof Theorem 3.3 shows that assertion (i) implies assertion (ii). Corollary 3.14 shows that (ii) implies the first two conditions of (iii); strict aperiodicity is deduced directly from $\overline{\phi(G)} \subseteq \sigma(M_\phi)$.

We now check that (iii) implies (i) and the first three statements will have been shown to be equivalent.

Assume now that assertion (iii) holds, so that $\phi=\phi_a+\phi_s$ with $\phi_a\in A(G)$ and $\|\phi_s\|<1$. Consider then $z\in\sigma(\phi)\cap\mathbb{T}$. There is then $\chi\in\Delta(B(G))$ such that $\chi(\phi)=z$. It follows that $\chi|_{A(G)}\neq\{0\}$, for, otherwise,

$$|\chi(\phi)|=|\chi(\phi_s)|\leqslant\|\phi_s\|<1.$$

Knowing that there is $x\in G$ such that $\chi(u)=u(x)$ for every $u\in A(G)$, we pick $u_x\in A(G)$ with $u_x(x)\neq 0$. And using that $\phi u_x\in A(G)$, we see that $\phi(x)=\chi(\phi)$, i.e., that $x\in E_\phi$. But this means $z=1$, ϕ being strictly aperiodic. We conclude that $\sigma(M_\phi)\cap\mathbb{T}\subseteq\{1\}$. Corollary 3.14 then ensures that we can apply statement (ii) of Theorem 3.3 and deduce that $(M_\phi^n)_n$ is norm convergent.

The equivalence of (iii) and (iv) follows from Corollary 3.14 and Proposition 2.6. □

Remark 3.16 It could be tempting to conjecture that, for a strictly aperiodic $\phi\in P^1(G)$, $\sigma(M_\phi)\cap\mathbb{T}=\{1\}$. In that way, the spread-out property would take care of uniform ergodicity and strict aperiodicity would take care of $\sigma(M_\phi)\cap\mathbb{T}=\{1\}$. That is, however, false. Let μ be a measure supported on an independent Cantor subset P of $\mathbb{T}$, μ is then strictly aperiodic. But powers of μ are mutually singular, [35, Theorem 5.3.2], and it is known that, in that case $\sigma(\mu)=\mathbb{T}$, [17, Theorem 6.1.1].

3.3 Uniform mean ergodicity and quasi-compact operators

The theorem of Yosida and Kakutani (Theorem 2.2) states that a quasi-compact operator with bounded powers is always uniformly mean ergodic. On the other hand, if G is discrete, Lau [27] characterizes compact multipliers on $A(G)$ as precisely those given by functions of $A(G)$. Hence, for a discrete amenable group and an adapted $\phi\in\mathcal{P}^1(G)$, M_ϕ is quasi-compact if and only if ϕ is spread-out. We observe next that the condition G discrete is required when discussing quasi-compactness.

Lemma 3.17 *Let* G *be a locally compact group and let* $\phi\in P^1(G)$ *with* H_ϕ *open. If* M_ϕ *is quasi-compact, then* G *is discrete.*

Proof Assume G is not discrete. Then, H_ϕ, being open, is not discrete either. For each $u\in A(H_\phi)$, we denote by $\overset{\circ}{u}\in A(G)$ the extension of u to G, by setting it to 0 in $G\setminus H_\phi$. By [22, Proposition 2.4.1], we have $\|\overset{\circ}{u}\|=\|u\|_{A(H_\phi)}$. Applying [6, Theorem 3.2] we can find a sequence $(u_n)_n\subseteq A(H_\phi)$ with $\|u_n\|_{A(H_\phi)}=1$ for every $n\in\mathbb{N}$, and $\|u_n-u_m\|_{A(H_\phi)}=2$, for $n\neq m$.

Since M_ϕ is quasi-compact, there exist $k\in\mathbb{N}$ and K a compact operator with $\|M_\phi^k-K\|<A<1$. Since K is compact we can assume, passing to a subsequence, if necessary, that $(K\overset{\circ}{u}_n)$

is a Cauchy sequence. Then for n, m big enough, $\|K \mathring{u}_n - K \mathring{u}_m\| < 2 - 2A$. We have then,

$$\begin{aligned} 2A &> \|M_\phi^k \mathring{u}_n - M_\phi^k \mathring{u}_m\| - \|K \mathring{u}_n - K \mathring{u}_m\| \\ &\geqslant \|M_\phi^k \mathring{u}_n - M_\phi^k \mathring{u}_m\| + 2A - 2 \\ &= \|\mathring{u}_n - \mathring{u}_m\| + 2A - 2. \end{aligned}$$

As $\|\mathring{u}\| = \|u\|$ for every $u \in A(H_\phi)$, we find that

$$2A > \|u_n - u_m\|_{A(H_\phi)} + 2A - 2 = 2A,$$

which is a contradiction, so G is discrete. □

One can characterize quasi-compactness of M_ϕ in the same way as quasi-compactness of the convolution operator $\lambda_1(\mu)$ was characterized in [16, Theorem 5.24]. Our proof here is slightly simpler.

Theorem 3.18 *Let G be a locally compact group and $\phi \in P^1(G)$. The following assertions are equivalent:*

(1) *The operator M_ϕ is quasi-compact.*
(2) *$(M_{\phi_{[n]}})_n$ is norm convergent to a finite dimensional projection.*
(3) *G is discrete, H_ϕ finite and $\tilde{\phi}$ spread-out.*

Proof Assertion (1) implies assertion (2) by Yosida-Kakutani Theorem 2.2.

Assume (2) holds, then $\tilde{\phi}$ is spread-out by Theorem 3.11. Since $(M_{\phi_{[n]}})_n$ converges to $M_{\mathbf{1}_{H_\phi}}$ we get that $\mathbf{1}_{H_\phi} \cdot A(G)$ is finite dimensional. Now, for any finite family $x_1, \ldots, x_N \in H_\phi$, we can find U_N an open neighbourhood of e, such that $x_i U_N \cap x_j U_N = \varnothing$, for $i \neq j$. This implies that $\{\mathbf{1}_{H_\phi}\mathbf{1}_{x_1 U_N}, \ldots, \mathbf{1}_{H_\phi}\mathbf{1}_{x_N U_N}\}$ are linearly independent. As $\mathbf{1}_{H_\phi} \cdot A(G)$ is finite dimensional, we see that H_ϕ has to be finite. Since it is also open (by Theorem 2.5) G is discrete.

Finally, when G is discrete, the functions of $A(G)$ define compact multipliers by [27, Lemma 6.8]. Taking into account that $A(G/H_\phi) \subseteq A(G)$ when H_ϕ is finite, if $\tilde{\phi}$ is spread-out, there are $k \in \mathbb{N}$ and $\phi_a \in A(G)$ such that $\|M_\phi^k - M_{\phi_a}\| < 1$ and M_{ϕ_a} is compact. Therefore assertion (1) holds. □

Example 3.19 Uniform convergence of the means of a multiplication operator M_ϕ, with $\phi \in B(G)$, to a finite dimensional operator does not necessarily imply that M_ϕ is quasi-compact, if ϕ is not positive-definite.

Proof It is indeed enough to consider the constant functions $\phi(x) = \alpha \in \mathbb{T}$, $\alpha \neq 1$, on an infinite group G. The sequence $(M_{\phi_{[n]}})_n$ is then convergent to 0. The operator M_ϕ is not quasi-compact, since if $\|M_\phi^n - K\| = \|\alpha^n Id - K\| < 1$, for some $n \in \mathbb{N}$ and some compact operator K, then K would be compact and invertible, which is impossible unless $A(G)$ is finite dimensional. This shows that, in Theorem 3.18, (2) does not imply (1) when $\phi \notin P(G)$. □

4 Dualizing random walks

If G is a locally compact group, a random walk whose transitions are determined by a probability measure μ is ergodic if and only if the operator $\lambda_1^0(\mu)$, obtained from restricting the convolution operator $\lambda_1(\mu)$ to the augmentation ideal $L_1^0(G) = \{f \in L_1(G) \colon \int f(x) \mathrm{d}m_G(x) = 0\}$, is mean ergodic and its means converge to 0, see [34].

In our context, we replace $L_1^0(G)$ by $A_0(G) = \{u \in A(G) \colon u(e) = 0\}$ and consider the operator M_ϕ^0 that results from restricting M_ϕ, $\phi \in \mathcal{P}^1(G)$, to $A_0(G)$. In this section, we address the problem of characterizing under which conditions the means of M_ϕ^0 converge to 0, both in the strong operator topology and the uniform norm.

4.1 The operator M_ϕ^0

The results in this subsection offer a glance into the nature of the operator M_ϕ^0. They follow exactly the same pattern of the results obtained in [16, Sections 3 and 5] for the operator $\lambda_1^0(\mu)$. We actually need only prove Theorem 4.2, as the rest of the proofs can be applied to this case.

Lemma 4.1 *Let G be a locally compact group and let $\phi \in B(G)$. If there are $u \in A(G) \cap P^1(G)$ and $\psi \in P^1(G)$ such that*

(1) $\|\phi u - \psi\| \geqslant M$ *for some* $M > 0$ *and*
(2) $\|\phi u \psi - \psi\| \leqslant \varepsilon$, *for some* $0 < \varepsilon < M$, *then*

$$\|M_\phi^0\| \geqslant \frac{M - \varepsilon}{2}.$$

Proof Since $u - \psi u \in A_0(G)$,

$$\|M_\phi^0\| \geqslant \frac{\|(\phi - \phi\psi)u\|}{\|u - \psi u\|}.$$

To achieve the estimate of this lemma one just needs to observe that

$$\|(\phi - \phi\psi)u\| \geqslant \|\phi u - \psi\| - \|\phi u \psi - \psi\|$$

and

$$\|u - \psi u\| \leqslant 2.$$

□

For our next proof, we need the concept of TI-net. A net $(u_\alpha)_\alpha \subseteq A(G) \cap P^1(G)$ is a TI-net if

$$\lim_\alpha \|u u_\alpha - u_\alpha\| = 0, \quad \text{for any } u \in A(G).$$

The existence of TI-nets in nondiscrete groups was shown in [33, Proposition 3], see also [6, 12].

Theorem 4.2 *If G is a nondiscrete locally compact group and $\phi \in P^1(G)$, then $\|M_\phi^0\| = 1$.*

Proof Let $(u_\alpha)_\alpha$ be a TI-net in $A(G) \cap P^1(G)$ and let u_0 be any element of $A(G) \cap P^1(G)$. By [6, Lemma 3.1], $\lim_\alpha \|\phi u_0 - u_\alpha\| = 2$, while $\lim_\alpha \|\phi u_0 u_\alpha - u_\alpha\| = 0$. Lemma 4.1 applied, for each α, to $u = u_0$ and $\psi = u_\alpha$ yields that $\|M_\phi^0\| = 1$. □

Corollary 4.3 *Let G be a locally compact group and let $\phi \in P^1(G)$. If ϕ is uniformly ergodic, then G is discrete.*

We do not know if the spectra of M^0_ϕ and M_ϕ are the same but they are definitely related. This is explored in the following theorem. It can be proved exactly as Proposition 5.8 and Corollary 5.9 of [16], we only have to replace [Proposition 3.2, loc. cit.] by Theorem 4.2 here. Recall that a complex number z is in $\sigma_{\mathrm{ap}}(T)$, the *approximate spectrum* of an operator $T \in \mathcal{L}(E)$, E a Banach space, if there exists a sequence x_n in the unit sphere of E such that $\lim_n \|T x_n - z x_n\| = 0$.

Theorem 4.4 *Let G be a locally compact group and let $\phi \in P^1(G)$. Then:*

(1) *If G is not discrete, then $\sigma_{\mathrm{ap}}(M^0_\phi) = \sigma_{\mathrm{ap}}(M_\phi)$.*
(2) *1 is isolated in $\sigma(M_\phi)$ if and only if 1 is isolated in $\sigma(M^0_\phi)$.*

4.2 Ergodicity of ϕ

In analogy with the convolution case, we have defined a function $\phi \in P^1(G)$ to be ergodic when $\phi_{[n]}u$ converges to 0 for every $u \in A_0(G)$. We prove in this subsection that ϕ is ergodic if and only if ϕ is adapted. For convolution operators the situation is more involved. In the case of commutative or compact groups the conclusion is the same, a measure is ergodic if and only if it is adapted. However, when the group is not amenable, no measure can be ergodic [34] and every group that is finitely generated and solvable, but is not virtually nilpotent, admits a nonergodic adapted measure [14].

The project of characterizing ergodicity in $P^1(G)$ has been taken up before, often in somewhat more general contexts. Kaniuth, Lau and Ülger [24, Theorem 3.4] work with multipliers on quite general Banach algebras, albeit requiring them to have bounded approximate identities. This condition is avoided in Theorem 5.1.1 of Guex's Ph.D. dissertation [19] but this theorem is not correct as stated. In [28, Theorem 3.3] Lau and Losert consider so-called *strongly ergodic sequences*. Their definition of strong ergodicity would then mean that $(\pi(\phi_{[n]}))_n$ is convergent, for every representation $\pi : A(G) \to \mathcal{L}(\mathbb{H})$ of $A(G)$ as operators on a Hilbert space $\mathbb{H}$. This requires the existence of a canonical way of extending representations from $A(G)$ to representations of $B(G)$, and for that, again, they have to restrict their characterization to amenable groups.

We provide next a short direct characterization of ergodicity in $P^1(G)$ that subsumes the ones mentioned in the previous paragraph. Our extension is a clean analog of the measure-theoretic concept and does not require amenability.

The following elementary fact will smooth our proof of Theorem 4.6.

Lemma 4.5 *Let G be a locally compact group and let $\phi \in P^1(G)$. If ϕ is adapted and M^0_ϕ is (uniformly) mean ergodic, then ϕ is (uniformly) ergodic.*

Proof Let P be the projection in $\mathcal{L}(A_0(G))$ such that $\lim_n M^0_{\phi_{[n]}} = P$ in the corresponding topology and let $u \in A_0(G)$. If $e \neq x$, then $(Pu)(x) = \lim_n \phi_{[n]}(x)u(x) = 0$. As $Pu \in A_0(G)$, this means that $Pu = 0$, and hence that $P = 0$. □

For our characterization, we need to recall the concept of support of an element of $\mathrm{VN}(G)$, which is based on the module action of $A(G)$ on $\mathrm{VN}(G)$, described in page 6, with $A(G)$ playing the role of $\mathcal{A}$. The support of T, denoted supp T, is then defined as the set of all points $a \in G$ satisfying that $\lambda_2(\delta_a)$ is the weak*-limit of operators of the form $T \cdot u$, $u \in A(G)$. See [11] or [22, Section 2.5] for all this.

Theorem 4.6 *Let G be a locally compact group and let $\phi \in P^1(G)$. Then ϕ is ergodic if and only it is adapted.*

Proof If ϕ is ergodic, then it must also be adapted, else there would exist $e \neq x \in H_\phi$ and $u \in A_0(G)$ with $u(x) \neq 0$, so $|\phi_{[n]}(x)u(x)| = |u(x)| \neq 0$, for all $n \in \mathbb{N}$, contradicting ergodicity.

For the converse, assume that ϕ is adapted and suppose that ϕ is not ergodic. Then M_ϕ^0 cannot be mean ergodic either, by the preceding Lemma, and the ergodic decomposition is not satisfied. Since $\ker(I - M_\phi^0) = \{0\}$, ϕ being adapted, this means that $A_0(G) \neq \overline{\{(I - M_\phi^0)(u) \colon u \in A_0(G)\}}$. There exists then $T \in \mathrm{VN}(G)$ with $T|_{A_0(G)} \neq 0$ such that $\langle T, u\rangle = \langle T, \phi u\rangle$, for every $u \in A_0(G)$.

Let $x \in \mathrm{supp}(T)$. Then $\lambda_2(\delta_x) = w^* - \lim_\alpha T \cdot u_\alpha$, for some net $u_\alpha \in A(G)$. If $x \neq e$, We can take $u \in A_0(G)$ with $u(x) \neq 0$ and

$$\begin{aligned} u(x) = \lim_\alpha \langle T \cdot u_\alpha, u\rangle &= \lim_\alpha \langle\, T, u_\alpha u \,\rangle \\ &= \lim_\alpha \langle\, T, u_\alpha \phi u \,\rangle \\ &= \lim_\alpha \langle T \cdot u_\alpha, \phi u\rangle = \phi(x)u(x). \end{aligned}$$

Therefore, $\phi(x) = 1$, but ϕ is adapted, so $x = e$.

So, $\mathrm{supp}(T) = \{e\}$ and [22, Corollary 2.5.9] proves that T is a multiple of $\lambda_2(\delta_e)$ and hence that $T|_{A_0(G)} \equiv 0$, a contradiction. □

The picture on ergodicity of $\phi \in \mathcal{P}^1(G)$ is completed by the solution to the complete mixing problem that follows from Theorem 2.1 of [24]. This problem is still open for convolution operators, see [16, Remark 2.10].

Theorem 4.7 (Kaniuth, Lau and Ülger) *Let $\phi \in P^1(G)$, then ϕ is strictly aperiodic if, and only if, it is completely mixing.*

Remark 4.8 The results of this section show that the mean ergodic properties of the operators M_ϕ and M_ϕ^0 are quite different. We see in the next subsection that the situation changes drastically when we study uniformly ergodic behaviour.

4.3 Uniform ergodicity

After the work already done in this and previous sections, uniform ergodicity can be characterized without effort.

We remark that all conditions in the statement of the following Theorem imply that G is discrete, for, when ϕ is adapted and 1 is isolated in $\overline{\phi(G)}$, H_ϕ must reduce to $\{e\}$ and be open.

Theorem 4.9 *Let G be a discrete group and let $\phi \in P^1(G)$. The following assertions are equivalent:*

(1) *ϕ is uniformly ergodic.*
(2) *ϕ is adapted and $(M_\phi)_{[n]}$ is uniformly mean ergodic.*
(3) *ϕ is adapted and spread-out.*
(4) *ϕ is adapted and M_ϕ is quasi-compact.*
(5) *ϕ is adapted and 1 is isolated in $\sigma(M_\phi)$.*
(6) *ϕ is adapted and 1 is isolated in $\sigma(M_\phi^0)$.*

Proof By Theorem 4.6, adaptedness is a necessary condition for uniform ergodicity, its appearance in items (2)–(6) needs not further mention.

Assertion (1) implies (2) because M_ϕ^0 is just the restriction of M_ϕ to the hyperplane $A_0(G)$ and uniform mean ergodicity of such a restriction implies uniform mean ergodicity of the operator, see [16, Proposition 4.4]. The converse follows from Lemma 4.5.

Corollary 3.14 and Theorem 3.18 prove that assertions (2), (3), (4) and (5) are equivalent. Finally, Theorem 4.4 proves that assertion (5) and (6) are equivalent. □

The same approach shows that the uniform completely mixing problem can be solved combining Corollary 3.15 With Theorem 4.9.

Theorem 4.10 *Let G be a locally compact group and $\phi \in P^1(G)$. Then ϕ is uniformly completely mixing if and only if it is strictly aperiodic and uniformly ergodic.*

We finish giving the results that we get if we proceed similarly but using Corollary 3.12 instead of Theorem 3.11. When G is amenable, this is nothing but putting together Theorem 4.9 and Theorem 4.10 above. Example 3.13 shows that when G is not amenable, the situation differs.

Theorem 4.11 *Let G be an amenable discrete locally compact group and let $\phi \in P_1(G)$ be adapted. Consider the following conditions:*

(i) $\lim_n \|\phi_{[n]} - \mathbf{1}_e\| = 0$.
(ii) *There is $k \in \mathbb{N}$ and $u \in A(G)$ such that $\|\phi^k - u\| < 1$.*
(iii) *1 is isolated in $\sigma(\phi)$.*

(a) $\lim_n \|\phi^n - \mathbf{1}_e\| = 0$.
(b) *1 is isolated in $\sigma(\phi)$ and ϕ is strictly aperiodic.*

Conditions (i), (ii) *and* (iii) *are equivalent. Conditions* (a) *and* (b) *are equivalent.*

Proof The proof of the equivalence between (i) and (ii) is completely analogous to the proof of Theorem 4.10, using Corollary 3.12. Using Proposition 3.3 we get that both are equivalent to (iii). It is trivial that (a) implies (b). If we assume (b), then (iii) holds and then also (ii), which yields $\|\widetilde{M}_\phi^k - \widetilde{M}_u\| < 1$. Note that, since $u \in A(G)$, $\widetilde{M}_u$ is the limit of finite range operators, which implies that $\widetilde{M}_\phi$ is quasicompact. We get (b) by Theorem 2.2, since $\sigma_p(\widetilde{M}_\phi) = \overline{\phi(G)}$. □

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