Stable Strategies in Repeated Games with Endogenous Separation

Segismundo S. Izquierdo*[a] and Luis R. Izquierdo[b]

[a] Department of Industrial Organization and BioEcoUva, Universidad de Valladolid, Spain. ORCID 0000-0002-7113-0633
[b] Department of Management Engineering, Universidad de Burgos, Spain. ORCID 0000-0003-1057-4465

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Abstract

Games with endogenous separation are repeated games where players have the option to leave their current partnership and keep on playing in a newly-formed partnership. Arguably, most repeated interactions in real life fall into this category. We present a general framework to analyze equilibria in games with endogenous separation, extending concepts from evolutionary game theory, and with a focus on neutrally stable strategies, i.e., stable strategies that are resistant to direct invasion by any alternative strategy. We introduce *path-protecting* strategies, which play a similar role to trigger strategies in standard (fixed-partnership) repeated games, and we provide a constructive proof of their existence. We also present a Folk Theorem for path-protecting strategies in these games. *JEL* classification numbers: C72, C73.

Keywords: Endogenous separation; conventions; neutral stability; path-protecting strategy; voluntarily repeated games

1 Introduction

Games with endogenous separation (Rob and Yang, 2010) are repeated games where players have the option to leave their current partnership and keep on playing in a newly-formed partnership with other players (Mailath and Samuelson, 2006, p. 152). In these games, partnerships may be broken for reasons that do not depend on the players' choices (exogenous separation), but also because players may decide to break

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*Correspondence to: EII, Universidad de Valladolid, Dr. Mergelina s/n, 47011 Valladolid, Spain. *e-mail*: segismundo.izquierdo@uva.es.

their partnership (endogenous separation).[1] The main feature of these games is that the strategies of a player's opponents are not fixed throughout time, because the opponents can change, with different strategy profiles presenting different expected durations.

As an example, consider a population of players who are randomly matched in pairs (partnerships) to play a Prisoner's Dilemma (the stage game) in discrete time. After each stage game, play may continue with the same partner with some probability, so that a partnership may last indefinitely, but may also be exogenously broken. In addition, any player who is unhappy with their partner's behavior can unilaterally break the partnership after each stage game, in which case both partners become single. Single players are then randomly matched again in a *pool of singles*, and continue playing in new partnerships. Choices at each stage (cooperate, defect, or leave) can be conditioned on the (potentially infinite) history of play within a partnership, so the set of possible strategies in the repeated game is infinite. Single players are anonymous: their past histories of play are lost (Schuessler, 1989; Ghosh and Ray, 1996).

At the level of bilateral interactions, games with endogenous separation can be interpreted as stochastic games (Solan and Vieille, 2015) where the state variable is given by the history of play within the current partnership, and transitions are driven both by players' actions (endogenous separation) and by exogenous breakup risk. The difficulty addressed in this paper does not lie in the within-partnership dynamics –which are Markovian– but in the aggregation of these interactions through a population-wide rematching process, which induces nonlinear payoff functions at the population level. The value of the option to leave in games with endogenous separation does not depend on the opponent's strategy, but on the distribution of strategies in the pool of singles.

Arguably, in most social and biological interactions in real life, individuals have the option to leave and change partners; however, the option to leave has not received much attention in the theory of repeated games. In the special case of the repeated Prisoner's Dilemma, it has been shown that the option to leave introduces fundamental changes in the strategies that can be sustained in equilibrium: while in the standard setting there are strategies, such as Grim trigger, which can sustain a fully cooperative equilibrium, in the repeated Prisoner's Dilemma with the option to leave –or *voluntarily repeated prisoner's Dilemma* (VRPD)– no strategy can sustain full cooperation (Mailath and Samuelson, 2006). The reason is that any population playing a *nice* strategy (Axelrod, 1984), which begins a new partnership cooperating, can be easily invaded by a defect-and-leave strategy, which begins defecting and then breaks the partnership, repeatedly meeting and exploiting new nice partners. Most of the literature on the option to leave has focused on variations of the VRPD[2], either considering the whole infinite set of potential strategies of the repeated game (Carmichael and MacLeod, 1997; Fujiwara-Greve and Okuno-Fujiwara, 2009) or focusing on some restricted subset of strategies (Vanberg and Congleton, 1992; Schuessler, 1989; Hayashi and Yamagishi, 1998; Izquierdo et al., 2010,

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[1] In most models with endogenous separation, one single player's decision to leave is sufficient to break the partnership, but other alternatives have also been considered (see e.g. Kurokawa (2022) or Křivan and Cressman (2020)).

[2] Two exceptions are Vesely and Yang (2010) and Izquierdo et al. (2021). See also Newton (2018) for a short summary of previous studies in the context of assortativity (Nax and Rigos, 2016).

2014; Zheng et al., 2017; Li and Lessard, 2021), sometimes in combination with a spatial setting (Aktipis, 2004; Premo and Brown, 2019). Even though full cooperation in the VRPD cannot be sustained in equilibrium, several studies indicate that the option to leave can increase average cooperation levels, both experimentally (Barclay and Raihani, 2016; Rand et al., 2011) and by simulation (Graser et al., 2025), and some authors have studied additional factors that may stabilize full cooperation, such as separation costs (Enquist and Leimar, 1993; Gutiérrez-Mielgo et al., 2025). A series of papers identify and discuss a special family of strategies that can constitute partially-cooperative equilibria in the VRPD: the family of *k-period trust-building* strategies (Datta, 1996; Ghosh and Ray, 1996; Carmichael and MacLeod, 1997; Kranton, 1996; Fujiwara-Greve and Okuno-Fujiwara, 2009). When a k-period trust-building strategy plays against itself, there is mutual defection (DD) for k periods and mutual cooperation (CC) from then on. If the partner of a trust-building strategy deviates from that pattern, the trust-building strategy breaks the partnership.

Here we present a general framework for the analysis of stable equilibria in symmetric two-player games with endogenous separation, with no restrictions on the set of strategies. To this end, we extend concepts from evolutionary game theory to games with endogenous separation. Our study contributes to explore the versatility of these tools, as well as their explanatory potential and their importance in the larger field of game theory.

Two useful concepts of stability in game theory are neutral stability and evolutionary stability. In standard two-player population games (with a finite strategy set and linear payoff functions), neutral stability implies Lyapunov stability under the replicator dynamics, and evolutionary stability ensures asymptotic stability (Taylor and Jonker, 1978; Thomas, 1985; Bomze and Weibull, 1995).

It is well known (Boyd and Lorberbaum, 1987) that in fixed-partnership infinitely repeated games there are no evolutionarily stable pure strategies (or mixed strategies with finite support). The same arguments can be used to show that there are no evolutionarily stable strategies in games with endogenous separation: given any equilibrium with finite support, there are other strategies that, when interacting with the equilibrium strategies, behave equivalently. Therefore, we focus on neutral stability.

Interestingly, extending the definition of neutral stability and the replicator dynamics to games with endogenous separation is not at all straightforward. Previous definitions of neutral stability for games with endogenous separation (Carmichael and MacLeod, 1997; Fujiwara-Greve and Okuno-Fujiwara, 2009; Izquierdo et al., 2021) can be considered unsatisfactory for different reasons (as discussed in section 3.4 and in appendix B). For instance, the definition by Fujiwara-Greve and Okuno-Fujiwara (2009) does not guarantee dynamic stability under the replicator dynamics.

The first challenge for games with endogenous separation consists in finding an appropriate characterization for the payoff functions $\mathbf{x} \mapsto F_i(\mathbf{x})$ which, based on the payoffs of the stage game and on the expected duration of each different partnership, provides the expected per-period payoff obtained by (a player using) strategy i of the repeated game when playing in a population whose strategy distribution is $\mathbf{x}$ (where $\mathbf{x}$ indicates which strategies are being used in the population and in which proportions). Note that single players are re-matched within a *pool of singles*, whose distribution of strategies, in general, is different from the distribution of strategies $\mathbf{x}$ in the whole population. This creates several difficulties when trying to calculate the payoff functions $\mathbf{x} \mapsto F_i(\mathbf{x})$. Alternatively, instead of using as a primitive the distribution of strategies in the population, some papers (most notably Fujiwara-Greve and Okuno-Fujiwara (2009)) are based on the distribution of strategies in the pool of singles, which leads to more manageable formulas, but this approach creates limitations when trying to extend evolutionary dynamics and stability concepts. Here we derive a relation between the distribution of strategies in the pool of singles and in the whole population that allows us to obtain a series of properties of the payoff functions (section 2.3) on which we base our results. Among these properties, Lipschitz continuity of the payoff functions allows us to extend the replicator dynamics (Taylor and Jonker, 1978) to games with endogenous separation.

The second difficulty stems from the fact that there are different definitions of neutral stability (Bomze and Weibull, 1995), which are equivalent if payoff functions are linear –as usually assumed for two-player games– but which are not equivalent when payoff functions are not linear –as it is the case in games with endogenous separation. With non-linear payoff functions, some definitions of neutral stability do not ensure Lyapunov stability in the replicator dynamics (Bomze and Weibull, 1995). Originally (Maynard Smith, 1982), a neutrally stable strategy was defined as a strategy q which (i) is a best response to itself, and (ii) if p is a (pure or mixed) alternative best response to q, then q against p obtains at least as much (expected payoff) as p against itself. In this paper we propose a definition of neutral stability for games with endogenous separation that is a direct adaptation for population games of the original concept, and which implies Lyapunov stability under the (extended) replicator dynamics. Neutrally stable strategies thus defined can be seen as potential *social conventions* (Hawkins et al., 2019) or *social customs* (Carmichael and MacLeod, 1997) in settings with the option to leave, i.e., behavioral regularities that serve as stable –but to some degree arbitrary– solutions to repeated coordination problems in a society.

After extending the concept of neutral stability to games with endogenous separation, we study necessary and sufficient conditions for strategies to be neutrally stable. It turns out that, in most games, a necessary condition for a strategy to be neutrally stable is that it never breaks up with itself (i.e., with a partner using the same strategy). We also identify a sufficient condition for neutral stability: *path-protection*. A *path-protecting* strategy never leaves a partner who mimics its behavior and, if adopted by all the players in a population, it guarantees that any player who deviates from the equilibrium path obtains a strictly lower payoff than the population's average.

Our main results are:

- A derivation of properties of payoff functions $\mathbf{x} \mapsto F_i(\mathbf{x})$ for games with endogenous separation. For more than three strategies these payoff functions do not admit a closed-form algebraic expression, but we can still derive several properties, including Lipschitz continuity.

- An extension of the concept of neutral stability to games with endogenous separation, and an extension of evolutionary replicator dynamics to these games. Our definition of neutral stability implies Lyapunov stability in the extended replicator dynamics.
- A definition and characterization of path-protecting strategies. These strategies guarantee that deviations from the equilibrium path are harmful. We prove that path-protecting strategies are neutrally stable and provide a constructive proof of existence. They constitute natural candidates to be adopted as stable equilibrium behavior (a social convention) in a population.
- A *Folk Theorem* for path-protecting strategies in repeated games with endogenous separation. This result establishes that, for large enough values of the (exogenous) continuation probability, any payoff between the pure minmax payoff and the maximum symmetric payoff of the stage game can be approximated arbitrarily closely as the equilibrium payoff of some path-protecting (neutrally stable) strategy.

Path-protecting strategies, which generalize the idea behind *trust-building* strategies, present some similarities and some differences with the classical *trigger* strategies which, in standard repeated games, prevent deviations from an equilibrium path by playing a minmax action after a deviation. In standard repeated games, trigger strategies protect a path by the threat of punishment, but such potential punishment does not materialize on the equilibrium path. In contrast, in games with endogenous separation, a player who deviates from the equilibrium path can avoid subsequent punishment from their current partners by breaking up the partnership. Therefore, any equilibrium path in a population sustaining repeated play of some non-Nash action profile of the stage game (such as mutual cooperation in the VRPD) needs to ensure that players who start new partnerships bear some initial cost, to prevent *reap-and-leave* behavior (section 3.2). In our setting, this cost can only take place through a painful *deviation-deterring* phase at the beginning of every new partnership. Furthermore, since there is no information flow between partnerships, every player must go through this initial deviation-deterring phase, so this unpleasant experience necessarily becomes part of the equilibrium path.

The rest of the paper is structured as follows. In section 2 we define games with endogenous separation derived from normal-form stage games, and we present their main elements: strategies, population states, pool states and payoff functions. In section 3 we provide definitions for Nash, evolutionarily stable and neutrally stable states in this framework. We discuss Nash states and the non-existence of evolutionarily stable strategies; we then focus on neutral stability. Having defined the payoff function for strategies and for distributions of strategies, it becomes natural to adapt a standard definition of neutral stability (Banerjee and Weibull, 2000) to repeated games with endogenous separation. We then show that neutral stability thus defined implies Lyapunov stability in the extended replicator dynamics for games with endogenous separation. Section 4 introduces *path-protecting* strategies, and shows how these strategies can be created, giving rise –for sufficiently high exogenous continuation probabilities– to monomorphic neutrally stable states. Here we also provide a *Folk Theorem* for path-protecting strategies in games with endogenous separation. Section 5 discusses subgame perfect strategies in games with endogenous separation, and shows that path-protecting strategies that leave after every history not occurring on the equilibrium path (*deviation-leaving* strategies) are subgame perfect. Section 6 presents the conditions that a deviation-leaving strategy needs to satisfy to be Nash (and subgame perfect) or path-protecting. Finally, in section 7 we present some conclusions.

The paper includes four appendices with proofs and additional results. Appendix A considers polymorphic neutrally stable states (mixtures of strategies). Here we show a strong limitation to the existence of polymorphic equilibria made up by different path-protecting strategies, and we extend the concept of path-protecting strategy to path-protecting state. Appendix B discusses some previous definitions of neutral stability that have been proposed for games with endogenous separation, and their limitations. Appendix C studies robustness against indirect invasions (van Veelen, 2012) in games with endogenous separation. Finally, appendix D contains most of the proofs.

2 Repeated games with endogenous separation

In this section we present repeated games with endogenous separation derived from normal-form stage games. We limit the presentation and the analysis to symmetric two-player stage games. Extensions to asymmetric or multi-population games present additional challenges and are left for future work.

We consider a unit-mass population of agents who are matched in couples or *partnerships* to play a symmetric two-player normal-form stage game. The stage game $G = \{A, U\}$ is defined by an action set $A = \{a_1, ..., a_n\}$, and a payoff function $U\colon A^2 \to \mathbb{R}$, where $U(a_k, a_l)$ represents the payoff obtained by a player using action a_k whose opponent plays action a_l. Every stage game G has an associated repeated game with endogenous separation G^{Ends}, which is characterized in this section. Following Mailath and Samuelson (2006), we refer to choices in the stage game G as *actions*, reserving *strategy* for behavior in the repeated game G^{Ends}.

2.1 Strategies

After playing a stage game G, partnerships may remain together and play the stage game again. A partnership is broken if either one of the players, according to their strategy, decides to break it (endogenous separation) or if some exogenous factor breaks the partnership, which happens with probability $(1-\delta) \in (0, 1)$ after every interaction (exogenous separation). Thus, δ is the continuation probability of the partnership assuming that both players decide to stay. At the beginning of every (discrete) time period, all single players are randomly (re-)matched in partnerships, and then all players play the stage game, i.e., every player plays the stage game at every period, either in newly-formed partnerships or in older ones. We assume that there is no information flow between partnerships (Ghosh and Ray, 1996), so there are no reputation effects: single players (those who make up new partnerships) are anonymous.[3]

Considering the sequence of action profiles taken in a partnership, let the stage-t game, with $t \in \{1, 2, ...\}$, be the t^{th} time that the stage game is played in that partnership, assuming the partnership has not been broken before. A strategy i for a player determines the choice that the player makes given any past history of play within a partnership. If the strategies followed by the two players in a partnership are i and j, the action profile played at stage t (assuming the partnership survives to play for the t^{th} time together), is $a_{ij}^{[t]} \equiv (a_i^{[t]}, a_j^{[t]}) \in A^2$, where $a_i^{[t]}$ is the action played by the player using strategy i (at stage t) and $a_j^{[t]}$ is the action played by the player using strategy j.

A *history of play of length* $t \geq 1$, $a^{[1,t]} = (a^{[1]}, ..., a^{[t]}) \in (A^2)^t$, is a sequence of t action profiles.[4] Let $a^{[1,0]}$ denote the empty history, or history of length 0. Let $\mathcal{H}$ be the set of all possible histories of any length (including the empty history) and let $\tilde{A} \equiv A \cup \{\textit{break}\}$ be the set of choices, where *break* represents the decision to break the current partnership. A strategy i for the repeated game is a mapping $i : \mathcal{H} \to \tilde{A}$, from the set of possible histories to the set of choices, that prescribes one choice $i(a^{[1,t]}) \in \tilde{A}$ for every possible history $a^{[1,t]}$, for every $t \geq 0$. As players in a new partnership are assumed to play the stage game at least once before deciding whether to break their partnership, we require $i(\emptyset) \in A$. Let Ω be the set of strategies.

Note that we assume $0 < \delta < 1$. The process for $\delta = 0$, where every partnership is exogenously broken after every stage game, would correspond to the standard framework for evolutionary non-repeated population games.

2.2 States and payoffs

We consider populations where the number of different strategies being played at any time is finite. Let x_i be the fraction of the population using strategy $i \in \Omega$. A (population) state $\mathbf{x}$ is a strategy distribution over Ω with finite support $\mathbb{S}(\mathbf{x}) \subset \Omega$, i.e., $\mathbf{x}$ is a function from Ω to $[0, 1]$ that:

i) assigns a positive value $x_i > 0$ to each strategy i in a finite set $\mathbb{S}(\mathbf{x})$,

ii) assigns the value 0 to strategies that are not in $\mathbb{S}(\mathbf{x})$, and

iii) satisfies $\sum_{i \in \mathbb{S}(\mathbf{x})} x_i = 1$.

Let $\mathbb{D}$ be the set of distributions with finite support, and let $\mathbf{e}_i$ represent the monomorphic state at which all players use strategy i (i.e., the distribution satisfying $x_i = 1$ and $x_j = 0$ for every $j \in \Omega \setminus \{i\}$).

Consider an index $\mathcal{T}$ for periods of play of the game in the population. At every period, single players are matched and every player plays a stage game. In contrast, index t refers to repetitions of the stage game within a partnership: at period $\mathcal{T}$, after

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[3] Fujiwara-Greve et al. (2012) consider a model where players may voluntarily provide information across partnerships in the context of the Prisoner's Dilemma.

[4] $a^{[1,t]}$ represents some sequence of t action profiles, while $a_{ij}^{[1,t]}$ represents the first t action profiles generated by strategy i when playing against strategy j, assuming they do not break up before stage t.

matching and before playing the stage game, every partnership has its own value for t, which, if the partnership has just been matched at that period, is set to 0 before playing the stage game and becomes 1 after playing the stage game. For any pair of strategies i and j, let their endogenous breakup period $T_{ij} \geq 1$ be the number of stages that an i-j partnership is to play together if the partnership is not broken by exogenous factors (i.e., the number of stage games they play together before one of them decides to break up). If an i-j partnership never breaks up endogenously, let $T_{ij} = \infty$.

To calculate the average (per player and per period) payoff $F_i(\mathbf{x})$ obtained by a player using strategy i when the population state is $\mathbf{x}$, we consider a stationary strategy distribution $\mathbf{p}$ in the pool of singles consistent with the population state $\mathbf{x}$. If the strategy distribution $\mathbf{p}$ in the pool of singles is stationary, then it should satisfy the following:

- Before matching, the mass of players in the pool of singles is a stationary value ϕ. The mass of single i-players in the pool is $\phi \, p_i$.

- After matching, the mass of i-players just matched to j-players, i.e., the mass of i-players in newly-formed (0-period-old) i-j partnerships, is $\phi \, p_i \, p_j$.

- For $1 \leq t \leq T_{ij}$, the mass of i-players in $(t-1)$-period-old i-j partnerships (after matching and before playing), is $\phi \, p_i \, p_j \, \delta^{t-1}$. These are the i-players that were matched in i-j partnerships $(t-1)$ periods ago and have survived exogenous (and endogenous) separation to play their t^{th} stage game in the current period $\mathcal{T}$. The total mass or fraction of i-players in the population is then

  $$x_i = \phi \sum_{j \in \mathbb{S}(\mathbf{x})} p_i \, p_j \sum_{t=1}^{T_{ij}} \delta^{t-1} = \phi \sum_{j \in \mathbb{S}(\mathbf{x})} p_i \, p_j \, \frac{1-\delta^{T_{ij}}}{1-\delta}$$

  and considering that $\sum_{j \in \mathbb{S}(\mathbf{x})} x_j = 1$, we have

  $$x_i = \frac{p_i \sum_{j \in \mathbb{S}(\mathbf{x})} p_j (1-\delta^{T_{ij}})}{\sum_{k,j \in \mathbb{S}(\mathbf{x})} p_k \, p_j (1-\delta^{T_{kj}})}. \tag{1}$$

  Technically, in eq. (1) we are assuming that the pool distribution has been stationary for at least as many periods as the longevity of the oldest partnership in the population.

  Equation (1) defines a function $f : \mathbb{D} \to \mathbb{D}$ such that $\mathbf{x} = f(\mathbf{p})$, which provides the population state $\mathbf{x}$ corresponding to pool state $\mathbf{p}$.

- Let $a_{ij}^{[t]} = (a_i^{[t]}, a_j^{[t]}) \in A^2$ be the action profile played at the t^{th} stage of an i-j partnership, with the first action in the profile corresponding to the player using strategy i and the second action in the profile corresponding to the player using strategy j. The total payoff obtained (at each and every period $\mathcal{T}$) by the mass of i-players is

$$\phi \sum_{j \in \mathbb{S}(\mathbf{x})} p_i \, p_j \sum_{t=1}^{T_{ij}} \delta^{t-1} \, U(a_{ij}^{[t]}),$$

so, dividing by the mass of i-players, we have that the per-period per-player average payoff to an i-player is

$$\hat{F}_i(\mathbf{p}) \equiv (1-\delta) \, \frac{\sum_{j \in \mathbb{S}(\mathbf{p})} p_j \, \sum_{t=1}^{T_{ij}} \delta^{t-1} \, U(a_{ij}^{[t]})}{\sum_{j \in \mathbb{S}(\mathbf{p})} p_j \, (1-\delta^{T_{ij}})}, \tag{2}$$

which is defined for every $i \in \Omega$.

From (2) we have a formula for $\hat{F}_i(\mathbf{p})$ that provides the payoff to strategy i corresponding to pool state $\mathbf{p}$, and from (1) we have a formula $\mathbf{x} = f(\mathbf{p})$, that provides the population state $\mathbf{x}$ corresponding to pool state $\mathbf{p}$. In order to use existing results and concepts from the literature in population games, it would be convenient to have payoff functions F_i that provide the payoff to strategy i corresponding to population state $\mathbf{x}$, i.e., $F_i(\mathbf{x})$. Considering $\mathbf{x} = f(\mathbf{p})$ as defined in (1), our first proposition shows that there is an inverse function f^{-1} such that $\mathbf{p} = f^{-1}(\mathbf{x})$, so we can define payoff functions F_i from population states as

$$F_i(\mathbf{x}) = \hat{F}_i(f^{-1}(\mathbf{x})). \tag{3}$$

Proposition 1 shows that f^{-1} is closely related to a symmetric matrix scaling problem (Idel, 2016; Brualdi, 1974). The symbol $\circ$ in proposition 1 denotes the Hadamard (entrywise) product.

Proposition 1. *Given a population state $\mathbf{x} \in \mathbb{D}$ with support S, there exists a unique pool state $\mathbf{p} \in \Delta(S)$ such that equation (1) holds. Let B be the symmetric positive matrix with entries $(B_{ij})_{i,j \in S} = (1-\delta^{T_{ij}})$. Then*

$$\mathbf{p} = \frac{\tilde{\boldsymbol{p}}}{\|\tilde{\boldsymbol{p}}\|_1},$$

where the vector $\tilde{\mathbf{p}}$ is the unique positive solution to the matrix scaling problem

$$\tilde{\boldsymbol{p}} \circ (B\tilde{\boldsymbol{p}}) = \mathbf{x}.$$

Proposition 2 shows that f^{-1} is Lipschitz. From this we can show that the payoff functions are Lipschitz, which will allow us to define replicator dynamics for games with endogenous separation.

Proposition 2. *Given a finite set of strategies S, the function $f : \Delta(S) \to \Delta(S)$ defined by equation (1), which relates the pool and population states with support in S via $\mathbf{x} = f(\mathbf{p})$, is a bi-Lipschitz homeomorphism. For more than three strategies, the inverse mapping f^{-1} does not admit a general algebraic expression.*

Finally, for a group of players with strategy distribution $\mathbf{y} \in \mathbb{D}$ entering a population with strategy distribution $\mathbf{x}$, we define the average *payoff of* $\mathbf{y}$ *against* $\mathbf{x}$, $E(\mathbf{y}, \mathbf{x})$, as:

$$E(\mathbf{y}, \mathbf{x}) \equiv \sum_{i \in \mathbb{S}(\mathbf{y})} y_i F_i(\mathbf{x}). \tag{4}$$

We can interpret this payoff as the average payoff obtained by a very small mass of players whose strategy distribution is $\mathbf{y}$ (sometimes called *mutants* or *entrants*) when they play in a population of players whose strategy distribution is $\mathbf{x}$.

Our results are based on a series of properties of the payoff functions $F_i(\mathbf{x})$ that we indicate in the following section.

2.3 Properties of the payoff functions

The payoff functions $F_i : \mathbb{D} \to \mathbb{R}$, defined in (3) for every $i \in \Omega$, satisfy the following properties:

- At monomorphic population states (where $\mathbf{x} = \mathbf{e}_j = \mathbf{p}$) we have, from (2):

  $$F_{ij} \equiv F_i(\mathbf{e}_j) = \frac{1-\delta}{1-\delta^{T_{ij}}} \sum_{t=1}^{T_{ij}} \delta^{t-1} U(a_{ij}^{[t]}). \tag{5}$$

  Note that the payoff F_{ij} to an i-player in a population of j-players is a convex combination of the stage payoffs $U(a_{ij}^{[t]})$ for $1 \leq t \leq T_{ij}$.

- It follows from (2), (3) and (5) that, for $\mathbf{p} = f^{-1}(\mathbf{x})$, we have

  $$F_i(\mathbf{x}) = \hat{F}_i(\mathbf{p}) = \sum_{j \in \mathbb{S}(\mathbf{x})} \frac{p_j\,(1-\delta^{T_{ij}})}{\sum_{k \in \mathbb{S}(\mathbf{x})} p_k\,(1-\delta^{T_{ik}})} F_{ij}, \tag{6}$$

  which shows that $F_i(\mathbf{x})$ is a convex combination of the payoffs F_{ij} for $j \in \mathbb{S}(\mathbf{x})$, with (strictly) positive coefficients for the convex combination.

- Given any finite set of strategies S, the payoff functions are Lipschitz continuous on $\mathbf{x} \in \Delta(S)$. This follows from (2), (3) and proposition 2.

Let the *path* $a_{ij}^{[1,T_{ij}]} = ((a_i^{[1]}, a_j^{[1]}), (a_i^{[2]}, a_j^{[2]}), ..., (a_i^{[T_{ij}]}, a_j^{[T_{ij}]}))$ be the series of T_{ij} action profiles that strategy i generates when playing with strategy j until they decide to break up. Let the *repeated path* $h_{ij}^{[\infty]}$ be the infinite series of action profiles that corresponds to (or is generated by) one i-player in a population of j-players, with no exogenous separation and with re-matching after each endogenous separation:

$$h_{ij}^{[\infty]} \equiv (a_{ij}^{[1,T_{ij}]}, a_{ij}^{[1,T_{ij}]}, ...). \tag{7}$$

For a sequence of T action profiles $a^{[1,T]}$, where the t^{th} action profile in the sequence is $a^{[t]} \in A^2$, let the *normalized discounted value* $V(a^{[1,T]})$ be

$$V(a^{[1,T]}) \equiv \frac{1-\delta}{1-\delta^T}\sum_{t=1}^{T}\delta^{t-1}U(a^{[t]}).$$

From the previous definitions and the properties of geometric series, we have:

$$F_{ij} = V\left(a_{ij}^{[1,T_{ij}]}\right) = V\left(h_{ij}^{[\infty]}\right) = (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}U(h_{ij}^{[t]}), \tag{8}$$

where $h_{ij}^{[t]}$ is the t^{th} action profile in $h_{ij}^{[\infty]}$. Formula (8) shows that F_{ij} coincides with $V(h_{ij}^{[\infty]})$, which is the normalized discounted value of the infinite sequence of action profiles in the repeated path $h_{ij}^{[\infty]}$.

Note that in the framework we have presented for games with endogenous separation there is no discounting, and F_{ij} is defined as a per-period per-player average payoff (averaged over individuals whose prevalence in t-period-old partnerships is proportional to δ^t). However, the definition of repeated path in (7) allows to establish an equivalence between F_{ij} and the normalized discounted value $V(h_{ij}^{[\infty]})$. It follows from this equivalence that any two strategies j_1 and j_2 that generate the same repeated path against i-players obtain the same payoff against i-players, even if they have different breakup periods, i.e., even if they have different paths (as long as these paths, when repeated, generate the same sequence), i.e.:

$$h_{j_1 i}^{[\infty]} = h_{j_2 i}^{[\infty]} \implies F_{j_1 i} = F_{j_2 i}. \tag{9}$$

3 Equilibria in games with endogenous separation: Nash, evolutionarily stable and neutrally stable states

In this section we adapt standard definitions of Nash state, evolutionarily stable state and neutrally stable state to games with endogenous separation. For completeness, and in order to introduce the notation, we begin with the definitions for the stage game G.

3.1 Definitions for the stage game

Here we present the main definitions and concepts for a stage game G that will be useful for the analysis of the repeated game with endogenous separation G^{Ends} derived from G.

The *best-response payoff* to action a is the best payoff that an action can obtain when playing against a, defined by

$$U^{BR}(a) \equiv \max_{a_l \in A} U(a_l, a).$$

The set of *best-response actions* to action a, $BR(a)$, is the set of actions that obtain the best-response payoff against a. If $a \in BR(a)$, i.e., if action a is a best-response to itself, we say that:

- (a, a) is a (symmetric) *Nash profile.*
- a is a *Nash action.*

The *pure minmax payoff* of G, m, is the minimum of the best-response payoffs to actions in A:

$$m \equiv \min_{a \in A} U^{BR}(a).$$

Every best-response payoff to an action is greater than or equal to m, i.e., $U^{BR}(a) \geq m \; \forall a \in A$. A *minmax action* $\tilde{a} \in A$ is an action such that $U^{BR}(\tilde{a}) = m$. By choosing a minmax action, a player can guarantee that her opponent's payoff does not exceed m.

Let $q \in \Delta(A) \equiv \{(q_k)_{k=1}^n \in \mathbb{R}_+^n : \sum_{k=1}^n q_k = 1\}$ be a distribution over actions or *mixture* of actions. The payoff of action a against q is defined by the linear function $U_a(q) \equiv \sum_{l=1}^n U(a, a_l) q_l$. With some abuse in notation, the payoff of mixture $p \in \Delta(A)$ against $q \in \Delta(A)$ is defined by

$$U(p, q) \equiv \sum_{k=1}^n p_k \, U_{a_k}(q) = \sum_{k,l} p_k \, q_l \, U(a_k, a_l)$$

The best-response payoff against q is defined by

$$U^{BR}(q) \equiv \max_{p \in \Delta(A)} U(p, q) = \max_{a \in A} U_a(q).$$

The set of best-response actions to q, $BR(q)$, is the set of actions that obtain the best-response payoff against q.

A (symmetric) Nash equilibrium of G is a distribution $q \in \Delta(A)$ such that

$$U(q, q) = U^{BR}(q).$$

The (mixed) *minmax payoff* $\underline{m}$ of G is the minimum of the best-response payoffs to mixtures in $\Delta(A)$:

$$\underline{m} \equiv \min_{q \in \Delta(A)} \max_{a \in A} U_a(q).$$

Every best-response payoff (to some mixture) is greater than or equal to $\underline{m}$: $U^{BR}(q) \geq \underline{m}$, i.e., independently of q, if a is a best response to q, then the payoff of a against q is at least $\underline{m}$. It follows from the definitions that $\underline{m} \leq m$.

A distribution over actions $q \in \Delta(A)$ is *evolutionarily stable* (Maynard Smith and Price, 1973) if, for every other distribution $p \in \Delta(A)$:

$$U(q, q) \geq U(p, q), \text{ i.e., } q \text{ is Nash, and}$$
$$U(p, q) = U(q, q) \implies U(q, p) > U(p, p).$$

An alternative definition of evolutionary stability only requires the condition $U(q, p) > U(p, p)$ to hold locally, i.e., in some punctured relative neighborhood of q. Evolutionary stability implies asymptotic stability under the replicator dynamics (Sandholm, 2010).

There are several definitions of neutral stability (Maynard Smith, 1982) that are equivalent in this setting of linear payoff functions (Bomze and Weibull, 1995). Here we adopt the following one: a distribution over actions $q \in \Delta(A)$ is *neutrally stable* if, for every distribution $p \in \Delta(A)$:

$$U(q,q) \geq U(p,q), \text{ i.e., } q \text{ is Nash, and}$$
$$U(p,q) = U(q,q) \implies U(q,p) \geq U(p,p).$$

Neutral stability requires that q is Nash and that it is robust to the introduction of (any combination of) alternative best responses to q, in the sense that q will not do worse than the average ($U(q,p) \geq U(p,p)$) when such alternative best responses are introduced. With the previous definition, neutral stability implies Lyapunov stability under the replicator dynamics (Thomas, 1985; Bomze and Weibull, 1995).

3.2 Nash states in the repeated game

A strategy j is a best response to state $\mathbf{x}$ if, when playing against $\mathbf{x}$, no other strategy (or distribution) can obtain a payoff greater than j's payoff, i.e., if and only if $F_j(\mathbf{x}) \geq F_k(\mathbf{x})$ for every $k \in \Omega$. Let $BR(\mathbf{x})$ be the set of best-response strategies to $\mathbf{x}$. A strategy distribution $\mathbf{y} \in \mathbb{D}$ is a best response to state $\mathbf{x}$ if and only if $E(\mathbf{y},\mathbf{x}) \geq E(\mathbf{z},\mathbf{x})$ for every $\mathbf{z} \in \mathbb{D}$. It follows from (4) that $\mathbf{y}$ is a best response to $\mathbf{x}$ if and only if every strategy in its support $\mathbb{S}(\mathbf{y})$ is a best response to $\mathbf{x}$.

Definition 1 (Nash equilibrium state)**.** *A state $\mathbf{x} \in \mathbb{D}$ is Nash (short for Nash equilibrium state) if $E(\mathbf{x},\mathbf{x}) \geq F_j(\mathbf{x})$ for every $j \in \Omega$. Equivalently, a state $\mathbf{x} \in \mathbb{D}$ is Nash if it is a best response to itself.*

If a monomorphic state $\mathbf{e}_i$ is Nash, we say that strategy i is a Nash strategy. Consequently, a strategy i is Nash if and only if $F_{ii} \geq F_{ji}$ for every $j \in \Omega$.

Let us now consider some implications of being a Nash strategy. The action profiles played at a monomorphic population $\mathbf{e}_i$ are always symmetric[5], i.e. in the set $\{(a,a)\}_{a \in A}$. Consequently, the payoff F_{ii} in a monomorphic population (see equation (5)) is a convex combination of the payoffs $\{U(a,a)\}_{a \in A}$ corresponding to the main diagonal of the payoff matrix of the stage game G. This implies that the maximum symmetric stage-game payoff $M \equiv \max_{a \in A} U(a,a)$ is an upper bound for F_{ii}.

If i is a Nash strategy, it cannot be beaten by any other strategy in its corresponding monomorphic population $\mathbf{e}_i$; in particular, strategy i cannot be beaten by what we call *reap-and-leave* strategies. Reap-and-leave strategies are those which, in a partnership with i, play exactly as i up to stage $T \leq T_{ii}$, at stage T adopt a best-response action to the action chosen by i, and then break the partnership. We say that such strategies reap-and-leave i at stage T.

The fact that being Nash implies robustness against reap-and-leave strategies allows us to derive simple conditions that must be satisfied by Nash strategies and Nash states

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[5] In the symmetric setting that we consider, it is assumed that there is no role asymmetry (like row-player and column-player) on which players could condition their actions.

in general. The next two propositions are based on robustness against strategies that reap-and-leave i at the first stage of an i-j partnership, while the third proposition considers robustness against a strategy that reaps-and-leaves i at stage T_{ii}.

Lemma 3.1. *The first action $a^{\emptyset}$ played by a Nash strategy in G^{Ends} must satisfy*

$$U^{BR}(a^{\emptyset}) \leq M,$$

where $U^{BR}(a^{\emptyset})$ is the best-response stage payoff to action $a^{\emptyset}$ and $M = \max_{a \in A} U(a, a)$ is the maximum symmetric stage-game payoff.

To illustrate some practical applications of each result, we will consider the Prisoner's Dilemma and the Hawk-Dove game (also known as Snowdrift), with actions C and D (table 1). In the Prisoner's Dilemma, C stands for cooperate and D for defect; in the Hawk-Dove game, C corresponds to Dove and D to Hawk. In both cases, coordinating on C is more efficient than on D (i.e., the maximum diagonal stage payoff is $M = U_{CC} > U_{DD}$), and D is the minmax action. For the examples, we use the simpler notation $U_{a_k a_l} \equiv U(a_k, a_l)$, and an action profile such as (D, D) is represented as DD.

In the Prisoner's Dilemma ($U_{CD} < U_{DD} < U_{CC} < U_{DC}$), D is a dominant action and DD is a Nash action profile. In the Hawk-Dove ($U_{DD} < U_{CD} < U_{CC} < U_{DC}$), the best-response to each action is the other action (this is an anti-coordination game). We will also consider the so-called 1-2-3 coordination game (table 1).

$$\begin{array}{c} \\ C \\ D \end{array} \!\!\begin{array}{c} \begin{array}{cc} C & D \end{array} \\ \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ C \\ D \end{array} \!\!\begin{array}{c} \begin{array}{cc} C & D \end{array} \\ \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \end{array}$$

Table 1: Left: A Prisoner's Dilemma game, with C for Cooperate and D for Defect. Middle: A Hawk-Dove game, with C for Dove and D for Hawk. Right: the 1-2-3 coordination game.

Example 1. *In the Prisoner's Dilemma, the only action that satisfies the condition in lemma 3.1 is action D. Consequently, every Nash strategy must begin a partnership by playing action D: no Nash strategy can be "nice" (Axelrod, 1984). This rules out strategies such as Tit for Tat.*

Similarly, for the Hawk-Dove, lemma 3.1 implies that every Nash strategy must begin a partnership by playing D (Hawk).

Lemma 3.2. *The minmax payoff $\underline{m}$ of a stage game G is a lower bound for the payoff at Nash states of G^{Ends}:*

$$\mathbf{x} \in \mathbb{D} \textit{ is Nash } \implies E(\mathbf{x}, \mathbf{x}) \geq \underline{m}.$$

The pure minmax payoff m of a stage game G is a lower bound for the payoff F_{ii} at a Nash strategy i of G^{Ends}, and $M \equiv \max_{a \in A} U(a, a)$ is an upper bound:

$$i \in \Omega \textit{ is Nash } \implies m \leq F_{ii} \leq M.$$

Example 2. *With the payoffs shown on table 1, lemma 3.2 implies that the payoffs to Nash strategies are: between* 2 *and* 3 *in the Prisoner's Dilemma; also between* 2 *and* 3 *in the Hawk-Dove; and between* 1 *and* 3 *in 1-2-3 coordination.*

Lemma 3.3. *If* i *is a Nash strategy with finite* T_{ii}*, then the action profile at the breakup stage* T_{ii} *of an* i*-*i *partnership is a Nash profile of the stage game* G*.*

Example 3. *In a Prisoner's Dilemma with endogenous separation, the action profile at the breakup stage of a Nash strategy with finite* T_{ii} *has to be* DD*.*

In a Hawk-Dove game, neither CC *nor* DD *are Nash profiles, so in a Hawk-Dove game with endogenous separation there is no Nash strategy* i *with finite* T_{ii}*.*

For the Prisoner's Dilemma, Observation 1 below strengthens the previous result.

Observation 1. *In the Prisoner's Dilemma with endogenous separation, Nash strategies with finite* T_{ii} *never play* C *on the equilibrium path.*

Observation 1 follows from considering that, in the Prisoner's Dilemma with endogenous separation, if a strategy i with finite T_{ii} ever plays the action profile CC in an i-i partnership, then there is a stage T_l in $[1, T_{ii}]$ at which CC is played for the last time, and a strategy j that reaps-and-leaves i at stage T_l beats i (in the sense $F_{ji} > F_{ii}$), so i cannot be Nash. Observation 1 can be extended to games G with only one symmetric Nash action profile which is the least efficient of the symmetric action profiles.

Lemma 3.4. *Let* (a^N, a^N) *be a Nash profile of* G*.*

- *Every strategy* i *that always chooses action* a^N *before breaking a partnership is a Nash strategy of* G^{Ends}*.*
- *Any mixture of strategies that satisfy the previous condition (for the same action* a^N*) is a Nash state of* G^{Ends}*.*

Example 4. *In a Prisoner's Dilemma with endogenous separation, any strategy* i *that for every history of length between* 0 *and* T_{ii} *(for some* $T_{ii} > 0$*) plays* D*, and breaks every partnership that gets to stage* T_{ii}*, is a Nash strategy (i.e.,* $\mathbf{e}_i$ *is a monomorphic Nash state). Any mixture of such strategies is a Nash (polymorphic) state.*

In a Hawk-Dove game, neither CC *nor* DD *are Nash profiles, so we cannot use lemma 3.4 to find Nash strategies for the game with endogenous separation.*

After discussing Nash strategies, we next extend and analyze stability concepts from evolutionary game theory: evolutionarily stable and neutrally stable strategies.

3.3 Evolutionarily stable strategies in the repeated game

Definition 2 (Evolutionarily stable strategy)**.** *A strategy* $i \in \Omega$ *is evolutionarily stable (Maynard Smith and Price, 1973) if*

$$\begin{aligned} F_{ii} &\geq F_{ji} && \text{for every } j \in \Omega, \ \text{i.e., } i \text{ is Nash, and} \\ F_i(\mathbf{y}) &> E(\mathbf{y}, \mathbf{y}) && \text{for every } \mathbf{y} \in \mathbb{D} \setminus \{\mathbf{e}_i\} \text{ such that } E(\mathbf{y}, \mathbf{e}_i) = F_{ii}. \end{aligned}$$

Evolutionary stability for a Nash strategy i requires that there are no alternative best-response strategies j with $F_{ij} = F_{jj}$. The concept of *path-equivalent strategy*, defined below, will be useful to show that, in games with endogenous separation, there are no evolutionarily stable strategies. The argument extends easily to polymorphic states with finite support (which, in the standard framework, are equivalent to mixed strategies), and is basically the same argument used to show that there are no evolutionarily stable strategies in standard repeated games (Selten and Hammerstein, 1984).

Definition 3 (Path-equivalent strategy)**.** *Strategy j is path-equivalent to strategy i if*

$$a_{jj}^{[1,T_{jj}]} = a_{ii}^{[1,T_{ii}]}.$$

Considering that the action profiles in $a_{ii}^{[1,T_{ii}]}$ are symmetric, it follows that if j is path-equivalent to i, then $a_{ii}^{[1,T_{ii}]} = a_{ji}^{[1,T_{ji}]} = a_{ij}^{[1,T_{ij}]} = a_{jj}^{[1,T_{jj}]}$ and, consequently, $F_{ii} = F_{ji} = F_{ij} = F_{jj}$. If i is Nash and j is path-equivalent to i, then j is an alternative best-response to i (i.e., $F_{ji} = F_{ii}$) that satisfies $F_{ij} = F_{jj}$. By modifying the choices made by strategy i after histories $a^{[1,t]}$ that do not belong to the set of histories $\{a_{ii}^{[1,t]}\}_{t \in [0,T_{ii}]}$ generated by an i-i partnership, one can create (an infinite number of) strategies that are path-equivalent to strategy i. This proves that no strategy is evolutionarily stable in a game with endogenous separation, given that evolutionary stability does not admit the existence of any (different) path-equivalent strategy.

For completeness, in appendix C we discuss another equilibrium concept stronger than neutral stability: robustness against indirect invasions (van Veelen, 2012). We show that in many games with endogenous separation, such as those whose stage game is the Prisoner's Dilemma or the Hawk-Dove game, no strategy can be robust against indirect invasions.

3.4 Neutrally stable strategies in the repeated game

After showing that there are no evolutionarily stable strategies, in this section we define neutral stability for games with endogenous separation.

There have been several attempts to define neutral stability in games with endogenous separation, but all of them present undesirable features (see discussion in appendix B). For instance, the definition in Fujiwara-Greve and Okuno-Fujiwara (2009) (Definition B.2 in appendix B) does not guarantee Lyapunov stability in the replicator dynamics because it only requires robustness to monomorphic invasions, i.e., it does not consider invasions by groups of players using a mix of different strategies. As an example (see also Izquierdo et al. (2021)), consider the one-shot game with payoff matrix (10).

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix} \end{array} \tag{10}$$

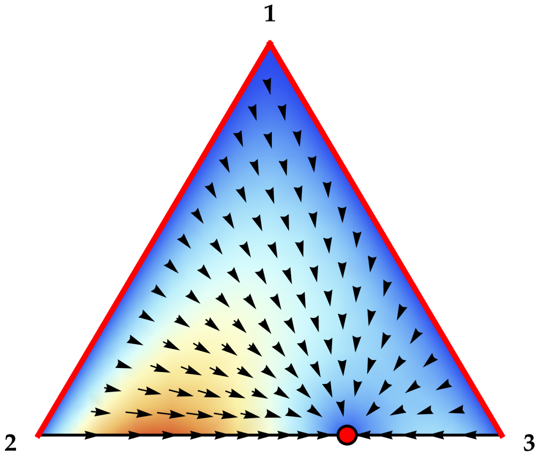

Figure 1: Phase portrait of the replicator dynamics in the game with payoff matrix (10). Rest points are shown in red: there is an isolated rest point at $(x_1, x_2, x_3) = (0, 1/3, 2/3)$ and a connected component of rest points along the edges where $x_2 = 0$ or $x_3 = 0$. Strategy 1 is neutrally stable according to Fujiwara-Greve and Okuno-Fujiwara (2009)'s definition because there are no exit routes along the edges from $\mathbf{e}_1$, but it is not neutrally stable according to the standard definition, and it is not Lyapunov stable. The background is colored according to the speed of motion: from blue (slowest) to brown (fastest). This figure has been generated with *EvoDyn-3s* (Izquierdo et al., 2018).

In this game, strategy 1 is neutrally stable according to Fujiwara-Greve and Okuno-Fujiwara (2009) because a monomorphic population of 1-players is robust to invasions by 2-players and also to invasions by 3-players (when considered separately). However, state $\mathbf{e}_1$ is not Lyapunov stable under the replicator dynamics (see figure 1). Strategy 1 is not neutrally stable according to the definition presented in section 3.1. The reason is that any mixture of 2-players and 3-players obtains a strictly greater payoff than 1-players at any interior state.

The definition we propose here is based on a standard definition of neutral stability (Banerjee and Weibull, 2000), adapted to population games: a state is neutrally stable if it is (i) a best response to itself, and also (ii) a weakly-better response to all its best-response states (than such states are to themselves).

Definition 4 (Neutrally stable state)**.** *A state* $\mathbf{x} \in \mathbb{D}$ *is neutrally stable if*

$$\begin{aligned} &E(\mathbf{x},\mathbf{x}) \geq E(\mathbf{y},\mathbf{x}) && \textit{for every } \mathbf{y} \in \mathbb{D},\ \textit{i.e., } \mathbf{x} \textit{ is Nash, and} \\ &E(\mathbf{x},\mathbf{y}) \geq E(\mathbf{y},\mathbf{y}) && \textit{for every } \mathbf{y} \in \mathbb{D} \textit{ such that } E(\mathbf{y},\mathbf{x}) = E(\mathbf{x},\mathbf{x}). \end{aligned}$$

A strategy i is said to be neutrally stable if and only if its associated monomorphic state $\mathbf{e}_i$ is neutrally stable.

Neutral stability requires strategy i to satisfy $F_i(\mathbf{y}) \geq E(\mathbf{y},\mathbf{y})$ whenever $\mathbf{y}$ is a mixture of alternative best-response strategies to $\mathbf{e}_i$. This robustness to every possible *mixture* of best response strategies is stronger than robustness against all best response strategies considered individually, as defined by the following condition:

$$F_{ij} \geq F_{jj} \text{ for every } j \in \Omega \text{ that is a best-response to } \mathbf{e}_i.$$

This latter condition is necessary but not sufficient for neutral stability. The reason is that, if j_1 and j_2 are two best-response strategies to $\mathbf{e}_i$, and $\mathbf{y}$ is a mixture of j_1 and j_2, then $E(\mathbf{y}, \mathbf{y})$ depends not only on the payoffs $F_{j_1 j_1}$ and $F_{j_2 j_2}$ of each best-response strategy against itself, but also on the payoffs $F_{j_1 j_2}$ and $F_{j_2 j_1}$ for the crossed interactions.

Considering a finite set of strategies and the properties of the payoff functions (section 2.3), we can define the replicator dynamics for games with endogenous separation as a direct adaptation of the standard replicator dynamics (Taylor and Jonker, 1978). Specifically, for a game with endogenous separation and a finite set of strategies $S \subset \Omega$, the replicator dynamics in $\Delta(S) = \{(x_i)_{i \in S} \in \mathbb{R}_+^{|S|} : \sum_{i \in S} x_i = 1\}$ is defined by the set of differential equations

$$\dot{x}_i = x_i [F_i(\mathbf{x}) - E(\mathbf{x}, \mathbf{x})] \tag{11}$$

for $i \in S$. It is easy to check that (11) keeps the simplex $\Delta(S)$ and its faces forward-invariant (because $\sum_{i \in S} \dot{x}_i = 0$ and $x_i = 0 \implies \dot{x}_i = 0$). Existence and uniqueness of the solution trajectories follows from the Lipschitz continuity of the payoff functions.

The replicator dynamics provides a good approximation to the (stochastic) dynamics of many reasonable evolutionary processes in large populations.[6] In our case, payoffs are calculated under the assumption of a stationary pool of singles. This means that, after any change in the composition of strategies in the population (caused, for instance, by reproduction or imitation), the pool of singles is assumed to approach its new stationary distribution (i.e., the payoffs are assumed to approach their new theoretical values) before new changes in the composition of strategies in the population take place.

Our next proposition shows that the definition of neutral stability that we adopt for games with endogenous separation (definition 4) guarantees Lyapunov stability in the replicator dynamics (11), independently of which strategies are included in S.

Proposition 3. *Let $\mathbf{x} \in \mathbb{D}$ be neutrally stable and let S be any finite set of strategies such that $\mathbf{x} \in \Delta(S)$. Then $\mathbf{x}$ is a Lyapunov stable rest point of the replicator dynamics (11) in $\Delta(S)$.*

Proposition 3 shows that, if $\mathbf{x}$ is neutrally stable according to definition 4, then it is a Lyapunov stable rest point in the replicator dynamics (11), considering any finite set of strategies S that includes the support of $\mathbf{x}$ (*incumbents*), and any other set of strategies (*potential invaders*), whatever those potential invaders may be. While the replicator dynamics (11) considers a specific finite set of strategies, proposition 3 guarantees Lyapunov stability of $\mathbf{x}$ for any chosen finite set of strategies, with the only obvious constraint that the strategies in (the support of) $\mathbf{x}$ must be in the chosen set. This highlights the difference between approaches that consider the whole strategy space Ω (as in this paper) vs. approaches that limit the strategy space to some subset $\Omega' \subset \Omega$.

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[6] See e.g. Weibull (1995, section 3.1.1), Sandholm (2010, examples 5.4.2-4), and Izquierdo et al. (2024, chapter V-1).

If the conditions for Nash, neutrally stable and evolutionarily stable strategy were to be weakened by using $\Omega' \subset \Omega$ instead of Ω in their definitions, then many additional strategies may satisfy the weaker conditions: trivially, every strategy is Nash, neutrally and evolutionarily stable (in the restricted space) if the strategy space is restricted to that strategy only. A condition of neutral stability restricted to Ω' would guarantee Lyapunov stability only for finite subsets of Ω'.

Next we study the existence of neutrally stable strategies. Lemma 3.5 below shows a strong limitation for the stability of strategies with finite breakup period: if the symmetric action profile of G with maximum payoff $M = \max_{a\in A} U(a,a)$ is not[7] Nash, then no strategy i with finite breakup period T_{ii} can be neutrally stable. Formally, let N_M^G be the (possibly empty) set of symmetric Nash profiles of stage game G that obtain the maximum symmetric payoff $M = \max_{a\in A} U(a,a)$.

Lemma 3.5. *If i is a neutrally stable strategy (of G^{Ends}) with finite breakup period T_{ii}, the action profiles played in an i-i partnership are in N_M^G.*

As a consequence of lemma 3.5, for games with $N_M^G = \emptyset$ (such as the Prisoner's Dilemma or the Hawk-Dove) no strategy i with finite breakup period can be neutrally stable. This result shows that, for many games, no neutrally stable strategy can display (endogenous) breakup on the equilibrium path.

After this first result in our search for stable strategies, we present *path-protecting* and *weakly path-protecting* strategies, which imply neutral stability and whose existence, for sufficiently large values of δ, is guaranteed for most games.

4 Path-protecting strategies

In this section we define *path-protecting* and *weakly path-protecting* strategies. Both concepts imply neutral stability. We also discuss their existence.

Definition 5 (Path-protecting strategy)**.** *A strategy $i \in \Omega$ is path-protecting if:*

$$a_{jj}^{[1,T_{jj}]} \neq a_{ii}^{[1,T_{ii}]} \implies F_{ji} < F_{ii}.$$

In words, a strategy i is path-protecting if, when playing against i, only strategies that are path-equivalent to i (those with $a_{jj}^{[1,T_{jj}]} = a_{ii}^{[1,T_{ii}]}$) obtain the same payoff as i, while every strategy j that is not path-equivalent to i obtains a strictly lower payoff.

Note that a necessary condition for a strategy i to be path-protecting is that $T_{ii} = \infty$. The reason is that, if T_{ii} is finite, then any strategy j with $T_{jj} > T_{ii}$ whose path of play up to stage T_{ii} coincides with that of i (i.e., $a_{jj}^{[1,T_{ii}]} = a_{ii}^{[1,T_{ii}]}$) satisfies $F_{ji} = F_{ii}$.

Considering that $a_{jj}^{[1,T_{jj}]} = a_{ii}^{[1,\infty]}$ if and only if $a_{ij}^{[1,T_{ij}]} = a_{ii}^{[1,\infty]}$, it is easy to see that a strategy i is path-protecting if and only if $T_{ii} = \infty$ and

$$a_{ji}^{[1,T_{ij}]} \neq a_{ii}^{[1,\infty]} \implies F_{ji} < F_{ii}.$$

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[7] are not, if there are several.

This alternative characterization shows that a path-protecting strategy i "protects" the equilibrium path against strategies that, when playing with i, deviate at some point from i's choice, either by choosing a different action or by breaking the partnership.

We now define a concept weaker than path-protecting strategy, which will turn out to be sufficient to guarantee neutral stability, namely *weakly path-protecting strategy.* Before doing so, for convenience, let us recall that $h_{ij}^{[\infty]} \equiv (a_{ij}^{[1,T_{ij}]})^{\infty}$ is the infinite sequence of action profiles generated by strategy i in a population of j-players with no exogenous separation (7), and F_{ij} coincides with $V(h_{ij}^{[\infty]})$, the normalized discounted value of (the action profiles in) $h_{ij}^{[\infty]}$.

Definition 6 (Weakly path-protecting strategy)**.** *A strategy $i \in \Omega$ with $T_{ii} = \infty$ is weakly path-protecting if:*

$$h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]} \implies F_{ji} < F_{ii}.$$

In words, a strategy i is weakly path-protecting if

- it never breaks a partnership with a partner who takes the same actions as i does, and
- if the repeated path $h_{ji}^{[\infty]}$ that strategy j generates with i-players is different from the path $a_{ii}^{[1,\infty]}$ that i generates, then j obtains a strictly lower payoff (in a population of i-players) than i.

Note that any strategy j that at some stage of an i-j partnership adopts a different action from the action adopted by i generates a different repeated path $h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]}$. Strategies j that, before breaking an i-j partnership at a finite stage T_{ij}, do not adopt different actions from i's, may still generate the same repeated path $h_{ji}^{[\infty]} = h_{ii}^{[\infty]}$, but only if $h_{ii}^{[\infty]}$ is an infinite repetition of the finite sequence of T_{ij} (symmetric) action profiles $a_{ii}^{[1,T_{ij}]} = a_{ji}^{[1,T_{ij}]}$.

For any strategy i with a path $a_{ii}^{[1,\infty]}$ that is not an infinite repetition $(a^{[1,T]})^{\infty}$ of some finite sequence $a^{[1,T]}$ of action profiles, being weakly path-protecting is equivalent to being path-protecting. By contrast, strategies i with a path $a_{ii}^{[1,\infty]}$ that is an infinite repetition of some finite sequence may be weakly path-protecting, but cannot be path-protecting.

Considering equation (9), it follows from definition 6 that if strategy i is weakly path-protecting, then:

- Strategy i is Nash, because strategies with the same repeated path $h_{ji}^{[\infty]} = h_{ii}^{[\infty]}$ obtain the same payoff $F_{ji} = F_{ii}$ and strategies with different repeated path obtain a lower payoff $F_{ji} < F_{ii}$, so $F_{ji} \leq F_{ii}$ for every j.
- Every best-response strategy j to $\mathbf{e}_i$ must generate the same (symmetric) repeated path $h_{ji}^{[\infty]} = h_{ii}^{[\infty]}$. This implies that, if j is a best-response to $\mathbf{e}_i$, then $F_{ij} = F_{ji} = F_{ii}$. It also implies that if $\mathbf{y}$ is a mixture of best-response strategies to $\mathbf{e}_i$, then $F_i(\mathbf{y}) = F_{ii}$.

Our next result states that (weakly) path-protecting strategies are neutrally stable. Its proof shows that, if strategy i is weakly path-protecting, then any mixture $\mathbf{y}$ of best-response strategies to $\mathbf{e}_i$ must satisfy $E(\mathbf{y},\mathbf{y}) = F_i(\mathbf{y})$. The reason is that every repeated path $h_{j_1 j_2}^{[\infty]}$ generated between any two best-response strategies (j_1 and j_2) to $\mathbf{e}_i$ must also be equal to $h_{ii}^{[\infty]}$, so if $\mathbf{y}$ is a mixture of best-response strategies to $\mathbf{e}_i$, then $E(\mathbf{y},\mathbf{y}) = F_{ii} = F_i(\mathbf{y})$.

Proposition 4. *(Weakly) path-protecting strategies are neutrally stable.*

Weakly path-protecting strategies can be easily found if the stage game has some strict Nash profile, as our next result shows.

Lemma 4.1. *If $(\hat{a},\hat{a})$ is a strict Nash profile of a stage game G, then any strategy of G^{Ends} that:*

- *chooses action $\hat{a}$ whenever it does not choose to break a partnership, and*
- *does not break a partnership while profile $(\hat{a},\hat{a})$ is played*

is weakly path-protecting (and, consequently, neutrally stable).

Example 5. *In the Prisoner's Dilemma, DD is a strict Nash profile. Consequently, any strategy that never plays C and never breaks up after a history of mutual defections is weakly path-protecting. For instance, the strategy "always play D and never leave", that maps every history to D, is weakly path-protecting and, consequently, neutrally stable.*

Much more generally than the case in which G has some strict Nash profile, Proposition 5 below shows that, for large enough δ, every stage game G with $M > m$ admits path-protecting strategies. Proposition 5 leads to a *Folk Theorem* for neutral stability which basically says that, for large enough δ, any payoff between m and M can be obtained, or approximated arbitrarily closely, as the equilibrium payoff of some path-protecting strategy.

Before stating proposition 5, let us define the average stage-payoff for a finite sequence of action profiles. Considering a sequence $\Phi = (\Phi^{[t]})_{t=1}^{T}$of T action profiles, where each $\Phi^{[t]} \in A^2$ is an action profile, let the average stage-payoff of sequence Φ be

$$\bar{U}_\Phi \equiv \frac{\sum_{t=1}^{T} U(\Phi^{[t]})}{T}.$$

The average stage payoff is specially relevant for large δ and for paths that end up repeating some sequence Φ of action profiles, because the normalized payoff of any infinite path $([...], \Phi, \Phi, \Phi, ...)$ which, after a finite number of periods, eventually repeats the finite sequence of outcomes Φ forever, converges to the average stage-payoff $\bar{U}_\Phi$ as δ goes to 1.

Proposition 5. *Let Φ be a finite sequence of symmetric action profiles with average stage payoff $\bar{U}_\Phi$ strictly greater than the pure minmax payoff. For large enough $\delta < 1$, there are path-protecting strategies whose equilibrium path, after a finite transient phase, is an infinite repetition of the sequence Φ, and whose equilibrium payoff converges to $\bar{U}_\Phi$ as $\delta \to 1$.*

Considering that $\bar{U}_\Phi$ can approximate any real payoff between m and M as much as desired, Proposition 5 has as a corollary the following Folk Theorem for path-protecting (neutrally stable) strategies.

Corollary 5.1. *(Folk Theorem). In a game with endogenous separation, for large enough values of the continuation probability δ, any payoff between the pure minmax payoff m and the maximum symmetric payoff M of the stage game can be obtained, or approximated as much as desired, as the equilibrium payoff of some path-protecting strategy.*

Note that the largest possible payoff in equilibrium in a VRPD corresponds to the one-period trust-building strategy, with path $DD(CC)^\infty$ and payoff $F_{Max} = (1-\delta)\,U_{DD} + \delta\,U_{CC}$. We have $F_{Max} < U_{CC}$ and $F_{Max} \to U_{CC}$ as $\delta \to 1$. Consequently, and in contrast to the standard setting, the set of feasible equilibrium payoffs is not compact, and it is not possible to find a threshold value of δ above which all payoffs in the set can be obtained in equilibrium.

The proof of proposition 5 is detailed in appendix D, but here we provide a sketch. The proof is constructive and considers a strategy i such that:

- It never breaks a partnership with a partner who takes the same actions as i does (i.e., $T_{ii} = \infty$).
- As soon as strategy j in an i-j partnership deviates from i's own action, strategy i breaks the partnership. Because of this condition, we know that an i-j partnership will not survive if j chooses a different action from the action chosen by i. Naturally, it will not survive either if j chooses to break the partnership. The only way in which an i-j partnership can survive indefinitely is if j chooses the same initial action as i does and, for every history $a_{ii}^{[1,t]}$ corresponding to an i-i partnership, j chooses the same action as i does.
- The path $a_{ii}^{[1,\infty]}$ is made up by three phases, each one associated to one finite sequence of symmetric action profiles (Φ_m, Φ_f and Φ_p), with

  $$a_{ii}^{[1,\infty]} = (\Phi_m, \Phi_f, (\Phi_p)^\infty),$$

  where Φ_m is a repetition of a minmax action profile, Φ_f is arbitrary (but finite), Φ_p (which corresponds to the infinitely repeated pattern Φ in proposition 5) has an average stage payoff greater than the pure minmax payoff m of the stage game, and $(\Phi_p)^\infty$ represents an infinite sequence of action profiles made up by repeating the sequence Φ_p infinitely.
- The first phase in $a_{ii}^{[1,\infty]}$ is a T_m-period-long phase, $T_m \geq 1$, during which a minmax action profile $(\tilde{a}, \tilde{a})$ is played, producing the sequence

  $$\Phi_m = a_{ii}^{[1,T_m]} = ((\tilde{a}, \tilde{a}), (\tilde{a}, \tilde{a}), ..., (\tilde{a}, \tilde{a})).$$

During this minmax or *deviation-deterring* phase, the stage payoff is $U(\tilde{a},\tilde{a}) \leq m$ and any strategy j that deviates in choice during this phase obtains a payoff $F_{ji} \leq m$.

- The second phase in the path $a_{ii}^{[1,\infty]}$ is an arbitrary finite sequence of $T_f \geq 0$ (symmetric) action profiles. This phase shows that path-protecting strategies can present a large variety of different paths.
- The last phase, or *pattern-playing* phase, in $a_{ii}^{[1,\infty]}$ is an infinite repetition of a finite sequence (pattern) Φ_p of $T_p \geq 1$ symmetric action profiles with average stage payoff $\bar{U}_{\Phi_p} > m$.

The proof of proposition 5 combines three intermediate results to create path-protecting strategies. These strategies are initially built to be weakly path-protecting, and then fine-tuned so the path when they play against themselves is not an infinite repetition of any finite sequence of action profiles, so they are also path-protecting.

- The first result (lemma D.1) shows that, in order to prove that the implication $h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]} \implies F_{ji} < F_{ii}$ holds for every strategy j, it is enough to prove that it holds for strategies j whose repeated path $h_{ji}^{[\infty]}$ differs or deviates from $h_{ii}^{[\infty]}$ before repetition of the pattern Φ_p begins, i.e., between periods $t = 1$ and $t = T_m + T_f + T_p$: if every deviation before and up to period $t = T_m + T_f + T_p$ is harmful, then every deviation (no matter when) is harmful.
- The second result states that, for any given Φ_f and Φ_p (with $\bar{U}_{\Phi_p} > m$), the deviation-deterring phase can be chosen to be long enough to guarantee that deviations in $h_{ji}^{[\infty]}$ from $h_{ii}^{[\infty]}$ at or before $t = T_m + T_f + T_p$ lead to payoffs F_{ji} close to or below m.
- The third result states that, for sufficiently large δ, the payoff F_{ii} is close to $\bar{U}_{\Phi_p} > m$.

Combining the three results shows that, given Φ_f and Φ_p, there is a length of the deviation-deterring phase T_m such that, for large enough δ, $h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]}$ implies $F_{ji} < F_{ii}$, so strategy i is weakly path protecting. Finally, by choosing Φ_p so that path $h_{ii}^{[\infty]}$ is not an infinite repetition of a pattern, we make sure that strategy i is also path-protecting.

Note in the proof of proposition 5 that the freedom in the second phase Φ_f and the relative freedom in the last phase Φ_p allow us to identify a very broad family of path-protecting strategies and outcome paths that can be sustained in stable equilibrium. For the VRPD, this family is much broader than the family of k-period trust-building strategies, which correspond to the special case $\Phi_m = (DD)^k$, $\Phi_f = \emptyset$ and $\Phi_p = CC$. Our next example discusses these strategies for the VRPD (and for the Hawk-Dove), and shows how to find values of δ that guarantee neutral stability, using the intermediate results we have just presented.

Example 6. *In a Prisoner's Dilemma or in a Hawk-dove game, the minmax profile is* DD*, so, when looking for path-protecting strategies as in the proof of proposition 5:*

- *The deviation-deterring or minmax phase* Φ_m *is a* T_m*-long series of* DD *action profiles:* $\Phi_m = (DD)^{T_m}$*.*
- *For the pattern-playing phase, the infinitely repeated finite pattern* Φ_p *can be any finite sequence of* DD *and* CC *action profiles with at least one* CC *in the sequence, which guarantees an average stage payoff* $\bar{U}_{\Phi_p} > m = U_{DD}$*.*

For instance, choosing $T_m = 3$*,* $\Phi_f = (CC, DD)$ *and* $\Phi_p = CC$*, we obtain a strategy* i *with path* $h_{ii}^{\infty} = ((DD)^3 \,|\, CC, DD \,|\, (CC)^{\infty})$*. For the stage payoffs shown on table 1 for the Prisoner's Dilemma, the sequence of payoffs corresponding to* h_{ii}^{∞} *is* $(2, 2, 2, 3, 2, (3)^{\infty})$*, where* $()^{\infty}$ *represents an infinite repetition of the payoffs in brackets, so*

$$F_{ii} = (1-\delta)(2 + 2\delta + 2\delta^2 + 3\delta^3 + 2\delta^4 + 3\frac{\delta^5}{1-\delta}) > 2.$$

The pattern $\Phi_p = CC$ *begins to be repeated after period 6. Strategies* j *with* $T_{ji} \le 3$ *obtain a payoff* F_{ji} *of at most the minmax payoff* $2 < F_{ii}$*. For* $T_{ji} = 4$ *the payoff* F_{ji} *is bounded by that of the series* $(2, 2, 2, 4)^{\infty}$*, and for* $5 \le T_{ji} \le 6$ *the payoff is bounded by that of the series* $(2, 2, 2, 3, 2, 4)^{\infty}$*. For* $\delta > 0.71$*,* F_{ii} *is greater than the payoffs corresponding to both series, so* i *is path-protecting.*

5 Subgame perfect equilibrium

While it is not evident how to define continuation payoffs in games with endogenous separation using the stationarity assumption, we can extend formula (8) and define the continuation payoff to strategy j in a population of i players after a non-empty history $a^{[1,t]}$ as the normalized discounted sum of the future series of outcomes with no exogenous separation:

$$F_{ji|a^{[1,t]}} = (1-\delta) \sum_{\tau=1}^{T_{ji|a^{[1,t]}}} \delta^{\tau-1} U(a^{[\tau]}_{ji|a^{[1,t]}}) + \delta^{T_{ji|a^{[1,t]}}} F_{ji} \tag{12}$$

where:

- $T_{ji|a^{[1,t]}} \ge 0$ is the number of stages that strategy i and j would play together after history $a^{[1,t]}$ before choosing to break up their partnership. If $T_{ji|a^{[1,t]}} = 0$ then the summation term in (12) is 0 and $F_{ji|a^{[1,t]}} = F_{ji}$.
- For $T_{ji|a^{[1,t]}} \ge 1$ and $1 \le \tau \le T_{ji|a^{[1,t]}}$, $a^{[\tau]}_{ji|a^{[1,t]}}$ is the stage game outcome that j and i generate the τ^{th} time they play together if they were to start playing with the non-empty history $a^{[1,t]}$.

Definition 7. *A strategy i is subgame perfect (equivalently, $\mathbf{e}_i$ is a subgame perfect equilibrium) if*

$$F_{ji|a^{[1,t]}} \leq F_{ii|a^{[1,t]}}$$

for every strategy $j \in \Omega$ and every history $a^{[1,t]} \in \mathcal{H}$.

Let us say that a strategy i is *deviation-leaving* if i breaks a partnership only after every history that deviates from the histories $a_{ii}^{[1,t]}$ that i generates against itself, i.e., i is deviation-leaving if

$$a^{[1,t]} \neq a_{ii}^{[1,t]} \iff i(a^{[1,t]}) = \mathit{break}$$

Deviation-leaving strategies are uniquely determined by their path $a_{ii}^{[1,\infty]}$. For histories $a^{[1,t]}$ that differ from $a_{ii}^{[1,t]}$ deviation-leaving strategies break the partnership, so for such histories $F_{ji|a^{[1,t]}} = F_{ji}$. Profitable deviations after histories that occur on the path $a_{ii}^{[1,t]}$ are linked to the existence of profitable deviations after the empty history. This leads to our next proposition.

Proposition 6. *If a deviation-leaving strategy is Nash then it is subgame perfect.*

Corollary 6.1. *Deviation-leaving path-protecting strategies are neutrally stable and subgame perfect.*

Note that proposition 5 can also be stated for deviation-leaving path-protecting strategies (the proof is the same), and, consequently, our Folk Theorem (corollary 5.1) guarantees neutrally stable subgame perfect equilibria supported by deviation-leaving path-protecting strategies.

6 Deviation-leaving strategies

In section 4 we showed that, for large enough δ, there are symmetric paths of the form $(\Phi_m, [], (\Phi_p)^\infty)$, (which start by playing a minmax action profile for some periods and end up repeating some pattern Φ_p) which can be sustained by path-protecting strategies. Here we consider any infinite symmetric path $a^{[1,\infty]}$ and analyze the conditions that its associated deviation-leaving strategy i needs to satisfy in order to be path-protecting (guaranteeing sub-game perfection and neutral stability).

To check whether a deviation-leaving strategy i is Nash it is enough to consider one-shot deviations j (strategies that map all but one histories to the same choice mapped by i) that make a different choice (than i) only after history $a_{ii}^{[1,T]}$, with one $T \in \{0, 1, 2, ...\}$ for each strategy. As such an i-j partnership does not last beyond stage $T + 1$, the subsequent behavior of j is actually irrelevant. If none of those one-shot deviations after histories that arise along the equilibrium path is profitable, the deviation-leaving strategy i is Nash (and subgame perfect). If all such deviations are harming, then i is path-protecting. If the path of the deviation-leaving strategy i ends up repeating some pattern, it is enough to check deviations up to (including) the first time the pattern is played (lemma D.1). In contrast, note that in standard repeated games this situation (non-existence of profitable one-shot deviations after histories that arise along the equilibrium path) does not imply Nash equilibrium (Mailath and Samuelson, 2006, p. 27), because in that framework behavior after a deviation is relevant.

In this section we assume that i is deviation-leaving. For $t = 0, 1, 2, ...$ it is useful to define

$$V_t(a^{[1,\infty]}) = (1-\delta)\sum_{k=1}^{\infty}\delta^{k-1}U(a^{[t+k]})$$

So $V_0(a_{ii}^{[1,\infty]}) = F_{ii}$ and $V_t(a_{ii}^{[1,\infty]}) = F_{ii|a_{ii}^{[1,t]}}$, the continuation payoff after history $a_{ii}^{[1,t]}$. Besides, $V_t(a_{ii}^{[1,\infty]}) = (1-\delta)U(a_{ii}^{[t+1]}) + \delta V_{t+1}(a_{ii}^{[1,\infty]})$.

It is easy to check that the condition "$F_{ji} \leq F_{ii}$ for (all) the strategies with $T_{ij} = t$" (for any chosen $t \in \{1, 2, ...\}$) is equivalent to

$$(1-\delta)U^{BR}(a_i^{[t]}) + \delta F_{ii} \leq V_{t-1}(a_{ii}^{[1,\infty]}) \tag{13}$$

which corresponds to the condition that playing a best response to $a_i^{[t]}$ and leaving (or being left) is not better than keeping playing as i does. The condition can also be stated as

$$(1-\delta)\left(U^{BR}(a_i^{[t]}) - U(a_{ii}^{[t]})\right) \leq \delta\left(V_t(a_{ii}^{[1,\infty]}) - F_{ii}\right)$$

which relates the advantage of playing a best-response action at stage t (instead of the prescribed action) with the advantage of complying and staying on the path instead of starting a new partnership. This last expression also shows that $V_t(a_{ii}^{[1,\infty]}) \geq F_{ii}$ for $t = 1, 2, ...$ is a necessary condition for a strategy with $T_{ii} = \infty$ to be Nash, as otherwise leaving after playing at some stage would be better than staying on the path. To summarize this section, conditions (13) (for $t = 1, 2, ...$) are sufficient and necessary for the deviation-leaving strategy i with path $a_{ii}^{[1,\infty]}$ to be Nash, and to be subgame perfect. With strict inequality, these conditions are sufficient and necessary for i to be path-protecting (so they are sufficient for neutral stability).

7 Conclusions

In the standard approach to repeated games, partners are tied to each other and do not have a say on whether they wish to stay together or whether they prefer to leave their current partner and meet a new one. For many real-life situations, the field of games with endogenous separation constitutes a natural and more realistic alternative.

Games with endogenous separation present significant challenges. Even computing expected payoffs for strategies –given the population composition and the stage-game payoffs– requires several intermediate steps (proposition 1), and, for more than three strategies, the payoff functions do not admit general closed-form algebraic expressions (proposition 2). Nevertheless, it is possible to derive a number of relevant properties of these payoff functions (section 2.3).

We apply concepts from evolutionary game theory to study games with endogenous separation and to explore the existence of strategies that may constitute stable socially extended behavior (social conventions). Our study highlights the versatility of these tools and hence contributes to clarifying their explanatory potential and broader relevance within game theory. Carrying out this analysis requires extending existing results on neutral stability and its relation to Lyapunov stability in the replicator dynamics to the specific modeling framework needed to study repeated games with endogenous separation (proposition 3). It also requires the development and exploration of novel concepts (*path-protecting strategy*) which provide a link between the theory of repeated games and evolutionary games with endogenous separation.

Path-protecting strategies ensure that deviations from the equilibrium path are harmful, and lead to neutrally stable states (proposition 4). They generalize the concept of *trust-building* strategies introduced in earlier work on the Prisoner's Dilemma (Carmichael and MacLeod, 1997; Fujiwara-Greve and Okuno-Fujiwara, 2009). Path-protecting strategies share both similarities and differences with the classical *trigger* strategies which, in standard repeated games, deter deviations from an equilibrium path by imposing a minmax *punishment* phase after a deviation. In games with endogenous separation, however, any punishment phase must be played at the beginning of a new partnership (to prevent reap-and-leave invasions), and all players –not only deviators– must go through it. Our results show that a large variety of equilibrium paths can be sustained by a path-protecting strategy (proposition 5). When applied to the Prisoner's Dilemma, this result identifies a much broader family of stable strategies than the family of trust-building strategies. We also show that deviation-leaving path-protecting strategies (which leave after every history that does not occur on the equilibrium path) are both subgame perfect and neutrally stable (proposition 6), and we analyze the conditions that any infinite symmetric path needs to satisfy for its associated deviation-leaving strategy to be Nash, or to be path-protecting.

Additionally, we provide a Folk Theorem for path-protecting strategies in games with endogenous separation (corollary 5.1).

In an appendix, we extend the concept of path-protecting strategy from strategies (monomorphic states) to mixtures of strategies in a population (polymorphic states). Extensions of the framework of games with endogenous separation to multiplayer asymmetric games or multi-population games present additional challenges and remain an open field of research.

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A Polymorphic neutrally stable states

Let us now consider polymorphic neutrally stable states, in which players in a population use different strategies (beyond those states already considered in lemma 3.4). In the standard setting of population games, polymorphic states can alternatively be interpreted as mixed strategies. In games with endogenous separation, the average payoff to a group of players with strategy distribution $\mathbf{y}$ in a population $\mathbf{x}$, equation (4), does not need to coincide with the payoff to an individual using mixed strategy $\mathbf{y}$ in a population $\mathbf{x}$ (because each of the pure strategies in the support of $\mathbf{y}$ may have different breakup periods with the strategies used in $\mathbf{x}$). In this setting, interpreting a strategy distribution as an individual's mixed strategy is not equivalent.

Looking for stable polymorphic states, the first candidate wold seem to be a mixture of path-protecting strategies. However, our next result shows that, if two path-protecting strategies i and j have different paths $a_{ii}^{[1,\infty]} \neq a_{jj}^{[1,\infty]}$, then they cannot both be in the support of a neutrally stable state. The result holds for weakly path-protecting strategies with different repeated path. Consequently, there are no neutrally stable states with more than one (weakly) path-protecting strategy, unless the different strategies are actually generating the same repeated path.

Proposition A.1. *If a neutrally stable state $\mathbf{x}$ has some (weakly) path-protecting strategy i in its support then all the repeated paths in $\mathbf{x}$ are equal to $h_{ii}^{[\infty]}$.*

Proposition A.1 shows that mixtures of path-protecting strategies with different paths do not satisfy definition 4 of neutral stability.

Next we present a series of definitions and a proposition that allow us to extend some of the results for monomorphic states to polymorphic states, and we conclude this section with an example of a polymorphic neutrally stable state.

Definition A.1 (Path-equivalent strategy in a set)**.** *Let S be a finite set of strategies satisfying $T_{ij} = \infty$ for every $i, j \in S$. We say that strategy k is path-equivalent in S to strategy $i \in S$ if, for every $j \in S$,*

$$T_{kj} = \infty \text{ and } a_{kj}^{[1,\infty]} = a_{ij}^{[1,\infty]}.$$

The idea here is that, with each of the strategies in S, strategy k behaves exactly as strategy i does, and there is no difference also between $a_{ii}^{[1,\infty]}$ and $a_{kk}^{[1,\infty]}$.

Definition A.2 (Path-protecting state)**.** *A population state* $\mathbf{x}$ *with finite support* $\mathbb{S}(\mathbf{x})$ *is path-protecting if:*

- $T_{ij} = \infty$ *for every* $i, j \in \mathbb{S}(\mathbf{x})$*, and*
- *If strategy* j *is not path-equivalent in* $\mathbb{S}(\mathbf{x})$ *to some strategy* $i \in \mathbb{S}(\mathbf{x})$*, then* $F_j(\mathbf{x}) < E(\mathbf{x}, \mathbf{x})$*.*

It follows from the definition that path-protecting states are Nash.

Definition A.3 (Internally neutrally stable state)**.** *A state* $\mathbf{x}$ *is internally neutrally stable if* $F_i(\mathbf{x}) = E(\mathbf{x}, \mathbf{x})$ *for every* $i \in \mathbb{S}(\mathbf{x})$ *and* $E(\mathbf{x}, \mathbf{y}) \geq E(\mathbf{y}, \mathbf{y})$ *for every* $\mathbf{y}$ *with support contained in* $\mathbb{S}(\mathbf{x})$*.*

This condition only considers strategies in the support of state $\mathbf{x}$, and it is clearly a necessary condition for neutral stability, which considers the whole strategy space.

Proposition A.2. *If a state is path-protecting and internally neutrally stable, then it is neutrally stable.*

A.1 Example of a bimorphic neutrally stable path-protecting equilibrium

Consider a Prisoner's Dilemma game with the payoffs shown in table 2. For the game with endogenous separation, let strategy 1 and strategy 2 be two strategies that generate the paths $a_{ij}^{[1,\infty]}$ shown in table 2, with the corresponding payoffs F_{ij} shown in table 3. Strategy 1 is such that, if an opposing strategy j generates in a j-1 partnership a history that is not coherent with either $a_{11}^{[1,\infty]}$ or $a_{21}^{[1,\infty]}$, strategy 1 breaks up the partnership. In the same way, strategy 2 breaks any j-2 partnership as soon as the history deviates from both $a_{12}^{[1,\infty]}$ and $a_{22}^{[1,\infty]}$.

	C	D
C	3	-1
D	5	0

	1	2
1	$(DD)^{T_1}\,(CC)^{\infty}$	$(DD)^{T_2}\,DC\,(CC)^{\infty}$
2	$(DD)^{T_2}\,CD\,(CC)^{\infty}$	$(DD)^{T_2}(CC)^{\infty}$

Table 2: Left: Stage game payoffs for a Prisoner's Dilemma, with C for Cooperate and D for Defect. Right: Paths $a_{ij}^{[1,\infty]}$ that strategy 1 and strategy 2 generate together, with i for the row strategy and j for the column. It is assumed that $T_1 > T_2$

	1	2
1	$\delta^{T_1}\,3$	$\delta^{T_2}[5\,(1-\delta) + 3\,\delta]$
2	$\delta^{T_2}[(-1)(1-\delta) + 3\,\delta]$	$\delta^{T_2}\,3$

Table 3: Payoffs F_{ij} corresponding to the paths shown in table 2.

Let us take $T_1 = 6$, $T_2 = 4$ and $\delta = 0.9$, leading to the F_{ij} payoffs shown in table 4.

	1	2
1	1.59	2.10
2	1.71	1.97

Table 4: Payoffs F_{ij} corresponding to the paths shown in table 2, for $T_1 = 6$, $T_2 = 4$ and $\delta = 0.9$.

At a population state made up by strategies 1 and 2 in proportions x_1 and x_2, considering that all paths have the same length, we have $F_1(\mathbf{x}) = x_1 F_{11} + x_2 F_{12}$ and $F_2(\mathbf{x}) = x_1 F_{21} + x_2 F_{22}$. These formulas together with the payoffs in Table 4 show that the internal or restricted game for strategies 1 and 2 has the structure of an anti-coordination game (such as a Hawk-Dove game), which presents an internally neutrally stable (in fact, internally evolutionarily stable) equilibrium $\hat{x}$ where $F_1(\hat{\mathbf{x}}) = F_2(\hat{\mathbf{x}})$, at $\hat{x}_1 = \frac{20}{37} \approx 0.54$ and $\hat{x}_2 = \frac{17}{37} \approx 0.46$, with $E(\hat{\mathbf{x}}, \hat{\mathbf{x}}) \approx 1.83$.

Let us check that $\hat{\mathbf{x}}$ is path-protecting.

- Strategies that do not get past history $(DD)^4$ when playing with strategies 1 or 2 (they break up or deviate in action before stage 5) obtain at most the minmax payoff $U_{DD} = 0 < E(\hat{\mathbf{x}}, \hat{\mathbf{x}})$.
- Strategies j that after history $(DD)^4$ play D (as strategy 1 does and strategy 2 does not) may go on generating with 1 and 2 the same paths $a_{11}^{[1,\infty]}$ and $a_{12}^{[1,\infty]}$ as strategy 1 does, may break up at stage 5 (after playing), or may deviate from $a_{11}^{[1,\infty]}$ at stage $T_{j1} > 5$ and from $a_{12}^{[1,\infty]}$ at stage $T_{j2} > 5$, obtaining a payoff (see (6), considering that the pool and population strategy distributions at $\hat{\mathbf{x}}$ are the same):

$$F_j(\hat{\mathbf{x}}) = \frac{\hat{x}_1(1-\delta^{T_{j1}})}{\hat{x}_1(1-\delta^{T_{j1}}) + \hat{x}_2(1-\delta^{T_{j2}})} F_{j1} + \frac{\hat{\mathbf{x}}_2(1-\delta^{T_{j2}})}{\hat{x}_1(1-\delta^{T_{j1}}) + \hat{x}_2(1-\delta^{T_{j2}})} F_{j2}.$$

Let us focus first on F_{j2} for deviations from $a_{12}^{[1,\infty]}$ after stage 5. Applying lemma D.1, we have that if $F_{j2} < F_{12}$ for every possible deviation at $T_{j2} = 6$ (first play of the repeated pattern CC in $h_{12}^{[\infty]}$), then $F_{j2} < F_{12}$ for every finite $T_{j2} > 6$. F_{j2} for a strategy j that breaks up at $T_{j2} = 5$ or deviates at $T_{j2} = 6$ is bounded by the payoff corresponding to the series of stage payoffs $(0, 0, 0, 0, 5, 5)$, which is $\frac{1-\delta}{1-\delta^6}\delta^4\,(5 + 5\,\delta) = 1.33 < F_{12}$, so $F_{j2} < F_{12}$ for $T_{j2} \geq 5$.

Let us focus now on F_{j1} for breakup at stage 5 or deviations from $a_{11}^{[1,\infty]}$ after stage 5. The payoff to these strategies is bounded by 0 for $5 \leq T_{j1} \leq 6$ and by the payoff corresponding to the series of stage payoffs $(0, 0, 0, 0, 0, 0, 3, ..., 3, 5)$, which is $\frac{1-\delta}{1-\delta^{T_{j1}}}\left[\frac{\delta^6\, 3(1-\delta^{T_{j1}-7})}{1-\delta} + \delta^{T_{j1}-1}\, 5\right]$ for $T_{j1} > 6$. Applying lemma D.1 for deviations at

$T_{j1} = 7$ (first play of the repeated pattern (CC) in $h_{11}^{[\infty]}$) shows $F_{j1} < F_1(\mathbf{e}_1) < F_{12}$. We can now state the following bound:

$$F_j(\hat{\mathbf{x}}) \leq \frac{\hat{x}_1(1-\delta^{T_{j1}})}{\hat{x}_1(1-\delta^{T_{j1}}) + \hat{x}_2(1-\delta^{T_{j2}})} F_{j1} + \frac{\hat{x}_2(1-\delta^{T_{j2}})}{\hat{x}_1(1-\delta^{T_{j1}}) + \hat{x}_2(1-\delta^{T_{j2}})} F_{12}.$$

Considering that $F_{j1} < F_1(\mathbf{e}_1) < F_{12}$, and that the weight multiplying F_{12} on the previous convex combination of F_{12} and F_{j1} increases with T_{j2}, we find that, for every T_{j1}, the maximum value of the bound corresponds to $T_{j2} = \infty$ (being smaller for finite T_{j2}). Thus, bearing in mind that $F_{j1} \leq 0$ for $5 \leq T_{j1} \leq 6$, we have:

$$F_j(\hat{x}) \leq \frac{\hat{x}_2}{\hat{x}_1(1-\delta^5) + \hat{x}_2} F_{12} \approx 1.42 < E(\hat{x}, \hat{x}), \textit{ for } 5 \leq T_{j1} \leq 6.$$

And, for $T_{j1} > 6$,

$$F_j(\hat{\mathbf{x}}) \leq \frac{\hat{x}_1(1-\delta)}{\hat{x}_1(1-\delta^{T_{j1}}) + \hat{x}_2} \left[\frac{\delta^6\, 3(1-\delta^{T_{j1}-7})}{1-\delta} + \delta^{T_{j1}-1}\, 5 \right] + \frac{\hat{x}_2\, F_{12}}{\hat{x}_1(1-\delta^{T_{j1}}) + \hat{x}_2}.$$

Note that the only variable in the previous bound is T_{j1}, with all the other terms being known numbers. By taking the derivative of this bound with respect to T_{j1} it can be checked that it is monotonic increasing with T_{j1} (for $T_{j1} > 6$), and its limit is $E(\hat{\mathbf{x}}, \hat{\mathbf{x}})$. Consequently, any strategy j that, when playing with strategies 1 and 2, gets to stage 5 and plays D there (as strategy 1 and its path-equivalent-in-$\{1,2\}$ strategies do) obtains a payoff $F_j(\hat{\mathbf{x}}) < E(\hat{\mathbf{x}}, \hat{\mathbf{x}})$ if j is not path-equivalent to strategy 1 in the set of strategies $\{1,2\}$.

- We now consider strategies j that after history $(DD)^4$ play C (as strategy 2 does and strategy 1 does not). Applying the same procedure that we followed before, it can be shown that any such strategy j that, when playing with strategies 1 and 2, gets to stage 5 and plays C there (as strategy 2 and its path-equivalent-in-$\{1,2\}$ strategies do), obtains a payoff $F_j(\hat{\mathbf{x}}) < E(\hat{\mathbf{x}}, \hat{\mathbf{x}})$ if j is not path-equivalent to strategy 2 in the set of strategies $\{1,2\}$.

B Other approaches to neutral stability in games with endogenous separation

Here we summarize previous definitions of neutral stability for games with endogenous separation.

Definition B.1. *Carmichael and MacLeod (1997). A Nash equilibrium population state* $\mathbf{x}$ *is a neutrally stable state* NSS_{CM} *if for every* $\mathbf{y} \in \mathbb{D}$ *there exists an* $\bar{\epsilon}_{\mathbf{y}} \in (0,1)$ *such that for every* $\epsilon \in (0, \bar{\epsilon}_{\mathbf{y}})$,

$$F_i(\,(1-\epsilon)\mathbf{x} + \epsilon\mathbf{y}) \geq F_j(\,(1-\epsilon)\mathbf{x} + \epsilon\mathbf{y})$$

for all $i \in supp(\mathbf{x})$ *and* $j \in supp(\mathbf{y})$.

Definition B.2. *Fujiwara-Greve and Okuno-Fujiwara (2009). A distribution in the matching pool* $\mathbf{p}$ *is a neutrally stable pool distribution* NSS_{FO} *if for every* $j \in \Omega$ *there exists an* $\bar{\epsilon}_j \in (0,1)$ *such that for every* $\epsilon \in (0, \bar{\epsilon}_j)$ *and every* $i \in supp(\mathbf{p})$,

$$\hat{F}_i(\,(1-\epsilon)\,\mathbf{p} + \epsilon\;\mathbf{e}_j) \geq \hat{F}_j(\,(1-\epsilon)\;\mathbf{p} + \epsilon\,\mathbf{e}_j)$$

These definitions are related to a standard condition for (neutral) stability from Taylor and Jonker (1978), which can be adapted as follows (Bomze and Weibull, 1995):

Definition B.3. *Considering a finite set of strategies* S*, a state* $\mathbf{x} \in \Delta(S)$ *is neutrally stable* NSS_{TJ} *in* $\Delta(S)$ *if for every* $\mathbf{y} \in \Delta(S)$ *there is some* $\bar{\epsilon}_y \in (0,1)$ *such that*

$$F(\mathbf{x}, \epsilon\,\mathbf{y} + (1-\epsilon)\,\mathbf{x}) \geq F(\mathbf{y}, \epsilon\,\mathbf{y} + (1-\epsilon)\,\mathbf{x})$$

for all $\epsilon \in (0, \bar{\epsilon}_y)$.

On the one hand, when considering a finite set of strategies, it is easy to see that the conditions for NSS_{CM} and NSS_{FO} are not equivalent to the standard condition NSS_{TJ}. Izquierdo et al. (2021, Appendix C) present simple examples of states that are neutrally stable (NSS_{TJ}) but are not NSS_{CM} or NSS_{FO}. They also show examples of states that are NSS_{FO} but are not neutrally stable NSS_{TJ} and are instable in the replicator dynamics (such as the example we presented in section 3.4). In short:

- Definitions NSS_{CM} and NSS_{FO} are not consistent with the standard definition NSS_{TJ} of neutral stability.
- NSS_{FO} does not guarantee Lyapunov stability under the replicator dynamics.

Considering behavioral strategies, Vesely and Yang (2010) provide a definition of neutral stability for games with endogenous separation that is close to NSS_{TJ}. However, if the payoff functions $F_i(\mathbf{x})$ are not linear (and this is generically the case in games with endogenous separation), then the different "standard" definitions of neutral stability, which are equivalent in the linear setting, are not equivalent any more (Bomze and Weibull (1995)), and being NSS_{TJ} does not guarantee Lyapunov stability in the replicator dynamics in $\Delta(S)$. In contrast, the condition that we use to define neutral stability does guarantee Lyapunov stability in the replicator dynamics in $\Delta(S)$.

Izquierdo et al. (2021) provide a definition of neutral stability that looks rather involved because it uses the population and pool distributions, related by the function f as defined by (1), as well as a function $\hat{E}(\mathbf{z}, \mathbf{p})$ that provides the payoff to a group of players with strategy distribution $\mathbf{z}$ entering a population with pool distribution $\mathbf{p}$.

Definition B.4. *A population-pool state* $\{\mathbf{x}, \mathbf{p}\}$ *with* $\mathbf{x} = f(\mathbf{p})$ *is neutrally stable* NNS_{IIV} *if* $\mathbf{x}$ *is a Nash equilibrium and for any finite set of strategies* $S \subset \Omega$ *such that* $\mathbb{S}(\mathbf{p}) \subseteq S$ *there is a neighborhood* O_S *of* $\mathbf{p}$ *in* $\Delta(S)$ *such that* $\hat{E}(\mathbf{x}, \mathbf{y}) \geq \hat{E}(f(\mathbf{y}), \mathbf{y})$ *for every* $\mathbf{y} \in O_S$ *satisfying* $\hat{E}(f(\mathbf{y}), \mathbf{p}) = \hat{E}(\mathbf{x}, \mathbf{p})$.

It can be shown that our condition for neutral stability (definition 4), which is a global condition, involves satisfaction of the condition NNS_{IIV} (which is actually a set of local conditions).

When comparing our results with those in Carmichael and MacLeod (1997), Fujiwara-Greve and Okuno-Fujiwara (2009) or Izquierdo et al. (2021), the reader should keep in mind the different definitions of neutral stability used in each of those papers. In particular, many of the polymorphic equilibria discussed by Fujiwara-Greve and Okuno-Fujiwara (2009) do not satisfy definition 4 of neutral stability, and can be destabilized by other strategies in the replicator dynamics (see also Vesely and Yang (2012)).

C Strategies robust against indirect invasions

Here we consider the equilibrium condition of robustness against indirect invasions or RAII (van Veelen, 2012) for a strategy in games with endogenous separation. It can be argued that any reasonable extension of this concept to games with endogenous separation would require at least neutral stability and that every weakly path-equivalent strategy is also neutrally stable, where j is said to be weakly path-equivalent to i if $h_{jj}^{[\infty]} = h_{ii}^{[\infty]} = h_{ij}^{[\infty]}$ (the second equality is implied by the first), which implies that any mixture $\mathbf{y}$ of strategies i and j satisfy $E(\mathbf{y}, \mathbf{e}_i) = F_{ii} = F_i(\mathbf{y}) = E(\mathbf{y}, \mathbf{y})$. With these minimum requirements, our results below show that, in many cases of interest, there are no RAII strategies in games with endogenous separation. We first show that being RAII requires playing Nash action profiles of the stage game and, in most cases of interest, it requires $T_{ii} = \infty$ and a sufficiently low value of δ. For (sufficiently) large values of δ, and unless the maximum payoff of the stage game is attained at a symmetric Nash action profile, no strategy is robust against indirect invasions. The reason is that every strategy i has a path-equivalent strategy j_1 that would let a potential invader j_2 who deviates in action (from i or j_1) at the first stage of an j_2-j_1 partnership obtain the maximum stage game payoff afterwards, in an infinite path $a_{j_2 j_1}^{[1,\infty]}$. The payoff $F_{j_2 j_1}$ to such a strategy j_2 converges to the maximum stage game payoff as $\delta \to 1$.

Proposition C.1. *A strategy $i \in \Omega$ can be robust against indirect invasions only if the action profiles played in the i-i equilibrium path are Nash profiles of the stage game.*

It follows from lemma 3.5 that, unless the maximum symmetric payoff of the stage game corresponds to a Nash profile, $T_{ii} = \infty$ is also a necessary condition for a strategy to be RAII, as it is a necessary condition for neutral stability.

Proof of proposition C.1. Suppose that the action profile $(a_i^{[t]}, a_i^{[t]}) = (a, a)$ is not Nash. Consider two strategies j and k such that:

- Strategy j is path-equivalent to i, so $F_{jj} = F_{ii}$.
- Strategy k behaves with j (or with i) like j up to stage t (i.e., $a_{kj}^{[1,t-1]} = a_{jj}^{[1,t-1]} = a_{ii}^{[1,t-1]}$ if $t > 1$) and deviates at t by playing a best response action to action a, obtaining at that stage a greater payoff than what j obtains in a j-j partnership.

- From stage t, strategies j and k do not break up and play the action profile that provides k the maximum possible payoff of the stage game.

Then $F_{kj} > F_{jj}$, so strategy j is not Nash. $\square$

Proposition C.2. *For stage games with a single Nash action profile which does not obtain the maximum symmetric payoff, such as the Prisoner's Dilemma, no strategy in the game with endogenous separation is RAII.*

Proof of proposition C.2. The only possible candidates to be RAII are strategies with $T_{ii} = \infty$ that always play the Nash action profile at the equilibrium. But any such strategy i has a weakly path-equivalent strategy j with finite T_{jj} that always plays the Nash action profile in $a_{jj}^{[1,T_{jj}]}$, and which, by lemma 3.5, is not neutrally stable. $\square$

Example C.1. *For the Prisoner's Dilemma, the only Nash action profile is DD and it does not obtain the maximum symmetric payoff U_{CC}, so there are no RAII strategies in the game with endogenous separation. For the Hawk-Dove game, no symmetric action profile is Nash, so there are no RAII strategies in the game with endogenous separation.*

D Proofs

Proof of proposition 1. Equation (1) can be stated as

$$\mathbf{x} = \frac{\mathbf{p} \circ (B\mathbf{p})}{||\mathbf{p} \circ (B\mathbf{p})||_1} \text{ with } \mathbf{p} \in \Delta(S), \tag{14}$$

where $\circ$ represents the Hadamard product. We first show that the vectors $\mathbf{p} \in \Delta(S)$ satisfying (14) are the same vectors satisfying

$$\mathbf{p} \circ (B\mathbf{p}) = \lambda \mathbf{x} \text{ for some } \lambda > 0, \text{ with } \mathbf{p} \in \Delta(S). \tag{15}$$

It is immediate that (14) $\implies$ (15), with $\lambda = ||\mathbf{p} \circ (B\mathbf{p})||_1$. To show that (15) $\implies$ (14), from the equality in (15) we obtain $||\mathbf{p} \circ (B\mathbf{p})||_1 = \lambda ||\mathbf{x}||_1 = \lambda$.

Now, let $\tilde{\mathbf{p}}$ be a positive vector such that $\tilde{\mathbf{p}} \circ (B\tilde{\mathbf{p}}) = \mathbf{x}$. By an extension of Sinkhorn's theorem known as the DAD theorem (Idel, 2016; Brualdi, 1974), this vector exists and it is unique. It is clear that the vector $\frac{\tilde{\mathbf{p}}}{||\tilde{\mathbf{p}}||_1}$ satisfies (15), with $\lambda = \frac{1}{||\tilde{\mathbf{p}}||_1^2}$. It is also the only vector in $\Delta(S)$ that satisfies (15), because if $\mathbf{q} \in \Delta(S)$ satisfies $\mathbf{q} \circ (B\mathbf{q}) = \lambda_q \mathbf{x}$, then, by uniqueness of the solution of the matrix scaling problem $\tilde{\mathbf{p}} \circ (B\tilde{\mathbf{p}}) = \mathbf{x}$, we have $\mathbf{q} = \sqrt{\lambda_q}\tilde{\mathbf{p}}$ which, together with $||\mathbf{q}||_1 = 1$, implies $\mathbf{q} = \frac{\tilde{\mathbf{p}}}{||\tilde{\mathbf{p}}||_1}$. $\square$

Proof of proposition 2. Note that $x_i = 0 \iff p_i = 0$ so, without loss of generality, we can assume that S is the support of $\mathbf{x}$ and both $\mathbf{x}$ and $\mathbf{p}$ are (positive) vectors in $\mathbb{R}^{|S|}_{>0}$. Proposition 1 proved the existence of the inverse function f^{-1}.

Given the symmetric positive matrix B with elements $B_{ij} = 1 - \delta^{T_{ij}}$, the function $\mathbf{x} \mapsto \tilde{\mathbf{p}}$ implicitly defined by $\tilde{\mathbf{p}} \circ (B\tilde{\mathbf{p}}) - \mathbf{x} = \mathbf{0}$ is Lipschitz for $\mathbf{x} \in \Delta(S)$ because:

- The Jacobian of $\tilde{\mathbf{p}} \circ (B\tilde{\mathbf{p}})$ has terms $J_{ij}(\tilde{\mathbf{p}}) = \tilde{p}_i B_{ij} + \delta_{ij} \sum_k B_{ik}\tilde{p}_k$, where δ_{ij} is the Kronecker delta. It is column strictly diagonally dominant in $\mathbb{R}^{|S|}_{>0}$ and non-singular in $\mathbb{R}^{|S|}_{\geq 0} \setminus \{\mathbf{0}\}$.
- $||\tilde{\mathbf{p}}||_1 \geq 1$ for $\mathbf{x} \in \Delta(S)$, which follows from $||\tilde{\mathbf{p}}||_1^2 \geq \frac{||\tilde{\mathbf{p}} \circ (B\tilde{\mathbf{p}})||_1}{Max(B_{ij})} = \frac{1}{Max(1-\delta^{T_{ij}})} \geq 1$.

Combining both results we have that the Jacobian determinant is bounded away from 0 for (values of $\tilde{\mathbf{p}}$ such that) $\mathbf{x} \in \Delta(S)$. The function $\mathbf{x} \mapsto \mathbf{p} = \frac{\tilde{\mathbf{p}}}{||\tilde{\mathbf{p}}||_1}$ is also Lipschitz because $||\tilde{\mathbf{p}}||_1 \geq 1$.

Let us last show that f^{-1} does not admit a general algebraic expression. For a given population state $\mathbf{x}$ with support S and a given value for δ, the system of equations (14) is a polynomial system in the components of $\mathbf{p}$. The solution of this system can be found using Gröbner basis (Cox et al., 2015). For four strategies in S, it is easy to find examples of (14) with rational coefficients (rational $\mathbf{x}_i$ and δ) which do not admit a solution in radicals, proving that, for more than three strategies, there is no general solution in radicals to (14), i.e., there is no general formula that allows to calculate $\mathbf{p}$ from $\mathbf{x}, \delta$ and (T_{ij}) using addition, subtraction, multiplication, division, and root extraction. For instance, for four strategies (numbered from 1 to 4), $\mathbf{x} = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$, $T_{ij} = \min(i, j)$ and $\delta = \frac{1}{2}$, the solution to (14) using Gröbner basis leads as an intermediate step to a univariate polynomial in p_4 (of degree 8) that is not solvable in radicals. This can be checked using a Computational Algebra System such as Magma (Bosma et al., 1997), by calculating the Galois group of the polynomial with rational coefficients and checking that the Galois group is not solvable. $\square$

Proof of lemma 3.1. Let i be a Nash strategy and let $a^{\emptyset} = i(\emptyset)$ be the first action played by i. Let j be a strategy that plays a best-response action to $a^{\emptyset}$ when starting a new partnership, i.e., $j(\emptyset) \in BR(a^{\emptyset})$, and then breaks the partnership. We have $F_{ji} = \max_l U(a_l, a^{\emptyset})$. Considering that M is an upper bound for F_{ii}, the Nash condition $F_{ii} \geq F_{ji}$ requires $M \geq \max_l U(a_l, a^{\emptyset})$ or, equivalently, $U^{BR}(a^{\emptyset}) \leq M$. $\square$

Proof of lemma 3.2. State $\mathbf{x}$ has an associated pool state $\mathbf{p} = f^{-1}(\mathbf{x})$. Any strategy arriving at the pool of singles $\mathbf{p}$ to be matched faces a distribution of initial actions $q \in \Delta(A)$ (given by $q_k = \sum_{i \in \mathbb{S}(\mathbf{x}):i(\emptyset)=a_k} p_i$). Given a state $\mathbf{x}$ and its associated q, consider a strategy j that at the beginning of a partnership plays a best response action to the distribution of actions q and then breaks the partnership. The payoff $F_j(\mathbf{x})$ to such a strategy is at least $\underline{m}$. Consequently, if $\mathbf{x}$ is Nash, then $E(\mathbf{x}, \mathbf{x})$ has to be greater than or equal to $\underline{m}$. For monomorphic states, we have that F_{ji} is at least m, while M is an upper bound for F_{ii}. $\square$

Proof of lemma 3.3. Suppose that i is a strategy with finite self-breakup period T_{ii} and the last action profile $(a_i^{[T_{ii}]}, a_i^{[T_{ii}]})$ in an i-i partnership is not a Nash profile of the stage game G. Consider a strategy j that when playing against i:

- behaves against i as i itself up to stage $T_{ii} - 1$, i.e., $j(a_{ii}^{[1,t]}) = i(a_{ii}^{[1,t]})$ for $0 \leq t < T_{ii} - 1$,

- at stage T_{ii} of an i-j partnership plays a best-response action against the action $a_i^{[T_{ii}]}$ played by i at that stage, and
- leaves i (i.e., breaks the partnership with i) after stage $T_{ij} = T_{ii}$.

Strategy j obtains the same stage payoff against i as i itself for the first $T_{ii} - 1$ stages of a partnership and a higher payoff at the last stage T_{ii}. Consequently, considering (8), $F_{ji} > F_{ii}$, so i cannot be a Nash strategy. $\square$

Proof of observation 1. Suppose that i is a Nash strategy with finite self-breakup period T_{ii} that plays the action profile CC at some stage (between stages $t = 1$ and $t = T_{ii}$) of an i-i partnership. Then we have $F_{ii} > U_{DD}$. Let t_l be the last stage at which CC is played. Consider a strategy j that when playing against i:

- behaves against i as i itself up to stage $t_l - 1$, i.e., $j(a_{ii}^{[1,t]}) = i(a_{ii}^{[1,t]})$ for $0 \leq t < t_l - 1$,
- at stage t_l of an i-j partnership plays action D, obtaining a stage payoff $U_{DC} > U_{CC}$, and
- breaks the partnership with i after stage t_l.

  Using formula (8), it can be seen that $F_{ji} > F_{ii}$, so i is not a Nash strategy (contradiction). The reason, comparing the sequence of payoffs to i in the infinite series $h_{ii}^{[\infty]}$ and to j in the infinite series $h_{ji}^{[\infty]}$ is that j obtains a higher payoff at stage t_l and (if $t_l < T_{ii}$) shortens the sequence of lowest payoffs U_{DD} until the next high payoffs U_{CC} or U_{DC}.

$\square$

Proof of lemma 3.4. With the conditions on i, the infinite series of actions that a j-player faces in a population of i-players (see 7) is $(a^N, a^N, \ldots)$. The best stage-payoff against a^N is obtained by a^N, and, considering that $F_{ji} = V(h_{ji}^{\infty}) = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} U(h_{ji}^{[t]})$, the best payoff against any strategy i satisfying the condition is obtained by strategies j that generate the path $h_{ji}^{[\infty]} = ((a^N, a^N), (a^N, a^N), \ldots)$, which obtain the payoff $F_{ji} = U(a^N, a^N)$. If i_1 and i_2 satisfy the conditions for i, we have $F_{i_1}(\mathbf{e}_{i_1}) = F_{i_1}(\mathbf{e}_{i_2}) = F_{i_2}(\mathbf{e}_{i_1}) = F_{i_2}(\mathbf{e}_{i_2}) = U(a^N, a^N)$. As $F_i(\mathbf{x})$ is a strictly convex combination of the payoffs F_{ij} for $j \in \mathbb{S}(\mathbf{x})$, we have proved the result: if $\mathbf{x}$ is a mixture of strategies satisfying the condition for i, we have $F(\mathbf{x}, \mathbf{x}) = U(a^N, a^N) \geq F_j(\mathbf{x})$ for every $j \in \Omega$.

$\square$

Proof of proposition 3. Given any finite set of strategies S, we can number the strategies and identify distributions $\mathbf{x} \in \Delta(S)$ with real vectors $\hat{\mathbf{x}} \in \Delta^{|S|-1} \equiv \{\hat{\mathbf{x}} \in \mathbb{R}_+^{|S|} : \sum_{k=1}^{|S|} \hat{x}_k = 1\}$. The restriction of F_i to distributions with support in S can then be identified with a function $F_{i|S} : \Delta^{|S|-1} \to \mathbb{R}$. By proposition 2, $F_{i|S}$ is Lipschitz continuous in $\Delta^{|S|-1}$.[8]

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[8] A function $f : \Delta^{|S|-1} \to \mathbb{R}$ is Lipschitz continuous in $\Delta^{|S|-1}$ if there exists a positive real constant K such that, for all $\mathbf{x}$ and $\mathbf{y}$ in $\Delta^{|S|-1}$, $||f(\mathbf{x}) - f(\mathbf{y})|| \leq K||\mathbf{x} - \mathbf{y}||$.

Given any finite set of strategies $S \subset \Omega$ and a neutrally stable state $\mathbf{x} \in \Delta(S)$, it follows from definition 4 and from the Lipschitz property of the payoff functions $F_{i|S}$ in $\Delta^{|S|-1}$ that the point $\hat{\mathbf{x}} \in \Delta^{|S|-1}$ associated to state $\mathbf{x}$ satisfies the conditions in Thomas (1985) [Theorem 1] to be a weakly evolutionarily stable state in $\Delta^{|S|-1}$ and, consequently, $\mathbf{x}$ is Lyapunov stable in the replicator dynamics (11) in $\Delta(S)$. □

Proof of lemma 3.5. Let $(a^M, a^M) \in N_M^G$ be one of the symmetric action profiles (there may be more than one) that attain the maximum symmetric payoff $M = \max_{a \in A} U(a, a)$.

Suppose that T_{ii} is finite and $F_{ii} < M$. This implies that $h_{ii}^{[\infty]}$ is a repetition of a pattern of length T_{ii}, and, for any fixed t_0, there is always $t > t_0$ with $U(h_{ii}^{[t]}) < M$. Consider a strategy j that when playing with i behaves like i up to period T_{ii}, but at that period does not break the partnership and turns to playing action a^M forever, without breaking the partnership. That would make play between strategy i and strategy j unfold in the same way as it does between two players that play strategy i, with $h_{ji}^{[\infty]} = h_{ii}^{[\infty]} = h_{ij}^{[\infty]}$, and hence $F_{ji} = F_{ii} = F_{ij}$. For $t \leq T_{ii}$, two players that play strategy j obtain a payoff $U(h_{jj}^{[t]}) = U(h_{ii}^{[t]}) = U(h_{ij}^{[t]})$. For $t > T_{ii}$, we have $U(h_{jj}^{[t]}) = M$, while $U(h_{ij}^{[t]}) = U(h_{ii}^{[t]}) \leq M$ and, for some $t > T_{ii}$, $U(h_{ij}^{[t]}) < M$. Consequently, considering (8), $F_{jj} > F_{ij}$, so i is not neutrally stable. Up to now we have proved that if i is neutrally stable with finite T_{ii} then $F_{ii} = M$, which implies $U(h_{ii}^{[t]}) = M$ for every t. Suppose that payoff M is obtained at time t_1 by some action profile $h_{ii}^{t_1}$ which is not a Nash equilibrium of the stage game. Then a strategy j that when playing with i chooses the same action as i up to period t_1 (obtaining M at every period up to t_1 if $t_1 \geq 1$), but at period t_1 plays a best response the action taken in $h_{ii}^{t_1}$ and breaks the partnership, obtains a payoff $F_{ji} > M = F_{ii}$, which cannot happen if i is neutrally stable. □

Proof of proposition 4. Let strategy i be weakly path-protecting and let j_1 and j_2 be two alternative best responses to $\mathbf{e}_i$, i.e., $\{j_1, j_2\} \in BR(\mathbf{e}_i)$. Considering that the action profiles in $h_{ii}^{[\infty]}$ are symmetric, we have $h_{j_1 i}^{[\infty]} = h_{j_2 i}^{[\infty]} = h_{ii}^{[\infty]} = h_{ij_1}^{[\infty]} = h_{ij_2}^{[\infty]}$. In a j_1-j_2 partnership, no strategy can take an action different from the one they take when playing with i until the split-up period $T_{j_1 j_2} = \min(T_{ij_1}, T_{ij_2})$, because the generated histories up to that point are the same as in an i-i partnership and, until they break the partnership, both j_1 and j_2 take the same action as i does given the history. Consequently, $h_{j_1 j_2}^{[\infty]}$ coincides either with $h_{j_1 i}^{[\infty]} = h_{ii}^{[\infty]}$ or with $h_{j_2 i}^{[\infty]} = h_{ii}^{[\infty]}$. Then we have $h_{j_1 j_2}^{[\infty]} = h_{ii}^{[\infty]} = h_{ij_1}^{[\infty]} = h_{ij_2}^{[\infty]}$, which implies that, for $\{j_1, j_2\} \in BR(\mathbf{e}_i)$, $F_{j_1 j_2} = F_{j_1 j_1} = F_{ii} = F_{ij_1} = F_{ij_2}$. Now, if $\mathbf{y}$ is a mixture of best responses to $\mathbf{e}_i$ and j is a best response to $\mathbf{e}_i$, we have $F_j(\mathbf{y}) = F_{ii} = F_i(\mathbf{y})$ and, consequently, $E(\mathbf{y}, \mathbf{y}) = F_i(\mathbf{y})$, proving that i is neutrally stable.

□

Proof of lemma 4.1. Let i be a strategy satisfying the conditions of the proposition. It is clear that $T_{ii} = \infty$, $h_{ii}^{[\infty]} = ((\hat{a}, \hat{a}), (\hat{a}, \hat{a}), \ldots)$ and $F_{ii} = U(\hat{a}, \hat{a})$. Any strategy j playing with i-players generates a repeated path $h_{ji}^{[\infty]}$ in which the action taken by i is always

$\hat{a}$, so, given that $(\hat{a}, \hat{a})$ is a (strict) Nash profile and that any deviation from the profile $(\hat{a}, \hat{a})$ is caused by strategy j (i always plays $\hat{a}$, so the second action in the profile is always $\hat{a}$), we have $U(h_{ji}^{[t]}) \leq U(\hat{a}, \hat{a})$ for every t. In fact, since $(\hat{a}, \hat{a})$ is strict Nash, we have $h_{ji}^{[t]} \neq (\hat{a}, \hat{a}) \implies U(h_{ji}^{[t]}) < U(\hat{a}, \hat{a})$, and, considering that F_{ji} is a strictly convex combination of the payoffs $U(h_{ji}^{[t]})$, it follows that $h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]} \implies F_{ji} < U(\hat{a}, \hat{a}) = F_{ii}$, proving that i is weakly path-protecting. $\square$

Proof of proposition 5. Consider a strategy i such that $T_{ii} = \infty$ and

$$h_{ii}^{[\infty]} = (\Phi_m, \Phi_f, (\Phi_p)^{\infty}),$$

where:

- Φ_m is a T_m-long repetition of a minmax action profile $(\tilde{a}, \tilde{a})$.
- Φ_f is a T_f-long sequence of symmetric action profiles.
- Φ_p is a T_p-long sequence of symmetric action profiles with average stage payoff $\bar{U}_{\Phi_p} > m$.

As soon as another strategy j in an i-j partnership deviates from i's own action, strategy i breaks the partnership.

Take Φ_f and Φ_p as fixed, and the length T_m of Φ_m as a parameter. We will show that, for large enough T_m and, then, for large enough δ,

$$h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]} \implies F_{ji} < F_{ii},$$

i.e., strategy i is weakly path-protecting. By choosing Φ_p in a way such that path $h_{ii}^{[\infty]}$ is not an infinite repetition of a pattern, strategy i is also path-protecting.

We will need some intermediate results. First, lemma D.1 implies that, in order to prove the implication $h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]} \implies F_{ji} < F_{ii}$, it is enough to prove that this statement holds for strategies j whose repeated path $h_{ji}^{[\infty]}$ differs or deviates from $h_{ii}^{[\infty]}$ before repetition of the pattern Φ_p begins, i.e., between periods $t = 1$ and $t = T_m + T_f + T_p$: if every deviation up to period $t = T_m + T_f + T_p$ is harmful, then every deviation (no matter when) is harmful. Consequently, it is enough to consider a finite number of possible deviating paths: those that deviate at some t not greater than $T_m + T_f + T_p$.

Second, the payoff to a strategy that deviates at $t \leq T_m$ is bounded above by the minmax payoff m (because i plays a minmax action up to stage T_m, so the stage payoff for a strategy j at every stage up to and including the deviating stage $t \leq T_m$ is bounded above by m). Let L be the maximum payoff in the stage game. Considering a repeated sequence $(m, ..., m, L, ..., L)$ of T_m payoffs m and $T_f + T_p$ payoffs L, we have that the payoff to a strategy that deviates not later than $T_m + T_f + T_p$ is bounded above[9] by

$$V_L \equiv \frac{m(1-\delta^{T_m}) + \delta^{T_m}(1-\delta^{T_f+T_p})L}{1-\delta^{T_m+T_f+T_p}},$$

and V_L is non-decreasing with δ (increasing if $L > m$).

Third, if an infinite sequence of action profiles Φ ends up repeating some finite pattern Φ_1, i.e., if $\Phi = (\Phi_0, (\Phi_1)^\infty)$ for some finite sequences Φ_0 and Φ_1, then[10]

$$\lim_{\delta \to 1} V(\Phi) = \bar{U}_{\Phi_1}.$$

This implies

$$V_L \leq \lim_{\delta \to 1} V_L = \alpha \equiv \frac{m\,T_m + L\,(T_f + T_p)}{T_m + T_f + T_p}, \tag{16}$$

with $\lim_{T_m \to \infty} \alpha = m$, and

$$\lim_{\delta \to 1} F_{ii} = \bar{U}_{\Phi_p} > m. \tag{17}$$

Choose some positive $\epsilon < \frac{\bar{U}_{\Phi_p} - m}{2}$. From (16), and considering that α approaches m as T_m grows, we can find a value for T_m such that $\alpha < m + \epsilon$, and then, fixing such T_m, we have $V_L < m + \epsilon$.

From (17), there is some $\delta_1 < 1$ such that, for $\delta > \delta_1$, $F_{ii} > \bar{U}_{\Phi_p} - \epsilon > m + \epsilon$. Consequently, for $\delta > \delta_1$,

$$V_L < F_{ii},$$

proving that strategy i is path-protecting.

□

For any finite series of action profiles Φ, let $(\Phi)^k$ represent the sequence made up by repeating k times the action profiles in Φ. Remember that $(\Phi)^\infty$ represents the infinite repetition.

Lemma D.1. *Consider two (not necessarily different) strategies j and i with $h^\infty_{ji} = (\Phi_0, (\Phi_p)^\infty)$, where Φ_0 and Φ_p are finite sequences of action profiles (and where Φ_0 may be empty). Let Φ_1 be another finite sequence of action profiles. If j_1 and j_2 are strategies such that*

$$h^\infty_{j_1 i} = (\Phi_0, \Phi_1)^\infty \text{ and}$$

$$h^\infty_{j_2 i} = (\Phi_0, (\Phi_p)^k, \Phi_1)^\infty \text{ for some } k \in \mathbb{N}$$

then

$$sgn(F_{j_1 i} - F_{ji}) = sgn(F_{j_2 i} - F_{ji}).$$

---

[9] It is easy to check that, for a fixed number of m payoffs T_m, V_L is non-decreasing with the number of L payoffs (V_L is a weighted average of m and $L \geq m$, with the weight of m decreasing if the number of L payoffs increases), so, by taking a number of L values equal to $T_f + T_p$, we can be sure that V_L is an upper bound for the payoff to any strategy that deviates up to $t = T_m + T_f + T_p$.

[10] This can be shown using L'Hopital rule.

Proof of lemma D.1. For any sequence Φ of length $T \geq 1$, let

$$V(\Phi) = \frac{1-\delta}{1-\delta^T} \sum_{t=1}^{T} \delta^{t-1}\, U(\Phi^{[t]}).$$

Let the respective lengths of Φ_0, Φ_p and Φ_1 be $T_0 \geq 0, T_p \geq 1$ and $T_1 \geq 1$. If $T_0 = 0$ let $V(\Phi_0) = 0$. Then

$$F_{ji} = (1-\delta^{T_0})V(\Phi_0) + \delta^{T_0} V(\Phi_p),$$

$$F_{j_1 i} = \frac{(1-\delta^{T_0})V(\Phi_0) + \delta^{T_0}(1-\delta^{T_1})V(\Phi_1)}{1-\delta^{T_0+T_1}}, \text{ and}$$

$$F_{j_2 i} = \frac{(1-\delta^{T_0})V(\Phi_0) + \delta^{T_0}(1-\delta^{k\,T_p})V(\Phi_p) + \delta^{T_0+k\,T_p}(1-\delta^{T_1})V(\Phi_1)}{1-\delta^{T_0+k\,T_p+T_1}}.$$

Any of the two conditions $F_{j_1 i} < F_{ji}$ or $F_{j_2 i} < F_{ji}$ can then be seen to be equivalent (rearranging and simplifying terms) to the condition

$$\delta^{T_1}(1-\delta^{T_0})V(\Phi_0) + (1-\delta^{T_1})V(\Phi_1) < (1-\delta^{T_0+T_1})V(\Phi_p).$$

And this result holds if the inequalities are reversed. $\square$

Proof of proposition 6. For histories that do not occur along the equilibrium path it is immediate that if i is Nash then $F_{ji|a^{[1,t]}} \leq F_{ii|a^{[1,t]}}$ because for such histories $F_{ji|a^{[1,t]}} = F_{ji}$ and $F_{ii|a^{[1,t]}} = F_{ii}$. For histories that occur along the equilibrium path, if there is some stage T at which some strategy j satisfies $F_{ji|a_{ii}^{[1,T]}} > F_{ii|a_{ii}^{[1,T]}}$ then the strategy j' defined by $j'(a^{[1,t]}) = i(a^{[1,t]})$ for $t \leq T$ and $j'(a^{[1,t]}) = j(a^{[T+1,t]})$ for $t > T$ satisfies

$$F_{j'i} = (1-\delta)\sum_{t=1}^{T} \delta^{t-1} U(a_{ii}^{[t]}) + \delta^T F_{ji|a_{ii}^{[1,T]}} > (1-\delta)\sum_{t=1}^{T} \delta^{t-1} U(a_{ii}^{[t]}) + \delta^T F_{ii|a_{ii}^{[1,T]}} = F_{ii},$$

which is not possible given that i is Nash. $\square$

Proof of proposition A.1. Suppose that a Nash equilibrium state $\mathbf{x}$ includes a weakly path protecting strategy i and some strategy j with $h_{ji}^{[\infty]} \neq h_{ii}^{[\infty]}$, then:

- $F_i(\mathbf{x}) = E(\mathbf{x}, \mathbf{x})$, because $\mathbf{x}$ is Nash and i is in its support, so $i \in BR(\mathbf{x})$, and
- $E(\mathbf{x}, \mathbf{e}_i) < F_{ii}$, because i is weakly path-protecting (so it is Nash) and $\mathbf{x}$ includes a strategy j that deviates from $h_{ii}^{[\infty]}$ when playing with i, obtaining a payoff $F_{ji} < F_{ii}$.

Consequently, $\mathbf{x}$ is not neutrally stable. $\square$

Proof of proposition A.2. Let $Eq(\mathbf{x})$ be the set of strategies that are path-equivalent in $\mathbb{S}(\mathbf{x})$ to some strategy in $\mathbb{S}(\mathbf{x})$, and let $\bar{E}q(\mathbf{x})$ be the complement of this set. As $\mathbf{x}$ is Nash and path-protecting, we have $F_i(\mathbf{x}) = E(\mathbf{x}, \mathbf{x})$ for $i \in Eq(\mathbf{x})$ and $F_i(\mathbf{x}) < E(\mathbf{x}, \mathbf{x})$ for $i \in \bar{E}q(\mathbf{x})$. Consequently, any state $\mathbf{y}$ that includes strategies both in $Eq(\mathbf{x})$ (for which $F_i(\mathbf{x}) = E(\mathbf{x}, \mathbf{x})$) and in $\bar{E}q(\mathbf{x})$ satisfies $E(\mathbf{y}, \mathbf{x}) < E(\mathbf{x}, \mathbf{x})$, and only mixtures of strategies in $Eq(\mathbf{x})$ can be (are) alternative best responses to $\mathbf{x}$. Because any strategy that is path-equivalent in $\mathbb{S}(\mathbf{x})$ to strategy $i \in \mathbb{S}(\mathbf{x})$ behaves like i does with strategies in $Eq(\mathbf{x})$, for any mixture $\mathbf{y}$ of strategies in $Eq(\mathbf{x})$ there is an "internal" state $\hat{\mathbf{y}}$ satisfying $\mathbb{S}(\hat{\mathbf{y}}) = \mathbb{S}(\mathbf{x})$ such that $E(\hat{\mathbf{y}}, \mathbf{x}) = E(\mathbf{x}, \mathbf{x})$, $E(\mathbf{x}, \mathbf{y}) = E(\mathbf{x}, \hat{\mathbf{y}})$ and $E(\mathbf{y}, \mathbf{y}) = E(\hat{\mathbf{y}}, \hat{\mathbf{y}})$. Consequently, internal neutral stability (which guarantees $E(\mathbf{x}, \hat{\mathbf{y}}) \geq E(\hat{\mathbf{y}}, \hat{\mathbf{y}})$) guarantees neutral stability ($E(\mathbf{x}, \mathbf{y}) \geq E(\mathbf{y}, \mathbf{y})$). $\square$