Minimizing the benefit risk in a target benefit stochastic pension plan

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RESUMEN

In this paper, we study the optimal management of a target benefit pension plan. The fund manager adjust the benefit to guarantee the plan stability. The fund can be invested in a rissless asset and a risky asset where the uncertainty comes from Brownian motion process. The manager minimizes the quadratic deviations between benefit and terminal fund with respect to their target values. A weighting factor included in the model indicates the importance of minimizing the deviation of the terminal fund. A stochastic control problem is considered and solved by the programming dynamic approach. Optimal benefit and investment strategies are analytically found and analyzed, both in finite and infinite horizon. An interesting particular case that receives special attention is when the contribution and the targets have an exponential form.

Palabras clave: [Target benefit pension plan; Portfolio optimization; Stochastic dynamic programming]
Área temática: [A4 - Matemáticas financieras y actuariales]

ABSTRACT

En este artículo, estudiamos la gestión óptima de un plan de pensiones con beneficio objetivo. El gestor del fondo ajusta la prestación para garantizar la estabilidad del plan. El fondo se puede invertir en un activo sin riesgo y en uno con riesgo, donde la incertidumbre proviene de un movimiento Browniano. El gestor minimiza las desviaciones cuadráticas entre la prestación y el fondo terminal y sus valores objetivo. Un factor de ponderación que se incluye en el modelo indica la importancia de minimizar la desviación del valor final del fondo. Se considera y resuelve un problema de control estocástico mediante el enfoque de la programación dinámica. Las estrategias óptimas de prestación e inversión se determinan explícitamente y analizan, tanto en horizonte finito como infinito. Un interesante caso particular que recibe especial atención es cuando la contribución y los valores objetivos tienen forma exponencial.

1 INTRODUCTION

Demographic changes have been observed in many developed countries, such as the increase in life expectancy and the reduction in the birth rate. This increases wealth awareness and concern after retirement. For this reason, it is of interest to find a type of pension system seeking financial sustainability and sharing risk between the fund manager and the participants. On the other hand, due to unexpected news, the evolution of the prices in the financial markets may be affected in the form of sudden changes. All this must be taken into account when designing the pension plan model to be analysed. The objective of this paper is to study a dynamic model of a risk sharing pension plan that takes into account those demographic and financial changes.

There are two major types of pension plans, defined benefit (DB therefore) and defined contribution (DC therefore). In a DB plan, the benefits are fixed in advance and the contributions are designed to maintain the fund in balance, that is, to fund employees' promised benefits. Usually, benefits are linked to salaries, and the contributions are shared by employer and employee. The fund manager bears the risk of funding the pension fund to assure future benefits, and the employee does not suffer possible investment losses. In contrast, in a DC plan, the individual builds his/her own pension fund, selecting a fixed contribution rate and an investment strategy across assets, such as equities and bonds. Benefits are not fixed anymore, but the inherent risk is entirely borne by the individual. The target benefit plan (TBP therefore) is a new type of collective pension plan that blend elements of DB and DC plans to provide benefits at retirement, that are linked with how well the pension plan performs. The contributions are fixed in advance and the benefits must be selected. For it, the fund manager can invest the fund in a financial market. This pension plan can be provide better risk sharing for participants, adequate benefits and to maintain stability of the plan.

Some hybrid pension plans which combine the features of DB and DC pension plans are proposed, in addition to the target benefit plan (see CIA (2015), in Canada), it can be considered the risk-sharing DB plan in Japan (cf. Puch and Yermo (2008)) and the collective DC plans in the Netherlands (Kortleve (2013)). Wang *et al.* (2018) propose a continuous investment and intergenerational risk sharing model for Canadian target benefit pension plans. In their model setting, TBPs are collective pension schemes with fixed contributions, and the corresponding target benefit level is calculated according to a formula usually linked to the participants' annual salaries. At the same time, all the risks are shared among different generations of plan participants. Except TBPs, other investment problems for hybrid pension plans are investigated recently. The hybrid pension plan whose contribution and benefit levels are adjusted simultaneously is considered in Wang and Lu (2019).

Wang *et al.* (2021) consider a robust optimal problem for TBP with exponential function maximization of wealth and benefit excess or minimization of wealth and benefit gap.

Zhao and Wang (2022) analize the optimal investment and benefit problem where the manager maximizes a Coob-Douglas and Epstein-Zin recursive utility when the fund is invested in a financial market with one risk free bond and one stock. Josa-Fombellida and López-casado (2025) consider a TBP model where study the optimal investment and benefit strategies where the fund manager maximizes a CRRA instantaneous utility function of the benefit and a final utility of the terminal fund when the fund is invested in a financial market with one risk free bond and one stock with Poisson jumps. Roch (2022) considers a pay-as-you-go pension system where the aim of the fund manager is to minimize the deviations of benefit and fund with respect to its target levels in a financial market with a riskless asset and several risky assets. Previously, Haberman and Zimbidis (2002) considered a similar model but minimizing the deviation of contribution with respect to its target instead the deviation of fund with respect to its target.

This basic framework has already been explored by us with dynamic programming methods, Josa–Fombellida and Rincón–Zapatero (2001, 2004, 2008, 2010, 2012, 2019) and Josa–Fombellida and López-Casado (2023, 2025), and by many other authors, such as Battocchio *et al* (2007), Cairns (2000), Chang (1999), Chang *et al* (2003), Haberman and Sung (1994, 2005), Haberman *et al* (2000), Taylor (2002) or Zhao and Wang (2022). In some of them quadratic proferences have been considered, see Josa–Fombellida and Rincón–Zapatero (2001) and Haberman and Sung (1994).

In this paper, we are interested in minimize quadratic deviations between benefit and a target benefit (that is to say, the benefit risk) and between the terminal fund and its target value, in an aggregated pension plan of TBP type, where the risky assets are stochastic with uncertainty given by the Brownian motion.

The main contributions with respect to other papers are described as follows. 1) Closed-form expressions for the optimal investment and benefit are obtained. 2) A general deterministic exponencial contribution is considered. 3) Bounded and unbounded horizon are considered.

We find that the optimal investment and benefit are linear of the fund. Parameters affect the optimal solutions and the optimal fund evolution. It is possible to obtain optimality with infinite horizon when the contribution and the benefit proportions are constant.

The paper is organized as follows. Section 2 defines the elements of the pension plan, describes the financial market and shows the fund wealth evolution. Section 3 analyzes the management of the TBP plan as a stochastic optimal control problem with the aim of minimization of the quadratic deviation between benefit and an objective benefit along the planning interval and the quadratic deviation between the terminal fund and a goal terminal fund. In this section, in order to simplify the calculations, we consider that the fund wealth is invested in a portfolio with a single risky asset and a bond. The infinite horizon case is also considered. The optimal benefit and the optimal investment strategy are provided, by means of dynamic programming techniques, together with the optimal fund and some properties. Finally, Section 4 establishes some conclusions. All proofs are relegated to Appendix A.

2 THE PENSION MODEL

Consider an aggregated target benefit pension plan where, at every instant of time, active participants coexist with retired participants. The benefit is a control variable for the fund manager, that adjust the investment also. The main elements intervening in the TBP are the following.

T : Planning horizon or date of the end of the pension plan, with $0 < T < \infty$;

$F(t)$: value of fund assets at time t;

$P(t)$: benefits promised to the participants at time t; they are related with the salary at the moment of retirement;

$C(t)$: contribution of the fund wealth made by the manager at time t to the funding process; it is a deterministic function;

ρ : positive constant rate of discount or time preference of the manager;

r : constant risk-free market interest rate.

An interesting form for the contribution is: $C(t) = c_1 e^{c_2 t}$, with $c_1 > 0$. We consider three interesting particular cases along the paper. When $c_2 = 0$ the contribution is constantly indexed to the fund wealth: $C(t) = c_1$, with $c_1 > 0$. When $c_2 > 0$, we assume a salary growth that is materialized in the contribution, see Roch (2022). When $c_2 < 0$, the manager allow reduce the contribution proportion without fund increase, in order to make the pension plan more attractive to the participants, see Zhao and Wang (2022).

The main objective of the manager is to minimize the benefit risk and the deviations between the terminal fund and a goal terminal fund.

The fund wealth is invested in a financial market composed of a riskless asset and a risky asset. In order to include the variations of the market, the uncertainty is modeled by the Brownian motion.

2.1 The financial market

Following Josa-Fombellida and Rincón-Zapatero (2004), we suppose that the risky asset is a diffusion process where the uncertainty is given by a Brownian motion process. To model the pension game, we consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where $\mathbb{P}$ is a probability measure on Ω and $\mathscr{F} = \{\mathscr{F}_t\}_{t \geq 0}$ is a complete and right continuous filtration generated by the one dimensional standard Brownian motion w, that is to say, $\mathscr{F}_t = \sigma\{(w(s);\, 0 \leq s \leq t\}$, $t \geq 0$.

The plan sponsor manages the fund in an unbounded planning horizon by means of a portfolio formed by one risky asset S, which is a geometric Brownian motion (GBM henceforth, stochastic processes extending the deterministic exponential function), and a riskless asset or bond S^0 (its price is an exponential function), as proposed Merton (1971), that is, whose evolutions are given by the equations:

$$dS^0(t) = rS^0(t)dt, \quad S^0(0) = 1, \tag{1}$$

$$dS(t) = S(t)\Big(bdt + \sigma dw(t)\Big), \quad S(0) = s_1 > 0. \tag{2}$$

Here $r > 0$ denotes the short risk-free rate of interest, $b > 0$ the mean rate of return of the risky asset S, and $\sigma > 0$ the uncertainty parameter. It is usual to assume that $b > r$, so the manager has incentives to invest with risk.

We denote by $\theta = \frac{b-r}{\sigma}$ the Sharpe ratio or the market price of risk of the portfolio. We will suppose that the symmetric matrix $\Sigma = \sigma\sigma^\top$ is positive definite.

2.2 The fund wealth

In order to provided the promised benefits at the retirement, the fund manager adopts an amortization scheme and proceeds actively in the financial market to form suitable portfolios. In this risk sharing scheme the contributions are fixed and the benefit is a control variable. The fund wealth $F > 0$ is invested in the riskless asset S^0 and the risky asset S. Let π the proportion of fund to be invested in S, so that $1 - \pi$ is invested in S^0. Borrowing and shortselling are allowed. A negative value of π means that the sponsor sells a part of her/his risky asset S short while, if π is larger than 1, he or she then gets into debt to purchase the corresponding stock, borrowing money at the riskless interest rate r.

Under the investment/benefit policy chosen, the dynamics of the fund F is driven by

$$dF(t) = \pi(t)F(t)\frac{dS(t)}{S(t)} + (1-\pi(t))F(t)\frac{dS^0(t)}{S^0(t)} + (C(t) - P(t))dt, \tag{3}$$

with $F(0) = F_0 > 0$. By substituting (1) and (2) in (3), the dynamic fund wealth evolution under the investment policy π is

$$dF(t) = \Big(rF(t) + \pi(t)(b-r)F(t) + C(t) - P(t)\Big)dt + \pi(t)F(t)\sigma\, dw(t), \tag{4}$$

with the initial condition $F(0) = F_0$.

We assume admissible strategies, that is to say, strategies to fulfill some technical conditions. A strategic profile (P, π) is called admissible if the extra benefits strategy $\{P(t) : t \geq 0\}$ and the investment strategy $\{\pi(t) : t \geq 0\}$ are Markovian processes and stationary, $P = P(t, F)$ and $\pi = \pi(t, F)$, adapted to filtration $\{\mathscr{F}_t\}_{t\geq 0}$, and $P(t)$ and $\Pi(t)$ are $\mathscr{F}_t$-measurable, $\forall t > 0$, and such that they satisfy the integrability condition

$$\mathbb{E}\int_0^T P(t)dt + \mathbb{E}\int_0^T \pi(t)^2 dt < \infty. \tag{5}$$

Thus, the stochastic differential equation (SDE henceforth) (4) admits a unique solution for every initial condition $F(0) = F_0$. We assume $P(t) > 0$. We denote by $\mathcal{A}$ the set of admissible strategy profiles.

3 THE OPTIMAL STRATEGIES

In this section, we analyze how the manager selects the optimal benefit and investment strategies, when the deviation with respect to the target benefit is minimized. We model the sponsor's preferences as quadratic, penalizing deviations from prescribed targets, one target along the planning interval related with the benefit and another is the terminal fund target. These quadratic deviations are clearly related with the practical objectives of a target benefit plan, that is to say, to provide benefits that are adequate, to maintain stability and to respect intergenerational equity. See Wang *et al* (2018). As in the previous section, in the optimization process, the sponsor faces one element of randomness due to the financial market, specifically the risky assets.

The objective functional to be minimized along a bounded planning horizon over the class of admissible controls $\mathcal{A}$, is given by

$$J((t,F);(P,\pi)) = \mathbb{E}_{t,F}\Big\{\int_t^T e^{-\rho(s-t)}\,(P(t)-\tilde{P}(s))^2 ds + \beta e^{-\rho(T-t)}(F(T)-\tilde{F}(T))^2\Big\}, \tag{6}$$

where ρ is the time preference of the manager. As in Wang *et al.* (2018) and in Josa-Fombellida *et al.* (2023), $\beta > 0$ is a weighting factor indicating the importance of minimizing the final deviation. The aim is to minimize the expected quadratic deviation of the benefit P and the target benefit $\tilde{P}$ along the planning interval and of the terminal fund wealth $F(T)$ and the goal terminal fund $\tilde{F}(T)$. We assume $\tilde{P}$ is a deterministic positive function; for instance $\tilde{P}(t) = Be^{\delta t}$, where B and δ are positive constants, and δ can be related with the inflation and other variables. As it is indicated in Wang *et al* (2018), a reasonable goal value of the terminal fund is the initial fund value growing exponentially with the riskless interest rate, that is $\tilde{F}(T) = F_0 e^{rT}$. We are considering P and π as control variables. Here, $\mathcal{A}$ denotes the set of Markovian processes (P,π), adapted to the filtration $\{\mathscr{F}_t\}_{t\geq 0}$ where (P,π) satisfies (5), and where F satisfies (4). In the above, $\mathbb{E}_{t,F}$ denotes conditional expectation with respect to the initial condition (t,F).

The value function is defined as

$$\widehat{V}(t,F) = \min_{(P,\pi)\in\mathcal{A}} \Big\{J((t,F);(P,\pi)) : \text{ s.t. } (4) \text{ and } F(t)=F\Big\}.$$

It is clear that the value function so defined is non-negative and strictly convex. The connection between value functions and optimal feedback controls in stochastic control theory under Brownian setting is accomplished by the HJB; see Fleming–

Soner (2006). The following result characterizes the solution.

Theorem 3.1 *The optimal benefit and the optimal investment proportion in the risky assets are given by*

$$P^* = \frac{a_1(t)}{2} + a_2(t)F + \tilde{P}(t); \tag{7}$$

and

$$\pi^* = \frac{-\theta}{\sigma}\Big(\frac{a_1(t)}{2a_2(t)F} + 1\Big), \tag{8}$$

where a_1 and a_2 are given by

$$a_2(t) = \frac{(2r-\rho-\theta^2)e^{(2r-\rho-\theta^2)(T-t)}}{\beta^{-1}(2r-\rho-\theta^2) + 1 - e^{(2r-\rho-\theta^2)(T-t)}}, \tag{9}$$

$$\begin{aligned} a_1(t) =& -2\beta\tilde{F}(T)e^{(-\rho+r-\theta^2)(T-t)-\int_t^T a_2(s)ds} \\ & - 2e^{(-\rho+r-\theta^2)(T-t)-\int_t^T a_2(s)ds}\int_t^T (C(s)-\tilde{P}(s))a_2(s)e^{(-\rho+r-\theta^2)(T-s)+\int_s^T a_2(u)du}ds, \end{aligned} \tag{10}$$

and the optimal fund wealth is given by

$$\begin{aligned} dF^*(t) =& \Big(\big(r-a_2(t)\big)F^*(t) - \theta^2\Big(\frac{a_1(t)}{2a_2(t)} + F^*(t)\Big) - \frac{1}{2}a_1(t) + C(t) - \tilde{P}(t)\Big)dt \\ & - \theta\Big(\frac{a_1(t)}{2a_2(t)} + F^*(t)\Big)\,dw(t), \end{aligned} \tag{11}$$

with $F^*(0) = F_0 > 0$.

The optimal benefit P^* and the optimal investment $\pi^* F$ are linear functions of the fund assets F. Both strategies depend on the parameters of the financial market and the contribution rate. Depending on the jump parameters, shorselling or borrowing could be necessary.

The expected fund value is given by

$$\begin{aligned} \mathbb{E}F^*(t) =& e^{(r-\theta^2)t-\int_0^t a_2(s)ds} \\ & \times \Big(F_0 + \Big(\int_0^t \Big(-\frac{1}{2}\frac{\theta^2}{a_2(s)} + 1\Big)a_1(s) + C(s) - \tilde{P}(s)\Big)e^{-(r-\theta^2)s+\int_0^s a_2(u)du}ds\Big). \end{aligned}$$

Remark 3.1 (Exponential case) Assume $C(t) = c_1 e^{c_2 t}$, $\tilde{P}(t) = Be^{\delta t}$ and $\tilde{F}(t) = F_0 e^{rT}$. The optimal strategies are $P^* = \frac{a_1(t)}{2} + a_2(t)F + Be^{\delta t}$ and $\pi^* = \frac{-\theta}{\sigma}\Big(\frac{a_1(t)}{2a_2(t)F} + 1\Big)$, and the optimal expected fund is

$$\mathbb{E}F^*(t) = e^{(r-\theta^2)t - \int_0^t a_2(s)ds}$$

$$\times\Big(F_0 + \Big(\int_0^t \Big(-\frac{1}{2}\frac{\theta^2}{a_2(s)} + 1\Big)a_1(s) + c_1 e^{c_2 s} - Be^{\delta s}\Big)e^{-(r-\theta^2)s + \int_0^s a_2(u)du} ds\Big).$$

Remark 3.2 (Infinite horizon case) When $T = \infty$, we assume that both the contribution proportion C and the benefit proportion g are constants. The objective functional to be minimized is given by

$$J(F_0; (P, \pi)) = \mathbb{E}_{F_0}\Big\{\int_0^\infty e^{-\rho s}\,(P(s) - \tilde{P}(s))^2 ds\Big\}, \tag{12}$$

and the value function is time independent, $V = V(F)$. The optimal benefit is given by

$$P^* = (2(r + C) - (\rho + \theta^2 + g))F, \tag{13}$$

the optimal investment proportion in the risky asset π^* is the constant

$$\pi^* = \frac{-\theta}{\sigma} \tag{14}$$

and the optimal fund wealth is given by the GBM

$$dF^*(t) = \Big(-r - C + \rho + \theta^2 + g - \frac{\theta(b-r)}{\sigma}\Big)F^*(t)dt - \theta F^*(t)\,dw(t), \tag{15}$$

with $F^*(0) = F_0 > 0$. The optimal expected fund is given by $\mathbb{E}F^*(t) = F_0 e^{(\rho - r - C - g)t}$. In order to check the optimality, it is necessary to add the transversality condition

$$C + r > \frac{1}{2}(\rho + \theta^2) + g. \tag{16}$$

Note that (16) implies that the optimal benefit is positive: $2(r + C) > \rho + \theta^2 + 2g > \rho + \theta^2 + g$. See proof in Appendix A.

Remark 3.3 (Alternative model) Instead to consider the minimization of the risk benefit along the time interval and the devitation between the fund and the goal fund at the end of the plan, the manager can consider to minimize a convex combination of both risks along the time interval as follows:

$$\min_{P,\Pi} \mathbb{E}_{F_0} \int_0^T e^{-\rho s}\Big(\theta(P(s)-\tilde{P}(s))^2 + (1-\theta)(F(s)-\tilde{F}(s))^2\Big)ds,$$

where $\theta \in [0,1]$ is a weight parameter that measure the importance of each risk in the objective function. Thus $\theta = 0.5$ gives the same importance to both deviations, but $\theta > 0.5$ gives more importance to minimize the benefit risk. See Roch (2020).

4 CONCLUSIONS

We have analyzed, by means of dynamic programming techniques, the management of an aggregated target benefit pension plan, where the benefit and the risky asset are jump diffusion processes. The objective is to determine the benefit and the investment strategies minimizing both the benefit risk and the quadratic deviation between the terminal fund and a target fund. We have found that there is a linear relationship between the optimal supplementary cost and the optimal investment strategy, and between both the optimal benefit and the optimal investment strategy with the optimal fund.

The parameters of the model intervene in the optimal strategies and in the optimal fund evolution. The optimal benefit and the optimal investment are linear functions of the fund. The optimal fund is given by a solution of a linear non homogeneous SDE.

Further research can be addressed to consider that the contribution is a stochastic process given by a GBM. Also it can be interesting to suppose that the risky asset is a GBM with a Poisson jump as in Josa-Fombellida and López-Casado (2025).

A APPENDIX

Proof of Theorem 3.1. For the problem of Section 3, the HJB equation is

$$-\rho V + \min_{P,\pi} \Big\{ V_t + (P-\tilde{P})^2 + \big(rF + \pi(b-r)F + C - P\big)V_F + \frac{1}{2}\pi^2\sigma^2F^2V_{FF} \Big\} = 0. \tag{17}$$

If there is a smooth solution V of the equation (17), strictly convex, then the minimizer values of the benefit and the investment proportion are given by

$$2(P-\tilde{P}) - V_F = 0 \Rightarrow P = \tfrac{1}{2}V_F + \tilde{P} \tag{18}$$

$$(b-r)FV_F(x) + \sigma^2\pi F^2 V_{FF} = 0. \tag{19}$$

The structure of the HJB equation obtained, once we have substituted these values for P and Π in (17), suggests a quadratic function $V(t,F) = a_0(t) + a_1(t)F + a_2(t)F^2$, with a_0, a_1 and a_2 suitable functions. From (18), we get that the benefit P is explicitly found in terms of the fund F, $P = \frac{1}{2}a_1 + a_2F + \tilde{P}$, where the functions a_1, a_2 must be determined with the HJB equation. From (19), we get that the investment proportion π is given by (8). Plugging into the HJB equation (17), the following nonlinear differential equations for a_0, a_1 and a_2 are obtained

$$a_2' = (\rho + \theta^2 - 2r)a_2 + a_2^2, \qquad a_2(T) = \beta, \tag{20}$$

$$a_1' = (\rho - r + \theta^2 + a_2)a_1 - 2(C-\tilde{P})a_2, \qquad a_1(T) = -2\beta\tilde{F}(T), \tag{21}$$

$$a_0' = \rho a_0 + \frac{1}{4}a_1^2 + \frac{1}{4}\theta^2\frac{a_1}{a_2} - (C-\tilde{P})a_1, \qquad a_0(T) = \beta\tilde{F}(T)^2. \tag{22}$$

Equation (20) is a Bernoulli equation, that can be solved linearizing it with the transformation $v = a_2^{-1}$. In terms of v (20) is the linear differential equation of first order

$$v'(t) - (2r - \rho - \theta^2)v(t) = -1,$$

with the final condition $v(T)=\beta^{-1}$. The solution is

$$v(t)=e^{-(2r-\rho-\theta^2)(T-t)}\Big(\beta^{-1}-\int_t^T e^{(2r-\rho-\theta^2)(T-s)}ds\Big),$$

and then a_2 is given by (9). One known a_2, equation (21) is linear, and one known a_1, a_2, equation (21) is linear also. By solving the linear differential equations, the functions a_1 and a_0 are given, respectively, by (10) and

$$a_0(t)=e^{-\rho(T-t)}\Big(\beta\tilde{F}(T)^2+\int_t^T\Big(\frac{1}{4}a_1^2(s)\Big(-1+\frac{\theta^2}{a_2(s)}\Big)+C(s)-\tilde{P}(s)\Big)e^{\rho(T-s)}ds\Big)\Big).$$

Finally, by susbtituting in (4) we obtain (11). □

Proof of Remark 3.2. For the problem of Section 3, the HJB equation is

$$-\rho V+\min_{P,\pi}\big\{(P-gF)^2+\big(rF+\pi(b-r)F+CF-P\big)V_F$$
$$+\frac{1}{2}\pi^2\sigma^2F^2V_{FF}\Big\}=0. \tag{23}$$

If there is a smooth solution V of the equation (23), strictly convex, then the minimizer values of the benefit and the investment are given by

$$2(P-gF)-V_F=0\Rightarrow P=\tfrac{1}{2}V_F+gF, \tag{24}$$

$$(b-r)FV_F+\sigma^2F^2\pi V_{FF}=0. \tag{25}$$

The structure of the HJB equation obtained, once we have substituted these values for P and π in (23), suggests a quadratic function $V(F)=\epsilon F^2$, with ϵ suitable constant. From (24), we get that the benefit P is explicitly found in terms of the fund F, $P=(\epsilon+g)F$, where the constant ϵ must be determined with the HJB equation. From (25), we get that the investment π is the constant proportion of surplus that solves the algebraic equation (25), that is to say, the expression (14). Plugging into the HJB equation (23), the following algebraic equation for ϵ is obtained

$$\rho=\epsilon+2(r+C-\epsilon-g-\theta^2)+\theta^2,$$

that allows to obtain $\epsilon = -\rho + 2(r + C - g) - \theta^2$, and then P^* given by (13). By susbtituting in (11) we obtain that the evolution of the optimal fund wealth is given by (15).

In order to prove the optimality, is sufficient to check the transversality condition in infinite

$$\lim_{t\to\infty} e^{-\rho t}\mathbb{E}_{F_0} V(F^*(t)) = \epsilon \lim_{t\to\infty} e^{-\rho t}\mathbb{E}_{F_0} F^*(t)^2 = 0 \tag{26}$$

is checked. From (15) and by the Ito's formula (see Arnold (1974)), we obtain

$$\begin{aligned} dF^*(t)^2 =& 2\Big(- r - C + \rho + \theta^2 + g\Big) F^*(t)^2 dt \\ & - \frac{2\theta}{\sigma} F^*(t)^2\, dw(t), \end{aligned} \tag{27}$$

where $F^*(0) = F_0 > 0$. Then

$$\mathbb{E}_{F_0} F^*(t)^2 = F_0^2 \exp\left\{ 2\Big(- r - C + \rho + g + \frac{1}{2}\theta^2\Big) t \right\}.$$

It is immediate to check that the transversality condition is $\rho > \theta^2 - 2r - 2C + 2\rho + 2g$, that is to say (16). □

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