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A three-point compact LOD-FDTD method for solving the 2D scalar wave equation

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Abstract

This letter introduces an unconditionally stable finite-difference time-domain (FDTD) method, based on the locally one-dimensional (LOD) technique, for the solution of the two-dimensional scalar wave equation (WE) in homogeneous media. The second spatial derivatives in the WE are discretized by using a three-point compact (implicit) finite-difference formula with a free parameter. This formula has second-order accuracy and becomes fourth-order by properly selecting the parameter value. Moreover, the resulting algorithm only involves tridiagonal matrices, as when using standard (explicit) second-order finite differences. Additionally, a stability analysis is performed and the numerical dispersion relation of the method is derived. The proposed compact LOD-WE-FDTD technique has been applied to the calculation of resonant frequencies in a metallic ridge cavity. The accuracy of the results obtained has been studied as a function of the parameter value.

KEYWORDS

compact finite differences, finite-difference time-domain method, locally one-dimensional, numerical dispersion, stability, wave equation

1 | INTRODUCTION

During the last two decades, a considerable effort has been made to develop efficient unconditionally stable finite-difference time-domain (FDTD) techniques for computational electromagnetics, such as the alternating-direction implicit FDTD (ADI-FDTD), the locally one-dimensional FDTD (LOD-FDTD) and the split-step FDTD (SS-FDTD) methods.[1–3] In comparison to the conventional FDTD method,[4] these techniques are able to remove the Courant–Friedrich–Levy limit on the time step size at the cost of solving sparse systems of linear equations in each time iteration. Specifically, tridiagonal systems arise when, as usual, the spatial derivatives in Maxwell's equations are approximated by explicit second-order FDs.

To improve the accuracy of the unconditionally stable FDTD methods, the use of explicit high-order FDs to approximate the spatial derivatives was proposed. However, this approach increases the bandwidth of the matrix equations involved. For instances, fourth-order unconditionally stable FDTD methods require the solution of heptadiagonal matrix equations.[5] To alleviate this problem, the explicit FDs were replaced by compact (implicit) FDs. As a result, compact fourth-order schemes, in which only pentadiagonal matrix equations need to be solved, were developed for the ADI-FDTD,[6] LOD-FDTD,[7] and SS-FDTD[8,9] methods.

As an alternative to the conventional FDTD method, the scalar wave-equation FDTD (WE-FDTD) technique has also been proposed for solving electromagnetic wave problems.[10,11] In this letter, we introduce a new compact LOD-WE-FDTD method for the solution of two-dimensional (2D) problems in homogenous media. The second spatial derivatives appearing in the scalar WE are directly approximated by using a three-point compact FD formula with a free parameter α. This FD formula is a second-order accurate approximation that becomes fourth-order accuracy by properly choosing the value of α. Moreover, the resulting algorithm only involves tridiagonal matrices. Additionally, a stability analysis is performed and the numerical dispersion relation of the method is derived. The proposed compact LOD-WE-FDTD technique has been applied to the calculation of resonant frequencies in a metallic ridge cavity.

2 | THE COMPACT LOD-WE-FDTD METHOD FOR $\mathrm{TM_Z}$ WAVES

The electric WE for TM_z waves in a source-free medium with permittivity ε and permeability μ can be written as

$$\frac{\partial^2 E_z}{\partial t^2} = \frac{1}{\varepsilon\mu}\left(\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2}\right). \tag{1}$$

By applying the Crank–Nicolson (CN) method to (1), the following semidiscrete equation is obtained

$$\frac{E_z^{n+1} - 2E_z^n + E_z^{n-1}}{\Delta_t^2} = \frac{1}{\varepsilon\mu}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{E_z^{n+1} + E_z^{n-1}}{2}\right), \tag{2}$$

where Δ_t is the time step. The right-hand side (RHS) of (2) has been approximated by using a single average in time as

$$E_z^n = \frac{E_z^{n+1} + E_z^{n-1}}{2} + O\left(\Delta_t^2\right).$$

The CN scheme given in (2) can be expressed as

$$\left(1 - \frac{\Delta_t^2}{2\varepsilon\mu}\frac{\partial^2}{\partial x^2}\right)\left(1 - \frac{\Delta_t^2}{2\varepsilon\mu}\frac{\partial^2}{\partial y^2}\right)\left(E_z^{n+1} + E_z^{n-1}\right) = 2E_z^n + \left(\frac{\Delta_t^2}{2\varepsilon\mu}\right)^2 \frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial x^2}\underbrace{\left(E_z^{n+1} + E_z^{n-1}\right)}_{O(1)}. \tag{3}$$

According to the LOD method, by dropping the second term of the RHS of (3), the resulting expression can be split into two substeps as

$$\left(1 - \frac{\Delta_t^2}{2\varepsilon\mu}\frac{\partial^2}{\partial x^2}\right)e_z^{(1)} = E_z^n, \tag{4a}$$

$$\left(1 - \frac{\Delta_t^2}{2\varepsilon\mu}\frac{\partial^2}{\partial y^2}\right)e_z^{(2)} = e_z^{(1)}, \tag{4b}$$

$$E_z^{n+1} = 2e_z^{(2)} - E_z^{n-1}, \tag{4c}$$

where $e_z^{(1)}$ and $e_z^{(2)}$ are auxiliary variables. This splitting procedure preserves the second-order accuracy in time of the original CN scheme given in (2).

To approximate the spatial derivatives appearing in (4), we use three-point compact FDs. Specifically, for the second partial derivative in the x-direction, the following expression is adopted[12]:

$$\alpha_x \left.\frac{\partial^2 E_z^n}{\partial x^2}\right|_{i-1,j} + \left.\frac{\partial^2 E_z^n}{\partial x^2}\right|_{i,j} + \alpha_x \left.\frac{\partial^2 E_z^n}{\partial x^2}\right|_{i+1,j} = \frac{1 + 2\alpha_x}{\Delta_x^2}\left(\left.E_z^n\right|_{i+1,j} - 2\left.E_z^n\right|_{i,j} + \left.E_z^n\right|_{i-1,j}\right) + T, \tag{5}$$

where α_x is a free parameter, I and j are spatial indices, Δ_x is the cell size in the x-direction and T is the truncation error, given by

$$T = \frac{\Delta_x^2}{12}(10\alpha_x - 1)\left.\frac{\partial^4 E_z^n}{\partial x^4}\right|_{i,j} + O\left(\Delta_x^4\right). \tag{6}$$

For $\alpha_x \neq .1$, Equation (5) is a family of implicit second-order FD approximations. For $\alpha_x = .1$ the second-order term in (6) vanish and, consequently, Equation (5) becomes a fourth-order accurate approximation. Additionally, notice that for $\alpha_x = 0$, Equation (5) reduces to the standard explicit second-order formula commonly used to approximate the second derivative.

It is useful to write (5) in operational form. To this end, we begin by considering the second-order central FD operator δ_x^2 defined as[13]

$$\delta_x^2 E_z^n \Big|_{i,j} = E_z^n \Big|_{i+1,j} - 2E_z^n \Big|_{i,j} + E_z^n \Big|_{i-1,j}.$$

Then, after dropping the error term in (5), the approximation for the second partial derivative with respect to x at the node (i, j) can be expressed in terms of δ_x^2 as

$$\frac{\partial^2 E_z^n}{\partial x^2}\Bigg|_{i,j} = \frac{1}{\Delta_x^2}\frac{\delta_x^2}{L_x}E_z^n\Big|_{i,j}, \tag{7}$$

where

$$L_x = 1 + \frac{\alpha_x}{1+2\alpha_x}\delta_x^2$$

is a FD operator that should be interpreted symbolically. The approximation for the second partial derivative with respect to y has an expression analogous to (7).

For the first LOD step in (4a), the second derivative with respect to x is replaced by (7) leading to

$$\left(1 - \frac{\Delta_t^2}{2\varepsilon\mu\Delta_x^2}\frac{\delta_x^2}{L_x}\right)e_z^{(1)}\Big|_{i,j} = E_z^n\Big|_{i,j}. \tag{8}$$

Multiplying now both sides of (8) by L_x we obtain

$$\left(1 + c_x\delta_x^2\right)e_z^{(1)}\Big|_{i,j} = \left(1 + d_x\delta_x^2\right)E_z^n\Big|_{i,j}, \tag{9}$$

where

$$c_x = \frac{\alpha_x}{1+2\alpha_x} - \frac{\Delta_t^2}{2\varepsilon\mu\Delta_x^2}, \tag{10a}$$

$$d_x = \frac{\alpha_x}{1+2\alpha_x}. \tag{10b}$$

For each j, (9) leads to a tridiagonal system of linear equations that allows one to compute $e_z^{(1)}$ in the x-direction, row by row.

Repeating the same procedure for the second LOD step in (4b), we get

$$\left(1 + c_y\delta_y^2\right)e_z^{(2)}\Big|_{i,j} = \left(1 + d_y\delta_y^2\right)e_z^{(1)}\Big|_{i,j}, \tag{11}$$

where c_y and d_y are defined analogously to c_x and d_x in (10). For each i, (11) also results in a tridiagonal system that is used to calculate $e_z^{(2)}$ in the y-direction. Finally, E_z^{n+1} is updated by using (4c).

Notice that previous LOD-FDTD formulations based on three-point FDs but directly applied to Maxwell's equations[7] involve the solution of pentadiagonal matrix equations, while the proposed LOD-WE-FDTD method involves tridiagonal matrices only. Moreover, as based on the scalar WE, the LOD-WE-FDTD method requires to update just one field component.

3 | STABILITY AND NUMERICAL DISPERSION

With the aim to study the stability and numerical dispersion of the compact FD scheme introduced in the preceding section, we begin by eliminating the auxiliary variables $e_z^{(1)}$ and $e_z^{(2)}$ in (4). After dropping the spatial indices, the following single-step scheme is obtained

$$\begin{aligned}\frac{E_z^{n+1} - 2E_z^n + E_z^{n-1}}{\Delta_t^2} &= \frac{1}{2\varepsilon\mu}\left(\frac{1}{\Delta_x^2}\frac{\delta_x^2}{L_x} + \frac{1}{\Delta_y^2}\frac{\delta_y^2}{L_y}\right)\left(E_z^{n+1} + E_z^{n-1}\right) \\ &\quad -\left(\frac{\Delta_t}{2\varepsilon\mu\Delta_x\Delta_y}\right)^2\frac{\delta_y^2\delta_x^2}{L_yL_x}\left(E_z^{n+1} + E_z^{n-1}\right).\end{aligned} \tag{12}$$

By applying the von Neumann method to (12),[14] we get the following stability polynomial

$$(A+1)Z^2 - 2Z + (A+1) = 0, \tag{13}$$

where $Z = E_z^{n+1}/E_z^n$ is the amplification factor and

$$A = \left(1+\nu_x^2\right)\left(1+\nu_y^2\right) - 1, \tag{14}$$

where

$$\nu_\xi^2 = \left(\frac{\Delta_t^2}{2\varepsilon\mu\Delta_\xi^2}\right)\frac{\sin^2\left(\frac{\tilde{k}_\xi\Delta_\xi}{2}\right)}{1 - \frac{4\alpha_\xi}{1+2\alpha_\xi}\sin^2\left(\frac{\tilde{k}_\xi\Delta_\xi}{2}\right)}, \tag{15}$$

with $\xi = x, y$. In (15), $\tilde{k}_x = \tilde{k}\cos\phi$ and $\tilde{k}_x = \tilde{k}\sin\phi$, where $\tilde{k}$ is the numerical wavenumber and ϕ the wave propagation angle.

The roots of (13) are

$$Z_{1,2} = \frac{1 \pm \sqrt{1-(A+1)^2}}{A+1}.$$

It is easy to see that, for $A \geq 0$, the LOD scheme is unconditionally stable since $|Z_{1,2}| = 1$.

To fulfill $A \geq 0$, we enforce the conditions $\nu_x^2, \nu_y^2 \geq 0$ in (14), which lead to the restriction $\alpha_x, \alpha_y \leq .5$ for the parameters α_x and α_y arising in the compact FD approximation of the second partial derivative in (5).

By doing $Z = \exp(\mathbf{j}\omega\Delta_t)$ in (13), where $\mathbf{j} = \sqrt{-1}$, we obtain the following numerical dispersion relation

$$\tan^2\left(\frac{\omega\Delta_t}{2}\right) = \frac{\left(1+\nu_x^2\right)\left(1+\nu_y^2\right)-1}{\left(1+\nu_x^2\right)\left(1+\nu_y^2\right)+1}, \qquad (16)$$

where ν_x^2 and ν_y^2 are given in (15).

4 | NUMERICAL RESULTS

With the aim of validating the numerical dispersion relation (16), we consider a line current source located at the center of a square computational domain and radiating in free space as shown in Figure 1. The wave phase constant is calculated far away from the source position, where the radiated fields are assumed to be plane waves. The procedure used for this calculation is analogous to the one described in Saehoon et al.[15] A working frequency $f = 10$ GHz is considered. The computational domain is discretized by using square cells of size $\Delta_x = \Delta_y = \lambda/40$, where $\lambda = c/f$ is the exact wavelength with c being the speed of light. We define the stability factor as

$$s = \frac{\Delta_t}{\Delta_{t,\max}^{(\mathrm{expl.})}},$$

where Δ_t is time step used in the simulations and $\Delta_{t,\max}^{(\mathrm{expl.})}$ is the maximum one permitted by the explicit FDTD method.[4]

Figure 2 depicts the phase relative error as a function of the wave propagation angle ϕ for $s = 2$ and for several values of the parameter $\alpha = \alpha_x = \alpha_y$. The theoretical results obtained by directly solving the numerical dispersion relation (16) have been plotted by lines and the results actually simulated by the compact LOD-WE-FDTD method have been denoted by symbols. Very good agreement can be seen between theory and simulation. It is worth noting in Figure 2 that the phase error decreases as α increases. Consequently, the fourth-order formulation ($\alpha = .1$) provides better accuracy than the standard explicit second-order one ($\alpha = 0$). However, compact second-order approximations with $\alpha > .1$ lead to more accurate solutions than in the fourth-order case.

Figure 3 shows the phase relative error as a function of the wave propagation angle ϕ for $\alpha = .5$ and for several values of the stability factor s. Again, an excellent agreement is observed between theory (lines) and simulations (symbols). Additionally, it can be seen that

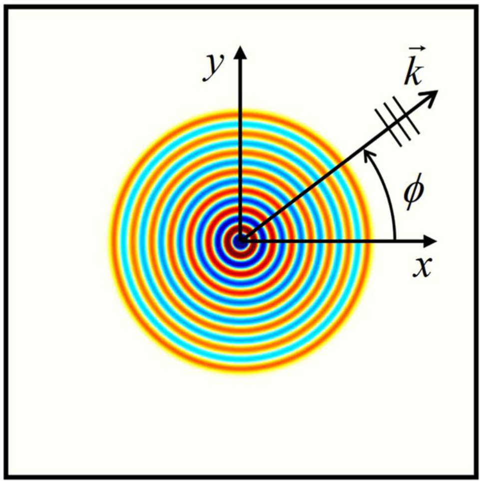

FIGURE 1 Electric field of a line current source radiating in free space.

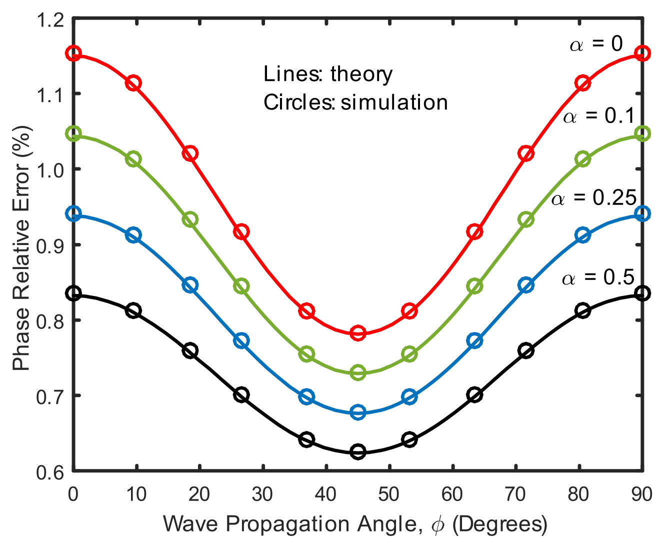

FIGURE 2 Relative phase error versus wave propagation angle for $s = 2$ and for several values of α.

the error increases with the stability factor s (i.e., with the time step), as expected.

As an application example, we consider a rectangular metallic ridge cavity as shown in Figure 4. The size of the cavity is $a \times b$ with $a = 20$ mm and $b = 1.5a$. The metallic ridge has length $a/2$ and zero thickness. The resonant frequency of the dominant mode was computed by using 40×60 spatial cells. This frequency was obtained from the first amplitude peak of the discrete Fourier transform of the electric field recorded at a selected point in the cavity. By using this setup, the problem was first solved by using the conventional FDTD method with $s = 1$. The result obtained was $f_{1,1} = 11.395$ GHz. Taking this result as a reference, Figure 5 plots the resonant frequency

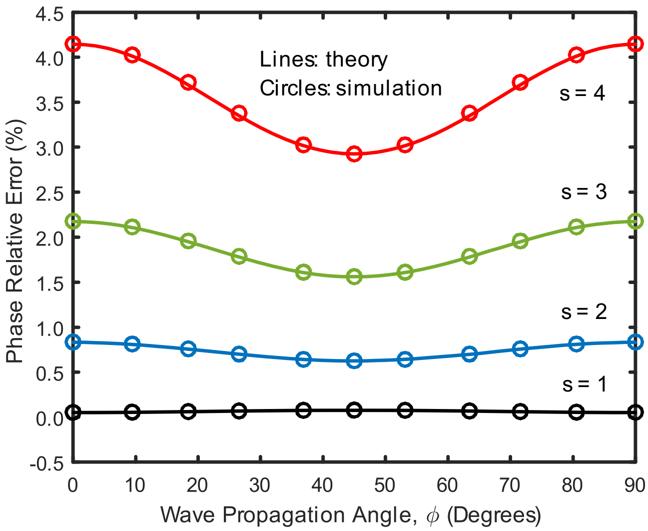

FIGURE 3 Relative phase error versus wave propagation angle for $\alpha = .5$ and for several values of the stability factor s.

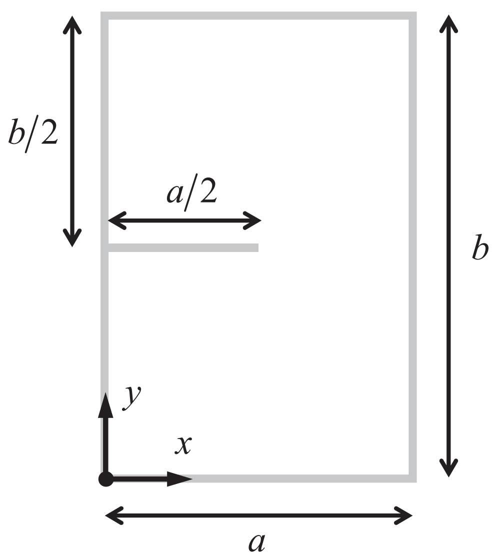

FIGURE 4 Rectangular metallic ridge cavity.

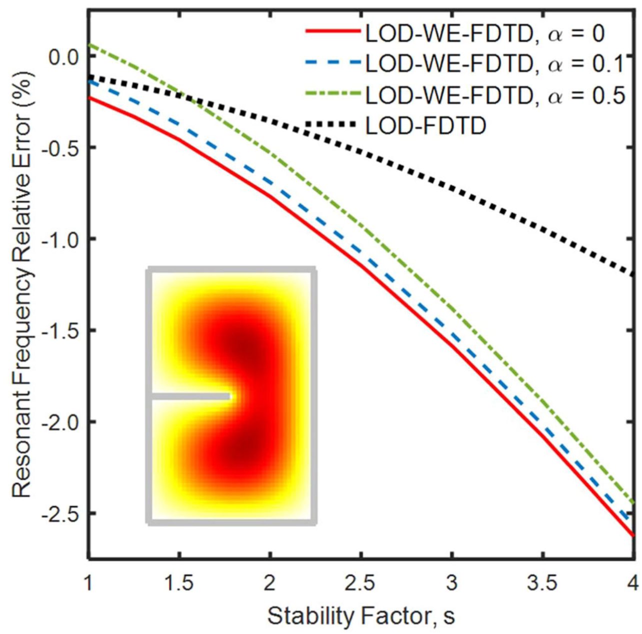

FIGURE 5 Resonant frequency relative error versus the stability factor s for the dominant mode of the metallic cavity depicted in Figure 4. The electric field pattern of this mode is shown in the inset. FDTD, finite-difference time-domain; LOD, locally one-dimensional; WE, wave equation.

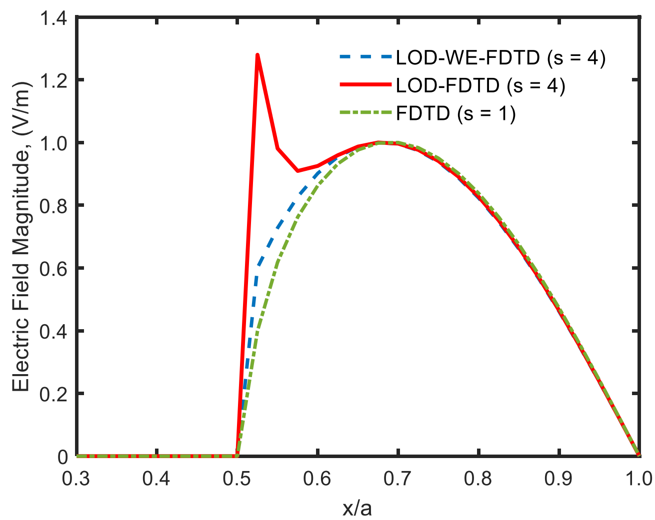

FIGURE 6 Electric field magnitude at the metal ridge position ($y = b/2$) for the first resonant mode of the cavity shown in Figure 4.

relative error against the stability factor s as computed by the compact LOD-WE-FDTD method for $\alpha = 0$, .1, and 0.5. Analogously to the results in Figure 2, it can be seen that the error decreases by increasing α. For the sake of comparison, the results obtained by the conventional LOD-FDTD method have been added to Figure 5. It can be seen that the conventional LOD-FDTD method provides somewhat better accuracy than the compact LOD-WAVE-FDTD method. The inset in Figure 5 shows the electric field pattern of the resonant mode under consideration.

For further comparison, Figure 6 depicts the electric field magnitude at $y = b/2$ (metal ridge position) for the first resonant mode of the cavity shown in Figure 4. These results have been computed by using the same setup as in Figure 5 for the conventional LOD-FDTD and the compact LOD-WAVE-FDTD methods with $\alpha = .1$. In both methods, the stability factor was $s = 4$. The electric field obtained by the conventional FDTD method with $s = 1$ is also included as a reference. A good agreement is observed between the compact LOD-WAVE-FDTD and the conventional FDTD method. However, it can be seen that the conventional LOD-FDTD method exhibits an anomalous field peak in the vicinity of the metal edge ($x/a = 0.5$).[16]

5 | CONCLUSION

A LOD-FDTD method for the solution of the 2D scalar WE in homogeneous media has been introduced. The second spatial derivatives in the WE have been discretized by using a three-point compact FD formula with a free parameter α. Even though this FD formula has fourth-order accuracy for $\alpha = .1$, the resulting formulation only involves tridiagonal matrices. A stability analysis has demonstrated that the proposed method is unconditionally stable for $\alpha \leq .5$. Additionally, a numerical dispersion study has shown that the dispersion error decreases by increasing α. The proposed compact LOD-WE-FDTD method has been applied to the calculation of resonant frequencies in a metallic ridge cavity confirming that the best accuracy is obtained for the maximum stable value of α, that is, $\alpha = .5$. Anyway, if fourth-order accuracy is required, the parameter α should be set to the value .1.

The proposed compact LOD-WE-FDTD method can also be formulated for three-dimensional (3D) problems by simply considering the 3D WE.[10]

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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