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Decision Support

Measuring consensus and voter influence in ternary preferences

Alessandro Albano [a,*], José Luis García-Lapresta [b], Antonella Plaia [a], Mariangela Sciandra [a]

[a] Department of Economics, Business and Statistics, University of Palermo, Palermo, Italy
[b] IMUVA, PRESAD Research Group, Departamento de Economía Aplicada, Universidad de Valladolid, Valladolid, Spain

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ABSTRACT

This paper explores the concept of consensus in the context of ternary preferences, an extension of dichotomous preference approvals, where alternatives are classified into three categories: acceptable, neutral, and unacceptable.We propose a novel distance-based measure to quantify consensus among voters and introduce a method for calculating the marginal contribution of each voter to the overall consensus, drawing parallels to the Banzhaf value in cooperative game theory. To handle large voter groups, we also present an estimation procedure based on sampling techniques to derive the marginal contributions. We performed comprehensive simulation studies to validate the statistical properties and computational efficiency of the proposed approach. Finally, empirical analyses using data from the Italian National Institute of Statistics (ISTAT) and the Balkan Barometer highlight its practical applicability.

1. Introduction

In voting and decision-making processes, accurately capturing the preferences of individuals is crucial for ensuring fair and representative outcomes.
However, as Dummett (1984) famously remarked: “If there are, say, twenty possible outcomes, the task of deciding the precise order of preference in which he ranks them may induce psychological paralysis in the voter".

The presence of many alternatives can make it challenging for voters to produce a full ranking, highlighting the need for simplified mechanisms that retain the essential structure of their preferences. One approach to mitigating these difficulties is to allow voters to classify outcomes into broad categories before ranking them within each class. A well-known solution to this problem is *approval voting*, which simplifies decision-making by requiring voters to classify alternatives into two groups {‘acceptable’, ‘unacceptable’} without needing to rank them in detail (Brams & Fishburn, 1978). However, as Balinski and Laraki argue, “Approve is not the opposite of Disapprove. Two grades are simply too few to adequately express voters’ opinions" (Balinski & Laraki, 2022). In order to capture the complexity of voter preferences, they support using at least three categories.

Several voting systems have adopted this principle. For instance, the Majority Judgment system employs a qualitative scale with multiple categories (e.g., {‘to reject’, ‘poor’, ‘acceptable’, ‘good’, ‘very good’, ‘excellent’}), allowing voters to classify alternatives without requiring a full ordering (see Balinski & Laraki, 2007, 2011). A related idea appears in Cooperative Game Theory, where different authors have considered three levels of approval (see Felsenthal & Machover, 1997, Musegaas et al., 2018 and Bilbao et al., 2010, among others).[1]

Unlike these multi-category systems, which typically do not impose any ordering among alternatives within the same category, the *preference-approvals* framework introduces a ranking component. Specifically, voters partition the set of alternatives into two categories—{*acceptable*, *unacceptable*}—and additionally rank all alternatives via a weak order (see (Brams, 2008, Chapter 3), Brams & Sanver, 2009, Sanver, 2011, Erdamar et al., 2014, Kamwa, 2019, 2023, Dong et al., 2021, Barokas, 2022, 2023, and Barokas & Sprumont, 2022).

Several authors have extended this approach to cover ternary preferences (under different names), where voters classify alternatives into three categories, such as {‘acceptable’, ‘neutral’, ‘unacceptable’} (see Felsenthal, 1989, Yılmaz, 1999 and Alcantud & Laruelle, 2014, among others). Recent work by Ye et al. (2024) further highlights the utility of ternary structures, proposing a three-way group consensus decision-making approach for medical diagnosis.

[1] However, differently from our purpose, the aforementioned game theory approach is dedicated to analyzing winning coalitions independently of any voting system.

* Corresponding author.
E-mail addresses: alessandro.albano@unipa.it (A. Albano), lapresta@uva.es (J.L. García-Lapresta), antonella.plaia@unipa.it (A. Plaia), mariangela.sciandra@unipa.it (M. Sciandra).

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Despite this rich literature, there remains a need for robust methods to measure consensus in the context of ternary preferences. In the field of Social Choice, Bosch (2005) introduced the concept of *consensus measure* as a mapping that assigns a number between 0 and 1 to every profile of linear orders, satisfying three properties: *unanimity* (in every subgroup of voters, the highest degree of consensus is only reached whenever all individuals have the same ranking), *anonymity* (the degree of consensus is not affected by any permutation of voters) and *neutrality* (the degree of consensus is not affected by any permutation of alternatives). While consensus measures in voting have been explored extensively in the context of linear orders (Bosch, 2005), weak orders (García-Lapresta & Pérez-Román, 2011), dichotomous preferences (Alcantud et al., 2013), and preference-approvals (Albano et al., 2023; Erdamar et al., 2014), the extension of these ideas to ternary preferences remains underdeveloped. This paper aims to fill that gap by proposing a novel distance-based measure of consensus specifically designed for ternary preferences.

Furthermore, this paper introduces the concept of *marginal contribution* to consensus, inspired by the Banzhaf value (Banzhaf, 1965) from cooperative game theory. The Banzhaf value, widely applied in various contexts such as voting power analysis, cost allocation, and coalition formation in political and economic settings, quantifies the power of individual players in cooperative games by measuring their marginal contribution to coalition formation, while our approach adapts this concept to group decision-making by measuring how much each voter contributes to the overall consensus. By quantifying the impact of each voter on the overall consensus, our method provides a new perspective on voter influence in ternary preference settings. In other words, if a voter's presence tends to raise the consensus score, that individual can be regarded as a "consensus driver", someone whose preferences align well with those of other voters. Conversely, if adding a voter consistently lowers the consensus score, that voter acts as a "consensus breaker", indicating that their preferences often introduce disagreement. We also develop an estimation procedure based on sampling techniques to make these calculations feasible in large-scale voter groups.

The remainder of the paper is structured as follows. Section 2 introduces the formal framework for preference-approvals and ternary preferences. In Section 3, we present the proposed consensus measure, outlining its theoretical foundation. Section 4 details the marginal contribution approach, including its theoretical properties and the estimation procedure for large voter groups. To empirically assess the statistical properties and the computational efficiency of the proposed methods, Section 5 presents extensive simulation studies. Following, Section 6 provides two case studies using data from the Italian National Institute of Statistics (ISTAT) and the Balkan Barometer to demonstrate the practical applicability of our methods. Finally, Section 7 concludes with a discussion of the implications of our findings and suggests potential directions for future research.

2. Preliminaries

Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set of alternatives with $n \geqslant 2$. A *weak order* (or *complete preorder*) on X is a binary relation that is both complete and transitive, while a *linear order* is a weak order that is also antisymmetric.
We write $W(X)$ for the set of weak orders on X and $L(X)$ for the set of linear orders. For any $R \in W(X)$, we denote its asymmetric part by $\succ$ and its symmetric part by $\sim$, so that $x_i \succ x_j$ if and only if not $x_j \, R \, x_i$ and $x_i \sim x_j$ if and only if $(x_i \, R \, x_j$ and $x_j \, R \, x_i)$. For an arbitrary set Y, $\mathcal{P}(Y)$ denotes its power set, i.e. $I \in \mathcal{P}(Y)$ if and only if $I \subseteq Y$. The notation $\#Y$ indicates the cardinality of Y.

Consider that a set of voters $V = \{v_1, v_2, \dots, v_m\}$, with $m \geqslant 2$, have to express their opinions over X. With $\mathcal{P}_2(V)$ we denote the set of all the subsets of V with at least two voters, i.e., $\mathcal{P}_2(V) = \{I \in \mathcal{P}(V) \mid \#I \geqslant 2\}$. Note that $\#\mathcal{P}_2(V) = \#\mathcal{P}(V) - \#V - 1 = 2^m - m - 1$.

2.1. Preference-approvals

Preference-approvals assume that each voter assesses each alternative as either acceptable or unacceptable by partitioning X into A, the set of *acceptable* alternatives, and $U = X \setminus A$, the set of *unacceptable* alternatives, where A and U may be empty and ranks the alternatives in X by means of a weak order.
To ensure coherence between approvals and weak orders a consistency condition is imposed: if x_j is acceptable and x_i is ranked above x_j, then x_i should be acceptable as well.

Definition 1. A *preference-approval* on X is a pair $(R, A) \in W(X) \times \mathcal{P}(X)$ satisfying the following condition:

$\forall x_i, x_j \in X \; \big((x_i \, R \, x_j \text{ and } x_j \in A) \Rightarrow x_i \in A\big).$

With $\mathcal{R}(X)$ we denote the set of preference-approvals on X.

Remark 1. If $(R, A) \in \mathcal{R}(X)$, then the following conditions are satisfied:

1. $\forall x_i, x_j \in X \; \big((x_i \in A \text{ and } x_j \in U) \Rightarrow x_i \succ x_j\big)$.
2. $\forall x_i, x_j \in X \; \big((x_i \, R \, x_j \text{ and } x_i \in U) \Rightarrow x_j \in U\big)$.

2.2. Ternary preferences

Hereinafter, we consider that each voter $v \in V$ ranks the set of alternatives through a weak order $R_v \in W(X)$ and, additionally, makes a partition of X in three categories: A_v (acceptable), N_v (neutral) and U_v (unacceptable).[2]

The three categories are mutually disjoint, i.e., $A_v \cap N_v = A_v \cap U_v = N_v \cap U_v = \emptyset$ and together they cover the entire set of alternatives, i.e., $A_v \cup N_v \cup U_v = X$. Whatever of these categories can be empty, but not all of them simultaneously. Notably, when $N_v = \emptyset$, the ternary preference reduces to a standard preference-approval.

Definition 2. A *ternary preference* on X is a 4-tuple $(R, A, N, U) \in W(X) \times \mathcal{P}(X) \times \mathcal{P}(X) \times \mathcal{P}(X)$ satisfying the following conditions:

1. $A \cap N = A \cap U = N \cap U = \emptyset$.
2. $A \cup N \cup U = X$.
3. $\forall x_i, x_j \in X \; \big((x_i \in A \text{ and } x_j \in N \cup U) \Rightarrow x_i \succ x_j\big)$.
4. $\forall x_i, x_j \in X \; \big((x_i \in N \text{ and } x_j \in U) \Rightarrow x_i \succ x_j\big)$.

With $\mathcal{T}(X)$ we denote the set of *ternary preferences* on X.

As an example, consider $T = (R, A, N, U) \in \mathcal{T}(\{x_1, \dots, x_9\})$ represented by

x_2
$x_1 \quad x_6$
———
$x_3 \quad x_5$
———
x_8
$x_4 \quad x_7 \quad x_9$

In our representation of ternary preferences, alternatives are arranged in rows according to a weak order. Alternatives on higher rows are strictly preferred to those on lower rows, while alternatives on the same row are tied in preference, i.e., indifferent. Two horizontal lines divide the alternatives into three categories. All alternatives above the

[2] From a behavioral standpoint, the ternary preference does not stem from an explicit expression of three distinct options, but rather emerges from a binary judgment, acceptable versus unacceptable, where everything else is treated as neutral.

first line are acceptable, those between the first and second line are neutral, and those below the second line are unacceptable. That implies that $A = \{x_1, x_2, x_6\}$, $N = \{x_3, x_5\}$ and $U = \{x_4, x_7, x_8, x_9\}$. The corresponding weak order is characterized by the relations

$$x_2 \succ (x_1 \sim x_6) \succ (x_3 \sim x_5) \succ x_8 \succ (x_4 \sim x_7 \sim x_9),$$

where $\succ$ and and $\sim$ denote strict preference and indifference between alternatives, respectively.

Remark 2. Drawing parallels with preference-approvals, conditions 3 and 4 in Definition 2 can be written as: given two alternatives x_i and x_j, if x_j is acceptable and x_i is ranked above x_j, then x_i must also be acceptable. Furthermore, if x_j is neutral and x_i is ranked above x_j, then x_i must be either acceptable or neutral.

1. If $x_j \in A$ and $x_i \, R \, x_j$, then $x_i \in A$.
2. If $x_j \in N$ and $x_i \, R \, x_j$, then $x_i \in A \cup N$.

In other words, any alternative labeled “acceptable" must be ranked strictly above any alternative labeled “neutral" or “unacceptable," ensuring consistency between approval status and ordering. In the same way, every “neutral" alternative is ranked strictly above any “unacceptable" alternative, establishing the hierarchy $A \succ N \succ U$. These conditions prevent any illogical ordering, such as placing an acceptable alternative below a neutral one, while still allowing complete freedom to rank or tie alternatives within each category as the weak order permits.

3. Consensus in ternary preferences

When voters express their evaluations using ternary preferences, measuring the degree of agreement among them becomes crucial for understanding the group's collective behaviour. To this aim, we extend the framework of Albano et al. (2023, 2024), originally developed for preference-approvals, by introducing two complementary measures of disagreement between voters. The first measure, denoted as p_{ij}^{kl} (Eq. (1)), captures *preference disagreement*, that is, how differently two voters compare a pair of alternatives. The second measure, a_{ij}^{kl} (Eq. (2)), captures *approval disagreement*, namely how differently two voters categorise alternatives into approved, neutral, or unapproved classes.

To formally define p_{ij}^{kl}, we first introduce a numerical representation of a voter's judgement between two alternatives. Given a weak order R over the set of alternatives X, we define:

$$O_R(x_i, x_j) = \begin{cases} 1, & \text{if } x_i \succ x_j \text{ (i.e., } x_i \text{ is strictly preferred to } x_j\text{)}, \\ 0, & \text{if } x_i \sim x_j \text{ (i.e., } x_i \text{ and } x_j \text{ are considered equivalent)}, \\ -1, & \text{if } x_j \succ x_i \text{ (i.e., } x_j \text{ is strictly preferred to } x_i\text{)}. \end{cases}$$

For $T_k, T_l \in \mathcal{T}(X)$, the preference-discordance between voters v_k and v_l over x_i and x_j is

$$p_{ij}^{kl} \; = \; \frac{\left| O_{R_k}(x_i, x_j) - O_{R_l}(x_i, x_j) \right|}{2}. \tag{1}$$

Note that $p_{ij}^{kl} \in \{0, 0.5, 1\}$ for all $k, l \in \{1, 2, \dots, m\}$ and $i, j \in \{1, 2, \dots, n\}$. Now, let us consider all the alternatives in each class, and define $P_A(x_i)$ as:

$$P_A(x_i) = \begin{cases} 1, & \text{if } x_i \in A, \\ 0, & \text{if } x_i \in N, \\ -1, & \text{if } x_i \in U. \end{cases}$$

For two ternary preferences $T_k, T_l \in \mathcal{T}(X)$, the approval-discordance between voters v_k and v_l over x_i and x_j) is defined as

$$a_{ij}^{kl} = \frac{\left| P_{A_k}(x_i) - P_{A_l}(x_i) \right| + \left| P_{A_k}(x_j) - P_{A_l}(x_j) \right|}{4}. \tag{2}$$

It is easy to check that $a_{ij}^{kl} \in \{0, 0.25, 0.5, 0.75, 1\}$ for all $k, l \in \{1, 2, \dots, m\}$ and $i, j \in \{1, 2, \dots, n\}$.

Having introduced two complementary measures of disagreement (p_{ij}^{kl} and a_{ij}^{kl}) between voters, we now turn to defining an overall distance metric between complete ternary preferences. To this end, we propose the metric d_λ on $\mathcal{T}(X)$, defined as the average of convex combinations of the two discordances across all pairs of alternatives.

Definition 3. Let $\lambda \in (0, 1)$, the function $d_\lambda : \mathcal{T}(X) \times \mathcal{T}(X) \longrightarrow [0, 1]$ is defined as

$$d_\lambda(T_k, T_l) = \frac{2}{n \cdot (n-1)} \cdot \sum_{\substack{i,j=1 \\ i<j}}^{n} \left(\lambda \cdot p_{ij}^{kl} + (1 - \lambda) \cdot a_{ij}^{kl} \right). \tag{3}$$

The parameter λ allows us to emphasize different aspects of agreement: values close to 1 focus on preference rankings, while values close to 0 focus on ternary approval classifications.

Note that d_λ is a metric on $\mathcal{T}(X)$ having the unit interval as codomain.

With D_λ we denote the matrix containing the pairwise distances in a set of voters. That is, $(D_\lambda)_{kl} = d_\lambda(T_k, T_l)$ for $k, l = 1, 2, \dots, m$, where T_k and T_l are the ternary preference of voters v_k and v_l, respectively. The matrix D_λ is clearly symmetric, and its main diagonal consists of zeros.

$$D_\lambda = \begin{pmatrix} 0 & \dots & d_\lambda(T_1, T_k) & \dots & d_\lambda(T_1, T_m) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_\lambda(T_k, T_1) & \dots & 0 & \dots & d_\lambda(T_k, T_m) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_\lambda(T_m, T_1) & \dots & d_\lambda(T_m, T_k) & \dots & 0 \end{pmatrix}.$$

The consensus among a set of voters is determined by computing the distances between their respective ternary preferences.

Definition 4. For $\lambda \in (0, 1)$, the mapping $C_\lambda : \mathcal{T}(X)^m \times \mathcal{P}_2(V) \longrightarrow [0, 1]$ defined as

$$C_\lambda(\boldsymbol{T}, I) = 1 - \frac{\displaystyle\sum_{\substack{v_k, v_l \in I \\ k<l}} d_\lambda(T_k, T_l)}{\dbinom{\#I}{2}}, \tag{4}$$

measures the consensus among the voters of I over the alternatives of X in the profile $\boldsymbol{T}$.

3.1. Basic properties

To establish some fundamental properties of the consensus measure, we first introduce the following notation. Let $\boldsymbol{T} = (T_1, T_2, \dots, T_m) \in \mathcal{T}(X)^m$ be a profile of ternary preferences of m voters, and let π be a permutation on the set $\{1, 2, \dots, m\}$. Define $\boldsymbol{T}_\pi = \left(T_{\pi(1)}, T_{\pi(2)}, \dots, T_{\pi(m)}\right)$, the profile obtained from $\boldsymbol{T}$ by relabeling the voters according to π. For any subset $I \subseteq V$ of voters, define $I_\pi = \{v_{\pi^{-1}(i)} \mid v_i \in I\}$, so that $v_j \in I_\pi$ if and only if $v_{\pi(j)} \in I$.

Example 1. Suppose we have a profile $\boldsymbol{T} = (T_1, T_2, T_3)$ and a permutation π swapping voter 1 and 3, i.e., $\pi = (1\,3)$. Then, $\boldsymbol{T}_\pi = (T_3, T_2, T_1)$. If $I = \{v_1, v_2\}$, then $I_\pi = \{v_{\pi^{-1}(1)}, v_{\pi^{-1}(2)}\} = \{v_3, v_2\}$.

Next, let σ be a permutation on the set of alternatives $\{1, 2, \dots, n\}$. Then the profile $\boldsymbol{T}^\sigma = \left(T_1^\sigma, T_2^\sigma, \dots, T_m^\sigma\right)$ is obtained from $\boldsymbol{T}$ by relabeling the alternatives according to σ. More precisely, for all $i, j \in \{1, 2, \dots, n\}$ and every voter k, $x_i \, R_k \, x_j \;\; \Leftrightarrow \;\; x_{\sigma(i)} \, R_k^\sigma \, x_{\sigma(j)}$, and similarly, $x_i \in A_k^\sigma \Leftrightarrow x_{\sigma(i)} \in A_k$, $\;x_i \in N_k^\sigma \Leftrightarrow x_{\sigma(i)} \in N_k$, $\;x_i \in U_k^\sigma \Leftrightarrow x_{\sigma(i)} \in U_k$.

Example 2. Let σ be a permutation on alternatives swapping x_1 and x_2. Then, for any voter v_k, the preference $x_1 \, R_k \, x_3$ becomes $x_2 \, R_k^\sigma \, x_3$, reflecting the renaming of alternatives, while preserving the structure of preferences.

Lastly, for a profile $\boldsymbol{T} = (T_1, T_2, \dots, T_m) \in \mathcal{T}(X)^m$, we define the inverse profile $\boldsymbol{T}^{-1} = \left(T_1^{-1}, T_2^{-1}, \dots, T_m^{-1}\right)$, obtained by inverting each voter's preferences. More specifically, for every voter $k \in \{1, 2, \dots, m\}$, $T_k^{-1} =$

$(R_k^{-1}, U_k, N_k, A_k)$ where R_k^{-1} is the inverse of the original preference relation R_k, and the acceptable and unacceptable sets are swapped. That is:

$$x_i \, R_k \, x_j \quad \text{if and only if} \quad x_j \, R_k^{-1} \, x_i.$$

Example 3. Suppose a voter v_k has the following ternary preference over three alternatives x_1, x_2, x_3:

$$x_1 \, R_k \, x_2 \, R_k \, x_3, \quad A_k = \{x_1\}, \quad N_k = \{\emptyset\}, \quad U_k = \{x_2, x_3\}.$$

That is, the voter strictly prefers x_1 over x_2, and x_2 over x_3; considers x_1 acceptable and x_2, x_3 unacceptable. Then, the inverted ternary preference T_k^{-1} is:

$$x_3 \, R_k \, x_2 \, R_k \, x_1, \quad A_k = \{x_2, x_3\}, \quad N_k = \{\emptyset\}, \quad U_k = \{x_1\}.$$

In this inverted ternary preference, the voter now strictly prefers x_3 over x_2, and x_2 over x_1; the formerly best alternative x_1 is now deemed unacceptable, the formerly worst alternatives x_2, x_3 are now acceptable. The set of neutral alternatives remains empty.

In the following Proposition 1, we show that the consensus measure C_λ satisfies *boundedness* (the degree of consensus is always between 0 and 1), *anonymity* (all voters are treated in the same way), *neutrality* (all alternatives are treated in the same way), *unanimity* (the maximum degree of consensus, 1, is only achieved whenever all voters have the same opinions), *maximum dissension* (when there are only two voters, the minimum degree of consensus, 0, is only achieved whenever voters rank order the alternatives in the opposite way; the acceptable alternatives of one voter are the unacceptable alternatives of the other; and none of the voters have neutral alternatives), and *reciprocity* (if all voters invert their opinions, the degree of consensus does not change).

Proposition 1. *For every $\lambda \in (0,1)$, the consensus measure C_λ satisfies the following properties:*

1. Boundedness*: For all $\boldsymbol{T} \in \mathcal{T}(X)^m$ and $I \in \mathcal{P}_2(V)$, it holds*

   $$0 \leqslant C_\lambda(\boldsymbol{T}, I) \leqslant 1.$$

2. Anonymity*: For all permutation π on $\{1,\dots,m\}$, $\boldsymbol{T} \in \mathcal{T}(X)^m$ and $I \in \mathcal{P}_2(V)$, it holds*

   $$C_\lambda(\boldsymbol{T}_\pi, I_\pi) = C_\lambda(\boldsymbol{T}, I).$$

3. Neutrality*: For all permutation σ on $\{1,\dots,n\}$, $\boldsymbol{T} \in \mathcal{T}(X)^m$ and $I \in \mathcal{P}_2(V)$, it holds*

   $$C_\lambda(\boldsymbol{T}^\sigma, I) = C_\lambda(\boldsymbol{T}, I).$$

4. Unanimity*: For all $\boldsymbol{T} \in \mathcal{T}(X)^m$ and $I \in \mathcal{P}_2(V)$, it holds*

   $$C_\lambda(\boldsymbol{T}, I) = 1 \;\Leftrightarrow\; (T_k = T_l \text{ for all } v_k, v_l \in I).$$

5. Maximum dissension*: For all $\boldsymbol{T} \in \mathcal{T}(X)^m$ and $v_k, v_l \in V$ such that $k \neq l$, it holds*

   $$C_\lambda(\boldsymbol{T}, \{v_k, v_l\}) = 0 \;\Leftrightarrow\; R_k, R_l \in L(X),\; R_l = R_k^{-1},$$

   $$A_l = U_k,\; N_k = N_l = \emptyset \text{ and } U_l = A_k.$$

6. Reciprocity*: For all $\boldsymbol{T} \in \mathcal{T}(X)^m$ and $I \in \mathcal{P}_2(V)$, it holds*

   $$C_\lambda(\boldsymbol{T}^{-1}, I) = C_\lambda(\boldsymbol{T}, I).$$

Proof.

- Boundedness, Anonymity and Neutrality: the proof is straightforward
- Unanimity: $C_\lambda(\boldsymbol{T}, I) = 1 \;\Leftrightarrow\; \sum_{\substack{v_k, v_l \in I \\ k<l}} d_\lambda(T_k, T_l) = 0 \;\Leftrightarrow\; (T_k = T_l$ for all $v_k, v_l \in I)$.
- Maximum dissension : $\Rightarrow$) If $C_\lambda(\boldsymbol{T}, \{v_k, v_l\}) = 0$, then $d_\lambda(T_k, T_l) = 1$. Taking into account Eq. (3), $p_{ij}^{kl} = a_{ij}^{kl} = 1$ for all $i, j \in \{1,\dots,n\}$ such that $i < j$. If $R_k \in W(X) \setminus L(X)$ or $R_l \in W(X) \setminus L(X)$, then $p_{ij}^{kl} < 1$ for some $i, j \in \{1,2,\dots,n\}$; consequently, $R_k, R_l \in L(X)$. Analogously, if $R_l \neq R_k^{-1}$, then $p_{ij}^{kl} < 1$ for some $i, j \in \{1,2,\dots,n\}$; consequently, $R_l = R_k^{-1}$. If $A_l \neq U_k$ or $N_k \neq \emptyset$ or $N_l \neq \emptyset$ or $U_l \neq A_k$, then $a_{ij}^{kl} < 1$ for some $i, j \in \{1,2,\dots,n\}$; consequently, $A_l = U_k$, $N_k = N_l = \emptyset$ and $U_l = A_k$.

  $\Leftarrow$) It is straightforward.
- Reciprocity: Since $O_{R_k}^{-1}(x_i, x_j) = -O_{R_k}(x_i, x_j)$, $O_{R_l}^{-1}(x_i, x_j) = -O_{R_l}(x_i, x_j)$, $P_{U_k}(x_i) = -P_{A_k}(x_i)$, $P_{U_l}(x_i) = -P_{A_l}(x_i)$, $P_{U_k}(x_j) = -P_{A_k}(x_j)$, $P_{U_l}(x_j) = -P_{A_l}(x_j)$, we have $d_\lambda(T_k^{-1}, T_l^{-1}) = d_\lambda(T_k, T_l)$ and, consequently, $C_\lambda(\boldsymbol{T}^{-1}, I) = C_\lambda(\boldsymbol{T}, I)$.

These properties ensure that the consensus measure is meaningful and interpretable in real-world scenarios: it is standardized and comparable across groups (boundedness), treats all voters and alternatives equally (anonymity and neutrality), detects full agreement or irreconcilable disagreement (unanimity and maximum dissension), and remains stable under global opinion reversals (reciprocity). Note that, for three or more voters, it is not possible to achieve a consensus value exactly equal to zero.

3.2. Group decomposability

The overall consensus of a group of voters can be broken down into contributions from consensus within subgroups, as well as the pairwise disagreements between voters in different subgroups.

As a matter of fact, given $I \in \mathcal{P}_2(V)$ and $\boldsymbol{T} = (T_1, T_2, \dots, T_m) \in \mathcal{T}(X)^m$, consider a partition of the voters of I into s disjoint subgroups: $I_1, I_2, \dots, I_s \subseteq I$, where $I_1 \cup I_2 \cup \dots \cup I_s = I$ and $I_p \cap I_q = \emptyset$ for $p \neq q$. Note that $\#I = \sum_{p=1}^{s} \#I_p$.

Let us consider the general formula for consensus given in Eq. (4). This formula relies on the sum of all pairwise distances among all voters in a group. When voters are partitioned into s disjoint subgroups, we can decompose this total sum of pairwise distances into two parts: the distances between voters within each subgroup I_p, and the distances between voters in different subgroups I_p and I_q (for $p \neq q$):

$$\sum_{\substack{v_k, v_l \in I \\ k<l}} d_\lambda(T_k, T_l) = \sum_{p=1}^{s} \sum_{\substack{v_k, v_l \in I_p \\ k<l}} d_\lambda(T_k, T_l) + \sum_{\substack{p,q=1 \\ p<q}}^{s} \sum_{k=1}^{\#I_p} \sum_{l=1}^{\#I_q} d_\lambda(T_k, T_l). \tag{5}$$

For any subgroup $I_p \subset I$, the consensus within I_p is given by:

$$C_\lambda(\boldsymbol{T}, I_p) = 1 - \frac{\sum_{\substack{v_k, v_l \in I_p \\ k<l}} d_\lambda(T_k, T_l)}{\binom{\#I_p}{2}}.$$

Rearranging this equation, the sum of distances within each subgroup I_p, $\sum_{\substack{v_k, v_l \in I_p \\ k<l}} d_\lambda(T_k, T_l)$, can be expressed in terms of the subgroup consensus:

$$\sum_{\substack{v_k, v_l \in I_p \\ k<l}} d_\lambda(T_k, T_l) = \binom{\#I_p}{2} \cdot \left(1 - C_\lambda(\boldsymbol{T}, I_p)\right). \tag{6}$$

By substituting Eq. (6) into Eq. (5), we obtain the following expression for the total sum of pairwise distances:

$$\sum_{\substack{v_k,v_l\in I\\k<l}} d_\lambda(T_k,T_l) = \sum_{p=1}^{s}\binom{\#I_p}{2}\cdot\big(1-C_\lambda(\boldsymbol{T},I_p)\big)+ \\ +\sum_{\substack{p,q=1\\p<q}}^{s}\sum_{k=1}^{\#I_p}\sum_{l=1}^{\#I_q} d_\lambda(T_k,T_l). \tag{7}$$

Finally, by substituting Eq. (7) into the general consensus formula in Eq. (4), we achieve the *Group Decomposability* property, i.e., the overall consensus $C_\lambda(\boldsymbol{T},I)$ can be expressed as:

$$C_\lambda(\boldsymbol{T},I) = 1-\frac{\sum_{p=1}^{s}\left(\binom{\#I_p}{2}\cdot\big(1-C_\lambda(\boldsymbol{T}_p,I_p)\big)\right)+\sum_{\substack{p,q=1\\p<q}}^{s}\sum_{k=1}^{\#I_p}\sum_{l=1}^{\#I_q} d_\lambda(T_k,T_l)}{\binom{m}{2}}, \tag{8}$$

where $\binom{\#I_p}{2}\cdot\big(1-C_\lambda(\boldsymbol{T}_p,I_p)\big)$ represents the contribution of the internal consensus within each subgroup I_p, while, $\sum_{\substack{p,q=1\\p<q}}^{s}\sum_{k=1}^{\#I_p}\sum_{l=1}^{\#I_q} d_\lambda(T_k,T_l)$ is the sum of pairwise distances between voters from different subgroups.

Example 4. Suppose four voters v_1,v_2,v_3,v_4 are grouped into two disjoint subgroups:

$$I_1=\{v_1,v_2\},\qquad I_2=\{v_3,v_4\},\qquad I=I_1\cup I_2.$$

Assume full agreement within each subgroup, so that:

$$d_\lambda(T_1,T_2)=0 \quad\text{and}\quad d_\lambda(T_3,T_4)=0.$$

Assume also full disagreement across subgroups:

$$d_\lambda(T_1,T_3)=d_\lambda(T_1,T_4)=d_\lambda(T_2,T_3)=d_\lambda(T_2,T_4)=1.$$

The total sum of pairwise distances among all voters in I is:

$$\sum_{\substack{k,l=1\\k<l}}^{4} d_\lambda(T_k,T_l)=$$

$$= d_\lambda(T_1,T_2)+d_\lambda(T_1,T_3)+d_\lambda(T_1,T_4)+$$

$$+ d_\lambda(T_2,T_3)+d_\lambda(T_2,T_4)+d_\lambda(T_3,T_4)=$$

$$=0+1+1+0+1+1+0=4.$$

There are $\binom{4}{2}=6$ total pairs, so the overall consensus is:

$$C_\lambda(\boldsymbol{T},I)=1-\frac{4}{6}=\frac{1}{3}.$$

Considering that the internal consensus of both subgroups is maximal (i.e., 1), the group decomposability formula yields:

$$C_\lambda(\boldsymbol{T},I)=1-\frac{\binom{2}{2}\cdot(1-C_\lambda(\boldsymbol{T},I_1))+\binom{2}{2}\cdot(1-C_\lambda(\boldsymbol{T},I_2))+4}{6}$$
$$=1-\frac{0+0+4}{6}=\frac{1}{3}.$$

4. Marginal contribution to consensus

In group decision-making, understanding each voter's influence is crucial for analyzing how individual preferences shape collective outcomes. A novel concept, *the marginal contribution to consensus*, is introduced as a measure of each voter's impact on group consensus. The approach builds on existing frameworks and is related to the Banzhaf value (Banzhaf, 1965) in the representation developed by Owen (1975) and characterized by Lehrer (1988).

Definition 5. Let $\boldsymbol{T}\in\mathcal{T}(X)^m$ be a profile and $\lambda\in(0,1)$. The *average marginal contribution to consensus* of voter $v_k\in V$ is defined as

$$c_k=\frac{\sum_{I\in S_k}\big(C_\lambda(\boldsymbol{T},I\cup\{v_k\})-C_\lambda(\boldsymbol{T},I)\big)}{\#S_k}, \tag{9}$$

where $S_k=\{I\in\mathcal{P}_2(V)\mid v_k\notin I\}$.

Remark 3. Note that $c_k\in[-1,1]$. However, in practice, the extreme values are impossible to achieve:

$c_k=-1$ means that $C_\lambda(\boldsymbol{T},I\cup\{v_k\})=0$ and $C_\lambda(\boldsymbol{T},I)=1$, for every $I\in S_k$.

$c_k=0$ means that $C_\lambda(\boldsymbol{T},I\cup\{v_k\})=C_\lambda(\boldsymbol{T},I)$ for every $I\in S_k$.

$c_k=1$ means that $C_\lambda(\boldsymbol{T},I\cup\{v_k\})=1$ and $C_\lambda(\boldsymbol{T},I)=0$, for every $I\in S_k$.

While in monotonic transferable utility cooperative games, each marginal contribution is always non-negative, in our approach $C_\lambda(\boldsymbol{T},I\cup\{v_k\})-C_\lambda(\boldsymbol{T},I)$ can be negative if the consensus in $I\cup\{v_k\}$ is smaller than in I. This reflects the fact that adding a voter decreases the overall consensus when the newly formed group $I\cup\{v_k\}$ is more heterogeneous than the original one I.

4.1. Standardized marginal contributions

As previously mentioned, achieving the theoretical minimum and maximum average marginal contributions, denoted as c_k, of 1 and -1 is impossible due to the pairwise additive nature of the consensus function. Specifically, for c_k to be equal to -1, the consensus among voters, excluding v_k, should be consistently 1 (i.e., $C_\lambda(\boldsymbol{T},I)=1$), while the consensus including v_k should be 0 (i.e., $C_\lambda(\boldsymbol{T},I\cup\{v_k\})=0$), which is infeasible (in the same way, reaching $c_k=1$ is impossible as well).

Remark 4. Recalling the *unanimity* property, the consensus within a coalition is maximized, i.e., $C_\lambda(\boldsymbol{T},I)=1$, if and only if all voters in the coalition share the same ternary preference. In contrast, according to the *total disagreement* property, the consensus within a coalition is minimized, i.e., $C_\lambda(\boldsymbol{T},I)=0$, if and only if the coalition consists of exactly two voters with opposing ternary preferences.

Example 5. Let us consider an example with three voters $V=\{v_1,v_2,v_3\}$ expressing their preferences on a set of three alternatives $X=\{x_1,x_2,x_3\}$. Assume that two voters, v_1,v_2, have the same ternary preference, while v_3 shows maximum disagreement with them. This setting could be displayed as:

$T_1=T_2$	T_3
x_1, x_2, x_3	$\emptyset$
	$\emptyset$
$\emptyset$	x_3
$\emptyset$	x_2, x_1

Which will produce the following distances:

$$d_\lambda(T_1,T_2)=0$$
$$d_\lambda(T_1,T_3)=d_\lambda(T_2,T_3)=1.$$

In this configuration, v_3 shows maximum disagreement with v_1 and v_2, while v_1 and v_2 are in complete agreement with each other. Consequently, this setup results in the most negative marginal contribution possible for v_3.

Let us compute the marginal contribution of v_3. First, we compute D_λ, the matrix containing the pairwise distances between voters' preferences:

$$D_\lambda=\begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix}.$$

To compute the marginal contribution of v_3, we need to consider how removing v_3 affects the consensus. In this case, Eq. (5) reduces to:

$$c_3 = C_\lambda(\boldsymbol{T}, I \cup \{v_3\}) - C_\lambda(\boldsymbol{T}, I)) = \left(1 - \frac{2}{3}\right) - 1 = -\frac{2}{3},$$

where $I = \{v_1, v_2\}$.

As observed, the marginal contribution does not reach -1, even in the case of maximum possible disagreement.

Based on the example results, it is reasonable to assume that the actual maximum and minimum values of c_k depend on the number of voters, m. The lowest possible value occurs when all other voters share the same ternary preference, a generic voter v_k exhibits maximum disagreement with them. Specifically:

$$\begin{cases} d_\lambda(T_p, T_k) = 1, & \text{if } p \neq k, \\ d_\lambda(T_p, T_q) = 0, & \text{if } p, q \neq k. \end{cases} \tag{10}$$

Without loss of generality, let v_k be the last element of the distance matrix, then D_λ becomes:

$$D_\lambda = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

It is straightforward to see that, given a coalition S such that $v_k \notin S$, the consensus will be $C_\lambda(\boldsymbol{T}, I) = 1$. On the other hand, the consensus including v_k will be:

$$C_\lambda(\boldsymbol{T}, I \cup \{v_k\}) = 1 - \frac{0 \cdot \binom{\#S-1}{2} + 1 \cdot (\#S - 1)}{\binom{\#S}{2}} = 1 - \frac{2}{\#S}.$$

Thus, the marginal contribution of v_k is given by:

$$C_\lambda(\boldsymbol{T}, I) - C_\lambda(\boldsymbol{T}, I \cup \{v_k\}) = -\frac{2}{\#S}.$$

Such result arises because, in the distance matrix D_λ, there are as many 0s as there are pairwise comparisons between voters excluding v_k, which is $\binom{\#S-1}{2}$, and there are as many 1s as there are comparisons between k and the other voters, which is $\#S - 1$.

Therefore, the *average* marginal contribution of v_k, derived by summing over all possible coalitions is given by:

$$c_k = -\sum_{\#S=3}^{m} \frac{\binom{m-1}{\#S-1} \cdot \frac{2}{\#S}}{2^{m-1} - m}. \tag{11}$$

Eq. (11) represents the minimum achievable average marginal contribution, which, as expected, depends on the value of m. To find the maximum average marginal contribution achievable, one might consider the inverse configuration where:

$$\begin{cases} d_\lambda(T_p, T_k) = 0, & \text{if } p \neq k, \\ d_\lambda(T_p, T_q) = 1, & \text{if } p, q \neq k. \end{cases} \tag{12}$$

In this hypothetical configuration, v_k is at a minimum distance (0) from all other voters, while all remaining pairwise distances reach the maximum value of 1. However, this configuration is not compatible with the ternary preference structures. It is impossible for all the other voters to be at maximum distance from each other (i.e., $d_\lambda(T_p, T_q) = 1$ for all $p \neq q$) while v_k maintains a minimum distance (0) from each of them. Identifying an explicit configuration that maximizes the marginal contribution is particularly challenging due to the complexity of these preference structures. Empirical methods can be employed to approximate the maximum c_k achievable. By running extensive simulations and evaluating a large number of random coalitions, one can approximate the upper bound of the average marginal contribution. Details of the simulation procedure are provided in Appendix B. The maximum and minimum c_k values are graphically illustrated in Fig. 1.

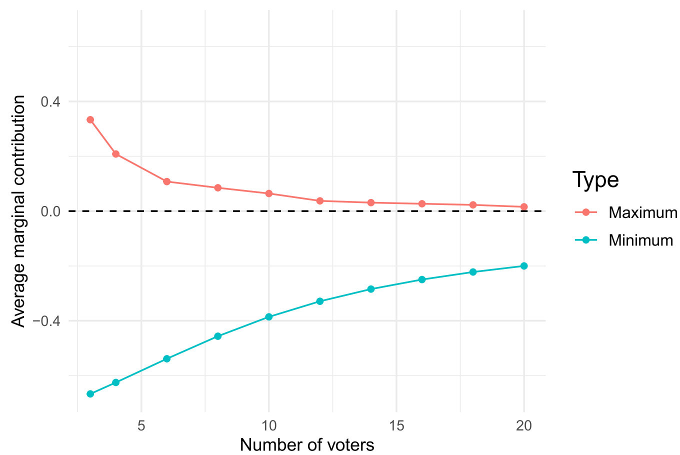


Fig. 1. Maximum and minimum average marginal contributions achievable as a function of the numbers of voters.

From the plot, we observe that both the maximum and minimum average marginal contributions tend to converge towards zero as the number of voters increases, reflecting a "dilution" effect of individual contributions in larger groups. Specifically, in a configuration with 20 voters, the maximum approaches a value of approximately 0.028, while the minimum approaches approximately -0.2. This convergence toward zero highlights a fundamental feature of large voting systems: as the group size increases, the influence of any individual voter diminishes. This mirrors a well-established principle in collective decision-making, where individual power declines with growing group size, ultimately approaching zero in the limit. Given this dilution effect, directly comparing marginal contributions across different group sizes becomes problematic. To overcome this limitation, we introduce the standardized marginal contributions c_k^*:

$$c_k^* = \begin{cases} \frac{c_k}{\max_m(c_k)}, & \text{if } c_k > 0, \\ \frac{c_k}{|\min_m(c_k)|}, & \text{if } c_k < 0, \\ 0, & \text{if } c_k = 0, \end{cases} \tag{13}$$

where $\max_m(c_k)$ and $\min_m(c_k)$ are the maximum and minimum values of c_k observed or calculated depending on the number of voters, m.
The quantity c_k^* scales the marginal contributions by the maximum absolute value observed for any group size m, allowing us to compare the relative influence of voters across different voting populations by removing the influence of group size from the calculation. In a setting with a large number of voters, the absolute impact of each voter is typically negligible. As a result, evaluating each voter's influence relative to their theoretical maximum and minimum contributions provides a more meaningful basis for comparison.

Remark 5. The standardized marginal contribution c_k^* does not replace the classical marginal contribution c_k, but rather complements it. Indeed, the standardization strips away the absolute scale, which remains important in contexts where the magnitude of influence matters. For instance, in small groups where individual voters have a more pronounced impact on the overall outcome, the raw c_k value is essential for understanding the actual shifts in group consensus caused by each voter's participation.

Remark 6. The proposed marginal contributions, c_k and c_k^*, can be used to detect outlying voters. By examining them, we can determine if any voter in our set has a substantially higher absolute impact on the consensus. This means that including or excluding such a voter greatly influences the consensus. Voters with a highly positive marginal contribution may have judgments that are central to the overall consensus, whereas those with a negative marginal contribution may have judgments that deviate significantly from others.

4.2. Estimating the average marginal contribution in large groups

From a computational perspective, two different sources of complexity must be distinguished: the number of alternatives n and the number of voters m. The main bottleneck comes from the exponential number of coalitions in m: computing the exact value of c_k requires considering $2^m - m - 1$ coalitions, which rapidly becomes infeasible as m grows. By contrast, the number of alternatives n affects the measure only indirectly, through the cost of computing the consensus function C_λ based on the distance d_λ (Eq. (3)). Each distance $d_\lambda(T_k, T_l)$ involves all $\binom{n}{2}$ pairs of alternatives, resulting in a computational cost of order $O(n^2)$.
As a result, the exact computation of the average marginal contribution c_k has overall complexity $O(2^m \cdot m^2 \cdot n^2)$, which is exponential in the number of voters m, but only quadratic in the number of alternatives n.

In other words, increasing the number of alternatives n only yields a computationally manageable growth even for moderately large n. By contrast, increasing the number of voters m leads to an exponential growth in the number of coalitions, making exact computation infeasible.

To address this issue, we now propose a general procedure for estimating the average marginal contribution using sampling techniques. Our proposal is adapted from the algorithm proposed by Saavedra-Nieves (2021), which introduces strategies for estimating the Banzhaf value and the Banzhaf-Owen value for general TU-games. This work is also related to Bachrach et al. (2010).

To estimate the average marginal contribution $\hat{c}_k$ of a voter v_k, we perform simple random sampling with replacement. This sampling method is chosen because it provides an *unbiased estimator*, is *computationally efficient* and *scalable* for large voter sets. Moreover, confidence intervals can be constructed using standard techniques (e.g., Central Limit Theorem). In contrast, more advanced sampling methods (e.g., stratified or importance sampling, see Saavedra-Nieves, 2020 and Saavedra-Nieves & Fiestras-Janeiro, 2021) often require prior knowledge about the distribution of marginal contributions or the structure of the game. First, we select a sample of ℓ coalitions with at least two voters not including v_k, $I_1, \dots, I_\ell$, where each $I_j \subseteq S_k$ for $j = 1, \dots, \ell$, and $1 < \ell \leqslant 2^m - m - 1$. Then, the estimator C_k of c_k is the average of the marginal contributions Δ_j over the ℓ sampled coalitions:

$$C_k = \frac{1}{\ell} \cdot \sum_{j=1}^{\ell} \Delta_j = \frac{1}{\ell} \cdot \sum_{j=1}^{\ell} \left(C_\lambda(\boldsymbol{T}, I_j \cup \{k\}) - C_\lambda(\boldsymbol{T}, I_j) \right). \tag{14}$$

The pseudocode to derive the estimate $\hat{c}_k$ is displayed below Algorithms 1 and 2.

Algorithm 1 Estimate average marginal contribution.

Input: (N^*, v^*), voter v_k, number of coalitions to be sampled ℓ
Output: The estimated average marginal contribution $\hat{c}_k$
Initialize $\hat{c}_k \leftarrow 0$
Initialize Marginal_Contributions ← empty list
Generate a sample $I = \{I_1, \dots, I_\ell\}$ of coalitions of S_k with replacement, where $I_j \subseteq S_k$ for $j = 1, \dots, \ell$, and $1 < \ell \leqslant 2^m - m - 1$
for each coalition $I_j \in I$ **do**
  Compute the jth marginal contribution $\delta_j = C_\lambda(\boldsymbol{T}, I_j \cup \{v_k\}) - C_\lambda(\boldsymbol{T}, I_j)$
  Append δ_j to Marginal_Contributions
end for
Compute $\hat{c}_k = \frac{1}{\ell} \cdot \sum_{j=1}^{\ell} \delta_j$
Return $\hat{c}_k$

Note that δ_j is the realization of Δ_j in the jth coalition.
This reduces the computational complexity to $O(\ell \cdot m^2 \cdot n^2)$, replacing the intractable exponential dependency on m with a tractable linear dependence on the sample size ℓ, while keeping the effect of n polynomial and thus computationally feasible.

Following the approaches outlined by Cochran (1977) and Saavedra-Nieves (2021), we review some known statistical properties of the C_k estimator for c_k. As an estimator for the mean, C_k is both unbiased (i.e., $\mathbb{E}[C_k] = c_k$) and consistent (i.e., $\forall \varepsilon > 0 \quad \lim_{\ell \to \infty} \Pr(|C_k - c_k| > \varepsilon) = 0$). The variance of the marginal contributions across all coalitions S_k is given by

$$\theta^2 = \frac{1}{2^m - m - 1} \cdot \sum_{I \subseteq S_k} \left(\Delta(\boldsymbol{T}, I) - c_k \right)^2.$$

Thus, the variance of the estimator C_k is $\mathrm{Var}(C_k) = \frac{\theta^2}{\ell}$. An unbiased estimate for this variance is provided by $\widehat{\mathrm{Var}}(c_k) = \hat{\theta}^2 = \frac{1}{\ell - 1} \cdot \sum_{j=1}^{\ell} \left(\delta_j - \hat{c}_k \right)^2$. Moreover, C_k is efficient, having the minimum variance among all unbiased estimators.
To construct confidence intervals and quantify estimation uncertainty, we rely on distribution-free methods. Specifically, for a given confidence level $(1 - \alpha)$ and desired precision $(\varepsilon > 0)$, we seek to ensure that $\Pr\left(|\hat{c}_k - c_k| \leqslant \varepsilon \right) \geqslant 1 - \alpha$. Hoeffding's inequality is particularly well-suited to this problem since the marginal contributions are bounded. Let the range for a fixed voter $(v_k \in V)$ be denoted by $w_k = \max_{I, I' \subseteq S_k} \left(\Delta(I)_k - \Delta(I')_k \right)$. Then, if $\ell \geqslant \frac{\ln(2/\alpha) \cdot w_k^2}{2\varepsilon^2}$, it follows $\Pr\left(|\hat{c}_k - c_k| \geqslant \varepsilon \right) \leqslant \alpha$.
Similarly, Chebyshev's inequality provides a more general bound that can be applied when the variance of the marginal contributions is known. Applying Chebyshev's inequality to the estimator C_k (which has variance θ^2/ℓ) gives

$$\Pr\left(|\hat{c}_k - c_k| \geqslant \varepsilon \right) \leqslant \frac{\mathrm{Var}(C_k)}{\varepsilon^2} = \frac{\theta^2}{\ell \cdot \varepsilon^2}.$$

Although this bound is typically looser than Hoeffding's inequality, it is useful since it only requires knowledge of the variance and holds under very general conditions.
Alternatively, non-parametric approaches such as the percentile bootstrap can be employed to construct confidence intervals Bootstrap methods naturally accommodate boundedness and potential skewness in the distribution of marginal contributions, and may therefore offer more accurate coverage in small-sample settings.

To perform the estimation, we use the R (Team, 2024) software to randomly sample indices of coalitions with replacement from S_k.

5. Simulation studies

In this section, we present a comprehensive empirical assessment of the proposed algorithm. Section 5.1 investigates its statistical properties by examining both the bias and the variance of the estimator, while Section 5.2 focuses on the computational performance and scalability of the approach.

5.1. Bias and variance

Here, we investigate the statistical properties of our proposed algorithm by evaluating both the bias and the variance of the estimator for the average marginal contribution via Monte Carlo simulations. First, we study the unbiasedness of our proposed estimator for the average marginal contribution. The experimental framework was designed to compare the estimates produced by our algorithm against the exact average marginal contribution computed via exhaustive enumeration (Eq. (9)). In these experiments, we considered small group sizes where the exact computation is tractable. Specifically, we set the number of the number of voters to $m = 12, 14, 16$ and the alternatives to $n = 5$.

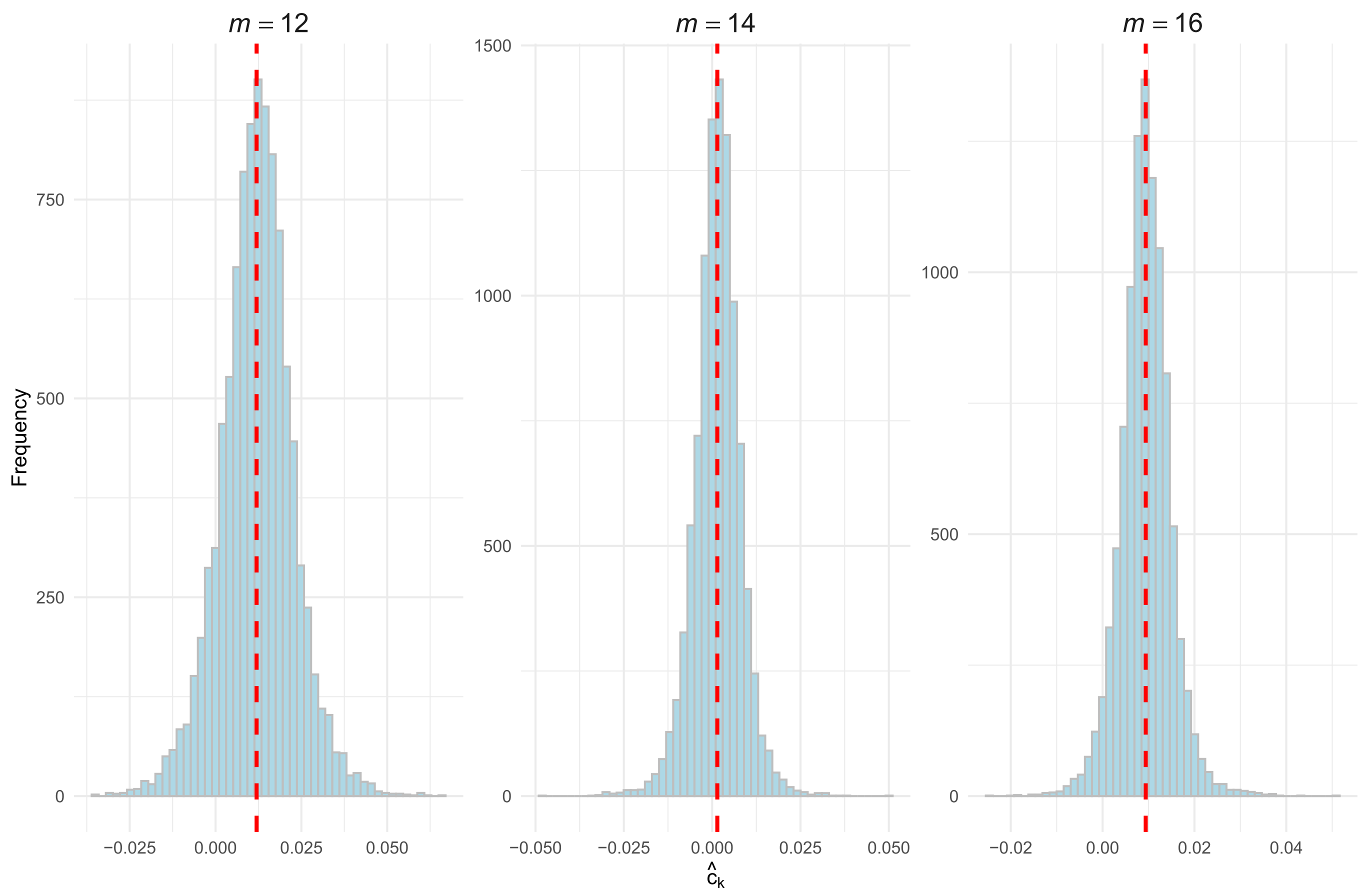


Fig. 2. Histograms displaying the distribution of the average marginal contribution estimates for different group sizes ($m = 12, 14, 16$) over 10 000 replications. In each panel, the red dashed vertical line denotes the true value c_k. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

For each configuration, we generated the corresponding preference profiles and computed the exact average marginal contribution, c_k, for a random voter. We then obtained 10 000 independent estimates of the average marginal contribution $\hat{c}_k$, by applying the proposed algorithm. The empirical bias of the estimator was computed by comparing the sample mean $\bar{\hat{c}}_k$ from the Monte Carlo replications with the true values c_k: $b = \bar{\hat{c}}_k - c_k$. In each replication, only $\ell = 10$ sub-samples were used to compute the estimate to show that our estimator is unbiased exactly, rather than asymptotically.

To assess statistical evidence for unbiasedness, we performed a one-sample t-test for each configuration. The null hypothesis H_0 posits that the true mean of the estimates is equal to the exact value, i.e., $H_0 : \mathbb{E}[C_k] = c_k$, while $H_1 : \mathbb{E}[C_k] \neq c_k$. The t-test is based on the observed t statistic: $t = \frac{\bar{\hat{c}}_k - c_k}{s/\sqrt{r}}$, where s is the sample standard deviation, and $r = 10\,000$ is the number of replications. Furthermore, the 95 % confidence intervals for the mean, calculated from the t-distribution with $r - 1$ degrees of freedom are reported.
Table 1 summarize the key results of our Monte Carlo simulations.

Table 1
Comparison between exact values, estimated average marginal contributions, and results of one-sample t-tests for unbiasedness.

m	c_k	$\bar{\hat{c}}_k$	Bias	t-statistic	p-value
12	0.01196	0.01203	7.48×10^{-5}	0.707	0.480
14	0.00146	0.00148	2.21×10^{-5}	0.332	0.740
16	0.00939	0.00938	-5.91×10^{-6}	−0.108	0.914

The results indicate that the empirical bias is negligible across all configurations. In the configuration with $m = 12$, the bias is 7.48×10^{-5}, while for $m = 14$ the bias is 2.21×10^{-5}, and for $m = 16$ the bias is even closer to zero. The one-sample t-tests further confirm that the estimated means are not statistically significantly different from the exact values, with p-values of 0.480, 0.740 and 0.914 for $m = 12$, 14 and 16, respectively; the 95 % confidence intervals for the mean estimates consistently contain the exact value. Thus, we conclude that the estimator is empirically unbiased. In addition, we examine how the empirical distribution of the estimates (Fig. 2) is shaped around the ground truth.

Although the experiments were conducted on medium-small voter groups (m), they demonstrate that the estimator is unbiased even with small sample size (ℓ). We now turn our attention to another critical statistical property: the variance of the estimator. In each experiment, we generated a set of ternary preferences by randomly sampling from the universe, considering group sizes $m = 20$, 50 and 100 with $n = 10$ alternatives. A range of sample sizes was defined, specifically $\ell \in \{10, 50, 100, 500, 1000\}$, and for each value of ℓ we performed 250 independent replications. In each replication, the average marginal contribution was estimated for a random voter v_k.

Fig. 3 shows the distributions of the estimates, for the random voter, as a function of ℓ for the three different values of m. The results confirm that the variance of the estimator decreases with increasing ℓ, which is in line with the theoretical properties of a mean estimator.

5.2. Computational time

We now investigate the computational efficiency of the proposed algorithm by evaluating how its execution time varies with both the num-

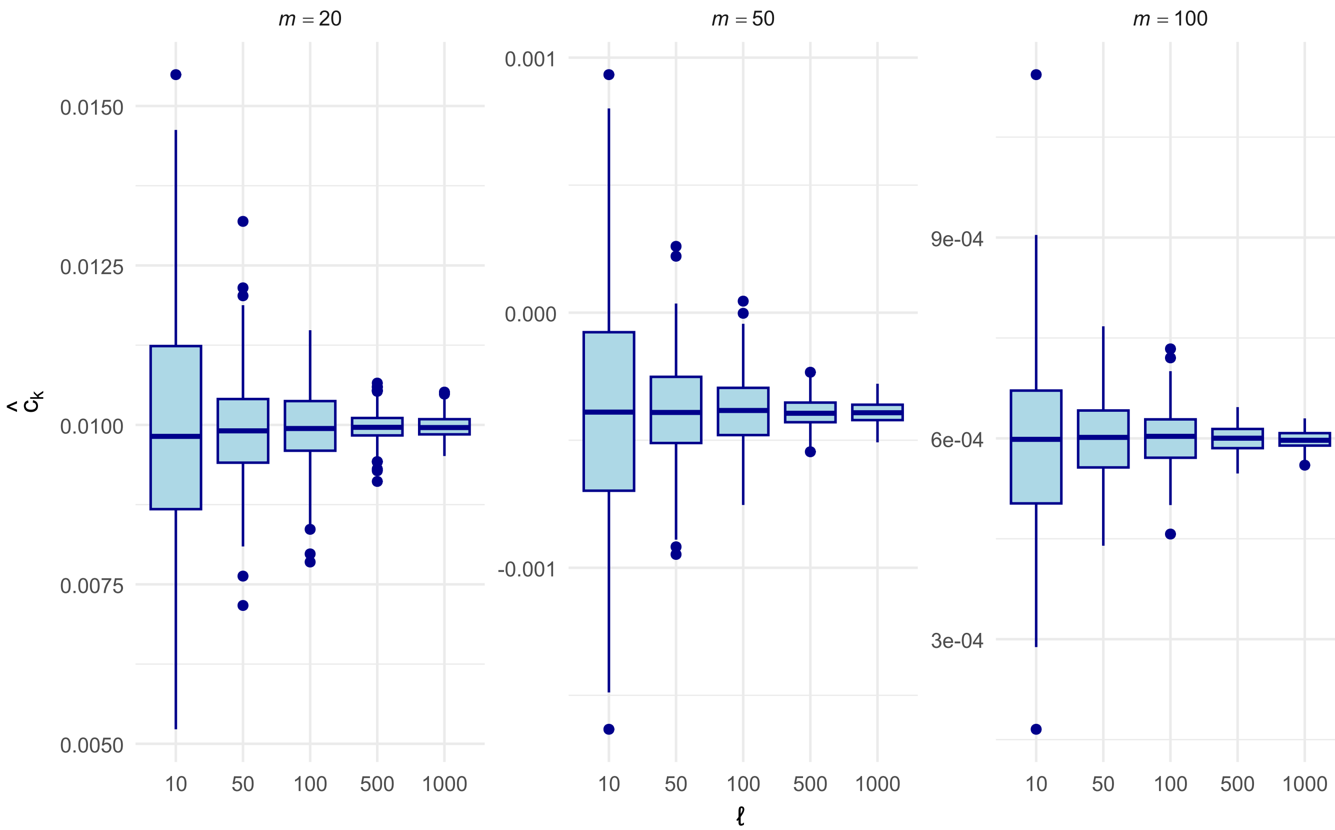


Fig. 3. Distribution of $\hat{c}_k$ as a function of ℓ for different values of m.

ber of voters m and the number of alternatives n. This investigation is motivated by two key considerations. Firstly, as the number of voters increases, the need for an optimized algorithm becomes critical due to the combinatorial explosion in the number of possible subgroups. Secondly, increasing n also results in higher computational costs, since the proposes algorithm relies on the computation of the consensus and the distance between ternary preferences (as defined in Eqs. (2) and (3)), and these functions scale with the size of the alternative set.

To systematically assess the scalability of our method, we considered a grid of configurations with $m \in \{20, 50, 100\}$ and $n \in \{5, 25, 50, 100, 200\}$. For each configuration, we generated synthetic preference profiles and executed the algorithm for a fixed number of iterations ($l = 500$), repeating the process 250 times to obtain robust estimates of the computational time. The total execution time (in seconds) was recorded for each run and the results are summarized in Fig. 4a–c.

As expected, the computational time increases with both m and n. For example, when $m = 20$, execution times range from approximately 0.25 s for $n = 5$ to 0.45 s for $n = 200$; for $m = 50$, the times increase to between 0.8 s and 1.4 s, and for $m = 100$ they range from 3.5 to 5.0 s. However, despite these increases, the algorithm remains computationally tractable across all configurations considered. Note that $n = 200$ already represents a relatively large number of alternatives. In most real-world surveys or decision-making contexts, asking individuals to rank or evaluate so many options is rare; typically, the number of alternatives ranges between 5 and 50.

6. Case studies

This section presents an empirical analysis using two datasets. First, we explore the ISTAT dataset to assess urban issues across different Italian regions. Next, we analyze the Balkan Barometer data to examine public opinion trends over time in six Western Balkan countries.

6.1. ISTAT dataset

The dataset employed in this study, already used in Albano and Plaia (2021), is derived from the Italian National Institute of Statistics[3] (ISTAT).

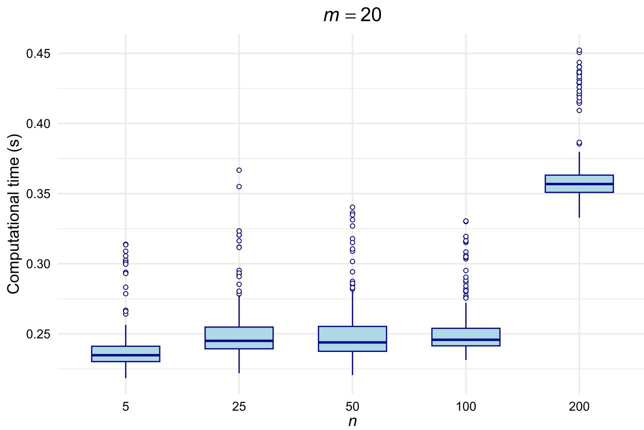
Specifically, data regard the survey titled “Aspetti della vita quotidiana" (Aspects of Daily Life). This annual survey, conducted each February since 2005, provides comprehensive insights into the everyday lives of individuals and families across Italy. The survey captures a wide range of thematic areas, offering detailed information about citizens’ habits, the challenges they face, and their overall satisfaction with various aspects of life, including their economic situation, local area conditions, and public services.

The original dataset consists of a 20×10 matrix, where the rows represent the 20 regions of Italy. The columns correspond to various urban issues, specifically: parking difficulties (x_1), inefficiency of public transport (x_2), traffic (x_3), poor street lighting (x_4), poor road conditions (x_5), dirty roads (x_6), air pollution (x_7), noise (x_8), risk of crime (x_9), and bad smell (x_{10}).

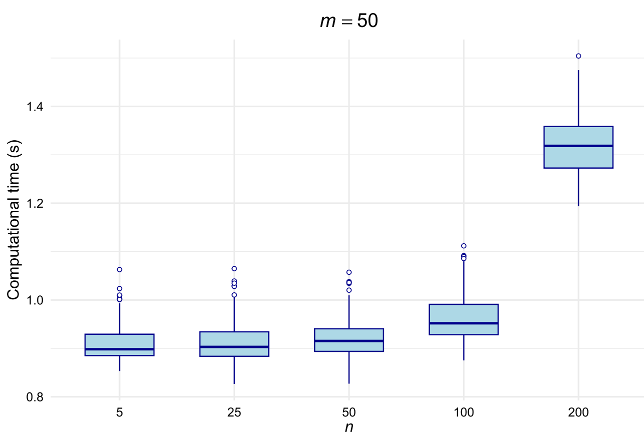
Each element n_{ij} in the original data matrix (included in the supplementary material) represents the percentage of individuals in the ith region who perceive the jth problem as a significant issue in their area.

In Albano and Plaia (2021), the data were transformed into a list of weak orders. Specifically, alternatives were ranked such that the prob-

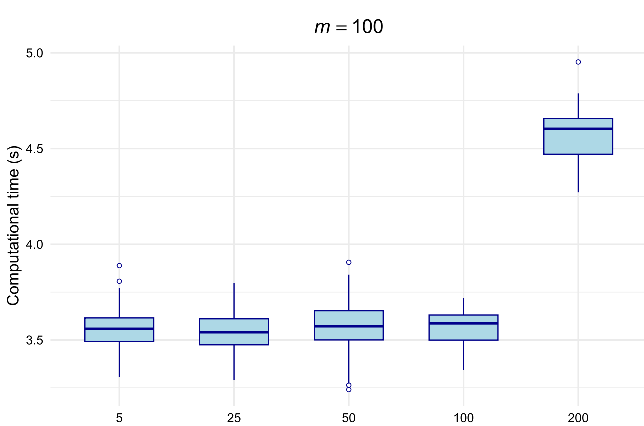
[3] https://www.istat.it/.



(a) $m = 20,\ n \in \{5, 25, 50, 100, 200\},\ l = 500$



(b) $m = 50,\ n \in \{5, 25, 50, 100, 200\},\ l = 500$



(c) $m = 100,\ n \in \{5, 25, 50, 100, 200\},\ l = 500$

Fig. 4. Computational time (in seconds) as a function of the number of alternatives m for different values of n.

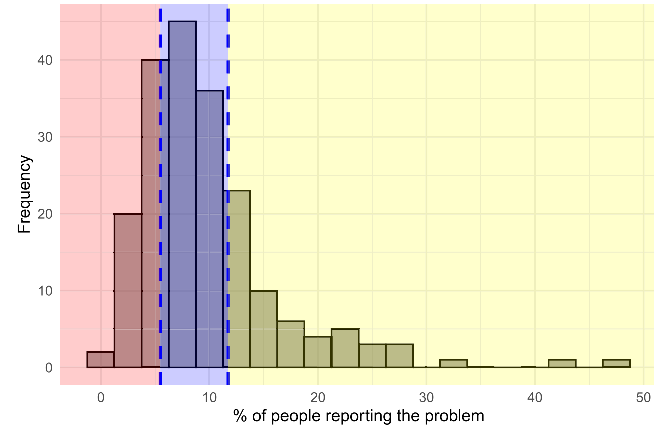


Fig. 5. Histogram of the distribution of the percentage of people reporting various problems as important (aggregated across all issues and regions in Italy). The dashed vertical lines indicate the first and third quartiles. The plot is divided into three shaded regions corresponding to three approval levels: red for U, blue for N, and yellow for A. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Table 2
Votes in sicily.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
Sicily	15.2	12.3	12.4	12.5	24.5	10.3	11.0	9.8	4.2	4.9

lem with the highest percentage was assigned rank 1, indicating it as the most prevalent issue, with subsequent ranks assigned in descending order of the percentages.

For our analysis, we extend this transformation to ternary preferences, by categorizing the alternatives into A, N and U (here representing important, neutral, and negligible problems, respectively).

Fig. 5 shows that the three categories are determinated using of distribution of the percentage of people reporting various problems as important (aggregated across all issues and regions in Italy). Alternatives ranked below the first quartile are categorized as U, those between the first and third quartiles are classified as N, and problems ranked above the third quartile are categorized as A. Aggregating across all regions instead of creating a separate distribution for each region allows us to apply consistent thresholds for all the regions.

As an example, let us consider the votes expressed in Sicily in Table 2. Note that the percentages in Table 2 do not sum to 100 % because respondents were allowed to select multiple options.

Since the first and third quartile of the global distribution are 5.5 % and 11.7 %, respectively, the ternary preference of Sicily is derived as follows:

$$\begin{array}{c} x_5 \\ x_1 \\ x_4 \\ x_3 \\ x_2 \\ \hline x_7 \\ x_6 \\ x_8 \\ \hline x_9 \\ x_{10} \end{array}$$

Here, x_5, x_1, x_4, x_3 and x_2 are categorized as important problems (A), while x_7, x_6 and x_8 are neutral (N), and x_9 and x_{10} are negligible (U).

Following this categorization, all regional votes are transformed into ternary preference. Given the number of voters ($m = 20$), the average

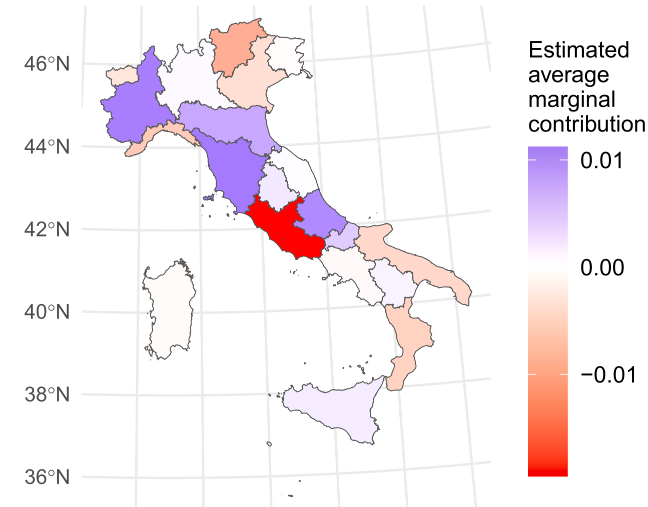


Fig. 6. Choropleth map of Italy illustrating the estimated average marginal contributions across different regions. Each region is shaded according to its estimate, with colors ranging from red (indicating negative contributions) through white (neutral) to violet (indicating positive contributions). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

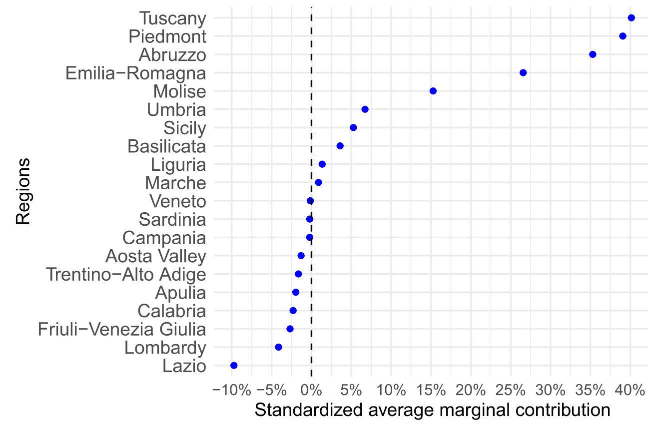


Fig. 7. Standardized estimates of marginal contribution for each region.

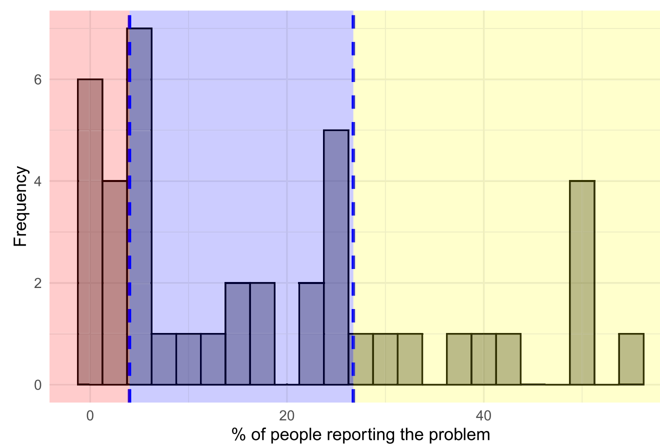


Fig. 8. Histogram of the distribution of the percentage of people reporting various problems as important (aggregated across all issues and Balkan countries). The dashed vertical lines indicate the first and third quartiles. The plot is divided into three shaded regions corresponding to three approval levels: red for U, blue for N and yellow for A. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

marginal contributions, shown in Fig. 6, are estimated according to the algorithm detailed in Section 4.2.

Fig. 6 illustrates that only a few regions, such as Piedmont, Emilia-Romagna, Tuscany and Abruzzo, show strong positive estimates ranging from 0.007 to 0.011. This suggests that these regions contribute positively to the overall consensus. In contrast, Lazio has a negative estimate of approximately of −0.02, indicating a more negative impact on the general consensus compared to other regions. Other regions, including Friuli Venezia Giulia, Lombardy, Calabria and Apulia, also show negative estimates, but these are less pronounced than Lazio's. Additionally, the estimates do not exhibit a clear geographic pattern across Northern, Central and Southern Italy. This lack of pattern suggests that regional impacts are more influenced by local factors than by geographic divisions. The confidence intervals were computed following the methodology described in Section 4.2 using Hoeffding's and Chebyshev's inequalities (details provided in the supplementary file). Specifically, most estimated marginal contributions are statistically significant under both methods (Chebyshev intervals are narrower), with the exception of Liguria, Veneto, Sardinia, and Campania, whose confidence intervals include zero in both approaches.

To facilitate interpretation further, regions are ranked according to their standardized marginal contributions, expressed in percentages, as shown in Fig. 7.

Piedmont, Emilia-Romagna, Tuscany and Abruzzo have standardized contributions ranging from approximately 25 % to 40 % of the maximum positive achievable in a group of 20 voters, remarking their substantial positive impact on the general consensus. Conversely, Lazio's negative estimate of −0.02 represents about 10 % of the lower bound, highlighting a moderate adverse impact on the consensus.

6.2. Balkan barometer

The data for this analysis comes from the Balkan Barometer,[4] an annual survey that measures public opinion and business sentiments across six Western Balkan economies. Commissioned by the Regional Cooperation Council (RCC), the Balkan Barometer surveys life and work aspirations, socio-economic and political trends, and regional integration. It collects data from over 6000 citizens and 1200 companies.

For this analysis, we used data from the section titled “Most important problems in your economy" within the broader category of “Life satisfaction and assessment of general trends". This survey, conducted annually since 2015, includes opinions from citizens of six Balkan countries: Albania, Bosnia and Herzegovina, Kosovo, North Macedonia, Montenegro and Serbia. Respondents are asked to identify the most pressing problems they perceive, choosing from a predefined list of issues.

To analyze the temporal evolution of marginal contributions in the Balkan countries, we focused exclusively on the alternatives included every year in the survey. This results in a total of seven consistent issues: “Unemployment"(x_1), “Economic situation"(x_2), “Crime"(x_3), “Corruption"(x_4), “Protection of human rights"(x_5), “Environmental change"(x_6), and “Security situation" (x_7). Moreover, since Bosnia and Herzegovina did not provide any opinions in 2018, we excluded that year from our analysis. The reason for this exclusion is that changes in the respondent pool can alter the significance of marginal contributions. Therefore, to accurately study the temporal evolution of marginal contributions, it is essential to ensure that the voter base remains the same across the years analyzed.

The true values recorded are displayed in the supplementary file. To transform these into ternary preferences, we used the same strategy as

[4] https://www.rcc.int/balkanbarometer/.

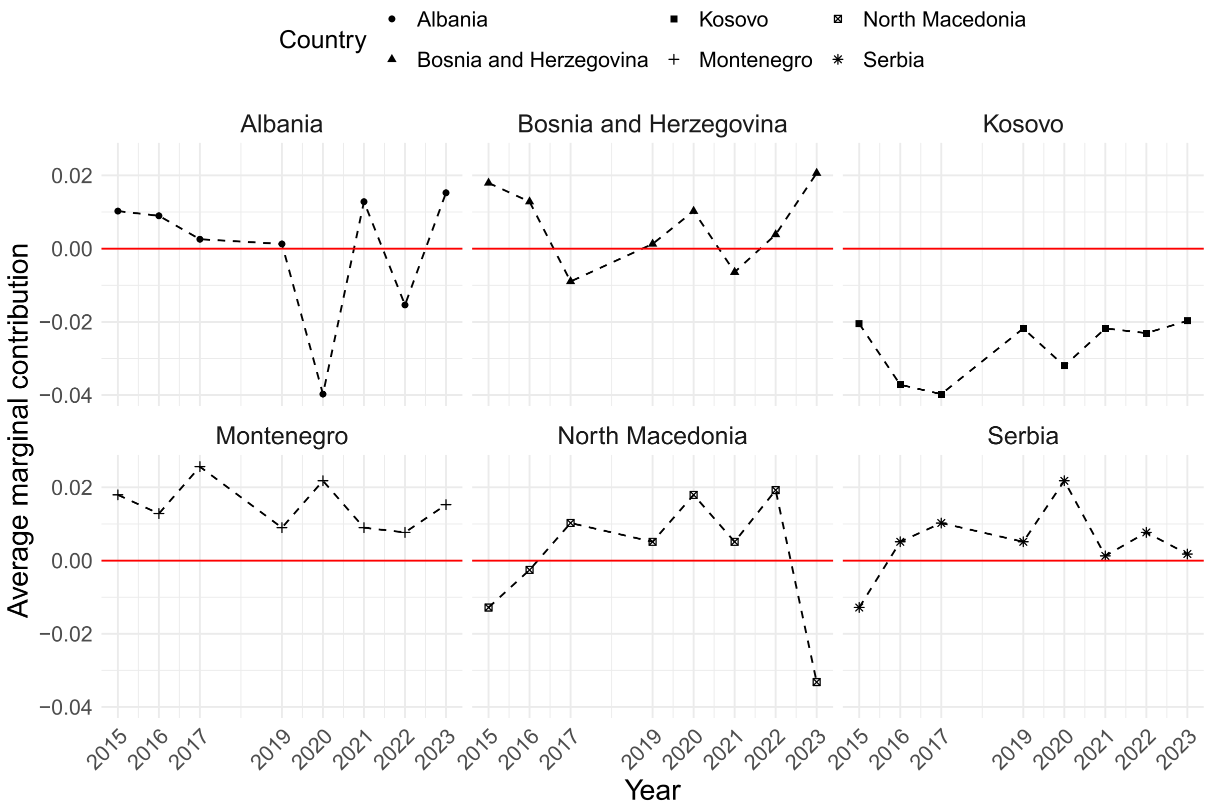


Fig. 9. Marginal contributions of Balkan countries across different years. Each country's data is represented by a dashed line and distinct points, with the horizontal red line indicating a zero marginal contribution. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

in the previous section. Fig. 8 illustrates the resulting thresholds, i.e. Q1 = 4.00 % and Q3 = 26.75 %.

Given the limited number of voters $n = 6$, marginal contributions are computed exactly using Eq. (5), and are reported in Fig. 9.

Fig. 9 shows fluctuations in marginal contributions for most countries, with values oscillating between positive and negative. Notably, Kosovo is unique among the six Balkan countries in consistently exhibiting a negative marginal contribution throughout the years from 2015 to 2023. This suggests that the opinions of Kosovo's citizens are consistently more divergent from the consensus of other countries, highlighting its distinct status within the region.

Conversely, Montenegro is the only country having always positive contribution during the period from 2015 to 2023. These consistent positive values suggest that the opinions of Montenegro citizens can be considered "central" to the whole area. Moreover, Albania contributions generally hover around zero, indicating minimal change relative to other countries, except for a significant dip below zero around 2020, highlighting a negative marginal contribution during that period. North Macedonia display some of the more pronounced positive contributions between 2017 and 2022. Bosnia and Herzegovina shows high variability, with its marginal contributions swinging markedly between positive and negative. In fact, half of the analyzed countries frequently cross the zero line, pointing to a complex and fluctuating pattern of consensus within the region.

To take into account the small number of voters (countries) in our group, let us analyze the standardized average marginal contribution, expressed in percentages, in Fig. 10 (which does not consider the temporal aspect).

Notably, Fig. 10 does not present any alarming or remarkably high extreme values, neither particularly negative nor particularly positive.

For instance, Kosovo, while consistently showing negative contributions, does not exhibit values far below −5%. This suggests that, although there is some disagreement, it is not particularly intense. Similarly, Montenegro, despite its consistently positive contributions, does not display exceptionally high values; the highest point is around 25 %, indicating a moderate positive impact on the consensus. The other countries, also show contained ranges of contributions, reflecting a mix of moderate positive and negative influences. For instance, Albania's contributions range from −5% to 15 %, reflecting some volatility but not extreme instability. North Macedonia and Serbia show similar moderate fluctuations around the zero mark, and Bosnia and Herzegovina's contributions generally remain positive without significant negative dips.

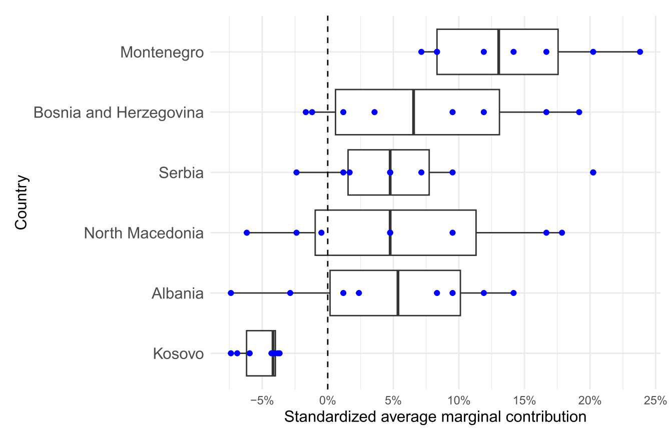


Fig. 10. Boxplot of standardized estimates by country.

The lack of particularly extreme values indicates that the Balkans regions, despite their diversity, do not show particularly different opinions of the alternatives analyzed. This suggests a stable environment with no significant polarization. However, some attention should be paid to Kosovo's consistent negative contributions. Although the negative values do not fall far below −5%, the persistent trend of negative contributions could indicate underlying disagreement that merits closer examination in the future.

7. Concluding remarks

In this paper, we examined consensus within ternary preferences, a framework in which voters rank alternatives using a weak order and classify them into three categories: acceptable, neutral and unacceptable.

We have proposed a distance-based measure to quantify the consensus among voters and introduced the notion of marginal contribution to consensus, which reflects the impact of each voter on the overall agreement within a group. This approach allowed us to estimate the influence of individual voters in scenarios involving ternary preferences.

Our findings demonstrate that as the number of voters increases, the marginal contribution of each voter tends to become more concentrated around zero, highlighting a dilution effect. This is particularly evident when standardizing marginal contributions, which provides a comparative framework across different coalition sizes. The standardized measure facilitates the identification of voters whose opinions significantly diverge from the group consensus, serving as a useful tool for detecting outliers.

To address the computational challenges inherent in calculating marginal contributions for large voter groups, we developed an estimation procedure using sampling techniques. Our comprehensive simulation studies validate both the statistical properties and computational efficiency of this approach. Specifically, we demonstrate that our estimator is empirically unbiased across different group sizes, with variance decreasing as sample size increases. Moreover, the computational performance analysis reveals that our algorithm remains tractable across various configurations.

The empirical analysis, based on real-world data from ISTAT and the Balkan Barometer, provided insights into the practical application of our theoretical framework. The results showed that our method could effectively capture and quantify the degree of consensus in diverse voting scenarios, even with complex preference structures. Specifically, the ISTAT dataset revealed significant regional variations in consensus, with Piedmont and Emilia-Romagna showing strong positive impacts and Lazio exhibiting a notable negative effect. Conversely, the Balkan Barometer dataset highlighted Kosovo's persistent negative contributions and Montenegro's consistently positive contributions, with other countries displaying less extreme fluctuations. The use of standardized marginal contributions revealed that the ISTAT dataset exhibited more pronounced impacts on consensus compared to the Balkan Barometer. This difference may be attributed to a higher overall consensus in the Balkan context, which led to less variation in marginal contributions across countries.

These findings underscore the effectiveness and versatility of our distance-based consensus measure and marginal contribution analysis for understanding and evaluating consensus within ternary preference settings. The proposed method has practical implications for decision-making processes in social choice, where understanding voter alignment and disagreement is essential.

Future research could build on this work by applying our voter influence measure to consensus-reaching processes (see Eklund et al., 2007 and Tang et al., among others). Since in consensus-reaching processes the agreement is often improved through iterative interactions, where individuals may adjust their preferences, our influence measure could be instrumental in identifying key voters who might guide or sway the group toward a stronger consensus. Additionally, expanding this framework to other preference structures, such as preference rankings, could reveal whether similar patterns of consensus and influence emerge across diverse voting paradigms. Finally, the present framework provides a foundation for future work aiming at an axiomatic characterization of the average marginal contribution of consensus.

CRediT authorship contribution statement

Alessandro Albano: Writing - original draft, Visualization, Software, Methodology, Formal analysis, Data curation, Conceptualization; **José Luis García-Lapresta:** Writing - original draft, Supervision, Methodology, Conceptualization; **Antonella Plaia:** Writing - original draft, Supervision; **Mariangela Sciandra:** Writing - original draft, Supervision.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Supplementary material

Supplementary material associated with this article can be found in the online version at 10.1016/j.ejor.2025.12.016.

Appendix A. List of symbols

Symbol	Description
X	Finite set of alternatives.
x_i, x_j	Generic alternative of X.
n	Number of alternatives.
$W(X)$	Set of weak orders on X.
$L(X)$	Set of linear orders on X.
R	Generic weak order.
$x_i \succ x_j$	x_i is strictly preferred to x_j in R.
$x_i \sim x_j$	x_i and x_j are indifferent in R.
$\mathcal{P}(Y)$	Power set of an arbitrary set Y.
$\#Y$	Cardinality of an arbitrary set Y.
V	Finite set of voters.
m	Number of voters.
$\mathcal{P}_2(V)$	Set of all the subsets of V with at least two voters.
A	Set of acceptable alternatives.
U	Set of unacceptable alternatives.
v, v_k, v_l	Generic voter.
R_v	Weak order of voter v.
A_v	Set of acceptable alternatives for v.
N_v	Set of neutral alternatives for v.
U_v	Set of unacceptable alternatives for v.
$T = (R, A, N, U)$	Ternary preference structure.
$\mathcal{T}(X)$	Set of ternary preferences on X.
$x_i \, R \, x_j$	x_i ranked at least as high as x_j in R.
O_R	Pairwise comparison between alternatives based on R.
p_{ij}^{kl}	Preference-discordance between v_k and v_l over x_i and x_j.
$P_A(x_i)$	Ternary approval score of alternative x_i.
a_{ij}^{kl}	Approval-discordance between v_k and v_l over x_i and x_j.
d_λ	Distance between ternary preferences.
D_λ	Matrix of pairwise distances between voters.
I	Generic subset of voters.
I_p	Subgroup in a voter partition of I.
$C_\lambda(\boldsymbol{T}, I)$	Consensus measure among voters of I in the profile $\boldsymbol{T}$.
$\boldsymbol{T}_\pi$	Profile relabeled by permutation π.
I_π	Voter subset via π^{-1} on I.
$\boldsymbol{T}^\sigma$	Profile relabeled by σ (alternatives).
$\boldsymbol{T}^{-1}$	Profile with inverted voter opinions.
c_k	Average marginal contribution to consensus by v_k.
S_k	Set of subsets of 2 voters not containing voter v_k: $S_k = \{I \in \mathcal{P}_2(V) \mid v_k \notin I\}$.
c_k^*	Standardized marginal contribution of v_k.
C_k	Estimator of c_k from sampled coalitions.
ℓ	Number of sampled coalitions.
$\hat{c}_k$	Estimated average marginal contribution of v_k.
θ^2	Variance of the marginal contributions across all coalitions.
$\Delta(\boldsymbol{T}, I)$	Change in consensus adding v_k to I.
$\mathrm{Var}(C_k)$	Variance of the estimator.
Pr	Probability.
w_k	Max range of marginal contributions for v_k.

Appendix B. Approximating the maximum c_k

The following algorithm is designed to approximate the maximum achievable c_k in a scenario involving m voters.

Algorithm 2 Approximate maximum average marginal contribution.

```
Input: Number of voters m, threshold = 10, iterations = 50,000
Output: Approximated maximum marginal contribution

maxMarginalContribution ← −∞
for i ← 1 to iterations do
  Extract m ternary preferences T = {T_1, T_2, …, T_m}
  if m ⩽ threshold then
    marginalContribution ← computes the exact c_k using formula (5)
  else
    marginalContribution ← estimates ĉ_k using the sampling-based Algorithm 1
  end if
  if marginalContribution > maxMarginalContribution then
    maxMarginalContribution ← marginalContribution
  end if
end for
Return maxMarginalContribution
```

References

Albano, A., García-Lapresta, J. L., Plaia, A., & Sciandra, M. (2023). A family of distances between preference-approvals. *Annals of Operations Research*, *323*, 1–29. https://doi.org/10.1007/s10479-022-05008-4

Albano, A., García-Lapresta, J. L., Plaia, A., & Sciandra, M. (2024). Clustering alternatives in preference-approvals via novel pseudometrics. *Statistical Methods & Applications*, *33*, 61–87.

Albano, A., & Plaia, A. (2021). Element weighted kemeny distance for ranking data. *Electronic Journal of Applied Statistical Analysis*, *14*(1), 117–145.

Alcantud, J. C. R., de Andrés Calle, R., & Cascón, J. M. (2013). On measures of cohesiveness under dichotomous opinions: Some characterizations of approval consensus measures. *Information Sciences*, *240*, 45–55.

Alcantud, J. C. R., & Laruelle, A. (2014). Dis & approval voting: A characterization. *Social Choice and Welfare*, *43*, 1–10.

Bachrach, Y., Markakis, E., Resnick, E., Procaccia, A. D., Rosenschein, J. S., & Saberi, A. (2010). Approximating power indices: Theoretical and empirical analysis. *Autonomous Agents and Multi-Agent Systems*, *20*(2), 105–122. https://doi.org/10.1007/s10458-009-9078-9

Balinski, M., & Laraki, R. (2007). A theory of measuring, electing and ranking. *Proceedings of the National Academy of Sciences of the United States of America*, *104*, 8720–8725.

Balinski, M., & Laraki, R. (2011). Majority Judgment: Measuring, Ranking, and Electing. Cambridge, MA: The MIT Press.

Balinski, M., & Laraki, R. (2022). Majority judgment vs. approval voting. *Operations Research*, *70*, 1296–1316.

Banzhaf, J. F. III (1965). Weighted voting doesn't work: A mathematical analysis. *Rutgers Law Review*, *19*(2), 317–343.

Barokas, G. (2022). Majority-approval social choice. *Journal of Mathematical Psychology*, *109*, 102694. https://doi.org/10.1016/j.jmp.2022.102694

Barokas, G. (2023). A non-lexicographic rule in the preference-approval setting. *Economics Letters*, *227*, 111124. https://doi.org/10.1016/j.econlet.2023.111124

Barokas, G., & Sprumont, Y. (2022). The broken borda rule and other refinements of approval ranking. *Social Choice and Welfare*, *58*(1), 187–199. https://doi.org/10.1007/s00355-021-01356-5

Bilbao, J. M., Fernández, J. R., Jiménez, N., & López, J. J. (2010). The Banzhaf power index for ternary bicooperative games. *Discrete Applied Mathematics*, *158*(9), 967–980. https://doi.org/10.1016/j.dam.2010.02.007

Bosch, R. (2005). Characterizations of voting rules and consensus measures. Ph.D. thesis. Tilburg University.

Brams, S. J. (2008). Mathematics and democracy: Designing better voting and fair-division procedures. Princeton: Princeton University Press.

Brams, S. J., & Fishburn, P. C. (1978). Approval voting. *American Political Science Review*, *72*, 831–857.

Brams, S. J., & Sanver, M. R. (2009). Voting systems that combine approval and preference. In S. J. Brams, W. V. Gherlein, & F. S. Roberts (Eds.), *The mathematics of preference, choice and order: Essays in honour of peter c. fishburn* Studies in Choice and Welfare (pp. 215–237). Berlin: Springer-Verlag.

Cochran, W. G. (1977). Sampling techniques. John Wiley & Sons.
Dong, Y., Li, Y., He, Y., & Chen, X. (2021). Preference-approval structures in group decision making: Axiomatic distance and aggregation. *Decision Analysis*, *18*, 273–295.
Dummett, M. (1984). Voting procedures. Oxford: Clarendon Press.
Eklund, P., Rusinowska, A., & De Swart, H. (2007). Consensus reaching in committees. *European Journal of Operational Research*, *178*(1), 185–193. https://doi.org/10.1016/j.ejor.2005.11.012
Erdamar, B., García-Lapresta, J. L., Pérez-Román, D., & Sanver, M. R. (2014). Measuring consensus in a preference-approval context. *Information Fusion*, *17*, 14–21.
Felsenthal, D. S. (1989). On combining approval with disapproval voting. *Behavioral Science*, *34*, 56–70.
Felsenthal, D. S., & Machover, M. (1997). Ternary voting games. *International Journal of Game Theory*, *26*(3), 335–351. https://doi.org/10.1007/BF01263275
García-Lapresta, J. L., & Pérez-Román, D. (2011). Measuring consensus in weak orders. In E. Herrera-Viedma, J. L. García-Lapresta, J. Kacprzyk, H. Nurmi, M. Fedrizzi, & S. Zadròzny (Eds.), *Consensual processes* STUDFUZZ 267 (pp. 213–234). Berlin: Springer-Verlag.
Kamwa, E. (2019). Condorcet efficiency of the preference approval voting and the probability of selecting the condorcet loser. *Theory and Decision*, *87*, 299–320.
Kamwa, E. (2023). On two voting systems that combine approval and preferences: Fallback voting and preference approval voting. *Public Choice*, *196*, 169–205.
Lehrer, E. (1988). An axiomatization of the Banzhaf value. *International Journal of Game Theory*, *17*, 89–99.
Musegaas, M., Borm, P. E. M., & Quant, M. (2018). Three-valued simple games. *Theory and Decision*, *85*(2), 201–224. https://doi.org/10.1007/s11238-017-9630-z
Owen, G. (1975). Multilinear extensions and the Banzhaf value. *Naval Research Logistics Quarterly*, *22*, 741–750.
Saavedra-Nieves, A. (2020). Assessing systematic sampling in estimating the Banzhaf-Owen value. *Operations Research Letters*, *48*, 725–731.
Saavedra-Nieves, A. (2021). Statistics and game theory: Estimating coalitional values in R software. *Operations Research Letters*, *49*(1), 129–135. https://doi.org/10.1016/j.orl.2020.12.005
Saavedra-Nieves, A., & Fiestras-Janeiro, M. G. (2021). Sampling methods to estimate the Banzhaf–Owen value. *Annals of Operations Research*, *301*(1), 199–223.
Sanver, M. R. (2011). Approval as an intrinsic part of preference. In J. F. Laslier, & M. R. Sanver (Eds.), *Handbook on approval voting* (pp. 469–481). Berlin: Springer-Verlag.
Team, R. C. (2024). R: A language and environment for statistical computing. R Foundation for Statistical Computing Vienna, Austria. https://www.R-project.org/.
Ye, J., Sun, B., Bai, J., Bao, Q., Chu, X., & Bao, K. (2024). A preference-approval structure-based non-additive three-way group consensus decision-making approach for medical diagnosis. *Information Fusion*, *101*, 102008.
Yılmaz, M. R. (1999). Can we improve upon approval voting? *European Journal of Political Economy*, *15*, 89–100.