Quantum, classical symmetries and action-angle variables by factorization of superintegrable systems

Şengül Kuru[a] *, Javier Negro[b] †, Sergio Salamanca[b] ‡

[a] Department of Physics, Faculty of Science, Ankara University, 06100 Ankara, Türkiye
[b] Departamento de Física Teórica, Atómica y Óptica, and IMUVA, Universidad de Valladolid, 47011 Valladolid, Spain

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Abstract

The purpose of this work is to present a method based on the factorizations used in one dimensional quantum mechanics in order to find the symmetries of quantum and classical superintegrable systems in higher dimensions. We apply this procedure to the harmonic oscillator and Kepler-Coulomb systems to show the differences with other more standard approaches. We have described in detail the basic ingredients to make explicit the parallelism of classical and quantum treatments. One of the most interesting results is the finding of action-angle variables as a natural component of the classical sysmmetries within this formalism.

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1 Introduction

The aim of this work is to introduce a method to find the symmetries of quantum and classical superintegrable systems by means of two paradigmatic examples: the harmonic oscillator (HO) and Kepler-Coulomb (KC). The technique we are going to apply is based on the well known factorization method [1], which was already applied in the early times of quantum mechanics [2], although the origin of this type of transformations goes back to Darboux; in that context it is referred to as Darboux transformations and has a wider range of applications in nonlinear equations [3]. Another development of factorizations is known as supersymmetric quantum mechanics (SUSY-qm) [4, 5, 6]. In principle, the factorization was designed to deal with one dimensional quantum problems; but in the present work we want to show how it can be extended to higher dimensions and the way that factorizations can also be used in the classical frame to obtain constants of motion (or "classical symmetries"; it happens that constants of motion are generating functions of canonical transformations that leave invariant the Hamiltonian). In fact, the symmetries that are obtained in this way for classical systems, can be naturally interpreted in terms of action-angle variables of Hamilton-Jacobi theory, what supplies an extra interest to this approach. In general, the methods used to obtain symmetries for quantum superintegrable systems and their classical analogs follow quite different routes; for instance, in the classical context often it is made use of Hamilton-Jacobi formalism [7, 8, 9] which is not valid for the quantum case. However, as we will show (see also Refs. [10, 11, 12, 13]), the factorization method supply a unified way towards the symmetries in classical and quantum mechanical systems.

Thus, in this work we will pay attention essentially to explain the ingredients of this method. We have chosen the HO and KC as two well known examples [14, 15, 16], where it is easy to check the consistency of our results and the standard

*kuru@science.ankara.edu.tr, ORCID: 0000-0001-6380-280X
†jnegro@fta.uva.es, ORCID: 0000-0002-0847-6420
‡sergio.salamanca@uva.es, ORCID: 0000-0003-0151-8373

expressions, to show how the method works, what are the key points and at the same time to appreciate the differences. For instance, the symmetries so obtained in the quantum case, are ready to compute eigenfunctions and degeneracy eigenspaces; while in the classical case the constants of motion here found lead directly to the open and closed trajectories or even to the motion, as well as their frequencies. The algebraic structure of symmetries in both quantum and classical contexts, can also be found inside this formalism and their similarity is immediate.

The method here presented is designed to deal with superintegrable systems [17, 18]. Recall that a system with n degrees of freedom that has n involutive symmetries in the quantum formalism (or constants of motion in classical mechanics), including the Hamiltonian if it does not depend explicitly on time, is said to be integrable; when there are additional symmetries (not commuting with all the involutive ones) it is called superintegrable; The name maximally superintegrable is reserved when there is the maximum number $2n-1$ of independent symmetries. In this scenario, the factorization method has been already put to work in a variety of configurations: systems defined on constant curvature surfaces [12, 13]; to find higher order symmetries [10, 19, 20, 21] or to a list of classical systems [22, 23]. In the near future we plan to apply it to further superintegrable models, such as Smorodinsky-Winternitz [24], Evans [25], or those of Darboux [26] and Perlick [21], in order to show its flexibility for a wide class of problems. Another point of interest regarding this method is that it allows for its generalization to any dimension in a straightforward way.

The organization of this work is as follows. In Section 2 we introduce the spherical coordinates appropriate to work with central potentials which is the case of HO and KC. The separation of variables leads to three reduced Hamiltonians in a fixed order, where the first two of them correspond to the angular variables. Then, the well known operators $\hat{L}^{\pm}$ corresponding to the $so(3)$ generators which span the angular symmetries are expressed as the product of two reduced operators in the separated variables. These reduced operators are called shift and ladder operators, since one of them changes the eigenvalues of the angular Hamiltonian $\hat{L}_z^2$, while the other changes a parameter of the potential of a second Hamiltonian, $\hat{L}^2$. At the same time, the reduced operators can be identified with factorization operators of reduced Hamiltonians. The key point is that the symmetries consist in the product of these two types of operators. This structure is repeated in all the other cases. In Sections 3 and 4 we construct the symmetries of KC and HO, respectively, following the same path. Section 5 is devoted to the classical symmetries of these two systems, and Section 6 to the relation of classical symmetries and action-angle variables. Some conclusions and remarks close this paper.

2 Symmetries in Spherical Coordinates of Quantum Central Potentials

In this section we will introduce the spherical coordinates (r, θ, ϕ), where r is the radius, θ is the polar and φ the azimuthal angles, to study the symmetries of both harmonic oscillator and Kepler-Coulomb systems under the same approach based on factorizations. Both systems are maximally superintegrable [17, 18] and therefore they admit five independent symmetries.

2.1 Spherical coordinates

The Schrödinger Hamiltonians for central potentials $V(r)$ have the form of a nested structure in spherical coordinates (r, θ, φ):

$$\hat{H} = -\partial_{rr} - \frac{2}{r}\partial_r + V(r) + \frac{1}{r^2}\left(-\partial_{\theta\theta} - \frac{1}{\tan\theta}\partial_\theta + \frac{1}{\sin^2\theta}\Big(-\partial_{\varphi\varphi}\Big)\right) \tag{2.1}$$

where $2m = \hbar = 1$ and $\partial_r = \frac{\partial}{\partial r}$, $\partial_{rr} = \frac{\partial^2}{\partial r^2}$, etc... From this structure we can see that the Hamiltonian $\hat{H}$, the total momentum operator $\hat{L}^2$ and the square of the z-component $\hat{L}_z^2$ given by

$$\hat{L}_z^2 = (-i\partial_\varphi)^2, \qquad \hat{L}^2 = -\partial_{\theta\theta} - \frac{1}{\tan\theta}\partial_\theta + \frac{\hat{L}_z^2}{\sin^2\theta}, \qquad \hat{H} = -\partial_{rr} - \frac{2}{r}\partial_r + V(r) + \frac{\hat{L}^2}{r^2} \tag{2.2}$$

constitute a sequence of three involutive operators, i.e. they commute among each other and may be interpreted as a kind of partial Hamiltonians. They are associated to the coordinate separation and determine a set of common eigenfunctions separated into a product of radial and angular components,

$$\Psi_{n,\ell,m}(r,\theta,\varphi) = R_n^\ell(r)\Phi_{\ell,m}(\theta,\varphi) = R_n^\ell(r)P_\ell^m(\theta)\phi_m(\varphi) \tag{2.3}$$

The physical solutions have additional boundary conditions to be specified later; at this moment we look at the above functions just as separated solutions of differential eigenvalue equations for this kind of Hamiltonians. Each component of the solution (2.3) satisfies a reduced eigenvalue equation in one variable, for each of the "reduced" partial Hamiltonians (2.2) consistent with the separation sequence:

$$\begin{aligned}
&(a) \qquad \hat{L}_z^2(\varphi)\phi_m(\varphi) := -\partial_{\varphi\varphi}\phi_m(\varphi) = m^2\phi_m(\varphi)\\
&(b) \qquad \hat{L}_m^2(\theta)P_\ell^m(\theta) := \left(-\partial_{\theta\theta} - \tfrac{1}{\tan\theta}\partial_\theta + \tfrac{m^2}{\sin^2\theta}\right)P_\ell^m(\theta) = \ell(\ell+1)\,P_\ell^m(\theta)\\
&(c) \qquad \hat{H}_\ell(r)R_n^\ell(r) := \left(-\partial_{rr} - \tfrac{2}{r}\partial_r + V(r) + \tfrac{\ell(\ell+1)}{r^2}\right)R_n^\ell(r) = E_n\,R_n^\ell(r)
\end{aligned} \tag{2.4}$$

where ℓ, m, n are quantum numbers to be specified later. For this reason the partial Hamiltonians (2.2) are also referred as "diagonal" operators in the separated basis (2.3).

Each component eigenfunction depends on the eigenvalue of the previous reduced Hamiltonian component; for instance in (2.4-b), the eigenfunction $P_\ell^m(\theta)$ of $\hat{L}^2$ carries the eigenvalue label ℓ of the current reduced operator $\hat{L}_m^2$ and the label m of the eigenfunction $\phi_m(\varphi)$, corresponding to the eigenvalue of the previous reduced Hamiltonian $\hat{L}_z^2$. We say that the involutive operators are in a nested construction: $\hat{L}_z^2 \subset \hat{L}^2 \subset \hat{H}$. Therefore, in order to build symmetry operators acting on the separated solutions of the total Hamiltonian by means of operators associated to the reduced Hamiltonians (2.4) they must keep this "matching" of eigenvalues of consecutive reduced partial Hamiltonians.

Our plan is the following. We start with the involutive symmetries $\hat{L}_z^2$, $\hat{L}^2$ and $\hat{H}$, mentioned above, then, we want to complete the number of independent symmetries (up to five in three dimensions) by computing two new pairs: i) $\hat{\mathcal{L}}^\pm_{\theta,\varphi}$ depending on the variables $\theta,\ \varphi$, which will commute with the Hamiltonians $\hat{L}^2$ and $\hat{H}$; and ii) $\hat{\mathcal{S}}^\pm_{r,\theta}$ depending on the variables $r,\ \theta$, which will commute with the Hamiltonians $\hat{H}$ and $\hat{L}_z^2$. We present all of them in the following natural ordering:

$$\hat{H}\,, \quad \hat{\mathcal{S}}^\pm_{r,\theta}\,, \quad \hat{L}^2\,, \quad \hat{\mathcal{L}}^\pm_{\theta,\varphi}\,, \quad \hat{L}_z^2 \tag{2.5}$$

We will arrive to these symmetries through the factorization of the reduced partial Hamiltonians. Of course not all of them will be independent, but we will check that five of them will be.

2.2 Angular states and symmetry operators $\hat{\mathcal{L}}^\pm_{\theta,\varphi}$

We will start by analyzing the symmetries of the total angular momentum operator $\hat{L}^2$ which are well known in quantum mechanics. Consider the lowering and raising momentum operators $\hat{L}^\pm$ (in our context we will use the notation $\hat{\mathcal{L}}^\pm_{\theta,\varphi}$ for them):

$$\hat{\mathcal{L}}^\pm_{\theta,\varphi} = e^{\pm i\varphi}(\pm\partial_\theta + i\frac{1}{\tan\theta}\partial_\varphi) \tag{2.6}$$

These operators commute with the total angular momentum but they modify the eigenvalues of $\hat{L}_z$,

$$\hat{\mathcal{L}}^\pm_{\theta,\varphi}\,\hat{L}^2 = \hat{L}^2\,\hat{\mathcal{L}}^\pm_{\theta,\varphi}\,, \qquad \hat{L}_z\,\hat{\mathcal{L}}^\pm_{\theta,\varphi} = \hat{\mathcal{L}}^\pm_{\theta,\varphi}\,(\hat{L}_z \pm 1) \tag{2.7}$$

Our goal is to showcase how this structure translate into the separated components of the angular eigenfunctions shown in (2.3):

$$\hat{L}^2\,\Phi_{\ell,m}(\theta,\varphi) = \ell(\ell+1)\,\Phi_{\ell,m}(\theta,\varphi), \qquad \Phi_{\ell,m}(\theta,\varphi) = P_\ell^m(\theta)\phi_m(\varphi) \tag{2.8}$$

We can split the $\hat{\mathcal{L}}^\pm_{\theta,\varphi}$ operators into a product of two one-dimensional factors.

(i) A factor is one of the pair of exponentials $e^{\pm i\varphi}$ which we rewrite in the form

$$\hat{\Lambda}^\pm_\varphi := e^{\pm i\varphi} \tag{2.9}$$

Then, we say that $\hat{\Lambda}^\pm_\varphi$ are **ladder operators** of the "diagonal" operator $\hat{L}_z$ in the variable φ. This is clear, since if $m \in \mathbb{Z}$ designs the eigenvalues of the eigenfunctions $\phi_m(\varphi) = e^{im\varphi}$ of $\hat{L}_z$, then

$$\hat{L}_z\phi_m(\varphi) = m\phi_m(\varphi), \qquad \hat{\Lambda}^\pm_\varphi\phi_m(\varphi) \propto \phi_{m\pm1}(\varphi), \qquad [\hat{L}_z, \hat{\Lambda}^\pm_\varphi] = \pm\hat{\Lambda}^\pm_\varphi \tag{2.10}$$

(ii) The second pair of factor operators in (2.6) are

$$\hat{\Sigma}^{\pm}_{\theta,\varphi} := \pm\partial_\theta + i\frac{\partial_\varphi}{\tan\theta} \tag{2.11}$$

If they act on separated eigenfunctions (2.8), they will give rise to the reduced operators $\hat{\Sigma}^{\pm}_{\theta,m}$ defined by:

$$\begin{aligned} \hat{\mathcal{L}}^{-}_{\theta,\varphi}\left(P^m_\ell(\theta)\phi_m(\varphi)\right) &:= \left(\hat{\Sigma}^{-}_{\theta,m}\,P^m_\ell(\theta)\right)\phi_{m-1}(\varphi)\,, & \hat{\Sigma}^{-}_{\theta,m} &:= -\partial_\theta - \tfrac{m}{\tan\theta} \\ \hat{\mathcal{L}}^{+}_{\theta,\varphi}\left(P^{m-1}_\ell(\theta)\phi_{m-1}(\varphi)\right) &:= \left(\hat{\Sigma}^{+}_{\theta,m}\,P^{m-1}_\ell(\theta)\right)\phi_m(\varphi)\,, & \hat{\Sigma}^{+}_{\theta,m} &:= \partial_\theta - \tfrac{m-1}{\tan\theta} \end{aligned} \tag{2.12}$$

where m is for the eigenvalues of L_z as mentioned above. The reduced operators $\hat{\Sigma}^{\pm}_{\theta,m}$ act on the components $P^m_\ell(\theta)$ of the eigenfunctions $\Phi^m_\ell(\theta,\varphi)$ as follows,

$$\hat{\Sigma}^{-}_{\theta,m}\,P^m_\ell(\theta) \propto P^{m-1}_\ell(\theta)\,, \qquad \hat{\Sigma}^{+}_{\theta,m}\,P^{m-1}_\ell(\theta) \propto P^m_\ell(\theta) \tag{2.13}$$

(iii) The total angular momentum, written as

$$\hat{L}^2(\theta,\varphi) = -\partial_{\theta\theta} - \frac{1}{\tan\theta}\partial_\theta - \frac{\partial_{\varphi\varphi}}{\sin^2\theta} \tag{2.14}$$

when it acts on angular eigenfunctions $\Phi_{\ell,m}(\theta,\varphi)$ leads to the reduced Hamiltonians $\hat{L}^2_m(\theta)$ (or $\hat{L}^2_{\theta,m}$) in θ,

$$\hat{L}^2_m(\theta) := -\partial_{\theta\theta} - \frac{1}{\tan\theta}\partial_\theta + \frac{m^2}{\sin^2\theta} \tag{2.15}$$

The reduced operators $\hat{\Sigma}^{\pm}_{\theta,m}$ in the variable θ are called **pure shift (or displacement) operators** of the reduced operator $\hat{L}^2_m$ given in (2.4-b) since when they act on an eigenfunction component $P^m_\ell(\theta)$, they change the parameter m according to (2.13), but the eigenvalue (determined by ℓ) is left invariant. This is shown by means of the intertwining and factorization relations

$$\hat{L}^2_m(\theta)\hat{\Sigma}^{+}_{\theta,m} = \hat{\Sigma}^{+}_{\theta,m}\hat{L}^2_{m-1}(\theta), \qquad \hat{\Sigma}^{-}_{\theta,m}\hat{L}^2_m(\theta) = \hat{L}^2_{m-1}(\theta)\hat{\Sigma}^{-}_{\theta,m}, \qquad \hat{\Sigma}^{-}_{\theta,m}\hat{\Sigma}^{+}_{\theta,m} = \hat{L}^2_{m-1}(\theta) - m(m-1) \tag{2.16}$$

The last expression of (2.16) means that, if the maximum value of $|m|$ is $m_{\max} = \ell$, and the eigenvalues of $\hat{L}^2_m$ are $\ell(\ell+1)$ then $-\ell \le m \le \ell$ and

$$\hat{\Sigma}^{-}_{\theta,-\ell}\,P^{-m=-\ell}_\ell(\theta) = 0\,, \qquad \hat{\Sigma}^{+}_{\theta,\ell+1}\,P^{m=\ell}_\ell(\theta) = 0 \tag{2.17}$$

Relations (2.16) for $\hat{\Sigma}^{\pm}_{\theta,m}$ are well known as standard factorization of $\hat{L}_m(\theta)$. Notice that the intertwining (2.16), can also be rewritten in terms of commutation rules,

$$[\hat{L}^2_m(\theta), \hat{\Sigma}^{+}_{\theta,m}] = \hat{\Sigma}^{+}_{\theta,m}\left(\hat{L}^2_{m-1}(\theta) - \hat{L}^2_m(\theta)\right), \qquad [\hat{\Sigma}^{-}_{\theta,m}, \hat{L}^2_m(\theta)] = \left(\hat{L}^2_{m-1}(\theta) - \hat{L}^2_m(\theta)\right)\hat{\Sigma}^{-}_{\theta,m} \tag{2.18}$$

(iv) In conclusion, we may consider the symmetry operators $\hat{\mathcal{L}}^{\pm}_{\theta,\varphi}$ of (2.6) in a reduced form as the product of ladder $\hat{\Lambda}^{\pm}_\varphi$ (of $\hat{L}_z(\varphi)$) and shift $\hat{\Sigma}^{\pm}_{\theta,m}$ (of the following partial Hamiltonian $L^2_m(\theta)$) operators

$$\hat{\mathcal{L}}^{\pm}_{\theta,m,\varphi} = \hat{\Sigma}^{\pm}_{\theta,m}\hat{\Lambda}^{\pm}_\varphi \tag{2.19}$$

In this case the shift operator can be simplified:

$$\hat{\Sigma}^{\pm}_{\theta,\varphi} = \pm\partial_\theta + i\frac{\partial_\varphi}{\tan\theta} \Rightarrow \hat{\Sigma}^{\pm}_{\theta,m} := \pm\partial_\theta + \frac{m}{\tan\theta}$$

Its action on separated solutions gives new separated solutions with the same eigenvalue (the same ℓ-value):

$$\hat{\mathcal{L}}^{+}_{\theta,\varphi}\left(P^m_\ell(\theta)\phi_m(\varphi)\right) \propto P^{m+1}_\ell(\theta)\phi_{m+1}(\varphi), \qquad \hat{\mathcal{L}}^{-}_{\theta,\varphi}\left(P^m_\ell(\theta)\phi_m(\varphi)\right) \propto P^{m-1}_\ell(\theta)\phi_{m-1}(\varphi)$$

We have thus shown that the angular momentum symmetries $\hat{\mathcal{L}}^{\pm}_{\theta,\varphi}$ have an structure shown in the scheme:

$$\hat{L}^2(\theta) \;\;\rightarrow\;\; \hat{\mathcal{L}}^{\pm}_{\theta,\varphi} = \hat{\Sigma}^{\pm}_{\theta}\,\hat{\Lambda}^{\pm}_{\varphi} \;\;\leftarrow\;\; \hat{L}^2_z(\varphi) \tag{2.20}$$

This will be a general property of the symmetries that we will find in the following: i) $\hat{\mathcal{L}}^{\pm}_{\theta,\varphi}$ depends on the variables φ and θ of two consecutive reduced Hamiltonians: $\hat{L}^2_m(\theta)$ and $\hat{L}^2_z(\varphi)$. ii) The operator $\hat{\mathcal{L}}^{\pm}_{\theta,\varphi}$ is a symmetry of $\hat{L}^2(\theta,\varphi)$, but its commutator with $\hat{L}_z(\varphi)$ is of ladder type (see (2.7)):

$$[\hat{L}^2, \hat{\mathcal{L}}^{\pm}_{\theta,\varphi}] = 0\,, \qquad [\hat{L}_z, \hat{\mathcal{L}}^{\pm}_{\theta,\varphi}] = \pm\hat{\mathcal{L}}^{\pm}_{\theta,\varphi} \tag{2.21}$$

2.3 Ladder operators of $\hat{L}^2$

Next, we will use the considerations of the previous section to find the symmetry operators $\hat{\mathcal{S}}^{\pm}_{r,\theta}$, mentioned in (2.5), which will be obtained from a ladder operator of the reduced $\hat{L}^2_m(\theta)$ and a shift operator of the reduced Hamiltonian $\hat{H}_\ell(r)$. Thus, in this case we follow the opposite way: Firstly we find the reduced operators by means of the factorization method and from them we construct a kind of symmetry operators.

We start by looking for ladder operators $\hat{\Lambda}^{\pm}_{\ell}(\theta)$ for $\hat{L}^2_{\theta,m}$ that play the same role as $\hat{\Lambda}^{\pm}_{\varphi}$ with respect to $\hat{L}_z$, i.e., modify the eigenvalues of the total angular momentum, ℓ. Thus, we will consider the reduced eigenvalue equation for $\hat{L}^2_m(\theta)$, given by (2.15), and write it in the following form

$$\hat{C}_\ell(\theta)P^m_\ell(\theta) := \Big(-\sin^2\theta\partial_{\theta\theta} - \sin\theta\cos\theta\partial_\theta - \ell(\ell+1)\sin^2\theta\Big)P^m_\ell(\theta) = -m^2P^m_\ell(\theta) \tag{2.22}$$

where equation (2.22) is an eigenvalue equation for a new operator $\hat{C}_\ell(\theta)$ with eigenvalues $-m^2$ and where ℓ plays the role of a parameter of such operator. Applying the factorization method to this equation, we obtain the factor operators [27, 28]

$$\hat{\Lambda}^{\pm}_{\ell}(\theta) = \pm\sin\theta\partial_\theta + \ell\cos\theta \tag{2.23}$$

which satisfy the following factorization properties

$$\hat{C}_\ell(\theta) = \hat{\Lambda}^{+}_{\ell,\theta}\hat{\Lambda}^{-}_{\ell,\theta} - \ell^2 = -m^2$$

$$\hat{\Lambda}^{-}_{\ell,\theta}\,\hat{C}_\ell = \hat{C}_{\ell-1}\,\hat{\Lambda}^{-}_{\ell,\theta}\,, \qquad \hat{\Lambda}^{+}_{\ell,\theta}\,\hat{C}_{\ell-1} = \hat{C}_\ell\,\hat{\Lambda}^{+}_{\ell,\theta}$$

$$[\hat{L}^2_m, \hat{\Lambda}^{\pm}_{\ell,\theta}]_{L^2\to\ell(\ell+1)} = \pm2\ell\,\hat{\Lambda}^{\pm}_{\ell,\theta}$$

This means that $\ell \geq |m|$, and that the action of $\hat{\Lambda}^{\pm}_{\ell,\theta}$ on eigenfunctions $\Phi_{\ell,m}(\theta,\varphi) = P^m_\ell(\theta)\phi_m(\varphi)$ (2.8) is to change the eigenvalue parameter ℓ of $\hat{L}^2_m(\theta)$ by $\ell \pm 1$, while the label m keeps unaltered:

$$\hat{\Lambda}^{-}_{\ell}(\theta)\,P^m_\ell(\theta) \propto P^m_{\ell-1}(\theta)\,, \qquad \hat{\Lambda}^{+}_{\ell+1}(\theta)\,P^m_\ell(\theta) \propto P^m_{\ell+1}(\theta)\,, \qquad \hat{\Lambda}^{-}_{\ell}(\theta)\,P^{m=\ell}_\ell(\theta) = 0$$

where the last equation determines the lowest ℓ-state, $\ell_{\min} = m$. The ladder operators $\hat{\Lambda}^{\pm}_{\ell}(\theta)$ of $\hat{L}^2$, together with the symmetry operators $\hat{\mathcal{L}}^{\pm}_{\theta,\varphi}$ generate a spectrum generating algebra (SGA) of $\hat{L}^2$.

These operators allow us to define the normalizable angular states of the system by acting on a fundamental state annihilated by ladder and symmetry operators $\hat{\Lambda}^{-}_{\ell=0}$ and $\hat{\mathcal{L}}^{-}_{\theta,\varphi}$,

$$\hat{\Lambda}^{-}_{\ell=0}\Phi_{\ell=0,m=0} = \hat{\mathcal{L}}^{-}_{\theta,\varphi}\Phi_{\ell=0,m=0} = 0 \implies \Phi_{0,0} \propto 1$$

$$\Phi_{\ell,m}(\theta,\varphi) = N\,(\hat{\mathcal{L}}^{+}_{\theta,\varphi})^m(\prod_{i=1}^{\ell}\hat{\Lambda}^{+}_{i,\theta})\Phi_{0,0}, \qquad 0 \leq m \leq \ell \tag{2.24}$$

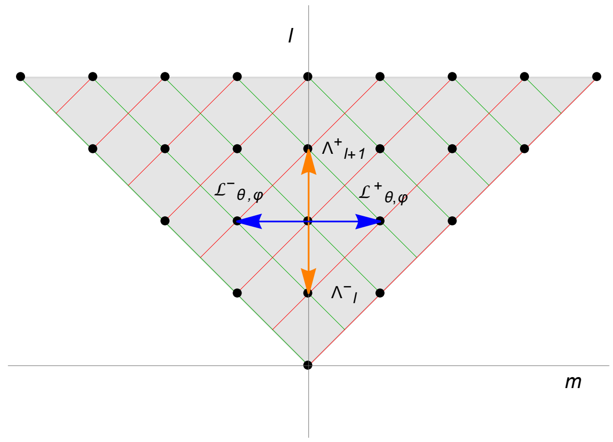

Figure 1: Schematic representation of the ladder operators $\hat{\Lambda}^{\pm}_{\ell}(\theta)$ and symmetries $\hat{\mathcal{L}}^{\pm}_{\theta,\varphi}$ for the angular Hamiltonian $\hat{L}^2$. Each state is represented by a dot in the plane (m, ℓ).

where N is a normalization constant. In other words, we can move on different eigenvalue (ℓ) states by means of ladder operators, and inside each energy level by symmetry operators (changing m). This is essentially the action of an spectrum generating algebra [29, 30] of the angular operator $\hat{L}^2$. The separated angular solutions obtained in this way are spherical harmonics, $\Phi_{\ell,m}(\theta,\varphi) = Y_{\ell}^{m}(\theta,\varphi)$. For instance,

$$Y_2^2(\theta,\varphi) \propto (\hat{\mathcal{L}}^{+}_{\theta,\varphi})^2 \hat{\Lambda}^{+}_{2}(\theta)\hat{\Lambda}^{+}_{1}(\theta)\Phi_{0,0}(\theta,\varphi).$$

These angular operators and the angular wavefunctions, are common to all central potential systems.

3 Radial Symmetries: The Harmonic Oscillator

To complete the symmetry analysis, we must consider the reduced radial eigenvalue problem in order to find the remaining shift operator $\hat{\Sigma}^{\pm}_{r,\ell}$; then together with the known ladder $\hat{\Lambda}^{\pm}_{\ell,\theta}$ operators (see (2.23)) we will form the second pair of symmetries following the pattern of the angular Hamiltonian. The radial shift operators will depend on the particular form of the radial potential $V(r)$; however, the ladder operators $\hat{\Lambda}^{\pm}_{\ell,\theta}$ of $\hat{L}^2_{\theta,m}$ (which are given above) are common to all central potentials. Thus, we have the following scheme for the remaining symmetry, $\hat{\mathcal{S}}^{\pm}_{r,\ell,\theta}$, similar to (2.20),

$$\hat{H}_\ell(r) \;\rightarrow\; \hat{\mathcal{S}}^{\pm}_{r,\ell,\theta} = \hat{\Sigma}^{\pm}_{r,\ell}\hat{\Lambda}^{\pm}_{\ell,\theta} \;\leftarrow\; \hat{L}^2_m(\theta) \tag{3.1}$$

In the following, we will apply the factorization method to the reduced radial Hamiltonian

$$\hat{H}_\ell(r) = -\partial_{rr} - \frac{2}{r}\partial_r + \frac{\ell(\ell+1)}{r^2} + V(r) \tag{3.2}$$

in order to find its shift $\hat{\Sigma}^{\pm}_{r,\ell}$ (and ladder $\hat{\Lambda}^{\pm}_{r}$) operators. The central potential $V(r)$ will be that of the HO or the KC potentials

$$V_{\rm HO} = \frac{\omega^2}{4}\,r^2, \qquad V_{\rm KC} = \frac{-k}{r} \tag{3.3}$$

where k is a positive real constant of the KC potential; $\omega > 0$ is an angular frequency and ℓ determines one of the eigenvalues of $\hat{L}^2$.

We will devote this section to finding the shift and ladder operators for the HO Hamiltonian $\hat{H}_{\rm HO}$ with potential $V_{\rm HO}(r)$. In the following section we will work out the same method for the KC potential $V_{\rm KC}(r)$.

3.1 Shift and ladder operators of harmonic oscilator

Basic operators $\hat a^\pm_\ell, \hat b^\pm_\ell$ of radial oscillator $\hat H_\ell(r)$

We will handle the following notation for the reduced HO Hamiltonian, and the corresponding discrete eigenfunctions and eigenvalues:

$$\hat H_\ell(r) R^\ell_n(r) = \left(-\partial_{rr} - \frac{2}{r}\partial_r + \frac{\ell(\ell+1)}{r^2} + \frac{\omega^2}{4} r^2\right) R^\ell_n(r) = E_n R^\ell_n(r) \tag{3.4}$$

Thus, we look for the factorizations of $\hat H_\ell(r) = \hat H_\ell$. There are two independent factorization sets of radial operators $\{\hat a^\pm_\ell, \hat b^\pm_\ell\}$ given by [31]:

$$\begin{aligned}
&\hat H_\ell = \hat a^+_\ell \hat a^-_\ell - \tfrac{\omega}{2}(2\ell-1), \qquad &&\begin{cases} \hat a^+_\ell = -\partial_r + \frac{\ell-1}{r} + \frac{\omega}{2} r, & \hat a^+_{\ell+1} R^\ell_n \propto R^{\ell+1}_{n+1} \\ \hat a^-_\ell = \partial_r + \frac{\ell+1}{r} + \frac{\omega}{2} r, & \hat a^-_\ell R^\ell_n \propto R^{\ell-1}_{n-1} \end{cases} \\
&\hat H_\ell = \hat b^-_{\ell+1} \hat b^+_{\ell+1} + \tfrac{\omega}{2}(2\ell+3), \qquad &&\begin{cases} \hat b^+_\ell = -\partial_r + \frac{\ell-1}{r} - \frac{\omega}{2} r, & \hat b^+_{\ell+1} R^\ell_n \propto R^{\ell+1}_{n-1} \\ \hat b^-_\ell = \partial_r + \frac{\ell+1}{r} - \frac{\omega}{2} r, & \hat b^-_\ell R^\ell_n \propto R^{\ell-1}_{n+1} \end{cases}
\end{aligned} \tag{3.5}$$

Both sets are related by the reflection $-\ell \to (\ell+1)$. The operators $\hat a^\pm_\ell$ and $\hat b^\pm_\ell$ close a kind of two independent Heisenberg algebras satisfying the following commutation relations,

$$\hat a^-_{\ell+1}\hat a^+_{\ell+1} - \hat a^+_\ell \hat a^-_\ell = 2\omega, \qquad \hat b^-_{\ell+1}\hat b^+_{\ell+1} - \hat b^+_\ell \hat b^-_\ell = -2\omega \tag{3.6}$$

These elementary factorization operators modify both the energy (n) and the angular momentum (ℓ) parameters of the radial states $R^\ell_n(r)$, as it is displayed in the commutation rules

$$\begin{aligned}
\hat a^-_\ell \hat H_\ell &= (\hat H_{\ell-1} + \omega)\hat a^-_\ell, \qquad & \hat a^+_{\ell+1}\hat H_\ell &= (\hat H_{\ell+1} - \omega)\hat a^+_{\ell+1} \\
\hat b^-_\ell \hat H_\ell &= (\hat H_{\ell-1} - \omega)\hat b^-_\ell, \qquad & \hat b^+_{\ell+1}\hat H_\ell &= (\hat H_{\ell+1} + \omega)\hat b^+_{\ell+1}
\end{aligned}$$

where one must be careful because $\hat a^-_\ell$ is an annihilation operator of energy quantum ω, while that role is played by $\hat b^+_\ell$ in the second set. The effect of these operators is summarized in the following diagram Fig.2 where each point (ℓ, n) represents the radial state R^ℓ_n.

Shift and ladder operators of the radial oscillator $\hat H_\ell(r)$

Next, from the previous operators we obtain pure shift, $\hat\Sigma^\pm_{r,\ell}$, that modify only the parameter ℓ of angular momentum and pure ladder, $\hat\Lambda^\pm_{\ell,r}$, operators, which change the label n of energy,

$$\begin{aligned}
\hat\Sigma^-_{r,\ell} &= \hat b^-_{\ell-1}\hat a^-_\ell\,, \qquad & \hat\Sigma^+_{r,\ell+2} &= \hat b^+_{\ell+2}\hat a^+_{\ell+1} \\
\hat\Lambda^-_{\ell,r} &= \hat b^+_\ell \hat a^-_\ell\,, \qquad & \hat\Lambda^+_{\ell,r} &= \hat b^-_{\ell+1}\hat a^+_{\ell+1}
\end{aligned} \tag{3.7}$$

They modify their respective parameter in two units, for instance:

$$\hat\Sigma^-_{r,\ell}\hat H_\ell = \hat H_{\ell-2}\hat\Sigma^-_{r,\ell}\,, \qquad \hat\Lambda^-_{\ell,r}\hat H_\ell = (\hat H_\ell + 2\omega)\hat\Lambda^-_{\ell,r}\,, \qquad [\hat H_\ell(r), \hat\Lambda^\pm_{\ell,r}] = \pm 2\omega \hat\Lambda^\pm_{\ell,r} \tag{3.8}$$

A representation of these operators can be found in a diagram Fig.3. For each energy, the state with the maximum value of ℓ is annihilated by $\hat b^+_{\ell+1}$ (the other options lead to no normalizable eigenstate). The expression $\hat H_\ell = \hat b^-_{\ell+1}\hat b^+_{\ell+1} + \frac{\omega}{2}(2\ell+3)$ in (3.5) leads to the energies: $E_n = \frac{\omega}{2}(2n+3)$, where $n \geq \ell$. If $\ell = 0$, $E_0 = \frac{3\omega}{2}$.

The ladder operators $\hat\Lambda^-_{\ell,r} = \hat b^+_\ell \hat a^-_\ell$ and $\hat\Lambda^+_{\ell,r} = \hat b^-_{\ell+1}\hat a^+_{\ell+1}$ depend on ℓ through the expression $\ell(\ell+1)$, which can be replaced by $\hat L^2$, therefore they may be rewritten simply as $\hat\Lambda^\pm_r$.

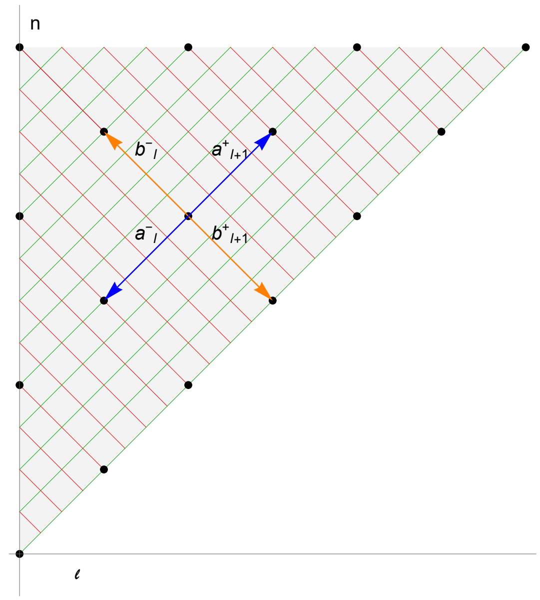

Figure 2: Schematic representation of the action of the auxiliary operators $\hat{a}^{\pm}_{\ell}(r)$ and $\hat{b}^{\pm}_{\ell}(r)$ on functions $R^{\ell}_{n}(r)$ represented by the points (ℓ, n) in the plane.

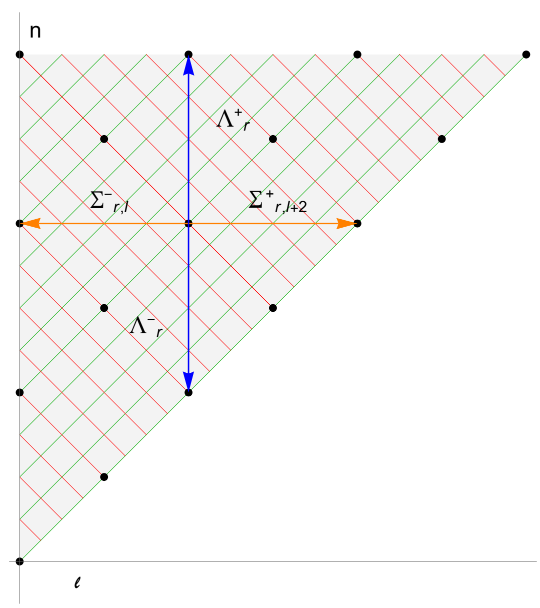

Figure 3: Schematic representation of the action of the shift and ladder operators $\hat{\Sigma}^{\pm}_{r,\ell}$ and $\hat{\Lambda}^{\pm}_{\ell,r}$ on functions $R^{\ell}_{n}(r)$ represented by the points (ℓ, n) in the plane.

Remark that the ladder operators for $\hat{H}_{\ell}$ can not be applied to get more symmetries, but they give us the spectrum and the connection of eigenfunctions of different eigenvalues. Therefore, they may be used to build the spectrum generating algebra [32] of $\hat{H}_{\text{HO}}$.

Symmetries $\hat{\mathcal{S}}^{\pm}_{r,\ell,\theta}$

We have all the ingredients to define the symmetries. For sake of clarity, let us write a complete separated eigenfunction in the form

$$\Psi_{n,\ell,m}(r,\theta,\varphi) = R^{\ell}_{n}(r) P^{m}_{\ell}(\theta) \phi_m(\varphi)$$

Thus, in this case the shift operators change two units the value of the parameter ℓ of the radial function,

$$\hat{\Sigma}^{+}_{r,\ell+2} : R_n^{\ell}(r)P_{\ell}^{m}(\theta)\phi_m(\varphi) \;\to\; R_n^{\ell+2}(r)P_{\ell}^{m}(\theta)\phi_m(\varphi)$$

Meanwhile, the ladder operators $\hat{\Lambda}^{\pm}_{\ell,\theta}$ constructed above acting on the eigenfunctions $P_{\ell}^{m}(\theta)$, change the eigenvalue $\ell \to \ell \pm 1$ therefore, we need two of them to match in a new eigenfunction:

$$\hat{\Lambda}^{+}_{\ell+2,\theta}\hat{\Lambda}^{+}_{\ell+1,\theta} : R_n^{\ell}(r)P_{\ell}^{m}(\theta)\phi_m(\varphi) \;\to\; R_n^{\ell}(r)P_{\ell+2}^{m}(\theta)\phi_m(\varphi)$$

The product of these second order radial shift and the two angular ladder operators supplies a second pair of symmetries:

$$\hat{\mathcal{S}}^{+}_{r,\ell+2,\theta} = \hat{\Sigma}^{+}_{r}\hat{\Lambda}^{+}_{\ell+2,\theta}\hat{\Lambda}^{+}_{\ell+1,\theta}, \qquad \hat{\mathcal{S}}^{-}_{r,\ell,\theta} = \hat{\Sigma}^{-}_{r}\hat{\Lambda}^{-}_{\ell-1,\theta}\hat{\Lambda}^{-}_{\ell,\theta}$$

The final action of the symmetries $\hat{\mathcal{S}}^{+}_{r,\ell,\theta}$ is

$$\hat{\mathcal{S}}^{+}_{r,\ell+2,\theta} : R_n^{\ell}(r)P_{\ell}^{m}(\theta)\phi_m(\varphi) \;\to\; R_n^{\ell+2}(r)P_{\ell+2}^{m}(\theta)\phi_m(\varphi)\,, \quad \hat{\mathcal{S}}^{-}_{r,\ell,\theta} : R_n^{\ell}(r)P_{\ell}^{m}(\theta)\phi_m(\varphi) \;\to\; R_n^{\ell-2}(r)P_{\ell-2}^{m}(\theta)\phi_m(\varphi)$$

They are not independent because their product, $\hat{\mathcal{S}}^{+}_{r,\ell,\theta}\hat{\mathcal{S}}^{-}_{r,\ell,\theta}$ depends on the involutive Hamiltonians. The same happens with $\hat{\mathcal{L}}^{\pm}_{\theta,m,\varphi}$, so at the end we have a total of five independent symmetries, as it should be. These symmetries are complemented with the ladder operators (3.8) which connect the eigenspaces with different energy.

Thus, the action of HO symmetries on the separated eigenfunctions is

$$\hat{\mathcal{S}}^{+}_{r,\ell,\theta}\Psi_{n,\ell-2,m} \propto \Psi_{n,\ell,m}, \qquad \hat{\mathcal{L}}^{+}_{\theta,m,\varphi}\Psi_{n,\ell,m-1} \propto \Psi_{n,\ell,m} \tag{3.9}$$

and the action of ladder operators:

$$\hat{\Lambda}^{\pm}_{r}\Psi_{n,\ell,m} \propto \Psi_{n\pm2,\ell,m} \tag{3.10}$$

The fundamental state is annihilated by $\hat{\Lambda}^{-}_{r}$ and $\hat{\mathcal{S}}^{-}_{r,\ell,\theta}$. This leads to two solutions (even and odd):

$$a) \quad \Psi_0^{\rm e} = \Psi_{n=0,\ell=0,m=0}\,, \qquad b) \quad \Psi_0^{\rm o} = \Psi_{n=1,\ell=1,m=0}$$

Then, in this case, there are two sublattices (even and odd) of states (ℓ, n) connected by ladder and symmetry operators. For instance, in the even sector,

$$\Psi_{2n,0,0} = \left(\hat{\Lambda}^{+}_{r}\right)^{n}\Psi_0, \qquad \Psi_{2n,2\ell,0} = \hat{\mathcal{S}}^{+}_{r,2\ell,\theta}\dots\hat{\mathcal{S}}^{+}_{r,2,\theta}\Psi_{n,0,0}, \qquad \Psi_{2n,2\ell,m} = (\hat{\mathcal{L}}^{+}_{\theta,\varphi})^{m}\Psi_{2n,2\ell,0}$$

Next, we will characterize the symmetries of KC in the same spirit.

4 Radial Symmetries: Kepler-Coulomb

In this case, the factorization applied to the reduced Hamiltonian $\hat{H}_{\ell}(r) = \hat{H}_{\ell}$

$$\hat{H}_{\ell} = -\partial_{rr} - \frac{2}{r}\partial_r + \frac{\ell(\ell+1)}{r^2} - \frac{k}{r} \tag{4.1}$$

lead us to only one pair of shift operators:

$$\hat{\Sigma}^{\pm}_{r,\ell} = \mp\partial_r + \frac{l \mp 1}{r} - \frac{k}{2\ell}$$

That modify the angular momentum, ℓ, keeping the energy, n,

$$\hat{H}_{\ell} = \hat{\Sigma}^{-}_{r,\ell+1}\hat{\Sigma}^{+}_{r,\ell+1} - \frac{k^2}{4(\ell+1)^2}, \qquad \begin{cases} \hat{\Sigma}^{+}_{r,\ell+1} = -\partial_r + \frac{\ell}{r} - \frac{k}{2(\ell+1)}, & \hat{\Sigma}^{+}_{r,\ell+1}R_n^{\ell} \propto R_n^{\ell+1} \\ \hat{\Sigma}^{-}_{r,\ell} = \partial_r + \frac{\ell+1}{r} - \frac{k}{2\ell}, & \hat{\Sigma}^{-}_{r,\ell}R_n^{\ell} \propto R_n^{\ell-1} \end{cases} \tag{4.2}$$

Then, for $\hat{H}_{\ell}$ the energy will be given by

$$E_n = -\frac{k^2}{4(n+1)^2}, \quad n \geq \ell$$

4.1 Coupling: Symmetries of Kepler-Coulomb

The factorization operators $\hat{\Sigma}^{\pm}_{r,\ell}$ modify the value of the total angular momentum from the radial states in one unit, this change is matched by the $\hat{\Lambda}^{\pm}_{\ell,\theta}$ operators,

$$\hat{\mathcal{S}}^-_{r,\ell,\theta} = \hat{\Sigma}^-_{r,\ell}\hat{\Lambda}^-_{\ell,\theta}, \qquad \hat{\mathcal{S}}^+_{r,\ell+1,\theta} = \hat{\Sigma}^+_{r,\ell+1}\hat{\Lambda}^+_{\ell+1,\theta}$$

These operators allow us to move through states of the Keppler-Coulomb system with the same energy since:

$$\hat{\mathcal{S}}^+_{r,\ell+1,\theta}\Psi_{n,\ell,m} \propto \Psi_{n,\ell+1,m}, \qquad \hat{\mathcal{S}}^-_{r,\ell,\theta}\Psi_{n,\ell,m} \propto \Psi_{n,\ell-1,m}$$

4.2 Ladder operators of Kepler-Coulomb

In order to get the ladder operators of KC system we follow Ref. [29]. Consider the eigenvalue solution of the reduced KC Schrödinger equation

$$\hat{H}_\ell\, \psi^n_\ell = E_n\, \psi^n_\ell, \quad E_n = -\frac{k^2}{4\,(n+1)^2} \tag{4.3}$$

where $n = \ell, \ell+1, \dots$. To obtain ladder operators for this potential, first we multiply the Schrödinger equation by r^2 to become

$$\hat{h}_n\psi^n_\ell = (-r^2\,\partial_r^2 - 2r\partial_r - r^2\,E_n - kr)\,\psi^n_\ell = -\ell(\ell+1)\,\psi^n_\ell \tag{4.4}$$

We propose the factorization

$$\hat{h}_n = -r^2\,\partial_r^2 - 2r\partial_r + \frac{k^2}{4(n+1)^2}r^2 - kr = \hat{\Lambda}^+_{r,n}\hat{\Lambda}^-_{r,n} + \omega_n \tag{4.5}$$

where

$$\hat{\Lambda}^+_{r,n} = (-r\,\partial_r + \tfrac{k}{2(n+1)}r - (n+1)\,\hat{D}_n^{-1}, \quad \hat{\Lambda}^-_{r,n} = \hat{D}_n(r\,\partial_r + \tfrac{k}{2(n+1)}r - n) \tag{4.6}$$

and

$$\omega_n = -n(n+1) \tag{4.7}$$

Now, we want this factorization to fulfill the fundamental property,

$$\hat{h}_n = \hat{\Lambda}^+_{r,n}\,\hat{\Lambda}^-_{r,n} + \omega_n = \hat{\Lambda}^-_{r,n+1}\,\hat{\Lambda}^+_{r,n+1} + \omega_{n+1} \tag{4.8}$$

This is true if we take $\hat{D}_n$ as a dilation operator that satisfy

$$\hat{D}_n\,r = \tfrac{n+1}{n}\,r\,\hat{D}_n, \qquad \hat{D}_n = \exp[\log\Big((n+1)/n\Big)\,r\partial_r] \tag{4.9}$$

Then, from the relation (4.8) we have the following intertwining relation

$$\hat{\Lambda}^-_{n,r}\,\hat{h}_n = \hat{h}_{n-1}\,\hat{\Lambda}^-_{n,r}\,, \quad \hat{\Lambda}^+_{n,r}\,\hat{h}_{n-1} = \hat{h}_n\,\hat{\Lambda}^+_{n,r} \tag{4.10}$$

which allow us to connect consecutive eigenfunctions by means of ladder operators,

$$\hat{\Lambda}^-_{n,r} : \psi^n_\ell \to \psi^{n-1}_\ell, \quad \hat{\Lambda}^+_{n,r} : \psi^{n-1}_\ell \to \psi^n_\ell \tag{4.11}$$

Then, the operators $\hat{\Lambda}^{\pm}_{\ell,n}$, are just the ladder operators we are looking for with the commutation rules:

$$\hat{\Lambda}^-_{r,n+1}\,\hat{\Lambda}^+_{r,n+1} - \hat{\Lambda}^+_{r,n}\,\hat{\Lambda}^-_{r,n} = 2(n+1)\,, \qquad [\hat{H}_\ell, \hat{\Lambda}^+_{r,n}] = \hat{\Lambda}^+_{r,n}(E^{n-1}_\ell - E^n_\ell) \tag{4.12}$$

In conclusion, the action of KC symmetries on the separated eigenfunctions

$$\hat{\mathcal{S}}^+_{r,\ell,\theta}\Psi_{n,\ell-1,m} \propto \Psi_{n,\ell,m}, \qquad \hat{\mathcal{L}}^+_{\theta,\varphi}\Psi_{n,\ell,m-1} \propto \Psi_{n,\ell,m} \tag{4.13}$$

together with the ladder operators

$$\hat{\Lambda}^+_{n,r}\Psi_{n-1,\ell,m} \propto \Psi_{n,\ell,m}, \qquad \hat{\Lambda}^-_{n,r}\Psi_{n,\ell,m} \propto \Psi_{n-1,\ell,m} \tag{4.14}$$

generate the eigentates from the ground state,

$$\hat{\mathcal{S}}^{\pm}_{r,0,\theta}\Psi_0 = \hat{\mathcal{L}}^{\pm}_{\theta,0,\varphi}\Psi_0 = \hat{\Lambda}^-_{0,r}\Psi_0 = 0$$

Then eigenfunction space $\Psi_{n,\ell,m}$ with energy E_n is given by:

$$\Psi_{n,0,0} = \hat{\Lambda}^+_{n,r}\dots\hat{\Lambda}^+_{1,r}\Psi_0, \qquad \Psi_{n,\ell,0} = \hat{\mathcal{S}}^+_{r,\ell,\theta}\dots\hat{\mathcal{S}}^+_{r,1,\theta}\Psi_{n,0,0}, \qquad \Psi_{n,\ell,m} = (\hat{\mathcal{L}}^+_{\theta,\varphi})^m\Psi_{n,\ell,0}$$

5 Symmetries of the Classical HO and KC Systems

Our next objective is to show that the same method based on factorizations can be applied to classical central Hamiltonians with an expression, similar to (2.1), but now in terms of canonical coordinates $(r, p_r, \theta, p_\theta, \varphi, p_\varphi)$,

$$H = p_r^2 + V(r) + \frac{1}{r^2}\left(p_\theta^2 + \frac{1}{\sin^2\theta}\left(p_\varphi^2\right)\right) \tag{5.1}$$

Its structure is the same as the quantum version (2.2) with three functions,

$$L_z = p_\varphi, \qquad L^2 = p_\theta^2 + \frac{L_z^2}{\sin^2\theta}, \qquad H = p_r^2 + V(r) + \frac{L^2}{r^2} \tag{5.2}$$

where H is the Hamiltonian, L^2 the total momentum, and L_z^2, the square of the angular z-component. These functions constitute a sequence of three involutive constants of motion, i.e. their Poisson brackets (PB), between any two of them, including H, vanishes. Their constant values are denoted by E, ℓ^2 and m^2, respectively. These constants of motion lead to a kind of reduced one dimensional Hamiltonians in one of the variables φ, θ, r when the rest of variables are replaced by the value of a previous constant of motion:

$$\begin{aligned}
&(a) \qquad L_z^2(\varphi, p_\varphi) := p_\varphi^2 = m^2 \\
&(b) \qquad L_m^2(\theta, p_\theta) := p_\theta^2 + \tfrac{m^2}{\sin^2\theta} \;= \ell^2 \\
&(c) \qquad H_\ell(r, p_r) := p_r^2 + V(r) + \tfrac{\ell^2}{r^2} = E
\end{aligned} \tag{5.3}$$

We start with these involutive constants of motion L_z^2, L^2 and H; then, we will construct additional constants of motion to reach the maximum number of five, corresponding to maximal superintegrable systems [18] as it is the case of HO and KC. We will search these extra constants in the form of two complex pairs: i) the first complex pair, $\mathcal{L}^\pm_{\theta,\varphi}$, will depend on the variables $\theta, p_\theta; \varphi, p_\varphi$. Its PB with L^2 will vanish and consequently its PB with H will also vanish (since H depends on angular variables through L^2); and ii) the second pair $\mathcal{S}^\pm_{r,\theta}$ depending on $r, p_r; \theta, p_\theta$, which will commute with H (and, trivially, with L_z^2, because it is independent of φ and p_φ); thus, the same ordering scheme (2.5) is valid here:

$$H\,, \quad \mathcal{S}^\pm_{r,\theta}\,, \quad L^2\,, \quad \mathcal{L}^\pm_{\theta,\varphi}\,, \quad L_z^2 \tag{5.4}$$

Of course, not all these constants will be independent; we will see how to choose five of them. We will arrive to these symmetries through the factorization of the reduced Hamiltonians (5.3), in terms of of ladder and shift functions. The question we have to clarify is the definition of "ladder and shift functions" in the classical frame. We will introduce them along the examples of HO and KC systems.

5.1 Angular symmetry functions $\mathcal{L}^\pm_{\theta,\varphi}$

Consider the classical version to the angular symmetries (5.7) of L^2,

$$\mathcal{L}^\pm_{\theta,\varphi} = e^{\pm i\varphi}(\pm i p_\theta - \frac{L_z}{\tan\theta}) \tag{5.5}$$

Similarly to the quantum case, these functions split into φ and θ components.

(i) The φ-factor: **angular ladder functions** $\Lambda^\pm_\varphi$

The first component corresponds to a pair of ladder functions for the constant of motion L_z:

$$\Lambda^\pm_\varphi := e^{\pm i\varphi}, \quad \{L_z, \Lambda^\pm_\varphi\} = \mp i\,\Lambda^\pm_\varphi \tag{5.6}$$

In the classical frame we say that a pair of functions (like $\Lambda^\pm_\varphi$) is a set of ladder functions for a constant of motion (like L_z) if a PB like (5.6) is satisfied (which is the classical analog of the commutator (2.10)).

(ii) The θ-factor: **angular shift functions** $\Sigma^{\pm}_{\theta,m}$

Let $L^2_m(\theta)$ be the reduced Hamiltonian given in (5.3), then we define the following functions

$$\Sigma^{\pm}_{\theta,m} := \pm i p_\theta - \frac{m}{\tan\theta} \tag{5.7}$$

These functions are called shift-functions of L^2_m since they are characterized by a factorization relation,

$$\Sigma^{+}_{\theta,m}\Sigma^{-}_{\theta,m} = L^2_m(\theta) - m^2 \tag{5.8}$$

which means that the constant of motion $L^2_m(\theta)$ must have a value ℓ^2 greater or equal than the constant m^2. The following type of PB, is similar to the commutation (2.18)

$$\{L^2, \Sigma^{\pm}_{\theta,m}\}_{p_\varphi\to m} = \pm i \frac{2m}{\sin^2\theta}\,\Sigma^{\pm}_{\theta,m} = \pm i \frac{\partial L^2_m}{\partial\, m}\,\Sigma^{\pm}_{\theta,m} \tag{5.9}$$

where the condition $p_\varphi \to m$ in the PB means that after taking the PB the replacement of the constant of motion p_φ by its value m sould be made.

The PB with L^2 will also be useful later,

$$\{L^2\,, \Lambda^{\pm}_{\varphi}\}_{p_\varphi\to m} = \mp i\, \frac{2m}{\sin^2\theta}\Lambda^{\pm}_{\varphi} \tag{5.10}$$

(iii) Then, the **angular constant of motion** $\mathcal{L}^{\pm}_{\theta\varphi}$ of (5.5) can be expressed as

$$\mathcal{L}^{\pm}_{\theta,m,\varphi} = \Sigma^{\pm}_{\theta,m}\Lambda^{\pm}_{\varphi} \tag{5.11}$$

when the classical system is characterized by $L_z = m$. In other words, it is the product of a ladder function of L_z^2 and a shift function of L^2_m. From (5.10) and (5.9) we check that $\mathcal{L}^{\pm}_{\theta,m,\varphi}$ will commute with L^2 and H, but they will act as ladder functions of L_z, due to (5.6),

$$\{L^2, \mathcal{L}^{\pm}_{\theta,m,\varphi}\}_{p_\varphi\to m} = \{H, \mathcal{L}^{\pm}_{\theta,m,\varphi}\}_{p_\phi\to m} = 0\,, \qquad \{L_z, \mathcal{L}^{\pm}_{\theta,m,\varphi}\}_{p_\varphi\to m} = \mp i \mathcal{L}^{\pm}_{\theta,m,\varphi}$$

We can say that the complex functions $\mathcal{L}^{\pm}_{\theta,m,\varphi}$ constitute a pair of complex symmetries for L^2 (and for H) when the value of the constant of motion L_z is m. These symmetries are complex conjugate, $\mathcal{L}^{-}_{\theta,m,\varphi} = (\mathcal{L}^{+}_{\theta,m,\varphi})^*$.

Let us remark that in this case, the constant of motion m is the value of p_φ, which is a polynomial function, therefore m can be replaced by p_φ simplifying the notation and PBs:

$$\Sigma^{\pm}_{\theta,m} \to \Sigma^{\pm}_{\theta,\varphi} := \pm i p_\theta - \frac{p_\varphi}{\tan\theta}, \qquad \mathcal{L}^{\pm}_{\theta,m,\varphi} \to \mathcal{L}^{\pm}_{\theta,\varphi} := \Sigma^{\pm}_{\theta,\varphi}\Lambda^{\pm}_{\varphi}$$

(iv) The **ladder functions of the total angular momentum** L^2 are obtained by another factorization of L^2_m (following the way shown in Sect-2.3) in the form,

$$\Lambda^{\pm}_{\ell,\theta} = \pm i \sin\theta p_\theta + \ell\cos\theta\,, \quad \Lambda^{+}_{\ell,\theta}\Lambda^{-}_{\ell,\theta} = L^2_m - m^2 = \ell^2 - m^2 \tag{5.12}$$

where ℓ^2 is the value of the constant of motion L^2. An important remark is that similarly to the quantum case the ladder character can be better expressed in terms of the square root $\sqrt{L^2}$:

$$\{L^2, \Lambda^{\pm}_{\ell,\theta}\}_{L^2\to\ell^2} = \mp i 2\ell\Lambda^{\pm}_{\ell,\theta}\,, \qquad \{\sqrt{L^2}, \Lambda^{\pm}_{\ell,\theta}\}_{L^2\to\ell^2} = \mp i \Lambda^{\pm}_{\ell,\theta} \tag{5.13}$$

5.2 Radial symmetries and ladder functions of the HO

Next, we will obtain the radial shift functions associated to the HO system H_ℓ in order to find the second pair of complex symmetries $\mathcal{S}^{\pm}_{r,\theta}$ in the variables r and θ. It will also be useful to find the ladder functions of H_ℓ, since they will allow us to find the motion, the frequency and a pair of action-angle variables.

(i) **Basic factorization functions** $a^{\pm}_\ell, b^{\pm}_\ell$. We will deal with the reduced HO Hamiltonian (where we have set $2m = 1$)

$$H_\ell(r) = p_r^2 + \frac{\ell^2}{r^2} + \frac{\omega^2}{4}\, r^2 \tag{5.14}$$

Then we look for the factorizations of this reduced Hamiltonian. There are two independent factorization sets of radial functions $\{a^{\pm}_\ell, b^{\pm}_\ell\}$,

$$\begin{aligned} H_{r,\ell} &= a^{+}_\ell a^{-}_\ell - \omega\, \ell\,, \qquad & a^{\pm}_\ell &= \mp i p_r + \tfrac{\ell}{r} + \tfrac{\omega}{2}\, r \\ H_{r,\ell} &= b^{-}_\ell b^{+}_\ell + \omega\, \ell\,, \qquad & b^{\pm}_\ell &= \mp i p_r + \tfrac{\ell}{r} - \tfrac{\omega}{2}\, r \end{aligned} \tag{5.15}$$

(ii) **Shift and ladder functions**. Next, from the previous auxiliary functions $\{a^{\pm}_\ell, b^{\pm}_\ell\}$ found in (i) we obtain **pure shift** functions, $\Sigma^{\pm}_{r,\ell}$, in a similar way to (3.7)

$$\Sigma^{\pm}_{r,\ell} = b^{\pm}_\ell a^{\pm}_\ell \tag{5.16}$$

The shift functions have the PBs of shift style (similar to the quantum version (3.8)):

$$\{H, \Sigma^{\pm}_{r,\ell}\}_{L^2\to\ell^2} = \pm i \frac{4\ell}{r^2}\, \Sigma^{\pm}_{r,\ell} = \pm 2i \frac{\partial H_\ell}{\partial \ell}\, \Sigma^{\pm}_{r,\ell}$$

They also allows to get **pure ladder** $\hat{\Lambda}^{\pm}_{\ell,r}$ radial functions, in the following way

$$\Lambda^{\pm}_{\ell,r} = a^{\pm}_\ell b^{\mp}_\ell\,, \tag{5.17}$$

The radial ladder functions have the following ladder-like PB:

$$\{H, \Lambda^{\pm}_{\ell,r}\}_{L^2\to\ell^2} = \mp 2i\omega\, \Lambda^{\pm}_{\ell,r} \tag{5.18}$$

This kind of ladder functions $\Lambda^{\pm}_{\ell,r}$ depend only on ℓ^2 which coincides with the eigenvalues of L^2. Thus, we can make this replacement in $\Lambda^{\pm}_{\ell,r}$; in this case we use the notation $\Lambda^{\pm}_r$. Although $\Lambda^{\pm}_r$ are not symmetries, their PBs (5.18) are quite important. This will provide the frequency of the system, and one pair of appropriate canonical variables, as we will see later.

(iii) **Symmetry functions**. The product of the second order radial shift, $\Sigma^{\pm}_r$, and angular ladder, $(\Lambda^{\pm}_\theta)^2$, functions, defined in (5.12), supply a second pair of radial symmetries:

$$\mathcal{S}^{\pm}_{r,\ell,\theta} = \Sigma^{\pm}_{r,\ell}(\Lambda^{\pm}_{\ell,\theta})^2 = b^{\pm}_\ell a^{\pm}_\ell \Big(\pm i \sin\theta\, p_\theta + \ell\cos\theta \Big)^2 \tag{5.19}$$

$$\{H, \mathcal{S}^{\pm}_{r,\ell,\theta}\}_{L^2\to\ell^2} = 0$$

Due to (5.13), these symmetries have the following PB with $\sqrt{L^2}$,

$$\{\sqrt{L^2}, \mathcal{S}^{\pm}_{r,\ell,\theta}\}_{L^2\to\ell^2} = \mp 2i \mathcal{S}^{\pm}_{r,\ell,\theta} \tag{5.20}$$

(iv) **Polynomial symmetries** The symmetries that we are calculating, for instance $\mathcal{S}^{\pm}_{r,\ell,\theta}$, have two drawbacks: a) They are not polynomial in the momenta (due to the square root $\ell = \sqrt{L^2}$); and b) they are complex. These two difficulties are solved by taking the linear symmetric and antisymmetric (and dividing by i) combination of the pair of symmetries (or, essentially, taking the real and the imaginary part of any of them):

$$\mathcal{S}^{s}_{r,\theta} = \frac{1}{2}\left(\mathcal{S}^{+}_{r,\ell,\theta} + \mathcal{S}^{-}_{r,\ell,\theta}\right), \qquad \mathcal{S}^{a}_{r,\theta} = \frac{1}{2i\ell}\left(\mathcal{S}^{+}_{r,\ell,\theta} - \mathcal{S}^{-}_{r,\ell,\theta}\right) \tag{5.21}$$

They the following form:

$$\mathcal{S}^{a}_{r,\theta} = (8m^2 p_r r + 8\ell^2 p_r p_\theta(-4\ell^2 + 4p^2 r^2 + \omega^2 r^4)\sin 2\theta))/(4r^2)$$

$$\mathcal{S}^{s}_{r,\theta} = ((2\ell^2 - Er^2)(m^2 + \ell^2\cos 2\theta) + 2\ell^2 p_r p_\theta r \sin 2\theta)/r^2$$

They are fourth ($\mathcal{S}^{s}_{r,\theta}$) and third ($\mathcal{S}^{a}_{r,\theta}$) degree polynomials in the momenta p_r, p_θ. They can also be expressed in terms of the Q_{ij} components of the Fradkin-Demkov tensor [14, 15]:

$$Q_{ij} = p_i p_j + \frac{\omega^2}{4} x_i x_j$$

Leading to the expressions:

$$\begin{aligned}\mathcal{S}^{s}_{r,\theta} &= (L^2 - L_z^2)H - 2L^2 Q_{zz}\\ \mathcal{S}^{a}_{r,\theta} &= 2(L_x Q_{yz} - L_y Q_{zx})\end{aligned} \tag{5.22}$$

5.3 Radial symmetries of the classical KC system

Next, we will write the classical functions relevant to form the specific symmetries $\mathcal{S}^{\pm}_{r,\ell,\theta}$ of the KC system.

(i) **Shift functions**. In this case, **the factorization** applied to the reduced Hamiltonian $H_\ell(r)$

$$H_\ell = p_r^2 + \frac{\ell^2}{r^2} - \frac{k}{r} \tag{5.23}$$

lead us to one pair of shift functions:

$$\Sigma^{\pm}_{r,\ell} = \mp i\, p_r + \frac{\ell}{r} - \frac{k}{2\ell}\,, \qquad \Sigma^{+}_{r,\ell}\,\Sigma^{-}_{r,\ell} = H + \frac{k^2}{4\ell^2} \tag{5.24}$$

They have shift-like PBs:

$$\{H, \Sigma^{\pm}_{r,\ell}\} = \pm i\,\frac{2\ell}{r^2}\Sigma^{\pm}_{r,\ell} = \pm i\,\frac{\partial H_\ell}{\partial \ell}\,\Sigma^{\pm}_{r} \tag{5.25}$$

(ii) **Symmetries**. Similarly to the quantum case,

$$\mathcal{S}^{\pm}_{r,\ell,\theta} = \Sigma^{\pm}_{r,\ell}\Lambda^{\pm}_{\ell,\theta}, \qquad \mathcal{S}^{\pm}_{r,\ell,\theta} = (\mp i p_r + \frac{\ell}{r} - \frac{k}{2\ell})\Big(\pm i\sin\theta p_\theta + \ell\cos\theta\Big)$$

(iii) **Ladder functions**, $\Lambda^{\pm}_{r}$, of the classical KC system. In the same way as we did for the quantum ladder operators, we obtain the ladder functions,

$$\Lambda^{\pm}_{r} = (\mp r\, i p_r + r\sqrt{-H} - \tfrac{k}{2\sqrt{-H}})\, D^{\mp}_{H}, \tag{5.26}$$

where

$$D^{\mp}_{H} = \exp[\mp 2 i r p_r \sqrt{-H}/k]$$

The functions $\Lambda^{\pm}_{r}$, are just the ladder functions we are looking for:

$$\{\Lambda^{+}_{r}, \Lambda^{-}_{r}\} = \frac{i\,k}{\sqrt{-H}}\,, \qquad \{H, \Lambda^{\pm}_{r}\} = \mp i\,4H\sqrt{-H}\,\Lambda^{\pm}_{r} \equiv \mp i\,\alpha(H)\Lambda^{\pm}_{r}, \tag{5.27}$$

where

$$\alpha(H) = 4(-H)\sqrt{-H}/k$$

In the same way as we mentioned in the ladder function of the HO, this function supply us with the frequency of the motion, $\alpha(H)$ as we will see later. Such frequency depends on the energy, as opposed to the HO, where it is constant.

(iv) **Polynomial symmetries** In principle the symmetries $\mathcal{S}^{\pm}_{r,\ell,\theta}$ are not polynomial in the momenta p_r, p_φ, due to the presence of $\ell = \sqrt{L^2}$, and they are complex. We can obtain real polynomial symmetries in the momentum functions by means of the symmetric and antisymmetric combination of $\mathcal{S}^{+}_{r,\theta}$ and $\mathcal{S}^{-}_{r,\theta}$:

$$\mathcal{S}^{s}_{r,\theta} = \frac{1}{2}\left(\mathcal{S}^{+}_{r,\theta} + \mathcal{S}^{-}_{r,\theta}\right), \qquad \mathcal{S}^{a}_{r,\theta} = \frac{\ell}{2i}\left(\mathcal{S}^{+}_{r,\theta} - \mathcal{S}^{-}_{r,\theta}\right) \tag{5.28}$$

$$\mathcal{S}^{s}_{r,\theta} = \cos(\theta)\left(\frac{\ell^2}{r} - \frac{k}{2}\right) + p_\theta p_r \sin(\theta)\,, \qquad \mathcal{S}^{a}_{r,\theta} = \sin(\theta)\left(\frac{\ell p_\theta}{r} - \frac{k p_\theta}{2\ell}\right) - \ell p_r \cos(\theta)$$

They are polynomials in the momenta p_r, p_φ of second and third degrees , respectively. Furthermore they are related to the well known Runge-Lenz $\mathbf{A}$ and angular momentum $\mathbf{L}$ vectors:

$$\mathcal{S}^{s}_{r,\theta} = A_z, \qquad \mathcal{S}^{a}_{r,\theta} = (L_x A_y - L_y A_x) = \mathbf{L}\times\mathbf{A}|_z \tag{5.29}$$

Where the Runge-Lenz vector is:

$$\mathbf{A} = \mathbf{p}\times\mathbf{L} - \frac{k}{2}\,\mathbf{r}$$

6 Action-Angle Variables and Symmetries

We collect in Table 1 some of the basic factorization functions (ladder, shift, or symmetries) and split them into their modulus and phase. We will make use of them in order to compute canonical variables that correspond to the solutions of the characteristic function of Hamilton-Jacobi and to action-angle variables. In this way, we want to stress that such canonical variables take a natural part in the symmetries (obtained by this method) of superintegrable systems, as we will show below by means of HO and KC systems.

Function	$\mathcal{L}^{\pm}_{\theta,m,\varphi}$	$\Lambda^{\pm}_{\ell,\theta}$	$a^{\pm}_{\ell}$ (HO)	$b^{\pm}_{\ell}$ (HO)	$\Sigma^{\pm}_{r,\ell}$ (KC)
Expression	$e^{\pm i\,\varphi}(\mp i\,p_\theta - \frac{m}{\tan\theta})$	$\pm i p_\theta + \ell\cos\theta$	$\mp i p_r + \frac{\ell}{r} + \frac{1}{2}\omega r$	$\mp i p_r + \ell/r - \frac{1}{2}\omega r$	$\mp i p_r r + \frac{\ell}{r} - \frac{k}{2\ell}$
$\lvert\cdot\rvert^2$	$\ell^2 - m^2$	$\ell^2 - m^2$	$E + \omega\ell$	$E - \omega\ell$	$E + \frac{k^2}{4\ell^2}$
Phase	$\varphi - \arccos\frac{m\cot\theta}{\sqrt{\ell^2-m^2}}$	$\arccos\frac{\ell}{\cos\theta}\sqrt{\ell^2-m^2}$	$\arccos\frac{2\ell+\omega r^2}{2r\sqrt{E+\omega\ell}}$	$\arccos\frac{2\ell-\omega r^2}{2r\sqrt{E-\omega\ell}}$	$\arccos\frac{\ell-kr}{r\sqrt{4\ell^2E+k^2}}$

Table 1: Some functions together with their modulus and phase, where $L^2 = \ell^2$, $L_z^2 = m^2$.

In this section, we want to interpret the constants of motion that we have previously obtained for the classical KC and HO systems. We have seen that, essentially, the complex constants of motion $\mathcal{S}^{\pm}_k$, which are related to each pair, H_k and H_{k+1}, of reduced Hamiltonians (in our case, $H_1 \equiv H_\varphi$, $H_2 \equiv H_\theta$, $H_3 \equiv H_r$), have the structure

$$\mathcal{S}^{\pm}_k = \Sigma^{\pm}_{k+1}\,\Lambda^{\pm}_k$$

where $\Sigma^{\pm}_{k+1}$ are shift functions of H_{k+1}, while $\Lambda^{\pm}_k$ are ladder functions of H_k. In polar coordinates we write

$$\mathcal{S}^{\pm}_k = |\mathcal{S}^{\pm}_k| e^{\pm i\,\xi_k} = |\Sigma^{\pm}_{k+1}| e^{\pm i\,\sigma_{k+1}}\,|\Lambda^{\pm}_k| e^{\pm\,i\lambda_k}$$

The modulus of the symmetry $|\mathcal{S}^{\pm}_k| = |\Sigma^{\pm}_{k+1}||\Lambda^{\pm}_k|$ depends on the involutive constants of motion, i.e., of the partial Hamiltonians, so it is not an independent symmetry; but the phase ξ_k is a symmetry which is independent of the reduced Hamiltonians. If we consider the pair

$$(H_k\,, \xi_k = \sigma_{k+1} + \lambda_k)$$

It happens that $\mathcal{S}^{\pm}_k$ is a symmetry of H_{k+1} but its PB with H_k has the form (see for instance (5.20),

$$\{H_k, \mathcal{S}^{\pm}_k\} = \mp i a \mathcal{S}^{\pm}_k \tag{6.1}$$

(where $a = 1$ for KC and $a = 2$ for HO) and, taking into account that $\mathcal{S}_k^\pm = |\mathcal{S}_k^\pm| e^{\pm i\, \xi_k}$ with $\{H_k, |\mathcal{S}_k^\pm|\} = 0$, then

$$\{\xi_k, H_k\} = a \tag{6.2}$$

Thus, based on these considerations, we expect that the pairs

$$(H_\varphi, \xi_\varphi),\;\; (H_\theta, \xi_\theta) \tag{6.3}$$

are a good choice as starting point to search for two pairs of canonical variables which are made of constants of motion. They can be modified slightly to get of a set of action-angle canonical coordinates. In fact the ξ_k variables will be identified with the angle, while partial Hamiltonian H_k will be a function of the action variable J_k. The remaining pair (J_r, ξ_r) comes from the reduced Hamiltonian H_r and its ladder functions $\Lambda_r^\pm$. Next, we will see the details for KC and HO systems.

6.1 Action-angle variables for the KC system

(1) From the pair $(L_z \equiv p_\varphi, \xi_\varphi)$ to the canonical variables $(J_\varphi \equiv p_\varphi, \xi_\varphi)$.

The constant of motion ξ_φ according to (5.11) and (5.6)-(5.7) is defined by

$$\xi_\varphi = \lambda_\varphi + \sigma_\theta = \varphi + \arccos\left(\frac{L_z}{\sqrt{L^2 - L_z^2}} \cot\theta\right) = \varphi + \arccos(\cot i \cot\theta) \tag{6.4}$$

where the angle i is that of the plane of the motion and the z axis. The canonical bracket $\{\xi_\varphi, p_\varphi\} = 1$ is fulfilled. Then, in this case, trivially, p_φ and J_φ coincide, and the φ canonical variables are

$$(J_\varphi := p_\varphi, \xi_\varphi) \tag{6.5}$$

(2) From the pair $(\sqrt{L^2}, \xi_\theta)$ to canonical variables $(J_\theta \equiv \sqrt{L^2}, \xi_\theta)$.

According to (5.24) and (5.25), we have

$$\xi_\theta = \lambda_\theta + \sigma_r = \arccos(\csc i\; \cos\theta) + \arccos\frac{2L^2/r - k}{\sqrt{4L^2H + k^2}}\,, \qquad \frac{\sqrt{L^2}}{\sqrt{L^2 - L_z^2}} = \csc i \tag{6.6}$$

From (5.13), we have the canonical bracket $\{\xi_\theta, \sqrt{L^2}\} = 1$. In conclusion, $J_\theta(L^2) = \ell - J_0$, such that for the minimum value of L^2, $J_\theta(m^2) = 0$, then the action is $J_\theta(L^2) = \ell - m$, where m is the action of the variable φ. Finally, the second pair of canonical variables is

$$(J_\theta = \ell - m,\; \xi_\theta)$$

The PBs of cross variables of both set of pairs will vanish.

(3) The pair $(H_{\rm KC}, \xi_r)$ and the canonical variables $(J_r(H_{\rm KC}), \xi_r)$.

In this case, we dispose only of a pair of ladder functions to build the last pair of canonical variables $(J_r(H_{\rm KC}), \xi_r)$. According to (5.26) they are given by

$$\Lambda_r^\pm = (\mp i\, r\, p_r + r\sqrt{-H_{\rm KC}} - \tfrac{k}{2\sqrt{-H_{\rm KC}}})\, \exp[\mp 2 i r p_r \sqrt{-H_{\rm KC}}/k]$$

therefore

$$\Lambda_r^\pm = |\Lambda_r^\pm| \exp[\pm i \xi_r], \quad |\Lambda_r^\pm| = \sqrt{\frac{k^2}{-4H_{\rm KC}} - L^2}, \quad \xi_r = 2 r p_r \sqrt{-H_{\rm KC}}/k + \arccos\frac{-k - H_{\rm KC} r}{\sqrt{k^2 - 4H_{\rm KC}L^2}}$$

Recall the commutator

$$\{H_{\rm KC}, \Lambda_r^{\pm}\} = \mp i\, 4(-H_{\rm KC})\sqrt{-H_{\rm KC}}\, \Lambda_r^{\pm} = \mp i\alpha(H_{\rm KC})\, \Lambda_r^{\pm}$$

then, since $\{H_{\rm KC}, e^{a\xi}\} = ae^{a\xi}\, \{H_{\rm KC}, \xi\}$, we get

$$\{\xi_r, H_{\rm KC}\} = \alpha(H_{\rm KC}) = -4H_{\rm KC}\sqrt{-H_{\rm KC}}/k \tag{6.7}$$

This PB leads to the canonical variable J_r:

$$\{\xi_r, J_r(H_{\rm KC})\} = 1 \implies \frac{d\,J_r(H_{\rm KC})}{d\,H_{\rm KC}}\,(-4H_{\rm KC}\sqrt{-H_{\rm KC}}/k) = 1 \implies J_r = \frac{k}{2\sqrt{-H_{\rm KC}}} - \ell\,,$$

where ℓ is the constant of integration associated to the value $J = 0$ at the minimum value of the potential well.

In summary, the symmetries allow us to find the canonical variables

$$(J_\varphi = p_\varphi, \xi_\varphi;\ J_\theta = \ell - m, \xi_\theta;\ J_r = \frac{1}{2\sqrt{-E}} - \ell, \xi_r) \tag{6.8}$$

such that all of them are constants of motion except ξ_r, whose evolution is given by (6.7),

$$\frac{d}{dt}\xi_r = \{\xi_r, H_{\rm KC}\} = 4(-H_{\rm KC})\sqrt{-H_{\rm KC}}/k$$

Thus $\alpha(H)$ is the frequency of the system which is equal to $\frac{dH(J)}{dJ}$. These results coincide with those given in [35, 37]. Notice that sometimes the dynamical variables take a different convention:

$$(J, \xi) \longrightarrow (J' = 2\pi\, J\,, \xi' = \xi/(2\pi))$$

6.2 Action-angle variables for the HO system

(1) The pair (p_ϕ, ξ_ϕ).

This is the same problem as before, for the KC potential, because the angular part is common to all central systems. Therefore, the first pair of canonical variables is the same,

$$(J_\varphi = p_\varphi\ ,\ \xi_\varphi = \varphi + \arccos(\cot i \cot\theta)\ ) \tag{6.9}$$

(2) The pair $(\sqrt{L^2}, \xi_\theta)$.

From the expression of the functions $a_\ell^{\pm}, b_\ell^{\pm}$ we get the explicit form of the shift function

$$\Sigma_r^{\pm} = a_\ell^{\pm} b_\ell^{\mp} = \Big(\mp i p_r + \frac{\sqrt{L^2}}{r} + \frac{\omega}{2} r\Big)\Big(\mp i p_r + \frac{\sqrt{L^2}}{r} - \frac{\omega}{2} r\Big) = \mp 2i\sqrt{L^2}\, p_r + (-H + 2L^2/r^2)$$

Therefore, λ_θ has the same form as in KC, while

$$\sigma_r = \arccos\frac{-H + 2L^2/r^2}{\sqrt{H^2 - \omega^2 L^2}}$$

In this case the contribution of λ_θ is two times that of KC, since the ladder function entering in the symmetry is $(\Lambda_{\ell,\theta}^{\pm})^2$:

$$\lambda_\theta = 2\arccos(\csc i\, \cos\theta)$$

We have for the HO,

$$\{\sqrt{L^2}, \mathcal{S}_{r,\ell,\theta}^{\pm}\} = \mp 2i\mathcal{S}_{r,\ell,\theta}^{\pm}$$

which implies

$$\{\sqrt{L^2}/2, \xi_\theta\} = 1$$

Therefore, the action variable will be $J_\theta = \sqrt{L^2}/2 - J_0$ where J_0 is a constant. Such a constant is determined from the fact that the action $J_\theta(L^2)$ must be zero for the minimum value of L^2. This minimum value is $L^2 = m^2$, (see the expression (5.3)), for $\theta = \pi/2$. Then, the action variable with $J_0 = m/2$ has the form:

$$J_\theta = \sqrt{L^2}/2 - m/2 \tag{6.10}$$

In this case, the angle variable is

$$\xi_\theta = 2\lambda_\theta + \sigma_r = 2\arccos(\sin i\ \cos\theta) + \arccos\frac{-H + 2L^2/r^2}{\sqrt{H^2 - \omega^2 L^2}} \tag{6.11}$$

This completes the second canonical pair (J_θ, ξ_θ).

(3) The pair (H, ξ_r).

In this case, we dispose only of a pair of ladder functions in order to build the last couple of canonical variables $(J_r(H), \xi_r)$. From the explicit expression

$$\Lambda_r^\pm = a^\pm b^\pm = \Big(\mp ip_r + \frac{\ell}{r} + \frac{\omega}{2}r\Big)\Big(\pm ip_r + \frac{\ell}{r} - \frac{\omega}{2}r\Big) = \mp i\,\frac{\omega}{2}\,p_r + (H - \frac{\omega^2}{2}r^2)$$

we get

$$\xi_r = \lambda_r = \arccos\frac{H - \frac{\omega^2}{2}r^2}{\sqrt{H^2 - \omega^2\ell^2}} \tag{6.12}$$

The PB in this case is

$$\{H_r, \xi_r\} = 2\omega \tag{6.13}$$

and the action has the form:

$$J_r(H_r) = \frac{H_r}{2\omega} - J_0, \qquad \{J_r, \xi_r\} = 1$$

where $J_0 = \ell/2$.

In conclusion, we have obtained via the symmetries, a set of canonical variables

$$(p_\varphi, \xi_\varphi;\ \ell - m, \xi_\theta;\ \frac{E}{2\omega} - \frac{\ell}{2}, \xi_r) \tag{6.14}$$

such that all of them are constants of motion except ξ_r, whose evolution is given by (6.13),

$$\frac{d}{dt}\xi_r = 2\omega$$

where the frequency 2ω in this case is constant and it comes from the ladder function of the KC problem.

The expressions of these action-angle for HO and KC systems agree with those given in [35], after making a correspondence: $(r' = r/\sqrt{2}, p'_r = p_r\sqrt{2}, \ell'^2(\equiv \mathcal{H}) = \ell^2/2, k'(\equiv \gamma) = k/\sqrt{2})$, where primed variables (and $\gamma, \mathcal{H}$) are from that reference.

7 Conclusions and remarks

Along this paper we have introduced in full detail a method based on factorizations to find the symmetries of quantum and classical superintegrable systems. We have chosen the KC and HO systems to prove the consistency and compare this approach with respect to other more conventional treatments. In the folllowing, we will point out some of the key points that are specific of this method. Remark that although we have worked out separable systems in spherical coordinates in this paper, it can also be implemented in other types of coordinates. The symmetries obtained are non Hermitian nor polynomial in the momenta (the polynomial character is generally assumed). We have shown how polynomial symmetries can also be obtained in a simple way within the factorization formalism, allowing us to express our symmetries in terms of the standard ones found in the literature (see (5.22) for HO or (5.29) for KC). Let us mention these above mentioned points in the quantum and classical contests.

- Quantum properties.
  The symmetries here obtained, in the quantum formalism, allow us to find the eigenfunctions of the degeneracy eigenspaces. Each symmetry operator is made of two types of elementary operators: (i) ladder operators (we use the notation $\hat{\Lambda}^{\pm}_{\ell,\theta}$) of each of the involutive symmetries which connect different eigenvalue levels (in this example the eigenvalue ℓ of $\hat{L}^2$, see for example (2.24)). In particular, the ladder operators of the Hamiltonian allows to jump among energy levels and they lead to spectrum generating algebras [33]. (ii) Shift operators (with the notation $\hat{\Sigma}^{\pm}_{r,\ell}$) which change a parameter of a partial Hamiltonian (in this case the parameter ℓ in $\hat{H}_\ell(r)$). These shift operators relate (partial) Hamiltonians with different potentials, and in the factorization style are called intertwining operators.

  The symmetries obtained in this formalism have non Hermitian form and they include square roots of differential operators (for instance ℓ, where $\ell(\ell+1) = L^2$), but polynomial symmetries in the momenta, are easily recovered. The symmetry algebra can also be determined by using this type of symmetry operators [36].

- Classical properties.
  We have shown how the factorizations can be defined in classical systems by means of PBs, and in particular the functions called shift and ladder (in correspondence with shift and ladder operators known from usual factorizations of quantum systems). Such functions do not have the same interpretation as in the quantum case, but they are most useful. For instance, while ladder operators can be applied in the building of coherent states [33, 34] or to span spectrum generating algebras [29, 30], the classical ladder functions give us algebraically the frequency of the motion, or allows to complete the action-angle variables [29], and also can be used to define a classical spectrum generating algebra.

- Connection between classical and quantum systems.
  The factorization method, here presented, to find the classical symmetries is completely new (developed along a list of references), and follows closely the steps of quantum systems. Some of the most interesting properties are that the symmetries obtained are complex functions, whose phase can be identified in terms of action-angle variables (see (6.8) and (6.14)). This is an example of the connection of symmetries with Hamilton-Jacobi formalism which was not studied yet in the literature. The orbits (open and closed) are directly obtained from this kind of symmetries, as well as the motion and frequency (corresponding to closed orbits) by means the ladder functions of the Hamiltonian.

  One of the main advantages of this method is that it shows the close connection of classical and quantum system analysis in a systematic way. We can follow at each step the quantum and classical expressions and interpret their form. In fact, coherent states is one application where our quantum and classical symmetries can be applied in a natural way to establish a bridge for quantum and classical formalisms [33, 34].

We have worked out the HO and KC systems in deep detail to make a clear exposition of the basic ingredients and to easily appreciate the differences that one may find with respect to the existing vast literature [35, 37]. Our aim in the near future is to showcase the versatility of this method considering its application to other models in quite different contexts such as non-central potential systems that maintain the nest structure [7, 9, 39, 40, 41], parabolic systems [38], systems with spin [42], non Euclidean systems [43], applications from exceptional orthogonal polynomials [44, 45] as well as a dimensional generalization [46].

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