SUSY hierarchies of Jaynes-Cummings Hamiltonians with different detuning parameters

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Abstract

The aim of this work is to show how supersymmetric (SUSY) quantum mechanics can be applied to the Jaynes-Cummings (JC) Hamiltonian of quantum optics. These SUSY transformations connect pairs of Jaynes-Cummings Hamiltonians characterized by different detuning parameters as well as Jaynes-Cummings to anti-Jaynes-Cummings Hamiltonians. Therefore, JC Hamiltonians can be classified in hierarchies or sequences which are connected through SUSY transformations. As a byproduct, the symmetries of JC Hamiltonians are found as well as the special case of a sequence of resonant-like interacting systems having the form of a simple shape invariant JC Hamiltonian hierarchy.

1 Introduction

The Jaynes-Cummings Hamiltonian is a simple theoretical model that describes the quantum interaction of a two-level atom and radiation (electromagnetic field quanta) [1]. It has given rise to a huge number of applications to multiple configurations of the interaction of matter and radiation [2, 3]. The JC Hamiltonian is an approximation in the Rotating Wave Approximation (RWA) of the quantum Rabi model of quantum optics. This approximation is good for low energy states and when the detuning (difference of atomic and radiation frequencies) is small with respect to the coupling radiation-atom parameter. In some coupling regimes the counter-rotating terms must be considered (for more details see Refs. [4, 5] for spectral classification of coupling regimes in the quantum Rabi model and [6, 7] for ultrastrong coupling regimes).

In this work we will consider the JC Hamiltonian corresponding to the Rabi quantum model in the RWA without any limitations on the parameters. This Hamiltonian may also be useful in condensed matter and solid-state physics. Thus, when the JC Hamiltonian is applied in quantum optics or in any other problem one should be aware of the corresponding restrictions on the parameters and energy

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values. Along this work we will deal with properties of the JC model which do not depend on these restrictions.

The wide interest of the JC model is based on the simplicity, as well as to its complete solvability in the RWA, when the frequency of the applied electromagnetic field is close to the atomic transition frequency of the atom (resonance regime) so that the high-frequency counter-rotating terms in the Hamiltonian are neglected. This model explains the quantum character [8] of properties such as purely quantum vacuum field effects, Rabi oscillations or quantum collapses and revivals [9] of some expected value evolutions.

The essentials of supersymmetric quantum mechanics methods, often referred to as Darboux [10], intertwining or factorization [11], depending on the context, can be found in many reviews and text-books [12, 13, 14]. The intertwining method can be applied differential, integral or matrix forms of Hamiltonians in quantum mechanics (or in mathematics) in order to provide a unified approach to constructing exactly solvable linear and nonlinear problems. It is also a way to obtain a whole hierarchy (a sequence) of exactly solvable Hamiltonians, starting from just one solvable system. The eigenvalues and eigenfunctions of the Hamiltonian operators in each hierarchy are connected by means of intertwining operators. These methods have been applied to a variety of scalar Hamiltonians, but here their relevance relies in the fact that they have also been extended to deal with matrix Hamiltonians. For example, in [15] the ingiedients of Darboux transformations for the JC Hamiltonian are given, in [16] generalized JC Hamiltonians are considered by shape-invariant hierarchies and their SUSY partners and in [17] Dirac-like Hamiltonians associated to Schrödinger factorizations are discussed. Thus, our aim in this work consists in translating SUSY techniques to the matrix JC model in a new original way.

There is already a considerable list of references where supersymmetry tools have been applied to study different aspects of the JC model. We will mention a number of representative references not exhaustive at all. For instance, Ref. [18] was devoted to the SUSY relation of JC and anti-JC Hamiltonians; in Refs. [19, 20] Jaynes-Cummings and anti-Jaynes-Cummings models (which include only the non-conserving interaction terms) were revisited by means of Lie algebras and superalgebras; differential realizations of superalgebras were applied in Refs. [21] and [22]. Refs. [23, 24] proposed an extension that includes nonlinear processes with qubits exchanges. Stark shift and nonlinear Kerr-like medium have also been considered, for instance in [25]. Other new similar systems have been introduced, for example [26] defines a system coupling a two-level atom with a two-dimensional supersymmetric system involving two shape invariant potentials. The coupling of two isospectral JC Hamiltonians is introduced in [27]; applications to quantum information technologies are frequent, for instance see [28, 29].

In this work, we want to present a novel approach of SUSY techniques to the JC model, which is essentially different to all of these previous references, up to our knowledge. We will apply in a systematic way the techniques presented in [15, 16] to the JC Hamiltonian. We find new intertwining operators depending on creation/annihilation operators made of atomic $\sigma^{\pm}$ and radiation field $a^{\pm}$ basic operators which are well defined in the quantum Hilbert space. These intertwining operators supply us with the relevant symmetries. We will also find original shape-invariance properties of the JC hierarchies. Thus, the main results presented here are completely new.

As a first step, we write the JC Hamiltonian in a suitable matrix form. Then, we will arrive to a hierarchy of JC Hamiltonians characterized by particular values of detuning parameters depending on the difference of atomic transition and field frequencies. The JC Hamiltonians in each hierarchy are connected by triangular intertwining operators, which are different from those applied earlier

in quantum optics. This means that the Hamiltonians in each sequence are quite similar (almost isospectral) and must share very similar properties. In addition, this procedure will allow us to obtain symmetries of JC Hamiltonians, such as the excitation number, in a natural way.

The scheme of the paper is as follows. In Section 2 we will discuss the form of the JC Hamiltonian adequate to the application of SUSY transformations. We consider physical as well as nonphysical states necessary for a complete understanding of SUSY transformations. Section 3 is devoted to a review of the ingredients of SUSY transformations that will be used later in the JC matrix context. Sections 4 and 5 show the results corresponding to several examples. Some of them are related to true JC Hamiltonians while others to the so called anti-JC Hamiltonians. Section 6 addresses the special case of a resonant shape invariant Hamiltonian hierarchy. This particular case is similar to other examples known for the two dimensional Dirac-Weyl Hamiltonians found in the continuous tight-binding approximation of graphene [17]. In this way, we show a connection between two areas far apart, quantum optics and condensed matter, where Hamiltonians with a similar structure are applied.

2 Matrix form of Jaynes-Cummings Hamiltonian

The Jaynes-Cummings Hamiltonian (in the RWA) has the following form [3]:

$$H_{JC} = H_A + H_F + H_{AF} = \frac{\hbar\omega_0}{2}\sigma_z + \hbar\omega a^+ a^- \sigma_0 + \hbar\mu\left(\sigma_- a^+ + \sigma_+ a^-\right) \tag{2.1}$$

where H_A is the Hamiltonian corresponding to a free two-level atom of energy difference $\hbar\omega_0$, H_F is for the field and H_{AF} for the interaction between field and atom in the electric dipole approximation [1, 3]. Here, $a^\pm$ are photon creation/annihilation operators of frequency ω, while the usual Pauli matrices (σ_z, $\sigma^\pm$ and the identity σ_0) describe the atomic part. The coupling coefficient μ depends on the interaction with the field.

Along this work we will consider the Hilbert space of the tensor product of the two-dimensional atomic space generated by the ground and excited states $|g\rangle$, $|e\rangle$, respectively, and the radiation quantum space of frequency ω, generated by the number states $|n\rangle$ (sometimes we write ψ_n instead of $|n\rangle$, when we use differential realizations). We make use of the standard notation:

$$|g\rangle \otimes |n\rangle := \begin{pmatrix} 0 \\ |n\rangle \end{pmatrix} := \begin{pmatrix} 0 \\ \psi_n \end{pmatrix}, \quad |e\rangle \otimes |n\rangle := \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix} := \begin{pmatrix} \psi_n \\ 0 \end{pmatrix}$$

For the operators a similar notation is applied, for instance

$$\sigma_x \otimes a^+ := \begin{pmatrix} 0 & a^+ \\ a^+ & 0 \end{pmatrix}$$

For all the systems of this work, the underlying Hilbert space of any JC Hamiltonian is the same and the intertwining operators between JC systems connect two isomorphic Hilbert spaces.

Let us introduce a new interacting parameter $\beta = \hbar\mu$ and a detuning parameter $\alpha = \hbar(\omega_0 - \omega)$ of the difference of atomic transition frequency ω_0 and field frequency ω. Then, we have the following matrix form for the Hamiltonian (2.1):

$$H_{JC} = \begin{pmatrix} \hbar\omega a^+ a^- + \frac{\hbar\omega_0}{2} & \beta a^- \\ \beta a^+ & \hbar\omega a^+ a^- - \frac{\hbar\omega_0}{2} \end{pmatrix} \tag{2.2}$$

If we use the commutator $[a^-, a^+] = 1$ in the first element of the above matrix, we can rewrite the Hamiltonian H_{JC} in the special form

$$H_{JC} = \begin{pmatrix} \hbar\omega a^- a^+ + \frac{\alpha}{2} & \beta a^- \\ \beta a^+ & \hbar\omega a^+ a^- - \frac{\alpha}{2} \end{pmatrix} - \frac{\hbar\omega}{2}\sigma_0 \qquad (2.3)$$

where σ_0 is the unit matrix. Since the constant term on the r.h.s. has no effect on the dynamics, it can be neglected.

2.1 Physical and nonphysical solutions of the HO

Hereafter, the above JC Hamiltonian will be re-expressed as $H_{\rm JC}$ in the following final notation

$$H_{JC} = \hbar\omega \begin{pmatrix} a^- a^+ + \delta & \lambda a^- \\ \lambda a^+ & a^+ a^- - \delta \end{pmatrix} := \hbar\omega H_{\rm JC} \qquad (2.4)$$

where

$$\lambda = \frac{\beta}{\hbar\omega}, \qquad \delta = \frac{\alpha}{2\hbar\omega} = \frac{\omega_0 - \omega}{2\omega} \qquad (2.5)$$

We will always deal with the Hamiltonian $H_{\rm JC}$, where we use the detuning parameter δ as well as the interacting coefficient λ in $\hbar\omega$-units as indicated in (2.5). The creation/annihilation operators $a^\pm$ of the HO satisfy the standard commutation

$$[a^-, a^+] = 1 \qquad (2.6)$$

We will make use of a the following differential realization for these operators,

$$a^\pm = \frac{1}{\sqrt{2}}\left(\mp\partial_x + x\right) \qquad (2.7)$$

The physical eigenfunctions $\psi_n := \psi_n(x)$ of the HO in this realization are defined in the usual way. These eigenfunctions for the number states (the energy eigenstates) of one-dimensional harmonic oscillator can be found in any standard quantum mechanics textbook. We will write them here explicitly to fix the notation:

$$a^- \psi_0 = 0 \implies \qquad \psi_0(x) = \frac{1}{\pi^{1/4}} e^{-x^2/2}$$

$$a^+ \psi_{n-1} = \sqrt{n}\,\psi_n, \qquad a^- \psi_n = \sqrt{n}\,\psi_{n-1}, \qquad (2.8)$$

$$a^+ a^- \psi_n = N\psi_n = n\psi_n, \qquad n = 0,1,2,3,...$$

In ket notation, this is

$$|0\rangle \xrightarrow{a^+} |1\rangle \dots \xrightarrow{a^+} |n-1\rangle \xrightarrow{a^+} |n\rangle \dots \qquad (2.9)$$

where $N = a^+ a^-$ is the number operator and $|n\rangle$ represents the state of ψ_n. We say that $|0\rangle$ is the physical ground state annihilated by a^-. We are also taking $\hbar\omega$-units for the eigenvalues of N, consistent the previous Hamiltonian $H_{\rm JC}$.

In future sections, we will make use of the following set of nonphysical eigenfunctions (they are not square integrable) of the HO,

$$a^+ \phi_{-1} = 0 \implies \qquad \phi_{-1}(x) := \frac{1}{\pi^{1/4}} e^{x^2/2}$$

$$a^- \phi_{-n} := \sqrt{-n}\,\phi_{-n-1}, \qquad a^+ \phi_{-n-1} := \sqrt{-n}\,\phi_{-n} \qquad (2.10)$$

$$a^+ a^- \phi_{-n} = N\phi_{-n} = -n\phi_{-n}, \qquad n = 1,2,3\dots$$

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$$\ldots \, |-n-1\rangle_{\text{np}} \xleftarrow{\;a^-\;} |-n\rangle_{\text{np}} \xleftarrow{\;a^-\;} \ldots \, |-2\rangle_{\text{np}} \xleftarrow{\;a^-\;} |-1\rangle_{\text{np}} \tag{2.11}$$

In this case, $|-1\rangle_{\text{np}}$ is the ket for the "highest" nonphysical state annihilated by a^+. There is a simple relation between physical and nonphysical eigenfunctions given by

$$\psi_n(ix) = i^n \phi_{-1-n}(x)\,, \qquad n = 0, 1, 2, 3 \ldots$$

2.2 Eigenfunctions and eigenvalues of JC Hamiltonian

It is convenient, at this stage, to make use of wave functions for the photon states. Later we will return to the states formulation. As usual, we can find the physical eigenstates (here they are spinorial eigenfunctions) of the JC Hamiltonian inside the subspace [3]

$$V_n = \begin{pmatrix} c_1 \psi_{n-1} \\ c_2 \psi_n \end{pmatrix}, \qquad c_1, c_2 \in \mathbb{C} \tag{2.12}$$

We will discuss later the connection of V_n with intertwining operators. Then, the eigenvalues and (not normalized) eigenfunctions are as follows ($\varepsilon_n = E_n/\hbar\omega$)

$$\varepsilon_n^{\pm} = n \pm \sqrt{\delta^2 + n\lambda^2}\,, \qquad \Psi_n^{\pm} = \begin{pmatrix} (\delta \pm \sqrt{\delta^2 + n\lambda^2})\,\psi_{n-1} \\ \sqrt{n}\,\lambda\,\psi_n \end{pmatrix}, \qquad n = 1, 2, \ldots$$

$$\varepsilon_0^- = -\delta\,, \qquad \Psi_0^- = \begin{pmatrix} 0 \\ \psi_0 \end{pmatrix}, \qquad n = 0 \tag{2.13}$$

For the values $n = 1, 2, \ldots$ there are two types of excited solutions $\Psi_n^{\pm}$, while only one ground single state, Ψ_0^-, is obtained for $n = 0$ (see Fig 1).

Next, we will consider some nonphysical eigenfunctions of the JC Hamiltonian. In order to find them, we make use of the previous nonphysical solutions of the HO. We construct this kind of formal eigenfunctions of the JC Hamiltonian inside the subspace

$$\mathcal{W}_{-n} = \begin{pmatrix} c_1 \phi_{-n-1} \\ c_2 \phi_{-n} \end{pmatrix}, \qquad c_1, c_2 \in \mathbb{C} \tag{2.14}$$

where ϕ_{-n} are the eigenfunctions defined above. These kind of solutions are as follows

$$\epsilon_{-n}^{\pm} = -n \pm \sqrt{\delta^2 - n\lambda^2}\,, \qquad \Phi_{-n}^{\pm} = \begin{pmatrix} (\delta \pm \sqrt{\delta^2 - n\lambda^2})\,\phi_{-n-1} \\ \sqrt{-n}\lambda\,\phi_{-n} \end{pmatrix}, \qquad n = 1, 2, \ldots$$

$$\epsilon_0^+ = \delta\,, \qquad \Phi_0^+ = \begin{pmatrix} \phi_{-1} \\ 0 \end{pmatrix}, \qquad n = 0 \tag{2.15}$$

Note that the energies $\epsilon_{-n}^{\pm}$ of the nonphysical solutions are real only for a finite number of n-values due to the condition $\delta^2 - n\lambda^2 \geq 0$. These nonphysical eigenfunctions with real eigenvalues will also be useful to get physical partner JC Hamiltonians. The physical and nonphysical eigenvalues above mentioned have been plotted in Fig 1.
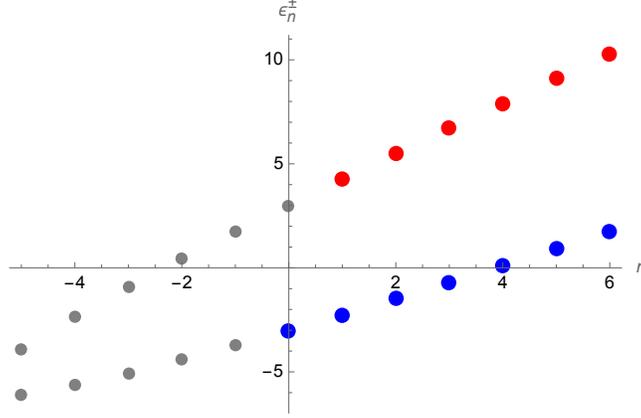
Figure 1: Eigenvalues of JC for $\delta = 3$, $\lambda = 1.25$. Blue (for ε_n^-) and red (for ε_n^+) points represent the (infinite) physical spectrum (2.13). Points in gray are for a (finite) nonphysical spectrum (2.15).

3 Construction of JC Hamiltonian partners

If we make use of the differential realization (2.7), the JC Hamiltonian has the following expression

$$
H_{\mathrm{JC}}^{(0)} := \begin{pmatrix} a^- a^+ + \delta & \lambda a^- \\ \lambda a^+ & a^+ a^- - \delta \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(-\partial_x^2 + x^2 + 1) + \delta & \frac{\lambda}{\sqrt{2}}(\partial_x + x) \\ \frac{\lambda}{\sqrt{2}}(-\partial_x + x) & \frac{1}{2}(-\partial_x^2 + x^2 - 1) - \delta \end{pmatrix} \tag{3.1}
$$

which can be rewritten as

$$
H_{\mathrm{JC}} = -\frac{1}{2}\partial_x^2 + V(x) + \frac{\lambda}{\sqrt{2}}\gamma\,\partial_x \tag{3.2}
$$

where

$$
V(x) = \begin{pmatrix} \frac{1}{2}(x^2 + 1) + \delta & \frac{\lambda}{\sqrt{2}}x \\ \frac{\lambda}{\sqrt{2}}x & \frac{1}{2}(x^2 - 1) - \delta \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.3}
$$

Then, in this context, we define the partner Hamiltonians $\widetilde{H}_{\mathrm{JC}}$ of the JC Hamiltonian H_{JC} by the following two conditions.

(i) The differential realization of $\widetilde{H}_{\mathrm{JC}}$ has the same structure (3.2):

$$
\widetilde{H}_{\mathrm{JC}} = -\frac{1}{2}\partial_x^2 + \widetilde{V}(x) + \frac{\tilde{\lambda}}{\sqrt{2}}\gamma\partial_x \tag{3.4}
$$

where $\widetilde{V}(x)$ has the same functional form as $V(x)$ but possibly with different parameters, $\tilde{\delta}, \tilde{\lambda}$. This means that they satisfy a shape invariance condition for these matrix Hamiltonians.

(ii) There is an intertwining differential operator

$$
L = \frac{1}{\sqrt{2}}\left(\partial_x - W(x)\right) \tag{3.5}
$$

where $W(x)$ is a matrix, connecting both of them

$$
L\,H_{\mathrm{JC}} = \widetilde{H}_{\mathrm{JC}}\,L \tag{3.6}
$$

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It can be shown that the W matrix can be obtained from two (linearly independent) eigenfunctions Ψ_{s1} and Ψ_{s2}, called seed functions, which may be physical or nonphysical, of the initial JC Hamiltonian,

$$H_{\text{JC}}\Psi_{\text{si}} = \varepsilon_i \Psi_{\text{si}}, \qquad i = 1, 2 \tag{3.7}$$

In a first step, we define an auxiliary matrix M obtained by arranging these two solutions as two columns $M(x) = (\Psi_{\text{s1}}, \Psi_{\text{s2}})$, such that $LM(x) = 0$. Then, W is given by

$$W(x) = M_x(x)M^{-1}(x) \tag{3.8}$$

where $M_x(x)$ is for the derivative $dM(x)/dx$. One can show that the new Hamiltonian $\widetilde{H}(x)$ has a new potential $\widetilde{V}(x)$ given by

$$\widetilde{V}(x) = V(x) + \Delta V(x) \tag{3.9}$$

with

$$\Delta V(x) = -W_x(x) - \frac{\lambda}{\sqrt{2}}\left[W(x), \gamma\right] \tag{3.10}$$

In the following, we will apply this program to find partners of H_{JC} by choosing different intertwining operators L, or equivalently, by choosing different pairs of seed solutions. The details can be found in [15, 16].

4 First example: anti-JC Hamiltonians

In this initial case, we choose the seed states Ψ_{s1} and Ψ_{s2} as follows

$$\Psi_{\text{s1}} \rightarrow \Psi_0^-, \qquad \Psi_{\text{s2}} \rightarrow \Phi_0^+ \tag{4.1}$$

In other words, one of them is the physical and the other is the nonphysical ground states, as defined in (2.13) and (2.15). Then, we obtain the following results (now we express them in terms of operators, leaving the differential realization):

(i) The intertwining operator has a diagonal form given by

$$L = \begin{pmatrix} -a^+ & 0 \\ 0 & a^- \end{pmatrix} \tag{4.2}$$

(ii) The partner Hamiltonian will be

$$\widetilde{H}_{\text{JC}} = H_{\text{aJC}} = \begin{pmatrix} a^+a^- + \delta & -\lambda a^+ \\ -\lambda a^- & a^-a^+ - \delta \end{pmatrix} \tag{4.3}$$

This partner Hamiltonian is a kind of anti-JC Hamiltonian H_{aJC} [18, 20, 28]. The aJC Hamiltonian is obtained when only the high frequency terms of the total H_{AF} are conserved. We can write this partner Hamiltonian in different equivalent forms by means of a unitary matrix R such that

$$\widetilde{H}_{\text{JC}}^R = R\widetilde{H}_{\text{JC}}R^{-1} \tag{4.4}$$

Let us consider just the following cases for R:

$$T = \sigma_z \quad \rightarrow \quad \widetilde{H}_{\text{JC}}^T = \begin{pmatrix} a^+a^- + \delta & \lambda a^+ \\ \lambda a^- & a^-a^+ - \delta \end{pmatrix} \tag{4.5}$$

$$S = \sigma_y \quad \rightarrow \quad \widetilde{H}_{\text{aJC}}^{(0)} := \widetilde{H}_{\text{JC}}^S = \begin{pmatrix} a^- a^+ - \delta & \lambda a^- \\ \lambda a^+ & a^+ a^- + \delta \end{pmatrix} = H_{\text{JC}}(-\delta) \tag{4.6}$$

The first equivalent partner (4.5), looks like the aJC Hamiltonian given in Ref. [18]. The second equivalent partner (4.6) is the same as the initial JC Hamiltonian except for the change $\delta \rightarrow -\delta$ (see Fig. 2). In other words, if the detuning coefficient δ change the sign to $-\delta$, we obtain a partner Hamiltonian, $H_{\text{aJC}}(\delta) = H_{\text{JC}}(-\delta)$ [20, 28]. This interpretation seems the most adequate and hereafter we will refer to this choice with the notation $\widetilde{H}_{\text{JC}}^{(0)}$ of (4.6).

The energies and eigenvalues of this partner Hamiltonian $\widetilde{H}_{\text{JC}}$ are

$$\widetilde{\varepsilon}_n^{\pm} = n \pm \sqrt{\delta^2 + n\lambda^2}, \qquad \widetilde{\Psi}_n^{\pm} = \begin{pmatrix} (-\delta \pm \sqrt{\delta^2 + n\lambda^2})\,\psi_n \\ \sqrt{n}\lambda\,\psi_{n-1} \end{pmatrix}$$

$$\widetilde{\varepsilon}_0 = \delta, \qquad \widetilde{\Psi}_0 = \begin{pmatrix} \psi_0 \\ 0 \end{pmatrix} \tag{4.7}$$

Note that the spectrum has the same energies as H_{JC} except for the fundamental states (see Fig. 2):

$$\varepsilon_0 = -\delta, \qquad \tilde{\varepsilon}_0 = \delta \tag{4.8}$$

The intertwining operator L transforms eigenfunctions of H_{JC} into eigenfunctions of H_{aJC} with the same eigenvalue, while the adjoint L^+ realizes the correspondence in the opposite way,

$$L\,\Psi_n^{\pm} \rightarrow \widetilde{\Psi}_n^{\pm}, \qquad L^+\,\widetilde{\Psi}_n^{\pm} \rightarrow \Psi_n^{\pm}, \qquad \varepsilon_n^{\pm} = \widetilde{\varepsilon}_n^{\pm}, \quad n = 1, 2, \dots \tag{4.9}$$

The action of L and L^+ on the respective ground states of H_{JC} and $\tilde{H}_{\text{JC}}$ is null:

$$L\Psi_0^- = 0 \qquad L^+\widetilde{\Psi}_0^+ = 0 \tag{4.10}$$

This implies that both Hamiltonians are isospectral, except for the ground states (see Fig. 2). The connection is carried out by the matrix intertwining operators L and L^+, where

$$L^+ = \begin{pmatrix} -a^- & 0 \\ 0 & a^+ \end{pmatrix} \tag{4.11}$$

The product L^+L gives a symmetry of H_{JC}; in fact it can be identified to the excitation number N_e [3]:

$$N_e := L^+L = \begin{pmatrix} a^- a^+ & 0 \\ 0 & a^+ a^- \end{pmatrix} \tag{4.12}$$

The invariant spaces V_n of (2.12) correspond to the eigenspaces of L^+L with eigenvalue n.

The product LL^+ will be a symmetry of $\widetilde{H}_{\text{JC}}^{(0)} = H_{\text{aJC}}$ which can be identified to the excitation number of the anti-Jaynes-Cummings Hamiltonian,

$$LL^+ = \begin{pmatrix} a^+ a^- & 0 \\ 0 & a^- a^+ \end{pmatrix} \tag{4.13}$$

If we compute the (anti) partner of H_{aJC}, we will recuperate H_{JC}. Therefore, the anti-anti-JC Hamiltonian is again H_{JC}.
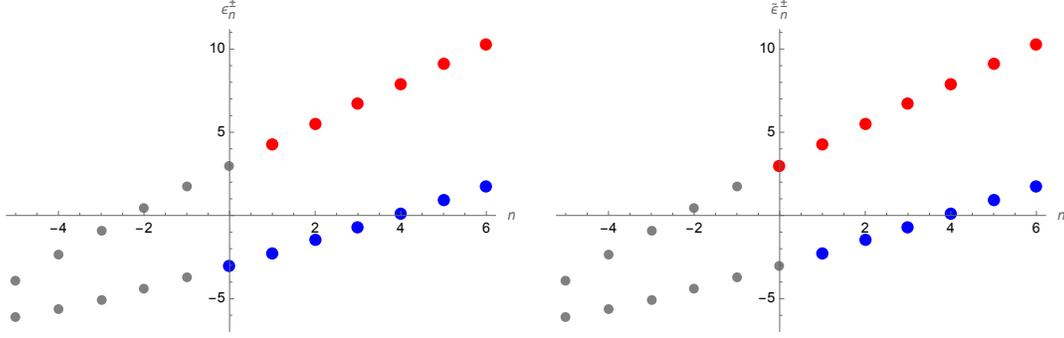
Figure 2: The eigenvalues of the initial JC Hamiltonian (left) and those of anti-JC (right) for $\delta = 3$, $\lambda = 1.25$. Blue and red points are for the physical spectrum, ε_n^- and ε_n^+, respectively, while gray are for nonphysical spectrum. Notice the difference of the ground states energies with respect to the initial JC Hamiltonian (left graphic) as mentioned after (4.10).

5 Second example: Detuned partner JC Hamiltonians

5.1 JC Hamiltonian partners by two nonphysical states

First option: Φ_0^+ and Φ_{-1}^-.

In this case, we choose the seed states $\Psi_{\mathrm{s}1}$ and $\Psi_{\mathrm{s}2}$ as follows

$$\Psi_{\mathrm{s}1} \; \to \; \Phi_0^+, \qquad \Psi_{\mathrm{s}2} \; \to \; \Phi_{-1}^- \tag{5.1}$$

In other words, one of them is the nonphysical ground state Φ_0^+ and the other is the nonphysical excited state Φ_{-1}^-, respectively. Then, we obtain a triangular intertwining operator

$$L_1 = \begin{pmatrix} a^+ & K_1 \\ 0 & a^+ \end{pmatrix}, \qquad K_1 = \frac{-\delta + \sqrt{\delta^2 - \lambda^2}}{\lambda} \tag{5.2}$$

The triangular form of the intertwining operator is a consequence of the seed functions which determine the SUSY transformation. Other choices of seed functions lead us to differential intertwining operators which do not admit an operator expression, so they were discarded. The partner Hamiltonian will be

$$\widetilde{H}_{\mathrm{JC}} = \begin{pmatrix} a^- a^+ - 1 + \sqrt{\delta^2 - \lambda^2} & \lambda a^- \\ \lambda a^+ & a^+ a^- - 1 - \sqrt{\delta^2 - \lambda^2} \end{pmatrix} \tag{5.3}$$

This partner Hamiltonian is called detuned JC Hamiltonian, in the sense that the change in $\widetilde{H}_{\mathrm{JC}}$ with respect to H_{JC} is that the detuning parameter δ has changed to $\sqrt{\delta^2 - \lambda^2}$, besides the subtraction of a global unit constant. This is possible if the parameters satisfy $\delta^2 \geq \lambda^2$. We can write this as follows

$$H_{\mathrm{JC}}(\delta) := H_{\mathrm{JC}}, \qquad \widetilde{H}_{\mathrm{JC}}^{(1)} := H_{\mathrm{JC}}(\sqrt{\delta^2 - \lambda^2}) - 1 \tag{5.4}$$

In this case we may interpret this transformation as a relation between two JC systems with the same radiation but different atom frequency between the two levels. The eigenvalues and eigenfunctions of

this partner Hamiltonian are as follows

$$\widetilde{\varepsilon}_n^\pm = (n-1) \pm \sqrt{\delta^2 + (n-1)\lambda^2}, \qquad \widetilde{\Psi}_n^\pm = \begin{pmatrix} \left(\sqrt{\delta^2 - \lambda^2} \pm \sqrt{\delta^2 + (n-1)\lambda^2}\right)\psi_{n-1} \\ \sqrt{n}\lambda\,\psi_n \end{pmatrix}$$

$$\widetilde{\varepsilon}_0 = -1 - \sqrt{\delta^2 - \lambda^2}, \qquad \widetilde{\Psi}_0 = \begin{pmatrix} 0 \\ \psi_0 \end{pmatrix}$$

(5.5)

The connection of the states and energies are as follows

$$L_1 : \Psi_n^\pm \to \widetilde{\Psi}_{n+1}^\pm, \qquad L_1^+ : \widetilde{\Psi}_{n+1}^\pm \to \Psi_n^\pm, \qquad \varepsilon_n^\pm = \widetilde{\varepsilon}_{n+1}^\pm, \qquad n = 1, 2, \ldots$$

(5.6)

But the action on the nonphysical states is null:

$$L_1 \Phi_0^+ = 0, \qquad L_1 \Phi_{-1}^- = 0$$

(5.7)

This means that both Hamiltonians are isospectral, except for two levels: One ground and one excited states. To be more precise, $\varepsilon_0 = \widetilde{\varepsilon}_1^-$, but $\widetilde{\varepsilon}_0^+$ and $\widetilde{\varepsilon}_1^+$ are two new levels of the partner Hamiltonian which were not present in the initial H_{JC}. The other energy levels are in one to one correspondence: $\varepsilon_n^\pm = \widetilde{\varepsilon}_{n+1}^\pm$, for $n = 1, 2, \ldots$ as mentioned in (5.6). The connection between the states of both Hamiltonians is carried out by the matrix intertwining operator L_1. In conclusion, we see that a smaller positive detuning δ leads to increase the spectrum with two more energy eigenvalues which are at the bottom of the spectrum, see Fig. 3e.

By using this intertwining operator we can find a symmetry, S_1 of the Hamiltonian:

$$S_1 = L_1^+ L_1 = \begin{pmatrix} a^- a^+ & a^- K_1 \\ a^+ K_1 & a^- a^+ + K_1^2 \end{pmatrix}, \qquad K_1 = \frac{-\delta + \sqrt{\delta^2 - \lambda^2}}{\lambda}$$

(5.8)

This symmetry is a linear combination of the previous symmetries:

$$S_1 = N_e + \frac{K_1}{\lambda}(H_{\mathrm{JC}} - N_e - \delta)$$

(5.9)

Second option: Φ_0^+ and Φ_{-1}^+.

In this case, we choose the nonphysical seed states $\Psi_{\mathrm{s}1}$ and $\Psi_{\mathrm{s}2}$ as follows

$$\Psi_{\mathrm{s}1} \to \Phi_0^+, \qquad \Psi_{\mathrm{s}2} \to \Phi_{-1}^+$$

(5.10)

In other words, one of them is the nonphysical ground state Φ_0^+ and the other is the nonphysical excited state, Φ_{-1}^+ respectively. Then, we obtain the intertwining operator

$$L_2 = \begin{pmatrix} a^+ & -K_2^+ \\ 0 & a^+ \end{pmatrix}, \qquad K_2^+ = \frac{\delta + \sqrt{\delta^2 - \lambda^2}}{\lambda}$$

(5.11)

The partner Hamiltonian will be

$$\widetilde{H}_{\mathrm{aJC}}^{(1)} := \widetilde{H}_{\mathrm{JC}} = \begin{pmatrix} a^- a^+ - 1 - \sqrt{\delta^2 - \lambda^2} & \lambda a^- \\ \lambda a^+ & a^+ a^- - 1 + \sqrt{\delta^2 - \lambda^2} \end{pmatrix}$$

(5.12)

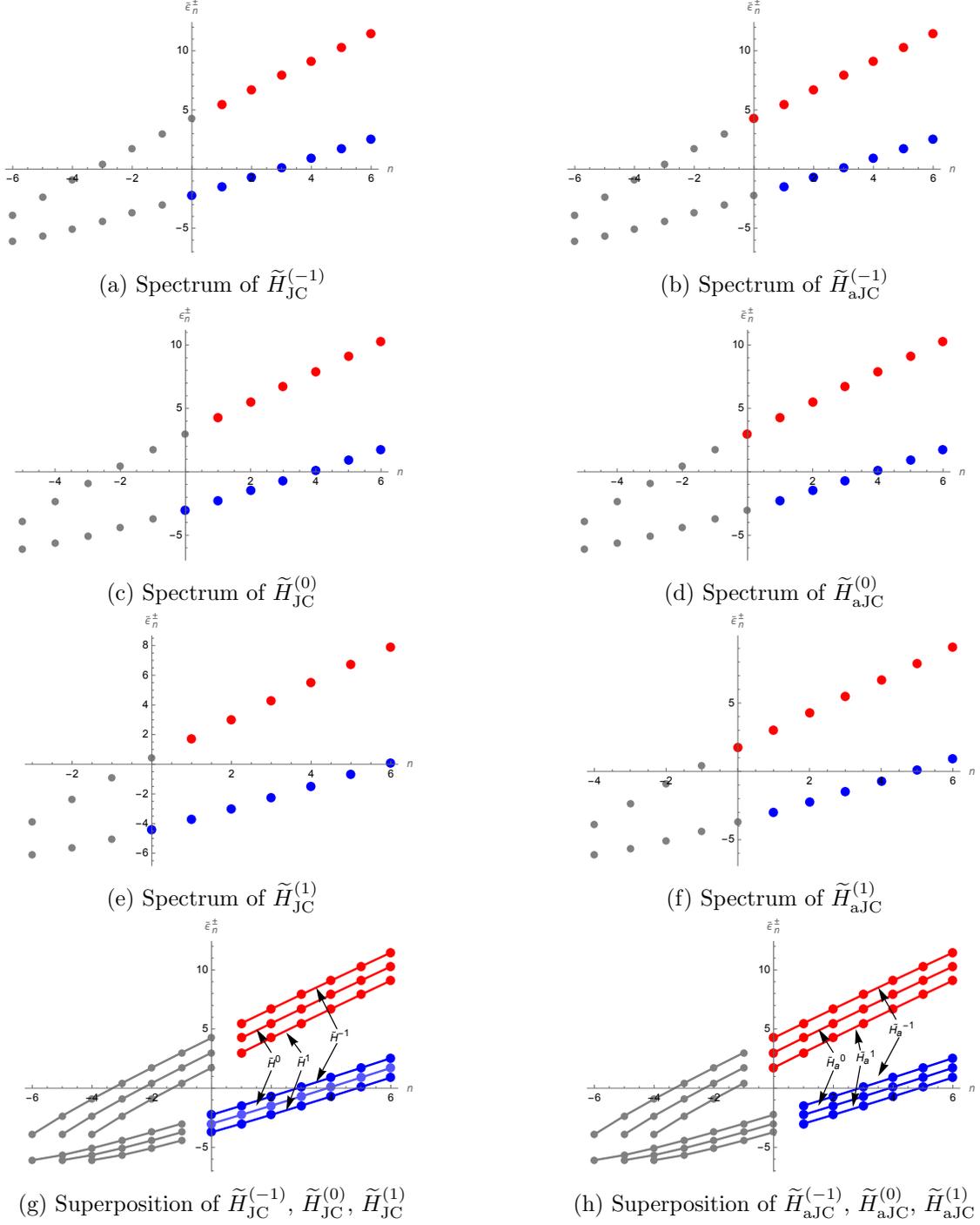(a) Spectrum of $\widetilde{H}_{\mathrm{JC}}^{(-1)}$

(b) Spectrum of $\widetilde{H}_{\mathrm{aJC}}^{(-1)}$

(c) Spectrum of $\widetilde{H}_{\mathrm{JC}}^{(0)}$

(d) Spectrum of $\widetilde{H}_{\mathrm{aJC}}^{(0)}$

(e) Spectrum of $\widetilde{H}_{\mathrm{JC}}^{(1)}$

(f) Spectrum of $\widetilde{H}_{\mathrm{aJC}}^{(1)}$

(g) Superposition of $\widetilde{H}_{\mathrm{JC}}^{(-1)}$, $\widetilde{H}_{\mathrm{JC}}^{(0)}$, $\widetilde{H}_{\mathrm{JC}}^{(1)}$

(h) Superposition of $\widetilde{H}_{\mathrm{aJC}}^{(-1)}$, $\widetilde{H}_{\mathrm{aJC}}^{(0)}$, $\widetilde{H}_{\mathrm{aJC}}^{(1)}$

Figure 3: Left column (for $\delta = 3$, $\lambda = 1.25$): spectrum of JC Hamiltonians $\tilde{H}_{\mathrm{JC}}^{(-1)}$, $H_{\mathrm{JC}}^{(0)}$, $\tilde{H}_{\mathrm{JC}}^{(1)}$. Points in red ($\tilde{\varepsilon}_n^+$) and in blue ($\tilde{\varepsilon}_n^-$) are for the physical spectrum, while gray is for nonphysical states. Right column: spectrum for the aJC Hamiltonians $\tilde{H}_{\mathrm{aJC}}^{(-1)}$, $\tilde{H}_{\mathrm{aJC}}^{(0)}$, $\tilde{H}_{\mathrm{aJC}}^{(1)}$. The values of the spectra for these three Hamiltonians are decreasing: $\tilde{\varepsilon}_n^{(-1)} > \tilde{\varepsilon}_n^{(0)} > \tilde{\varepsilon}_n^{(1)}$.

This partner Hamiltonian is a detuned anti-JC Hamiltonian. If we make an equivalence transforma-

tion

$$\widetilde{H}_{\mathrm{JC}} \;\rightarrow\; \widetilde{H}_{\mathrm{aJC}} = T\widetilde{H}_{\mathrm{JC}}T^{-1}\,, \quad \text{where} \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.13}$$

we have

$$\widetilde{H}_{\mathrm{aJC}} = \begin{pmatrix} a^{+}a^{-} - 1 + \sqrt{\delta^2 - \lambda^2} & -\lambda a^{+} \\ -\lambda a^{-} & a^{-}a^{+} - 1 - \sqrt{\delta^2 - \lambda^2} \end{pmatrix} \tag{5.14}$$

Therefore, $\widetilde{H}_{\mathrm{aJC}}$ has the form of an anti-JC Hamiltonian like (4.3) where the detuning parameter δ has changed to $\sqrt{\delta^2 - \lambda^2}$, besides a global constant.

In this case, we have also the following interpretation: We are relating two JC systems with of opposite detuning sign as seen in (5.12). The eigenvalues and eigenfunctions of this partner Hamiltonian are as follows

$$\widetilde{\varepsilon}_n^{\pm} = (n-1) \pm \sqrt{\delta^2 + (n-1)\lambda^2}, \qquad \widetilde{\Psi}_n^{\pm} = \begin{pmatrix} \left(-\sqrt{\delta^2 - \lambda^2} \pm \sqrt{\delta^2 + (n-1)\lambda^2}\right)\psi_{n-1} \\ \sqrt{n}\,\lambda\,\psi_n \end{pmatrix}$$

$$\widetilde{\varepsilon}_0^{+} = -1 + \sqrt{\delta^2 - \lambda^2}, \qquad\qquad \widetilde{\Psi}_0 = \begin{pmatrix} 0 \\ \psi_0 \end{pmatrix} \tag{5.15}$$

The connection of the states and energies are as follows

$$L_2 : \Psi_n^{\pm} \;\rightarrow\; \widetilde{\Psi}_{n+1}^{\pm}, \qquad \varepsilon_n^{\pm} = \widetilde{\varepsilon}_{n+1}^{\pm} \tag{5.16}$$

But the action on the nonphysical states is null:

$$L_2\Phi_0^{+} = 0, \qquad L_2\Phi_{-1}^{+} = 0 \tag{5.17}$$

This means that both Hamiltonians are isospectral, except for two lower energy levels: $\widetilde{\varepsilon}_0^{+}$ and $\widetilde{\varepsilon}_{-1}^{+}$, see Fig. 3f.

The connection between the states of both Hamiltonians is carried out by the matrix intertwining operator L_2. In this case, the symmetry is the following linear combination of the previous ones:

$$S_2 = L_2^{+}L_2 = N_e - \frac{K_2^{+}}{\lambda}(H_{\mathrm{JC}} - N_e - \delta) \tag{5.18}$$

5.2 JC Hamiltonian partners by two physical states

First option: Ψ_0^{-} and Ψ_1^{+}.

In this case, we choose the seed states $\Psi_{\mathrm{s}1}$ and $\Psi_{\mathrm{s}2}$ as follows

$$\Psi_{\mathrm{s}1} \;\rightarrow\; \Psi_0, \qquad \Psi_{\mathrm{s}2} \;\rightarrow\; \Psi_1^{+} \tag{5.19}$$

So, one of them is the physical ground state Ψ_0 an the other is the physical first excited state, Ψ_1^{+} respectively. Then, we obtain the intertwining operator

$$L_3 = \begin{pmatrix} a^{-} & 0 \\ K_3 & a^{-} \end{pmatrix}, \qquad K_3 = \frac{\delta - \sqrt{\delta^2 + \lambda^2}}{\lambda} \tag{5.20}$$

The partner Hamiltonian will be

$$\widetilde{H}_{\mathrm{JC}}^{(-1)} := \widetilde{H}_{JC} = \begin{pmatrix} a^- a^+ + 1 + \sqrt{\delta^2 + \lambda^2} & \lambda a^- \\ \lambda a^+ & a^+ a^- + 1 - \sqrt{\delta^2 + \lambda^2} \end{pmatrix} \quad (5.21)$$

The eigenvalues and eigenfunctions are

$$\widetilde{\varepsilon}_n^\pm = (n+1) \pm \sqrt{\delta^2 + (n+1)\lambda^2}, \qquad \widetilde{\Psi}_n^\pm = \begin{pmatrix} \left(\sqrt{\delta^2 + \lambda^2} \pm \sqrt{\delta^2 + (n+1)\lambda^2}\right) \psi_{n-1} \\ \sqrt{n}\lambda\,\psi_n \end{pmatrix}$$

$$\widetilde{\varepsilon}_0 = 1 - \sqrt{\delta^2 + \lambda^2}, \qquad \widetilde{\Psi}_0 = \begin{pmatrix} 0 \\ \psi_0 \end{pmatrix} \quad (5.22)$$

In this case, the new detuning parameter is bigger: $\delta \to \sqrt{\delta^2 + \lambda^2}$. The connection of the states and energies are as follows

$$L_3 \Psi_n^\pm \to \widetilde{\Psi}_{n-1}^\pm, \qquad \varepsilon_n^\pm = \widetilde{\varepsilon}_{n-1}^\pm \quad (5.23)$$

But the action on the physical states is null:

$$L_3 \Psi_0 = 0, \qquad L_3 \Psi_1^+ = 0 \quad (5.24)$$

This implies that this partner Hamiltonian has lost two energy levels (see Fig. 3a) corresponding to these annihilated states.

By using this intertwining operator we can find a symmetry, S_3 of the Hamiltonian:

$$S_3 = L_3^+ L_3 = \begin{pmatrix} a^+ a^- + K_3^2 & a^- K_3 \\ a^+ K_3 & a^+ a^- \end{pmatrix}, \qquad K_3 = \frac{\delta - \sqrt{\delta^2 + \lambda^2}}{\lambda} \quad (5.25)$$

This symmetry is a linear combination of the previous symmetries:

$$S_3 = L_3^+ L_3 = N_e + \frac{K_3}{\lambda}(H_{\mathrm{JC}} - N_e + \delta) \quad (5.26)$$

Second option: Ψ_0 and Ψ_1^-.

In this case, we choose the seed states $\Psi_{\mathrm{s}1}$ and $\Psi_{\mathrm{s}2}$ as follows

$$\Psi_{\mathrm{s}1} \to \Psi_0, \qquad \Psi_{\mathrm{s}2} \to \Psi_1^- \quad (5.27)$$

So, one of then is the physical ground state Ψ_0 an the other is the physical first excited state, Ψ_1^- respectively. Then, we obtain the intertwining operator

$$L_4 = \begin{pmatrix} a^- & 0 \\ K_4 & a^- \end{pmatrix}, \qquad K_4 = \frac{\delta + \sqrt{\delta^2 + \lambda^2}}{\lambda} \quad (5.28)$$

The partner Hamiltonian will be

$$\widetilde{H}_{\mathrm{aJC}}^{(-1)} := \widetilde{H}_{JC} = \begin{pmatrix} a^- a^+ + 1 - \sqrt{\delta^2 + \lambda^2} & \lambda a^- \\ \lambda a^+ & a^+ a^- + 1 + \sqrt{\delta^2 + \lambda^2} \end{pmatrix} \quad (5.29)$$

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This parter Hamiltonian has changed δ by $-\sqrt{\delta^2 + \lambda^2}$, so it is of aJC type. The eigenvalues and eigenfunctions are

$$\widetilde{\varepsilon}_n^{\pm} = (n+1) \pm \sqrt{\delta^2 + (n+1)\lambda^2}, \qquad \widetilde{\Psi}_n^{\pm} = \begin{pmatrix} (-\sqrt{\delta^2 - \lambda^2} \pm \sqrt{\delta^2 + (n+1)\lambda^2})\,\psi_{n-1} \\ \sqrt{n}\lambda\,\psi_n \end{pmatrix}$$

$$\widetilde{\varepsilon}_0 = 1 + \sqrt{\delta^2 + \lambda^2}, \qquad \widetilde{\Psi}_0 = \begin{pmatrix} 0 \\ \psi_0 \end{pmatrix} \tag{5.30}$$

The connection of the states and energies are as follows

$$L_4 \Psi_n^{\pm} \to \widetilde{\Psi}_{n-1}^{\pm}, \qquad \varepsilon_n^{\pm} = \widetilde{\varepsilon}_{n-1}^{\pm} \tag{5.31}$$

But the action on the physical seed states is null:

$$L_4 \Psi_0 = 0, \qquad L_4 \Psi_1^- = 0 \tag{5.32}$$

This implies that the partner Hamiltonian has also lost two energy levels (see Fig. 3b) corresponding to these annihilated states.

By using this intertwining operator we can find a symmetry, S_4 of the Hamiltonian:

$$S_4 = L_4^+ L_4 = \begin{pmatrix} a^+ a^- + K_4^2 & a^- K_4 \\ a^+ K_4 & a^+ a^- \end{pmatrix}, \qquad K_4 = \frac{\delta + \sqrt{\delta^2 + \lambda^2}}{\lambda} \tag{5.33}$$

This symmetry is a linear combination of the previous symmetries:

$$S_4 = L_4^+ L_4 = N_e + \frac{K_4}{\lambda}(H_{\mathrm{JC}} - N_e + \delta) \tag{5.34}$$

5.3   Summary of JC Hamiltonians SUSY partners

As we have seen above, the SUSY transformations constructed in this work act on a JC Hamiltonian characterized by a detuning parameter δ giving rise to a kind of anti-JC Hamiltonian and four types of partner Hamiltonians with different detuning parameters, as shown in the diagram of Fig. 4.
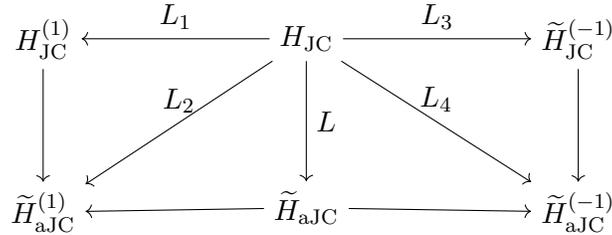

Figure 4: This diagram shows all the partner Hamiltonians of $H_{\mathrm{JC}}(\delta)$ obtained along Sections 4 and 5. The intertwining operators are L, L_1, L_2, L_3, L_4.

We can obtain new Hamiltonians by consecutive SUSY transformations, in this way we arrive at a sequence of JC Hamiltonians starting from $H_{\mathrm{JC}}(\delta)$, as follows,

$$(i) \qquad \leftarrow H_{\mathrm{JC}}^{(-n)} \leftarrow \ldots H_{\mathrm{JC}}^{(-1)} \leftarrow H_{\mathrm{JC}}(\delta) \to H_{\mathrm{JC}}^{(1)} \ldots \to H_{\mathrm{JC}}^{(n)} \to \tag{5.35}$$

where

$$H_{\mathrm{JC}}^{(n)} = H_{\mathrm{JC}}(\sqrt{\delta^2 - n\lambda^2}) - n, \qquad H_{\mathrm{JC}}^{(-n)} = H_{\mathrm{JC}}(\sqrt{\delta^2 + n\lambda^2}) + n \tag{5.36}$$

From $H_{\mathrm{JC}}(\delta)$ we obtained an anti-JC Hamiltonian, H_{aJC}, which was equivalent to $H_{\mathrm{JC}}(-\delta)$. Applying sucessive SUSY transformation to H_{aJC} we get a second sequence

$$(ii) \qquad \leftarrow H_{\mathrm{aJC}}^{(-n)} \leftarrow \dots H_{\mathrm{aJC}}^{(-1)} \leftarrow H_{\mathrm{aJC}}(\delta) \rightarrow H_{\mathrm{aJC}}^{(1)} \dots \rightarrow H_{\mathrm{aJC}}^{(n)} \rightarrow \tag{5.37}$$

where

$$H_{\mathrm{aJC}}^{(n)} \approx H_{\mathrm{JC}}(-\sqrt{\delta^2 - n\lambda^2}) - n, \qquad H_{\mathrm{aJC}}^{(-n)} \approx H_{\mathrm{JC}}(-\sqrt{\delta^2 + n\lambda^2}) + n \tag{5.38}$$

6 The resonant hierarchy of JC Hamiltonians

In the particular case where $\delta^2 = k\lambda^2$, for k a positive integer, after k SUSY transformations, we can obtain a partner Hamiltonian $H_{\mathrm{JC}}(\delta' = 0)$, which is a resonant JC Hamiltonian. This type of Hamiltonians, which will be denoted H_{JC}^k, constitute a "resonant hierarchy" and they have a very special form:

$$H_{\mathrm{JC}}^k = \begin{pmatrix} a^- a^+ + \lambda\sqrt{k} & \lambda a^- \\ \lambda a^+ & a^+ a^- - \lambda\sqrt{k} \end{pmatrix} + kI, \qquad k = 0, 1, 2 \dots \tag{6.1}$$

The spectrum of one example in this class, including the unphysical part, is given in Fig. 5, where it is shown that the two sequences of physical and nonphysical spectrum end in one point (not two as it is the generic case). These Hamiltonians are intertwined to the resonant Hamiltonian H_{JC}^0 by means of the operators having the form

$$L_k = \begin{pmatrix} -a^+ & \sqrt{k} - \sqrt{k-1} \\ 0 & -a^+ \end{pmatrix} \tag{6.2}$$

and the intertwining relations

$$L_k H_{\mathrm{JC}}^k = H_{\mathrm{JC}}^{k-1} L_k, \qquad k = 1, 2, \dots \tag{6.3}$$

The Hamiltonians H_{JC}^k have $2k$ states less than the resonant one (each transformation looses two states). We can express this class of Hamiltonians in the form

$$H_{\mathrm{JC}}^k = \begin{pmatrix} a^- a^+ & 0 \\ 0 & a^+ a^- \end{pmatrix} + \lambda \begin{pmatrix} \sqrt{k} & a^- \\ a^+ & -\sqrt{k} \end{pmatrix} = N_e + \lambda H_{\mathrm{HO}}^k \tag{6.4}$$

where N_e is excitation number operator, while H_{HO}^k is a first order Hamiltonian associated to the harmonic oscillator. We find that

$$\left(H_{\mathrm{HO}}^k\right)^2 = N_e + k \tag{6.5}$$

The Hamiltonians H_{HO}^k constitute a hierarchy of shape invariant Dirac Hamiltonians which was considered in Ref. [17]. Needless to say, there is also an anti-JC Hamiltonian resonant hierarchy with $\delta = -\sqrt{n}\lambda$.
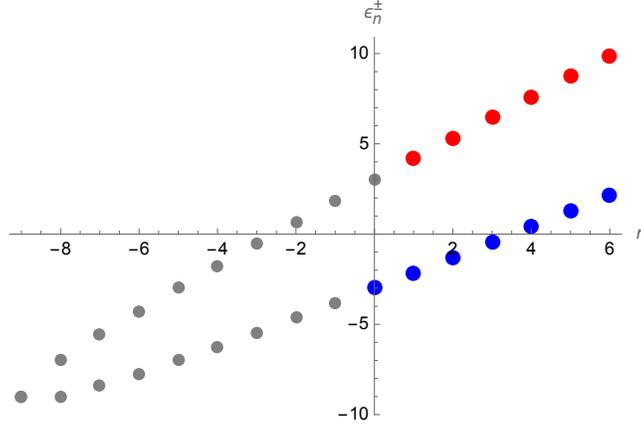
Figure 5: Eigenvalues of JC Hamiltonian of the resonant type, for $\delta = 3$, $\lambda = 1$. Blue and red points are for the (infinite) physical spectrum, while gray for a (finite) nonphysical spectrum. It is characterized by a common energy eigenvalue for both initial ground states Ψ_0^+ and Ψ_0^-, which corresponds to the left gray point in the figure.

7 Conclusions

In this work, we have shown how SUSY transformations can be applied to JC Hamiltonians. These JC Hamiltonians are characterized by a pair of parameters: The interacting coefficient λ and the detuning δ; what we have seen is that the Hamiltonian $H_{\mathrm{JC}}(\delta, \lambda)$ can be transformed into other two $\widetilde{H}_{\mathrm{JC}}^{(\pm 1)} = H_{\mathrm{JC}}(\delta'_\pm, \lambda) \pm 1$, with different detuning parameters $\delta'_\pm = \sqrt{\delta^2 \pm \lambda^2}$. These two Hamiltonians have the same spectra except for two of the lowest points of the spectrum.

We find that there is the possibility to change the sign of the detuning by means of a SUSY transformation, so that we can transform $H_{\mathrm{JC}}(\delta, \lambda)$ into $H_{\mathrm{JC}}(-\delta, \lambda)$ or into $H_{\mathrm{JC}}(\delta', \lambda)$ with $\delta' = -\sqrt{\delta^2 \pm \lambda^2}$). In this way, we find two sequences of JC Hamiltonians, the so called JC and anti-JC hierarchies corresponding to positive or negative δ values, connected by special SUSY transformations. Note that the transformations here involved were constructed by means of appropriate physical as well as nonphysical eigenstates called seed functions.

Thus, any sequence or hierarchy is connected by SUSY transformations, such that any two of their consecutive Hamiltonians differ in two eigenvalues and they are related by intertwining operators annihilating their corresponding eigenfunctions. The product of two intertwining operators gives a symmetry of each JC Hamiltonian. In particular, the excitation number is a product of intertwining operators, i.e., it is one of such symmetries. Each Hamiltonian has a particular factorized symmetry. In fact, many properties of these partner Hamiltonians should have common features due to this relationship. The program to show the similarities and differences related with expected values of atomic operators (such as σ_z for the atomic inversion) or photon operators (such as quadratures), is postponed to a future work to avoid a too lengthy paper.

There is a particular hierarchy, where one of the JC Hamiltonian has a vanishing detuning parameter $\delta = 0$. This is a very interesting situation where all the intertwining properties have very simplified expressions. This case is similar to previous results on the shape invariance of matrix Hamiltonians of two-dimensional relativistic equations applied in graphene [17]. This is another example of a quite appealing relation between quantum optics (through JC Hamiltonians) and condensed matter (of low energy electronic states in graphene ruled by the Dirac-Weyl matrix equation).

Data availability statement

No new data were created or to be shared in this study.

Acknowledgments

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