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SYZYGIES, REGULARITY, AND THEIR INTERPLAY WITH ADDITIVE COMBINATORICS

Presentada por Mario González Sánchez para optar al grado de
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Abstract

In this thesis, we study some interactions between commutative algebra and additive combinatorics. Based on recent works by Eliahou and Mazumdar [30], Elias [32], and Colarte-Gómez, Elias and Miró-Roig [18], we associate with each finite set $\mathcal{A} \subset \mathbb{N}^d$ a projective toric variety $\mathcal{X} \subset \mathbb{P}^n_{\mathbb{k}}$, where $\mathbb{k}$ is an infinite field and $n = |\mathcal{A}|-1$. We focus on the study of the sumsets of $\mathcal{A}$ and the Castelnuovo-Mumford regularity of $\mathbb{k}[\mathcal{X}]$, the coordinate ring of $\mathcal{X}$. In particular, we look at the cases when $\mathcal{X}$ is a curve, a smooth variety, and a surface with a single singular point. Moreover, when $\mathcal{X}$ is a curve $\mathcal{C}$, we study the relation between the Betti numbers of $\mathbb{k}[\mathcal{C}]$ and its affine charts. Finally, we provide an explicit method to compute the minimal graded free resolution of R/I as A-module, where $I \subset R = \mathbb{k}[x_1, \ldots, x_n]$ is a weighted homogeneous ideal and $A = \mathbb{k}[x_{n-d+1}, \ldots, x_n]$, whenever the variables are in Noether position.

Resumen

En esta tesis, estudiamos algunas interacciones entre el álgebra conmutativa y la combinatoria aditiva. Basándonos en los recientes trabajos de Eliahou [31], Elias [32], y Colarte-Gómez, Elias y Miró-Roig [18], a cada conjunto finito $\mathcal{A} \subset \mathbb{N}^d$ le asociamos una variedad tórica proyectiva $\mathcal{X} \subset \mathbb{P}_{\Bbbk}^n$, donde $\Bbbk$ es un cuerpo infinito y $n = |\mathcal{A}| - 1$. Nos centramos en el estudio de los conjuntos suma de $\mathcal{A}$ y la regularidad de Castelnuovo-Mumford de $\Bbbk[\mathcal{X}]$, el anillo de coordenadas de $\mathcal{X}$. En particular, nos fijamos en los casos en que $\mathcal{X}$ es una curva, una variedad lisa o una superficie con un único punto singular. Además, cuando $\mathcal{X}$ es una curva $\mathcal{C}$, estudiamos la relación entre los números de Betti de $\Bbbk[\mathcal{C}]$ y sus cartas afines. Por último, proporcionamos un método explícito para construir la resolución libre minimal graduada de R/I como A-módulo, donde $I \subset R = \Bbbk[x_1, \ldots, x_n]$ es un ideal homogéneo para unos ciertos pesos y $A = \Bbbk[x_{n-d+1}, \ldots, x_n]$, suponiendo que las variables están en posición de Noether.

Contents

Introduction

"Let no one ignorant of algebra enter here."
Adapted from the inscription at Plato's Academy

Graded free resolutions were introduced by Hilbert to compute the so-called Hilbert function of a homogeneous ideal in the polynomial ring. Using resolutions and Hilbert's Syzygy Theorem, one gets that the Hilbert function becomes a polynomial for sufficiently large values of the input, and this polynomial contains valuable geometric information about the ideal. Moreover, graded free resolutions can be used to compute other invariants such as the depth (and, equivalently, using the Auslander-Buchsbaum formula, the projective dimension) or the Castelnuovo-Mumford regularity.

The first result to compute graded free resolutions using Gröbner bases was obtained by Buchberger, who proved that the reductions of the S-polynomials of a Gröbner basis provide a finite generating set of the first syzygy module, i.e., the first step in a resolution. Applying this result repeatedly (and using Gröbner bases for modules), one can construct a graded free resolution through several Gröbner bases computations. Later, Schreyer introduced a monomial order for which the generating set of the syzygy module is indeed a Gröbner basis, and hence, one only needs one Gröbner basis computation to construct a graded free resolution (that may not be minimal), the so-called Schreyer resolution.

The study of graded Betti numbers has attracted a lot of attention and is a classical problem in commutative algebra, since they encode the numerical information in any minimal graded free resolution of I, and hence they are enough to compute the Hilbert function and other invariants. However, the Hilbert function does not determine the Betti numbers. For instance, Eisenbud proves in his book [27] that the Hilbert function of the coordinate ring of seven points in general position in $\mathbb{P}^3$ does not depend on the relative position of the points. But the Betti numbers depend on whether the points lie on a curve of degree 3 or not.

On the other hand, given G an abelian semigroup with identity and a finite set $\mathcal{A} \subset G$, additive combinatorics studies the sets of sums of elements in $\mathcal{A}$ and their cardinality. For all $s \in \mathbb{Z}_{>0}$, the *s-fold iterated sumset of* $\mathcal{A}$ is defined as the set of sums of s elements in $\mathcal{A}$, with the convention $0A := \{0\}$. In 1992, Khovanskii proved in [57] that the function $\mathbb{N} \to \mathbb{N}$, $s \mapsto |s\mathcal{A}|$ is asymptotically polynomial. This is the same situation that appears in the study of the Hilbert function. Indeed, Khovanskii's proof is based on the existence of the Hilbert polynomial of a certain finitely generated graded module. This is the first interaction between the two fields, commutative algebra and additive combinatorics.

Khovanskii's theorem has recently attracted the attention of some researchers. In 2022, Eliahou and Mazumdar gave a new proof of this result in [30]. In their proof, they associate with $\mathcal{A}$ a standard graded $\Bbbk$-algebra $R(\mathcal{A})$, whose Hilbert function is $s \mapsto |s\mathcal{A}|$. A geometric counterpart when $\mathcal{S} = \mathbb{N}^d$ can be found in the paper [18] by Colarte-Gómez, Elias and Miró-Roig. The special case $d = 1$ is treated in the paper [32] by Elias. In [18] and [32], the authors associate with $\mathcal{A}$ a certain projective toric variety.

The main objective of the thesis is to exploit this relation to obtain new results on the Betti numbers and the Castelnuovo-Mumford regularity of projective toric varieties and, on the other hand, to obtain and improve known results in additive combinatorics. We treat the following four topics:

- Betti numbers of projective and affine monomial curves (Chapter 2).
- Structure theorems for sumsets in additive combinatorics (Chapter 3).
- Castelnuovo-Mumford regularity of simplicial projective toric curves and surfaces, and its relation to sumsets (Chapter 4).
- Short resolution of a weighted homogeneous ideal (Chapter 5).

In Chapter 1, we introduce the background and notation necessary for the rest of the thesis. We start with an introduction to numerical and affine semigroups in Section 1.1. In Section 1.2 we study free resolutions; in Section 1.3, we introduce toric ideals and toric varieties, the basic objects in this thesis. Finally, in Section 1.4 we explain the relation between commutative algebra and additive combinatorics. Although it is an introductory chapter, this chapter contains two novel results: Theorem 1.51, which gives a precise relation between the Castelnuovo-Mumford regularity and the regularity of the Hilbert function in terms of some Betti numbers; and Proposition 1.78, which provides the specific form of the parametrization of a simplicial projective toric surface with a single singular point.

In Chapter 2, we study the Betti numbers of projective and affine monomial curves. Consider a set $\mathcal{A} = \{a_0 = 0 < a_1 < \cdots < a_{n-1} < a_n = D\} \subset \mathbb{N}$ such that $\gcd(a_1, \ldots, a_n) = 1$, and let $\underline{\mathbf{a}}_i = (D - a_i, a_i) \in \mathbb{N}^2$ for all $i = 0, \ldots, n$ and $\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \ldots, \underline{\mathbf{a}}_n\}$. Fix an infinite field $\mathbb{k}$ and let $\mathcal{C} \subset \mathbb{P}^n_{\mathbb{k}}$ be the projective monomial curve determined by $\underline{\mathcal{A}}$. One of the two affine charts of $\mathcal{C}$ is $\mathcal{C}_1 \subset \mathbb{A}^n_{\mathbb{k}}$, the affine monomial curve determined by $\mathcal{A}_1 = \{a_1, \ldots, a_n\}$. Let $\mathbb{k}[\mathcal{C}]$ and $\mathbb{k}[\mathcal{C}_1]$ be the coordinate rings of $\mathcal{C}$ and $\mathcal{C}_1$, respectively. In Section 2.1, we define the Apéry set and posets of the semigroups $\mathcal{S} = \langle \underline{\mathcal{A}} \rangle$ and $\mathcal{S}_1 = \langle \mathcal{A}_1 \rangle$, and we characterize the Cohen-Macaulayness of $\mathbb{k}[\mathcal{C}]$ in terms of the Apéry poset of $\mathcal{S}$. The main result of Section 2.2 (and of this chapter) is Theorem 2.12, which provides a combinatorial contidion on the Apéry posets of $\mathcal{S}$ and $\mathcal{S}_1$ that ensures the equality of the Betti numbers of $\mathbb{k}[\mathcal{C}]$ and $\mathbb{k}[\mathcal{C}_1]$. In Propositions 2.18 and 2.23, we give two families for which the condition in Theorem 2.12 is satisfied: arithmetic sequences and their first projections. Using Theorem 2.12, in Theorem 2.26 we improve Vu's bound for the equality of the Betti numbers of $\mathbb{k}[\mathcal{C}]$ and $\mathbb{k}[\mathcal{C}_1]$ for the shifts of $\mathcal{A}_1$, and in Section 2.4, we provide a method to construct an arithmetically Gorenstein projective monomial curve $\mathcal{C}$ starting from a symmetric semigroup $\mathcal{S}_1$; see Theorem 2.32. Finally, in Section 2.5, we study the Betti numbers of $\mathcal{C}_1$ for a certain class of numerical semigroups defined by Kunz and Waldi. The main results in this section are Theorem 2.49, where we characterize when the defining ideal of $\mathcal{C}_1$ is determinantal; and Theorem 2.53, where we provide the Betti numbers of some of the semigroups in the Kunz-Waldi class.

In Chapters 3 and 4, we explore the relations between additive combinatorics and commutative algebra initiated in the papers [18, 30, 32]. In particular, in Chapter 3 we show how additive combinatorics benefits from commutative algebra, and in Chapter 4 we show how commutative algebra benefits from additive combinatorics.

Given a finite set $\mathcal{A} = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$, the s-fold iterated sumset of $\mathcal{A}$ is defined by $s\mathcal{A} = \{\mathbf{a}_{i_1} + \cdots + \mathbf{a}_{i_s} \mid 0 \leq i_1 \leq \ldots \leq i_s \leq n\}$ for $s \in \mathbb{Z}_{>0}$ and $0\mathcal{A} = \{\mathbf{0}\}$. In Chapter 3, we study the structure of the sumsets of $\mathcal{A}$ for $s \gg 0$. In Section 3.1 we consider the case $d = 1$. We start this section recalling the classical structure theorem by Nathanson (Theorem 3.1). Fix an infinite field $\mathbb{k}$ and consider a set $\mathcal{A} \subset \mathbb{N}$ and the same projective monomial curve as in Chapter 2, $\mathcal{C} \subset \mathbb{P}^n_{\mathbb{k}}$. Elias provided in [32, Prop. 3.4] a characterization of the elements in the structure theorem in terms of the curve $\mathcal{C}$. In this section, we complete this characterization defining the *sumsets regularity* of $\mathcal{A}$, $\sigma(\mathcal{A})$, and expressing it in terms of the curve $\mathcal{C}$ in Theorem 3.7. The rest of the section is devoted to improve the known upper bounds on $\sigma(\mathcal{A})$. We propose a new upper bound (3.5) and compare it with the existing ones. Using these results, in Section 3.2 we describe the structure of the sumsets of $\underline{\mathcal{A}}$. In Section 3.3, we study the sumsets of $\mathcal{A}$ when $d \geq 2$. Take $D := \max\{|\mathbf{a}_i|: i = 0, \ldots, n\}$ and consider $\underline{\mathbf{a}}_i = (D - |\mathbf{a}_i|, \mathbf{a}_i) \in \mathbb{N}^{d+1}$ for all $i = 0, \ldots, n$, and $\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \ldots, \underline{\mathbf{a}}_n\}$. Let $\mathcal{X} = \mathcal{X}_{\underline{\mathcal{A}}} \subset \mathbb{P}^n_{\Bbbk}$ be the projective toric variety defined by $\underline{\mathcal{A}}$ and assume that $\mathcal{X}$ is simplicial. In this context, the structure theorem by Curran and Goldmakher [23, Thm. 1.3] provides a value s_0 such that for all $s \geq s_0$, the sumsets $s\mathcal{A}$ can be explicitly described. We improve this result in two particular cases: $\mathcal{X}$ smooth and $\mathcal{X}$ a surface with a single singular point. In Theorem 3.26 we characterize the sets $\mathcal{A}$ for which $\mathcal{X}$ is a smooth variety, in terms of the shape of $\mathcal{A}$ and also in terms of its sumsets. We define a sumsets regularity for $\mathcal{A}$ and provide a tight upper bound in Theorem 3.29. Similarly, in Theorem 3.35 we characterize the sets $\mathcal{A}$ for which $\mathcal{X}$ is a surface with a single singular point, in terms of the shape of $\mathcal{A}$ and also in terms of its sumsets. We define a sumsets regularity for $\mathcal{A}$ and provide an upper bound on it in Proposition 3.40. Finally, in Theorem 3.41 we improve the previous bound in some cases.

In Chapter 4, we study the Castelnuovo-Mumford regularity of projective monomial curves and simplicial projective monomial surfaces, with a special emphasis on the Eisenbud-Goto conjecture. In Section 4.1, we provide a combinatorial formula to compute the Castelnuovo-Mumford regularity of $\Bbbk[\mathcal{C}]$ in terms of the Apéry and exceptional sets of $\mathcal{S}$ in Theorem 4.2. Moreover, in Theorem 4.9 we determine the step of a minimal graded free resolution of $\Bbbk[\mathcal{C}]$ in which the regularity is attained. Using the combinatorial formula for the Castelnuovo-Mumford regularity of $\Bbbk[\mathcal{C}]$, in Theorem 4.13 we provide upper and lower bounds on it in terms of the sumsets regularity of $\mathcal{A}$. These bounds give a new combinatorial proof of the Eisenbud-Goto conjecture for projective monomial curves (Gruson-Lazarsfeld-Peskine's Theorem for projective monomial curves). In Section 4.2, we study the regularity of simplicial projective monomial surfaces $\mathcal{X}$. In Theorem 4.25, we provide a combinatorial formula to compute the Castelnuovo-Mumford regularity of $\Bbbk[\mathcal{X}]$ in terms of the Apéry and exceptional sets of $\mathcal{S}$. In the special case of surfaces with a single singular point, in Theorem 4.27 we prove that $\operatorname{reg}(\Bbbk[\mathcal{X}]) \leq \sigma(\mathcal{A}) + 1$. Using this relation, in Theorem 4.29 we prove the Eisenbud-Goto conjecture for $\mathcal{X}$ whenever the degree of $\mathcal{X}$ is either minimal or maximal.

In Chapter 5, we consider a ω-homogeneous ideal $I \subset R = \Bbbk[x_1, \ldots, x_n]$ for some weight vector $\omega = (\omega_1, \ldots, \omega_n) \in (\mathbb{Z}_{>0})^n$ and assume that $A = \Bbbk[x_{n-d+1}, \ldots, n]$ is a Noether normalization of R/I. In this context, we study the so-called *short resolution* of R/I ([3, 75, 78]), i.e., the minimal graded free resolution of R/I as A-module. In Section 5.1, we provide a general method to compute the short resolution of any R/I as before. The main results are Proposition 5.2 and Theorem 5.7, where we provide a system of generators of the first syzygy module and prove that it is indeed the reduced Gröbner basis for a certain monomial order that we call the Schreyer-like order. In Section 5.2, we describe the short resolution of R/I when it is a simplicial toric ring of dimension 3 in terms of the combinatorics of the associated semigroup (Theorem 5.15). In Section 5.3, we provide an algorithm to compute the short resolution for 3-dimensional simplicial toric rings. This algorithm first constructs a graded free resolution that may not be minimal (Algorithm 5.2), and then minimalizes/prunes it to obtain the short one by applying Theorems 5.24 and 5.26 (Algorithm 5.3). Finally, in Section 5.4, we show an example or a simplicial toric ring whose short resolution depends on the characteristic of the field $\mathbb{k}$.

The results of this thesis have given rise to the following publications (sorted in chronological order):

[39] P. Gimenez and M. González-Sánchez. Castelnuovo–Mumford regularity of projective monomial curves via sumsets. *Mediterr. J. Math.*, 20(287), 2023. https://doi.org/10.1007/s00009-023-02482-3

[36] I. García-Marco, P. Gimenez, and M. González-Sánchez. Projective Cohen-Macaulay monomial curves and their affine charts. *Ric. Mat.*, pages 1–22, 2025. https://doi.org/10.1007/s11587-025-00929-1

[42] M. González-Sánchez, S. Singh, and H. Srinivasan. The Betti numbers of Kunz-Waldi semigroups. *Proc. Amer. Math. Soc.*, 153:4215-4224, 2025. https://doi.org/10.1090/proc/17338

[35] I. García-Marco, P. Gimenez, and M. González-Sánchez. Computational aspects of the short resolution. *ArXiv preprint*, 2025. https://doi.org/10.48550/arXiv.2504.12019

[41] M. González-Sánchez. ShortRes: A Sage package to compute the short resolution of a weighted homogeneous ideal. *GitHub Repository*, 2025. Available online: https://github.com/mgonzalezsanchez/shortRes

Chapter 1

Preliminaries

"The theory of syzygies offers a microscope for looking at systems of equations, and helps to make their subtle properties visible."
D. Eisenbud

In this chapter, we include the concepts and results that we will use in the next chapters. Section 1.1 contains the background on semigroups, with an emphasis on numerical and affine semigroups; Section 1.2 focuses on graded free resolutions, Betti numbers, and Castelnuovo–Mumford regularity; Section 1.3 is a survey on toric ideals and toric varieties; finally, in Section 1.4 we present some recent results on the interface between Additive Combinatorics and Commutative Algebra.

1.1 Semigroups

This first section contains some concepts about numerical and affine semigroups. For more details, we refer the reader to the books [80], [81], and [82].

A *semigroup* is a pair $(\mathcal{S}, +)$, where $\mathcal{S}$ is a nonempty set, and $+$ is a binary operation on $\mathcal{S}$ that is associative. When the operation $+$ is commutative, we say that the semigroup is *abelian*, and if there exists an identity element $0 \in \mathcal{S}$, $\mathcal{S}$ is called a *monoid*. In this thesis, all the semigroups will be abelian monoids, and we will call them just semigroups.

As it occurs for all algebraic structures, a subset $\mathcal{H} \subset \mathcal{S}$ is a *subsemigroup* of $\mathcal{S}$ if $(\mathcal{H}, +)$ is a semigroup, where $+$ is the restriction of the operation in $\mathcal{S}$ to $\mathcal{H}$. Given a semigroup $\mathcal{S}$ and a subset $\mathcal{A} \subset \mathcal{S}$, the *subsemigroup generated by* $\mathcal{A}$, $\langle \mathcal{A} \rangle$, is the smallest subsemigroup of $\mathcal{S}$ containing $\mathcal{A}$, that is

$$\langle \mathcal{A} \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{Z}_{>0}, \lambda_i \in \mathbb{N}, a_i \in \mathcal{A}, 1 \leq i \leq n\}.$$

A semigroup $\mathcal{S}$ is *finitely generated* if there exists a finite subset $\mathcal{A} \subset \mathcal{S}$ such that $\mathcal{S} = \langle \mathcal{A} \rangle$.

1.1.1 Numerical semigroups

For $\mathcal{A} \subset \mathbb{N}$ a nonempty subset, we say that $\mathcal{S} = \langle \mathcal{A} \rangle$ is a *numerical semigroup* if $\mathbb{N} \setminus \mathcal{S}$ is finite. A system of generators of a numerical semigroup is said to be a *minimal system of generators* if none of its proper subsets generates the numerical semigroup. By [82, Thm. 2.7], every numerical semigroup has a unique minimal system of generators, and it is finite. We denote by $\mathrm{MSG}(\mathcal{S})$ the minimal system of generators of $\mathcal{S}$. Given a nonempty finite subset $\mathcal{A} \subset \mathbb{N}$, $\langle \mathcal{A} \rangle$ is a numerical semigroup if and only if $\gcd(\mathcal{A}) = 1$ ([82, Lem. 2.1]). If $\mathcal{S}$ is a numerical semigroup minimally generated by $\mathcal{A} = \{a_1, \ldots, a_n\}$, and $s \in \mathcal{S}$, a *factorization of* s is a n-tuple $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$, such that $s = \sum_{i=1}^n \lambda_i a_i$. The *length* of the factorization $\boldsymbol{\lambda}$ is $\ell(\boldsymbol{\lambda}) = \sum_{i=1}^n \lambda_i$.

Let us now introduce some basic concepts that arise in the study of numerical semigroups.

Definition 1.1. Let $\mathcal{S}$ be a numerical semigroup and $\mathrm{MSG}(\mathcal{S}) = \{a_1, \ldots, a_n\}$ be its minimal system of generators that we assume without loss of generality ordered as $a_1 < a_2 < \cdots < a_n$.

(1) The *multiplicity* of $\mathcal{S}$ is $m(\mathcal{S}) = a_1 = \min(\mathcal{S} \setminus \{0\})$.

(2) The *embedding dimension* of $\mathcal{S}$ is $e(\mathcal{S}) = n = |\mathrm{MSG}(\mathcal{S})|$.

(3) The *set of gaps* of $\mathcal{S}$ is $G(\mathcal{S}) = \mathbb{N} \setminus \mathcal{S}$. It is finite and its cardinality is called the *genus* of $\mathcal{S}$.

(4) The *Frobenius number* of $\mathcal{S}$ is $F(\mathcal{S}) = \max(\mathbb{N} \setminus \mathcal{S})$, and the *conductor* of $\mathcal{S}$ is $c(\mathcal{S}) = F(\mathcal{S}) + 1$. It is the smallest element $s \in \mathcal{S}$ such that any $x \geq s$ belongs to $\mathcal{S}$.

(5) Given $s \in \mathcal{S} \setminus \{0\}$, the *Apéry set of* $\mathcal{S}$ *with respect to* s is $\mathrm{Ap}(\mathcal{S}, s) = \{x \in \mathcal{S} \mid x - s \notin \mathcal{S}\}$. By default, if we do not specify the element $s \in \mathcal{S}$, the *Apéry set* of $\mathcal{S}$ is $\mathrm{Ap}(\mathcal{S}) = \mathrm{Ap}(\mathcal{S}, m(\mathcal{S}))$.

(6) The *pseudo Frobenius set* of $\mathcal{S}$ is $\mathrm{PF}(\mathcal{S}) = \{x \in \mathbb{Z}\setminus\mathcal{S} \mid x+s \in \mathcal{S}, \forall s \in \mathcal{S}\setminus\{0\}\}$, the *pseudo Frobenius numbers* of $\mathcal{S}$ are the elements in $\mathrm{PF}(\mathcal{S})$, and the *type* of $\mathcal{S}$ is $t(\mathcal{S}) = |\mathrm{PF}(\mathcal{S})|$.

Given a numerical semigroup $\mathcal{S} \subset \mathbb{N}$, the problem of determining the Frobenius number of $\mathcal{S}$ is NP-hard [79]. In the next proposition, we include two upper bounds on the Frobenius number of $\mathcal{S}$. The first one is due to Schur, while the second one to Erdös and Graham.

Proposition 1.2 ([80, Thm. 3.1.1 and 3.1.12]). *Let $a_1, \ldots, a_n$ be positive integers such that $\gcd(a_1, \ldots, a_n) = 1$ and $a_1 < \cdots < a_n$, and consider the numerical semigroup $\mathcal{S} = \langle a_1, \ldots, a_n\rangle$. Then,*

(1) $F(\mathcal{S}) \leq (a_1 - 1)(a_n - 1) - 1$.

(2) $F(\mathcal{S}) \leq 2a_{n-1} \left\lfloor \frac{a_n}{n} \right\rfloor - a_n$.

For all $s \in \mathcal{S}$, one has that $F(\mathcal{S}) = \max(\mathrm{Ap}(\mathcal{S}, s)) - s$. Moreover, $\mathrm{Ap}(\mathcal{S}, s) \cup \{s\} \setminus \{0\}$ generates $\mathcal{S}$. In particular, $\mathrm{Ap}(\mathcal{S}) \cup \{m(\mathcal{S})\} \setminus \{0\}$ generates $\mathcal{S}$, and hence the embedding dimension of $\mathcal{S}$ verifies $e(\mathcal{S}) \leq m(\mathcal{S})$.

Definition 1.3. Let $\mathcal{S}$ be a numerical semigroup. We say that $\mathcal{S}$ has *maximal embedding dimension* if $e(\mathcal{S}) = m(\mathcal{S})$.

Example 1.4. Consider the numerical semigroup $\mathcal{S} = \langle 5, 9, 11\rangle$. The multiplicity of $\mathcal{S}$ is $m(\mathcal{S}) = 5$. The embedding dimension of $\mathcal{S}$ is $e(\mathcal{S}) = 3$, since $\mathcal{S}$ is minimally generated by $\{5, 9, 11\}$. The set of gaps of $\mathcal{S}$ is $G(\mathcal{S}) = \{1, 2, 3, 4, 6, 7, 8, 12, 13, 17\}$. Therefore, its genus is $g(\mathcal{S}) = 10$ and its Frobenius number is $F(\mathcal{S}) = 17$. The Apéry set of $\mathcal{S}$ (with respect to 5) is $\mathrm{Ap}(\mathcal{S}) = \{0, 11, 22, 18, 9\}$. The Pseudo Frobenius set of $\mathcal{S}$ is $\mathrm{PF}(\mathcal{S}) = \{13, 17\}$, and hence its type is $t(\mathcal{S}) = 2$. All these invariants can be computed using the package `NumericalSgps` [25] of GAP.

Proposition 1.5 ([82, Lem. 2.4]). *Let $\mathcal{S}$ be a numerical semigroup and $s \in \mathcal{S}$ a nonzero element. Then, $\mathrm{Ap}(\mathcal{S}, s) = \{w_0 = 0, w_1, \ldots, w_{s-1}\}$, where w_i is the least element of $\mathcal{S}$ congruent to i modulo s, for $1 \leq i \leq s$. Hence, the Apéry set $\mathrm{Ap}(\mathcal{S}, s)$ is a complete set of residues modulo s.*

In particular, the Apéry set of $\mathcal{S}$ with respect to s is finite for all $s \in \mathcal{S}$. Set $m = m(\mathcal{S})$ the multiplicity of $\mathcal{S}$, and denote the elements of the Apéry set as in the previous proposition, $\mathrm{Ap}(\mathcal{S}) = \{w_0 = 0, w_1, \ldots, w_{m-1}\}$.

Definition 1.6. The *Apéry coordinate vector* of $\mathcal{S}$ is the tuple $(w_1, w_2, \ldots, w_{m-1})$. We will also refer to this vector as the *Kunz coordinates* of $\mathcal{S}$.

From the definition of the Apéry set, one can easily deduce that $w_i + w_j \geq w_{i+j}$, for all $1 \leq i \leq j \leq m-1$ such that $i+j \neq 0$, where the sum of indices is interpreted modulo m. This is the idea that Kunz used to define the so-called Kunz cone in [60].

Definition 1.7. For each $m \in \mathbb{N}$, $m \geq 2$, the *Kunz cone* $\mathfrak{C}_m \subset \mathbb{R}^{m-1}_{\geq 0}$ is the cone with defining inequalities $z_i + z_j \geq z_{i+j}$ whenever $i + j \neq 0$, where $i, j \in \mathbb{Z}_m \setminus \{0\}$,

$$\mathfrak{C}_m = \bigcap_{\substack{i,j \in \mathbb{Z}_m \setminus \{0\} \\ i+j \neq 0}} \left\{ (z_1, \ldots, z_{m-1}) \in \mathbb{R}^{m-1}_{\geq 0} \mid z_i + z_j \geq z_{i+j} \right\}.$$

Proposition 1.8 ([9, Prop. 2.5])**.** *A vector* $\mathbf{z} = (z_1, \ldots, z_{m-1}) \in \mathbb{R}^{m-1}_{\geq 1}$ *with* $z_i \equiv i \pmod{m}$ *for all* i *lies in* $\mathfrak{C}_m$ *if and only if* $\mathbf{z}$ *is the Apéry coordinate vector of a numerical semigroup* $\mathcal{S}$*. Moreover,* $\mathbf{z}$ *is in the interior of* $\mathfrak{C}_m$ *if and only if* $\mathcal{S}$ *has maximal embedding dimension.*

By the previous result, we can identify a numerical semigroup with its Kunz coordinates. By an abuse of language, for a numerical semigroup $\mathcal{S}$ of multiplicity m, we will say that it *lies* in the interior of $\mathfrak{C}_m$ if the Kunz coordinates of $\mathcal{S}$ are in the interior of the Kunz cone $\mathfrak{C}_m$. Similarly, we will say that $\mathcal{S}$ lies in (the interior of) a certain face of $\mathfrak{C}_m$ if its Kunz coordinate are in (the interior of) that face of $\mathfrak{C}_m$.

Example 1.9. Let $m = 4$. The Kunz cone $\mathfrak{C}_4$ is defined by

$$\mathfrak{C}_4 = \left\{ \mathbf{z} = (z_1, z_2, z_3) \in \mathbb{R}^3_{\geq 0} \mid 2z_1 \geq z_2,\ z_1 + z_2 \geq z_3,\ z_2 + z_3 \geq z_1,\ 2z_3 \geq z_2 \right\},$$

and it is generated by the rays $(1, 0, 1)$, $(1, 2, 3)$, $(1, 2, 1)$, and $(3, 2, 1)$. Note that the integer points in $\mathfrak{C}_4$ with all their coordinates ≥ 1 correspond to all the numerical semigroups of multiplicity 4, by Proposition 1.8.

- For $\mathcal{S} = \langle 4, 7, 9 \rangle$, $\mathrm{Ap}(\mathcal{S}) = \{0, 9, 14, 7\}$, and hence its Kunz coordinates are $(9, 14, 7)$. Note that $\mathcal{S}$ lies on the face $\mathfrak{F}$ of $\mathfrak{C}_4$,

  $$\mathfrak{F} = \left\{ \mathbf{z} \in \mathfrak{C}_4 \mid 2z_1 \geq z_2,\ z_1 + z_2 \geq z_3,\ z_2 + z_3 \geq z_1,\ 2z_3 = z_2 \right\}.$$

  In fact, it lies in the interior of $\mathfrak{F}$, since the Kunz coordinates of $\mathcal{S}$ satisfy the inequalities $2z_1 > z_2$, $z_1 + z_2 > z_3$, and $z_2 + z_3 > z_1$.

- For $\mathcal{S}' = \langle 4, 6, 7, 9 \rangle$, $\mathrm{Ap}(\mathcal{S}') = \{0, 9, 6, 7\}$ and its Kunz coordinates are $(9, 6, 7)$. Therefore, the semigroup $\mathcal{S}'$ lies in the interior of the Kunz cone $\mathfrak{C}_4$, since in this case all the inequalities are strict for $(9, 6, 7)$. Note that $\mathcal{S}'$ has maximal embedding dimension, as follows from Proposition 1.8.

Let $\mathcal{S}$ be a numerical semigroup of multiplicity m and $(w_1, \ldots, w_{m-1})$ be its Kunz coordinates. We now consider a poset structure on the set $\mathbb{Z}_m$, based on the relations between the elements in $\mathrm{Ap}(\mathcal{S}) = \{0, w_1, \ldots, w_{m-1}\}$.

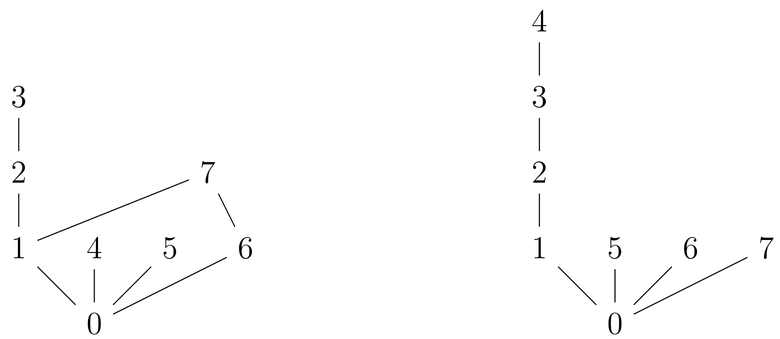
Definition 1.10. The *Apéry poset* of $\mathcal{S}$ is $\mathcal{P}(\mathcal{S}) = (\mathbb{Z}_m, \preceq)$, where $i \preceq j$ if and only if $w_j - w_i \in \mathcal{S}$ for $i, j \in \mathbb{Z}_m$. We write $i \prec\!\cdot\, j$ and say j *covers* i if $i \prec j$ and there is no k such that $i \prec k \prec j$.

The following lemma characterizes the covering relations in terms of the minimal generators of $\mathcal{S}$. This result allows us to construct the Hasse diagram of the poset $\mathcal{P}(\mathcal{S})$.

Lemma 1.11. *For all $i, j \in \mathbb{Z}_m$, $i \prec\!\cdot\, j$ if and only if $w_j - w_i$ is a minimal generator of $\mathcal{S}$.*

Proof. Being ($\Leftarrow$) trivial, let us prove ($\Rightarrow$). Let $i, j \in \mathbb{Z}_m$ such that $i \prec\!\cdot\, j$ and write $w_j = w_i + \alpha + \beta$ for some $\alpha, \beta \in \mathcal{S}$ with α a minimal generator of $\mathcal{S}$. Note that $w_i \prec w_i + \alpha \preceq w_j$. Since $w_j \in \mathrm{Ap}(\mathcal{S})$, then $w_i + \alpha \in \mathrm{Ap}(\mathcal{S})$, so $w_j = w_i + \alpha$ as j covers i. □

Example 1.12. Consider the numerical semigroups $\mathcal{S} = \langle 8, 17, 60, 69, 78 \rangle$ and $\mathcal{S}' = \langle 8, 17, 53, 62, 55 \rangle$, whose Apéry sets are $\mathrm{Ap}(\mathcal{S}) = \{0, 17, 34, 51, 60, 69, 78, 95\}$ and $\mathrm{Ap}(\mathcal{S}') = \{0, 17, 34, 51, 68, 53, 62, 55\}$, respectively. The Hasse diagrams of the Apéry posets $\mathcal{P}(\mathcal{S})$ and $\mathcal{P}(\mathcal{S}')$ are shown in Figure 1.1.



(a) $\mathcal{P}(\mathcal{S})$ (b) $\mathcal{P}(\mathcal{S}')$

Figure 1.1: Apéry posets in Example 1.12.

The following theorem characterizes when two semigroups $\mathcal{S}$ and $\mathcal{S}'$ lie in the interior of the same face of the Kunz cone $\mathfrak{C}_m$ in terms of their Apéry posets.

Theorem 1.13 ([10, Thm. 3.10]). *Let $\mathcal{S}$ and $\mathcal{S}'$ be two numerical semigroups of multiplicity m. Then, $\mathcal{S}$ and $\mathcal{S}'$ lie on the interior of the same face of the Kunz cone $\mathfrak{C}_m$ if and only if $\mathcal{P}(\mathcal{S}) = \mathcal{P}(\mathcal{S}')$.*

Let $\mathcal{S}$ be a numerical semigroup and $F(\mathcal{S})$ its Frobenius number. For all $x \in \mathcal{S}$, one has that $F(\mathcal{S})-x \notin \mathcal{S}$. When the other implication holds for all x, the semigroup is called symmetric.

Definition 1.14. Let $\mathcal{S}$ be a numerical semigroup. We say that $\mathcal{S}$ is *symmetric* if for all $x \in \mathbb{Z}$, $x \in \mathcal{S}$ if and only if $F(\mathcal{S}) - x \notin \mathcal{S}$.

Proposition 1.15 ([82, Prop. 4.4, Cor. 4.5 and 4.11]). *The following conditions are equivalent:*

(a) $\mathcal{S}$ *is symmetric.*

(b) The genus of $\mathcal{S}$ *is* $g(\mathcal{S}) = \frac{F(\mathcal{S})+1}{2}$.

(c) The Pseudo Frobenius set of $\mathcal{S}$ *is* $\mathrm{PF}(\mathcal{S}) = \{F(\mathcal{S})\}$.

(d) The type of $\mathcal{S}$ *is* $t(\mathcal{S}) = 1$.

1.1.2 Affine semigroups

Definition 1.16. An *affine semigroup* is a finitely generated subsemigroup of $\mathbb{N}^d$, for some $d \geq 1$.

Remark 1.17. More generally, a semigroup $\mathcal{S}$ is said to be an affine semigroup if it is isomorphic to a finitely generated subsemigroup of $\mathbb{N}^d$ for some d. By [81, Thm. 3.11], $\mathcal{S}$ is an affine semigroup if and only if it is finitely generated, cancellative, torsion free, and reduced:

- $\mathcal{S}$ is *cancellative* if for all $a, b, c \in \mathcal{S}$, $a + b = a + c \Rightarrow b = c$.
- $\mathcal{S}$ is *torsion-free* if for all $a, b \in \mathcal{S}$, $n \in \mathbb{Z}_{>0}$, $na = nb \Rightarrow a = b$.
- $\mathcal{S}$ is *reduced* if $\mathcal{S} \cap (-\mathcal{S}) = \{0\}$.

Also, note that some authors define affine semigroups as those isomorphic to a finitely generated subsemigroup of $\mathbb{Z}^d$, for some $d \geq 1$; see, e.g., [11]. Here we restrict to $\mathbb{N}^d$ because we are only interested in reduced affine semigroups and, by Grillet's theorem [81, Thm. 3.11], every reduced affine semigroup is isomorphic to a finitely generated subsemigroup of $\mathbb{N}^d$, for some $d \geq 1$.

Given an affine semigroup $\mathcal{S} \subset \mathbb{N}^d$, one can always consider a natural partial order $\leq_{\mathcal{S}}$ in $\mathcal{S}$ as follows:

$$\text{for } \mathbf{s}_1, \mathbf{s}_2 \in \mathcal{S},\ \mathbf{s}_1 \leq_{\mathcal{S}} \mathbf{s}_2 \text{ if and only if } \mathbf{s}_2 - \mathbf{s}_1 \in \mathcal{S}\,. \tag{1.1}$$

This partial order will be useful to characterize some properties of semigroup algebras (see Section 1.3) and it will appear in Chapter 2.

Definition 1.18. Given an affine semigroup $\mathcal{S} \subset \mathbb{N}^d$ and an element $\mathbf{s} \in \mathcal{S}$, the *Apéry set of* $\mathcal{S}$ *with respect to* $\mathbf{s}$ is $\mathrm{Ap}(\mathcal{S}, \mathbf{s}) = \{\mathbf{x} \in \mathcal{S} \mid \mathbf{x} - \mathbf{s} \notin \mathcal{S}\}$. If $\mathcal{B} \subset \mathcal{S}$ is a finite subset, the *Apéry set of* $\mathcal{S}$ *with respect to* $\mathcal{B}$ is defined as $\mathrm{Ap}(\mathcal{S}, \mathcal{B}) = \cap_{\mathbf{b} \in \mathcal{B}} \mathrm{Ap}(\mathcal{S}, \mathbf{b})$.

Unlike in the case of numerical semigroups, the Apéry set $\mathrm{Ap}(\mathcal{S}, \mathbf{s})$ is not finite in general. Proposition 1.20 characterizes when the Apéry set of $\mathcal{S}$ with respect to a finite subset is finite.

Definition 1.19. Let $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{N}^d$ be a finite subset. The *rational cone spanned by* $\mathcal{A}$ is $\mathrm{Pos}(\mathcal{A}) := \{\sum_{i=1}^n \lambda_i \mathbf{a}_i \mid \lambda_i \in \mathbb{Q}_{\geq 0}\} \subset \mathbb{Q}^d_{\geq 0}$. The *dimension of* $\mathrm{Pos}(\mathcal{A})$ is the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q}\mathcal{A} = \{\sum_{i=1}^n \mu_i \mathbf{a}_i \mid \mu_i \in \mathbb{Q}\} \subset \mathbb{Q}^d$.

Proposition 1.20 ([38, Thm. 2.6])**.** *Let* $\mathcal{A} \subset \mathbb{N}^d$ *be a finite set of nonzero vectors, and let* $\mathcal{S} = \langle \mathcal{A} \rangle$ *be the affine semigroup generated by* $\mathcal{A}$. *If* $\mathcal{B} \subset \mathcal{S}$ *is a finite subset, then* $\mathrm{Ap}(\mathcal{S}, \mathcal{B})$ *is finite if and only if* $\mathrm{Pos}(\mathcal{B}) = \mathrm{Pos}(\mathcal{A})$.

Definition 1.21. Let $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{N}^d$, and $\mathcal{S} = \langle \mathcal{A} \rangle$ be the affine semigroup generated by $\mathcal{A}$. We say that $\mathcal{S}$ is *simplicial* when the rational cone spanned by $\mathcal{A}$, $\mathrm{Pos}(\mathcal{A})$, has dimension d and is minimally generated by d rays.

If $\mathcal{S} = \langle \mathcal{A} \rangle \subset \mathbb{N}^d$ is simplicial, take $\{\mathbf{v}_1, \dots, \mathbf{v}_d\} \subset \mathbb{Q}^d_{\geq 0}$ a generating set of $\mathrm{Pos}(\mathcal{A})$. For $i = 1, \dots, d$, consider $\mathbf{e}_i \in \mathcal{A} \cap \mathrm{Pos}(\mathbf{v}_i)$ the element with the smallest norm. We call $\mathbf{e}_1, \dots, \mathbf{e}_d$ the *extremal rays* of the cone $\mathrm{Pos}(\mathcal{A})$.

Remark 1.22. (1) For $d \in \{1, 2\}$, every affine semigroup $\mathcal{S} \subset \mathbb{N}^d$ is simplicial.

(2) Let $\mathcal{S}$ be a simplicial semigroup and $\mathcal{E}$ the set of extremal rays. By Proposition 1.20, the Apéry set of $\mathcal{S}$ with respect to $\mathcal{E}$ is finite. We will call $\mathrm{Ap}(\mathcal{S}, \mathcal{E})$ the *Apéry set of* $\mathcal{S}$, and denote it just by $\mathrm{Ap}(\mathcal{S})$ or $\mathrm{AP}_{\mathcal{S}}$.

As mentioned before, when $d = 1$, the affine semigroups with finite complement in $\mathbb{N}$ are called numerical semigroups. We now define an analogous concept for $d \geq 2$.

Definition 1.23. Let $\mathcal{S}$ be a subsemigroup of $\mathbb{N}^d$. We say that $\mathcal{S}$ is a *generalized numerical semigroup* if $\mathbb{N}^d \setminus \mathcal{S}$ is finite.

By [17, Prop 2.3], every generalized numerical semigroup is finitely generated, i.e., it is an affine semigroup. Moreover, in the same paper the authors characterize generalized numerical semigroups in terms of the generators of $\mathcal{S}$.

Theorem 1.24 ([17, Thm. 2.8, Rem. 2.13])**.** *Let $d \geq 2$, $\mathcal{A} \subset \mathbb{N}^d$ a finite subset, and $\mathcal{S}$ the affine semigroup generated by $\mathcal{A}$. Then $\mathcal{S}$ is a generalized numerical semigroup if and only if $\mathcal{A}$ fulfills the following two conditions:*

(1) For all $j = 1, 2, \ldots, d$, the j-th coordinates of the elements of $\mathcal{A}$ generate a numerical semigroup.

(2) For all $j = 1, 2, \ldots, d$, if $\mathcal{A}^{(j)} \subset \mathbb{N}^{d-1}$ is the set obtained from the elements of $\mathcal{A}$ removing the j-th component, the affine semigroup $\langle \mathcal{A}^{(j)} \rangle$ is $\mathbb{N}^{d-1}$.

Example 1.25. Let $\mathcal{A} = \{(1,0),(2,0),(3,0),(0,2),(0,3),(2,1)\}$, and $\mathcal{S} \subset \mathbb{N}^2$ the affine semigroup generated by $\mathcal{A}$. By Theorem 1.24, $\mathcal{S}$ is a generalized numerical semigroup, since $\langle 1,2,3 \rangle = \mathbb{N}$ is a numerical semigroup. Moreover, one can easily prove that $\mathcal{S} = \mathbb{N}^2 \setminus \{(0,1),(1,1)\}$.

1.2 Free resolutions, Betti numbers, and Castelnuovo–Mumford regularity

In this section, we introduce the algebraic background on free resolutions of finitely generated graded modules over the polynomial ring $\Bbbk[x_1, \ldots, x_n]$. Since we will mainly work with free resolutions of toric ideals, we treat here the general case of multigradings. Moreover, we introduce the Castelnuovo-Mumford regularity for finitely generated standard graded modules. In this case, we work over the polynomial ring $\Bbbk[x_0, \ldots, x_n]$. We refer the reader to [20], [27], [28], [58], and [69] for the details.

We assume that the reader is familiarized with Gröbner basis for ideals and modules. See [19, Chap. 2] and [20, Chap. 5] for an overview of Gröbner bases.

1.2.1 Free resolutions and Betti numbers

Let $\Bbbk$ be a field and $R = \Bbbk[\mathbf{x}] = \Bbbk[x_1, \ldots, x_n]$ the polynomial ring in n variables, $n \in \mathbb{Z}_{>0}$. Consider a set $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d \setminus \{\mathbf{0}\}$, $d \in \mathbb{Z}_{>0}$, and denote by $\mathcal{S} \subset \mathbb{N}^d$ the affine semigroup generated by $\mathcal{A}$. A natural way to define a grading on $R = \Bbbk[x_1, \ldots, x_n]$ is to assign *multidegree* (also called $\mathcal{S}$*-degree*) $\mathbf{a}_i$ to the variable x_i, $|x_i|_{\mathcal{S}} = \deg_{\mathcal{S}}(x_i) = \mathbf{a}_i$, $1 \leq i \leq n$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ the multidegree or $\mathcal{S}$-degree of the monomial $\mathbf{x}^\alpha$ is $|\mathbf{x}^\alpha|_{\mathcal{S}} = \deg_{\mathcal{S}}(\mathbf{x}^\alpha) = \sum_{i=1}^n \alpha_i \mathbf{a}_i \in \mathcal{S}$.

For every $\mathbf{s} \in \mathcal{S}$, let $R_{\mathbf{s}}$ denote the $\Bbbk$-vector space spanned by the set of all monomials $\mathbf{x}^\alpha$ of multidegree $\mathbf{s}$. Then, R has the direct sum decomposition $R = \oplus_{\mathbf{s} \in \mathcal{S}} R_{\mathbf{s}}$, where $R_{\mathbf{s}} R_{\mathbf{s}'} \subset R_{\mathbf{s}+\mathbf{s}'}$ for all $\mathbf{s}, \mathbf{s}' \in \mathcal{S}$. We say that R is $\mathcal{S}$*-graded*, and we call this grading an $\mathcal{S}$-*grading* or *multigrading* in R. The polynomials $f \in R_{\mathbf{s}}$ are called *homogeneous of degree* $\mathbf{s}$.

If necessary, we can extend this grading to $\mathbb{N}^d$ (resp. $\mathbb{Z}^d$) by setting $R_{\mathbf{s}} = \{0\}$ for all $\mathbf{s} \in \mathbb{N}^d \setminus \mathcal{S}$ (resp. $\mathbf{s} \in \mathbb{Z}^d \setminus \mathcal{S}$).

When $d = 1$, we usually denote the degrees of the variables $x_1, \ldots, x_n$ by $\omega_1, \ldots, \omega_n \in \mathbb{Z}_{>0}$, respectively, and say that the ω-degree of the monomial $\mathbf{x}^\alpha$ is $|\mathbf{x}^\alpha|_\omega = \deg_\omega(\mathbf{x}^\alpha) = \sum_{i=1}^n \alpha_i\omega_i$. If $\omega_1 = \cdots = \omega_n = 1$, we get the *standard grading* of R. Throughout this subsection, all the results apply for multigradings, gradings given by weights and the standard grading; and we use the word multigrading to include all cases. We treat the standard graded case in more detail in Subsection 1.2.2.

Given an R-module M, we say that M is $\mathcal{S}$-*graded* if $M = \oplus_{\mathbf{s}\in\mathcal{S}} M_{\mathbf{s}}$, where $M_{\mathbf{s}} \subset M$ is an additive subgroup for all $\mathbf{s} \in \mathcal{S}$, such that $R_{\mathbf{s}} M_{\mathbf{s}'} \subset M_{\mathbf{s}+\mathbf{s}'}$ for all $\mathbf{s}, \mathbf{s}' \in \mathcal{S}$.

Example 1.26. (1) Let $I \subset R = \Bbbk[\mathbf{x}]$ be an ideal. Then, I is $\mathcal{S}$-graded (as an R-module) if and only if there exist homogeneous polynomials $f_1, \ldots, f_r$ such that $I = \langle f_1, \ldots, f_r \rangle$. In this case, $I = \oplus_{\mathbf{s}\in\mathcal{S}} I_{\mathbf{s}}$, where $I_{\mathbf{s}} = I \cap R_{\mathbf{s}}$. The R-module R/I is also $\mathcal{S}$-graded, $R/I = \oplus_{\mathbf{s}\in\mathcal{S}} (R/I)_{\mathbf{s}}$, where $(R/I)_{\mathbf{s}} = R_{\mathbf{s}}/I_{\mathbf{s}}$ for all $\mathbf{s} \in \mathcal{S}$.

(2) For $m \in \mathbb{Z}_{>0}$, consider R^m the free R-module of rank m. Then,

$$R^m = \oplus_{\mathbf{s}\in\mathcal{S}} (R^m)_{\mathbf{s}}\,,$$

where $(R^m)_{\mathbf{s}} := (R_{\mathbf{s}})^m$ for all $\mathbf{s} \in \mathcal{S}$. This decomposition makes R^m an $\mathcal{S}$-graded module.

For all $\mathbf{s}_0 \in \mathcal{S}$, one can define a new $\mathcal{S}$-grading in R by shifting the degrees: $R(-\mathbf{s}_0) = \oplus_{\mathbf{s}\in\mathcal{S}} R(-\mathbf{s}_0)_{\mathbf{s}}$, where $R(-\mathbf{s}_0)_{\mathbf{s}} = R_{\mathbf{s}-\mathbf{s}_0}$ for all $\mathbf{s} \in \mathcal{S}$. If $m \in \mathbb{Z}_{>0}$ and $\mathbf{s}_1, \ldots, \mathbf{s}_m \in \mathcal{S}$, one can also shift the $\mathcal{S}$-grading in R^m by $\mathbf{s}_1, \ldots, \mathbf{s}_m$:

$$R(-\mathbf{s}_1) \oplus \ldots \oplus R(-\mathbf{s}_m) = \oplus_{\mathbf{s}\in\mathcal{S}} \left(R(-\mathbf{s}_1) \oplus \ldots \oplus R(-\mathbf{s}_m)\right)_{\mathbf{s}}\,,$$

where $\left(R(-\mathbf{s}_1) \oplus \ldots \oplus R(-\mathbf{s}_m)\right)_{\mathbf{s}} = R(-\mathbf{s}_1)_{\mathbf{s}} \oplus \ldots \oplus R(-\mathbf{s}_m)_{\mathbf{s}}$ for all $\mathbf{s} \in \mathcal{S}$. A way of thinking of this new $\mathcal{S}$-grading is as follows: if $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ denotes the standard basis of R^m, we are assigning degree $\mathbf{s}_i$ to the vector $\mathbf{e}_i$, $i = 1, \ldots, m$. Thus, for $\mathbf{f} = (f_1, \ldots, f_m) \in R^m$, the degree of $f_i\mathbf{e}_i$ is $\deg_{\mathcal{S}}(f_i) + \mathbf{s}_i$, and

$$\mathbf{f} \in \left(R(-\mathbf{s}_1) \oplus \ldots \oplus R(-\mathbf{s}_m)\right)_{\mathbf{s}} \Longleftrightarrow f_i \in R_{\mathbf{s}-\mathbf{s}_i} \text{ for all } i = 1, \ldots, m\,.$$

Remark 1.27. One can show that every $\mathcal{S}$-graded free R-module F of rank m has the form $F \cong R(-\mathbf{s}_1) \oplus \ldots R(-\mathbf{s}_m)$ for some $\mathbf{s}_1, \ldots, \mathbf{s}_m \in \mathcal{S}$; see, e.g., [27, Ex. 4.11]. Hence, if $M = \langle \mathbf{f}_1, \ldots, \mathbf{f}_m \rangle$ is a finitely generated $\mathcal{S}$-graded R-module, where $\mathbf{f}_i \in M$ is homogeneous of degree $\mathbf{s}_i \in \mathcal{S}$ for all $i = 1, \ldots, m$, then the graded epimorphism of R-modules $\psi : R(\mathbf{s}_1) \oplus \ldots \oplus R(-\mathbf{s}_m) \to M$ defined by $\psi(\mathbf{e}_i) = \mathbf{f}_i$ induces an isomorphism of $\mathcal{S}$-graded R-modules $M \cong (R(-\mathbf{s}_1) \oplus \ldots \oplus R(-\mathbf{s}_m)) / \ker(\psi)$.

Definition 1.28. Let M be a finitely generated $\mathcal{S}$-graded R-module. A *graded free resolution of* M (as R-module) is an exact sequence of $\mathcal{S}$-graded R-modules and homomorphisms of R-modules

$$\mathcal{F} : \ldots \to F_i \xrightarrow{\varphi_i} F_{i-1} \to \ldots \to F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0 \tag{1.2}$$

satisfying the following properties:

- $F_i = R(-\mathbf{s}_{i,1}) \oplus \ldots \oplus R(-\mathbf{s}_{i,r_i})$ is a free $\mathcal{S}$-graded free R-module of finite rank for all $i \geq 0$.
- The morphism φ_i is graded for all $i \geq 0$, i.e., it maps elements of degree $\mathbf{s}$ in F_i to elements of degree $\mathbf{s}$ in F_{i-1}, for all $\mathbf{s} \in \mathcal{S}$.

The elements $\mathbf{s}_{i,j}$, $i \geq 0$, $1 \leq j \leq r_i$, are called the *shifts* of the resolution $\mathcal{F}$. The *i-th syzygy module of* M is $\varphi_{i+1}(F_{i+1}) = \ker(\varphi_i) \subset F_i$. If for some $\ell \in \mathbb{N}$, $F_\ell \neq 0$ and $F_r = 0$ for all $r > \ell$, we say that the resolution is *finite of length* ℓ.

To compute graded free resolutions, one can use Gröbner bases. Let $F = R^m$ be a free R-module, $m \in \mathbb{Z}_{>0}$, and $M = \langle \mathbf{g}_1, \ldots, \mathbf{g}_t \rangle$ an $\mathcal{S}$-graded submodule of F, where for all $i \in \{1, \ldots, t\}$, $\mathbf{g}_i$ is homogeneous of $\mathcal{S}$-degree $\mathbf{s}_i \in \mathcal{S}$. Set $\{\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_m\}$ the canonical basis of F. Assume that $\mathcal{G} = \{\mathbf{g}_1, \ldots, \mathbf{g}_t\}$ is a Gröbner basis of M with respect to a certain monomial order $>$ on F. By Buchberger's criterion, for all $1 \leq i < j \leq t$, the S-polynomial $S(\mathbf{g}_i, \mathbf{g}_j)$ either is $\mathbf{0}$, or reduces to $\mathbf{0}$ modulo $\mathcal{G}$. Hence, there exists an expression $S(\mathbf{g}_i, \mathbf{g}_j) = M_{ji}\mathbf{g}_i - M_{ij}\mathbf{g}_j = \sum_{k=1}^{t} f_k^{(ij)} \mathbf{g}_k$, where $M_{ji}, M_{ij} \in R$ are monomials, $f_k^{(ij)} \in R$ for all $k \in \{1, \ldots, t\}$, and $\operatorname{in}(f_k^{(ij)} \mathbf{g}_k) \leq \operatorname{in}(S(\mathbf{g}_i, \mathbf{g}_j))$ for all $k \in \{1, \ldots, t\}$, where $\operatorname{in}(-)$ denotes the leading monomial for $>$. Each one of the relations

$$M_{ji}\mathbf{g}_i - M_{ij}\mathbf{g}_j - \sum_{k=1}^{t} f_k^{(ij)} \mathbf{g}_k = 0$$

provides a syzygy which can be represented as a vector $\boldsymbol{\tau}_{ij} := M_{ji}\boldsymbol{\epsilon}_i - M_{ij}\boldsymbol{\epsilon}_j - \sum_{k=1}^{t} f_k^{(ij)} \boldsymbol{\epsilon}_k \in R^t$. Indeed, if $\boldsymbol{\tau}_{ij} \neq \mathbf{0}$, one has that $\boldsymbol{\tau}_{ij} \in R(-\mathbf{s}_1) \oplus \ldots \oplus R(-\mathbf{s}_t)$ is homogeneous of a certain $\mathcal{S}$-degree $\mathbf{s}_{ij}$.

Theorem 1.29 (Schreyer's Theorem, [27, Thm. 15.10]). *With notations as above, suppose that* $\mathbf{g}_1, \ldots, \mathbf{g}_t$ *is a Gröbner basis of* M *with respect to a monomial order* $>$, *and consider the graded homomorphism of* R*-modules* $\varphi : R(-\mathbf{s}_1) \oplus \ldots \oplus R(-\mathbf{s}_t) \to R^m$ *defined by* $\varphi(\boldsymbol{\epsilon}_i) = \mathbf{g}_i$, $i \in \{1, \ldots, t\}$. *Then,* $\ker(\varphi) = \langle \boldsymbol{\tau}_{ij} \mid 1 \le i < j \le t \rangle$. *Indeed,* $\{\boldsymbol{\tau}_{ij} \mid 1 \le i < j \le t\}$ *forms a Gröbner basis of* $\ker(\varphi)$ *for the monomial order* $>_S$ *on* $R(-\mathbf{s}_1) \oplus \ldots \oplus R(-\mathbf{s}_t)$ *defined by taking*

$$M\boldsymbol{\epsilon}_i >_S M'\boldsymbol{\epsilon}_j \iff \begin{cases} \mathrm{in}(M\mathbf{g}_i) > \mathrm{in}(M'\mathbf{g}_j) \text{ for the monomial order } > \text{ on } R^m, \text{ or} \\ \mathrm{in}(M\mathbf{g}_i) = \mathrm{in}(M'\mathbf{g}_j) \text{ and } i < j\,. \end{cases}$$

Remark 1.30. The initial term of $\boldsymbol{\tau}_{ij}$ for the monomial order $>_S$ is $\mathrm{in}(\boldsymbol{\tau}_{ij}) = M_{ji}\boldsymbol{\epsilon}_i$.

Repeated use of Theorem 1.29 provides a graded free resolution of M. The next result shows how to sort the elements in the Gröbner basis to obtain a resolution that finishes in at most n steps.

Corollary 1.31 ([27, Cor. 15.11]). *With notations as in Theorem 1.29, suppose that the* $\mathbf{g}_i$ *are arranged so that whenever* $\mathrm{in}(\mathbf{g}_i)$ *and* $\mathrm{in}(\mathbf{g}_j)$ *involve the same vector of the canonical basis of* F, *say* $\mathrm{in}(\mathbf{g}_i) = M_i\boldsymbol{\epsilon}$ *and* $\mathrm{in}(\mathbf{g}_j) = M_j\boldsymbol{\epsilon}$ *with* $M_i, M_j \in R$ *monomials, we have*

$$i < j \Rightarrow M_i >_{LEX} M_j\,,$$

where $>_{LEX}$ *is the lexicographic order on* R *with* $x_1 > x_2 > \cdots > x_n$. *If the variables* $x_1, \ldots, x_s$ *are missing from the initial terms of the* $\mathbf{g}_i$, *then the variables* $x_1, \ldots, x_{s+1}$ *are missing from the* $\mathrm{in}(\boldsymbol{\tau}_{ij})$ *and* $F/\langle \mathbf{g}_1, \ldots, \mathbf{g}_t\rangle$ *has a resolution of length* $\le n - s$.

Theorem 1.29 and Corollary 1.31 provide the so-called Schreyer's algorithm to compute graded free resolutions based on Gröbner bases computations when M is a submodule of a free R-module F. This resolution is called the *Schreyer resolution* and it is always of length $\le n$ by Corollary 1.31.

Example 1.32 (Schreyer's resolution). Take $R = \mathbb{Q}[x, y, z, t]$ and consider in R the multigrading defined by $\deg_{\mathcal{S}}(x) = (2, 3)$, $\deg_{\mathcal{S}}(y) = (1, 4)$, $\deg_{\mathcal{S}}(z) = (0, 5)$ and $\deg_{\mathcal{S}}(t) = (5, 0)$, where $\mathcal{S} = \langle (2, 3), (1, 4), (0, 5), (5, 0)\rangle \subset \mathbb{N}^2$. The ideal $I = \langle g_1, g_2, g_3\rangle$, where $g_1 = x^3 - yzt$, $g_2 = x^2y - z^2t$, and $g_3 = y^2 - xz$, is $\mathcal{S}$-graded, since g_1, g_2, g_3 are homogeneous. Fix $>$ the degree reverse lexicographic on R with $x > y > z > t$, i.e., for $\mathbf{x}^\alpha, \mathbf{x}^\beta \in R$ two distinct monomials,

$$\mathbf{x}^\alpha > \mathbf{x}^\beta \iff \begin{cases} \deg(\mathbf{x}^\alpha) > \deg(\mathbf{x}^\beta), \text{ or} \\ \deg(\mathbf{x}^\alpha) = \deg(\mathbf{x}^\beta) \text{ and the last nonzero entry of } \alpha - \beta \text{ is } < 0. \end{cases}$$

Observe that the elements g_1, g_2, g_3 satisfy $\mathrm{in}(g_1) > \mathrm{in}(g_2) > \mathrm{in}(g_3)$ for the lexicographic order with $x > y > z > t$. Let us construct a graded free resolution of I as

R-module. Note that g_1 is homogeneous of degree $(6,9)$, g_2 of degree $(5,10)$, and g_3 of degree $(2,8)$. Set $F_0 \coloneqq R(-(6,9)) \oplus R(-(5,10)) \oplus R(-(2,8))$, and compute the S-polynomials between g_1, g_2, g_3:

- $S(g_1, g_2) = yg_1 - xg_2 = -ztg_3$, and hence we have $\boldsymbol{\tau}_{12} = y\boldsymbol{\epsilon}_1 - x\boldsymbol{\epsilon}_2 + zt\boldsymbol{\epsilon}_3$, where $\{\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_3\}$ is the canonical basis of R^3. Note that $\boldsymbol{\tau}_{12} \in F_0$ is homogeneous of degree $(7,13)$, i.e., $\boldsymbol{\tau}_{12} \in (F_0)_{(7,13)}$.
- $S(g_1, g_3) = y^2 g_1 - x^3 g_3 = xzg_1 - yztg_3$, and hence $\boldsymbol{\tau}_{13} = (y^2 - xz)\boldsymbol{\epsilon}_1 + (-x^3 + yzt)\boldsymbol{\epsilon}_3$. Note that $\boldsymbol{\tau}_{13} \in (F_0)_{(8,17)}$.
- $S(g_2, g_3) = yg_2 - x^2 g_3 = zg_1$, and hence $\boldsymbol{\tau}_{23} = -z\boldsymbol{\epsilon}_1 + y\boldsymbol{\epsilon}_2 - x^2\boldsymbol{\epsilon}_3$. Note that $\boldsymbol{\tau}_{23} \in (F_0)_{(6,14)}$.

By Buchberger's criterion, $\{g_1, g_2, g_3\}$ is a Gröbner basis of I with respect to the degrevlex order. The initial term of the $\boldsymbol{\tau}_{ij}$ for the monomial order $>_S$ are $\mathrm{in}(\boldsymbol{\tau}_{12}) = y\boldsymbol{\epsilon}_1$ $\mathrm{in}(\boldsymbol{\tau}_{13}) = y^2\boldsymbol{\epsilon}_1$, and $\mathrm{in}(\boldsymbol{\tau}_{23}) = y\boldsymbol{\epsilon}_2$, by Remark 1.30. We sort the $\boldsymbol{\tau}_{ij}$ following Corollary 1.31; $\mathbf{f}_1 = \tau_{13}$, $\mathbf{f}_2 = \boldsymbol{\tau}_{12}$ and $\mathbf{f}_3 = \boldsymbol{\tau}_{23}$. By Theorem 1.29, the sequence

$$0 \to \begin{array}{c} R(-(8,17)) \oplus R(-(7,13)) \\ \oplus R(-(6,14)) \end{array} \xrightarrow{\varphi_1} \begin{array}{c} R(-(6,9)) \oplus R(-(5,10)) \\ \oplus R(-(2,8)) \end{array} \xrightarrow{\varphi_0} I \to 0$$

is exact and the morphisms φ_0 and φ_1 are graded, where $\varphi_0(\boldsymbol{\epsilon}_i) = g_i$, $\varphi_1(\boldsymbol{\epsilon}'_i) = \mathbf{f}_i$ for all $i = 1,2,3$, and $\{\boldsymbol{\epsilon}'_1, \boldsymbol{\epsilon}'_2, \boldsymbol{\epsilon}'_3\}$ is the canonical basis of $R(-(8,17)) \oplus R(-(7,13)) \oplus R(-(6,14))$. Moreover, by Theorem 1.29, $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is a Gröbner basis of $\ker(\varphi_0)$ for the monomial order $>_S$. Now, $S(\mathbf{f}_1, \mathbf{f}_3) = S(\mathbf{f}_2, \mathbf{f}_3) = \mathbf{0}$, and

$$S(\mathbf{f}_1, \mathbf{f}_2) = \mathbf{f}_1 - y\mathbf{f}_2 = x\mathbf{f}_3,$$

so $\boldsymbol{\tau}'_{12} = \boldsymbol{\epsilon}'_1 - y\boldsymbol{\epsilon}'_2 - x\boldsymbol{\epsilon}'_3$. Note that $\boldsymbol{\tau}'_{12}$ is homogeneous of degree $(8,17)$, and it generates $\ker(\varphi_1)$ by Theorem 1.29. Hence, one has that

$$\begin{aligned} \mathcal{F} : 0 \to R(-(8,17)) &\xrightarrow{\varphi_2} R(-(8,17)) \oplus R(-(7,13)) \oplus R(-(6,14)) \\ &\xrightarrow{\varphi_1} R(-(6,9)) \oplus R(-(5,10)) \oplus R(-(2,8)) \xrightarrow{\varphi_0} I \to 0 \end{aligned}$$

is a graded free resolution of I, where φ_2 is defined by $\varphi_2(1) = \boldsymbol{\tau}'_{12}$. The resolution $\mathcal{F}$ is finite of length 2.

By Remark 1.27, every finitely generated $\mathcal{S}$-graded module M is isomorphic to the quotient of a free R-module, and hence, the Schreyer's resolution of M has length $\leq n$. This provides an easy proof of the graded version of Hilbert's syzygy theorem. Moreover, this proof is constructive and shows a method to compute a graded free resolution of any finitely generated $\mathcal{S}$-graded R-module M, providing that we know how to compute $\ker(\psi)$ in Remark 1.27.

Theorem 1.33 (Graded Hilbert's syzygy theorem, [58, Thm. 4.8.4])**.** *Every finitely generated $\mathcal{S}$-graded R-module has a graded free resolution of length at most n.*

If $\mathcal{F}$ is a graded free resolution of M, as in (1.2), every map φ_i, $i \in \mathbb{N}$, can be represented by a matrix Φ_i with coefficients in $\Bbbk[x_1, \ldots, x_n]$. Moreover, since (1.2) is an exact sequence, then $\Phi_{i+1}\Phi_i = 0$ for all $i \in \mathbb{N}$. When there are not nonzero constants in any of the matrices Φ_i, we say that the resolution is minimal. This can be reformulated in the following way.

Definition 1.34. A graded free resolution $\mathcal{F}$ of M is *minimal* if $\mathrm{Im}(\varphi_i) \subset \mathfrak{m}F_{i-1}$ for all $i \geq 1$, where $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$.

By the graded version of Nakayama's lemma, this condition is equivalent to saying that φ_i maps a basis of F_i to a minimal generating set of $\mathrm{Im}(\varphi_i)$, for all $i \geq 1$ (see [58, p. 151]). Since one can always obtain a *minimal graded free resolution (m.g.f.r.)* from a graded free resolution ([58, Thm. 4.8.6]), it follows from Theorem 1.33 that every finitely generated $\mathcal{S}$-graded R-module has a m.g.f.r. of length at most n.

Example 1.35. Consider the same R and I as in Example 1.32, and the graded free resolution $\mathcal{F}$ that we computed in that example. The matrices Φ_i representing the morphisms φ_i for $i = 0, 1, 2$ are

$$\Phi_0 = (g_1 \; g_2 \; g_3)\,, \quad \Phi_1 = \begin{pmatrix} y^2 - xz & y & -z \\ 0 & -x & y \\ -x^3 + yzt & zt & -x^2 \end{pmatrix}, \text{ and } \; \Phi_2 = \begin{pmatrix} 1 \\ -y \\ -x \end{pmatrix}.$$

Since the matrix Φ_2 contains a nonzero constant, the resolution $\mathcal{F}$ is not minimal. The 1 in Φ_2 comes from the relation $\mathbf{f}_1 = y\mathbf{f}_2 + x\mathbf{f}_3$. To get a m.g.f.r. from $\mathcal{F}$, just note that $\ker(\varphi_0) = \langle \mathbf{f}_2, \mathbf{f}_3 \rangle$, so one can remove the first column of Φ_1. If one does that, then $\ker(\varphi_1) = 0$, and the resolution ends at that point. Hence, a m.g.f.r. of I is as follows:

$$0 \to R(-(7,13)) \oplus R(-(6,14)) \to R(-(6,9)) \oplus R(-(5,10)) \oplus R(-(2,8)) \to I \to 0.$$

From a m.g.f.r. of I, one can get a m.g.f.r. of R/I

$$0 \to R(-(7,13)) \oplus R(-(6,14)) \to R(-(6,9)) \oplus R(-(5,10)) \oplus R(-(2,8)) \\ \to R \to R/I \to 0. \quad (1.3)$$

Given a finitely generated $\mathcal{S}$-graded R-module M, any two m.g.f.r. of M are isomorphic; see, e.g., [58, Thm. 4.8.9]. Here, isomorphic means that at each step, there is a graded isomorphism of modules between the corresponding modules in the two resolutions. As a consequence, one can extract some invariants of the module M from any m.g.f.r. of M. In particular, all the m.g.f.r. of M have the same length, and this length is the *projective dimension of* M, $\mathrm{pd}(M)$. Theorem 1.33 ensures that $\mathrm{pd}(M) \leq n$, and the Auslander-Buchsbaum formula provides the exact relation between these two numbers.

Theorem 1.36 (Auslander-Buchsbaum formula, [13, Thm. 1.3.3])**.** *Let M be a finitely generated $\mathcal{S}$-graded R-module. Then,*

$$\mathrm{pd}(M) + \mathrm{depth}(M) = n\,,$$

where $\mathrm{depth}(M)$ *denotes the depth of M as R-module.*

Since $\mathrm{depth}(M) \leq \dim(M)$, one has that $\mathrm{pd}(M) \leq n - \dim(M)$, and the equality holds if and only if M is *Cohen-Macaulay*. Hence, the Cohen-Macaulay property of a module can be checked in terms of its projective dimension. For a fixed dimension d, Cohen-Macaulay modules are the ones with the shortest resolutions.

Let M be a finitely generated $\mathcal{S}$-graded R-module. Consider a minimal $\mathcal{S}$-graded free resolution of M as R-module,

$$\mathcal{F} : 0 \to F_p \xrightarrow{\varphi_p} F_{p-1} \to \ldots \to F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0\,. \tag{1.4}$$

and write $F_i = \oplus_{\mathbf{s}\in\mathcal{S}} R(-\mathbf{s})^{\beta_{i,\mathbf{s}}}$ for all $i = 0, \ldots, p$. Note that for all i, $\beta_{i,\mathbf{s}} = 0$ for all but finitely many $\mathbf{s} \in \mathcal{S}$, since F_i has finite rank.

Definition 1.37. The *i-th multigraded (or $\mathcal{S}$-graded) Betti number of M in degree* $\mathbf{s}$ is the number $\beta_{i,\mathbf{s}}(M) = \beta_{i,\mathbf{s}}$ of summands $R(-\mathbf{s})$ in F_i. The *i-th Betti number of* M is the rank of the free module F_i, $\beta_i(M) = \beta_i = \sum_{\mathbf{s}\in\mathcal{S}} \beta_{i,\mathbf{s}}$. The *Betti sequence* of M is $(\beta_0, \beta_1, \ldots, \beta_p)$.

Moreover, if M is a Cohen-Macaulay module, the *(Cohen-Macaulay) type of M* is $\mathrm{type}(M) = \beta_p$. We say that M is *Gorenstein* if it is Cohen-Macaulay of type 1.

Usually, we will apply the results of this section to the finitely generated R-module $M = R/I$, where $I \subset R$ is a homogeneous ideal. In particular, if $\mathcal{X} \subset \mathbb{A}^n_{\Bbbk}$ is an affine algebraic variety and $I = I(\mathcal{X})$ is its vanishing ideal, we will work with the coordinate ring of $\mathcal{X}$, $\Bbbk[\mathcal{X}] = R/I$. We will say that $\mathcal{X}$ is *arithmetically Cohen-Macaulay* (resp. *Gorenstein*) if $\Bbbk[\mathcal{X}]$ is Cohen-Macaulay (resp. Gorenstein). The same applies to projective varieties.

Remark 1.38. As in the standard graded case, the i-th multigraded Betti number of M in multidegree $\mathbf{s}$ can also be computed, by [69, Lem. 1.32], as the vector space dimension

$$\beta_{i,\mathbf{s}}(M) = \dim_{\Bbbk}\left(\mathrm{Tor}_i^R(M, \Bbbk)_{\mathbf{s}}\right),$$

where Tor denotes the Tor functor (see, e.g., [83, Chap. 6 and 7] for a reference). To understand this formula note that if M_1, M_2 are two $\mathcal{S}$-graded R-modules, then $M_1 \otimes M_2 = \oplus_{\mathbf{s}\in\mathcal{S}}(M_1 \otimes M_2)_{\mathbf{s}}$, where for all $\mathbf{s} \in \mathcal{S}$, $(M_1 \otimes M_2)_{\mathbf{s}}$ is generated by all elements $f_1 \otimes f_2$ such that $f_1 \in M_{\mathbf{s}_1}$ and $f_2 \in M_{\mathbf{s}_2}$, with $\mathbf{s}_1 + \mathbf{s}_2 = \mathbf{s}$.

Example 1.39. Consider R and I as in Example 1.32. From the m.g.f.r (1.3), the multigraded Betti numbers of R/I are $\beta_{0,(0,0)} = 1$, $\beta_{1,(6,9)} = 1$, $\beta_{1,(5,10)} = 1$, $\beta_{1,(2,8)} = 1$, $\beta_{2,(7,13)} = 1$, $\beta_{2,(6,14)} = 1$, and $\beta_{i,\mathbf{s}} = 0$ otherwise. Then, the Betti sequence of R/I is $(1, 3, 2)$, and $\mathrm{pd}(R/I) = 2$.

Definition 1.40. Let M be a finitely generated $\mathcal{S}$-graded R-module. The *(multigraded) Hilbert function* of M is the map $\mathrm{HF}_M : \mathcal{S} \to \mathbb{N}$ given by

$$\mathrm{HF}_M(\mathbf{s}) = \dim_{\Bbbk} M_{\mathbf{s}}\,,$$

where $\dim_{\Bbbk} M_{\mathbf{s}}$ denotes the dimension of $M_{\mathbf{s}}$ as a $\Bbbk$-vector space (which is finite for all $\mathbf{s} \in \mathcal{S}$). The *(multigraded) Hilbert series* of M is

$$\mathrm{HS}_M(\mathbf{t}) = \sum_{\mathbf{s}\in\mathcal{S}} \mathrm{HF}_M(\mathbf{s}) \cdot \mathbf{t}^{\mathbf{s}} \in \mathbb{Z}[\![t_1, \ldots, t_d]\!]\,,$$

where $\mathbf{t}^{\mathbf{s}} = t_1^{s_1} \ldots t_d^{s_d}$.

It is easy to prove that the Hilbert series of R (as R-module) can be written as a rational function,

$$\mathrm{HS}_R(\mathbf{t}) = \frac{1}{(1 - \mathbf{t}^{\mathbf{a}_1}) \ldots (1 - \mathbf{t}^{\mathbf{a}_n})}\,;$$

see, e.g., [69, Lem. 8.16] for a proof. Moreover, if $\mathbf{s} \in \mathcal{S}$, then $\mathrm{HS}_{R(-\mathbf{s})}(\mathbf{t}) = \frac{\mathbf{t}^{\mathbf{s}}}{(1-\mathbf{t}^{\mathbf{a}_1})\ldots(1-\mathbf{t}^{\mathbf{a}_n})}$. As a consequence, we get that if $F = R(-\mathbf{s}_1) \oplus \ldots \oplus R(-\mathbf{s}_m)$ for some $\mathbf{s}_1, \ldots, \mathbf{s}_m \in \mathcal{S}$, then

$$\mathrm{HS}_F(\mathbf{t}) = \frac{\mathbf{t}^{\mathbf{s}_1} + \cdots + \mathbf{t}^{\mathbf{s}_m}}{(1 - \mathbf{t}^{\mathbf{a}_1}) \ldots (1 - \mathbf{t}^{\mathbf{a}_n})}\,. \tag{1.5}$$

A proof of this result can be found in [69, Thm. 8.20].

Proposition 1.41 ([69, Lem. 8.19])**.** *Let M be a finitely generated $\mathcal{S}$-graded R-module, and consider a graded free resolution $\mathcal{F}$ of M,*

$$\mathcal{F} : 0 \to F_r \xrightarrow{\varphi_p} \ldots \to F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0\,.$$

The multigraded Hilbert function and series of M can be computed as

$$\mathrm{HF}_M = \sum_{j=0}^{r} (-1)^j\, \mathrm{HF}_{F_j}\ , \quad \text{and } \mathrm{HS}_M = \sum_{j=0}^{r} (-1)^j \mathrm{HS}_{F_j}\,.$$

Combining Proposition 1.41 and Equation (1.5), one can compute the Hilbert function and series of any finitely generated $\mathcal{S}$-graded R-module M. Indeed, if p denotes the projective dimension of M and $\beta_{i,\mathbf{s}}$ are its multigraded Betti numbers, then

$$\mathrm{HS}_M(\mathbf{t}) = \frac{\sum_{i=0}^{p}\sum_{\mathbf{s}\in\mathcal{S}}(-1)^i\beta_{i,\mathbf{s}}\mathbf{t}^{\mathbf{s}}}{(1-\mathbf{t}^{\mathbf{a}_1})\dots(1-\mathbf{t}^{\mathbf{a}_n})}\,. \tag{1.6}$$

Example 1.42. Consider R and I as in Example 1.32. By Equation (1.6) and the minimal graded free resolution (1.3), the multigraded Hilbert series of R/I is

$$\mathrm{HS}_{R/I}(t_1,t_2) = \frac{1-t_1^6t_2^9-t_1^5t_2^{10}-t_1^2t_2^8+t_1^7t_2^{13}+t_1^{16}t_2^{14}}{(1-t_1^2t_2^3)(1-t_1t_2^4)(1-t_2^5)(1-t_1^5)}\,.$$

1.2.2 The standard graded case: the Castelnuovo–Mumford regularity

Consider the polynomial ring $R=\Bbbk[x_0,\dots,x_n]$, endowed with the standard grading (i.e., $\deg(x_i)=1$, for all $i=0,\dots,n$). Let M be a finitely generated graded R-module. All the results of Subsection 1.2.1 apply here.

Let $0\to F_p\to\dots\to F_1\to F_0\to M\to 0$ be a minimal graded free resolution of M, where $F_i=\oplus_j R(-j)^{\beta_{i,j}}$ for all $i=0,\dots,p$.

Definition 1.43. Let M be a finitely generated (standard) graded R-module. The *Castelnuovo–Mumford regularity* of M is

$$\mathrm{reg}(M)=\max\{j-i \mid \beta_{i,j}\neq 0\}\,.$$

Note that $\beta_{i,j}=0$ if $i>p$ or $j-i>\mathrm{reg}(M)$. Moreover, if $\beta_{i,j}=0$ for all $j\le j_0$, then $\beta_{i+1,j}=0$ for all $j\le j_0+1$, by the minimality of the resolution (see Def. 1.34). Hence, the graded Betti numbers $\beta_{i,j}=\beta_{i,j}(M)$ are usually presented in a table, called the *Betti table* or *Betti diagram* of M. In this table, the entry corresponding to the i-th column and the j-th row is $\beta_{i,i+j}$:

	0	1	$\dots$	p
0	$\beta_{0,0}$	$\beta_{1,1}$	$\dots$	$\beta_{p,p}$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\dots$	$\beta_{p,p+1}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
reg	$\beta_{0,\mathrm{reg}}$	$\beta_{1,\mathrm{reg}+1}$	$\dots$	$\beta_{p,\mathrm{reg}+p}$

In the table above, reg $=\operatorname{reg}(M)$ is the Castelnuovo-Mumford regularity of M, and $p=\operatorname{pd}(M)$ is its projective dimension.

Note that the Castelnuovo-Mumford regularity of M is the label of the last nonzero row of the Betti table. From Remark 1.38, it follows that $\operatorname{reg}(M)=\max_i\{b_i(M)-i\}$, where $b_i(M)=\max\{\mu \mid \operatorname{Tor}_i^R(M,\Bbbk)_\mu\neq 0\}$ if $\operatorname{Tor}_i^R(M,\Bbbk)\neq 0$ and $b_i(M)=-\infty$ otherwise.

By [85, Thm. 3.11], the regularity is always determined by the tail of a minimal graded free resolution. In other words, Definition 1.43 can be simplified as

$$\operatorname{reg}(M)=\max\{j-i \mid \beta_{i,j}\neq 0,\, n+1-\dim(M)\leq i\leq n+1-\operatorname{depth}(M),\, j\geq 0\}\,. \tag{1.7}$$

As a consequence, when M is Cohen-Macaulay, the regularity is always attained at the last step of a m.g.f.r., a general and well-known fact.

Example 1.44. Take $R=\mathbb{Q}[x,y,z,t]$, with the standard grading, and $I=\langle g_1,g_2,g_3\rangle$, where $g_1=x^3-yzt$, $g_2=x^2y-z^2t$, and $g_3=y^2-xz$ the ideal in Example 1.32, which is homogeneous for the standard grading. With the same computations as in Examples 1.32 and 1.35, one can compute a m.g.f.r. of R/I:

$$R(-4)\oplus R(-4)\to R(-3)\oplus R(-3)\oplus R(-2)\to R\to R/I\to 0,$$

where the maps are the same as in (1.3). The Betti diagram of R/I is

	0	1	2
0:	1	-	-
1:	-	1	-
2:	-	2	2
total:	1	3	2

The Castelnuovo-Mumford regularity of R/I is $\operatorname{reg}(R/I)=2$.

There is an equivalent definition of the Castelnuovo-Mumford regularity in terms of the *local cohomology modules of M*. Let $\mathfrak{m}=\langle x_0,\dots,x_n\rangle$ be the homogeneous maximal ideal of R, called the *irredundant ideal*, and set $H^0_{\mathfrak{m}}(M)\coloneqq\{z\in M \mid \exists \ell\in\mathbb{N} \text{ with } z\mathfrak{m}^\ell=0\}$. For every positive integer i, $H^i_{\mathfrak{m}}(-)$ is defined as the i-th right derived functor of $H^0_{\mathfrak{m}}(-)$.

Grothendieck's theorem states that $H^i_{\mathfrak{m}}(M)=0$ for $i>\dim(M)$ and $i<\operatorname{depth}(M)$, and $H^i_{\mathfrak{m}}(M)$ does not vanish for $i=\dim(M)$ and $i=\operatorname{depth}(M)$. For all $i\in\mathbb{N}$, denote $\operatorname{end}(H^i_{\mathfrak{m}}(M))=\max\{\mu \mid H^i_{\mathfrak{m}}(M)_\mu\neq 0\}$ if $H^i_{\mathfrak{m}}(M)\neq 0$, and $\operatorname{end}(H^i_{\mathfrak{m}}(M))=-\infty$, otherwise.

Theorem 1.45 ([16, Cor. 2.2]). *Let M be a finitely generated graded R-module. Then, the Castelnuovo-Mumford regularity of M is*

$$\operatorname{reg}(M) = \max\left\{\operatorname{end}\left(H^i_{\mathfrak{m}}(M)\right) + i \mid \operatorname{depth}(M) \leq i \leq \dim(M)\right\} .$$

Moreover,

$$\max\{j \mid \beta_{n+1-\operatorname{depth}(M),j} \neq 0\} = \operatorname{end}\left(H^{\operatorname{depth}(M)}_{\mathfrak{m}}(M)\right) + n + 1 .$$

Unlike the projective dimension, which is upper bounded by n, the Castelnuovo-Mumford regularity of M cannot be bounded in general in a simple way. Hence, it has been a topic of research for many years, and has been object of many conjectures. Probably, the most famous one is the Eisenbud-Goto conjecture (see [29]), which was thought to be true for many years (1984-2017), until McCullough and Peeva provided in [68] a family of counterexamples to this long-standing conjecture. However, the conjecture remains still open in some interesting cases, such as simplicial projective toric ideals (see Section 1.3).

Conjecture 1.46 (Eisenbud-Goto, [29]). *Suppose that $\Bbbk$ is an algebraically closed field. Let $R = \Bbbk[x_0, \ldots, x_n]$ and $I \subset R$ be a homogeneous prime ideal such that $I \subset \langle x_1, \ldots, x_n\rangle^2$. Then,*

$$\operatorname{reg}(R/I) \leq \deg(R/I) - \operatorname{codim}(I) ,$$

where $\deg(R/I)$ is the multiplicity of R/I (also called the degree of R/I, see Remark 1.49), and $\operatorname{codim}(I) = n + 1 - \dim(R/I)$ is the codimension (also called height) of I.

Although the conjecture was proved to be false in general, it is known that the conjecture holds in some cases and is still believed to hold for the ideals I defining "nice" projective varieties. For example, it is true for arithmetically Cohen-Macaulay varieties, arithmetically Buchsbaum varieties [88], projective curves [46], smooth surfaces [63], smooth threefolds in $\mathbb{P}^5$ [55] and toric varieties of codimension two [77]. When it comes to projective toric varieties, the conjecture is known to be true for simplicial smooth toric varieties [48]. In fact, the authors gave a bound that improves Eisenbud-Goto's one in that case. For toric varieties of dimension 1, there is a combinatorial proof for Eisenbud-Goto conjecture [73].

Now, consider the Hilbert function of M. Combining Proposition 1.41 and the fact $\operatorname{HF}_{R(-j)}(s) = \binom{s-j+n}{n}$, one has that

$$\operatorname{HF}_M(s) = \sum_{i,j \mid \beta_{ij} \neq 0} (-1)^i \beta_{ij} \binom{s-j+n}{n} \tag{1.8}$$

for all $s \in \mathbb{N}$. If one considers the polynomial

$$\mathrm{HP}_M(t) = \sum_{i,j|\beta_{ij}\neq 0} (-1)^i \beta_{ij} \frac{(t-j+n)(t-j+n-1)\dots(t-j+1)}{n!} \in \mathbb{Q}[t]\,,$$

then it is clear that $\mathrm{HF}_M(s) = \mathrm{HP}_M(s)$ for all $s \in \mathbb{N}$ such that $s - j + n \geq 0$ for all j. Hence, we have proved the following.

Theorem 1.47 ([20, Chap. 6, Prop. 4.7])**.** *Let $R = \Bbbk[x_0, \dots, x_n]$ and M be a finitely generated graded R-module. Then, there exists a (unique) polynomial $\mathrm{HP}_M(t) \in \mathbb{Q}[t]$, such that*

$$\mathrm{HP}_M(s) = \mathrm{HF}_M(s)$$

for all $s \in \mathbb{N}$ sufficiently large.

Definition 1.48. In the conditions of the previous theorem, the polynomial HP_M is called the *Hilbert polynomial* of M, and the minimum $s \in \mathbb{N}$ such that $\mathrm{HP}_M(s') = \mathrm{HF}_M(s')$ for all $s' \geq s$ is called the *regularity of the Hilbert function* of M. We denote it by $\mathrm{r}(M)$.

By Equation (1.6), the Hilbert series of M can be written as

$$\mathrm{HS}_M(t) = \frac{\sum_{i,j|\beta_{i,j}\neq 0}(-1)^i \beta_{i,j} t^j}{(1-t)^{n+1}}.$$

Let q be the maximal power such that $(1-t)^q$ divides the numerator of $\mathrm{HS}_M(t)$. Then, one can write

$$\mathrm{HS}_M(t) = \frac{h(t)}{(1-t)^{n+1-q}},$$

where $h(t) \in \mathbb{Z}[t]$ is called the *h-polynomial* of M. By [76, Thm. 16.7(1)], $n + 1 - q = \dim(M)$ is the Krull dimension of M. The *multiplicity of M* is defined as $e(M) = h(1)$.

Remark 1.49. Let M be a finitely generated graded R-module. The multiplicity and dimension of M can also be read from the Hilbert polynomial of M as follows:

- The degree of the polynomial HP_M is $d - 1$, where $d = \dim(M)$ is the Krull dimension of M, by [13, Thm. 4.1.3] .
- The leading term of HP_M is $\frac{e}{(d-1)!}t^{d-1}$, where $e = e(M)$ is the multiplicity of M, by [76, Thm. 16.7(2)].

When $I \subset R$ is a homogeneous ideal and $M = R/I$, we will also refer to the multiplicity of R/I as the *degree of R/I*, or the *degree of the algebraic set $V(I) \subset \mathbb{P}^n_{\Bbbk}$*.

Let M be a finitely generated graded R-module. We want to relate precisely the Castelnuovo-Mumford and the Hilbert function regularity of M in terms of the Betti numbers of M, β_{ij}. If we set $\mathrm{reg} := \mathrm{reg}(M)$ the Castelnuovo-Mumford regularity of M, then for all $s \in \mathbb{N}$

$$\mathrm{HF}_M(s) = \sum_{i=0}^{n+1} \sum_{j=0}^{\mathrm{reg}} (-1)^i \beta_{i,i+j} \binom{s-(i+j)+n}{n}, \text{ and}$$
$$\mathrm{HP}_M(s) = \frac{1}{n!} \sum_{i=0}^{n+1} \sum_{j=0}^{\mathrm{reg}} (-1)^i \beta_{i,i+j} \prod_{\ell=1}^{n} (s-(i+j)+\ell)\,.$$

Taking into account the roots of the polynomial $\prod_{\ell=1}^{n}(s-(i+j)+\ell)$, it is easy to prove that $\mathrm{HF}_M(s) = \mathrm{HP}_M(s)$ for all $s \geq \mathrm{reg}+1$, that is

$$\mathrm{r}(M) \leq \mathrm{reg}(M) + 1\,. \tag{1.9}$$

To determine precisely the difference δ between the two regularities, $\delta := \mathrm{reg}(M) - \mathrm{r}(M)$, we need to evaluate the difference $\mathrm{HP}_M(\mathrm{reg}+1-\lambda) - \mathrm{HF}_M(\mathrm{reg}+1-\lambda)$ for $1 \leq \lambda \leq \mathrm{reg}+1$. For $\lambda \geq 1$ and $k \geq 0$, set

$$A_k^{(\lambda)} := \binom{\mathrm{reg}+1+n-(\lambda+1)-k}{n}, \text{ and}$$
$$B_k^{(\lambda)} := \frac{1}{n!} \prod_{\ell=1}^{n} (\mathrm{reg}+1-\lambda-k+\ell)\,.$$

Using these notations, for all λ, $1 \leq \lambda \leq \mathrm{reg}+1$, we can write

$$\begin{aligned} \mathrm{HF}_M(\mathrm{reg}+1-\lambda) - \mathrm{HP}_M(\mathrm{reg}+1-\lambda) &= \sum_{i=0}^{n+1} \sum_{j=0}^{\mathrm{reg}} (-1)^i \beta_{i,i+j} \left(A_{i+j}^{(\lambda)} - B_{i+j}^{(\lambda)}\right) \\ &= \sum_{i+j=0}^{\mathrm{reg}+n+1} (-1)^i \beta_{i,i+j} \left(A_{i+j}^{(\lambda)} - B_{i+j}^{(\lambda)}\right). \end{aligned} \tag{1.10}$$

The following lemma establishes when $A_k^{(\lambda)}$ and $B_k^{(\lambda)}$ coincide.

Lemma 1.50. *Consider $\lambda \geq 1$ and k, such that $0 \leq k \leq \mathrm{reg}+n+2-\lambda$.*

(1) If $0 \leq k \leq \mathrm{reg}-\lambda+1$, then $A_k^{(\lambda)} = B_k^{(\lambda)} \neq 0$.

(2) If $\mathrm{reg}-\lambda+2 \leq k \leq \mathrm{reg}+n-\lambda+1$, then $A_k^{(\lambda)} = B_k^{(\lambda)} = 0$.

(3) If $k = \mathrm{reg}+n+2-\lambda$, then $A_k^{(\lambda)} = 0$ and $B_k^{(\lambda)} = (-1)^n$.

Proof. If $k \leq \mathrm{reg} - \lambda + 1$, then $\mathrm{reg} + n + 1 - \lambda - k \geq n$, so $A_k^{(\lambda)} = B_k^{(\lambda)} \neq 0$ and (1) follows. Otherwise, $A_k^{(\lambda)} = 0$ and we distinguish two cases. If $k \leq \mathrm{reg} + n - \lambda + 1$, then $1 \leq k + \lambda - \mathrm{reg} - 1 \leq n$, and hence, $B_k^{(\lambda)} = 0$ and (2) follows. Finally, if $k = \mathrm{reg} + n + 2 - \lambda$, then

$$B^{(\lambda)}_{\mathrm{reg}+n+2-\lambda} = \frac{1}{n!}\prod_{\ell=1}^{n}(\ell - n - 1) = (-1)^n,$$

and we are done. □

By Lemma 1.50 (3) and Equation (1.10), $\mathrm{HF}_M(\mathrm{reg}) - \mathrm{HP}_M(\mathrm{reg}) = \beta_{n+1,\mathrm{reg}+n+1}$ so if $\beta_{n+1,\mathrm{reg}+n+1} \neq 0$, one gets that $\mathrm{r}(M) = \mathrm{reg}(M) + 1$, i.e., $\delta = -1$. And the reciprocal statement also holds. This is a particular case of the following result that relates precisely δ to some of the Betti numbers.

Theorem 1.51. *Let M be a finitely generated graded module over $\Bbbk[x_0, \ldots, x_n]$, and denote by δ the difference between the Castelnuovo-Mumford regularity and the regularity of the Hilbert function of M, i.e., $\delta := \mathrm{reg}(M) - \mathrm{r}(M)$ ($\delta \geq -1$). Then, $\delta = \lambda - 2$, where $\lambda \geq 1$ is the least integer, such that $\sum_i (-1)^i \beta_{i,\mathrm{reg}(M)+n+2-\lambda} \neq 0$, where the β_{ij} are the graded Betti numbers of M.*

Proof. The case $\lambda = 1$ is proved just before the proposition, so assume that $\lambda \geq 2$. Since, for all $\mu = 1, 2, \ldots, \lambda - 1$, $\sum_i (-1)^i \beta_{i,\mathrm{reg}+n+2-\mu} = 0$, by Equation (1.10) one gets that $\mathrm{HP}_M(s) = \mathrm{HF}_M(s)$ for all $s \geq \mathrm{reg} - \lambda + 2$, i.e., $\mathrm{r}(M) \leq \mathrm{reg}(M) - \lambda + 2$. Moreover, by applying Lemma 1.50 (3) to Equation (1.10), we obtain that

$$\begin{aligned}\mathrm{HF}_M(\mathrm{reg} - \lambda + 1) - \mathrm{HP}_M(\mathrm{reg} - \lambda + 1) &= \sum_i (-1)^{i+1} \beta_{i,\mathrm{reg}+n+2-\lambda} B^{(\lambda)}_{\mathrm{reg}+n+2-\lambda} \\ &= \sum_i (-1)^{n+i+1} \beta_{i,\mathrm{reg}+n+2-\lambda} \neq 0\,,\end{aligned}$$

and hence $\mathrm{r}(M) = \mathrm{reg}(M) - \lambda + 2$. □

Remark 1.52. (1) If we focus on the secondary diagonals of the Betti diagram starting from the bottom right of the table, the number λ in the previous theorem is the label of the first diagonal, such that the alternating sum of the Betti numbers on this diagonal is not 0; see Table 1.1.

(2) If p denotes the projective dimension of the module M, the previous result implies that $\beta_{p,\mathrm{reg}(M)+p} \neq 0$, i.e., the regularity is attained at the last step of a m.g.f.r. of M, if and only if $\lambda = n - p + 2$, i.e., $\delta = n - p$. This occurs, in particular, whenever M is a Cohen-Macaulay module, so, in this case, $\mathrm{reg}(M) - \mathrm{r}(M) = n - p$ which is a well-known fact; see, e.g., [28, Cor. 4.8].

j/i	0	1	$\dots$	$p-1$	p	$p+1$	$\dots$	$n+1$
0	1	$\beta_{1,1}$	$\dots$	$\beta_{p-1,p-1}$	$\beta_{p,p}$	0	$\dots$	0
1	$-$	$\beta_{1,2}$	$\dots$	$\beta_{p-1,p}$	$\beta_{p,p+1}$	0	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
reg $-$ 1	$-$	$\beta_{1,\text{reg}}$	$\dots$	$\beta_{p-1,\text{reg}+p-2}$	$\beta_{p,\text{reg}+p-1}$	0	$\dots$	0
reg	$-$	$\beta_{1,\text{reg}+1}$	$\dots$	$\beta_{p-1,\text{reg}+p-1}$	$\beta_{p,\text{reg}+p}$	0	$\dots$	0

$\lambda = n-p+3$ $\lambda = n-p+2$ $\lambda = 1$

Table 1.1: Betti diagram in Remark 1.52 (1).

(3) If $\operatorname{depth}(M) \geq 1$, i.e., $p \leq n$ (by Theorem 1.36), then $\mathrm{r}(M) \leq \operatorname{reg}(M)$.

(4) Let $\lambda \geq 1$ be the least integer such that $\sum_i (-1)^i \beta_{i,\operatorname{reg}(M)+n+2-\lambda} \neq 0$. By Equation (1.6) and Theorem 1.51, the degree of the numerator in the Hilbert series of M (before simplifying) is $\operatorname{reg}(M) + n + 2 - \lambda = \mathrm{r}(M) + 2$. Hence, $\mathrm{r}(M) - 1$ is the difference of the degrees of the polynomials in the numerator and denominator of the Hilbert series of M.

Example 1.53. Take R and $I \subset R$ as in Example 1.44. By (1.8), the Hilbert function of R/I is given by

$$\mathrm{HF}_{R/I}(s) = \binom{s+3}{3} - \binom{s+1}{3} - \binom{s}{3} + \binom{s-1}{3},$$

for all $s \in \mathbb{N}$, and the Hilbert polynomial of R/I is $\mathrm{HP}_{R/I}(t) = \frac{1}{2}t^2 - \frac{7}{2}t$. By Theorem 1.51, $\operatorname{reg}(R/I) - \mathrm{r}(R/I) = 1$, and hence $\mathrm{r}(R/I) = 1$, i.e., $\mathrm{HF}_{R/I}(s) = \mathrm{HP}_{R/I}(s)$ for all $s \geq 1$ and $\mathrm{HF}_{R/I}(0) \neq \mathrm{HP}_{R/I}(0)$.

1.3 Toric ideals and toric varieties

In this section, we introduce toric ideals and toric varieties since they will play an important role in this thesis. Toric varieties appear in the literature in several different ways; see [21, Chap. 1-2]. For us, a toric variety will be the zero set of a toric ideal.

Let $\mathbb{k}$ be a field, $\mathbb{k}[\mathbf{x}] := \mathbb{k}[x_1, \dots, x_n]$ and $\mathbb{k}[\mathbf{t}] := \mathbb{k}[t_1, \dots, t_d]$ two polynomial rings over $\mathbb{k}$. Given $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{N}^d$ a set of nonzero vectors, each element

$\mathbf{a}_i = (a_{i1}, \ldots, a_{id}) \in \mathbb{N}^d$ corresponds to the monomial $\mathbf{t}^{\mathbf{a}_i} = t_1^{a_{i1}} \ldots t_d^{a_{id}}$ in $\Bbbk[t_1, \ldots, t_d]$. Set $\mathcal{S}_\mathcal{A} \subset \mathbb{N}^d$ the affine semigroup generated by $\mathcal{A}$.

Definition 1.54. The *toric ideal determined by* $\mathcal{A}$, $I_\mathcal{A}$, is the kernel of the homomorphism of $\Bbbk$-algebras $\varphi_\mathcal{A} : \Bbbk[\mathbf{x}] \to \Bbbk[\mathbf{t}]$ defined by $\varphi(x_i) = \mathbf{t}^{\mathbf{a}_i}$. The *toric ring determined by* $\mathcal{A}$ is $\Bbbk[\mathbf{x}]/I_\mathcal{A}$.

The image of $\varphi_\mathcal{A}$ is the *semigroup algebra of* $\mathcal{S}_\mathcal{A}$, $\Bbbk[\mathcal{S}_\mathcal{A}] = \mathrm{Im}(\varphi_\mathcal{A})$, and the surjective homomorphism $\varphi_\mathcal{A}$ endows $\Bbbk[\mathcal{S}_\mathcal{A}]$ with a structure of $\Bbbk[\mathbf{x}]$-module. Note that this structure of $\Bbbk[\mathbf{x}]$-module depends not only on the semigroup $\mathcal{S}_\mathcal{A}$, but also on the generating set $\mathcal{A}$ that we have fixed for $\mathcal{S}_\mathcal{A}$.

If necessary, we can assume that $\delta(\mathcal{A}) := \gcd(\{a_{ij} \mid 1 \le i \le n, 1 \le j \le d\}) = 1$. If this is not the case, consider the set $\mathcal{A}' = \frac{1}{\delta(\mathcal{A})}\mathcal{A}$, since one has that $I_{\mathcal{A}'} = I_\mathcal{A}$ and $\delta(\mathcal{A}') = 1$.

Remark 1.55. Given a set $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$ of nonzero vectors, one can compute the toric ideal $I_\mathcal{A}$ as follows. Consider the ideal $L_\mathcal{A} = \langle x_1 - \mathbf{t}^{\mathbf{a}_1}, \ldots, x_n - \mathbf{t}^{\mathbf{a}_n} \rangle$ of $\Bbbk[\mathbf{x}, \mathbf{t}] = \Bbbk[x_1, \ldots, x_n, t_1, \ldots, t_d]$. By [1, Thm. 2.3.4], the toric ideal $I_\mathcal{A} = L_\mathcal{A} \cap \Bbbk[x_1, \ldots, x_n]$. Hence, if $\mathcal{G}$ is a Gröbner basis of $L_\mathcal{A}$ for an elimination monomial order $>$ in $\Bbbk[\mathbf{x}, \mathbf{t}]$ such that $t_j > x_i$ for all $1 \le i \le n$, $1 \le j \le d$, then $\mathcal{G} \cap \Bbbk[\mathbf{x}]$ is a Gröbner basis of $I_\mathcal{A}$. In particular, $\mathcal{G} \cap \Bbbk[\mathbf{x}]$ generates $I_\mathcal{A}$.

Although the described algorithm is theoretically feasible, sometimes it is not the best for computations. We refer the reader to [8] and [52] for other algorithms that exploit the structure of toric ideals to compute $I_\mathcal{A}$. These algorithms mainly use lattices and saturation.

The toric ideal $I_\mathcal{A}$ is prime and, if one sets the $\mathcal{S}_\mathcal{A}$-degree of a monomial $\mathbf{x}^\alpha \in \Bbbk[x_1, \ldots, x_n]$ as $|\mathbf{x}^\alpha|_{\mathcal{S}_\mathcal{A}} := \alpha_1 \mathbf{a}_1 + \cdots + \alpha_n \mathbf{a}_n \in \mathcal{S}_\mathcal{A}$, it is $\mathcal{S}_\mathcal{A}$-homogeneous, i.e., $I_\mathcal{A}$ is homogeneous for the grading induced by $|\mathbf{x}^\alpha|_{\mathcal{S}_\mathcal{A}}$. Indeed, by [89, Cor. 4.3], $I_\mathcal{A}$ is the binomial ideal

$$I_\mathcal{A} = \langle\, \mathbf{x}^\alpha - \mathbf{x}^\beta : |\mathbf{x}^\alpha|_{\mathcal{S}_\mathcal{A}} = |\mathbf{x}^\beta|_{\mathcal{S}_\mathcal{A}} \,\rangle\,.$$

Moreover, if one considers $\omega_i = \sum_{j=1}^d a_{ij}$, $i = 1, \ldots, n$, and sets $\deg_\omega(x_i) = \omega_i$, the ideal $I_\mathcal{A}$ is also ω-homogeneous for the weight vector $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{Z}^n_{>0}$.

Note that for both gradings, the homomorphism of $\Bbbk[\mathbf{x}]$-modules $\varphi_\mathcal{A} : \Bbbk[\mathbf{x}] \to \Bbbk[\mathbf{t}]$ defined by $x_i \mapsto \mathbf{t}^{\mathbf{a}_i}$ is graded, and hence $\Bbbk[\mathcal{S}_\mathcal{A}]$ and the toric ring $\Bbbk[x_1, \ldots, x_n]/I_\mathcal{A}$ are isomorphic as graded $\Bbbk[\mathbf{x}]$-modules.

Example 1.56. Consider $\mathcal{A} = \{\mathbf{a}_1 = (2,3), \mathbf{a}_2 = (1,4), \mathbf{a}_3 = (0,5), \mathbf{a}_4 = (5,0)\} \subset \mathbb{N}^2$, and let $I_\mathcal{A} \subset \mathbb{Q}[x_1, x_2, x_3, x_4]$ be the toric ideal determined by $\mathcal{A}$. By Remark 1.55,

$$I_\mathcal{A} = \langle x_1 - t_1^2 t_2^3, x_2 - t_1 t_2^4, x_3 - t_2^5, x_4 - t_1^5 \rangle \cap \mathbb{Q}[x_1, x_2, x_3, x_4]\,.$$

Note that $I_\mathcal{A}$ is the ideal from Example 1.32 (after relabeling the variables). One has that $I_\mathcal{A}$ is homogeneous for the $\mathcal{S}_\mathcal{A}$-grading induced by $|x_i|_{\mathcal{S}_\mathcal{A}} = \mathbf{a}_i$, $i = 1, \ldots, 4$, where $\mathcal{S}_\mathcal{A} = \langle \mathcal{A} \rangle$. Moreover, $I_\mathcal{A}$ is ω-homogeneous for the weight vector $\omega = (5, 5, 5, 5)$, and hence, it is also homogeneous for the standard grading.

The next proposition characterizes when the toric ideal $I_\mathcal{A}$ is homogeneous for the standard grading, i.e., homogeneous when one sets $\deg(x_i) = 1$ for all $i = 1, \ldots, n$.

Proposition 1.57 ([89, Lem. 4.14]). *Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$, and $I_\mathcal{A}$ be the toric ideal determined by $\mathcal{A}$. Then, $I_\mathcal{A}$ is homogeneous (for the standard grading) if and only if there exists a vector $\mathbf{v} \in \mathbb{Q}^d$, such that $\mathbf{a}_i \cdot \mathbf{v} = \sum_{j=1}^n a_{ij} v_j = 1$ for all $i = 1, \ldots, n$.*

Note that, by the previous result, one has that the toric ideal $I_\mathcal{A}$ is homogeneous if and only if $\mathcal{A}$ is contained in a hyperplane H of $\mathbb{Q}^d$ not passing through the origin, $H = \{(z_1, \ldots, z_d) \in \mathbb{Q}^d \mid \sum_{i=1}^d v_i w_i = 1\}$, for some $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{Q}^n$, $\mathbf{v} \neq \mathbf{0}$.

By [89, Lem. 4.2], the Krull dimension of $R/I_\mathcal{A}$, $\dim(R/I_\mathcal{A})$, equals the dimension of the cone $\mathrm{Pos}(\mathcal{A})$, i.e., the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q}\mathcal{A}$. If this dimension is $d' < d$, one can easily show that there exists a set $\mathcal{A}' = \{\mathbf{a}'_1, \ldots, \mathbf{a}'_n\} \subset \mathbb{N}^{d'}$ such that $I_\mathcal{A} = I_{\mathcal{A}'}$. Hence, without loss of generality, we will always assume that the cone $\mathrm{Pos}(\mathcal{A})$ has dimension d, and $\dim(R/I_\mathcal{A}) = d$ if $\mathcal{A} \subset \mathbb{N}^d$.

Definition 1.58. (1) Given an ideal $I \subset \Bbbk[x_1, \ldots, x_n]$ of height $n - d$, we say that I is a *toric ideal* if there exists a finite set $\mathcal{A} \subset \mathbb{N}^d$ such that $I = I_\mathcal{A}$.

(2) The toric ideal $I_\mathcal{A}$ is *simplicial* if the affine semigroup $\mathcal{S}_\mathcal{A}$ is simplicial.

Example 1.59. Let $\mathcal{A} = \{\mathbf{a}_1 = (0, 5), \mathbf{a}_2 = (1, 3), \mathbf{a}_3 = (2, 1)\} \subset \mathbb{N}^2$, and $\Bbbk = \mathbb{Q}$. The toric ideal determined by $\mathcal{A}$ is simplicial, since every affine semigroup in $\mathbb{N}^2$ is simplicial, and, by Proposition 1.57, it is homogeneous for the standard grading, because $\mathbf{a}_i \cdot (\frac{2}{5}, \frac{1}{5}) = 1$ for $i = 1, 2, 3$. One has that

$$I_\mathcal{A} = \langle x_1 - t_2^5, x_2 - t_1 t_2^3, x_3 - t_1^2 t_2 \rangle \cap \mathbb{Q}[x_1, x_2, x_3] = \langle x_2^2 - x_1 x_3 \rangle .$$

Note that $I_\mathcal{A}$ is also homogeneous for the multigrading given by $\deg(x_i) = \mathbf{a}_i$ since, for this grading, f is homogeneous of degree $\deg_{\mathcal{S}_\mathcal{A}}(f) = (2, 6)$. Moreover, $I_\mathcal{A}$ is ω-homogeneous for the weight vector $\omega = (5, 4, 3)$.
Note that if $\mathcal{B} = \{(2, 0), (1, 1), (0, 2)\} \subset \mathbb{N}^2$, one has that $I_\mathcal{B} = I_\mathcal{A}$.

Simplicial toric ideals can always be seen as the toric ideal determined by a set $\mathcal{A} \subset \mathbb{N}^d$ whose cone $\mathrm{Pos}(\mathcal{A}) = \mathbb{N}^d$, and all the extremal rays of the cone are of the same length.

Proposition 1.60 ([48, Sect. 2]). *Let $I \subset \Bbbk[x_1, \ldots, x_n]$ be an ideal. Then, I is a simplicial toric ideal if and only if there exists a set $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$ such that $I = I_{\mathcal{A}}$ and $\{D\boldsymbol{\epsilon}_1, \ldots, D\boldsymbol{\epsilon}_d\} \subset \mathcal{A}$, where $\{\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_d\}$ is the canonical basis of $\mathbb{N}^d$, and $D \in \mathbb{Z}_{>0}$.*

Proof. Being ($\Leftarrow$) straightforward, let us prove ($\Rightarrow$). Let $I \subset \Bbbk[\mathbf{x}]$ be a simplicial toric ideal, and $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subset \mathbb{N}^d$ a set of nonzero vectors such that $I = I_{\mathcal{B}}$. By hypothesis, $\langle \mathcal{B} \rangle$ is simplicial, and hence we can suppose without loss of generality that $\operatorname{Pos}(\mathcal{B})$ is minimally generated by $\{\mathbf{b}_1, \ldots, \mathbf{b}_d\}$. Let M be the $d \times d$ matrix whose i-th column is $\mathbf{b}_i$, and M^* the adjoint of M. Note that $\det(M) \neq 0$ since the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_d$ are linearly independent; indeed, we can assume $\det(M) > 0$. Then, $M^* M = \det(M) I_d$, where I_d is the $d \times d$ identity matrix. Consider the set $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, where $\mathbf{a}_i = M^* \cdot \mathbf{b}_i$ for all $i = 1, \ldots, n$. Then, $\mathbf{a}_i = \det(M)\boldsymbol{\epsilon}_i$ for $i = 1, \ldots, d$, one can easily check that $\mathcal{A} \subset \mathbb{N}^d$, and $I = I_{\mathcal{A}}$ since the matrix M is invertible. □

For general toric ideals $I = I_{\mathcal{A}} \subset \Bbbk[\mathbf{x}]$, it is known that binomial generating sets and Gröbner bases of $I_{\mathcal{A}}$ (see, e.g., [89]) and also several Betti numbers of $\Bbbk[\mathcal{S}_{\mathcal{A}}] \cong \Bbbk[\mathbf{x}]/I_{\mathcal{A}}$ (see [12, Thm. 1.3]) are independent of $\Bbbk$. Nevertheless, the Gorenstein, Cohen-Macaulay and Buchsbaum properties of $\Bbbk[\mathcal{S}_{\mathcal{A}}]$ depend on the characteristic of $\Bbbk$ (see [51], [94] and [50], respectively). This situation changes in the context of simplicial semigroup rings, since the Gorenstein, Cohen-Macaulay, and Buchsbaum properties can be entirely described in terms of the combinatorics of the semigroup $\mathcal{S}_{\mathcal{A}}$ and, as a consequence, they do not depend on $\Bbbk$ (see [43], [87] and [37], respectively). We present here the combinatorial characterization of the Cohen-Macaulay and Gorenstein properties for simplicial toric rings.

Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$ be a set of nonzero vectors, and suppose that the semigroup $\mathcal{S}_{\mathcal{A}} = \langle \mathcal{A} \rangle$ is simplicial. Denote by $\mathcal{E} = \{\mathbf{e}_1, \ldots, \mathbf{e}_d\} \subset \mathcal{A}$ the set of extremal rays of the cone $\operatorname{Pos}(\mathcal{A})$.

For $d = 1$, the semigroup algebra $\Bbbk[\mathcal{S}_{\mathcal{A}}]$ is always Cohen-Macaulay, because $\dim(\Bbbk[\mathcal{S}_{\mathcal{A}}]) = \operatorname{depth}(\Bbbk[\mathcal{S}_{\mathcal{A}}]) = 1$ in this case.

Proposition 1.61 ([15, Lem. 2.6], [59]). *Let $\mathcal{A} = \{a_1, \ldots, a_n\} \subset \mathbb{N}$ such that $\gcd(a_1, \ldots, a_n) = 1$, and denote by $\mathcal{S}_{\mathcal{A}}$ the numerical semigroup generated by $\mathcal{A}$.*

(1) The semigroup algebra $\Bbbk[\mathcal{S}_{\mathcal{A}}]$ is Cohen-Macaulay, and its Cohen-Macaulay type is equal to the type of $\mathcal{S}_{\mathcal{A}}$,

$$\operatorname{type}(\Bbbk[\mathcal{S}_{\mathcal{A}}]) = t(\mathcal{S}_{\mathcal{A}}) = |\operatorname{PF}(\mathcal{S}_{\mathcal{A}})|.$$

(2) $\Bbbk[\mathcal{S}_{\mathcal{A}}]$ *is Gorenstein if and only if the semigroup* $\mathcal{S}_{\mathcal{A}}$ *is symmetric.*

For $d \geq 2$, one can check if $\Bbbk[\mathcal{S}_{\mathcal{A}}]$ has the Cohen-Macaulay property in terms of the semigroup $\mathcal{S}_{\mathcal{A}}$, by the following result due to Goto, Suzuki, and Watanabe.

Theorem 1.62 ([43, Thm. 2.6])**.** *The simplicial semigroup algebra* $\Bbbk[\mathcal{S}_{\mathcal{A}}]$ *is Cohen-Macaulay if and only if for all* $\mathbf{s} \in \mathbb{N}^d$

$$\text{if } \mathbf{s}+\mathbf{e}_i \in \mathcal{S}_{\mathcal{A}} \text{ and } \mathbf{s}+\mathbf{e}_j \in \mathcal{S}_{\mathcal{A}} \text{ for some } 1 \leq i < j \leq d, \text{ then } \mathbf{s} \in \mathcal{S}_{\mathcal{A}}\,.$$

When $\Bbbk[\mathcal{S}_{\mathcal{A}}]$ is Cohen-Macaulay, one can compute its type by counting the number of maximal elements in the Apéry set $\mathrm{AP}_{\mathcal{S}_{\mathcal{A}}} = \mathrm{Ap}\,(\mathcal{S}_{\mathcal{A}}, \mathcal{E})$ for the natural order $\leq_{\mathcal{S}_{\mathcal{A}}}$ defined in (1.1). Hence, one can characterize combinatorially the Gorenstein property.

Theorem 1.63 ([53, Prop. 3.3], [14])**.** *Suppose that* $\Bbbk[\mathcal{S}_{\mathcal{A}}]$ *is Cohen-Macaulay. Then, its type is the number of maximal elements in the Apéry set* $\mathrm{AP}_{\mathcal{S}_{\mathcal{A}}}$ *for the order* $\leq_{\mathcal{S}_{\mathcal{A}}}$*. Hence,* $\Bbbk[\mathcal{S}_{\mathcal{A}}]$ *is Gorenstein if and only if the poset* $(\mathrm{AP}_{\mathcal{S}_{\mathcal{A}}}, \leq_{\mathcal{S}_{\mathcal{A}}})$ *has a unique maximal element.*

Now, we introduce the geometric counterpart of toric ideals: the toric sets and toric varieties.

Definition 1.64. Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$ be a set of nonzero vectors.

(1) The *toric set* determined by $\mathcal{A}$ is the set $\Gamma_{\mathcal{A}} \subset \mathbb{A}^n_{\Bbbk}$ defined parametrically by

$$\Gamma_{\mathcal{A}} = \{(u_1^{a_{11}} \ldots u_d^{a_{1d}}, \ldots, u_1^{a_{n1}} \ldots u_d^{a_{nd}}) \in \mathbb{A}^n_{\Bbbk} \mid u_1, \ldots, u_n \in \Bbbk\}\,.$$

(2) The *affine toric variety* determined by $\mathcal{A}$, also called *affine monomial variety*, is the zero set of the toric ideal $I_{\mathcal{A}}$, $\mathcal{X}_{\mathcal{A}} = V(I_{\mathcal{A}}) \subset \mathbb{A}^n_{\Bbbk}$. We say that the toric variety $\mathcal{X}_{\mathcal{A}}$ is *simplicial* if $I_{\mathcal{A}}$ is simplicial.

(3) We say that $\mathcal{X}_{\mathcal{A}}$ is an *affine toric curve* (resp. *surface*) or an *affine monomial curve* (resp. *surface*) if the dimension of $\mathcal{X}_{\mathcal{A}}$ is 1 (resp. 2).

Remark 1.65. It is clear that $\Gamma_{\mathcal{A}} \subset \mathcal{X}_{\mathcal{A}}$, where $\mathcal{X}_{\mathcal{A}} = V(I_{\mathcal{A}})$. A natural question one can ask is when the Zariski closure of $\Gamma_{\mathcal{A}}$ is equal to $\mathcal{X}_{\mathcal{A}}$. By [96, Cor. 8.4.13], if $\Bbbk$ is infinite, then the defining ideal of $\Gamma_{\mathcal{A}}$ is $I(\Gamma_{\mathcal{A}}) = I_{\mathcal{A}}$, and $\mathcal{X}_{\mathcal{A}} = \overline{\Gamma_{\mathcal{A}}}$ is the Zariski closure of $\Gamma_{\mathcal{A}}$. Hence, the vanishing ideal of $\mathcal{X}_{\mathcal{A}}$ is the toric ideal $I_{\mathcal{A}}$, and the *coordinate ring of* $\mathcal{X}_{\mathcal{A}}$ is $\Bbbk[\mathcal{X}_{\mathcal{A}}] = \Bbbk[x_1, \ldots, x_n]/I_{\mathcal{A}}$.

Under certain additional hypotheses, the toric set $\Gamma_{\mathcal{A}}$ is equal to the toric variety $\mathcal{X}_{\mathcal{A}}$. This is the case of the simplicial toric varieties when $\Bbbk$ is algebraically closed and one chooses an appropriate parametrization of $\mathcal{X}_{\mathcal{A}}$, or when $\mathcal{A} = \{a_1, \ldots, a_n\} \subset \mathbb{N}$ and $\gcd(a_1, \ldots, a_n) = 1$ (for any field).

Proposition 1.66 ([56, Cor. 2], [34, Prop. 2.1.4]). *Let $\Bbbk$ be an algebraically closed field. If $\mathcal{A} \subset \mathbb{N}^d$ is a finite set of nonzero vectors such that $\{\omega_1\boldsymbol{\epsilon}_1, \ldots, \omega_d\boldsymbol{\epsilon}_d\} \subset \mathcal{A}$, where $\{\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_d\}$ is the canonical basis of $\mathbb{N}^d$, then $\mathcal{X}_\mathcal{A} = \Gamma_\mathcal{A}$.*

Proposition 1.67 ([31, Lem. 3.4]). *Let $\Bbbk$ be any field. If $\mathcal{A} = \{a_1, \ldots, a_n\} \subset \mathbb{N}$ is such that $\gcd(a_1, \ldots, a_n) = 1$, then $\mathcal{X}_\mathcal{A} = \Gamma_\mathcal{A}$.*

Example 1.68. Consider $\mathcal{A} = \{2, 3\} \subset \mathbb{N}$ and $\Bbbk = \mathbb{F}_2$, the finite field with 2 elements. The toric set determined by $\mathcal{A}$ is $\Gamma_\mathcal{A} = \{(t^2, t^3) \mid t \in \mathbb{F}_2\} = \{(0, 0), (1, 1)\}$. The toric ideal determined by $\mathcal{A}$ is $I_\mathcal{A} = \langle x+t^2, y+t^3\rangle \cap \mathbb{F}_2[x, y] = \langle x^2+y^3\rangle$, and the affine toric variety determined by $\mathcal{A}$ is $\mathcal{X}_\mathcal{A} = V(I_\mathcal{A}) = \{(0, 0), (1, 1)\}$, which is equal to $\Gamma_\mathcal{A}$ by Proposition 1.67. However, the defining ideal of $\mathcal{X}_\mathcal{A}$ is $I(\mathcal{X}_\mathcal{A}) = \langle x+y\rangle \neq I_\mathcal{A}$. This happens because $\mathbb{F}_2$ is not an infinite field.

The next result characterizes when an affine toric variety $\mathcal{X}_\mathcal{A}$ is smooth in terms of the semigroup $\mathcal{S}_\mathcal{A} = \langle\mathcal{A}\rangle$.

Lemma 1.69 ([34, Thm. 1.1.11]). *Let $\Bbbk$ be an algebraically closed field and $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$ a finite set of nonzero vectors. Consider the affine toric variety $\mathcal{X}_\mathcal{A} = V(I_\mathcal{A}) \subset \mathbb{A}^n_\Bbbk$ determined by $\mathcal{A}$. The following statements are equivalent:*

(a) *$\mathcal{X}_\mathcal{A}$ is smooth.*

(b) *$\mathbf{0} = (0, \ldots, 0) \in \mathbb{A}^n_\Bbbk$ is a regular point of $\mathcal{X}_\mathcal{A}$.*

(c) *The affine semigroup $\mathcal{S}_\mathcal{A} = \langle\mathcal{A}\rangle$ admits a system of generators with $\dim(\mathbb{Q}\mathcal{A})$ elements.*

When the toric ideal $I_\mathcal{A}$ is homogeneous for the standard grading (i.e., when $\mathcal{A}$ is contained in a hyperplane of $\mathbb{Q}^d$ not passing through the origin, by Proposition 1.57), the affine toric variety $\mathcal{X}_\mathcal{A} = V(I_\mathcal{A})$ is a cone, i.e., it consists of lines passing through the origin $\mathbf{0} \in \mathbb{A}^n_\Bbbk$. Therefore, one can consider $\mathcal{X}_\mathcal{A}$ as a projective variety, $\mathcal{X}_\mathcal{A} \subset \mathbb{P}^{n-1}_\Bbbk$ of dimension $d-1$. If we are in this case, we will assume that $\mathcal{A} \subset \mathbb{N}^{d+1}$ and has $n+1$ elements.

Definition 1.70. Let $\mathcal{A} = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^{d+1}$ be a set of nonzero vectors contained in a hyperplane of $\mathbb{Q}^{d+1}$ not passing through the origin, and $I_\mathcal{A} \subset \Bbbk[x_0, \ldots, x_n]$ the toric ideal determined by $\mathcal{A}$, which is homogeneous.

(1) The *projective toric variety* determined by $\mathcal{A}$, also called *projective monomial variety*, is the zero set of the toric ideal $I_\mathcal{A}$, $\mathcal{X}_\mathcal{A} = V(I_\mathcal{A}) \subset \mathbb{P}^n_\Bbbk$. We say that the toric variety $\mathcal{X}_\mathcal{A}$ is *simplicial* if $I_\mathcal{A}$ is simplicial.

(2) We say that $\mathcal{X}_\mathcal{A}$ is a *projective toric curve* (resp. *surface*) or a *projective monomial curve* (resp. *surface*) if the Krull dimension of $\Bbbk[\mathcal{X}_\mathcal{A}]$ is 2 (resp. 3).

When the field $\Bbbk$ is infinite, the *(homogeneous) coordinate ring of* $\mathcal{X}_\mathcal{A}$ is $\Bbbk[\mathcal{X}_\mathcal{A}] = \Bbbk[x_0, \ldots, x_n]/I_\mathcal{A}$, by Remark 1.65.

Remark 1.71. By Propositions 1.60 and 1.57, when I is a homogeneous simplicial toric ideal, there exists a set $\mathcal{A} = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^{d+1}$ such that $\mathbf{a}_i = D\boldsymbol{\epsilon}_i$ for $i = 0, \ldots, d$, and $|\mathbf{a}_i| = \sum_{j=0}^{d} a_{ij} = D$ for all $i = 0, \ldots, n$, for some $D \in \mathbb{Z}_{>0}$, where $\{\boldsymbol{\epsilon}_0, \ldots, \boldsymbol{\epsilon}_d\}$ denotes the canonical basis of $\mathbb{N}^{d+1}$.

Suppose $\Bbbk$ is an algebraically closed field. Set $\mathcal{A} = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^{d+1}$ as above, and $\mathcal{X}_\mathcal{A} = V(I_\mathcal{A}) \subset \mathbb{P}^n_\Bbbk$ the projective toric variety determined by $\mathcal{A}$. Consider the affine charts of $\mathcal{X}_\mathcal{A}$, $\{\mathcal{X}_\mathcal{A} \cap \mathcal{U}_i\}_{i=0}^n$, where $\mathcal{U}_i = \mathbb{P}^n_\Bbbk \setminus V(x_i) \simeq \mathbb{A}^n_\Bbbk$ for all $i = 0, \ldots, n$; note that $\mathcal{X}_\mathcal{A} \cap \mathcal{U}_i$ is an open set of $\mathcal{X}_\mathcal{A}$ for all i. Since $\mathcal{X}_\mathcal{A}$ is simplicial, one has that

$$\mathcal{X}_\mathcal{A} = \cup_{i=0}^{d} (\mathcal{X}_\mathcal{A} \cap \mathcal{U}_i) \ . \tag{1.11}$$

Indeed, suppose that $P = (p_0 : \cdots : p_n) \in \mathcal{X}_\mathcal{A}$ and $p_i \notin \mathcal{U}_i$ for all $i = 0, \ldots, d$. Then, $p_0 = \cdots = p_d = 0$. For all $j = d+1, \ldots, n$, consider the binomial $f_j = x_j^D - \prod_{k=0}^{d} x_k^{a_{jk}} \in I_\mathcal{A}$. Since $f_j(P) = 0$, then $p_j = 0$ for all $j = d+1, \ldots, n$, which is impossible. This proves (1.11). For all $i = 0, \ldots, d$ and all $j = d+1, \ldots, n$, denote

$$a_j^{(i)} := (a_{j,1}, \ldots, a_{j,i-1}, a_{j,i+1}, \ldots, a_{j,d}) \in \mathbb{N}^d \,,$$

and $\mathcal{A}^{(i)} = \{D\boldsymbol{\epsilon}'_1, \ldots, D\boldsymbol{\epsilon}'_d, a_{d+1}^{(i)}, \ldots, a_n^{(i)}\} \subset \mathbb{N}^d$, where $\{\boldsymbol{\epsilon}'_1, \ldots, \boldsymbol{\epsilon}'_d\}$ is the canonical basis of $\mathbb{N}^d$. With these notations, one has that the affine chart $\mathcal{X}_\mathcal{A} \cap \mathcal{U}_i$ is homeomorphic to the simplicial affine toric variety $\mathcal{Y}_i := V(I_{\mathcal{A}^{(i)}})$, for all $i = 0, \ldots, d$.

Thus, the projective toric variety $\mathcal{X}_\mathcal{A}$ is smooth if and only if $\mathcal{Y}_i$ is smooth for all $i = 0, \ldots, d$. The following result by Herzog characterizes simplicial projective toric varieties which are smooth.

Theorem 1.72 ([48, Thm. 2.1])**.** *Fix an algebraically closed field $\Bbbk$, and let $\mathcal{X} \subset \mathbb{P}^n_\Bbbk$ be a simplicial projective toric variety of dimension d, and denote by $\{\boldsymbol{\epsilon}_0, \ldots, \boldsymbol{\epsilon}_d\}$ the canonical basis of $\mathbb{N}^{d+1}$.*

Then, $\mathcal{X}$ is smooth if and only if there exist a number $D \in \mathbb{Z}_{>0}$ and a set $\mathcal{A} = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^{d+1}$, such that $|\mathbf{a}_i| = D$ for all $i = 0, \ldots, n$,

$$\{\boldsymbol{\epsilon}_i + (D-1)\boldsymbol{\epsilon}_j \mid 0 \le i, j \le d\} \subset \mathcal{A}\,,$$

and $\mathcal{X} = \mathcal{X}_\mathcal{A}$.

The degree of simplicial projective toric varieties can be computed using the following result.

Theorem 1.73 ([74, Thm. 2.13, Thm. 4.5]). *Let $\mathcal{X}_{\mathcal{A}}$ be a simplicial projective toric variety, where $\mathcal{A} = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^{d+1}$, $|\mathbf{a}_i| = D \in \mathbb{Z}_{>0}$ for all $i = 0, \ldots, n$ and $\mathbf{a}_i = D\boldsymbol{\epsilon}_i$ for $i = 0, \ldots, d$, where $\{\boldsymbol{\epsilon}_0, \ldots, \boldsymbol{\epsilon}_d\}$ is the canonical basis of $\mathbb{N}^{d+1}$. The degree of the projective variety $\mathcal{X}_{\mathcal{A}}$ can be computed as*

$$\deg(\mathcal{X}_{\mathcal{A}}) = \frac{(d+1)! \cdot \mathrm{vol}\,(\mathrm{conv}\,(\mathcal{A} \cup \{\mathbf{0}\}))}{\theta_{d+1}} = \frac{D^{d+1}}{\theta_{d+1}},$$

where

- $\mathrm{vol}\,(\mathrm{conv}\,(\mathcal{A} \cup \{\mathbf{0}\}))$ *denotes the volume of the convex hull of $\mathcal{A} \cup \{\mathbf{0}\} \subset \mathbb{R}^{d+1}$, and*
- θ_{d+1} *is the greatest common divisor of the $(d+1) \times (d+1)$ minors of the $(d+1) \times (n+1)$ matrix $M_{\mathcal{A}}$, whose columns are the vectors $\mathbf{a}_0, \ldots, \mathbf{a}_n$.*

Example 1.74. Let $D \in \mathbb{Z}_{>0}$ be a positive integer and $\Bbbk = \mathbb{C}$. Consider $\mathcal{A}$ the set whose elements are the column vectors of the matrix $M_{\mathcal{A}}$:

$$M_{\mathcal{A}} = \begin{pmatrix} D & 0 & 0 & D-1 & D-1 & 1 & 0 & 1 & 0 \\ 0 & D & 0 & 1 & 0 & D-1 & D-1 & 0 & 1 \\ 0 & 0 & D & 0 & 1 & 0 & 1 & D-1 & D-1 \end{pmatrix}.$$

By Theorem 1.72, the projective toric surface determined by $\mathcal{A}$, $\mathcal{X}_{\mathcal{A}} \subset \mathbb{P}^8_{\mathbb{C}}$ is smooth. Let Δ_3 be the g.c.d. of the 3×3 minors of $M_{\mathcal{A}}$. Since the sum of the rows of $\mathcal{A}$ is D, then D divides Δ_3. Moreover, since $\begin{vmatrix} 0 & 0 & 1 \\ D & D-1 & D-1 \\ 0 & 1 & 0 \end{vmatrix} = D$, then $\Delta_3 = D$. On the other hand, $\mathrm{conv}\,(\mathcal{A} \cup \{\mathbf{0}\})$ is the simplex with vertices $(0,0,0), (D,0,0), (0,D,0), (0,0,D)$, and hence it has volume $\frac{D^3}{3!}$. Therefore, by Theorem 1.73, the degree of the surface $\mathcal{X}_{\mathcal{A}}$ is $\deg(\mathcal{X}_{\mathcal{A}}) = D^2$.

To finish this section, we recall some properties of projective monomial curves, and characterize simplicial projective monomial surfaces with exactly one singular point. These surfaces will appear in Chapters 3 and 4.

Projective monomial curves

Consider an integer $D > 0$ and a sequence $a_0 = 0 < a_1 < \cdots < a_n = D$ of relatively prime integers, i.e., $\gcd(a_1, \ldots, a_n) = 1$. Set $\mathcal{A} = \{a_0, a_1, \ldots, a_n\}$ and $\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \ldots, \underline{\mathbf{a}}_n\} \subset \mathbb{N}^2$, where $\underline{\mathbf{a}}_i = (D - a_i, a_i)$ for all $i = 0, \ldots, n$. Let $\mathcal{C}$ be the projective monomial curve determined by $\underline{\mathcal{A}}$, and denote $\Bbbk[\mathcal{C}] = \Bbbk[x_0, \ldots, x_n]/I_{\underline{\mathcal{A}}}$. If $\Bbbk$ is infinite, $\Bbbk[\mathcal{C}]$ is the homogeneous coordinate ring of $\mathcal{C}$.

Proposition 1.75 (Folklore, see, e.g., [32]). *Let $\mathcal{C}$ be a projective monomial curve as above.*

(1) $\mathcal{C} \subset \mathbb{P}^n_{\Bbbk}$ *is a projective curve of degree* D*, i.e., the multiplicity of the two-dimensional ring* $\Bbbk[x_0, \ldots, x_n]/I_{\mathcal{A}}$ *is* D*.*

(2) $\mathcal{C}$ *has at most two singular points, which are* $P_1 = (1 : 0 \cdots : 0) \in \mathbb{P}^n_{\Bbbk}$ *and* $P_2 = (0 : \cdots : 0 : 1) \in \mathbb{P}^n_{\Bbbk}$*.* P_1 *is non-singular if and only if* $a_1 = 1$*, and* P_2 *is non-singular if and only if* $D - a_{n-1} = 1$*.*

(3) If $\delta(\mathcal{C}, P_i)$ *denotes the singularity order of* P_i*,* $i = 1, 2$*, then* $\delta(\mathcal{C}, P_1) = |\mathbb{N} \setminus \mathcal{S}_1|$ *and* $\delta(\mathcal{C}, P_2) = |\mathbb{N} \setminus \mathcal{S}_2|$*, where* $\mathcal{S}_1 = \langle a_1, \ldots, a_n \rangle$ *and* $\mathcal{S}_2 = \langle D - a_0, \ldots, D - a_{n-1} \rangle$*.*

(4) The arithmetic genus of $\mathcal{C}$ *is* $p_a(\mathcal{C}) = 1 - \delta(\mathcal{C}, P_1) - \delta(\mathcal{C}, P_2)$*. Therefore, the Hilbert polynomial of* $\mathcal{C}$ *is* $\mathrm{HP}_{\Bbbk[\mathcal{C}]}(t) = Dt + 1 - \delta(\mathcal{C}, P_1) - \delta(\mathcal{C}, P_2)$*, and the regularity of the Hilbert function of* $\Bbbk[\mathcal{C}]$ *is*

$$\mathrm{r}(\Bbbk[\mathcal{C}]) = \min\{s \in \mathbb{N} : \mathrm{HF}_{\Bbbk[\mathcal{C}]}(s' + 1) - \mathrm{HF}_{\Bbbk[\mathcal{C}]}(s') = D, \forall s' \geq s\}\,.$$

The Castelnuovo-Mumford regularity of $\Bbbk[\mathcal{C}]$ satisfies the Eisenbud-Goto conjecture; see Theorem 1.76 (1). This was proved by Gruson, Lazarsfeld and Peskine in [46]. Indeed, their result is more general, and they proved it before Eisenbud-Goto conjecture was stated. Later, L'Vovsky provided another bound on the Castelnuovo-Mumford regularity of $\Bbbk[\mathcal{C}]$; see Theorem 1.76 (2).

Theorem 1.76 ([46], [22, Prop. 3.1], [67, Prop. 5.5])**.** *Let* $\mathcal{A} = \{a_0 = 0 < a_1 < \cdots < a_n = D\} \subset \mathbb{N}$ *be such that* $\gcd(a_1, \ldots, a_n) = 1$*, and set* $\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \ldots, \underline{\mathbf{a}}_n\} \subset \mathbb{N}^2$*, where* $\underline{\mathbf{a}}_i = (D - a_i, a_i)$ *for all* i*. Consider* $\mathcal{C} = \mathcal{C}_{\underline{\mathcal{A}}}$ *the projective monomial curve determined by* $\underline{\mathcal{A}}$*, and* $\Bbbk[\mathcal{C}] = \Bbbk[x_0, \ldots, x_n]/I_{\underline{\mathcal{A}}}$*. We have the following bounds for the Castelnuovo-Mumford regularity of* $\Bbbk[\mathcal{C}]$*.*

(1) $\mathrm{reg}(\Bbbk[\mathcal{C}]) \leq D - n + 1$*. Moreover,* $\mathrm{reg}(\Bbbk[\mathcal{C}]) = D - n + 1$ *if and only if* $\mathcal{A}$ *or* $D - \mathcal{A}$ *belongs to one of the following two families:*

- $\mathcal{A} = [0, D] \setminus \{a\}$*, for some* a*, such that* $1 \leq a \leq D - 1$*;*
- $\mathcal{A} = [0, 1] \sqcup [a + 1, D]$*, for some* a*, such that* $2 \leq a \leq D - 2$*.*

(2) $\mathrm{reg}(\Bbbk[\mathcal{C}]) \leq \max_{1 \leq 1 < j \leq n}\{(a_i - a_{i-1}) + (a_j - a_{j-1})\} - 1$*.*

Example 1.77 (Macaulay's curve)**.** Set $\Bbbk = \mathbb{C}$ and let $\mathcal{C}$ be the projective monomial curve determined by $\underline{\mathcal{A}} = \{(4, 0), (3, 1), (1, 3), (0, 4)\}$. By Proposition 1.75, $\mathcal{C}$ is smooth and its Hilbert polynomial is $\mathrm{HP}_{\Bbbk[\mathcal{C}]}(t) = 4t + 1$. The Castelnuovo-Mumford regularity of $\Bbbk[\mathcal{C}]$ is $\mathrm{reg}(\Bbbk[\mathcal{C}]) = 2$, by Theorem 1.76 (1).

In the next proposition, we characterize the simplicial projective surfaces with exactly one singular point.

Projective monomial surfaces with a single singular point

Proposition 1.78. *Let $\Bbbk$ be an algebraically closed field and $\mathcal{A} = \{\mathbf{a}_0, \dots, \mathbf{a}_n\} \subset \mathbb{N}^3$ a set of nonzero vectors, $\mathbf{a}_i = (a_{i0}, a_{i1}, a_{i2})$ for all i, and set $\{\boldsymbol{\epsilon}_0, \boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2\}$ the canonical basis of $\mathbb{N}^3$. Suppose that $\mathbf{a}_0 = D\boldsymbol{\epsilon}_0$, $\mathbf{a}_1 = D\boldsymbol{\epsilon}_1$, $\mathbf{a}_2 = D\boldsymbol{\epsilon}_2$, and $|\mathbf{a}_i| = D$ for all $i = 0, \dots, n$, for some $D \in \mathbb{Z}_{>0}$, and denote by $\mathcal{X} = V(I_{\mathcal{A}}) \subset \mathbb{P}^n_{\Bbbk}$ the simplicial projective toric variety determined by $\mathcal{A}$.*

(1) If $\mathcal{X}$ has exactly one single singular point, that point is $P_0 = (1 : 0 : \cdots : 0)$, $P_1 = (0 : 1 : 0 : \cdots : 0)$, or $P_2 = (0 : 0 : 1 : 0 \cdots : 0)$.

(2) If the only singular point of $\mathcal{X}$ is P_0, then $n \geq 4$ and

$$\{(0, D-1, 1), (0, 1, D-1), (e, D-e, 0), (e, 0, D-e)\} \subset \mathcal{A},$$

where $e \in \mathbb{Z}_{>0}$ is a divisor of D that divides a_{i0} for all $i \in \{0, \dots, n\}$, and if $e = 1$ then either $(D-1, 1, 0) \notin \mathcal{A}$ or $(D-1, 0, 1) \notin \mathcal{A}$.
Conversely, if $\mathcal{X} = \mathcal{X}_{\mathcal{A}}$ with $\mathcal{A} \subset \mathbb{N}^3$ as before, then $\mathcal{X}$ has a single singular point, P_0.

Remark 1.79. In part (2) of the previous proposition we distinguish two different behaviors.

(i) <u>$e < D$</u>: In this case, $(e, D-e, 0) \neq (e, 0, D-e)$, and $n \geq 6$.

(ii) <u>$e = D$</u>: In this case, $(e, D-e, 0) = (e, 0, D-e) = (D, 0, 0)$ and for all $\mathbf{a}_i \in \mathcal{A}$, such that $\mathbf{a}_i \neq (D, 0, 0)$, one has that $a_{i0} = 0$. Hence, $\mathcal{A} = \{D\boldsymbol{\epsilon}_0\} \cup (\{0\} \times \mathcal{A}')$, and $\{(D, 0), (D-1, 1), (1, D-1), (0, D)\} \subset \mathcal{A}'$. Observe that $I_{\mathcal{A}'} \subset \Bbbk[x_1, \dots, x_n]$ is the defining ideal of a smooth projective monomial curve, and $I_{\mathcal{A}} = I_{\mathcal{A}'}.\Bbbk[x_0, \dots, x_n]$ is the extension of $I_{\mathcal{A}'}$. Therefore, the resolutions of $\Bbbk[x_0, \dots, x_n]/I_{\mathcal{A}}$ and $\Bbbk[x_1, \dots, x_n]/I_{\mathcal{A}'}$ are identical. This observation will be useful in Chapter 4.

Proof of Prop. 1.78. For $i \in \{0, 1, 2\}$, let $\mathcal{Y}_i$ be the i-th affine chart of $\mathcal{X}$, i.e. $\mathcal{Y}_i = V(I_{\mathcal{A}^{(i)}})$, where $\mathcal{A}^{(i)} \subset \mathbb{N}^2$ is the set defined in the paragraph before Theorem 1.72. If there are two affine charts that are not smooth, then $\mathcal{X}$ has at least two singular points by Lemma 1.69 (b). Thus, there is only one singular affine chart. Again, by Lemma 1.69 (b), the singular affine chart is $\mathcal{Y}_k$ if and only if the only singular point is P_k, for $k = 0, 1, 2$. This proves (1).

Assume now that $\mathcal{X}$ has a single singular point, and it is P_0. Moreover, suppose that $\gcd(\{a_{ij} \mid 0 \leq i \leq n, 0 \leq j \leq 2\}) = 1$. For all $0 \leq i, j \leq 2$, $i \neq j$, let $\lambda_{ij} := \min\{k \in \mathbb{Z}_{>0} \mid (D-k)\boldsymbol{\epsilon}_i + k\boldsymbol{\epsilon}_j \in \mathcal{A}\}$. We have to show that $\lambda_{12} = \lambda_{21} = 1$, $\lambda_{10} = \lambda_{20}$, and λ_{10} divides a_{i0} for all $0 \leq i \leq n$.

Since $\mathcal{Y}_1$ is smooth, by Lemma 1.69 (c) one has that $\{(\lambda_{10}, 0), (0, \lambda_{12})\}$ is the minimal generating set of $\langle \mathcal{A}^{(1)} \rangle$. Hence, λ_{12} divides a_{i2} for all $0 \le i \le n$. In particular, $\lambda_{12} \mid D$ and $\lambda_{12} \mid D - \lambda_{21}$, so $\lambda_{12} \mid \lambda_{21}$. A similar argument with $\mathcal{Y}_2$ shows that $\lambda_{21} \mid a_{i1}$ for all $0 \le i \le n$. Hence, $\lambda_{12} \mid a_{ij}$ for all $0 \le i \le n$, $1 \le j \le 2$. Since $\lambda_{12} \mid D$ and $a_{i0} = D - a_{i1} - a_{i2}$ for all i, then $\lambda_{12} \mid a_{ij}$ for all $0 \le i \le n$, $0 \le j \le 2$. Therefore, $\lambda_{12} = 1$.

Working with $\mathcal{Y}_2$, one gets that $\lambda_{21} \mid \lambda_{12}$, and hence $\lambda_{21} = 1$. Analogously, one has that $\lambda_{10} \mid a_{i0}$ and $\lambda_{20} \mid a_{i0}$ for all $0 \le i \le n$. In particular, $\lambda_{10} \mid \lambda_{20}$ and $\lambda_{20} \mid \lambda_{10}$, so $\lambda_{10} = \lambda_{20}$. Since $\lambda_{10} \mid D$, then there exists $e \in \mathbb{Z}_{>0}$ a divisor of D such that $\lambda_{10} = \lambda_{20} = e$ and $e \mid a_{i0}$ for all $i = 0, \dots, n$. Hence, we have proved $\{(0, D-1, 1), (0, 1, D-1), (e, D-e, 0), (e, 0, D-e)\} \subset \mathcal{A}$ and e divides a_{i0} for all i. Finally, if $e = 1$, note that if $(D-1, 1, 0) \in \mathcal{A}$ and $(D-1, 0, 1) \in \mathcal{A}$, then $\mathcal{X}$ is smooth by Theorem 1.72.

Conversely, assume that $\{(0, D-1, 1), (0, 1, D-1), (e, D-e, 0), (e, 0, D-e)\} \subset \mathcal{A}$. By Lemma 1.69 (c), the affine charts $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are smooth, since $\langle \mathcal{A}^{(1)} \rangle = \langle \mathcal{A}^{(2)} \rangle = \langle (e, 0), (0, 1) \rangle$. Moreover, $\mathcal{Y}_0$ is the affine toric surface determined by $\mathcal{A}^{(0)}$, and $\{(D, 0), (0, D), (D-1, 1), (1, D-1), (D-e, 0), (0, D-e)\} \subset \mathcal{A}^{(0)}$. A direct computation shows that $(0, \dots, 0) \in \mathcal{Y}_0$ is the only singular point of $\mathcal{Y}_0$. Hence, $\mathcal{X}$ has a single singular point, which is $P_0 = (1 : 0 : \cdots : 0)$.

□

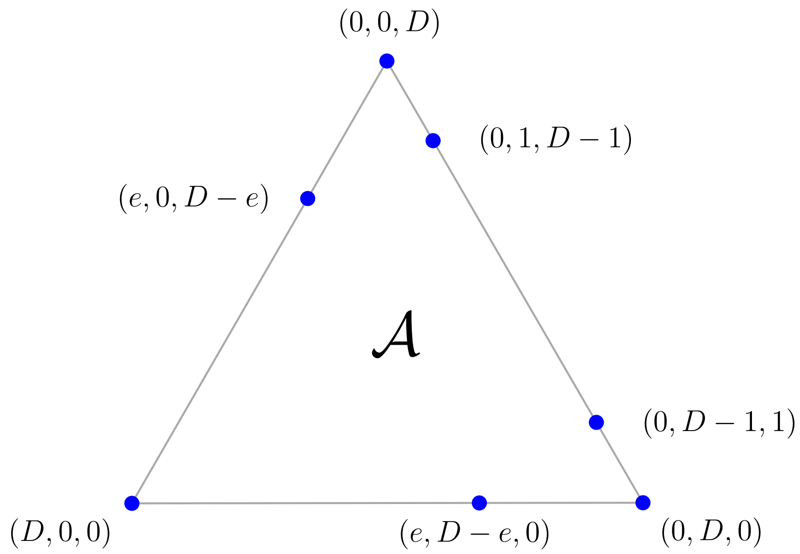


Figure 1.2: Shape of a set $\mathcal{A}$ in Proposition 1.78 if $e \neq D$.

1.4 Sumsets and commutative algebra

This last section contains some basic results on additive combinatorics and its connection with commutative algebra. For more results on additive combinatorics, see [71] and [91].

Let $\mathcal{S}$ be a semigroup (abelian with identity) and $\mathcal{A}, \mathcal{B} \subset \mathcal{S}$ finite nonempty subsets, the *sumset* $\mathcal{A} + \mathcal{B}$ is defined as

$$\mathcal{A} + \mathcal{B} = \{a + b \mid a \in \mathcal{A}, b \in \mathcal{B}\}.$$

Similarly, if $\mathcal{A}_1, \ldots, \mathcal{A}_s$ are s finite nonempty subsets of $\mathcal{S}$, $s \geq 1$, one can define

$$\mathcal{A}_1 + \ldots + \mathcal{A}_s = \{a_1 + \cdots + a_s \mid a_i \in \mathcal{A}_i, 1 \leq i \leq s\}.$$

The most interesting case is $\mathcal{A}_i = \mathcal{A}$ for all i. In this case, we denote the set $\mathcal{A}_1 + \cdots + \mathcal{A}_s$ by $s\mathcal{A}$,

$$\begin{aligned} &s\mathcal{A} = \{a_1 + \ldots + a_s \mid a_i \in \mathcal{A}, 1 \leq i \leq s\},\ s \in \mathbb{Z}_{>0};\ \text{and} \\ &0\mathcal{A} = \{0\}. \end{aligned}$$

Definition 1.80. Let $\mathcal{S}$ be a semigroup and $\mathcal{A} \subset \mathcal{S}$ a finite nonempty subset. For all $s \in \mathbb{N}$, the set $s\mathcal{A}$ is called the *s-fold iterated sumset* of $\mathcal{A}$.

If $\mathcal{S}_\mathcal{A}$ denotes the subsemigroup of $\mathcal{S}$ generated by $\mathcal{A}$, one has that $\mathcal{S}_\mathcal{A} = \cup_{s=0}^{\infty} s\mathcal{A}$. When the set $\mathcal{A}$ contains the identity $0 \in \mathcal{S}$, the sumsets of $\mathcal{A}$ form a nested sequence, i.e., $s\mathcal{A} \subset (s+1)\mathcal{A}$ for all $s \in \mathbb{N}$. Hence, the sequence $(s\mathcal{A})_{s=0}^{\infty}$ is increasing and converges to $\mathcal{S}_\mathcal{A}$.

Additive combinatorics studies the sumsets of $\mathcal{A}$ and their cardinality. One central problem in additive combinatorics is the study of the function $\mathbb{N} \to \mathbb{N}$ defined by $s \mapsto |s\mathcal{A}|$. Khovanskii proved in 1992 that this function is asymptotically polynomial.

Theorem 1.81 ([57, Thm. 1])**.** *Let $\mathcal{S}$ be a semigroup and $\mathcal{A} \subset \mathcal{S}$ be a nonempty finite subset. Then, there exists a polynomial $p_\mathcal{A}(t) \in \mathbb{Q}[t]$ of degree at most $|\mathcal{A}|$ such that $|s\mathcal{A}| = p_\mathcal{A}(s)$ for all $s \in \mathbb{N}$ sufficiently large enough.*

Khovanskii's proof relates the function $s \mapsto |s\mathcal{A}|$ to the Hilbert function of a certain graded module M over the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$, where $|\mathcal{A}| = n+1$, and so agrees with the Hilbert polynomial of M once $s \geq \mathrm{r}(M)$, the regularity of the Hilbert function of M. Later, Nathanson and Rusza gave a combinatorial proof of the same result in [72].

Khovanskii's theorem has recently attracted the attention of some researchers. In 2022, Eliahou and Mazumdar gave a new proof of this result in [30], which we now present. In their proof, they associate with $\mathcal{A}$ a standard graded $\Bbbk$-algebra $R(\mathcal{A})$, whose Hilbert function is $s \mapsto |s\mathcal{A}|$. A geometric counterpart when $\mathcal{S} = \mathbb{N}^d$ can be found in the paper [18] by Colarte-Gómez, Elias and Miró-Roig. The special case $d = 1$ is treated in the paper [32] by Elias.

Construction by Eliahou and Mazumdar

Let $\mathcal{S}$ be a semigroup[1] and $\mathcal{A} = \{a_0, a_1, \ldots, a_n\} \subset \mathcal{S}$ a finite set. Fix a field $\Bbbk$ and consider the semigroup algebra $\Bbbk[\mathcal{S}]$, which is spanned by $\{t^s \mid s \in \mathcal{S}\}$ as a $\Bbbk$-vector space. Let $T = \Bbbk[\mathcal{S}][w]$ be the polynomial ring in the variable w with coefficients in $\Bbbk[\mathcal{S}]$, graded via $\deg(t) = 0$, $\deg(w) = 1$. A basis of T as a $\Bbbk$-vector space is $\mathcal{B} = \{t^s w^n \mid s \in \mathcal{S}, n \in \mathbb{N}\}$. The grading defined on T gives it a structure of graded $\Bbbk$-algebra, $T = \oplus_{i \in \mathbb{N}} T_i$, where T_i is the k-vector space spanned by $\{t^s w^i \mid s \in \mathcal{S}\}$.

Consider the $\Bbbk$-subalgebra $R(\mathcal{A})$ of T generated by $\{t^{a_0} w, \ldots, t^{a_n} w\}$,

$$R(\mathcal{A}) = \Bbbk[t^{a_0} w, \ldots, t^{a_n} w] \subset T\,.$$

We have that $R(\mathcal{A}) = \oplus_{i \in \mathbb{N}} R(\mathcal{A})_i$, where $R(\mathcal{A})_i$ is the $\Bbbk$-vector space with basis $\{t^b w^i \mid b \in i\mathcal{A}\}$. Hence, $R(\mathcal{A})$ is a standard graded $\Bbbk$-algebra and $\dim_{\Bbbk}(R(\mathcal{A})_i) = |i\mathcal{A}|$ for all $i \in \mathbb{N}$. Theorem 1.81 follows then from Theorem 1.47.

Proposition 1.82 ([30, Sect. 6])**.** *Let $\mathcal{S}$ be a semigroup and $\mathcal{A} = \{a_0, \ldots, a_n\} \subset \mathcal{S}$ a finite set. Then, $R(\mathcal{A})$ is isomorphic to $\Bbbk[x_0, \ldots, x_n]/\ker\varphi$ as graded $\Bbbk$-algebras, where $\varphi : \Bbbk[x_0, \ldots, x_n] \to R(\mathcal{A})$ is the morphism of $\Bbbk$-algebras defined by $\varphi(x_i) = t^{a_i} w$, $i = 0, \ldots, n$. Moreover,*

$$\ker\varphi = \langle \mathbf{x}^\alpha - \mathbf{x}^\beta \mid \sum_{i=0}^{n} \alpha_i = \sum_{i=0}^{n} \beta_i, \text{ and } \sum_{i=0}^{n} \alpha_i a_i = \sum_{i=0}^{n} \beta_i a_i \rangle\,.$$

When $\mathcal{S} = \mathbb{N}^d$, for some positive integer d, one can interpret $R(\mathcal{A})$ geometrically in terms of the toric varieties introduced in Section 1.3. The case $d = 1$ is treated in [32], and the case $d \geq 2$, in [18].

[1] In their article, the authors only consider the case when $\mathcal{S}$ is a group. However, their results can be generalized to any semigroup $\mathcal{S}$.

Sumsets of $\mathbb{N}$ and projective monomial curves

Let $\mathcal{A} = \{a_0, \ldots, a_n\} \subset \mathbb{N}$ be a finite set, and suppose $a_0 < a_1 < \cdots < a_n$. To study the sumsets of $\mathcal{A}$, we can always reduce to the case $a_0 = 0$ and $\gcd(a_1, \ldots, a_n) = 1$. Let us show how.

Consider $\delta(\mathcal{A}) = \gcd(a_1 - a_0, a_2 - a_0, \ldots, a_n - a_0)$. For all $i = 0, \ldots, n$, denote $a'_i = (a_i - a_0)/\delta(\mathcal{A})$, and define

$$\mathcal{A}^{(N)} = \{a'_0, a'_1, \ldots, a'_n\}.$$

Then, one has that $0 = a'_0 < a'_1 < \cdots < a'_n$, $\delta\left(A^{(N)}\right) = \gcd(a'_1, \ldots, a'_n) = 1$, and $\mathcal{A} = a_0 + \delta(\mathcal{A}) \cdot \mathcal{A}^{(N)}$. Therefore,

$$s\mathcal{A} = \{sa_0\} \cup \delta(\mathcal{A}) \cdot sA^{(N)}$$

and, in particular, $|s\mathcal{A}| = |s\mathcal{A}^{(N)}|$ for all $s \in \mathbb{N}$. The set $\mathcal{A}^{(N)}$ is called the *normal form of* $\mathcal{A}$. When $\mathcal{A} = \mathcal{A}^{(N)}$, we will say that $\mathcal{A}$ is *in normal form*.

Given a set $\mathcal{A} = \{a_0, a_1, \ldots, a_n\} \subset \mathbb{N}$ in normal form, denote $D = a_n$ and consider the set $\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \underline{\mathbf{a}}_1, \ldots, \underline{\mathbf{a}}_n\} \subset \mathbb{N}^2$, where $\underline{\mathbf{a}}_i = (D - a_i, a_i) \in \mathbb{N}^2$, for all $i = 0, \ldots, n$. Fix an infinite field $\Bbbk$ and denote by $\mathcal{C} = \mathcal{C}_{\underline{\mathcal{A}}}$ the projective monomial curve determined by $\underline{\mathcal{A}}$. The homogeneous coordinate ring of $\mathcal{C}$ is $\Bbbk[\mathcal{C}] = \Bbbk[x_0, \ldots, x_n]/I_{\underline{\mathcal{A}}}$, where $I_{\underline{\mathcal{A}}}$ is the toric ring determined by $\underline{\mathcal{A}}$.

Proposition 1.83 ([32, Prop. 2.6])**.** *The Hilbert function of* $\Bbbk[\mathcal{C}]$ *satisfies* $|s\mathcal{A}| = \mathrm{HF}_{\Bbbk[\mathcal{C}]}(s)$ *for all* $s \in \mathbb{N}$.

Example 1.84. Consider the set $\mathcal{A} = \{0, 2, 4, 6, 9\} \subset \mathbb{N}$. Fix $\Bbbk = \mathbb{Q}$ and let $\mathcal{C}$ be the projective monomial curve determined by $\underline{\mathcal{A}} = \{(9,0), (7,2), (5,4), (3,6), (0,9)\}$. By Proposition 1.83, $|s\mathcal{A}| = \mathrm{HF}_{\Bbbk[\mathcal{C}]}(s)$ for all $s \in \mathbb{N}$. If one computes the Hilbert function (and polynomial) of $\Bbbk[\mathcal{C}]$, one gets $\mathrm{HF}_{\Bbbk[\mathcal{C}]}(0) = 1$, $\mathrm{HF}_{\Bbbk[\mathcal{C}]}(1) = 5$, and $\mathrm{HF}_{\Bbbk[\mathcal{C}]}(s) = 9s - 6$ for all $s \geq 2$. Hence, $|s\mathcal{A}| = 9s - 6$ for all $s \geq 2$.

Sumsets of $\mathbb{N}^d$ and projective monomial varieties

Let $\mathcal{A} = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$ be a set of nonzero vectors, $d \geq 2$, where $\mathbf{a}_i = (a_{i1}, \ldots, a_{id})$ for all $i = 0, \ldots, n$. Consider $D = \max\{|\mathbf{a}_i| \colon i = 0, \ldots, n\}$, and define $\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \ldots, \underline{\mathbf{a}}_n\} \subset \mathbb{N}^{d+1}$, where $\underline{\mathbf{a}}_i = (D - |\mathbf{a}_i|, a_{i1}, \ldots, a_{id}) \in \mathbb{N}^{d+1}$ for all i. Fix an infinite field $\Bbbk$, and let $\mathcal{X} = \mathcal{X}_{\underline{\mathcal{A}}}$ be the projective toric variety determined by $\underline{\mathcal{A}}$. The homogeneous coordinate ring of $\mathcal{X}$ is $\Bbbk[\mathcal{X}] = \Bbbk[x_0, \ldots, x_n]/I_{\underline{\mathcal{A}}}$, where $I_{\underline{\mathcal{A}}}$ is the toric ideal determined by $\underline{\mathcal{A}}$.

For all $s \in \mathbb{N}$, denote $H_s \coloneqq \{(z_0, \ldots, z_d) \in \mathbb{N}^{d+1} \mid z_0 + \cdots + z_d = sD\}$. Note that since $\underline{\mathcal{A}} \subset H_1$, then $s\underline{\mathcal{A}} \subset H_s$ for all $s \in \mathbb{N}$. Moreover, if $\mathcal{S} = \langle \underline{\mathcal{A}} \rangle$ is the semigroup generated by $\underline{\mathcal{A}}$, then $s\underline{\mathcal{A}} = \mathcal{S} \cap H_s$ for all $s \in \mathbb{N}$. Also, it is important to observe that

$$s\underline{\mathcal{A}} = \{(sD - |\mathbf{b}|, b_1, \ldots, b_d) \mid \mathbf{b} = (b_1, \ldots, b_d) \in s\mathcal{A}\} \tag{1.12}$$

for all $s \in \mathbb{N}$. In particular, one has that $|s\mathcal{A}| = |s\underline{\mathcal{A}}|$ for all $s \in \mathbb{N}$.

Proposition 1.85 ([18, Prop. 3.3])**.** *The Hilbert function of* $\Bbbk[\mathcal{X}]$ *satisfies* $|s\mathcal{A}| = \mathrm{HF}_{\Bbbk[\mathcal{X}]}(s)$ *for all* $s \in \mathbb{N}$*.*

In Chapter 3, we will study more precisely the structure of the sumsets of sets $\mathcal{A} \subset \mathbb{N}^d$ when $d = 1$, and when $d \geq 2$ and $\mathcal{A}$ has a special structure.

Chapter 2

The Betti numbers of projective and affine monomial curves

"Everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions."
F. Klein

Fix an infinite field $\Bbbk$. Consider an integer $D > 0$ and a sequence $a_0 = 0 < a_1 < \cdots < a_n = D$ of relatively prime integers, i.e., $\gcd(a_1, \ldots, a_n) = 1$. Set $\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \ldots, \underline{\mathbf{a}}_n\} \subset \mathbb{N}^2$, where $\underline{\mathbf{a}}_i = (D - a_i, a_i)$ for all $i = 0, \ldots, n$. Denote by $\mathcal{C}$ the *projective monomial curve* $\mathcal{C} \subset \mathbb{P}^n_{\Bbbk}$ determined by $\underline{\mathcal{A}}$, $\mathcal{C} = V(I_{\underline{\mathcal{A}}})$, where $I_{\underline{\mathcal{A}}}$ is the toric ideal determined by $\underline{\mathcal{A}}$. Since $\Bbbk$ is inifinite, by [96, Cor. 8.4.13], $\mathcal{C}$ is the Zariski closure of

$$\{(t_0^{D-a_0} t_1^{a_0} : \cdots : t_0^{D-a_i} t_1^{a_i} : \cdots : t_0^{D-a_n} t_1^{a_n}) \in \mathbb{P}^n_{\Bbbk} \mid (t_0 : t_1) \in \mathbb{P}^1_{\Bbbk}\},$$

and the defining ideal of $\mathcal{C}$ is $I_{\underline{\mathcal{A}}}$. Hence, the coordinate ring of $\mathcal{C}$ is the two-dimensional ring $\Bbbk[\mathcal{C}] = \Bbbk[x_0, \ldots, x_n]/I_{\underline{\mathcal{A}}}$. Note that $\Bbbk[\mathcal{C}]$ is isomorphic to $\Bbbk[\mathcal{S}]$ as $\mathcal{S}$-graded $\Bbbk[x_0, \ldots, x_n]$-modules, $\Bbbk[\mathcal{C}] \cong \Bbbk[\mathcal{S}]$, where $\mathcal{S} = \mathcal{S}_{\underline{\mathcal{A}}}$ denotes the affine semigroup generated by $\underline{\mathcal{A}}$.

The projective curve $\mathcal{C}$ has two affine charts, the affine monomial curves $\mathcal{C}_1 = \{(t_1^{a_1}, \ldots, t_1^{a_n}) \in \mathbb{A}^n_{\Bbbk} \mid t_1 \in \Bbbk\}$ and $\mathcal{C}_2 = \{(t_0^{D-a_0}, t_0^{D-a_1}, \ldots, t_0^{D-a_{n-1}}) \in \mathbb{A}^n_{\Bbbk} \mid t_0 \in \Bbbk\}$, associated with the sequences $a_1 < \cdots < a_n$ and $D - a_{n-1} < \cdots < D - a_1 < D - a_0$, respectively. The second sequence is sometimes called the *dual* of the first one. Set $\mathcal{S}_1 \coloneqq \mathcal{S}_{\mathcal{A}_1}$ the numerical semigroup generated by $\mathcal{A}_1 = \{a_1, \ldots, a_n\}$. The vanishing ideal of $\mathcal{C}_1$ is $I_{\mathcal{A}_1} \subset \Bbbk[x_1, \ldots, x_n]$, and hence, its coordinate ring is the one-dimensional ring $\Bbbk[\mathcal{C}_1] = \Bbbk[x_1, \ldots, x_n]/I_{\mathcal{A}_1} \cong \Bbbk[\mathcal{S}_1]$. Moreover, $I_{\underline{\mathcal{A}}}$ is the homogenization of $I_{\mathcal{A}_1}$ with respect to the variable x_0. Similarly, denoting by $\mathcal{S}_2 \coloneqq \mathcal{S}_{\mathcal{A}_2}$ the numerical semigroup generated by $\mathcal{A}_2 \coloneqq \{D - a_0, D - a_1, \ldots, D - a_{n-1}\}$, the vanishing ideal of $\mathcal{C}_2$ is $I_{\mathcal{A}_2} \subset \Bbbk[x_0, \ldots, x_{n-1}]$, its coordinate ring is $\Bbbk[\mathcal{C}_2] = \Bbbk[x_0, \ldots, x_{n-1}]/I_{\mathcal{A}_2} \cong \Bbbk[\mathcal{S}_2]$, and $I_{\underline{\mathcal{A}}}$ is the homogenization of $I_{\mathcal{A}_2}$ with respect to x_n.

One has that $\beta_i(\Bbbk[\mathcal{C}]) \geq \beta_i(\Bbbk[\mathcal{C}_1])$ for all i, and the main goal of this chapter is to understand when the Betti sequences of $\Bbbk[\mathcal{C}]$ and $\Bbbk[\mathcal{C}_1]$ coincide. A necessary condition is that $\Bbbk[\mathcal{C}]$ is Cohen-Macaulay. Indeed, affine monomial curves are always arithmetically Cohen-Macaulay while projective ones may be arithmetically Cohen-Macaulay or not, and $\mathrm{pd}(\Bbbk[\mathcal{C}]) = \mathrm{pd}(\Bbbk[\mathcal{C}_1])$ if and only if $\mathcal{C}$ is arithmetically Cohen-Macaulay. Then, $\Bbbk[\mathcal{C}]$ is Cohen-Macaulay if and only if the Betti sequences of $\Bbbk[\mathcal{C}]$ and $\Bbbk[\mathcal{C}_1]$ have the same length (and hence it is a necessary condition for the two Betti sequences to coincide).

In the recent paper [84], the authors give a sufficient condition that ensures the equality of the Betti numbers in terms of Gröbner bases.

Theorem 2.1 ([84, Thm. 4.1]). *Let $\mathcal{G}$ be the reduced Gröbner basis of $I_{\mathcal{A}_1}$ with respect to the degree reverse lexicographic (degrevlex) order with $x_1 > x_2 > \cdots > x_n$. If $\mathcal{C}$ is arithmetically Cohen-Macaulay and x_n is involved in all non-homogeneous binomials of $\mathcal{G}$, then $\beta_i(\Bbbk[\mathcal{C}]) = \beta_i(\Bbbk[\mathcal{C}_1])$ for all $i = 0, \ldots, n-1$.*

We address the same problem, but with a combinatorial approach. In Section 2.1, we recall some concepts on the Apéry sets of the semigroups $\mathcal{S}$ and $\mathcal{S}_1$, and define the Apéry posets $(\mathrm{Ap}_1, \leq_1)$ and $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$. In Section 2.2, we provide a combinatorial sufficient condition for having equality between the Betti sequences of $\Bbbk[\mathcal{C}]$ and $\Bbbk[\mathcal{C}_1]$ by means of the posets Ap_1 and $\mathrm{AP}_{\mathcal{S}}$. This is the content of Theorem 2.12, which is the main result of this chapter. In Propositions 2.18 and 2.23, we use our main result to provide explicit families of curves where $\beta_i(\Bbbk[\mathcal{C}]) = \beta_i(\Bbbk[\mathcal{C}_1])$ for all i. In Section 2.3, we apply our results to study the shifted family of monomial curves, i.e., the family of curves associated to the sequences $j + a_1 < \cdots < j + a_n$ parametrized by $j \in \mathbb{N}$. In this setting, Vu proved in [97] that the Betti numbers in the shifted family become periodic in j for $j > N$ for an integer N explicitly given. A key step in his argument is to prove that for $j > N$ one has equality between the Betti numbers of the affine and projective curves. Using our results, we substantially improve this latter bound in Theorem 2.26. In Section 2.4, we show how to construct arithmetically Gorenstein projective curves from a symmetric numerical semigroup (Theorem 2.32). Finally, in Section 2.5, we compute the Betti sequence of certain affine monomial curves coming from a class of semigroups defined by Kunz and Waldi in [61]. The main results of this section are Theorem 2.49, in which we characterize the semigroups in this family whose defining ideal is generated by the 2×2 minors of a $2 \times n$ matrix; and Theorem 2.53, where we provide the whole Betti sequence of some of these curves.

The results included in this chapter are part of [36] and [42].

2.1 Apéry sets and their poset structure

Fix an infinite field $\Bbbk$. Let $a_0 = 0 < a_1 < \cdots < a_n = D$ be a sequence of relatively prime integers. For each $i = 0, \ldots, n$, set $\underline{\mathbf{a}}_i := (D - a_i, a_i) \in \mathbb{N}^2$, and consider the three sets $\mathcal{A}_1 = \{a_1, \ldots, a_n\}$, $\mathcal{A}_2 = \{D, D - a_1, \ldots, D - a_{n-1}\}$ and $\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \ldots, \underline{\mathbf{a}}_n\} \subset \mathbb{N}^2$. We denote by $\mathcal{C} \subset \mathbb{P}^n_{\Bbbk}$ the projective monomial curve determined by $\underline{\mathcal{A}}$, and by $\mathcal{C}_1$ and $\mathcal{C}_2$ its affine charts, i.e., the affine monomial curves given by $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. We denote the vanishing ideal of $\mathcal{C}_i$ by $I_{\mathcal{A}_i}$ for $i = 1, 2$ and the vanishing ideal of $\mathcal{C}$ by $I_{\underline{\mathcal{A}}}$; these are the toric ideals determined by $\mathcal{A}_1$, $\mathcal{A}_2$, and $\underline{\mathcal{A}}$, respectively. Consider $\mathcal{S}_1$ and $\mathcal{S}_2$ the numerical semigroups generated by $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, and $\mathcal{S}$ the affine semigroup generated by $\underline{\mathcal{A}}$.

As already mentioned, $\Bbbk[\mathcal{C}_1]$ and $\Bbbk[\mathcal{C}_2]$ are always Cohen-Macaulay, while $\Bbbk[\mathcal{C}]$ can be Cohen-Macaulay or not. There are many ways to determine when a projective monomial curve is arithmetically Cohen-Macaulay; see, e.g., [14, Cor. 4.2], [15, Lem. 4.3, Thm. 4.6], [43, Thm. 2.6] or [49, Thm. 2.2]. We recall some of them in Proposition 2.4, but let us previously recall the notion of Apéry set (Section 1.1), since it is involved in some of those charaterizations. For $i = 1, 2$, the *Apéry set of* $\mathcal{S}_i$ *(with respect to* D*)* is

$$\mathrm{Ap}_i := \{y \in \mathcal{S}_i \,|\, y - D \notin \mathcal{S}_i\}\,.$$

By Proposition 1.5, Ap_i is a complete set of residues modulo D, i.e., $\mathrm{Ap}_1 = \{v_0 = 0, v_1, \ldots, v_{D-1}\}$ and $\mathrm{Ap}_2 = \{u_0 = 0, u_1, \ldots, u_{D-1}\}$ for some positive integers u_i and v_i such that $u_i \equiv v_i \equiv i \pmod{D}$ for all $i = 0, \ldots, D-1$.

Definition 2.2. The *Apéry set* $\mathrm{AP}_{\mathcal{S}}$ of $\mathcal{S}$ and the *exceptional set* $E_{\mathcal{S}}$ of $\mathcal{S}$ are defined as follows:

- $\mathrm{AP}_{\mathcal{S}} := \{\mathbf{s} \in \mathcal{S} : \mathbf{s} - \underline{\mathbf{a}}_0 \notin \mathcal{S}, \mathbf{s} - \underline{\mathbf{a}}_n \notin \mathcal{S}\}$.
- $E_{\mathcal{S}} := \{\mathbf{s} \in \mathcal{S} : \mathbf{s} - \underline{\mathbf{a}}_0 \in \mathcal{S}, \mathbf{s} - \underline{\mathbf{a}}_n \in \mathcal{S}, \mathbf{s} - \underline{\mathbf{a}}_0 - \underline{\mathbf{a}}_n \notin \mathcal{S}\}$.

By Proposition 1.20, $\mathrm{AP}_{\mathcal{S}}$ is finite, and we will see in Chapter 3 that $E_{\mathcal{S}}$ is also finite.

Figure 2.1: An element $\mathbf{s} \in \mathrm{AP}_{\mathcal{S}}$ and an element $\mathbf{s}' \in E_{\mathcal{S}}$.

Lemma 2.3. *For all $i = 1, \ldots, D-1$, the following claims hold:*

(1) If $(u_{D-i}, v_i) \in \mathcal{S}$, then $(u_{D-i}, v_i) \in \mathrm{AP}_{\mathcal{S}}$.

(2) If $(u_{D-i}, v_i) \notin \mathcal{S}$, then there exist natural numbers $x > u_{D-i}$ and $y > v_i$, such that $(x, v_i) \in \mathrm{AP}_{\mathcal{S}}$ and $(u_{D-i}, y) \in \mathrm{AP}_{\mathcal{S}}$.

Proof. (1) is trivial. To prove (2), take $i \in \{1, 2, \ldots, D-1\}$. Since $v_i \in \mathcal{S}_1$, there exists a natural number $x > u_{D-i}$, such that $(x, v_i) \in \mathcal{S}$, and if we choose the smallest $x \in \mathbb{N}$ satisfying this property, then $(x, v_i) \in \mathrm{AP}_{\mathcal{S}}$. The proof of the existence of y is analogous. □

As a consequence of the previous lemma, one has that $|\mathrm{AP}_{\mathcal{S}}| \geq D$. Denote by G the subgroup of $\mathbb{Z}^2$ generated by $\mathcal{S}$ and set $\mathcal{S}' = G \cap (\mathcal{S}_2 \times \mathcal{S}_1)$.

Proposition 2.4. *The following statements are equivalent:*

(a) $\mathcal{C}$ is arithmetically Cohen-Macaulay, i.e., the ring $\Bbbk[\mathcal{C}]$ is Cohen-Macaulay.

(b) For all $i = 1, \ldots, D-1$, $(u_{D-i}, v_i) \in \mathcal{S}$. In other words, if $v \in \mathrm{Ap}_1$, $u \in \mathrm{Ap}_2$ and $u + v \equiv 0 \pmod{D}$, then $(u, v) \in \mathrm{AP}_{\mathcal{S}}$.

(c) $\mathrm{AP}_{\mathcal{S}} = \{(0,0)\} \cup \{(u_{D-i}, v_i) : 1 \leq i < D\}$.

(d) $\mathrm{AP}_{\mathcal{S}}$ has exactly D elements.

(e) The exceptional set $E_{\mathcal{S}}$ is empty.

(f) $\mathcal{S}' = \mathcal{S}$.

(g) The variable x_n does not divide any minimal generator of $\mathrm{in}(I_{\mathcal{A}_1})$, the initial ideal of $I_{\mathcal{A}_1}$ for the degrevlex order in $\Bbbk[x_1, \ldots, x_n]$ with $x_1 > \cdots > x_n$.

Proof. The equivalences (a) ⇔ (e), (a) ⇔ (f) and (a) ⇔ (g) are well known; see, e.g., [15, Lem. 4.3, Thm. 4.6] and [49, Thm. 2.2]. Moreover, the implications (c) ⇒ (b) and (c) ⇒ (d) are trivial and (d) ⇒ (c) is a direct consequence of Lemma 2.3, so let us prove (b) ⇔ (e) ⇒ (c).
(e) ⇔ (b): Suppose that there exists an index i, $1 \leq i < D$, such that $(u_{D-i}, v_i) \notin \mathcal{S}$. By Lemma 2.3 (2), there exist $x > u_{D-i}$ and $y > v_i$, such that $(x, v_i) \in \mathrm{AP}_{\mathcal{S}}$ and

$(u_{D-i}, y) \in \mathrm{AP}_\mathcal{S}$. Then, there exist $x' \le x$ and $y' \le y$, such that $(x', y') \in E_\mathcal{S}$, so $E_\mathcal{S}$ is not empty. Conversely, suppose that there exists $(x, y) \notin \mathcal{S}$ such that $(x + D, y) \in \mathcal{S}$ and $(x, y + D) \in \mathcal{S}$ and let i be the index, $1 \le i \le D - 1$, such that $y \equiv i \equiv v_i \pmod D$ and $x \equiv D - i \equiv u_{D-i} \pmod D$. As $(x, y + D) \in \mathcal{S}$, $x \in \mathcal{S}_2$; and $y \in \mathcal{S}_1$, because $(x + D, y) \in \mathcal{S}$, so $u_{D-i} \le x$ and $v_i \le y$. This implies that $(u_{D-i}, v_i) \notin \mathcal{S}$.

(e)+(b) $\Rightarrow$ (c): Assuming that (b) holds, one gets that $\{(0, 0)\} \cup \{(u_{D-i}, v_i) : 1 \le i < D\} \subset \mathrm{AP}_\mathcal{S}$ by Lemma 2.3 (1). To prove the equality, take $(x, y) \in \mathrm{AP}_\mathcal{S}$. If $y \notin \mathrm{Ap}_1$, then $y - D \in \mathcal{S}_1$, so there exists $x' > x$, such that $(x', y - D) \in \mathcal{S}$ and choosing x' minimum with this property, one gets that $(x', y - D) \in \mathcal{S}$, $(x' - D, y) \in \mathcal{S}$ and $(x' - D, y - D) \notin \mathcal{S}$, a contradiction with (e). This implies that $y \in \mathrm{Ap}_1$, and we prove that $x \in \mathrm{Ap}_2$ using a similar argument. Thus, $(x, y) = (u_{D-i}, v_i)$ for some i, $1 \le i < D$, and we are done. □

Example 2.5. Let $\mathcal{A} = \{0, 1, 2, 3, 8\} \subset \mathbb{N}$. One can check that the Apéry sets of $\mathcal{S}_1$ and $\mathcal{S}_2$ are $\mathrm{Ap}_1 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and $\mathrm{Ap}_2 = \{0, 17, 10, 11, 12, 5, 6, 7\}$, respectively, and $\mathrm{AP}_\mathcal{S} = \{(0, 0), (7, 1), (6, 2), (5, 3), (12, 4), (11, 5), (10, 6), (17, 7)\}$. Hence, $\Bbbk[\mathcal{C}]$ is Cohen-Macaulay.

Remark 2.6. If $\Bbbk[\mathcal{S}]$ is not Cohen-Macaulay, the ring $\Bbbk[\mathcal{S}']$ is called the *Cohen-Macaulayfication* of $\Bbbk[\mathcal{S}]$. This is because $\mathcal{S} \ne \mathcal{S}'$ by Proposition 2.4 (f) and $\Bbbk[\mathcal{S}']$ is the least Cohen-Macaulay intermediate between $\Bbbk[\mathcal{S}]$ and its field of fractions; see, e.g., [15, Remark 4.7].

For $i = 1, 2$, one can consider the order relation $\le_i$ in $\mathcal{S}_i$ given by $y \le_i z \iff z - y \in \mathcal{S}_i$. Similarly, in $\mathcal{S}$ one can consider the order relation $\le_\mathcal{S}$ defined by $\mathbf{y} \le_\mathcal{S} \mathbf{z} \iff \mathbf{z} - \mathbf{y} \in \mathcal{S}$. The Apéry sets Ap_i and $\mathrm{AP}_\mathcal{S}$ inherit a poset structure from $(\mathcal{S}_i, \le_i)$ and $(\mathcal{S}, \le_\mathcal{S})$, respectively. We will denote these posets by $(\mathrm{Ap}_i, \le_i)$ and $(\mathrm{AP}_\mathcal{S}, \le_\mathcal{S})$.

Since $\mathcal{S} \subset \mathcal{S}_2 \times \mathcal{S}_1$, it follows that if $(x, y) \le_\mathcal{S} (x', y')$, then $x \le_2 x'$ and $y \le_1 y'$. Using Proposition 2.4, one can prove that the poset structure of $(\mathrm{AP}_\mathcal{S}, \le_\mathcal{S})$ is completely determined by those of $(\mathrm{Ap}_1, \le_1)$ and $(\mathrm{Ap}_2, \le_2)$ when $\mathcal{C}$ is arithmetically Cohen-Macaulay.

Proposition 2.7. *If $\mathcal{C}$ is arithmetically Cohen-Macaulay, then for all $(x, y), (x', y') \in \mathrm{AP}_\mathcal{S}$,*

$$(x, y) \le_\mathcal{S} (x', y') \iff x \le_2 x' \text{ and } y \le_1 y'.$$

Proof. As observed before stating the proposition, $(\Rightarrow)$ always holds. Let us prove $(\Leftarrow)$ when $\mathcal{C}$ is arithmetically Cohen-Macaulay. Since $(x, y), (x', y') \in \mathrm{AP}_\mathcal{S}$, one has that $y, y' \in \mathrm{Ap}_1$, $x, x' \in \mathrm{Ap}_2$ by Proposition 2.4 (c), and $x + y \equiv x' + y' \equiv 0 \pmod D$.

Assume that $y \leq_1 y'$ and $x \leq_2 x'$, then $w := y' - y \in \mathcal{S}_1$ and $z := x' - x \in \mathcal{S}_2$. Moreover, $w \in \mathrm{Ap}_1$ and $z \in \mathrm{Ap}_2$; otherwise, $y' \notin \mathrm{Ap}_1$ and $x' \notin \mathrm{Ap}_2$. Since $z + w = x' + y' - x - y \equiv 0 \pmod{D}$, then $(z, w) \in \mathcal{S}$ by Proposition 2.4 (b), and we are done. □

Let us recall now some notions about posets that will be used in the sequel for the posets $(\mathrm{Ap}_1, \leq_1)$, $(\mathrm{Ap}_2, \leq_2)$ and $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$.

Definition 2.8. Let $(P, \preceq)$ be a finite poset.

(1) For $y, z \in P$, we say that z *covers* y, and denote it by $y \prec\!\!\cdot\, z$, if $y \prec z$ and there is no $w \in P$ such that $y \prec w \prec z$.

(2) We say that P is *graded* if there exists a function $\rho : P \to \mathbb{N}$, called *rank function*, such that $\rho(z) = \rho(y) + 1$ whenever $y \prec\!\!\cdot\, z$.

The following result shows that the poset $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ is always graded while $(\mathrm{Ap}_1, \leq_1)$ may be graded or not. Observe that, since $(\mathrm{Ap}_1, \leq_1)$ has a minimum element which is 0, whenever it is graded, the corresponding rank function is completely determined by the value of the rank function at 0 that we fix to 0. In the following proposition, we characterize the covering relation in Ap_1 and in $\mathrm{AP}_{\mathcal{S}}$, and describe the rank functions of $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$, and of $(\mathrm{Ap}_1, \leq_1)$ when it is graded.

Proposition 2.9. *(a.1) For all $y, z \in \mathrm{Ap}_1$, $y <\!\!\cdot_1\, z \Longleftrightarrow z = y + a_i$ for some $a_i \in \mathrm{MSG}(\mathcal{S}_1) \setminus \{a_n\}$.*

(a.2) Ap_1 is graded if and only if, for all $y \in \mathrm{Ap}_1$, all the factorizations of y have the same length. When it is graded, the rank function $\rho_1 : \mathrm{Ap}_1 \to \mathbb{N}$ is given by the length of the factorizations of the elements in Ap_1.

(b.1) For all $\mathbf{y} = (y_1, y_2), \mathbf{z} = (z_1, z_2) \in \mathrm{AP}_{\mathcal{S}}$, $\mathbf{y} <\!\!\cdot_{\mathcal{S}}\, \mathbf{z} \Longleftrightarrow \mathbf{z} = \mathbf{y} + \underline{\mathbf{a}}_i$ for some $i \in \{1, \ldots, n-1\}$.

(b.2) $\mathrm{AP}_{\mathcal{S}}$ is graded by the rank function $\rho : \mathrm{AP}_{\mathcal{S}} \to \mathbb{N}$ defined by $\rho(y_1, y_2) := (y_1 + y_2)/D$.

Proof. In (a.1) and (b.1), $(\Leftarrow)$ is trivial. Let us prove $(\Rightarrow)$.

(a.1) Consider $y, z \in \mathrm{Ap}_1$ such that $y <\!\!\cdot_1\, z$. Since $z - y \in \mathcal{S}_1$, there exists $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ such that $z = y + \sum_{i=1}^{n} \alpha_i a_i$, and $\alpha_n = 0$ because $z \in \mathrm{Ap}_1$. If $|\alpha| > 1$, then there exists $j \in \{1, \ldots, n-1\}$ such that $\alpha_j \neq 0$ and $y + a_j \neq z$. Thus, $y + a_j \in \mathrm{Ap}_1$ because $z \in \mathrm{Ap}_1$, and $y <_1 y + a_j <_1 z$, a contradiction because $y <\!\!\cdot_1\, z$, so $|\alpha| = 1$.

(b.1) Consider $\mathbf{y}, \mathbf{z} \in \mathrm{AP}_{\mathcal{S}}$ such that $\mathbf{y} \prec\!\cdot_{\mathcal{S}} \mathbf{z}$. Since $\mathbf{z} - \mathbf{y} \in \mathcal{S}$, there exists $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$ such that $\mathbf{z} - \mathbf{y} = \sum_{i=0}^{n} \alpha_i \underline{\mathbf{a}}_i$, and $\alpha_0 = \alpha_n = 0$ because $\mathbf{z} \in \mathrm{AP}_{\mathcal{S}}$. Again, if $|\alpha| = \sum_{i=1}^{n-1} \alpha_i > 1$, we can choose any $\alpha_j \neq 0$ and get $\mathbf{y} \prec_{\mathcal{S}} \mathbf{y} + \underline{\mathbf{a}}_j \prec_{\mathcal{S}} \mathbf{z}$ (one has that $\mathbf{y} + \underline{\mathbf{a}}_j \in \mathrm{AP}_{\mathcal{S}}$ because $\mathbf{z} \in \mathrm{AP}_{\mathcal{S}}$), a contradiction because $\mathbf{y} \prec\!\cdot_{\mathcal{S}} \mathbf{z}$.

Now (a.2) and (b.2) are direct consequences of (a.1) and (b.1), respectively. □

Remark 2.10. By Proposition 2.9 (b.2), the fiber of 1 under the rank function ρ is $\rho^{-1}(1) = \{\underline{\mathbf{a}}_i : 1 \leq i \leq n-1\}$, and hence $|\rho^{-1}(1)| = n - 1$. On the other hand, when Ap_1 is graded, the fiber of 1 under ρ_1 is $\rho_1^{-1}(1) = \mathrm{MSG}(\mathcal{S}_1) \setminus \{a_n\}$, by Proposition 2.9 (a.1).

Set $\mathcal{A}_1' := \mathrm{MSG}(\mathcal{S}_1) \setminus \{a_n\}$ and $\mathrm{Ap}_1^{(s)} := \mathrm{Ap}_1 \cap s\mathcal{A}_1'$ for each $s \in \mathbb{N}$. Since Ap_1 is finite, consider $N := \max\{s \in \mathbb{N} : \mathrm{Ap}_1^{(s)} \neq \emptyset\} \in \mathbb{N}$. As a direct consequence of Proposition 2.9 (a.2), we get a characterization of the graded property for $(\mathrm{Ap}_1, \preceq_1)$.

Corollary 2.11. $(\mathrm{Ap}_1, \preceq_1)$ *is graded if and only if* $\sum_{s=0}^{N} |\mathrm{Ap}_1^{(s)}| = D$.

2.2 Equality between the Betti numbers

Recall that $I_{\mathcal{A}_1} \subset \Bbbk[x_1, \ldots, x_n]$ and $I_{\mathcal{A}} \subset \Bbbk[x_0, \ldots, x_n]$ are the vanishing ideals of $\mathcal{C}_1$ and $\mathcal{C}$, respectively. When $\mathcal{C}$ is arithmetically Cohen-Macaulay, $\mathrm{pd}(\Bbbk[\mathcal{C}]) = \mathrm{pd}(\Bbbk[\mathcal{C}_1])$. Moreover, by Proposition 2.4 (d), in this case, one has that $|\mathrm{AP}_{\mathcal{S}}| = |\mathrm{Ap}_1| = D$. The main result in this section is Theorem 2.12 where we give a sufficient condition in terms of the poset structures of the Apéry sets Ap_1 and $\mathrm{AP}_{\mathcal{S}}$ for the Betti sequences of $\Bbbk[\mathcal{C}_1]$ and $\Bbbk[\mathcal{C}]$ to coincide. We postpone its proof after Propositions 2.14 and 2.16.

Theorem 2.12. *If* $(\mathrm{AP}_{\mathcal{S}}, \preceq_{\mathcal{S}}) \simeq (\mathrm{Ap}_1, \preceq_1)$, *then* $\beta_i(\Bbbk[\mathcal{C}]) = \beta_i(\Bbbk[\mathcal{C}_1])$ *for all* i.

Note that the converse of this result does not hold, as the following example shows.

Example 2.13. For the sequence $1 < 2 < 4 < 8$, one can check using, e.g., [24], that both $\Bbbk[\mathcal{C}_1]$ and $\Bbbk[\mathcal{C}]$ are complete intersections with Betti sequence $(1, 3, 3, 1)$. However, the posets $(\mathrm{Ap}_1, \preceq_1)$ and $(\mathrm{AP}_{\mathcal{S}}, \preceq_{\mathcal{S}})$ are not isomorphic since $\preceq_1$ is a total order on Ap_1, while $\preceq_{\mathcal{S}}$ is not. More generally, for $a_1 = 1 < a_2 < \cdots < a_n = D$ with a_i a divisor of a_{i+1} for all $i \in \{1, \ldots, n-1\}$, one has that both $\Bbbk[\mathcal{C}_1]$ and $\Bbbk[\mathcal{C}]$ are complete intersections; see [4, Theorem 5.3]. Thus, both Betti sequences are $(1, \binom{n-1}{1}, \ldots, \binom{n-1}{i}, \ldots, \binom{n-1}{n-2}, 1)$, by [92, Thm. 6]. However, again the posets $(\mathrm{Ap}_1, \preceq_1)$ and $(\mathrm{AP}_{\mathcal{S}}, \preceq_{\mathcal{S}})$ are not isomorphic since $\preceq_1$ is a total order on Ap_1, while $\preceq_{\mathcal{S}}$ is not.

Proposition 2.14. *The following two claims are equivalent:*

(a) The posets $(\mathrm{Ap}_1, \leq_1)$ *and* $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ *are isomorphic;*

(b) $\Bbbk[\mathcal{C}]$ *is Cohen-Macaulay,* $(\mathrm{Ap}_1, \leq_1)$ *is graded, and* $\{a_1, \ldots, a_{n-1}\}$ *is contained in the minimal system of generators of* $\mathcal{S}_1$.

Proof. (a) $\Rightarrow$ (b): If $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}}) \simeq (\mathrm{Ap}_1, \leq_1)$, then Ap_1 and $\mathrm{AP}_{\mathcal{S}}$ have the same number of elements, and hence $\Bbbk[\mathcal{C}]$ is Cohen-Macaulay by Proposition 2.4 (d). Moreover, since $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ is graded by Proposition 2.9 (b.2), $(\mathrm{Ap}_1, \leq_1)$ is graded. Finally, $|\rho_1^{-1}(1)| = |\rho^{-1}(1)|$ so, by Remark 2.10, $|\mathrm{MSG}(\mathcal{S}_1) \setminus \{a_n\}| = n-1$, and hence $\{a_1, \ldots, a_{n-1}\} \subset \mathrm{MSG}(\mathcal{S}_1)$.
(b) $\Rightarrow$ (a): If $\Bbbk[\mathcal{C}]$ is Cohen-Macaulay, then $|\mathrm{AP}_{\mathcal{S}}| = |\mathrm{Ap}_1|$ by Proposition 2.4 (d), and hence the map $\varphi : \mathrm{AP}_{\mathcal{S}} \to \mathrm{Ap}_1$ defined by $\varphi(u_{D-j}, v_j) = v_j$ for all $j = 0, \ldots, D-1$, is bijective. Let us prove that it is an isomorphism of posets. By Proposition 2.7, φ is an order-preserving map, so one just has to show that φ^{-1} is also order-preserving. Consider $y, y' \in \mathrm{Ap}_1$ such that $y \lessdot_1 y'$. Then, there exists $i \in \{1, \ldots, n-1\}$ such that $y' = y + a_i$, by Proposition 2.9 (a.1). Set $(x, y) = \varphi^{-1}(y)$ and $(x', y') = \varphi^{-1}(y')$. One has that $x + D - a_i \geq x'$ since $x' \in \mathrm{Ap}_2$ and $x + D - a_i \in \mathcal{S}_2$. Note that $\rho(x, y) = \rho_1(y)$ (and the same holds for (x', y')). This is because if we write $(x, y) = \sum_{i=1}^{n-1} \alpha_i (D - a_i, a_i)$ for some $\alpha_i \in \mathbb{N}$, then $y = \sum_{i=1}^{n-1} \alpha_i a_i$ provides a factorization of y of length $\sum_{i=1}^{n-1} \alpha_i$, and hence $\rho(x, y) = \sum_{i=1}^{n-1} \alpha_i = \rho_1(y)$, by Proposition 2.9 (a.2) and (b.2). If $x' < x + D - a_i$, then $\rho_1(y') = \rho(x', y') \leq \rho(x, y) = \rho_1(y)$, a contradiction since $y \lessdot_1 y'$. Therefore, $y + D - a_i = y'$ and we are done. □

Note that Ap_1 can be a graded poset even if $(\mathrm{Ap}_1, \leq_1)$ and $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ are not isomorphic as the following example shows.

Example 2.15. For the sequence $a_1 = 5 < a_2 = 11 < a_3 = 13$, the Apéry set of the numerical semigroup $\mathcal{S}_1 = \langle a_1, a_2, a_3 \rangle$ is $\mathrm{Ap}_1 = \{0, 27, 15, 16, 30, 5, 32, 20, 21, 22, 10, 11, 25\}$. This Apéry set is graded with the rank function $\rho_1 : \mathcal{S}_1 \to \mathbb{N}$ defined below (see Figure 2.2):

- $\rho_1(0) = 0$,
- $\rho_1(5) = \rho(11) = 1$,
- $\rho_1(10) = \rho_1(16) = \rho_1(22) = 2$,
- $\rho_1(15) = \rho_1(21) = \rho_1(27) = 3$,
- $\rho_1(20) = \rho_1(32) = 4$,
- $\rho_1(25) = 5$,
- $\rho_1(30) = 6$.

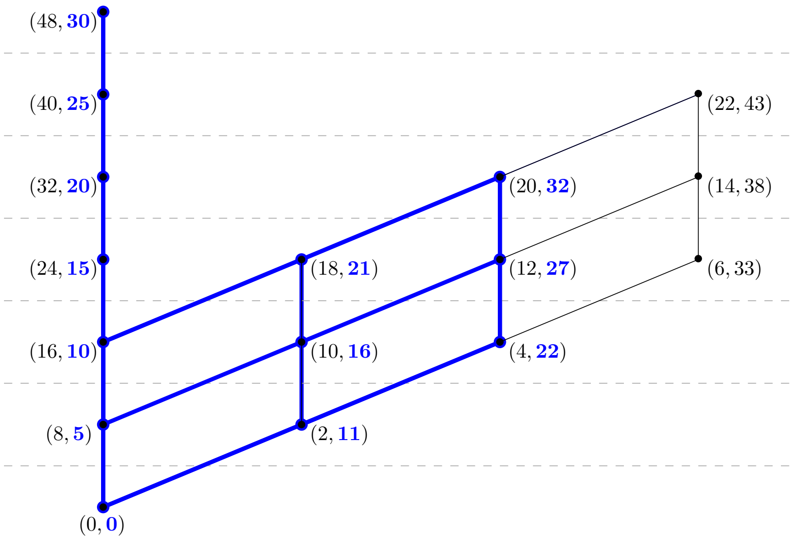


Figure 2.2: The posets $(\mathrm{Ap}_1, \leq_1)$ (in blue) and $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ (in black) for $\mathcal{S}_1 = \langle 5, 11, 13\rangle$.

Moreover, since $\mathrm{AP}_{\mathcal{S}}$ has 16 elements, $\Bbbk[\mathcal{C}]$ is not Cohen-Macaulay, and hence $(\mathrm{Ap}_1, \leq_1)$ and $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ are not isomorphic by Proposition 2.14.

We now relate the condition in Proposition 2.14 to the criterion in Theorem 2.1, which uses Gröbner bases.

Proposition 2.16. *Consider the following two claims:*

(a) $(\mathrm{Ap}_1, \leq_1)$ *is graded and* $\{a_1, \ldots, a_{n-1}\}$ *is contained in the minimal system of generators of* $\mathcal{S}_1$*.*

(b) The variable x_n *appears in every non-homogeneous binomial of* $\mathcal{G}_>$*, the reduced Gröbner basis of* $I_{\mathcal{A}_1}$ *with respect to the degree reverse lexicographic order with* $x_1 > x_2 > \cdots > x_n$*.*

Then (b) $\Rightarrow$ *(a), and (a)* $\Rightarrow$ *(b) holds if* $\Bbbk[\mathcal{C}]$ *is Cohen-Macaulay.*

Proof. (a) $\Rightarrow$ (b) when $\Bbbk[\mathcal{C}]$ is Cohen-Macaulay: Assume that there exists a non-homogeneous binomial $f = \mathbf{x}^\alpha - \mathbf{x}^\beta \in \mathcal{G}_>$ with $\mathrm{in}(f) = \mathbf{x}^\alpha$ such that x_n does not appear in the binomial f, i.e. $|\alpha| > |\beta|$ and $\alpha_n = \beta_n = 0$, and consider $s = \sum_{i=1}^{n-1} \alpha_i a_i = \sum_{i=1}^{n-1} \beta_i a_i \in \mathcal{S}_1$. Let us prove that $s - a_n \notin \mathcal{S}_1$. If $s - a_n \in \mathcal{S}_1$, we can write s as $s = \sum_{i=1}^{n} \gamma_i a_i + a_n$ for some $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n$, and consider the binomial $g = \mathbf{x}^\gamma x_n - \mathbf{x}^\beta \in I_{\mathcal{A}_1}$. Note that $\mathbf{x}^\beta - \mathbf{x}^\gamma x_n \neq 0$ since $\beta_n = 0$. As $f \in \mathcal{G}_>$ and $\mathcal{G}_>$ is reduced, one has that $\mathbf{x}^\beta \notin \mathrm{in}(I_{\mathcal{A}_1})$ and hence $\mathrm{in}(g) = \mathbf{x}^\gamma x_n$.

Therefore, $\mathbf{x}^\gamma x_n \in \text{in}(I_{\mathcal{A}_1})$ and, by Proposition 2.4 (g), $\mathbf{x}^\gamma \in \text{in}(I_{\mathcal{A}_1})$. The remainder of the division of $\mathbf{x}^\gamma$ by $\mathcal{G}_>$ is a monomial $\mathbf{x}^\delta$ such that $\mathbf{x}^\delta \notin \text{in}(I_{\mathcal{A}_1})$, and one has that the binomial $\mathbf{x}^\beta - \mathbf{x}^\delta x_n \in I_{\mathcal{A}_1}$ is the difference of two binomials that do not belong to $\text{in}(I_{\mathcal{A}_1})$ using again Proposition 2.4 (g), a contradiction. Thus, $s - a_n \notin \mathcal{S}_1$ and hence $s \in \text{Ap}_1$. But $s = \sum_{i=1}^{n-1} \alpha_i a_i = \sum_{i=1}^{n-1} \beta_i a_i \in \mathcal{S}_1$ with $|\alpha| > |\beta|$ so if $\{a_1, \ldots, a_{n-1}\} \subset \text{MSG}(\mathcal{S}_1)$, one gets by Proposition 2.9 (a.2) that Ap_1 is not graded.
(b) $\Rightarrow$ (a): If $\{a_1, \ldots, a_{n-1}\} \not\subset \text{MSG}(\mathcal{S}_1)$, select $i \in \{2, \ldots, n-1\}$ such that a_i is not a minimal generator. Then, there exists $\alpha = (\alpha_1, \ldots, \alpha_{i-1}) \in \mathbb{N}^{i-1}$ with $|\alpha| > 2$ such that $x_i - \prod_{j<i} x_j^{\alpha_j} \in I_{\mathcal{A}_1}$. Note that any set of generators of $I_{\mathcal{A}_1}$ contains an element of this form. Thus, $\mathcal{G}_>$ contains a non-homogeneous binomial that does not involve the variable x_n, a contradiction, and hence $\{a_1, \ldots, a_{n-1}\} \subset \text{MSG}(\mathcal{S}_1)$.

If $(\text{Ap}_1, \leq_1)$ is not graded, by Proposition 2.9 (a.2), there exists $s \in \text{Ap}_1$ which has two factorizations of different length , i.e., $s = \sum_{i=1}^{n-1} \alpha_i a_i = \sum_{i=1}^{n-1} \beta_i a_i$ with $|\alpha| > |\beta|$. Note that $\alpha_n = \beta_n = 0$ since $s \in \text{Ap}_1$. We can choose $\beta = (\beta_1, \ldots, \beta_{n-1})$ such that $|\beta| > 0$ is the least possible value, and $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ such that, for this election of β, $\mathbf{x}^\alpha$ is the smallest possible monomial for the degree reverse lexicographic order. Then $f = \mathbf{x}^\alpha - \mathbf{x}^\beta \in I_{\mathcal{A}_1}$ and $\text{in}(f) = \mathbf{x}^\alpha$. Since $\mathbf{x}^\alpha \in \text{in}(I_{\mathcal{A}_1})$, there exists a binomial $h = \mathbf{x}^\lambda - \mathbf{x}^\mu \in \mathcal{G}_>$ such that $\mathbf{x}^\lambda$ divides $\mathbf{x}^\alpha$. Let us see that h is not homogeneous and that the variable x_n is not involved in h. If h is homogeneous, dividing $\mathbf{x}^\alpha$ by h, we get $\mathbf{x}^\alpha = \mathbf{x}^{\alpha-\lambda}(\mathbf{x}^\lambda - \mathbf{x}^\mu) + \mathbf{x}^{\alpha-\lambda+\mu}$. Then, $s = \sum_i (\alpha_i - \lambda_i + \mu_i) a_i = \sum_i \alpha_i a_i$ with $|\alpha - \lambda + \mu| = |\alpha|$ and $\mathbf{x}^{\alpha-\lambda+\mu} < \mathbf{x}^\alpha$, a contradiction with the choice of α, so h is not homogeneous. On the other hand, since $\mathbf{x}^\lambda$ divides $\mathbf{x}^\alpha$ and $\alpha_n = 0$, if x_n appears in $\mathbf{x}^\lambda - \mathbf{x}^\mu$, it must be in the support of $\mathbf{x}^\mu$. If we write $\mathbf{x}^\mu = \mathbf{x}^{\mu'} x_n$, then $\mathbf{x}^\alpha = \mathbf{x}^{\alpha-\lambda}(\mathbf{x}^\lambda - \mathbf{x}^\mu) + \mathbf{x}^{\alpha-\lambda+\mu'} x_n$ and hence $s = \sum_i (\alpha_i - \lambda_i + \mu'_i) a_i + a_n$ which is impossible because $s \in \text{Ap}_1$. Therefore, we have found a non-homogeneous binomial $h = \mathbf{x}^\lambda - \mathbf{x}^\mu \in \mathcal{G}_>$ where the variable x_n is not involved, a contradiction. Thus, $(\text{Ap}_1, \leq_1)$ is graded. □

Remark 2.17. (1) In our proof of (a) $\Rightarrow$ (b), we strongly use that $\Bbbk[\mathcal{C}]$ is Cohen-Macaulay but we could not find any non-Cohen-Macaulay example where this implication is wrong.

(2) In general, if one considers the Apéry set of $\mathcal{S}_1$ with respect to a_i, $i \in \{1, \ldots, n\}$, one can determine when $\text{Ap}(\mathcal{S}_1, a_i)$ is graded in terms of the generators of I_1. By [54, Cor. 3.10], $\text{Ap}(\mathcal{S}_1, a_i)$ is graded if and only if there exists a minimal set of generators of I_1 such that x_i appears in all of its non-homogeneous binomials.

Proof of Theorem 2.12. By Propositions 2.14 and 2.16, the Apéry posets $(\text{AP}, \leq_\mathcal{S})$ and $(\text{Ap}_1, \leq_1)$ are isomorphic if and only if the variable x_n appears in every non-homogeneous binomial of $\mathcal{G}_>$, the reduced Gröbner basis of $I_{\mathcal{A}_1}$ with respect to the degrevlex order with $x_1 > x_2 > \cdots > x_n$. Hence, the result follows from Theorem 2.1. □

Families of curves where the Betti sequences coincide

In Propositions 2.18 and 2.23 below, we provide sequences $a_1 < \cdots < a_n$ for which the condition in Theorem 2.12 is satisfied.

Let us start with arithmetic sequences, i.e., sequences $a_1 < \cdots < a_n$ such that $a_i = a_1 + (i-1)e$ for some positive integer e with $\gcd(a_1, e) = 1$. For this family, we refine [84, Cor. 4.2] that considers $a_1 > n-1$.

Proposition 2.18. *Let $a_1 < \ldots < a_n = D$ be an arithmetic sequence of relatively prime integers, i.e., for all $i = 1, \ldots, n$, $a_i = a_1 + (i-1)e$ for some integers $a_1, e > 0$ such that $\gcd(a_1, e) = 1$. Then, $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}}) \simeq (\mathrm{Ap}_1, \leq_1)$ if and only if $a_1 > n-2$. Therefore, if $a_1 > n-2$, the Betti sequences of $\Bbbk[\mathcal{C}_1]$ and $\Bbbk[\mathcal{C}]$ coincide.*

Proof. We use Proposition 2.14 to characterize when $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ and $(\mathrm{Ap}_1, \leq_1)$ are isomorphic. When $a_1 < \cdots < a_n$ is an arithmetic sequence, $\Bbbk[\mathcal{C}]$ is always Cohen-Macaulay by [2, Cor. 2.3]. Moreover, one can easily check that $\{a_1, \ldots, a_{n-1}\} \subset \mathrm{MSG}(\mathcal{S}_1)$ if and only if $a_1 > n-2$. Therefore, if $a_1 \leq n-2$, then $(\mathrm{Ap}_1, \leq_1)$ is not isomorphic to $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$. Conversely, if $a_1 > n-2$, it is sufficient to prove that $(\mathrm{Ap}_1, \leq_1)$ is graded. By [66, Thm. 3.4], the Apéry set of $\mathcal{S}_1$ is described as follows: if, for all $b \in \{0, \ldots, D-1\}$, q_b and $-r_b$ denote respectively the quotient and the reminder of the division with negative remainder of b by $n-1$, i.e., $q_b = \lceil b/(n-1) \rceil$ and $r_b = q_b(n-1) - b$ with $0 \leq r_b \leq n-2$, then

$$\mathrm{Ap}_1 = \{q_b a_1 + r_b e\,,\ 0 \leq b \leq D-1\}\ .$$

We claim that the grading is given by the function $\rho_1 : \mathrm{Ap}_1 \to \mathbb{N}$ defined by $\rho_1(q_b a_1 + r_b e) = q_b$. Consider $y, y' \in \mathrm{Ap}_1$ such that $y \lessdot_1 y'$, an let us prove that $\rho_1(y') = \rho_1(y) + 1$. By Proposition 2.9 (a.1), there exist natural numbers $b \in \{0, \ldots, D-1\}$ and $i \in \{1, \ldots, n-1\}$ such that $y = q_b a_1 + r_b e$ and $y' = y + a_i = (q_b+1)a_1 + (r_b+i-1)e$. If $i \geq n - r_b$, then $y' - D = q_b a_1 + (r_b + i - 1 - (n-1))\, e \in \mathcal{S}_1$, contradicting the fact $y' \in \mathrm{Ap}_1$. Hence, $i \leq n - r_b - 1$. Set $b' := (q_b+1)(n-1) - (r_b + i - 1)$. As $0 \leq r_b + i - 1 \leq n-2$, on the one hand one has that $0 \leq b' \leq D-1$, on the other $q_{b'} = q_b + 1$ and $r_{b'} = r_b + i - 1$. Therefore $y' = q_{b'} a_1 + r_{b'} e$, and hence $\rho_1(y') = \rho_1(y) + 1$. □

Remark 2.19. Let $a_1 < a_2 < \cdots < a_n = D$ be an arithmetic sequence of relatively prime integers. Set $1 \leq \ell \leq n-1$ such that $a_1 \equiv \ell \pmod{n-1}$ (this number ℓ will appear in Lemma 2.21). By [40, Thm. 4.1], the Betti numbers of $\mathbb{k}[\mathcal{C}_1]$ are $\beta_0 = 1$ and

$$\beta_j = j\binom{n-1}{j+1} + \begin{cases} (n-\ell+1-j)\binom{n-1}{j-1} & 1 \leq j \leq n-\ell, \\ (j-n+\ell)\binom{n-1}{j} & n-\ell < j \leq n-1. \end{cases}$$

Note that they depend only on the remainder of a_1 modulo $n-1$. Hence, when $a_1 > n-2$, the Betti numbers of $\mathbb{k}[\mathcal{C}]$ are also given by the previous formula, by Proposition 2.18.

Example 2.20. For the sequence $5 < 6 < 7 < 8 < 9 < 10$, one has that $a_1 = 5 > 4 = n-2$. Therefore, the Apéry sets $(\mathrm{Ap}_1, \leq_1)$ and $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ are isomorphic by Proposition 2.18. The Betti sequences of $\mathbb{k}[\mathcal{C}_1]$ and $\mathbb{k}[\mathcal{C}]$ coincide and one can check, using, e.g., [24], that both sequences are $(1, 11, 30, 35, 19, 4)$. This also follows from Remark 2.19 and Proposition 2.18. The isomorphic posets $(\mathrm{Ap}_1, \leq_1)$ and $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ in this example are shown in Figure 2.3.

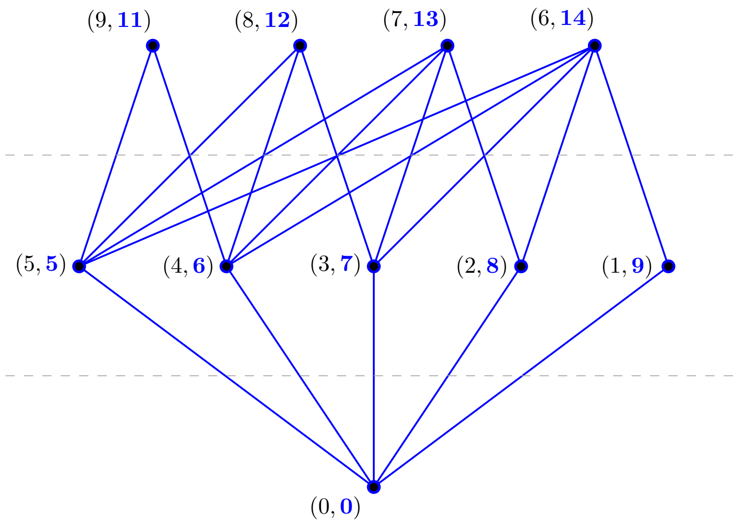


Figure 2.3: The posets $(\mathrm{Ap}_1, \leq_1)$ (in blue) and $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ (in black) for $\mathcal{S}_1 = \langle 5, 6, 7, 8, 9, 10\rangle$.

The next family that we now consider are monomial curves defined by an arithmetic sequence in which we have removed one term. In [3, Sect. 6], the authors study the canonical projections of the projective monomial curve $\mathcal{C}$ defined by an arithmetic sequence $a_1 < \cdots < a_n$ of relatively prime integers, i.e., the curve $\pi_r(\mathcal{C})$ obtained as the Zariski closure of the image of $\mathcal{C}$ under the r-th canonical projection $\pi_r : \mathbb{P}^n_{\mathbb{k}} \dashrightarrow \mathbb{P}^{n-1}_{\mathbb{k}}$, $(p_0 : \cdots : p_n) \mapsto (p_0 : \cdots : p_{r-1} : p_{r+1} : \cdots : p_n)$. We know that $\pi_r(\mathcal{C})$ is the projective monomial curve associated to the sequence

$a_1 < \cdots < a_{r-1} < a_{r+1} < \cdots < a_n$.

If one removes either the first or the last term from an arithmetic sequence, the sequence is still arithmetic. Moreover, note that if an arithmetic sequence $a_1 < \cdots < a_n$ satisfies the condition $a_1 > n - 2$ in Proposition 2.18, then the arithmetic sequence obtained by removing either the first or the last term also satisfies the condition in Proposition 2.18 because the number of terms in the new sequence is smaller, and its first term may have increased. Thus, we will only focus here on sequences obtained from an arithmetic sequence $a_1 < \cdots < a_n$ by removing a_r for $r \in \{2, \ldots, n-1\}$. Set $\mathcal{A}_1 := \{a_1, \ldots, a_n\} \setminus \{a_r\}$, and consider the numerical semigroup $\mathcal{S}_1 = \mathcal{S}_{\mathcal{A}_1}$ and its homogenization $\mathcal{S}$. We characterize in Proposition 2.23 when the posets $(\mathrm{Ap}_1, \leq_1)$ and $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ are isomorphic. Two main ingredients in the proof are the following two results in [3] that we recall for convenience. The first one is a technical lemma, while the second describes the Apéry set of $\mathcal{S}_1$.

Lemma 2.21 ([3, Lem. 2]). *Let $a_1 < \cdots < a_n$ be an arithmetic sequence of relatively prime integers with common difference e. Set $q := \lfloor (a_1 - 1)/(n-1) \rfloor \in \mathbb{N}$ and $\ell := a_1 - q(n-1) \in \{1, \ldots, n-1\}$. Then,*

(1) $(q+e)a_1 + a_i = a_{\ell+i} + qa_n$, for all $i \in \{1, \ldots, n-\ell\}$, and

(2) $q + e + 1 = \min\{m > 0 \mid ma_1 \in \langle a_2, \ldots, a_n\rangle\}$.

Lemma 2.22 ([3, Cor. 4]). *Let $a_1 < \cdots < a_n$ be an arithmetic sequence of relatively prime integers with common difference e. Denote by A the Apéry set of $\bar{\mathcal{S}}_1 = \langle a_1, \ldots, a_n\rangle$ with respect to a_n, $q := \lfloor (a_1-1)/(n-1) \rfloor$, and, for all $\mu \in \mathbb{N}$, set $v_\mu := \mu a_1 + a_2$. Given $r \in \{2, \ldots, n-1\}$, consider $\mathcal{A}_1 = \{a_1, \ldots, a_n\} \setminus \{a_r\}$, and the semigroup $\mathcal{S}_1$ generated by $\mathcal{A}_1$. When $a_1 \geq r$, the Apéry set of $\mathcal{S}_1$ with respect to a_n is described as follows:*

(1) If $r = 2$,

$$\mathrm{Ap}_1 = \begin{cases} (A \setminus \{v_\mu \mid 0 \leq \mu \leq q+e\}) \cup \{v_\mu + a_n \mid 0 \leq \mu \leq q+e\}, & \text{if } n-1 \mid a_1, \\ (A \setminus \{v_\mu \mid 0 \leq \mu \leq q+e-1\}) \\ \qquad \cup \{v_\mu + a_n \mid 0 \leq \mu \leq q+e-1\}, & \text{otherwise.} \end{cases}$$

(2) If $r \in \{3, \ldots, n-2\}$,

$$\mathrm{Ap}_1 = (A \setminus \{a_r\}) \cup \{a_r + a_n\}.$$

(3) If $r = n-1$,

$$\mathrm{Ap}_1 = \begin{cases} (A \setminus \{a_{n-1}\}) \cup \{a_{n-1} + (q+1)a_n\}, & \text{if } n-1 \mid a_1, \\ (A \setminus \{a_{n-1}\}) \cup \{a_{n-1} + qa_n\}, & \text{otherwise.} \end{cases}$$

Proposition 2.23. *Consider $a_1 < \ldots < a_n$ an arithmetic sequence of relatively prime integers with $n \geq 4$, and take $r \in \{2, \ldots, n-1\}$. Set $\mathcal{A}_1 := \{a_1, \ldots, a_n\} \setminus \{a_r\}$, and let $\mathcal{S}_1$ be the numerical semigroup generated by $\mathcal{A}_1$, and $\mathcal{S}$ its homogenization. Then,*

$$(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}}) \simeq (\mathrm{Ap}_1, \leq_1) \Longleftrightarrow \begin{cases} a_1 > n-2 \text{ and } a_1 \neq n, & \text{if } r = 2, \\ a_1 \geq n \text{ and } r \leq a_1 - n + 1, & \text{if } 3 \leq r \leq n-2, \\ a_1 \geq n-2, & \text{if } r = n-1. \end{cases}$$

Consequently, if the previous condition holds, then $\beta_i(\Bbbk[\mathcal{C}_1]) = \beta_i(\Bbbk[\mathcal{C}])$, for all i.

Proof. Denote by $\bar{\mathcal{S}}_1$ the numerical semigroup generated by the whole arithmetic sequence $a_1 < \cdots < a_n$. Again, we use Proposition 2.14 to characterize when the posets $(\mathrm{Ap}_1, \leq_1)$ and $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ are isomorphic. Note that $\{a_1, \ldots, a_n\} \setminus \{a_r\} \subset \mathrm{MSG}(\mathcal{S}_1)$ if and only if

$$\text{either } r \neq n-1 \text{ and } a_1 > n-2, \text{ or } r = n-1 \text{ and } a_1 \geq n-2. \tag{2.1}$$

On the other hand, by [3, Cor. 5], $\Bbbk[\mathcal{C}]$ is Cohen-Macaulay if and only if

$$r \leq a_1 \text{ or } r = n-1. \tag{2.2}$$

Finally, by Proposition 2.9 (a.2), $(\mathrm{Ap}_1, \leq_1)$ is graded if and only if

$$\forall b \in \mathrm{Ap}_1,\ b = \sum_{i \notin \{r,n\}} \alpha_r a_i = \sum_{i \notin \{r,n\}} \beta_i a_i \implies \sum_{i \notin \{r,n\}} \alpha_i = \sum_{i \notin \{r,n\}} \beta_i. \tag{2.3}$$

We split the proof in three cases depending on the value of r.

- $r = 2$.

By (2.1), if $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}}) \simeq (\mathrm{Ap}_1, \leq_1)$, then $a_1 > n-2$. If $a_1 = n$, the element $a_3 + a_{n-1} = a_2 + a_n$ of Ap_1 can be written as $(2+e)a_1$, and hence $(\mathrm{Ap}_1, \leq_1)$ is not graded by (2.3). Assume now that $a_1 > n-2$ and $a_1 \neq n$, and let us prove that $(\mathrm{Ap}_1, \leq_1)$ is graded in this case. By Lemma 2.22 (1),

$$\mathrm{Ap}_1 = (A \setminus \{v_\mu \mid 0 \leq \mu \leq t\}) \cup \{v_\mu + a_n \mid 0 \leq \mu \leq t\},$$

for $t \in \{q+e-1, q+e\}$. Every $b \in A \cap \mathrm{Ap}_1$ satisfies (2.3) by Proposition 2.18, so consider $b_\mu := \mu a_1 + a_2 + a_n = \mu a_1 + a_3 + a_{n-1} \in \mathrm{Ap}_1$, with $0 \leq \mu \leq t$. Let us prove that whenever $b_\mu = \sum_{i \notin \{2,n\}} \alpha_i a_i$, with $\alpha_i \in \mathbb{N}$, then $\sum_{i \notin \{2,n\}} \alpha_i = \mu + 2$.

Using iteratively the relations $a_i + a_j = a_{i-1} + a_{j+1}$ in $\bar{\mathcal{S}}_1$, we get that

$$b_\mu = \sum_{i \notin \{2,n\}} \alpha_i a_i = \beta_1 a_1 + \epsilon a_m + \beta_n a_n$$

for some m, $2 \leq m \leq n-1$, $\epsilon \in \{0,1\}$, and $\beta_1, \beta_n \in \mathbb{N}$ such that $\sum_{i \notin \{2,n\}} \alpha_i = \beta_1 + \epsilon + \beta_n$.

If $\epsilon = 0$ or $m \neq 2$, then a_2 is not involved in the expression $b_\mu = \beta_1 a_1 + \epsilon a_m + \beta_n a_n$, so $\beta_n = 0$ since $b_\mu \in \mathrm{Ap}_1$. Thus, $b_\mu = \mu a_1 + a_2 + a_n = \beta_1 a_1 + \epsilon a_m$, and hence

$$(\beta_1 - \mu) a_1 = a_2 + a_n - \epsilon a_m. \tag{2.4}$$

If $\epsilon = 0$, a_1 divides $a_2 + a_n = 2a_1 + ne$, and hence a_1 divides n which is impossible since $a_1 \geq n-1$ and $a_1 \neq n$. Now if $\epsilon = 1$ and $m \neq 2$, (2.4) implies that $(\beta_1 - \mu)a_1 = a_2 + a_n - a_m = a_1 + (n-m+1)e$, and hence $a_1 \mid n-m+1$, a contradiction since $a_1 \geq n-1 > n-m+1$. Thus, $\epsilon = 1$ and $m = 2$, i.e., $b_\mu = \sum_{i \notin \{2,n\}} \alpha_i a_i = \mu a_1 + a_2 + a_n = \beta_1 a_1 + a_2 + \beta_n a_n$.

Note that since $\beta_1 a_1 + a_2$ cannot be transformed into $\sum_{i \notin \{2,n\}} \alpha_i a_i$ using the relations $a_i + a_j = a_{i-1} + a_{j+1}$ in $\bar{\mathcal{S}}_1$, we have that $\beta_n \neq 0$. Moreover, $(\mu - \beta_1)a_1 = (\beta_n - 1)a_n$ and $\mu - \beta_1 < q + e + 1$ since $\mu \leq t \leq q + e$. By Lemma 2.21 (2), this implies that $\mu = \beta_1$ and $\beta_n = 1$, and we have shown that $\sum_{i \notin \{2,n\}} \alpha_i = \beta_1 + \beta_n + 1 = \mu + 2$.

- $3 \leq r \leq n-2$.

By (2.1) and (2.2), the conditions $a_1 \geq n-1$ and $r \leq a_1$ are necessary for $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ and $(\mathrm{Ap}_1, \leq_1)$ to be isomorphic, and by Lemma 2.22 (2), $\mathrm{Ap}_1 = (A \setminus \{a_r\}) \cup \{a_r + a_n\}$. Using Proposition 2.18, we get that $(\mathrm{Ap}_1, \leq_1)$ is graded if and only if all the factorizations of $a_r + a_n$ have the same the length, which is two since $a_r + a_n = a_{r+1} + a_{n-1}$.

Now, if $a_1 = n-1$, then $a_{r+1} + a_{n-1} = ea_1 + a_2 + a_{r-1}$, and if $r > a_1 - n + 1$, then $a_{r+1} + a_{n-1} = (2+e)a_1 + (r - a_1 + n - 2)e = (1+e)a_1 + a_{r-a_1+n-1}$. Thus, in both cases $(\mathrm{Ap}_1, \leq_1)$ is not graded. Conversely, assume that $a_1 \geq n$ and $r \leq a_1 - n + 1$. If $a_r + a_n = 2a_1 + (n+r-2)e$ can be written using more than 2 minimal generators of $\mathcal{S}_1$, then there exists $\mu \geq 3$ (the number of minimal generators involved), and $m \geq 0$, such that $a_r + a_n = \mu a_1 + me$. Then, $m \leq n + r - 3$ and a_1 divides $n + r - 2 - m$, a contradiction since $a_1 > n + r - 2 \geq n + r - 2 - m$.

- $r = n-1$.

By (2.1) and (2.2), we only have to show in this case that if $a_1 \geq n-2$, then $(\mathrm{Ap}_1, \leq_1)$ is graded, i.e., using Lemma 2.22 (3) and Proposition 2.18, that (2.3) holds for $b = a_{n-1} + (q+1)a_n$ when $n-1 \mid a_1$, and $b = a_{n-1} + qa_n$ otherwise.

Assume that $n-1$ does not divide a_1, and consider the element $b = a_{n-1} + qa_n$ in Ap_1. By Lemma 2.21 (1), there exists $j \in \{1, \ldots, n-2\}$ such that $b = (q+e)a_1 + a_j$, and hence we have to show that whenever $b = \sum_{i=1}^{n-2} \alpha_i a_i$ with $\alpha_i \in \mathbb{N}$, then $\sum_{i=1}^{n-2} \alpha_i = q + e + 1$. As in the case $r = 2$, using iteratively the equalities

$a_i + a_j = a_{i-1} + a_{j+1}$ in $\bar{\mathcal{S}}_1$, we get that

$$b = \sum_{i=1}^{n-2} \alpha_i a_i = \beta_1 a_1 + \epsilon a_m + \beta_n a_n$$

for some m, $2 \leq m \leq n-1$, $\epsilon \in \{0,1\}$, and $\beta_1, \beta_n \in \mathbb{N}$ such that $\sum_{i=1}^{n-2} \alpha_i = \beta_1+\epsilon+\beta_n$.

If $\beta_n > 0$, since $b \in \mathrm{Ap}_1$, we have that $b - a_n = \beta_1 a_1 + \epsilon a_m + (\beta_n - 1)a_n \notin \mathcal{S}_1$, and hence $\epsilon = 1$ and $m = n-1$, i.e., $b - a_n = \beta_1 a_1 + a_{n-1} + (\beta_n - 1)a_n$. But this is also equal to $(\beta_1 - 1)a_1 + a_2 + a_{n-2} + (\beta_n - 1)a_n$ so $\beta_1 = 0$ (otherwise $b - a_n \in \mathcal{S}_1$). Thus, $b = a_{n-1} + \beta_n a_n$, which cannot be transformed into $\sum_{i=1}^{n-2} \alpha_i a_i$ using the relations $a_i + a_j = a_{i-1} + a_{j+1}$ in $\bar{\mathcal{S}}_1$, a contradiction. This shows that $\beta_n = 0$.

Then $b = \beta_1 a_1 + \epsilon a_m = (q+e)a_1 + a_j$. Since $\{a_1, \ldots, a_{n-2}\} \subset \mathrm{MSG}(\mathrm{S}_1)$, we deduce that $\epsilon = 1$, $m = j$, and $\beta_1 = q + e$. Hence, $\sum_{i=1}^{n-2} \alpha_i = \beta_1 + \epsilon + \beta_n = q + e + 1$, and we are done in this case.

When $n-1$ divides a_1, consider $b = a_{n-1} + (q+1)a_n$ in Ap_1, and the relation $b = (q+e+1)a_1 + a_{n-1}$ given by Lemma 2.21 (1), and an analogue argument works. □

Example 2.24. For the arithmetic sequence $9 < 10 < 11 < 12 < 13$, the parameters are $a_1 = 9$, $e = 1$ and $n = 5$. By Proposition 2.18, the Betti sequences of $\Bbbk[\mathcal{C}_1]$ and $\Bbbk[\mathcal{C}]$ coincide. Indeed, it is $(1, 10, 20, 15, 4)$ for both curves. Now the Betti sequences of $\Bbbk[\pi_r(\mathcal{C}_1)]$ and $\Bbbk[\pi_r(\mathcal{C})]$ also coincide for all values of r, $1 \leq r \leq 5$: they coincide for $r = 1$ and 5 as observed before Lemma 2.21, and for $r = 2, 3, 4$ by Proposition 2.23. One can check that the sequence is $(1, 6, 8, 3)$ for $r = 1$, $(1, 5, 6, 2)$ for $r = 2$ and 4, $(1, 8, 12, 5)$ for $r = 3$, and $(1, 4, 5, 2)$ for $r = 5$.

Example 2.25. Consider the arithmetic sequence $9 < 10 < 11 < 12 < 13 < 14 < 15$, whose parameters are $a_1 = 9$, $e = 1$ and $n = 7$. By Proposition 2.18, the Betti sequences of $\Bbbk[\mathcal{C}_1]$ and $\Bbbk[\mathcal{C}]$ coincide, and one can check that it is $(1, 19, 58, 75, 44, 11, 2)$ for both the affine and the projective monomial curves. Using the notations in Proposition 2.23, one has that Ap_1 and $\mathrm{AP}_{\mathcal{S}}$ are isomorphic if and only if $r \in \{2, 3, 6\}$. Hence, the Betti sequences of $\Bbbk[\pi_r(\mathcal{C}_1)]$ and $\Bbbk[\pi_r(\mathcal{C})]$ coincide for those values of r by Theorem 2.12, and also for $r = 1$ and 7 as observed before Lemma 2.21. On the other hand, one can check using [24] that the Betti sequences of $\Bbbk[\pi_r(\mathcal{C}_1)]$ and $\Bbbk[\pi_r(\mathcal{C})]$ do not coincide for $r \in \{4, 5\}$. Table 2.1 shows the Betti sequences of $\Bbbk[\pi_r(\mathcal{C}_1)]$ and $\Bbbk[\pi_r(\mathcal{C})]$ for all r, $1 \leq r \leq 7$.

Table 2.1: Betti sequences in Example 2.25.

r	$\Bbbk[\pi_r(\mathcal{C}_1)]$	$\Bbbk[\pi_r(\mathcal{C})]$
1	$(1, 11, 30, 35, 19, 4)$	$(1, 11, 30, 35, 19, 4)$
2	$(1, 12, 25, 21, 10, 3)$	$(1, 12, 25, 21, 10, 3)$
3	$(1, 13, 30, 29, 14, 3)$	$(1, 13, 30, 29, 14, 3)$
4	$(1, 12, 27, 27, 14, 3)$	$(1, 12, 29, 29, 14, 3)$
5	$(1, 12, 25, 21, 10, 3)$	$(1, 13, 30, 29, 14, 3)$
6	$(1, 12, 25, 21, 10, 3)$	$(1, 12, 25, 21, 10, 3)$
7	$(1, 12, 25, 25, 14, 3)$	$(1, 12, 25, 25, 14, 3)$

2.3 Improving Vu's bound for the equality of the Betti numbers

Take a sequence of nonnegative integers $0 = c_1 < \cdots < c_n$, not necessarily relatively prime, and consider, for all $j > 0$, the shifted set of integers $\mathcal{A}_1^j = \{c_1+j, \ldots, c_n+j\}$, and the semigroup $\mathcal{S}_1^j$ generated by the sequence $a_0 := 0 < a_1 := c_1 + j < \cdots < a_n := c_n + j$. Herzog and Srinivasan conjectured that the Betti numbers of $\Bbbk[\mathcal{S}_1^j]$ eventually become periodic with period c_n. In [97], Vu provides a proof of this conjecture together with an explicit value N such that this periodic behavior occurs for all $j > N$. One of the key steps in Vu's argument is [97, Thm. 5.7] where he proves that, for all $j > N$, the Betti numbers of the affine and projective monomial curves defined by $c_1+j < \cdots < c_n+j$ coincide. In the following theorem we provide a smaller value of N such that this occurs.

Theorem 2.26. *Let $0 = c_1 < \cdots < c_n$ be a sequence of nonnegative integers and set $N := (c_n - 1)(\sum_{i=2}^{n-1} c_i)$. Then, for all $j \geq N$, the affine and projective monomial curves defined by the sequence $a_0 = 0 < a_1 = c_1 + j < \cdots < a_n = c_n + j$ have the same Betti numbers.*

Proof. Take $j \geq N$. Let $\mathcal{G}_>^j$ be the reduced Gröbner basis of $I_{\mathcal{A}_1^j}$ with respect to the degrevlex order with $x_1 > \cdots > x_n$, and consider $f = \mathbf{x}^\alpha - \mathbf{x}^\beta \in \mathcal{G}_>^j$ with $\mathbf{x}^\alpha > \mathbf{x}^\beta$. If we show that

(i) x_n does not divide $\mathbf{x}^\alpha$, and

(ii) if f is not homogeneous, then x_n divides $\mathbf{x}^\beta$,

then the result follows from Theorem 2.1. Note that this result is true even if the generators of the semigroup are not relatively prime since the defining ideal does not change when we divide them by a common divisor.

If x_n divides $\mathbf{x}^\alpha$, then x_n does not divide $\mathbf{x}^\beta$, and hence $|\alpha|> |\beta|$. Thus,

$$N = (c_n - 1)(\sum_{i=2}^{n-1} c_i) \leq j \leq (|\alpha|-|\beta|)j < \sum_{i=1}^{n}(\alpha_i - \beta_i)j + \sum_{i=2}^{n} \alpha_i c_i = \sum_{i=2}^{n-1} \beta_i c_i\,.$$

This implies that there exists $i \in \{2,\ldots,n-1\}$ such that $\beta_i \geq c_n$. If we consider the monomial $\mathbf{x}^\gamma := \frac{\mathbf{x}^\beta x_1^{c_n - c_i} x_n^{c_i}}{x_i^{c_n}}$, then the homogeneous binomial $g = \mathbf{x}^\beta - \mathbf{x}^\gamma$ belongs to $I_{\mathcal{A}_1^j}$ because the homogeneous binomial $x_i^{c_n} - x_1^{c_n - c_i} x_n^{c_i}$ belongs to $I_{\mathcal{A}_1^j}$. As x_n divides $\mathbf{x}^\gamma$ and does not divide $\mathbf{x}^\beta$, $\mathrm{in}(g) = \mathbf{x}^\beta \in \mathrm{in}(I_{\mathcal{A}_1^j})$, a contradiction because $\mathcal{G}_>^j$ is reduced and $f \in \mathcal{G}_>^j$. This shows that that x_n does not divide $\mathbf{x}^\alpha$, and (i) is proved.

Now assume that f is not homogeneous, i.e., $|\alpha|> |\beta|$, and that x_n does not divide $\mathbf{x}^\beta$. By (i), x_n does not divide $\mathbf{x}^\alpha$ either, and hence

$$N = (c_n - 1)(\sum_{i=2}^{n-1} c_i) \leq j \leq (|\alpha|-|\beta|)j < \sum_{i=1}^{n-1}(\alpha_i - \beta_i)j + \sum_{i=2}^{n-1} \alpha_i c_i = \sum_{i=2}^{n-1} \beta_i c_i\,.$$

Thus, there exists $i \in \{2,\ldots,n-1\}$ such that $\beta_i \geq c_n$. Using exactly the same argument as before for (i), we get a contradiction, and hence (ii) is proved. □

Corollary 2.27. *Let $a_1 < \cdots < a_n$ be a sequence of positive integers, and set $M := a_n + (a_n - 1)(\sum_{i=1}^{n-1}(a_n - a_i))$. Then, for all $j \geq M$, the projective monomial curve defined by the sequence $a_1 < \cdots < a_n < j$ is arithmetically Cohen-Macaulay.*

Proof. Consider the sequence $b_0 := 0 < b_1 := a_n - a_{n-1} < \cdots < b_{n-1} := a_n - a_1 < b_n := a_n$. By Theorem 2.26, one has that the projective monomial curve defined by $l < l + b_1 < \cdots < l + b_n$ is arithmetically Cohen-Macaulay for all $l \geq (b_n - 1)(\sum_{i=1}^{n-1} b_i) = (a_n - 1)(\sum_{i=1}^{n-1}(a_n - a_i))$. To finish the proof, it suffices to observe that the dual sequence of $0 < a_1 < \cdots < a_n < l+a_n$ is $l < l+b_1 < \cdots < l+b_n$ and take $l + a_n = j$. □

2.4 Construction of Gorenstein projective monomial curves

Since $\beta_i(\Bbbk[\mathcal{C}]) \geq \max\left(\beta_i(\Bbbk[\mathcal{C}_1]), \beta_i(\Bbbk[\mathcal{C}_2])\right)$ for all i, whenever $\Bbbk[\mathcal{C}]$ is Gorenstein, then so are $\Bbbk[\mathcal{C}_1]$ and $\Bbbk[\mathcal{C}_2]$. The converse of this statement is false; indeed, it could happen that $\mathcal{C}_1$ and $\mathcal{C}_2$ are both arithmetically Gorenstein and that $\mathcal{C}$ is not even arithmetically Cohen-Macaulay, as can be seen in Example 2.28 (1). Actually, even if $\Bbbk[\mathcal{C}]$ is Cohen-Macaulay, it may happen that $\Bbbk[\mathcal{C}]$ is not Gorenstein, as Example 2.28 (2) shows.

Example 2.28. (1) The affine monomial curve $\mathcal{C}_1$ defined by the sequence $4 < 9 < 10$ is an (ideal-theoretic) complete intersection and, thus, $\Bbbk[\mathcal{C}_1]$ is Gorenstein with Betti sequence $(1, 2, 1)$. The corresponding projective monomial curve is not arithmetically Cohen-Macaulay, indeed, the Betti sequence of $\Bbbk[\mathcal{C}]$ is $(1, 5, 6, 2)$.

(2) The affine monomial curve $\mathcal{C}_1$ defined by the sequence $10 < 14 < 15 < 21$ is an (ideal-theoretic) complete intersection and, thus, $\Bbbk[\mathcal{C}_1]$ is Gorenstein with Betti sequence $(1, 3, 3, 1)$. The corresponding projective monomial curve is arithmetically Cohen-Macaulay but not Gorenstein, indeed, the Betti sequence of $\Bbbk[\mathcal{C}]$ is $(1, 4, 5, 2)$.

Recall from Proposition 1.61 that $\Bbbk[\mathcal{C}_1]$ is Gorenstein if and only if $\mathcal{S}_1$ is symmetric. In this section we show how to construct an arithmetically Gorenstein projective monomial curve from a symmetric numerical semigroup $\mathcal{T}$. We begin with the following result, which provides a necessary and sufficient condition for $\mathcal{C}$ to be arithmetically Gorenstein and is a consequence of the results in [15].

Proposition 2.29. *Let $\mathcal{C}$ be the projective monomial curve defined by the sequence $a_0 = 0 < a_1 < \cdots < a_n = D$ of relatively prime integers. Then, $\mathcal{C}$ is arithmetically Gorenstein if and only if $\mathcal{C}$ is arithmetically Cohen-Macaulay, both $\mathcal{S}_1$ and $\mathcal{S}_2$ are symmetric, and D divides $F(\mathcal{S}_1) + F(\mathcal{S}_2)$.*

Proof. ($\Rightarrow$) If $\mathcal{C}$ is arithmetically Gorenstein, $\mathcal{C}$ is arithmetically Cohen-Macaulay and both $\mathcal{S}_1$ and $\mathcal{S}_2$ are symmetric by Proposition 1.61 (2). Assume now that D does not divide $F(\mathcal{S}_1) + F(\mathcal{S}_2)$. By Proposition 2.4 (c), there exist $x \in \mathcal{S}_2$ and $y \in \mathcal{S}_1$ such that $(x, F(\mathcal{S}_1) + D)$ and $(F(\mathcal{S}_2) + D, y)$ are two different elements of $\mathrm{AP}_{\mathcal{S}}$. Moreover, by Proposition 2.7, they are both maximal in the poset $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$, and hence, $\mathcal{C}$ is not arithmetically Gorenstein by Theorem 1.63.
($\Leftarrow$) If D divides $F(\mathcal{S}_1)+F(\mathcal{S}_2)$, then by Proposition 2.4 (c), $(F(\mathcal{S}_2)+D, F(\mathcal{S}_1)+D) \in \mathrm{AP}_{\mathcal{S}}$ and by Proposition 2.7, this element is the maximum of $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$. Hence, $\mathcal{C}$ is arithmetically Gorenstein by Theorem 1.63. □

Note that in the previous result, one cannot remove the condition of $\Bbbk[\mathcal{C}]$ being Cohen-Macaulay as Example 2.30 shows.

Example 2.30. For the sequence $6 < 7 < 8 < 15 < 16$, one has that the numerical semigroup $\mathcal{S}_1 = \langle 6, 7, 8, 15, 16 \rangle$ is symmetric, and $\mathcal{S}_2 = \mathbb{N}$ is also symmetric. Moreover, $F(\mathcal{S}_1) = 17$ and $F(\mathcal{S}_2) = -1$, so $D = 16$ divides $F(\mathcal{S}_1) + F(\mathcal{S}_2)$. But $\Bbbk[\mathcal{C}]$ is not Cohen-Macaulay, so it cannot be Gorenstein either.

The following example provides a family of arithmetically Gorenstein projective curves. This example gives some insights on Theorem 2.32, which is the main result of this section and shows how to construct a projective Gorenstein curve from a symmetric numerical semigroup.

Example 2.31. If $m > 3$ is an odd integer, one has that

$$\mathcal{S}_1 = \langle (m+1)/2, \dots, m-1 \rangle = \{0, (m+1)/2, \dots, m-1, m+1, \rightarrow\}$$

is a symmetric numerical semigroup with $F(\mathcal{S}_1) = m$. Hence the ring $\Bbbk[\mathcal{C}_1]$ is Gorenstein. The sequence $\frac{m+1}{2} < \cdots < m-1$ defines a projective curve of degree $D = m - 1 = F(\mathcal{S}_1) - 1$. We claim that $\Bbbk[\mathcal{C}]$ is Gorenstein. Note that $\text{Ap}_1 = \{0\} \cup [\frac{m+1}{2}, m-2] \cup [m+1, \frac{3}{2}(m-1)] \cup \{2m-1\}$. Since $\mathcal{S}_2 = \mathbb{N}$, we have that $F(\mathcal{S}_2) = -1$ and $\text{Ap}_2 = [0, m-1]$. By Proposition 2.29, it only remains to check that $\Bbbk[\mathcal{C}]$ is Cohen-Macaulay. By Proposition 2.4 (b), $\Bbbk[\mathcal{C}]$ is Cohen-Macaulay because $B \subset \mathcal{S}$, where $B \subset \mathbb{N}^2$ is the following set with D elements:

$$\{(0,0)\} \cup \{(D-a, a) \mid \frac{m+1}{2} \le a \le m-2\} \\ \cup \{(D-g, D+g) \mid 1 < g < \frac{m+1}{2}\} \cup \{(D-1, 2D+1)\}\,.$$

We now generalize this to any symmetric numerical semigroup $\mathcal{T}$ such that $\mathcal{T} \neq \mathbb{N}$ and $\mathcal{T} \neq \langle 2, a \rangle$ for some a odd or, in other words, such that $2 \notin \mathcal{T}$. The idea under this construction is to consider the projective closure of the affine monomial curve parametrized by the so-called *small elements* in the semigroup, that is, all the elements in the numerical semigroup that are smaller than the Frobenius number. The precise statement of the result is the following.

Theorem 2.32. *Let $\mathcal{T} \subseteq \mathbb{N}$ be a symmetric numerical semigroup such that $2 \notin \mathcal{T}$ and consider $\mathcal{T} \cap [0, F(\mathcal{T}) - 1] = \{0, a_1, \dots, a_n\}$ with $0 < a_1 < \cdots < a_n$. Then, the projective monomial curve defined by the sequence $a_1 < \cdots < a_n$ is arithmetically Gorenstein.*

To prove this theorem we use the following two lemmas.

Lemma 2.33. *Let $\mathcal{T} \subset \mathbb{N}$ be a symmetric numerical semigroup and consider $a_1 < \cdots < a_n$ its minimal set of generators. If $2 \notin \mathcal{T}$, then $a_n < F(\mathcal{T})$.*

Proof. We prove that every $y \in \mathcal{T}$ such that $y > F(\mathcal{T})$ can be written as $y = z_1 + z_2$ with $z_1, z_2 \in \mathcal{T} \setminus \{0\}$ and, hence, $y \notin \text{MSG}(\mathcal{T})$.

- For $y = F(\mathcal{T}) + 1$, we take $z_1 = a_1 \in \mathcal{T}$ and $z_2 = F(\mathcal{T}) - a_1 + 1$. We have that $z_2 \in \mathcal{T}$ because $F(\mathcal{T}) - z_2 = a_1 - 1 \notin \mathcal{T}$ and $\mathcal{T}$ is symmetric.
- For $y = F(\mathcal{T}) + 2$, we take $z_1 = a_1 \in \mathcal{T}$ and $z_2 = F(\mathcal{T}) - a_1 + 2$. We have that $z_2 \in \mathcal{T}$ because $F(\mathcal{T}) - z_2 = a_1 - 2 \notin \mathcal{T}$ (because $a_1 > 2$) and $\mathcal{T}$ is symmetric.

- For $y = F(\mathcal{T}) + 3$. If $y/2 \in \mathcal{T}$, we take $z_1 = z_2 = y/2$. Otherwise, we observe that

$$|[1, y-1] \cap \mathcal{T}| = |[1, F(\mathcal{T})] \cap \mathcal{T}| + y - F(\mathcal{T}) - 1 = y - \frac{F(\mathcal{T}) + 3}{2} = \frac{y}{2}.$$

Thus, there exists $1 \leq i < y/2$ such that $i, y - i \in \mathcal{T}$ and we are done.

- For $y > F(\mathcal{T}) + 3$, we observe that

$$|[1, y-1] \cap \mathcal{T}| = |[1, F(\mathcal{T})] \cap \mathcal{T}| + y - F(\mathcal{T}) - 1 = y - \frac{F(\mathcal{T}) + 3}{2} > \frac{y}{2}.$$

Thus there exists $1 \leq i \leq y/2$ such that $i, y - i \in \mathcal{T}$ and we are done.

□

Lemma 2.34. *Let $\mathcal{S}_1 = \langle a_1, \ldots, a_n \rangle \subsetneq \mathbb{N}$ be a numerical semigroup with $a_1 < \cdots < a_n$, and set $a := \min\{b \in \mathcal{S}_1 : a_1 \nmid b\}$. If $y \in \mathbb{N}$ satisfies that $y + i \notin \mathcal{S}$ for all $i \in \{0, \ldots, a-1\}$ such that $a_1 \nmid i$, then $y = 0$.*

Proof. Since $y + 1, \ldots, y + a_1 - 1 \notin \mathcal{S}_1$, we deduce that a_1 divides y, so $y \in \mathcal{S}_1$. Moreover, $a - a_1$ is not a multiple of a_1, so $y + a - a_1 \notin \mathcal{S}_1$ and $y + a - a_1 \equiv a \pmod{a_1}$. Thus, we get that $y + a - a_1 \leq a - a_1$, and hence $y = 0$. □

Proof of Theorem 2.32. Since $\mathcal{T}$ is symmetric and $2 \notin \mathcal{T}$, then by Lemma 2.33 we have that $\mathrm{MSG}(\mathcal{T}) \subset \{a_1, \ldots, a_n\}$. Hence, $\mathcal{S}_1 = \mathcal{T}$ and $\mathcal{S}_1$ is symmetric. Moreover, since $1, 2 \notin \mathcal{S}_1$, then $D = a_n = F(\mathcal{S}_1) - 1$ and $a_{n-1} = F(\mathcal{S}_1) - 2$. Thus $\mathcal{S}_2 = \mathbb{N}$ and we get that $F(\mathcal{S}_2) = -1$ and $F(\mathcal{S}_1) + F(\mathcal{S}_2) = D$. By Proposition 2.29, it is enough to prove that $\mathcal{C}$ is arithmetically Cohen-Macaulay to conclude that it is arithmetically Gorenstein.

One can easily check that $\mathrm{Ap}_1 = \{a \in \mathcal{S}_1 \,|\, 0 \leq a < D\} \cup \{g + D \,|\, g \notin \mathcal{S}_1, 1 < g < D\} \cup \{2D + 1\}$, and $\mathrm{Ap}_2 = \{0, 1, \ldots, D - 1\}$. Consider now the following set $B \subset \mathbb{N}^2$ with D elements:

$$B = \{(0,0)\} \cup \{(D - a, a) \,|\, a \in \mathcal{S}_1, 1 < a < D\}$$
$$\cup \{(D - g, D + g) \,|\, g \notin \mathcal{S}_1, 1 < g < D\} \cup \{(D - 1, 2D + 1)\}.$$

By Proposition 2.4 (b), $\mathcal{C}$ is arithmetically Cohen-Macaulay if and only if $B \subset \mathcal{S}$, and in this case $\mathrm{AP}_{\mathcal{S}} = B$. Let us prove that $B \subset \mathcal{S}$. Clearly $(0,0) \in \mathcal{S}$ and $\{(D - a, a) \,|\, a \in \mathcal{S}_1, 0 < a < D\} = \{(D - a_i, a_i) \,|\, 1 \leq i < n\} \subset \mathcal{S}$, and one has to show that $(D - g, D + g) \in \mathcal{S}$ for all $g \notin \mathcal{S}_1, 1 < g < D$ and $(D - 1, 2D + 1) \in \mathcal{S}$. Let $a \in \mathcal{S}_1$ be the minimum element in $\mathcal{S}_1$ which is not a multiple of a_1. We distinguish between two cases.

(1) $D > g > F(\mathcal{S}_1) - a = D + 1 - a$. We claim that $g + 1 \in \mathcal{S}_1$. Otherwise, by the symmetry of $\mathcal{S}_1$ one has that $F(\mathcal{S}_1) - g$ and $F(\mathcal{S}_1) - g - 1$ are two consecutive elements of $\mathcal{S}_1$ which are both smaller than a, and this is not possible. Then, $(1, D-1), (D-g-1, g+1) \in \mathcal{S}$ and we get that $(D-g, D+g) = (1, D-1) + (D-g-1, g+1) \in \mathcal{S}$.

(2) $1 < g \leq F(\mathcal{S}_1) - a = D + 1 - a$. We claim that there exists $j \in \{0, \ldots, a-1\}$ such that both $D+1-j$ and $g-1+j$ belong to $\mathcal{S}_1$. Assume by contradiction that this statement does not hold. Whenever $j \in \{0, \ldots, a-1\}$ is not a multiple of a_1, we have that $j \notin \mathcal{S}_1$ and, by the symmetry of $\mathcal{S}_1$, $F(\mathcal{S}_1) - j = D + 1 - j \in \mathcal{S}_1$ and hence $g - 1 + j \notin \mathcal{S}_1$. By Lemma 2.34, this means that $g = 1$, a contradiction. Now, we take $j \in \{0, \ldots, a-1\}$ such that $D+1-j, g-1+j \in \mathcal{S}_1$ (clearly $j \neq 0$ because $D + 1 = F(\mathcal{S}_1) \notin \mathcal{S}_1$). Then $(D+1-g-j, g-1+j), (j-1, D+1-j) \in \mathcal{S}$, and hence $(D-g, D+g) = (D+1-g-j, g-1+j) + (j-1, D+1-j) \in \mathcal{S}$.

Finally, taking any $g \notin \mathcal{S}_1$, $1 < g < D$, we have that $F(\mathcal{S}_1) - g = D + 1 - g \in \mathcal{S}_1$. Thus, $(D-1, 2D+1) = (D-g, D+g) + (g-1, D+1-g) \in \mathcal{S}$. □

Following the construction in Theorem 2.32, one gets an arithmetically Gorenstein projective curve $\mathcal{C}$. However, the Betti numbers of $\Bbbk[\mathcal{C}_1]$ and $\Bbbk[\mathcal{C}]$ can be very different, as the following example shows.

Example 2.35. Consider the symmetric numerical semigroup $\mathcal{T} = \langle 4, 9, 10 \rangle$. One has that the Frobenius number of $\mathcal{T}$ is $F(\mathcal{T}) = 15$ and, hence, $\mathcal{T} \cap [0, 14] = \{0, 4, 8, 9, 10, 12, 13, 14\}$. By Theorem 2.32 we have that the projective monomial curve defined by the sequence $4 < 8 < 9 < 10 < 12 < 13 < 14$ is Gorenstein. A computation with [24] shows that the Betti sequence of $\Bbbk[\mathcal{C}_1]$ is $(1, 6, 15, 20, 15, 6, 1)$, while the Betti sequence of $\Bbbk[\mathcal{C}]$ is $(1, 15, 39, 50, 39, 15, 1)$.

2.5 The Betti numbers of Kunz–Waldi semigroups

In this last section, we compute the Betti numbers of affine monomial curves coming from a class of numerical semigroups defined by Kunz and Waldi in [61]. This class of semigroups has been studied later in [62] and [86].

2.5.1 Definition of the KW class

Let $3 \leq p < q$ be two relatively prime integers and consider the numerical semigroup $\langle p, q \rangle$, which is symmetric and has Frobenius number $F_{pq} = pq - p - q$. Therefore, all the gaps of $\langle p, q\rangle$ are of the form $pq - up - vq$ for $u, v \in \mathbb{Z}_{>0}$, with F_{pq} being the largest. If we associate with $pq - up - vq$ the point $(u-1, v-1) \in \mathbb{N}^2$, then the gaps of $\langle p, q\rangle$ are in one-to-one correspondence with the lattice points in $\mathbb{N}^2$ below the line $p(U+1) + q(V+1) = pq$.

Example 2.36. Consider the semigroup $\langle p, q\rangle$ for $p = 5$ and $q = 8$. The gaps of this semigroup are shown in Figure 2.4. Note that they are in one-to-one correspondence with the lattice points below the line $5(U+1) + 8(V+1) = 40$.

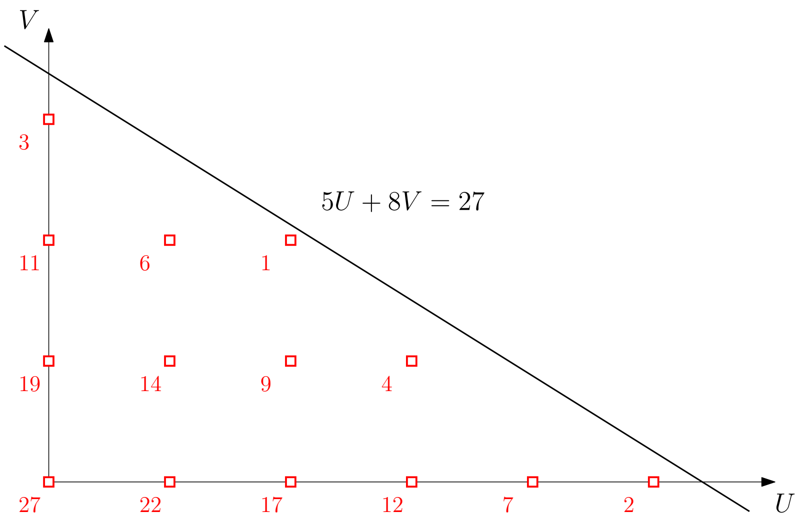


Figure 2.4: Gaps of the semigroup $\langle 5, 8\rangle$ in Example 2.36.

In their paper [61], Kunz and Waldi build numerical semigroups of the same multiplicity p by filling in some gaps of $\langle p, q\rangle$ in a sistematic way, such that the type of the resulting semigroups is one less than their embedding dimension.

Definition 2.37. Let $3 \leq p < q$ be two relatively prime integers. The class of Kunz–Waldi semigroups associated to $p < q$, $KW(p, q)$, is the set of all numerical semigroups $\mathcal{S}_1$, such that $\langle p, q\rangle \subsetneq \mathcal{S}_1 \subset \langle p, q, r\rangle$, where

$$r = \begin{cases} p/2 & \text{if } p \text{ is even,} \\ q/2 & \text{if } q \text{ is even, and} \\ (p+q)/2 & \text{otherwise.} \end{cases}$$

Proposition 2.38 ([62, Cor. 3.1, Ex. 4.6])**.** *Let $\mathcal{S}_1 \in KW(p, q)$ of embedding dimension n, $n \geq 4$. Then, the type of $\mathcal{S}_1$ is $t(\mathcal{S}_1) = n - 1$.*

Using the terminology of numerical semigroups, the semigroups in $KW(p,q)$ are obtained from $\langle p,q\rangle$ by closing only gaps from the *fractional ideal* $r+p\mathbb{N}+q\mathbb{N}$.

By [61, p. 673], the semigroups of $KW(p,q)$ are in one-to-one correspondence to the lattice paths, with right and downward steps, in the rectangle $\mathbf{R}\subset\mathbb{R}^2$ with vertices $(0,0)$, $(0,p'-1)$, $(q'-1,p'-1)$, and $(q'-1,0)$, where $p'=\lfloor p/2\rfloor$ and $q'=\lfloor q/2\rfloor$.

Proposition 2.39 ([86, Rem. 1])**.** *Let $\mathcal{S}_1=\langle p,q,a_1,\ldots,a_{n-2}\rangle$ be a numerical semigroup of embedding dimension n, $n\geq 3$. Then, $\mathcal{S}_1\in KW(p,q)$ if and only if there exist natural numbers $0<u_1<\cdots<u_{n-2}\leq q/2$ and $p/2\geq v_1>\cdots>v_{n-2}>0$, such that $a_i=pq-u_ip-v_iq$ for $1\leq i\leq n-2$.*

For a semigroup $\mathcal{S}_1=\langle p,q,a_1,\ldots,a_{n-2}\rangle\in KW(p,q)$, with $a_i=pq-u_ip-v_iq$, the corresponding lattice path in the rectangle $\mathbf{R}$ has vertices (u_i-1,v_i-1), $1\leq i\leq n-2$, and one has that

$$\mathcal{S}_1=\langle p,q\rangle\cup\{pq-up-vq \mid u\leq u_i, v\leq v_i \text{ for some } 1\leq i\leq n-2\}\,. \tag{2.5}$$

This explains the correspondence between the lattice paths ℓ with right and downward steps in $\mathbf{R}$ and the semigroups $\mathcal{S}_1\in KW(p,q)$.

Example 2.40. Consider the semigroup $\mathcal{S}_1=\langle 5,8,9,12\rangle$. One has that $\mathcal{S}_1\in KW(5,8)$, since $\langle 5,8\rangle\subsetneq\mathcal{S}_1\subset\langle 4,5\rangle$. Moreover, $9=40-5u_1-8v_1$ and $12=40-5u_2-8v_2$ for $u_1=3<u_2=4$ and $v_1=2>v_2=1$. Hence, the lattice path ℓ defining $\mathcal{S}$ has vertices $(2,1)$ and $(3,0)$. This is shown in Figure 2.5.

In their paper [62], Kunz and Waldi characterize the toric ideal of any semigroup in $KW(p,q)$. Let $\mathcal{S}_1=\langle p,q,a_1,\ldots,a_{n-2}\rangle\in KW(p,q)$ and fix a field $\Bbbk$. Consider the polynomial ring $R:=\Bbbk[x,y,x_1,\ldots,x_{n-2}]$, graded via $\deg(x)=p$, $\deg(y)=q$, and $\deg(x_i)=a_i$, $i=1,\ldots,n-2$. Set $\mathcal{A}_1=\{p,q,a_1,\ldots,a_{n-2}\}$, and let $I_{\mathcal{A}_1}\subset R$ be the toric ideal determined by $\mathcal{A}_1$.

Proposition 2.41 ([62, App. A])**.** *The toric ideal $I_{\mathcal{A}_1}$ is minimally generated by the $\binom{n}{2}$ $\mathcal{S}_1$-graded homogeneous binomials*

$$\begin{aligned}
f_{ij}&=x_ix_j-x^{q-u_i-u_j}y^{p-v_i-v_j}, && 1\leq i\leq j\leq n-2\\
g_i&=y^{v_i-v_{i+1}}x_i-x^{u_{i+1}-u_i}x_{i+1}, && 1\leq i\leq n-3\\
\eta_1&=y^{p-v_1}-x^{u_1}x_1\\
\eta_2&=y^{v_{n-2}}x_{n-2}-x^{q-u_{n-2}}.
\end{aligned}$$

We now show how to construct the Apéry set (with respect to the multiplicity, p) and poset of any semigroup $\mathcal{S}_1\in KW(p,q)$ from the lattice path defining $\mathcal{S}_1$.

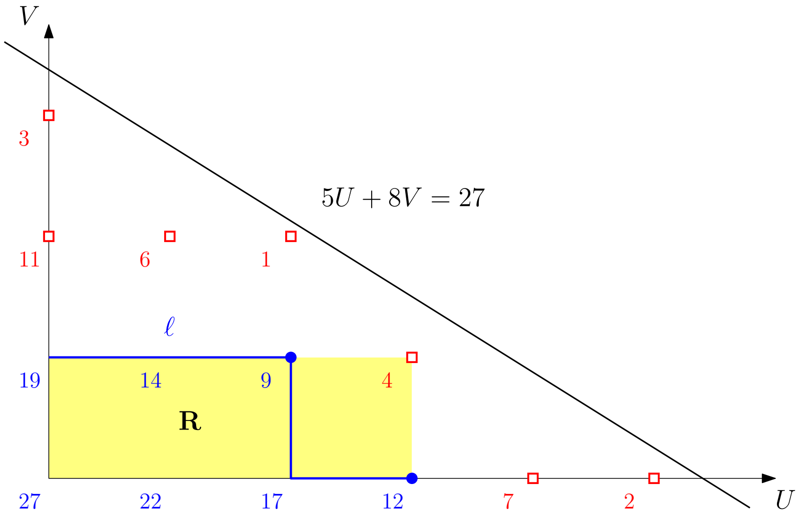


Figure 2.5: Lattice path corresponding to the semigroup $\langle 5, 8, 9, 12\rangle \in KW(5, 8)$ in Example 2.40.

Proposition 2.42. *Let $\mathcal{S}_1 = \langle p, q, a_1, \ldots, a_{n-2}\rangle \in KW(p, q)$ be the semigroup defined by the sequences $u_1 < u_2 < \cdots < u_{n-2} \le q/2$ and $p/2 \ge v_1 > v_2 > \cdots > v_{n-2}$, and set $v_{n-1} := 0$.*

(1) The Apéry set of $\mathcal{S}_1$ (with respect to p) is

$$\mathrm{Ap}(\mathcal{S}_1) = \{\lambda q \mid 0 \le \lambda < p - v_1\} \cup \left(\cup_{i=1}^{n-2}\{a_i + \lambda q \mid 0 \le \lambda < v_i - v_{i+1}\}\right) .$$

(2) $\mathrm{Ap}(\mathcal{S}_1)$ is graded, i.e., for all $z \in \mathrm{Ap}(\mathcal{S}_1)$, all the factorizations of z have the same length.

Proof. Let A_{pq} be the Apéry set of the numerical semigroup $\langle p, q\rangle$, $A_{pq} = \{0, q, 2q, \ldots, (p-1)q\}$. We can obtain the Apéry set $\mathrm{Ap}(\mathcal{S}_1)$ from A_{pq} as follows. Recall from Equation (2.5) that

$$\mathcal{S}_1 = \langle p, q\rangle \cup \{pq - up - vq \mid u \le u_i, v \le v_i \text{ for some } 1 \le i \le n-2\} .$$

Note that $pq - up - vq \equiv -vq \pmod p$, so the elements that we have to replace in A_{pq} are the ones congruent to $-vq$ modulo p for $0 < v \le v_1$. For each one of these congruence classes, we choose the smallest element in $\mathcal{S}_1$, i.e., the one with the largest u. This corresponds to the element in the lattice path defining $\mathcal{S}_1$ whose second coordinate is $v - 1$. Therefore,

$$\mathrm{Ap}(\mathcal{S}_1) = \{\lambda q \mid 0 \le \lambda < p - v_1\} \cup \left(\cup_{i=1}^{n-3}\{a_i + \lambda q \mid 0 \le \lambda < v_i - v_{i+1}\}\right) \\ \cup \{a_{n-2} + \lambda q \mid 0 \le \lambda < v_{n-2}\} .$$

By Remark 2.17 (2) and Proposition 2.41, $\mathrm{Ap}(\mathcal{S}_1)$ is graded since the variable x appears in all non-homogeneous binomials of a minimal generating set of the toric ideal $I_{\mathcal{A}_1}$. □

Using the previous result, let us show what the Apéry poset of $\mathcal{S}_1$, $\mathcal{P}(\mathcal{S}_1)$, looks like (see Def. 1.10). For $0 \leq \lambda < p - v_1$ set $i_{0,\lambda}$ the label of λq in $\mathcal{P}(\mathcal{S}_1)$, i.e. $0 \leq i_{0,\lambda} < p$ and $i_{0,\lambda} \equiv \lambda q \pmod p$. Similarly, for each $1 \leq j \leq n-2$ and $0 \leq \lambda < v_j - v_{j+1}$, let $i_{j,\lambda}$ be the label of $a_j + \lambda q$ in $\mathcal{P}(\mathcal{S}_1)$. Note that $\mathbb{Z}_p = \{i_{0,\lambda} \mid 0 \leq \lambda < p - v_1\} \cup \{i_{j,\lambda} \mid 1 \leq j \leq n-2, 0 \leq \lambda < v_j - v_{j+1}\}$.

Proposition 2.43. *Let $3 \leq p < q$ be relatively prime and $\mathcal{S}_1 \in KW(p,q)$.*

(1) The covering relations in $\mathcal{P}(\mathcal{S}_1)$ are the following. For all $i_{j_1,k_1}, i_{j_2,k_2} \in \mathbb{Z}_p$,

$$i_{j_1,k_1} \prec\!\cdot\; i_{j_2,k_2} \Leftrightarrow \begin{cases} j_2 = j_1 & \text{and} \quad k_2 = k_1 + 1, \text{ or} \\ j_1 = 0 & \text{and} \quad k_2 = k_1 \,. \end{cases}$$

Thus, the Hasse diagram of $\mathcal{P}(\mathcal{S}_1)$ is as shown in Figure 2.6.

(2) The poset $\mathcal{P}(\mathcal{S}_1)$ is graded for the rank function $\rho : \mathcal{P}(\mathcal{S}_1) \to \mathbb{N}$ defined by $\rho(i_{j,k}) = j + k$.

Proof. Note that $(\lambda+1)q - \lambda q = q$, $(a_j + (\lambda+1)q) - (a_j + \lambda q) = q$, $(a_j + \lambda q) - \lambda q = a_j$, so $i_{j,\lambda} \prec\!\cdot\; i_{j,\lambda+1}$ and $i_{0,\lambda} \prec\!\cdot\; i_{j,\lambda}$ for all j, λ by Proposition 1.11. Let us prove that there are no more covering relations. By Propositions 2.42 (2) and 1.11, it suffices to show that $(a_i + (\lambda+1)q) - (a_j + \lambda q) = a_i - a_j + q$ is not a minimal generator of $\mathcal{S}_1$ when $i \neq j$ and $\lambda \geq 0$. Note that $a_i - a_j + q \neq q$ since $i \neq j$, and $a_i - a_j + q \neq p$ because $a_i + q, a_j \in \mathrm{Ap}(\mathcal{S}_1)$. Now suppose that $a_i - a_j + q = a_k$ for some $1 \leq k \leq n-2$, $k \neq i, j$. Then

$$(u_j + u_k - u_i)p + (v_j + v_k - v_i + 1)q = pq\,.$$

Thus, q divides $u_j + u_k - u_i$ and since $3 - q/2 \leq u_j + u_k - u_i \leq q - 2$, then $u_j + u_k - u_i = 0$, so $u_i > u_j$ and hence $i > j$. With a similar argument, one can prove that $v_i = v_j + v_k + 1 > v_j$, so $i < j$, a contradiction. This completes the proof of part (1), and part (2) is a direct consequence of part (1) and Proposition 2.42 (2). □

Finally, we compute the set of Pseudo Frobenius elements for any $\mathcal{S}_1 \in KW(p,q)$.

Proposition 2.44. *Let $\mathcal{S}_1 \in KW(p,q)$ of embedding dimension n. Then* $\mathrm{PF}(\mathcal{S}_1) = \{g_i := pq - (u_i+1)p - (v_{i+1}+1)q \mid 0 \leq i \leq n-2\}$, *where $u_0 = v_{n-1} = 0$.*

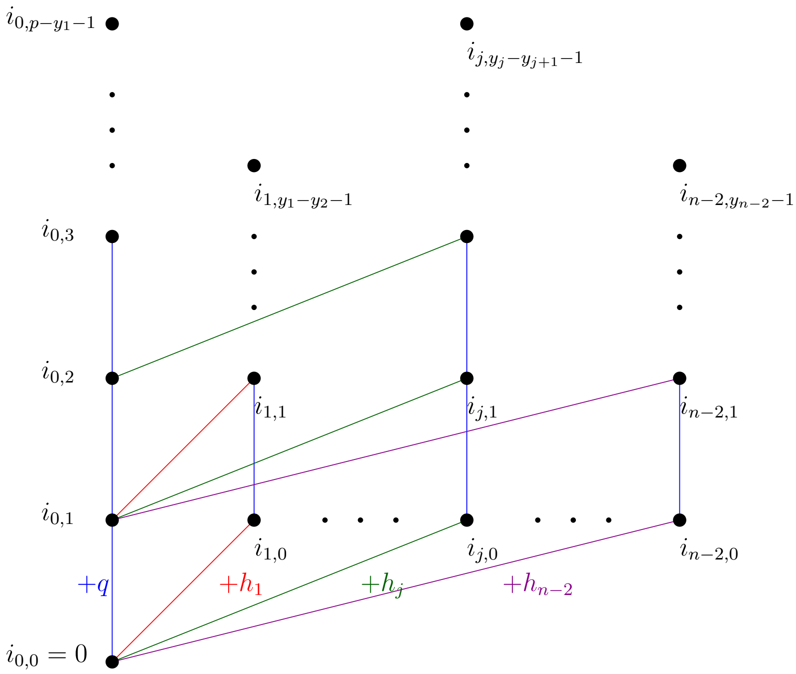


Figure 2.6: Hasse diagram of the Apéry poset of a semigroup $\mathcal{S}_1 \in KW(p,q)$.

Proof. Set $u_0 = v_{n-1} = 0$ and let $\mathcal{H} = \{g_i \coloneqq pq - (u_i+1)p - (v_{i+1}+1)q \mid 0 \leq i \leq n-2\}$. Recall from Proposition 2.39 that $\mathcal{S}_1 = \langle p, q, a_1, \ldots, a_{n-2}\rangle$ is in one-to-one correspondence to a lattice path ℓ in the rectangle $\mathbf{R}$. In particular, each a_i corresponds to the point $(u_i - 1, v_i - 1)$ under the line $pq - p - q = pU + qV$. To see that $\mathcal{H} = \mathrm{PF}(\mathcal{S}_1)$, first note that $\mathcal{H} = \{s \in \mathbb{N} \setminus \mathcal{S}_1 \mid s + p \in \mathcal{S}_1, s + q \in \mathcal{S}_1\}$. This is clear from Figure 2.7. Hence, $\mathrm{PF}(\mathcal{S}_1) \subset \mathcal{H}$, and the equality $\mathrm{PF}(\mathcal{S}_1) = \mathcal{H}$ follows from the fact $|\mathcal{H}| = n - 1 = |\mathrm{PF}(\mathcal{S}_1)|$, by Proposition 2.38. □

Example 2.45. Consider $p = 5$, $q = 8$, and the semigroup $\mathcal{S}_1 = \langle 5, 8, 9, 12\rangle \in KW(5,8)$ from Example 2.40. The Apéry set of $\langle 5, 8\rangle$ is $A_{58} = \{0, 8, 16, 24, 32\}$. By Proposition 2.42 (1), $\mathrm{Ap}(\mathcal{S}_1) = \{0, 8, 16, 9, 12\}$. By Proposition 2.43, the Hasse diagram of the Apéry poset $\mathcal{P}(\mathcal{S}_1)$ is the one shown in Figure 2.8. Finally, by Proposition 2.44, one has that the Pseudo Frobenius set of $\mathcal{S}_1$ is

$$\mathrm{PF}(\mathcal{S}_1) = \{pq - p - 3q, pq - 4p - 2q, pq - 5p - q\} = \{11, 4, 7\}.$$

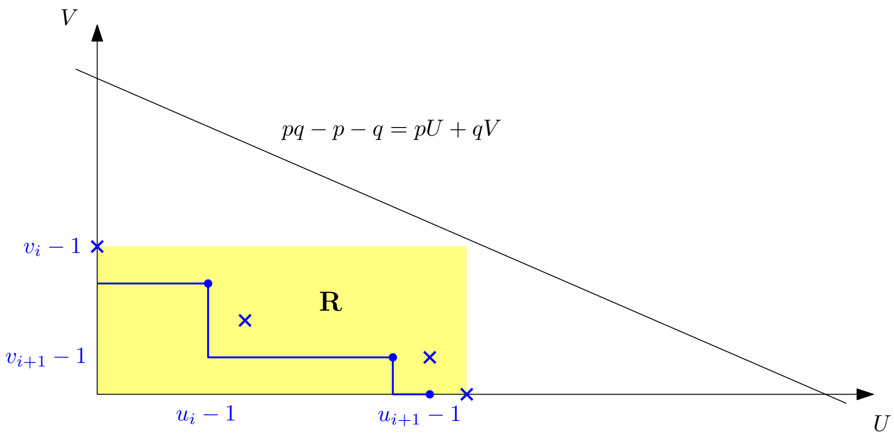


Figure 2.7: Pseudo Frobenius elements of $\mathcal{S}_1 \in KW(p,q)$ correspond to the points $\times$.

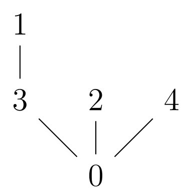


Figure 2.8: Hasse diagram of $\mathcal{P}(\mathcal{S}_1)$ for $\mathcal{S}_1 = \langle 5, 8, 9, 12 \rangle$ in Example 2.45.

2.5.2 Betti numbers

Let $\mathcal{S}_1 = \langle p, q, a_1, \ldots, a_{n-2}\rangle \in KW(p, q)$ and fix a field $\Bbbk$. Consider the polynomial ring $R = \Bbbk[x, y, x_1, \ldots, x_{n-2}]$, graded via $\deg(x) = p$, $\deg(y) = q$, and $\deg(x_i) = a_i$, $i = 1, \ldots, n-2$. Set $\mathcal{A}_1 = \{p, q, a_1, \ldots, a_{n-2}\}$, and let $I_{\mathcal{A}_1} \subset R$ be the toric ideal determined by $\mathcal{A}_1$. By Proposition 1.67, $I_{\mathcal{A}_1}$ is the defining ideal of the affine monomial curve parametrized by $\mathcal{A}_1$. Hence, its coordinate ring is $R/I_{\mathcal{A}_1}$, and it is isomorphic to the semigroup algebra $\Bbbk[\mathcal{S}_1]$.

We already know some of the Betti numbers of $\Bbbk[\mathcal{S}_1]$, $\beta_0 = 1$, $\beta_1 = \binom{n}{2}$ by Proposition 2.41, and $\beta_{n-1} = n-1$ by Proposition 2.38. Our goal is to obtain the whole Betti sequence of $\Bbbk[\mathcal{S}_1]$. To achieve this, first we characterize all Kunz–Waldi semigroups whose defining ideals are determinantal.

Definition 2.46. Let $3 \le p < q$ be relatively prime. We define a subclass $KW_D(p, q)$ of $KW(p, q)$ as follows. A semigroup $\mathcal{S}_1 = \langle p, q, a_1, \ldots, a_{n-2}\rangle \in KW(p, q)$ is in $KW_D(p, q)$ if there exist $u, v \in \mathbb{Z}_{>0}$ such that $(n-2)u \le q/2$, $(n-2)v \le p/2$, and $a_i = pq - u_i p - v_i q$, where $u_i = iu$ and $v_i = (n-1-i)v$ for all $1 \le i \le n-2$.

The notation chosen to denote this class will be justified after Theorem 2.49, where we prove that the semigroups in $KW_D(p, q)$ are precisely all the semigroups in $KW(p, q)$ with determinantal defining ideal.

Remark 2.47. Let $\mathcal{S}_1 \in KW_D(p, q)$ for some $u, v \in \mathbb{Z}_{>0}$. Note that $a_{i+1} - a_i = vq - up$ for all $1 \le i \le n-3$. Thus, the generators $a_1, a_2, \ldots, a_{n-2}$ necessarily form an arithmetic sequence with common difference $vq - up$. This sequence is increasing if $vq > up$ and decreasing otherwise.

Proposition 2.48. *If $\mathcal{S}_1 \in KW_D(p, q)$ for some $u, v, n \in \mathbb{Z}_{>0}$, then $I_{\mathcal{A}_1}$ is generated by the 2×2 minors of the following $2 \times n$ matrix*

$$M = \begin{pmatrix} x_{n-2} & x^u & y^{p-(n-1)v} & x_1 & x_2 & \cdots & x_{n-4} & x_{n-3} \\ x^{q-(n-1)u} & y^v & x_1 & x_2 & x_3 & \cdots & x_{n-3} & x_{n-2} \end{pmatrix}.$$

Proof. Let m_{ij} denote the determinant of the submatrix of M obtained from the i- and j-th columns, and $I_2(M)$ be the ideal generated by all the m_{ij}. Each m_{ij} is a $\mathcal{S}_1$-homogeneous binomial and thus, is in $I_{\mathcal{A}_1}$. Therefore, $I_2(M) \subset I_{\mathcal{A}_1}$. Let us prove the other inclusion. Note that for all $1 \le i \le n-3$, $g_i = -m_{2,i+3}$; $\eta_1 = -m_{23}$, and $\eta_2 = m_{12}$, so all of them are in $I_2(M)$. To finish the proof, it suffices to show that $f_{ij} \in I_2(M)$ for all $1 \le i \le j \le n-2$. We have $f_{1,n-2} = m_{13} \in I_2(M)$ and for all $1 \le j \le n-3$, one can check that

$$f_{1,j} = -m_{3,j+3} - \sum_{k=0}^{n-j-4} \left(x^{ku} y^{p-(n+k)v} m_{2,j+4+k}\right) + x^{(n-j-3)u} y^{p-(2n-j-3)v} m_{12} \in I_2(M)\,.$$

Moreover, for all $2 \le i \le n-2$,

$$f_{i,n-2} = m_{1,i+2} + \sum_{k=0}^{i-2} \left(x^{q-(n+k)u} y^{kv} m_{2,i+1-k} \right) \in I_2(M)\,.$$

Finally, note that for all $2 \le i \le j \le n-3$, $f_{i,j} = f_{i-1,j+1} - m_{i+2,j+3}$, and hence all the f_{ij} belong to $I_2(M)$. □

Theorem 2.49. *Let $\mathcal{S}_1 \in KW(p,q)$ of embedding dimension n. Then the following are equivalent:*

(a) $\mathcal{S}_1 \in KW_D(p,q)$.

(b) There exist positive integers u, v such that $(n-2)v \le p/2$, $(n-2)u \le q/2$ and $I_{\mathcal{A}_1}$ is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} x_{n-2} & x^u & y^{p-(n-1)v} & x_1 & x_2 & \cdots & x_{n-4} & x_{n-3} \\ x^{q-(n-1)u} & y^v & x_1 & x_2 & x_3 & \cdots & x_{n-3} & x_{n-2} \end{pmatrix}.$$

(c) There exist $\mathcal{S}_1$-homogeneous polynomials $F_i, G_i \in \langle x, y, x_1, \ldots, x_{n-2}\rangle$ such that $I_{\mathcal{A}_1}$ is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} F_1 & F_2 & \ldots & F_n \\ G_1 & G_2 & \ldots & G_n \end{pmatrix}.$$

(d) $\mathrm{PF}(\mathcal{S}_1) = \{z+k, z+2k, \ldots, z+(n-1)k\}$ *for some $z \ge 0$ and $k > 0$.*

When this is the case, $\mathrm{PF}(\mathcal{S}_1) = \{pq-(iu+1)p-((n-i-2)v+1)\,q \mid 0 \le i \le n-2\}$, *for the numbers $u, v \in \mathbb{Z}_{>0}$ defining $\mathcal{S}_1 \in KW_D(p,q)$, and hence $k = |up - vq|$.*

Proof. (a) ⇒ (b) is Proposition 2.48, (b) ⇒ (c) is trivial, and (c) ⇒ (d) is proved in general for any numerical semigroup in [95, Sect. 2].
(d) ⇒ (a): By Proposition 2.44, $PF(\mathcal{S}_1) = \{g_i := pq - (u_i+1)p - (v_{i+1}+1)q \mid 0 \le i \le n-2\}$, where $u_0 = v_{n-1} = 0$. Since $\mathrm{PF}(\mathcal{S}_1)$ satisfies (d) by our hypothesis, we must have that the difference between any two consecutive elements is constant. Firstly, for any $0 \le i \le n-2$,

$$\begin{aligned} g_i - g_{i+1} &= pq - (u_i+1)p - (v_{i+1}+1)q - [pq - (u_{i+1}+1)p - (v_{i+2}+1)q] \\ &= p(u_{i+1} - u_i) + q(v_{i+2} - v_{i+1})\,. \end{aligned}$$

Let $\alpha_i = u_{i+1} - u_i$ and $\beta_i = v_{i+2} - v_{i+1}$. Now we must have that for any $0 \le i < j \le n-3$,

$$p\alpha_i + q\beta_i = p\alpha_j + q\beta_j \implies p(\alpha_i - \alpha_j) = q(\beta_j - \beta_i)\,.$$

Since p and q are relatively prime, there exists $\ell \in \mathbb{Z}$ such that $\alpha_i - \alpha_j = q\ell$. But $|\alpha_i - \alpha_j| = |u_{i+1} + u_j - u_i - u_{j+1}| \leq q - 2$, so $\ell = 0$. Thus, since $u_{i+1} > u_i$, there is some $u \in \mathbb{Z}_{>0}$ such that $\alpha_i = \alpha_j = u$. The recursive definition of α_i now gives us $u_i = iu$ for $1 \leq i \leq n-2$. If $u > \frac{q}{2(n-2)}$, then $u_{n-2} = (n-2)u > q/2$, a contradiction. Similarly, $\beta_i = \beta_j = -v$, for some $v \in \mathbb{Z}_{>0}$ since $v_{i+2} < v_{i+1}$. This implies $v_{n-2} = v, v_{n-3} = 2v, ..., v_1 = (n-2)v$. If $v > \frac{p}{2(n-2)}$ then $v_1 > p/2$, a contradiction.

We have shown that $\mathcal{S}_1$ must be as in Definition 2.46. In addition, when (d) holds, we have shown that $\mathrm{PF}(\mathcal{S}_1) = \{pq - (iu+1)p - ((n-i-2)v+1)\,q \mid 0 \leq i \leq n-2\}$, which implies $k = |up - vq|$. □

Remark 2.50. Let $3 \leq p < q$ relatively prime integers and denote $p' = \lfloor p/2 \rfloor$, and $q' = \lfloor q/2 \rfloor$. By Theorem 2.49, the cardinality of $KW_D(p,q)$ is

$$|KW_D(p,q)| = \sum_{n=3}^{p'+2} \left\lfloor \frac{p'}{n-2} \right\rfloor \left\lfloor \frac{q'}{n-2} \right\rfloor = \sum_{n=1}^{p'} \left\lfloor \frac{p'}{n} \right\rfloor \left\lfloor \frac{q'}{n} \right\rfloor ,$$

and the cardinality of $KW(p,q)$ is

$$|KW(p,q)| = \sum_{n=3}^{p'+2} \binom{p'}{n-2}\binom{q'}{n-2} = \sum_{n=1}^{p'} \binom{p'}{n}\binom{q'}{n} = \binom{p'+q'}{p'} - 1\,.$$

Thus, the proportion of semigroups in $KW(p,q)$ whose defining ideal is determinantal is

$$\rho_D(p,q) := \frac{|KW_D(p,q)|}{|KW(p,q)|} = \frac{\sum_{n=1}^{p'} \left\lfloor \frac{p'}{n} \right\rfloor \left\lfloor \frac{q'}{n} \right\rfloor}{\binom{p'+q'}{p'} - 1}\,.$$

Since the ideal $I_{\mathcal{A}_1} \subset R$ for any $\mathcal{S}_1 \in KW_D(p,q)$ is generated by the 2×2 minors of a $2 \times n$ matrix and its height is $n-1$ $(= n-2+1)$, it is resolved by the Eagon-Northcott complex (see [26, Thm. 2]). In particular, the Betti numbers are given by the formula $\beta_i = i\binom{n}{i+1}$ for all $1 \leq i \leq n-1$. We want to expand this to compute the Betti sequence of more KW-semigroups. One way to do this is by using the fact that any two numerical semigroups of multiplicity p lying in the interior of the same face of the Kunz cone $\mathfrak{C}_p$ have the same Betti sequence. This is a result of Kunz.

Theorem 2.51 ([60, Prop. 2.6], [9, Thm. 2.7])**.** *Let* $\mathcal{A}_1 = \{a_1 = p < a_2 < \cdots < a_n\} \subset \mathbb{N}$ *and* $\mathcal{A}'_1 = \{a'_1 = p < a'_2 < \cdots < a'_m\} \subset \mathbb{N}$ *such that* $\gcd(a_1, \ldots, a_n) = 1$ *and* $\gcd(a'_1, \ldots, a'_m) = 1$. *Denote by* $\mathcal{S}_1$ *and* $\mathcal{S}'_1$ *the numerical semigroups generated by* $\mathcal{A}_1$ *and* $\mathcal{A}'_1$*, respectively. If* $\mathcal{S}_1$ *and* $\mathcal{S}'_1$ *lie in the interior of the same face of the Kunz cone* $\mathfrak{C}_p$*, then* $\beta_i(\Bbbk[\mathcal{S}_1]) = \beta_i(\Bbbk[\mathcal{S}'_1])$ *for all* i.

Let $\mathcal{S}_1, \mathcal{S}'_1 \in KW(p,q)$ be of the same embedding dimension n. Write $\mathcal{S}_1 = \langle p, q, a_1, ..., a_{n-2}\rangle$ and $\mathcal{S}'_1 = \langle p, q, a'_1, ..., a'_{n-2}\rangle$. Further, write $a_i = pq - u_i p - v_i q$ and $a'_i = pq - u'_i p - v'_i q$, $1 \leq i \leq n-2$.

Proposition 2.52. *Let $\mathcal{S}_1, \mathcal{S}'_1 \in KW(p,q)$. Then $\mathcal{S}_1$ and $\mathcal{S}'_1$ belong to the interior of the same face of the Kunz cone $\mathfrak{C}_p$ if and only if*

(1) $e(\mathcal{S}_1) = e(\mathcal{S}'_1) = n$, *and*

(2) $v_i = v'_i$ *for all* $1 \leq i \leq n-2$.

Proof. By Proposition 2.43, the Apéry poset of any semigroup $\mathcal{S}_1 \in KW(p,q)$ is completely determined by the embedding dimension of $\mathcal{S}_1$, $e(\mathcal{S}_1) = n$, and the sequence $p/2 \geq v_1 > \cdots > v_{n-2} > 0$. Thus, the result follows from Theorem 1.13. □

As a consequence of Proposition 2.52 and Theorem 2.49, we get the following result that provides the whole Betti sequence of $\Bbbk[\mathcal{S}_1]$ for some $\mathcal{S}_1 \in KW(p,q)$.

Theorem 2.53. *Let $\mathcal{S}_1 \in KW(p,q)$ of embedding dimension n be such that $v_i = (n-i-1)v$ for some $v \in \mathbb{Z}_{>0}$ with $(n-2)v \leq p/2$. Then the Betti numbers of $\Bbbk[\mathcal{S}_1]$ are*

$$\beta_i = i\binom{n}{i+1},\ 1 \leq i \leq n-1\,.$$

Proof. Consider the semigroup $\mathcal{S}'_1 \in KW_D(p,q)$ defined by the sequence $u_1 = 1 < u_2 = 2 < \cdots < u_{n-2} = n-2$ and the sequence $v_1 > v_2 > \cdots > v_{n-2}$, $v_i = (n-1-i)v$. By Proposition 2.52, $\mathcal{S}_1$ and $\mathcal{S}'_1$ lie in the interior of the same face of the Kunz cone $\mathfrak{C}_p$. Therefore, $\Bbbk[\mathcal{S}_1]$ and $\Bbbk[\mathcal{S}'_1]$ have the same Betti sequence by Theorem 2.51, that is, $\beta_i = i\binom{n}{i+1}$, $1 \leq i \leq n-1$ by the construction of the Eagon-Northcott complex. □

Example 2.54. Consider the semigroup $\mathcal{S}_1 = \langle 8, 17, 53, 62, 55\rangle \in KW(8,17)$. Note that $\mathcal{S}_1$ is not in $KW_D(8,17)$ as $53, 62, 55$ are not in an arithmetic sequence, by Remark 2.47. However, $\mathcal{S}_1$ is in the same face of $\mathfrak{C}_8$ as $\mathcal{S}'_1 = \langle 8, 17, 69, 70, 71\rangle \in KW(8,17)$. Note that for both $\mathcal{S}_1$ and $\mathcal{S}'_1$, $v_1 = 3$, $v_2 = 2$, $v_3 = 1$. Hence, their Betti sequence is $(4, 15, 20, 10, 1)$, by Theorem 2.53, and the Hasse diagram of $\mathcal{P}(\mathcal{S}_1)$ and $\mathcal{P}(\mathcal{S}'_1)$ is the one shown in Figure 2.9.

Remark 2.55. By Theorem 2.53, the number of numerical semigroups in the class $KW(p,q)$ whose Betti numbers are $\beta_i = i\binom{n}{i+1}$, $1 \leq i \leq n-1$, is at least $\sum_{n=1}^{p'} \left\lfloor \frac{p'}{n} \right\rfloor \binom{q'}{n}$, where $p' = \lfloor p/2 \rfloor$ and $q' = \lfloor q/2 \rfloor$.

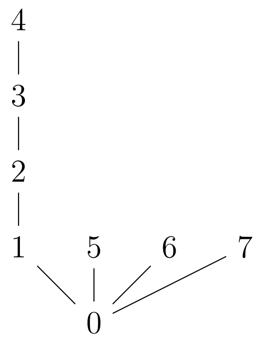


Figure 2.9: Hasse diagram of $\mathcal{P}(\mathcal{S}_1)$ and $\mathcal{P}(\mathcal{S}_1')$ in Example 2.54.

It is interesting to note that even if $\mathcal{S}_1 \in KW(p,q)$ does not satisfy the hypothesis of Theorem 2.53, the conclusion still seems to hold. In particular, if the $v_1, ..., v_{n-2}$ are not in an arithmetic sequence, is it still true that $\beta_i = i\binom{n}{i+1}$ for $1 \leq i \leq n-1$? We give an example to support a positive answer:

Example 2.56. Consider $\mathcal{S}_1 = \langle 8, 17, 36, 45, 63 \rangle \in KW(8,17) \setminus KW_D(8,17)$. The Betti sequence of $\Bbbk[\mathcal{S}_1]$ is $(4, 15, 20, 10, 1)$, and yet, $\mathcal{S}_1$ is not even in the same face as some $\mathcal{S}_1' \in KW_D(8,17)$.

Chapter 3

The structure of the sumsets

"... the art of combinations is mastered through algebra."
W. von Tschirnhaus

Let $\mathcal{A} = \{\mathbf{a}_0, \dots, \mathbf{a}_n\} \subset \mathbb{N}^d$ be a finite set, $n, d \in \mathbb{Z}_{>0}$. Additive combinatorics studies the sumsets of $\mathcal{A}$ and their cardinality. When $d = 1$, the part of additive combinatorics that studies the sumsets of $\mathcal{A}$ is called additive number theory. As stated in Section 1.4, to study the sumsets of such a set $\mathcal{A}$, one can always assume that it is in normal form, i.e., $\mathcal{A} = \{a_0 = 0 < a_1 < \cdots < a_n = D\}$, and $\gcd(a_1, \dots, a_n) = 1$. Associated with $\mathcal{A}$, one has the set $\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \dots, \underline{\mathbf{a}}_n\} \subset \mathbb{N}^2$, where $\underline{\mathbf{a}}_i = (D - a_i, a_i)$ for all $i = 0, \dots, n$, and we call $\underline{\mathcal{A}}$ the *homogenization* of $\mathcal{A}$. Also, fixed an infinite field $\Bbbk$, we consider the projective monomial curve $\mathcal{C} = \mathcal{C}_{\underline{\mathcal{A}}} \subset \mathbb{P}^n_{\Bbbk}$ determined by $\underline{\mathcal{A}}$.

In Section 3.1, we study the structure of the sumsets of $\mathcal{A}$, $(s\mathcal{A})_{s=0}^{\infty}$, starting with the so-called structure theorem by Nathanson (Theorem 3.1). In Proposition 3.4, we recall the characterization of the elements that appear in the structure theorem in terms of the curve $\mathcal{C}$, given by Elias. We define the sumsets regularity of $\mathcal{A}$, $\sigma(\mathcal{A})$, as the least integer such that, for all larger integers, the decomposition in the structure theorem holds, and in Theorem 3.7 we show how to interpret it in terms of the curve $\mathcal{C}$. Several upper bounds on $\sigma(\mathcal{A})$ that appear in the literature are recalled. Indeed, we provide a new proof of Granville-Walker's bound using the characterization for $\sigma(\mathcal{A})$ and a bound on the Castelnuovo-Mumford regularity of the coordinate ring of $\mathcal{C}$. Moreover, we provide a new upper bound on $\sigma(\mathcal{A})$ in (3.5) and compare it with the other known bounds, showing that it improves them in most cases. In Section 3.2, we analyze the structure of the sumsets of $\underline{\mathcal{A}}$, and observe that the sumsets regularity of $\mathcal{A}$ defined in the previous section could also be called the sumsets regularity of $\underline{\mathcal{A}}$. We recall the definition of the Apéry and exceptional sets of $\mathcal{S}$ given in Chapter 2 and give a relation on the size of these sets and the size of the sumsets of $\mathcal{A}$ (or $\underline{\mathcal{A}}$) in Proposition 3.19.

When $d \geq 2$, consider $D = \max\{|\mathbf{a}_i|: i = 0, \dots, n\}$ and $\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \dots, \underline{\mathbf{a}}_n\}$, where $\underline{\mathbf{a}}_i = (D - |\mathbf{a}_i|, \mathbf{a}_i) \in \mathbb{N}^{d+1}$ for all $i = 0, \dots, n$. We call $\underline{\mathcal{A}}$ the *homogenization* of $\mathcal{A}$. Fix an algebraically closed field $\Bbbk$ and consider the projective toric variety determined by $\underline{\mathcal{A}}$, $\mathcal{X} = \mathcal{X}_{\underline{\mathcal{A}}} \subset \mathbb{P}^n_{\Bbbk}$. Whenever $\mathcal{X}$ is simplicial, in Theorem 3.24 we recall a structure theorem for the sumsets of $\mathcal{A}$. In Section 3.3, we make the structure theorem more explicit in two particular cases, when $\mathcal{X}$ is smooth and when it is a surface with exactly a single singular point. In Subsection 3.3.1 we treat the smooth case, giving the structure of the sumsets of $\mathcal{A}$ in Proposition 3.25 and Theorem 3.26. We define the sumsets regularity and provide a better upper bound on the sumsets regularity in Theorem 3.29. In Subsection 3.3.2 we perform a similar study for surfaces with a single singular point. We study the structure of the sumsets of $\mathcal{A}$ in Proposition 3.34 and Theorem 3.35. Then, we define the sumsets regularity and give an upper bound on it in Theorem 3.41.

Most of the results included in Section 3.1 are part of [39].

3.1 Structure theorem in $\mathbb{N}$

Let $\mathcal{A} = \{a_0, a_1, \dots, a_n\} \subset \mathbb{N}$ be a finite set. Recall from Section 1.4 that, to study the sumsets of $\mathcal{A}$, one can always assume that $\mathcal{A}$ is in normal form, i.e., $a_0 = 0$ and $\gcd(a_1, \dots, a_n) = 1$. In this section, we study the structure of the sumsets of $\mathcal{A}$. From now on, $\mathcal{A} = \{a_0 = 0 < a_1 < \cdots < a_n =: D\} \subset \mathbb{N}$ will be a set in normal form. Note that, if $n = 1$, then $\mathcal{A} = \{0, 1\}$, and hence $s\mathcal{A} = [0, s]$ for all $s \in \mathbb{N}$. Therefore, throughout this section, we will assume $n \geq 2$.

In 1972, Nathanson proved the so-called structure theorem, one of the main results in additive number theory, which shows that the sumsets $s\mathcal{A}$ always have a fixed structure, for s sufficiently large.

Theorem 3.1 (Structure Theorem, [70], [71, Thm. 1.1])**.** *If $\mathcal{A} = \{a_0 = 0 < a_1 < \cdots < a_n = D\} \subset \mathbb{N}$ is a finite set in normal form, $n \geq 1$, then there exist integers $c_1, c_2 \in \mathbb{N}$ and finite subsets $C_i \subset [0, c_i - 2]$, $i = 1, 2$, such that*

$$s\mathcal{A} = C_1 \sqcup [c_1, sD - c_2] \sqcup (sD - C_2) \tag{3.1}$$

for all $s \geq \max\{1, s_0^N\}$, where $s_0^N := (n-1)(D-1)D$.

Nathanson proved the theorem by induction on $s \geq s_0^N$, and his proof is constructive in the sense that the numbers $c_1, c_2 \in \mathbb{N}$ and the sets $C_1, C_2 \subset \mathbb{N}$ can be read on the sumset $s_0^N\mathcal{A}$. But there is no nice interpretation of these elements in terms of the set $\mathcal{A}$.

Remark 3.2. In the conditions of Theorem 3.1, one has that that $c_1 = 0 \Leftrightarrow a_1 = 1$, and $c_2 = 0 \Leftrightarrow D - a_{n-1} = 1$. Moreover, if $D - \mathcal{A} := \{D - a \mid a \in \mathcal{A}\} = \mathcal{A}$, then $c_1 = c_2$ and $C_1 = C_2$. This can be directly proved, but it can also be obtained as an easy consequence of Proposition 3.4.

Example 3.3. (1) Consider $\mathcal{A} = \{0, 1, 3, 4\}$. It is straightforward to show that $2\mathcal{A} = [0, 8]$, $3\mathcal{A} = [0, 12]$ and, in general, $s\mathcal{A} = [0, 4s]$ for all $s \in \mathbb{N}$, $s \geq 2$. The elements that appear in the structure theorem are $c_1 = c_2 = 0$, $C_1 = C_2 = \emptyset$, and $s_0^N = 24$.

(2) Consider $\mathcal{A} = \{0, 2, 3, 5\}$. A direct computation shows that $2\mathcal{A} = \{0\} \sqcup [2, 8] \sqcup \{10\}$, $3\mathcal{A} = \{0\} \sqcup [2, 13] \sqcup \{15\}$. Indeed, one can prove that $s\mathcal{A} = \{0\} \sqcup [2, 5s - 2] \sqcup \{5s\}$, for all $s \in \mathbb{N}$, $s \geq 1$. The elements that appear in Theorem 3.1 are $c_1 = c_2 = 2$, $C_1 = C_2 = \{0\}$, and $s_0^N = 40$.

Consider the homogenization of $\mathcal{A}$, $\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \ldots, \underline{\mathbf{a}}_n\} \subset \mathbb{N}^2$, where $\underline{\mathbf{a}}_i = (D - a_i, a_i)$ for all $i = 0, \ldots, n$. Fix an infinite field $\Bbbk$, and let $\mathcal{C} = \mathcal{C}_{\underline{\mathcal{A}}} \subset \mathbb{P}^n_{\Bbbk}$ be the projective monomial curved determined by $\underline{\mathcal{A}}$. The coordinate ring of $\mathcal{C}$ is $\Bbbk[\mathcal{C}] = \Bbbk[x_0, \ldots, x_n]/I_{\underline{\mathcal{A}}}$, where $I_{\underline{\mathcal{A}}}$ is the toric ideal determined by $\underline{\mathcal{A}}$. By Proposition 1.83, one has that $|s\mathcal{A}| = \mathrm{HF}_{\Bbbk[\mathcal{C}]}(s)$ for all $s \in \mathbb{N}$.

The elements in the structure theorem have recently been characterized in [32, Prop. 3.4] in terms of the curve $\mathcal{C}$ and some of its invariants. Recall by Proposition 1.75 that $\mathcal{C}$ has at most two singular points, $P_1 = (1 : 0 : \cdots : 0) \in \mathbb{P}^n_{\Bbbk}$ and $P_2 = (0 : \cdots : 0 : 1) \in \mathbb{P}^n_{\Bbbk}$. Moreover, if $\delta(\mathcal{C}, P)$ denotes the singularity order of P, then $\delta(\mathcal{C}, P_1) = |\mathbb{N} \setminus \mathcal{S}_1|$ and $\delta(\mathcal{C}, P_2) = |\mathbb{N} \setminus \mathcal{S}_2|$, where $\mathcal{S}_1$ and $\mathcal{S}_2$ denote the numerical semigroups generated by $\mathcal{A}_1 = \{a_1, \ldots, a_n\}$ and $\mathcal{A}_2 = \{D - a_{n-1}, \ldots, D - a_1, D\}$, respectively. By Proposition 1.75 (4), one gets that for all $s \geq \mathrm{r}(\Bbbk[\mathcal{C}])$,

$$|s\mathcal{A}| = \mathrm{HF}_{\Bbbk[\mathcal{C}]}(s) = sD + 1 - \delta(\mathcal{C}, P_1) - \delta(\mathcal{C}, P_2)\,, \tag{3.2}$$

where $\mathrm{r}(\Bbbk[\mathcal{C}])$ denotes the regularity of the Hilbert function of $\Bbbk[\mathcal{C}]$ (see Definition 1.48).

Proposition 3.4 ([32, Prop. 3.4])**.** *Following notations in Theorem 3.1, for $i = 1, 2$, the following claims hold:*

(1) c_i is the conductor of $\mathcal{S}_i$;

(2) $C_i = \mathcal{S}_i \cap [0, c_i - 2]$; and

(3) $\delta(\mathcal{C}, P_i) = c_i - |C_i|$.

Proof. Let us prove the result for $i = 1$. Since $0 \in \mathcal{A}$, the sumsets of $\mathcal{A}$ form a nested sequence, i.e., $s\mathcal{A} \subset (s+1)\mathcal{A}$ for all $s \in \mathbb{N}$. Hence, the sequence $(s\mathcal{A})_{s=0}^{\infty}$ converges to $\cup_{s=0}^{\infty} s\mathcal{A} = \mathcal{S}_1$. By Theorem 3.1, we can write

$$s\mathcal{A} = C_1 \sqcup [c_1, sD - c_2] \sqcup (sD - C_2)$$

for all $s \geq s_0^N$, where $C_i \subset [0, c_i - 2]$ for $i = 1, 2$. Since $D > 0$, $\lim_{s\to\infty}(sD - c_2) = \infty$, and hence $[c_1, sD - c_2]$ converges to $[c_1, \infty)$ when $s \to \infty$. Therefore, $\mathcal{S}_1 = C_1 \sqcup [c_1, \infty)$. From this expression, it is clear that c_1 is the conductor of $\mathcal{S}_1$, and $C_1 = \mathcal{S}_1 \cap [0, c_1 - 2]$. Finally, by Proposition 1.75 (3), one has that

$$\delta(\mathcal{C}, P_i) = |\mathbb{N} \setminus \mathcal{S}_1| = (c_1 - 1) - |C_1| + 1 = c_1 - |C_1|\,.$$

For $i = 2$, apply the previous reasoning to the set $\mathcal{A}^* = D - \mathcal{A} = \{D - a \mid a \in \mathcal{A}\}$. □

Using Proposition 3.4, one can interpret (and compute) the elements that appear in the structure theorem in terms of the curve $\mathcal{C}$. However, as we have seen in Example 3.3, the number s_0^N such that the structure theorem holds for all $s \geq s_0^N$ is far from being tight. With this in mind, we give the following definition.

Definition 3.5. The least integer σ, such that the decomposition (3.1) in Theorem 3.1 holds for all $s \geq \sigma$, will be called the *sumsets regularity* of $\mathcal{A}$ and we will denote it by $\sigma(\mathcal{A})$, or simply σ if there is no confusion.

Theorem 3.1 provides an upper bound on $\sigma(\mathcal{A})$ that is generally far from its real value: $\sigma(\mathcal{A}) \leq (n-1)(D-1)D = s_0^N$. After Nathanson's proof, other proofs of Theorem 3.1 have been published, [98, 44, 45]. In these articles, the authors give the following improved upper bounds on $\sigma(\mathcal{A})$:

- [98, Thm. 2] (Wu, Chen, Chen; 2011) $\sigma(\mathcal{A}) \leq \left(\sum_{i=2}^{n-1} a_i\right) + D - n =: s_0^{WCC}$.
- [44, Thm. 1] (Granville, Shakan; 2020) $\sigma(\mathcal{A}) \leq 2\lfloor \frac{D}{2} \rfloor =: s_0^{GS}$.
- [45, Thm. 1] (Granville, Walker; 2021) $\sigma(\mathcal{A}) \leq D - n + 1 =: s_0^{GW}$.

Note that in [98, 44, 45], the union in Equation (3.1) is not shown to be disjoint, but this is shown in [65] for s_0^{GW} and, as $s_0^{WCC} > s_0^{GW}$ and $s_0^{GS} > s_0^{GW}$ if $n \geq 3$, the above claims hold for $n \geq 3$. For $n = 2$, one has that $s_0^{WCC} < s_0^{GW} \leq s_0^{GS}$, but the above inequalities hold (see Example 3.8).

Besides giving a great upper bound on $\sigma(\mathcal{A})$, Granville and Walker also characterize the sets $\mathcal{A}$ for which this bound is attained.

Theorem 3.6 ([45, Thm. 2])**.** *Let $n \in \mathbb{N}$, $n \geq 2$, and $\mathcal{A} = \{a_0 = 0 < a_1 < \cdots < a_n = D\} \subset \mathbb{N}$ be a set in normal form. Then, $\sigma(\mathcal{A}) \leq D - n + 1$. Moreover, the equality holds if and only if either $\mathcal{A}$ or $D - \mathcal{A}$ belongs to one of the following two families:*

- $\mathcal{A} = [0, D] \setminus \{a\}$, *for some* a, *such that* $2 \leq a \leq D-2$;
- $\mathcal{A} = [0, 1] \sqcup [a+1, D]$, *for some* a, *such that* $2 \leq a \leq D-2$.

Note for any $\mathcal{A}$ belonging to one of the two families in Theorem 3.6, the monomial curve $\mathcal{C}$ is smooth, by Proposition 1.75 (2).

We now give a characterization of σ, the sumsets regularity of $\mathcal{A}$, in terms of the curve $\mathcal{C}$ and its invariants. This result concludes the characterization of the elements in the structure theorem given in Proposition 3.4.

Theorem 3.7. *The least integer* σ*, such that the decomposition* (3.1) *in Theorem 3.1 holds for all* $s \geq \sigma$*, i.e., the sumsets regularity of* $\mathcal{A}$*, is*

$$\sigma = \max\left\{ \mathrm{r}(\Bbbk[\mathcal{C}]), \left\lceil \frac{c_1 + c_2}{D} \right\rceil \right\},$$

where $\mathrm{r}(\Bbbk[\mathcal{C}])$ *is the regularity of the Hilbert function of* $\Bbbk[\mathcal{C}]$ *and* c_i *is the conductor of the numerical semigroup* $\mathcal{S}_i$ *for* $i = 1, 2$.

Proof. If $s \in \mathbb{N}$ is such that $sD - c_2 < c_1$, then $[c_1, sD - c_2] = \emptyset$. Hence, $\sigma \geq \lceil \frac{c_1+c_2}{D} \rceil$. Moreover, for all $s \geq 0$, $\mathrm{HF}_{\Bbbk[\mathcal{C}]}(s) = |s\mathcal{A}|$ by Proposition 1.83, and if $s \geq \sigma$, then

$$\begin{aligned} \mathrm{HF}_{\Bbbk[\mathcal{C}]}(s) &= |s\mathcal{A}| = sD + 1 - (c_1 - |C_1| + c_2 - |C_2|) \\ &= sD + 1 - \delta(\mathcal{C}, P_1) - \delta(\mathcal{C}, P_2) = \mathrm{HP}_{\Bbbk[\mathcal{C}]}(s) \end{aligned} \tag{3.3}$$

by Proposition 3.4, so $\sigma \geq \mathrm{r}(\Bbbk[\mathcal{C}])$. Therefore, $\sigma \geq \max\left\{\mathrm{r}(\Bbbk[\mathcal{C}]), \lceil \frac{c_1+c_2}{D} \rceil\right\}$. Conversely, for $s \geq \max\left\{\mathrm{r}(\Bbbk[\mathcal{C}]), \lceil \frac{c_1+c_2}{D} \rceil\right\}$, one has that (3.3) is satisfied by applying (3.2). Moreover, since $sD - c_2 \geq c_1$, one has that

$$\begin{aligned} s\mathcal{A} &= (s\mathcal{A} \cap C_1) \sqcup (s\mathcal{A} \cap [c_1, sD - c_2]) \sqcup (s\mathcal{A} \cap (sD - C_2)) \\ &\subset C_1 \sqcup [c_1, sD - c_2] \sqcup (sD - C_2). \end{aligned}$$

Since both sets $s\mathcal{A}$ and $C_1 \sqcup [c_1, sD - c_2] \sqcup (sD - C_2)$ are finite and have the same cardinality, they are equal, so $\max\left\{\mathrm{r}(\Bbbk[\mathcal{C}]), \lceil \frac{c_1+c_2}{D} \rceil\right\} \geq \sigma$ and the resut follows. □

Example 3.8. Take $a, D \in \mathbb{Z}_{>0}$ such that $a < D$ and $\gcd(a, D) = 1$, and consider $\mathcal{A} = \{0, a, D\}$. Set $\mathcal{S}_1 = \langle a, D \rangle$ and $\mathcal{S}_2 = \langle D - a, D \rangle$. Then, the elements that appear in Theorem 3.1 are $c_1 = (a-1)(D-1)$, $c_2 = (D-a-1)(D-1)$ and $C_1 = \mathcal{S}_1 \cap [0, c_1 - 2]$, $C_2 = \mathcal{S}_2 \cap [0, c_2 - 2]$, by Proposition 3.4. Moreover, we have that $\lceil \frac{c_1+c_2}{D} \rceil = D - 2$ and $\mathrm{r}(\Bbbk[\mathcal{C}]) = \mathrm{reg}(\Bbbk[\mathcal{C}]) - 1 \leq D - 2$, by Remark 1.52 (2) and Theorem 1.76 (1), since $\Bbbk[\mathcal{C}]$ is Cohen-Macaulay in this case. Hence, the sumsets regularity of $\mathcal{A}$ is $\sigma(\mathcal{A}) = D - 2$, by Theorem 3.7.

Given a set $\mathcal{A} \subset \mathbb{N}$ in normal form, it is not easy to know in advance whether $\sigma = \mathrm{r}(\Bbbk[\mathcal{C}])$ or $\sigma = \lceil \frac{c_1+c_2}{D} \rceil$. However, in some cases it is, as Proposition 3.9 shows.

Proposition 3.9. *(1) If $\mathcal{C}$ is smooth, then $\sigma = \mathrm{r}(\Bbbk[\mathcal{C}]) \geq \lceil \frac{c_1+c_2}{D} \rceil = 0$.*

(2) If $\mathcal{C}$ is arithmetically Cohen-Macaulay, then $\sigma = \lceil \frac{c_1+c_2}{D} \rceil \geq \mathrm{r}(\Bbbk[\mathcal{C}])$.

Proof. If $\mathcal{C}$ is smooth, then $c_1 = c_2 = 0$ and (1) follows. Now, for $s = \lceil \frac{c_1+c_2}{D} \rceil$, the sumset $s\mathcal{A}$ decomposes as the union of three disjoint subsets

$$s\mathcal{A} = (s\mathcal{A} \cap C_1) \sqcup (s\mathcal{A} \cap [c_1, sD - c_2]) \sqcup (s\mathcal{A} \cap (sD - C_2)) \,.$$

If either $s\mathcal{A} \cap C_1 \neq C_1$, or $s\mathcal{A} \cap [c_1, sD - c_2] \neq [c_1, sD - c_2]$, or $s\mathcal{A} \cap (sD - C_2) \neq (sD - C_2)$, then $E_{\mathcal{S}} \neq \emptyset$, where $\mathcal{S} \subset \mathbb{N}^2$ is the affine semigroup generated by $\underline{\mathcal{A}}$ and $E_{\mathcal{S}}$ is as in Definition 2.2. Thus, if $\mathcal{C}$ is arithmetically Cohen-Macaulay, by applying Proposition 2.4 (e), one gets that $s\mathcal{A} = C_1 \sqcup [c_1, sD - c_2] \sqcup (sD - C_2)$ and (2) follows. ☐

As a direct consequence of Proposition 3.9, we recover the well-known fact that for any $n \geq 3$, the rational normal curve, i.e., the curve $\mathcal{C}$ given by $\mathcal{A} = [0, n]$, is the only projective monomial curve in $\mathbb{P}^n_{\Bbbk}$ which is both smooth and arithmetically Cohen-Macaulay.

Example 3.10. (1) If $\mathcal{A} = [0, D] \setminus \{a\}$ for some $2 \leq a \leq D - 2$, then $c_1 = c_2 = 0$ and $\sigma = 2$ by Theorem 3.6. In this example, $\sigma = \mathrm{r}(\Bbbk[\mathcal{C}]) > \lceil \frac{c_1+c_2}{D} \rceil$.

(2) For $\mathcal{A} = \{0, 2, 5, 6, 9\}$, one has $c_1 = 4$, $c_2 = 6$ and $\mathrm{r}(\Bbbk[\mathcal{C}]) = 1$, so $\sigma = \lceil \frac{c_1+c_2}{D} \rceil = 2 > \mathrm{r}(\Bbbk[\mathcal{C}])$.

In Proposition 3.4 and Theorem 3.7, we have characterized all the elements that appear in the structure theorem in terms of the projective monomial curve $\mathcal{C}$. Applying these results, we can recover the bound on σ given by Granville and Walker, s_0^{GW}, using known bounds on $\lceil \frac{c_1+c_2}{D} \rceil$ and $\mathrm{r}(\Bbbk[\mathcal{C}])$.

Lemma 3.11. *Let $\mathcal{A} = \{a_0 = 0 < a_1 < \cdots < a_n = D\} \subset \mathbb{N}$ be a set in normal form. If $\mathcal{S}_1$ is the semigroup generated by $\mathcal{A}_1 = \{a_1, \ldots, a_n\}$, $\mathcal{S}_2$ is the semigroup generated by $\mathcal{A}_2 = \{D - a_{n-1}, \ldots, D - a_1, D\}$, and c_i is the conductor of $\mathcal{S}_i$ for $i = 1, 2$, then*

$$\left\lceil \frac{c_1 + c_2}{D} \right\rceil \leq D - n \,.$$

Proof. By Proposition 1.2 (2), $c_1 \leq (a_1 - 1)(D - 1)$ and $c_2 \leq (D - a_{n-1} - 1)(D - 1)$. Therefore,

$$c_1 + c_2 \leq (a_1 + D - a_{n-1} - 2)(D - 1) \leq (D - n)(D - 1)$$

because $a_{n-1} \geq a_1 + n - 2$. Thus, the result follows dividing by D in the previous equation. □

Using the known fact $\mathrm{r}(\Bbbk[\mathcal{C}]) \leq \mathrm{reg}(\Bbbk[\mathcal{C}])$ (Remark 1.52 (3)), the bound $\sigma \leq s_0^{GW}$ follows from Theorem 3.7, Lemma 3.11, and Theorem 1.76 (1).

In their article [45], Granville and Walker give a whole new proof for the existence of the elements in the structure theorem, obtaining the improved bound $\sigma \leq s_0^{GW}$. Note that we have obtained the same result using Theorem 3.1, the formula for σ in Theorem 3.7, and appropriate bounds on c_1, c_2, and $\mathrm{r}(\Bbbk[\mathcal{C}])$. Now, we apply the same idea to obtain a new bound on σ that improves s_0^{GW}.

A new bound on the sumsets regularity

By combining the Erdös-Graham bound on the condutor of a numerical semigroup (Proposition 1.2 (2)) and the bound on the Castelnuovo-Mumford regularity of a projective monomial curve given by L'vovsky (Theorem 1.76 (2)), we obtain the following new bound on the sumsets regularity. This bound is different from the already known bounds $s_0^N, s_0^{WCC}, s_0^{GS}, s_0^{GW}$.

Proposition 3.12. *If $\mathcal{A} = \{a_0 = 0 < a_1 < \cdots < a_n = D\} \subset \mathbb{N}$ is a finite set in normal form, set*

- $s_0^{EG} := \left\lceil 2\left(\left\lfloor \frac{D}{n} \right\rfloor (1 + \frac{a_{n-1} - a_1}{D}) - 1 + \frac{1}{D}\right)\right\rceil$, *and*
- $s_0^{L} := \max_{1 \leq i < j \leq n} \{(a_i - a_{i-1}) + (a_j - a_{j-1})\} - 1$.

Then, the smallest integer σ such that the decomposition (3.1) in Theorem 3.1 holds for all $s \geq \sigma$, i.e., the sumsets regularity of $\mathcal{A}$, satisfies

$$\sigma \leq \max\{s_0^{EG}, s_0^{L}\}.$$

Proof. By Proposition 1.2 (2), one has that $c_1 \leq 2a_{n-1}\lfloor \frac{D}{n} \rfloor - D + 1$ and $c_2 \leq 2(D - a_1)\lfloor \frac{D}{n} \rfloor - D + 1$. Combining these two bounds, one gets that $\lceil \frac{c_1 + c_2}{D} \rceil \leq s_0^{EG}$. On the other hand, by Remark 1.52 (3) and Theorem 1.76 (2), one has that $\mathrm{r}(\Bbbk[\mathcal{C}]) \leq \mathrm{reg}(\Bbbk[\mathcal{C}]) \leq s_0^L$. Hence, the upper bound follows from Theorem 3.1. □

Example 3.13. (1) Consider the set $\mathcal{A} = \{0, 2, 5, 6, 9\}$ from Example 3.10 (2), and recall that $\sigma = 2$ in this case. The previously known bounds on σ are $s_0^N = 216$, $s_0^{WCC} = 16$, $s_0^{GS} = 8$ and $s_0^{GW} = 6$. The new bound given in Proposition 3.12 is 5, since $s_0^{EG} = 4$ and $s_0^L = 5$. Although the bound is not sharp in this case, it is better than the other known bounds on σ.

(2) For $\mathcal{A} = \{0, 1, 5, 6\}$, one has that $\mathcal{A}$ is one of the families in Theorem 3.6. Hence, $\sigma = s_0^{GW} = 4$. On the other hand, $s_0^L = 4$ and $s_0^{EG} = 5$. Hence, the upper bound on the sumsets regularity given in Proposition 3.12 is $\sigma \leq 5$, which in this case is worse than s_0^{GW}.

When $a_1 = 1$ and $a_{n-1} = D - 1$, it can happen that $s_0^{GW} < \max\{s_0^{EG}, s_0^L\}$, as Example 3.13 (2) shows. If $a_1 \neq 1$ or $a_{n-1} \neq D - 1$, and $n \geq 6$, then $s_0^{EG} < s_0^{GW}$. We prove this in Proposition 3.14, and we compare the bounds s_0^{GW} and s_0^L in Proposition 3.15.

Proposition 3.14. *Let $\mathcal{A} = \{a_0 = 0 < a_1 < \cdots < a_n = D\} \subset \mathbb{N}$ be a finite set in normal form. If $a_1 \neq 1$ or $a_{n-1} \neq D - 1$, then $s_0^{EG} \leq D - n + 1 = s_0^{GW}$, where*

$$s_0^{EG} = \left\lceil 2\left(\left\lfloor \frac{D}{n} \right\rfloor \left(1 + \frac{a_{n-1} - a_1}{D}\right) - 1 + \frac{1}{D}\right)\right\rceil .$$

Indeed, if $n \geq 6$ and both $a_1 \neq 1$ and $a_{n-1} \neq D - 1$, then $s_0^{EG} < s_0^{GW}$.

Proof. If $a_1 \neq 1$ or $a_{n-1} \neq D-1$, then $a_{n-1} - a_1 \leq D-3$ and $D \geq n+1$. Therefore,

$$2\left(\left\lfloor \frac{D}{n} \right\rfloor \left(1 + \frac{a_{n-1} - a_1}{D}\right) - 1 + \frac{1}{D}\right) \leq 2\left(\frac{D}{n}\left(1 + \frac{D-3}{D}\right) - 1 + \frac{1}{D}\right)$$
$$= 2\left(2\frac{D}{n} - \frac{3}{n} - 1 + \frac{1}{D}\right),$$

and the rightmost part of the previous equation is $\leq D - n + 1$ if and only if $(n-4)D^2 + (6 + 3n - n^2)D - 2n \geq 0$. The largest root of this degree-2 polynomial in D is

$$\alpha(n) = \frac{n^2 - 3n - 6 + \sqrt{(6 + 3n - n^2)^2 + 4(n-4)2n}}{2(n-4)},$$

and one has that $\alpha(n) \leq n+1$ for all $n \in \mathbb{N}$, since the real function $x \mapsto \alpha(x) - (x+1)$ is negative for all $x \geq 0$. Therefore, $(n-4)D^2 + (6 + 3n - n^2)D - 2n \geq 0$ for all $D \geq n + 1$. We have proved

$$2\left(\left\lfloor \frac{D}{n} \right\rfloor \left(1 + \frac{a_{n-1} - a_1}{D}\right) - 1 + \frac{1}{D}\right) \leq D - n + 1,$$

and hence $s_0^{EG} \leq D - n + 1$.

Assume now that $a_1 \neq 1$, $a_{n-1} \neq D - 1$ and $n \geq 6$. Hence, $D \geq n + 2$. If $D = n + 2$, then $\mathcal{A} = \{0\} \sqcup [2, n] \sqcup \{n + 2\}$, and one has that $s_0^{GW} = 3$ and $s_0^{EG} = \lceil 2 - \frac{6}{n+2} \rceil \leq 2$. Suppose $D \geq n + 3$. Using the same argument as before, now we have to check the inequality $2\left(2\frac{D}{n}-\frac{4}{n}-1+\frac{1}{D}\right)\leq D-n$, and it holds if and only if $(n-4)D^2+(8+2n-n^2)D-2n\geq 0$. The largest root of this degree-2 polynomial is

$$\beta(n)=\frac{n^2-2n-8+\sqrt{(8+2n-n^2)^2+4(n-4)2n}}{2(n-4)},$$

and one can show that $\beta(n)\leq n+3$ for all $n\geq 6$. Hence, $(n-4)D^2+(8+2n-n^2)D-2n\geq 0$ for all $D\geq n+3$, so $s_0^{EG}\leq D-n$ in this case. □

On the other hand, one always has that $s_0^L\leq D-n+1$, and we can also determine when the strict inequality holds. For a set $\mathcal{A}=\{a_0=0<a_1<\cdots<a_n=D\}\subset\mathbb{N}$ in normal form, set $\beta_0:=0$, $\alpha_{\ell+1}:=D$, and write

$$\mathcal{A}=[\beta_0,\alpha_1]\sqcup[\beta_1,\alpha_2]\sqcup\ldots\sqcup[\beta_{\ell-1},\alpha_l]\sqcup[\beta_\ell,\alpha_{\ell+1}]\,, \tag{3.4}$$

with $\ell\geq 0$, $\alpha_{i+1}-\beta_i\geq 0$ for all $i\in\{0,\ldots,\ell\}$ and if $\ell\geq 1$, then $\beta_i-\alpha_i\geq 2$ for all $i\in\{1,\ldots,\ell\}$. Note that this way of writing $\mathcal{A}$ is unique.

Proposition 3.15. *Let $\mathcal{A}=\{a_0=0<a_1<\cdots<a_n=D\}\subset\mathbb{N}$ be a set in normal form. Then, $s_0^L\leq s_0^{GW}$. Indeed, if one writes $\mathcal{A}$ as in Equation (3.4), then $s_0^L=s_0^{GW}$ if $0\leq\ell\leq 2$, and $s_0^L<s_0^{GW}$ if $\ell\geq 3$.*

Proof. Write $\mathcal{A}=[\beta_0,\alpha_1]\sqcup[\beta_1,\alpha_2]\sqcup\ldots\sqcup[\beta_{\ell-1},\alpha_l]\sqcup[\beta_\ell,\alpha_{\ell+1}]$ with $\ell\geq 0$, $\alpha_{i+1}-\beta_i\geq 0$ for all $i\in\{0,\ldots,\ell\}$ and if $\ell\geq 1$, then $\beta_i-\alpha_i\geq 2$ for all $i\in\{1,\ldots,\ell\}$.
If $\ell=0$, then $\mathcal{A}=[0,D]$, and $s_0^L=s_0^{GW}=1$. If $\ell_1=1$, then $s_0^L=s_0^{GW}=\beta_1-\alpha_1$, and if $\ell=2$, then $s_0^L=s_0^{GW}=(\beta_1-\alpha_1)+(\beta_2-\alpha_2)-1$.
Assume $\ell\geq 3$, and let $1\leq j<k\leq\ell$ be such that $s_0^L=(\beta_j-\alpha_j)+(\beta_k-\alpha_k)-1$. On the other hand,

$$\begin{aligned}
s_0^{GW}&=D-n+1=D-|\mathcal{A}|+2=D-\left(\sum_{i=0}^{\ell}(\alpha_{i+1}-\beta_i+1)\right)+2\\
&=D-\left(D-\sum_{i=1}^{\ell}(\beta_i-\alpha_i)+\ell+1\right)+2=\sum_{i=1}^{\ell}(\beta_i-\alpha_i)-\ell+1\\
&=(\beta_j-\alpha_j)+(\beta_k-\alpha_k)+\sum_{\substack{1\leq i\leq\ell\\ i\neq j,k}}(\beta_i-\alpha_i)-\ell+1\\
&\geq(\beta_j-\alpha_j)+(\beta_k-\alpha_k)+2(\ell-2)-\ell+1\\
&\geq(\beta_j-\alpha_j)+(\beta_k-\alpha_k)=s_0^L+1\,.
\end{aligned}$$

Therefore, we have proved that $s_0^L\leq s_0^{GW}$, and the inequality is strict if and only if $\ell\geq 3$. □

With Propositions 3.14 and 3.15 in mind, we propose the following bound on the sumsets regularity of $\mathcal{A}$,

$$s_0^* := \begin{cases} s_0^L & \text{if } a_1 = 1 \text{ and } a_{n-1} = D-1, \\ \max\{s_0^L, s_0^{EG}\} & \text{otherwise,} \end{cases} \tag{3.5}$$

where $s_0^L = \max_{1\le i<j\le n}\{(a_i - a_{i-1}) + (a_j - a_{j-1})\} - 1$ and

$$s_0^{EG} = \left\lceil 2\left(\left\lfloor \frac{D}{n} \right\rfloor (1 + \frac{a_{n-1} - a_1}{D}) - 1 + \frac{1}{D}\right)\right\rceil.$$

According to the previous results and Proposition 3.12, one has that $\sigma(\mathcal{A}) \le s_0^* \le s_0^{GW}$. Indeed, if $n \ge 6$, $a_1 \ne 1$, $a_{n-1} \ne n-1$, and $\ell \ge 3$, then $\sigma(\mathcal{A}) \le s_0^* < s_0^{GW}$.

To conclude this section, we present a precise statement of the structure theorem, which summarizes the majority of the results in this section.

Theorem 3.16 (Refined Structure Theorem)**.** *Let $\mathcal{A} = \{a_0 = 0 < a_1 < \cdots < a_n = D\} \subset \mathbb{N}$ be a finite set in normal form. Denote by $\mathcal{S}_1$ the numerical semigroup generated by $\mathcal{A}_1 = \{a_1, \ldots, a_n\}$ and by $\mathcal{S}_2$ the numerical semigroup generated by $\mathcal{A}_2 = \{D - a_{n-1}, \ldots, D - a_1, D\}$. For $i = 1, 2$, set c_i the conductor of $\mathcal{S}_i$ and $C_i = \mathcal{S}_i \cap [0, c_i - 2]$. Then,*

$$s\mathcal{A} = C_1 \sqcup [c_1, sD - c_2] \sqcup (sD - C_2)$$

for all $s \ge \sigma = \max\{\lceil \frac{c_1+c_2}{D} \rceil, \mathrm{r}(\Bbbk[\mathcal{C}])\}$, where $\mathrm{r}(\Bbbk[\mathcal{C}])$ is the regularity of the Hilbert function of the projective monomial curve $\mathcal{C}$ determined by $\mathcal{A}$. Moreover, $\sigma \le s_0^*$, where s_0^* is the number defined in Equation (3.5).

3.2 Homogeneous sets in $\mathbb{N}^2$

Let $\mathcal{A} = \{a_0 = 0 < a_1 < \cdots < a_n = D\} \subset \mathbb{N}$ be a set in normal form. As already observed, associated with $\mathcal{A}$, one has the set

$$\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \underline{\mathbf{a}}_1, \ldots, \underline{\mathbf{a}}_n\} \subset \mathbb{N}^2,$$

where $\underline{\mathbf{a}}_i = (D - a_i, a_i)$ for all $i = 0, \ldots, n$. A semigroup $\mathcal{S}$ in $\mathbb{N}^2$ generated by a set $\underline{\mathcal{A}}$ of this form will be said to be *homogeneous of degree D*.

It is straightforward to verify that the sumsets of $\underline{\mathcal{A}}$ are completely determined by those of $\mathcal{A}$, since, for each $s \in \mathbb{N}$,

$$s\underline{\mathcal{A}} = \{(sD - y, y) : y \in s\mathcal{A}\}.$$

In particular, for any $s \in \mathbb{N}$, $|s\mathcal{A}| = |s\underline{\mathcal{A}}|$. Furthermore, the affine semigroup $\mathcal{S}$ generated by $\underline{\mathcal{A}}$ satisfies that $\mathcal{S} = \sqcup_{s=0}^{\infty} s\underline{\mathcal{A}}$. Note that each $s\underline{\mathcal{A}}$ lies on the "line" $L_s := \{(x, y) \in \mathbb{N}^2 : x + y = sD\}$.

We can apply the structure theorem from the previous section to expand our understanding of the sumsets of $\underline{\mathcal{A}}$ and the semigroup $\mathcal{S}$. By Theorem 3.1 and Proposition 3.4, we have that for all $s \geq \sigma(\mathcal{A})$, the sumsets regularity of $\mathcal{A}$, $s\underline{\mathcal{A}}$ consists of a central interval and, outside that interval, a copy of the non-trivial part of the semigroups $\mathcal{S}_1$ and $\mathcal{S}_2$, i.e., for all $s \geq \sigma(\mathcal{A})$,

$$s\underline{\mathcal{A}} = \{(sD - i, i) : i \in \mathcal{S}_1 \cap [0, c_1 - 2]\} \sqcup \{(sD - i, i) : i \in [c_1, sD - c_2]\} \sqcup \{(i, sD - i) : i \in \mathcal{S}_2 \cap [0, c_2 - 2]\}. \quad (3.6)$$

Furthermore, $\sigma(\mathcal{A})$ is the least integer such that this decomposition is satisfied for all $s \geq \sigma(\mathcal{A})$. More precisely, for $s \geq \sigma(\mathcal{A})$, when we go from $s\underline{\mathcal{A}}$ to $(s+1)\underline{\mathcal{A}}$, gaps coming from $\mathcal{S}_1$ move to the right while gaps coming from $\mathcal{S}_2$ move up, and there are no other gaps in $(s+1)\underline{\mathcal{A}}$ than the ones coming from $s\underline{\mathcal{A}}$, as shown in Figure 3.1. And $\sigma(\mathcal{A})$ is the least integer such that this occurs. For this reason, the sumsets regularity of $\mathcal{A}$, $\sigma(\mathcal{A})$, could also be called the *sumsets regularity* of $\underline{\mathcal{A}}$, and denoted by $\sigma(\underline{\mathcal{A}})$. If no confusion arises, from now on, we will simply denote this number by σ, i.e., $\sigma = \sigma(\mathcal{A}) = \sigma(\underline{\mathcal{A}})$.

Example 3.17. Take $D \in \mathbb{N}$, $D \geq 4$, and consider $\mathcal{A} = \{0, 1, D-1, D\}$. The homogenization of $\mathcal{A}$ is $\underline{\mathcal{A}} = \{(D, 0), (D-1, 1), (1, D-1), (0, D)\}$. By Theorem 3.1, Proposition 3.4, and Theorem 3.6, $s\mathcal{A} = [0, sD]$ for all $s \geq D - 2$. Indeed, the sumsets regularity of $\mathcal{A}$ is $\sigma(\mathcal{A}) = D - 2$. Hence, one has that $s\underline{\mathcal{A}} = \{(sD - i, i) \mid i \in [0, sD]\} = L_s$ for all $s \geq D - 2$.

Recall from Section 2.1 the definition of the Apéry $\mathrm{AP}_{\mathcal{S}}$ and the exceptional set $E_{\mathcal{S}}$ of $\mathcal{S}$ (Definition 2.2):

- $\mathrm{AP}_{\mathcal{S}} := \{\mathbf{s} \in \mathcal{S} : \mathbf{s} - \underline{\mathbf{a}}_0 \notin \mathcal{S}, \mathbf{s} - \underline{\mathbf{a}}_n \notin \mathcal{S}\}$, and
- $E_{\mathcal{S}} := \{\mathbf{s} \in \mathcal{S} : \mathbf{s} - \underline{\mathbf{a}}_0 \in \mathcal{S}, \mathbf{s} - \underline{\mathbf{a}}_n \in \mathcal{S}, \mathbf{s} - \underline{\mathbf{a}}_0 - \underline{\mathbf{a}}_n \notin \mathcal{S}\}$.

Moreover, for each $s \in \mathbb{N}$, set $\mathrm{AP}_s := \mathrm{AP}_{\mathcal{S}} \cap L_s = \mathrm{AP}_{\mathcal{S}} \cap s\underline{\mathcal{A}}$ and $E_s := E_{\mathcal{S}} \cap L_s = E_{\mathcal{S}} \cap s\underline{\mathcal{A}}$. Figure 3.2 shows what points in $E_{\mathcal{S}}$ and $\mathrm{AP}_{\mathcal{S}}$ look like when one draws the semigroup $\mathcal{S}$ taking into account the levels determined by the sumsets of $\underline{\mathcal{A}}$.

Remark 3.18. As a consequence of Theorem 3.1 and, more precisely, Equation (3.6), one gets that, if σ is the sumsets regularity of $\underline{\mathcal{A}}$, then

$$\forall s \geq \sigma + 2,\ \mathrm{AP}_s = E_s = \emptyset\,.$$

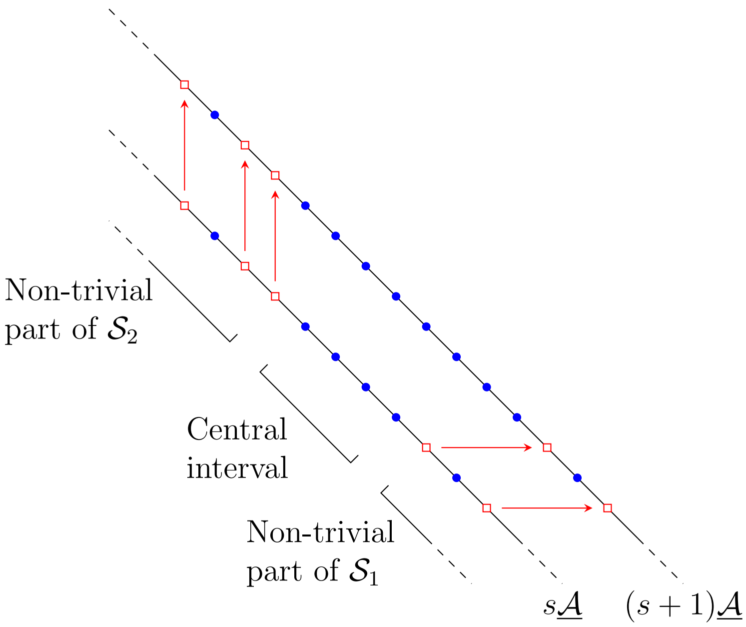


Figure 3.1: Structure of the sumsets of $\underline{\mathcal{A}}$. For $s \geq \sigma$, we distinguish three disjoint areas: the central interval and the copies of the non-trivial parts of $\mathcal{S}_1$ and $\mathcal{S}_2$.

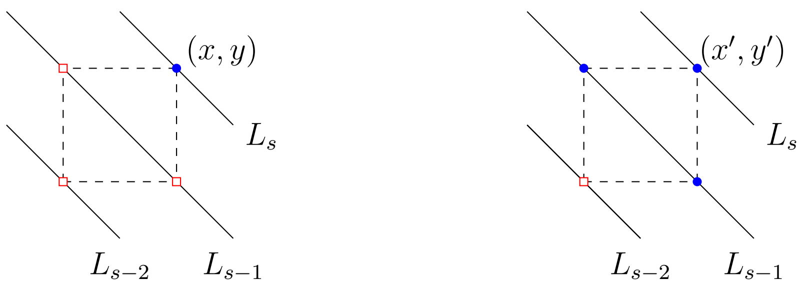


Figure 3.2: An element (x, y) in AP_s and an element (x', y') in E_s.

In particular, this shows that $\mathrm{AP}_{\mathcal{S}}$ and $E_{\mathcal{S}}$ are finite sets.

Now, we focus on the distribution of the elements (x,y) in $\mathrm{AP}_{\mathcal{S}}$ and $E_{\mathcal{S}}$ on the levels given by the sumsets of $\underline{\mathcal{A}}$ (or $\mathcal{A}$).

Proposition 3.19. *For all $s \in \mathbb{N}$, one has that*

$$|\mathrm{AP}_s| - |E_s| = |s\mathcal{A}| - 2|(s-1)\mathcal{A}| + |(s-2)\mathcal{A}| \,.$$

Proof. Let us count the number of elements in AP_s for all $s \in \mathbb{N}$. Note that $|\mathrm{AP}_0| = 1 = |0\mathcal{A}|$ and $|\mathrm{AP}_1| = |\underline{\mathcal{A}}| - 2 = |\mathcal{A}| - 2|0\mathcal{A}|$. Since $E_0 = E_1 = \emptyset$ and $s\mathcal{A} = \emptyset$ if $s < 0$, the formula holds for $s \leq 1$. Now, consider $s \geq 2$. For each element $\mathbf{s} \in (s-1)\underline{\mathcal{A}}$, neither $\mathbf{s} + \underline{\mathbf{a}}_0$ nor $\mathbf{s} + \underline{\mathbf{a}}_n$ belongs to AP_s. Thus, every element in $(s-1)\underline{\mathcal{A}}$ provides two elements in $s\underline{\mathcal{A}}$ that do not belong to AP_s and any other element in $s\underline{\mathcal{A}}$ belongs to AP_s. However, we are counting some of those elements twice, precisely the $\mathbf{s} \in s\underline{\mathcal{A}}$, such that $\mathbf{s} - \underline{\mathbf{a}}_0 \in (s-1)\underline{\mathcal{A}}$ and $\mathbf{s} - \underline{\mathbf{a}}_n \in (s-1)\underline{\mathcal{A}}$. Now, for such an element $\mathbf{s}$, either $\mathbf{s} - \underline{\mathbf{a}}_0 - \underline{\mathbf{a}}_n \notin (s-2)\underline{\mathcal{A}}$, and hence, $\mathbf{s} \in E_s$, or $(x,y) - \underline{\mathbf{a}}_0 - \underline{\mathbf{a}}_n \in (s-2)\underline{\mathcal{A}}$. This provides the following formula:

$$|\mathrm{AP}_s| = |s\underline{\mathcal{A}}| - 2|(s-1)\underline{\mathcal{A}}| + \left(|(s-2)\underline{\mathcal{A}}| + |E_s|\right),$$

and the result follows. □

Remark 3.20. As a consequence of the previous theorem and Remark 3.18, we obtain that $|\mathrm{AP}_{\mathcal{S}}| = |E_{\mathcal{S}}| + D$, since

$$\begin{aligned}|\mathrm{AP}_{\mathcal{S}}| &= \sum_{s=0}^{\sigma+1} |\mathrm{AP}_s| = \sum_{s=0}^{\sigma+1} \left(|s\mathcal{A}| - 2|(s-1)\mathcal{A}| + |(s-2)\mathcal{A}|\right) + \sum_{s=0}^{\sigma+1} |E_s| \\ &= \left(|(\sigma+1)\mathcal{A}| - |\sigma\mathcal{A}|\right) + |E_{\mathcal{S}}| = |E_{\mathcal{S}}| + D\,,\end{aligned}$$

where we have that $|(\sigma+1)\mathcal{A}| - |\sigma\mathcal{A}| = D$ since $\sigma \geq \mathrm{r}(\Bbbk[\mathcal{C}])$ by Theorem 3.7. In particular, $|\mathrm{AP}_{\mathcal{S}}| \geq D$, and we recover that (d) $\Leftrightarrow$ (e) in Proposition 2.4.

Corollary 3.21. *If $\mathcal{C}$ is arithmetically Cohen-Macaulay, then the sequence $(|s\mathcal{A}| - |(s-1)\mathcal{A}|)_{s=0}^{\infty} \subset \mathbb{N}$ is increasing (and it stabilizes at D).*

Proof. For each $s \in \mathbb{N}$, we observe that

$$|s\mathcal{A}| - |(s-1)\mathcal{A}| = \sum_{j=0}^{s} \left(|j\mathcal{A}| - 2|(j-1)\mathcal{A}| + |(j-2)\mathcal{A}|\right) = \sum_{j=0}^{s} |\mathrm{AP}_j|\,,$$

by Proposition 3.19, and the result follows. □

Remark 3.22. The result in Corollary 3.21 holds in a more general setting. For a graded (or local) $\Bbbk$-algebra R of Krull dimension two, the differences between two consecutive elements in the sequence $(\mathrm{HF}_R(s) - \mathrm{HF}_R(s-1))_{s=0}^{\infty}$ are the coefficients of its h-polynomial (the polynomial in the numerator of its Hilbert series) that are known to be non-negative when R is Cohen-Macaulay [87]. Thus, the sequence $(\mathrm{HF}_R(s) - \mathrm{HF}_R(s-1))_{s=0}^{\infty}$ is increasing.

Note that if one removes the Cohen-Macaulay hypothesis, then the result in Corollary 3.21 may be wrong, as the first example below shows. However, this property does not characterize arithmetically Cohen-Macaulay curves, as the second example shows.

Example 3.23. (1) For $\mathcal{A} = \{0, 1, 3, 11, 13\}$, $(|s\mathcal{A}| - |(s-1)\mathcal{A}|)_{s=0}^{\infty} = (1, 4, 9, 14, 17, 15, 13, 13, \dots)$ is not increasing, and hence, $\Bbbk[\mathcal{C}]$ is not Cohen-Macaulay by Corollary 3.21.

(2) [6, Ex. 4.3] For $\mathcal{A} = \{0, 5, 9, 11, 20\}$, $(|s\mathcal{A}| - |(s-1)\mathcal{A}|)_{s=0}^{\infty} = (1, 4, 9, 15, 20, 20, \dots)$ is increasing, but $\Bbbk[\mathcal{C}]$ is not Cohen-Macaulay.

3.3 Higher dimension. The simplicial case

Consider a finite set $\mathcal{A} = \{\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{N}^d$, $d \geq 2$, with $\mathbf{a}_i = (a_{i1}, \dots, a_{id})$ for $i \in \{0, \dots, n\}$, and set $D = \max\{|\mathbf{a}_i| \colon i = 0, \dots, n\}$. For all $i = 0, \dots, n$, set $\underline{\mathbf{a}}_i = (D - |\mathbf{a}_i|, \mathbf{a}_i)$, and consider $\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \dots, \underline{\mathbf{a}}_n\} \subset \mathbb{N}^{n+1}$. Let $\Bbbk$ be an algebraically closed field, and $\mathcal{X} = \mathcal{X}_{\underline{\mathcal{A}}} \subset \mathbb{P}^n_{\Bbbk}$ be the projective toric variety determined by $\underline{\mathcal{A}}$. We assume that $\mathcal{X}$ is simplicial. Hence, by Remark 1.71, we can assume, without loss of generality, that $\{D\boldsymbol{\epsilon}_0, \dots, D\boldsymbol{\epsilon}_d\} \subset \underline{\mathcal{A}}$, where $\{\boldsymbol{\epsilon}_0, \dots, \boldsymbol{\epsilon}_d\}$ denotes the canonical basis of $\mathbb{N}^{d+1}$. Note that this is equivalent to saying that $\mathcal{A} = \{\mathbf{a}_0, \dots, \mathbf{a}_n\} \subset \mathbb{N}^d$ has the following properties:

- $\mathbf{0} = (0, \dots, 0) \in \mathcal{A}$;
- for all $i = 1, \dots, d$, $\mathbf{e}'_i \coloneqq D\boldsymbol{\epsilon}'_i \in \mathcal{A}$, where $\{\boldsymbol{\epsilon}'_1, \dots, \boldsymbol{\epsilon}'_d\}$ denotes the canonical basis of $\mathbb{N}^d$; and
- for all $i = 0, \dots, n$, $|\mathbf{a}_i| = \sum_{j=1}^{d} a_{ij} \leq D$.

The following result describes the structure of the sumsets of such a set $\mathcal{A} \subset \mathbb{N}^d$. Granville and Shakan first obtained this result in [44], but later Curran and Goldmakher provided a bound on s for which the result holds.

Theorem 3.24 ([23, Thm. 1.3], [44, Thm. 2]). *Let $\mathcal{A} \subset \mathbb{N}^d$ be a finite set such that $\mathcal{A} - \mathcal{A}$ generates $\mathbb{Z}^d$ additively, $\mathbf{0} \in \mathcal{A}$, $\mathbf{e}'_i \in \mathcal{A}$ for all $i \in \{1, \ldots, d\}$, and $|\mathbf{a}| \leq D$ for all $\mathbf{a} \in \mathcal{A}$. For all $i \in \{1, \ldots, d\}$, set*

$$T_i(\mathcal{A}) \coloneqq \langle \mathcal{A} - \mathbf{e}'_i \rangle \subset \mathbb{Z}^d ,$$

the subsemigroup of $\mathbb{Z}^d$ generated by $\mathcal{A} - \mathbf{e}'_i$. Then, for all $s \in \mathbb{N}$ such that $s \geq (d+1)D^d - 2 - 2d$, one has that

$$s\mathcal{A} = \langle \mathcal{A} \rangle \cap \left(\bigcap_{i=1}^{d} (s\mathbf{e}'_i + T_i(\mathcal{A})) \right) .$$

In this section, our aim is to provide a more explicit formulation of Theorem 3.24 for certain classes of sets $\mathcal{A}$, together with improved bounds for s such that this result holds. More specifically, we are interested in the sets $\mathcal{A}$ such that the simplicial projective toric variety $\mathcal{X} = \mathcal{X}_{\underline{\mathcal{A}}}$ is smooth (Subsection 3.3.1), or it is a surface with a single singular point (Subsection 3.3.2).

3.3.1 The smooth case

Let $\mathcal{A} = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$ be a finite set, and suppose that $\underline{\mathcal{A}}$ defines a smooth simplicial projective toric variety $\mathcal{X} = \mathcal{X}_{\underline{\mathcal{A}}} \subset \mathbb{P}^n_{\Bbbk}$. Hence, without loss of generality, we can assume that $\underline{\mathcal{A}} \subset \mathbb{N}^{d+1}$ satisfies the conditions of Theorem 1.72. This is equivalent to saying that $\mathcal{A}$ has the following properties:

- $\mathbf{0} = (0, \ldots, 0) \in \mathcal{A}$;
- for all $i \in \{0, \ldots, n\}$, $|\mathbf{a}_i| \leq D$;
- for all $i \in \{1, \ldots, d\}$, $\boldsymbol{\epsilon}'_i \in \mathcal{A}$ and $(D-1)\boldsymbol{\epsilon}'_i \in \mathcal{A}$; and
- for all $1 \leq i \leq j \leq d$, $\boldsymbol{\epsilon}'_i + (D-1)\boldsymbol{\epsilon}'_j \in \mathcal{A}$;

where $\{\boldsymbol{\epsilon}'_1, \ldots, \boldsymbol{\epsilon}'_d\}$ denotes the canonical basis of $\mathbb{N}^d$. For all $i = 1, \ldots, d$, set $\mathbf{e}'_i \coloneqq D\boldsymbol{\epsilon}'_i$.

For all $s \in \mathbb{N}$, denote $\Delta_s \coloneqq \{(y_1, \ldots, y_d) \in \mathbb{N}^d \mid y_1 + \cdots + y_d \leq sD\}$. Note that for all $s \geq 1$, Δ_s is the set of lattice points of a simplex. Moreover, one has that $s\mathcal{A} \subset \Delta_s$ for all $s \in \mathbb{N}$. In this context, Theorem 3.24 can be rewritten as follows.

Proposition 3.25. *Let $\mathcal{A} \subset \mathbb{N}^d$ be a finite set as above. Then, $s\mathcal{A} = \Delta_s$ for all $s \in \mathbb{N}$, $s \geq (d+1)D^d - 2 - 2d$.*

Proof. Let us compute the elements that appear in Theorem 3.24. Since $\{\boldsymbol{\epsilon}'_1, \ldots, \boldsymbol{\epsilon}'_d\} \subset \mathcal{A}$, then $\langle \mathcal{A} \rangle = \mathbb{N}^d$. Take $i \in \{1, \ldots, d\}$ and let us describe the set $T_i(\mathcal{A})$. Consider the following subset $\mathcal{A}(i)$ of $\mathcal{A}$

$$\mathcal{A}(i) = \{(D-1)\boldsymbol{\epsilon}'_i\} \cup \{\boldsymbol{\epsilon}'_j + (D-1)\boldsymbol{\epsilon}'_i \mid 1 \leq j \leq d\} \subset \mathcal{A},$$

and set $\mathcal{B}_i := \mathcal{A}(i) - D\boldsymbol{\epsilon}'_i$. One has that $\mathcal{B}_i = \{-\boldsymbol{\epsilon}'_i\} \cup \{\boldsymbol{\epsilon}'_j - \boldsymbol{\epsilon}'_i \mid 1 \leq j \leq d\}$. The semigroup generated by $\mathcal{B}_i$ is given by the lattice points in the cone with rays $\boldsymbol{\epsilon}'_j - \boldsymbol{\epsilon}'_i$, $j \neq i$, and $-\boldsymbol{\epsilon}'_i$, and it can be written as

$$\langle \mathcal{B}_i \rangle = \{(y_1, \ldots, y_d) \in \mathbb{Z}^d \mid y_i \leq 0, y_j \geq 0, \forall j \neq i, y_1 + \cdots + y_d \leq 0\}.$$

Since $\mathcal{B}_i \subset \mathcal{A} - D\boldsymbol{\epsilon}'_i$ and $\mathcal{A} - D\boldsymbol{\epsilon}'_i \subset \langle \mathcal{B}_i \rangle$, then $T_i(\mathcal{A}) = \langle \mathcal{A} - D\boldsymbol{\epsilon}'_i \rangle = \langle \mathcal{B}_i \rangle$. Therefore,

$$s\mathbf{e}'_i + T_i(\mathcal{A}) = \{(y_1, \ldots, y_d) \in \mathbb{Z}^d \mid y_i \leq sD,\ y_j \geq 0, \forall j \neq i,\ y_1 + \cdots + y_d \leq sD\},$$

and hence,

$$\bigcap_{i=1}^{d} (T_i(\mathcal{A}) + s\mathbf{e}'_i) = \{(y_1, \ldots, y_d) \in \mathbb{Z}^d \mid y_i \geq 0, \forall i,\ y_1 + \cdots + y_d \leq sD\} = \Delta_s \,.$$

By Theorem 3.24, $s\mathcal{A} = \mathbb{N}^d \cap \Delta_s = \Delta_s$ for all $s \geq (d+1)D^d - 2 - 2d$. $\square$

Theorem 3.26. *Let* $\mathcal{A} = \{\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$ *be a finite set such that* $\{\mathbf{0}, \mathbf{e}'_1, \ldots, \mathbf{e}'_d\} \subset \mathcal{A}$ *and* $|\mathbf{a}_i| \leq D$ *for all* $i \in \{0, \ldots, n\}$. *Consider* $\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \ldots, \underline{\mathbf{a}}_n\} \subset \mathbb{N}^{d+1}$ *the homogenization of* $\mathcal{A}$, *where* $\underline{\mathbf{a}}_i = (D - |\mathbf{a}_i|, \mathbf{a}_i)$ *for all* $i \in \{0, \ldots, n\}$. *If* $\mathbb{k}$ *is an algebraically closed field, then the following conditions are equivalent:*

- *(a) The simplicial projective toric variety defined by* $\underline{\mathcal{A}}$, $\mathcal{X} = \mathcal{X}_{\underline{\mathcal{A}}} \subset \mathbb{P}^n_{\mathbb{k}}$ *is smooth.*
- *(b)* $\{\boldsymbol{\epsilon}_i + (D-1)\boldsymbol{\epsilon}_j \mid 0 \leq i, j \leq d\} \subset \underline{\mathcal{A}}$, *where* $\{\boldsymbol{\epsilon}_0, \ldots, \boldsymbol{\epsilon}_d\}$ *is the canonical basis of* $\mathbb{N}^{d+1}$.
- *(c)* $\{\mathbf{0}, \boldsymbol{\epsilon}'_i, (D-1)\boldsymbol{\epsilon}'_i, \boldsymbol{\epsilon}'_i + (D-1)\boldsymbol{\epsilon}'_j \mid 1 \leq i, j \leq d\} \subset \mathcal{A}$, *where* $\{\boldsymbol{\epsilon}'_1, \ldots, \boldsymbol{\epsilon}'_d\}$ *is the canonical basis of* $\mathbb{N}^d$.
- *(d) There exists* $s_0 \in \mathbb{N}$ *such that* $s\mathcal{A} = \Delta_s$ *for all* $s \geq s_0$.

Proof. The equivalence (a) $\Leftrightarrow$ (b) is Proposition 1.78, (b) $\Leftrightarrow$ (c) is direct from the construction of $\underline{\mathcal{A}}$, and the implication (c) $\Rightarrow$ (d) is Proposition 3.25. Let us prove (d) $\Rightarrow$ (c). First, if $s\mathcal{A} = \Delta_s$ for all $s \gg 0$, then $\langle \mathcal{A} \rangle = \cup_{s=0}^{\infty} s\mathcal{A} = \mathbb{N}^d$. For all $i = 1, \ldots, d$, set $T_i(\mathcal{A}) = \langle \mathcal{A} - \mathbf{e}'_i \rangle$. By Theorem 3.24 and (d), we have that for all $s \gg 0$,

$$s\mathcal{A} = \bigcap_{i=1}^{d} (s\mathbf{e}'_i + T_i(\mathcal{A}))\,.$$

Thus, for all i, one must have

$$T_i(\mathcal{A}) = \{(y_1, \dots, y_d) \in \mathbb{Z}^d \mid y_i \le 0, y_j \ge 0, \forall j \ne i, y_1 + \cdots + y_d \le 0\}. \qquad (3.7)$$

The inclusion $\subset$ is clear. To show the reverse inclusion, suppose that there exists a point $\mathbf{y} = (y_1, \dots, y_d)$ in the right-hand size of the equation that does not belong to $T_i(\mathcal{A})$. Then $\mathbf{y} + s\mathbf{e}'_i \notin s\mathcal{A}$ for all $s \ge \max\{s_0, (|y_1| + \cdots + |y_d|)/D\}$, a contradiction, and hence the equality in (3.7) holds. To conclude, note that (3.7) is equivalent to $\{(D-1)\boldsymbol{\epsilon}'_i\} \cup \{\boldsymbol{\epsilon}'_j + (D-1)\boldsymbol{\epsilon}'_i \mid 1 \le j \le d\} \subset \mathcal{A}$. Therefore, $\mathcal{A}$ is as in (c). □

Example 3.27. Consider $\mathcal{A} = \{(0,0),(1,0),(2,0),(3,0),(0,1),(0,2),(0,3),(1,2),(2,1)\} \subset \mathbb{N}^2$. By Proposition 3.25, one has that $s\mathcal{A} = \Delta_s = \{(x,y) \in \mathbb{N}^2 \mid x + y \le 3s\}$ for all $s \in \mathbb{N}$, $s \ge 19$. However, one can check that the equality $s\mathcal{A} = \Delta_s$ holds for all $s \ge 2$.

As Example 3.27 shows, the bound $s \ge (d+1)D^d - 2 - 2d$ in Proposition 3.25 is usually far from being tight. This observation motivates the following definition.

Definition 3.28. Let $\mathcal{A} \subset \mathbb{N}^d$ be a set satisfying Theorem 3.26 (c). The *sumsets regularity of* $\mathcal{A}$ is defined as $\sigma(\mathcal{A}) = \min\{s \in \mathbb{N} \mid s'\mathcal{A} = \Delta_{s'}, \forall s' \ge s\}$.

Theorem 3.29. *Let $\mathcal{A} \subset \mathbb{N}^d$ be a set satisfying Theorem 3.26 (c). Then, the sumsets regularity of $\mathcal{A}$ is $\sigma(\mathcal{A}) \le d(D-2)$.*

Proof. Note that it is sufficient to prove that $s\mathcal{A} = \Delta_s$ for $s = d(D-2)$, since it implies $s\mathcal{A} = \Delta_s$ for all $s \ge d(D-2)$. We prove $d(D-2)\mathcal{A} = \Delta_{d(D-2)}$ by induction on $d \ge 1$. For $d = 1$, the result follows from Example 3.17. Suppose that the result holds for $d-1$, and let us prove it for d. Take $\mathbf{y} = (y_1, \dots, y_d) \in \mathbb{N}^d$, such that $|\mathbf{y}| \le d(D-2)D$, and let us show that $\mathbf{y} \in d(D-2)\mathcal{A}$.
<u>Case 1:</u> If $|\mathbf{y}| \le (d-1)(D-2)D$, there exists an index j, $1 \le j \le d$, such that $y_j \le (D-2)D$. We write $\mathbf{y} = \mathbf{y}^{(j)} + \mathbf{y}_j$, where $\mathbf{y}^{(j)} = (y_1, \dots, y_{j-1}, 0, y_{j+1}, \dots, y_d)$ and $\mathbf{y}_j = y_j\boldsymbol{\epsilon}'_j$. Since $y_j \le (D-2)D$, then $y_j \in (D-2)\{0, 1, D-1, D\}$, by Example 3.17, so $\mathbf{y}_j \in (D-2)\mathcal{A}$. On the other hand, by the inductive hypothesis, one has that $\mathbf{y}^{(j)} \in (d-1)(D-2)\mathcal{A}$, by considering $(y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_d) \in \mathbb{N}^{d-1}$. Hence, $\mathbf{y} = \mathbf{y}^{(j)} + \mathbf{y}_j \in d(D-2)\mathcal{A}$.
<u>Case 2:</u> Suppose now $|\mathbf{y}| > (d-1)(D-2)D$. We distinguish between two cases. First, let us prove the following claim.
Claim: If $|\mathbf{y}| = (d-1)(D-2)D$, then $\mathbf{y} \in (d-1)(D-2)\mathcal{A}$.
Proof of the claim: If $d = 1$, the result is clear, so assume $d \ge 2$. Let $\pi : \mathbb{N}^d \to \mathbb{N}^{d-1}$ be the projection $\pi(z_1, \dots, z_d) = (z_2, \dots, z_d)$, and consider $\pi(\mathbf{y}) \in \mathbb{N}^{d-1}$. Set $\tilde{\mathcal{A}} := \pi(\{\mathbf{a} \in \mathcal{A} : |\mathbf{a}| = D\}) \subset \mathbb{N}^{d-1}$, and note that $\tilde{\mathcal{A}}$ satisfies Theorem 3.26 (c). Since

$|\pi(\mathbf{y})|\leq (d-1)(D-2)D$, then by the inductive hypothesis $\pi(\mathbf{y}) \in (d-1)(D-2)\tilde{\mathcal{A}}$. Thus, homogenizing with respect to the first coordinate, one gets that

$$\mathbf{y} = ((d-1)(D-2)D - |\pi(\mathbf{y})|, y_2, \ldots, y_d) \in (d-1)(D-2)\mathcal{A},$$

and this concludes the proof of the claim
Case 2.1: Suppose $|\mathbf{y}|= (d-1)(D-2)D + \lambda D - \mu$ for some $1 \leq \lambda \leq D-2$ and $0 \leq \mu \leq \lambda$. Take $\mathbf{b}_1, \ldots, \mathbf{b}_\mu \in \mathcal{A}$ with $|\mathbf{b}_i|= D-1$ for all i, and $\mathbf{c}_1, \ldots, \mathbf{c}_{\lambda-\mu} \in \mathcal{A}$ with $|\mathbf{c}_j|= D$ for all j, such that $\mathbf{z} = \mathbf{y} - \sum_{i=1}^{\mu} \mathbf{b}_i - \sum_{j=1}^{\lambda-\mu} \mathbf{c}_j \in \mathbb{N}^d$. Then, $|\mathbf{z}|= (d-1)(D-2)D$, and hence $\mathbf{z} \in (d-1)(D-2)\mathcal{A}$ by the previous claim. Therefore, $\mathbf{y} \in s\mathcal{A}$ for $s = (d-1)(D-2) + \mu + (\lambda-\mu) \leq d(D-2)$, so $\mathbf{y} \in d(D-2)\mathcal{A}$.
Case 2.2: Suppose $|\mathbf{y}|= (d-1)(D-2)D + \lambda D - \mu$ for some $1 \leq \lambda \leq D-2$ and $\lambda < \mu \leq D-1$. Take $\mathbf{b}_1, \ldots, \mathbf{b}_{D-\mu} \in \mathcal{A}$ with $|\mathbf{b}_i|= 1$ for all i, and $\mathbf{c}_1, \ldots, \mathbf{c}_{\lambda-1} \in \mathcal{A}$ with $|\mathbf{c}_j|= D$ for all j, such that $\mathbf{z} = \mathbf{y} - \sum_{i=1}^{D-\mu} \mathbf{b}_i - \sum_{j=1}^{\lambda-1} \mathbf{c}_j \in \mathbb{N}^d$. Then, $|\mathbf{z}|= (d-1)(D-2)D$, and hence $\mathbf{z} \in (d-1)(D-2)\mathcal{A}$ by the previous claim. Therefore, $\mathbf{y} \in s\mathcal{A}$ for $s = (d-1)(D-2) + D - \mu + \lambda - 1 \leq d(D-2)$, since $-\mu + \lambda \leq -1$, so $\mathbf{y} \in d(D-2)\mathcal{A}$. □

The bound on $\sigma(\mathcal{A})$ obtained in Theorem 3.29 is sharp, as the following result shows.

Corollary 3.30. *Let $\mathcal{A} = \{\mathbf{0}, \boldsymbol{\epsilon}'_i, (D-1)\boldsymbol{\epsilon}'_i, \boldsymbol{\epsilon}'_i + (D-1)\boldsymbol{\epsilon}'_j \mid 1 \leq i, j \leq d\} \subset \mathbb{N}^d$, where $\{\boldsymbol{\epsilon}'_1, \ldots, \boldsymbol{\epsilon}'_d\}$ is the canonical basis of $\mathbb{N}^d$. Then, $\sigma(\mathcal{A}) = d(D-2)$.*

Proof. By Theorem 3.29, $\sigma(\mathcal{A}) \leq d(D-2)$. Moreover, note that $(D-2, D-2, \ldots, D-2) \notin (d(D-2)-1)\,\mathcal{A}$, since the only way of writing it as a sum of nonzero elements in $\mathcal{A}$ is $(D-2, \ldots, D-2) = \sum_{i=1}^{d}(D-2)\boldsymbol{\epsilon}'_i$. Then, $(D-2, \ldots, D-2) \notin \Delta_{d(D-2)-1}$. Thus, $\sigma(\mathcal{A}) = d(D-2)$. □

3.3.2 Surfaces with one singular point

Let $\mathcal{A} = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^2$ be a finite set. Fix an algebraically closed field $\Bbbk$, and suppose that its homogenization $\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \ldots, \underline{\mathbf{a}}_n\} \subset \mathbb{N}^3$ defines a simplicial projective monomial surface with a single singular point. Proposition 1.78 characterizes such sets $\underline{\mathcal{A}}$. Hence,

$$\{(0,0), (D,0), (0,D), (D-1,1), (1,D-1), (D-e,0), (0,D-e)\} \subset \mathcal{A}, \qquad (3.8)$$

where $a_{i1} + a_{i2} \leq D$, $1 \leq e \leq D$ is a divisor of D that divides $a_{i1} + a_{i2}$ for all $i \in \{0, \ldots, n\}$; and if $e = 1$, then either $(1,0) \notin \mathcal{A}$ or $(0,1) \notin \mathcal{A}$.
We denote $\mathbb{Z}^2_e := \{(x,y) \in \mathbb{Z}^2 \mid e \text{ divides } x+y\}$ and $\mathbb{N}^2_e := \mathbb{Z}^2_e \cap \mathbb{N}^2$. Clearly, $\langle \mathcal{A} \rangle \subset \mathbb{N}^2_e$, out next goal is to prove that $\mathbb{N}^2_e \setminus \langle \mathcal{A} \rangle$ is finite. Set $\mathbf{e}'_1 := (D,0)$ and $\mathbf{e}'_2 := (0,D)$.

Remark 3.31. If $e = D$, we can write $\mathcal{A} = \{(0,0)\} \cup \underline{\mathcal{A}}'$, where $\mathcal{A}' \subset \mathbb{N}$ is a set that defines a smooth projective monomial curve, and $\underline{\mathcal{A}}'$ is the homogenization of $\mathcal{A}'$. From the expression $\mathcal{A} = \{(0,0)\} \cup \underline{\mathcal{A}}'$, it follows that for all $s \in \mathbb{N}$, $s\mathcal{A} = \cup_{i=0}^{s} i\underline{\mathcal{A}}'$. By Theorem 3.29, one has that $s\underline{\mathcal{A}}' = \{(x,y) \in \mathbb{N}^2 : x+y = sD\}$, for all $s \geq D-2$. Therefore, for all $(x,y) \in \mathbb{N}^2$ such that $x+y \geq (D-2)D$ and $x+y \equiv 0 \pmod{D}$, $(x,y) \in s\mathcal{A}$ for all $s \geq \frac{x+y}{D}$.

Lemma 3.32. *Assume $1 \leq e < D$. If $(x,y) \in \mathbb{N}_e^2$ and $x+y \geq D\left(D + \frac{D}{e} - 4\right)$, then $(x,y) \in s\mathcal{A}$ for all $s \in \mathbb{N}$, $s \geq \frac{x+y}{D}$, and, in particular, $(x,y) \in \langle \mathcal{A} \rangle$.*

Proof. Assume that $x+y \geq D\left(D + \frac{D}{e} - 4\right)$. Let $\lambda \in \mathbb{N}$, $\lambda \leq D/e - 1$ such that $x+y \equiv -\lambda e \pmod{D}$, and $\lambda_1, \lambda_2 \in \mathbb{N}$ such that $\lambda_1 + \lambda_2 = \lambda$, $x \geq \lambda_1(D-e)$, and $y \geq \lambda_2(D-e)$. Set $(x',y') = (x,y) - \lambda_1(D-e,0) - \lambda_2(0,D-e)$. Then $(x',y') \in \mathbb{N}^2$ satisfies $x'+y' \equiv 0 \pmod{D}$ and $x'+y' = x+y-\lambda(D-e)$. If $\lambda = 0$, then $x'+y' = x+y \geq D(D+\frac{D}{e}-4) \geq (D-2)D$; and if $\lambda \geq 1$, then $x+y \geq D(D+\frac{D}{e}-4)+e$ and hence $x'+y' = x+y-\lambda(D-e) \geq (D-2)D$. Thus, (x',y') can be written using the elements $(0,D), (1,D-1), (D-1,1), (D,0)$ in $\mathcal{A}$, by Example 3.17, so $(x,y) \in \langle \mathcal{A} \rangle$. We have proved that $(x,y) = \sum_i \mu_i(x_i,y_i) + \lambda_1(D-e,0) + \lambda_2(0,D-e)$ for some $\mu_i \in \mathbb{N}$, where $x_i + y_i = D$ for all i and $\lambda_1, \lambda_2 \in \mathbb{N}$ are as before. Then, $(x,y) \in \left(\sum_i \mu_i + \lambda\right)\mathcal{A}$. Take $s \in \mathbb{N}$ such that $s \geq \frac{x+y}{D}$ and let us prove that $\sum_i \mu_i + \lambda \leq s$. Since $sD \geq x+y = \left(\sum_i \mu_i\right)D + \lambda(D-e) = \left(\sum_i \mu_i + \lambda\right)D - \lambda e$, then

$$\sum_i \mu_i + \lambda \leq s + \lambda\frac{e}{D} \leq s + \left(\frac{D}{e} - 1\right)\frac{e}{D} < s+1\,.$$

Thus, $\sum_i \mu_i + \lambda \leq s$, and hence $(x,y) \in s\mathcal{A}$. □

Proposition 3.33. *The semigroup $\langle \mathcal{A} \rangle$ is contained in $\mathbb{N}_e^2$, and $|\mathbb{N}_e^2 \setminus \langle \mathcal{A} \rangle| < \infty$.*

Proof. For all $(x,y) \in \mathcal{A}$, $x+y$ is a multiple of e. Thus, $\langle \mathcal{A} \rangle \subset \mathbb{N}_e^2$. If $e = D$, the result follows from Remark 3.31. For $1 \leq e < D$, the result follows from Lemma 3.32. □

Set $\mathcal{H} := \mathbb{N}_e^2 \setminus \langle \mathcal{A} \rangle$. By Proposition 3.33, $\mathcal{H}$ is a finite set. Moreover, $\mathcal{H} \neq \emptyset$ if $e = 1$, since in this case either $(1,0) \notin \mathcal{A}$ or $(0,1) \notin \mathcal{A}$. For all $s \in \mathbb{N}$, denote $\mathcal{T}_{s,e} = \{(x,y) \in \mathbb{N}_e^2 \mid x+y \leq sD\}$. One has that $\mathcal{A} \subset \mathcal{T}_{1,e}$, and $s\mathcal{A} \subset \mathcal{T}_{s,e}$ for all $s \in \mathbb{N}$. Moreover, if $1 \leq e < D$, then $\mathcal{H} \subset \mathcal{T}_{D+D/e-4,e}$ by Lemma 3.32.

Proposition 3.34. *There exists $s_0 \in \mathbb{N}$ such that, for all $s \geq s_0$, $s\mathcal{A} = \mathcal{T}_{s,e} \setminus \mathcal{H}$.*

Proof. If $e = D$, the result follows from Remark 3.31. Suppose that $1 \leq e < D$. Then, the result follows from Lemma 3.32. □

Indeed, the condition $s\mathcal{A} = \mathcal{T}_{s,e} \setminus \mathcal{H}$ for all $s \gg 0$ characterizes the sets $\mathcal{A} \subset \mathbb{N}^2$ of the form (3.8), as the following result shows.

Theorem 3.35. *Let $\mathcal{A} = \{\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^2$ be a finite set, $\mathbf{a}_i = (a_{i1}, a_{i2})$, such that $\{(0,0), (D,0), (0,D)\} \subset \mathcal{A}$ and $|\mathbf{a}_i| = a_{i1} + a_{i2} \le D$ for all $i \in \{0, \ldots, n\}$. Consider $\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \ldots, \underline{\mathbf{a}}_n\} \subset \mathbb{N}^3$ the homogenization of $\mathcal{A}$, where $\underline{\mathbf{a}}_i = (D - a_{i1} - a_{i2}, a_{i1}, a_{i2})$ for all $i \in \{0, \ldots, n\}$. If $\Bbbk$ is an algebraically closed field, then the following conditions are equivalent:*

(a) The simplicial projective toric surface determined by $\underline{\mathcal{A}}$, $\mathcal{X} = \mathcal{X}_{\underline{\mathcal{A}}} \subset \mathbb{P}^n_{\Bbbk}$, has a single singular point, and it is $(1 : 0 : 0 : \cdots : 0)$.

(b) $\underline{\mathcal{A}}$ contains $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, (0, D-1, 1), (0, 1, D-1), (e, D-e, 0), (e, 0, D-e)\}$, where $\mathbf{e}_i = D\boldsymbol{\epsilon}_i$, $\{\boldsymbol{\epsilon}_0, \boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2\}$ is the canonical basis of $\mathbb{N}^3$, $1 \le e \le D$ is a divisor of D that divides a_{i0} for all $i \in \{0, \ldots, n\}$, and if $e = 1$ then either $(D-1, 1, 0) \notin \underline{\mathcal{A}}$ or $(D-1, 0, 1) \notin \underline{\mathcal{A}}$.

(c) $\mathcal{A}$ contains $\{(0,0), (D,0), (0,D), (D-1,1), (1,D-1), (D-e,0), (0,D-e)\}$, where $1 \le e \le D$ is a divisor of D that divides $a_{i1} + a_{i2}$ for all $i \in \{0, \ldots, n\}$, and if $e = 1$ then either $(1,0) \notin \mathcal{A}$ or $(0,1) \notin \mathcal{A}$.

(d) There exist a finite set $\mathcal{H} \subset \mathbb{N}^2_e$, with $\mathcal{H} \neq \emptyset$ if $e = 1$, and a number $s_0 \in \mathbb{N}$ such that $s\mathcal{A} = \mathcal{T}_{s,e} \setminus \mathcal{H}$ for all $s \ge s_0$.

Proof. The equivalence (a) $\Leftrightarrow$ (b) is Proposition 1.78, (b) $\Leftrightarrow$ (c) is direct from the construction of $\underline{\mathcal{A}}$, and the implication (c) $\Rightarrow$ (d) is Proposition 3.34. Let us prove (d) $\Rightarrow$ (c). Take $s \ge s_0$ such that $\mathcal{H} \subset \mathcal{T}_{s-1,e}$. Since $(sD-1, 1) \in s\mathcal{A}$, then $(D-1, 1) \in \mathcal{A}$. Moreover, since $(sD-e, 0) \in s\mathcal{A}$, then $(D-e, 0) \in \mathcal{A}$. Similarly, one can show that $(1, D-1) \in \mathcal{A}$ and $(0, D-e) \in \mathcal{A}$, and (c) follows. □

Definition 3.36. Let $\mathcal{A} \subset \mathbb{N}^2_e$ be a set as in (3.8). The *sumsets regularity of* $\mathcal{A}$, $\sigma(\mathcal{A})$, is the smallest number $s_0 \in \mathbb{N}$ such that $\mathcal{H} \subset \mathcal{T}_{s_0,e}$ and $s\mathcal{A} = \mathcal{T}_{s,e} \setminus \mathcal{H}$ for all $s \ge s_0$, i.e.

$$\sigma(\mathcal{A}) = \min\{s_0 \in \mathbb{N} \mid \mathcal{H} \subset \mathcal{T}_{s_0,e}, \text{ and } s\mathcal{A} = \mathcal{T}_{s,e} \setminus \mathcal{H}, \forall s \ge s_0\}\,.$$

When there is no confusion, we will denote it just by $\sigma = \sigma(\mathcal{A})$.

Remark 3.37. (1) If one allows $e = 1$ and $\{(1,0), (0,1)\} \subset \mathcal{A}$ in the previous definition, then we are under the hypothesis of Subsection 3.3.1. Note that in this case $\mathcal{H} = \emptyset$ and Definitions 3.28 and 3.36 coincide for such a set $\mathcal{A}$.

(2) For all $s \ge \sigma$, $(s+1)\mathcal{A} \setminus s\mathcal{A} = \mathcal{T}_{s+1,e} \setminus \mathcal{T}_{s,e}$. In fact, it is easy to show that

$$\sigma = \min\{s_0 \in \mathbb{N} \mid (s+1)\mathcal{A} \setminus s\mathcal{A} = \mathcal{T}_{s+1,e} \setminus \mathcal{T}_{s,e}, \forall s \ge s_0\}\,.$$

(3) If $e = D$, we can write $\mathcal{A} = \{(0,0)\} \cup \underline{\mathcal{A}}'$ as in Remark 3.31. From the fact $s\mathcal{A} = \cup_{i=0}^{s} i\underline{\mathcal{A}}'$ for all $s \in \mathbb{N}$, it follows that $\sigma(\mathcal{A}) = \sigma(\mathcal{A}')$, where $\sigma(\mathcal{A}')$ denotes the sumsets regularity of $\mathcal{A}'$. By Theorem 3.6, $\sigma(\mathcal{A}') \le D - |\mathcal{A}'| + 2 = D - n + 2$, and hence $\sigma(\mathcal{A}) \le D - n + 2$.

Example 3.38. (1) Let $D \in \mathbb{Z}_{>0}$ and $1 < e \le D$ a divisor of D. For $\mathcal{A} = \{(x,y) \in \mathbb{N}_e^2 \mid x + y \le D\}$, one has that $s\mathcal{A} = \mathcal{T}_{s,e}$ for all $s \in \mathbb{N}$, and hence $\sigma(\mathcal{A}) = 0$.

(2) For $\mathcal{A} = \{(0,0),(3,0),(0,3),(2,1),(2,1),(2,0),(0,2)\}$, one has that $\langle \mathcal{A} \rangle$ is a generalized numerical semigroup, by Theorem 1.24, and $\mathcal{H} = \mathbb{N}^2 \setminus \langle \mathcal{A} \rangle = \{(1,0),(0,1),(1,1),(1,3),(3,1)\}$. Hence, $\mathcal{H} \subset \mathcal{T}_{2,1}$. Moreover, one can check that $s\mathcal{A} = \mathcal{T}_{s,1} \setminus \mathcal{H}$ for all $s \ge 2$. Thus, $\sigma(\mathcal{A}) = 2$.

The next proposition provides a method for finding bounds on the sumsets regularity, $\sigma(\mathcal{A})$, which will be useful in the rest of the chapter.

Proposition 3.39. *Let $\mathcal{A} \subset \mathbb{N}_e^2$ be a set as in (3.8). Suppose that there exist positive integers ν, s_ν such that for all $(x,y) \in \mathbb{N}_e^2$,*

(i) if $(x,y) \in \langle \mathcal{A} \rangle$ and $x + y < \nu \cdot D$, then $(x,y) \in s_\nu \mathcal{A}$,

(ii) if $x + y \ge \nu \cdot D$, then $(x,y) \in s\mathcal{A}$ for all $s \ge \lceil \frac{x+y}{D} \rceil$.

Then, the sumsets regularity of $\mathcal{A}$ satisfies $\sigma(\mathcal{A}) \le \max\{s_\nu, \nu\}$.

Proof. Let $s \in \mathbb{N}$ such that $s \ge \max\{s_\nu, \nu\}$. By the above hypotheses, for all $(x,y) \in \langle \mathcal{A} \rangle$ such that $x + y \le sD$, one has that $(x,y) \in s\mathcal{A}$. Moreover, by (ii) $\mathbb{N}_e^2 \setminus \langle \mathcal{A} \rangle \subset \mathcal{T}_{s,e}$. Therefore, $\sigma(\mathcal{A}) \le s$, and the result follows. □

Applying the previous result and Lemma 3.32, we obtain a bound on $\sigma(\mathcal{A})$.

Proposition 3.40. *Let $\mathcal{A} \subset \mathbb{N}_e^2$ be a set as in (3.8). Then, the sumsets regularity of $\mathcal{A}$, $\sigma(\mathcal{A})$, satisfies*

$$\sigma(\mathcal{A}) \le \begin{cases} \frac{D}{e}\left(D + \frac{D}{e} - 4\right) & \text{if } 1 \le e < D \\ D - 2 & \text{if } e = D. \end{cases}$$

Proof. If $e = D$, the result follows from Remark 3.37 (3). Suppose $1 \le e < D$. By Lemma 3.32, for all $(x,y) \in \mathbb{N}_e^2$ such that $x + y \ge D(D + \frac{D}{e} - 4)$, one has that $(x,y) \in s\mathcal{A}$ for all $s \ge \lceil \frac{x+y}{D} \rceil$. Take $(x,y) \in \langle \mathcal{A} \rangle$ such that $x+y \le D(D+\frac{D}{e}-4)$, and write $(x,y) = \sum_{i=0}^{n} \mu_i \mathbf{a}_i$ for some $\mu_i \in \mathbb{N}$. Then, $(x,y) \in s\mathcal{A}$ for all $s \ge \sum_i \mu_i$ and, on the other hand, $x+y = \sum_i \mu_i|\mathbf{a}_i| \geq (\sum_i \mu_i)\, e$. Hence, $\sum_i \mu_i \leq \frac{x+y}{e} \leq \frac{D}{e}\left(D+\frac{D}{e}-4\right)$, so $(x,y) \in s\mathcal{A}$ for $s = \frac{D}{e}\left(D+\frac{D}{e}-4\right)$. By Proposition 3.39, one has that

$$\sigma(\mathcal{A}) \leq \max\left\{\frac{D}{e}\left(D+\frac{D}{e}-4\right), D+\frac{D}{e}-4\right\} = \frac{D}{e}\left(D+\frac{D}{e}-4\right),$$

and this concludes the proof. □

In the special case $e = 1$, we can improve it even more, as shown in the following theorem.

Theorem 3.41. *Let* $\mathcal{A} = \{\mathbf{a}_0 = (0,0), \mathbf{a}_1 = (D,0), \mathbf{a}_2 = (0,D), \mathbf{a}_3 = (D-1,1), \mathbf{a}_4 = (1,D-1), \mathbf{a}_5 = (D-1,0), \mathbf{a}_6 = (0,D-1), \mathbf{a}_7, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^2$ *such that* $0 < a_{i1}+a_{i2} \leq D$ *for all* $7 \leq i \leq n$*, and assume that either* $(1,0) \notin \mathcal{A}$*, or* $(0,1) \notin \mathcal{A}$*. Then, the sumsets regularity of* $\mathcal{A}$ *satisfies* $\sigma(\mathcal{A}) \leq D^2 - n + 1 = D^2 - |\mathcal{A}| + 2$.

Proof of Theorem 3.41

Before proving the theorem, we include some results on the sumsets of the skeleton of $\mathcal{A}$, the set $\mathcal{A}_0 := \{\mathbf{a}_0, \ldots, \mathbf{a}_6\}$, i.e., $\mathcal{A}_0 = \{(0,0), (D,0), (0,D), (D-1,0), (0,D-1), (1,D-1), (D-1,1)\}$. Set $T_{00} := \{(0,0)\}$ and, for all $i, j \in \mathbb{N}$, $i \geq 1$, $0 \leq j \leq i$,

$$T_{ij} := \{(x,y) \in \mathbb{N}^2 \mid x \geq j(D-1),\, y \geq (i-j)(D-1),\, x+y \leq iD\}.$$

Figure 3.3 shows what T_{ij} looks like.

Remark 3.42. (1) The sumset $0\mathcal{A}_0 = T_{00}$ and $\mathcal{A}_0 = T_{00} \cup (T_{10} \cup T_{11})$.

(2) For all $i \in \mathbb{N}$ and $j \in \{0, \ldots, i\}$, $T_{ij} + T_{10} = T_{i+1,j}$, and $T_{ij} + T_{11} = T_{i+1,j+1}$.

Proposition 3.43. *For all* $s \in \mathbb{N}$*, the* $(s+1)$*-fold sumset of* $\mathcal{A}_0$ *is given by*

$$(s+1)\mathcal{A}_0 = s\mathcal{A}_0 \cup \left(\bigcup_{j=0}^{s+1} T_{s+1,j}\right).$$

Proof. For $s = 0$, the formula is the one in Remark 3.42 (1). For $s \geq 1$, apply induction on s, note that $(s+1)\mathcal{A}_0 = s\mathcal{A}_0 + \mathcal{A}_0$, and then the result follows from Remark 3.42 (2). □

Corollary 3.44. *For all* $s \in \mathbb{N}$, $s\mathcal{A}_0 = \cup_{i=0}^{s} \cup_{j=0}^{i} T_{ij}$*, and* $\langle \mathcal{A}_0 \rangle = \cup_{s=0}^{\infty} \cup_{j=0}^{s} T_{sj}$.

Proposition 3.45. *Let* $(x,y) \in \langle \mathcal{A}_0 \rangle$ *and* $s \in \mathbb{N}$ *such that* $x+y \leq sD$*. Then,* $(x,y) \in s\mathcal{A}_0$.

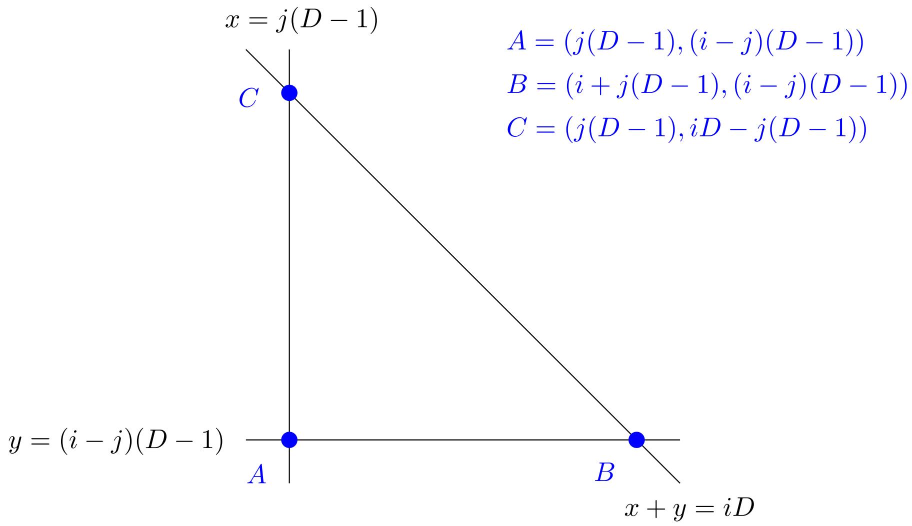


Figure 3.3: T_{ij} consists of the lattice points of the triangle defined above.

Proof. By Corollary 3.44, there exist indices i, j such that $(x, y) \in T_{ij}$. If $s \geq i$, then $(x, y) \in T_{ij} \subset i\mathcal{A}_0 \subset s\mathcal{A}_0$ by Corollary 3.44. Suppose that $s < i$ and let us prove that there exists $j' \leq s < i$ such that $(x, y) \in T_{sj'}$, and hence $(x, y) \in s\mathcal{A}_0$ by Corollary 3.44.
If $j \leq s$, since $(x, y) \in T_{ij}$, then $x \geq j(D-1)$, $y \geq (i-j)(D-1) \geq (s-j)(D-1)$, and $x + y \leq sD$. Thus, $(x, y) \in T_{sj} \subset s\mathcal{A}_0$.
If $j > s$, let us prove that $(x, y) \in T_{s,j-(i-s)}$. First, note that $0 \leq j - (i-s) \leq s$ since $j \leq i$ and $(i-1)D \leq sD$. Moreover, $x \geq j(D-1) \geq (j-(i-s))(D-1)$, $y \geq (s-(j-(i-s)))(D-1)$ and $x + y \leq sD$. Therefore, $(x, y) \in T_{s,j-(i-s)} \subset s\mathcal{A}_0$. $\square$

As a consequence, we get that the sumsets of $\mathcal{A}_0$ are filled in a nice way. This is the content of the following result, which is straightforward from Proposition 3.45.

Corollary 3.46. *Let $(x, y) \in \mathbb{N}^2$ and $s \in \mathbb{N}$ such that $x + y \leq sD$. Then, $(x, y) \in \langle \mathcal{A}_0 \rangle$ if and only if $(x, y) \in s\mathcal{A}_0$.*

Remark 3.47. By Proposition 3.45 and Corollary 3.46, the definition of the sumsets regularity of $\mathcal{A}_0$ reduces to $\sigma(\mathcal{A}_0) = \min\{s \in \mathbb{N} \mid \mathcal{H} \subset \mathcal{T}_{s,1}\}$, where $\mathcal{H} = \mathbb{N}^2 \setminus \langle \mathcal{A}_0 \rangle$. Taking into account the definition of the triangles T_{ij} and Corollary 3.44, it is easy to show that $\mathcal{H} \subset \mathcal{T}_{2(D-2),1}$, but $\mathcal{H} \not\subset \mathcal{T}_{2D-3,1}$, and hence $\sigma(\mathcal{A}_0) = 2(D-2)$.

Now, let us consider a set $\mathcal{A} \subset \mathbb{N}^2$ such that $a_{i1} + a_{i2} \leq D$ for all i and $\mathcal{A}_0 \subset \mathcal{A}$. The following result shows that if there is an element $(x_0, y_0) \in \mathcal{A}$ with $x_0 + y_0 = m > 0$, then we can improve Lemma 3.32.

Proposition 3.48. *Let $\mathcal{A} = \{\mathbf{a}_0, \dots, \mathbf{a}_n\} \subset \mathbb{N}^2$ be a finite set such that $a_{i1}+a_{i2} \leq D$ for all $i = 0, \dots, n$ and $\mathcal{A}_0 \subset \mathcal{A}$. Suppose that there exists $(x_0, y_0) \in \mathcal{A}$ with $m := x_0 + y_0 > 0$. Then, for all $(x, y) \in \mathbb{N}^2$ such that $x + y \geq (D + m - 3)D$, one has that $(x, y) \in \langle \mathcal{A} \rangle$. Indeed, if $s = \lceil \frac{x+y}{D} \rceil$, then $(x, y) \in (s + \lambda - 1)\mathcal{A}$, where $\lambda = \min\{\lambda' \in \mathbb{N} \mid x + y - \lambda' m \leq (s - 1)D\}$.*

As a consequence, if $(x, y) \in \mathbb{N}^2$ and $(D + m - 3)D \leq x + y \leq 2(D - 2)D$, then $(x, y) \in 2(D - 2)\mathcal{A}$.

Proof. Let $(x, y) \in \mathbb{N}^2$ such that $x + y \geq (D + m - 3)D$. If $x + y = (D + m - 3)D$, the result is straightforward, so assume $x + y > (D + m - 3)D$. Moreover, suppose that $(x, y) \notin \langle \mathcal{A}_0 \rangle$, otherwise (x, y) is trivially in $\langle \mathcal{A} \rangle$ and $(x, y) \in s\mathcal{A}$ for $s = \lceil \frac{x+y}{D} \rceil$, by Proposition 3.45. Since $(x, y) \notin \langle \mathcal{A}_0 \rangle$, by Proposition 3.43 if we take j such that $j(D - 1) \leq x < (j + 1)(D - 1)$, then $(x, y) \notin T_{s,j}$ for $s = \lceil \frac{x+y}{D} \rceil$, and hence $y < (s-j)(D-1)$. Note that $s \geq D+m-2$, since we are assuming $x+y > (D+m-3)D$. Let $\lambda \in \mathbb{N}$ be

$$\lambda = \min\{\lambda' \in \mathbb{N} \mid x + y - \lambda' m \leq (s - 1)D\},$$

and let us show that $(x, y) - \lambda(x_0, y_0) \in T_{s-1,j}$, which implies that $(x, y) \in \langle \mathcal{A} \rangle$. That is, we have to show that $x - \lambda x_0 \geq j(D - 1)$, $y - \lambda y_0 \geq (s - 1 - j)(D - 1)$ and $x + y - \lambda m \leq (s - 1)D$. Since $x + y - \lambda m > (s - 1)D - m$ by the election of λ, $y < (s - j)(D - 1)$, $x < (j + 1)(D - 1)$ and $s \geq D + m - 2$, we get

$$\begin{aligned} x - \lambda x_0 &\geq (s - 1)D - m - y + \lambda y_0 + 1 \\ &\geq (s - 1)D - m + ((j - s)(D - 1) + 1) + \lambda y_0 + 1 \\ &\geq j(D - 1) + (s - D - m + 2 + \lambda y_0) \geq j(D - 1)\,, \text{ and} \\ y - \lambda y_0 &\geq (s - 1)D - m - x + \lambda x_0 + 1 \\ &\geq (s - 1)D - m - (j + 1)(D - 1) + 1 + \lambda x_0 + 1 \\ &\geq (s - 1 - j)(D - 1) + (s - D - m + 2 + \lambda x_0) \geq (s - 1 - j)(D - 1)\,, \end{aligned}$$

as desired. Thus, we have shown $(x, y) - \lambda(x_0, y_0) \in T_{s-1,j}$, and hence $(x, y) \in (s - 1 + \lambda)\mathcal{A}$.

Finally, note that $\lambda \leq \frac{x+y-(s-1)D}{m} \leq (j+1)(D-1)+(s-j)(D-1)-2-(s-1)D = 2D - 3 - s$, and the last statement follows directly from this fact. □

We are now in conditions to prove Theorem 3.41. Let $\mathcal{A} = \{\mathbf{a}_0 = (0, 0), \mathbf{a}_1 = (D, 0), \mathbf{a}_2 = (0, D), \mathbf{a}_3 = (D - 1, 1), \mathbf{a}_4 = (1, D - 1), \mathbf{a}_5 = (D - 1, 0), \mathbf{a}_6 = (0, D - 1), \mathbf{a}_7, \dots, \mathbf{a}_n\} \subset \mathbb{N}^2$ such that $0 < a_{i1} + a_{i2} \leq D$ for all $7 \leq i \leq n$, and assume $(0, 1) \notin \mathcal{A}$. Every element $(x, y) \in \langle \mathcal{A} \rangle$ can be written as

$$(x, y) = \sum_{1 \leq i+j \leq d} \alpha_{ij}(i, j) \tag{3.9}$$

for some $\alpha_{ij} \in \mathbb{N}$, where $\alpha_{ij} = 0$ if $(i,j) \notin \mathcal{A}$. In particular, $\alpha_{01} = 0$. We can always assume that $\alpha_{ij} \leq D-2$ for all i,j such that $i+j \leq D-2$ since $(D-1,0),(0,D-1) \in \mathcal{A}$. Using (3.9), one has that $(x,y) \in s\mathcal{A}$ for $s = \sum_{i,j} \alpha_{ij}$.

For all $0 \leq i \leq D$, consider $\mathcal{A}^{\langle i \rangle} := \{(x,y) \in \mathcal{A} \mid x+y = i\}$. One has $\mathcal{A} = \sqcup_{i=0}^{D} \mathcal{A}^{\langle i \rangle}$, and hence $|\mathcal{A}| = \sum_{i=0}^{D} |\mathcal{A}^{\langle i \rangle}|$. In the proof of the theorem, we will use the following bound on the size of $\mathcal{A}$:

$$|\mathcal{A}| \leq 1 + |\mathcal{A}^{\langle 1 \rangle}| + |\mathcal{A}^{\langle 2 \rangle}| + (4 + 5 + \cdots + (D+1)) = |\mathcal{A}^{\langle 1 \rangle}| + |\mathcal{A}^{\langle 2 \rangle}| + \frac{D^2+3D}{2} - 4\,. \quad (3.10)$$

Equivalently, one has that $n = |\mathcal{A}| - 1 \leq |\mathcal{A}^{\langle 1 \rangle}| + |\mathcal{A}^{\langle 2 \rangle}| + \frac{D^2+3D}{2} - 5$.

We split the proof of Theorem 3.41 into several cases. For each case, we find positive constants ν, s_ν as in Proposition 3.39 and apply that result together with Lemma 3.32 and Proposition 3.48 (if necessary). To make the proof lighter, we include a lemma for each case.

Proof of Theorem 3.41. If $D = 3$ or $D = 4$, we check the result on a computer. There are 4 possibilities for $\mathcal{A}$ when $D = 3$, and 128 possibilities when $D = 4$. For each value of D and n, we compute $\sigma(\mathcal{A})$ for all $\mathcal{A}$ with $|\mathcal{A}| = n+1$. Table 3.1 shows the maximum value of $\sigma(\mathcal{A})$ for such sets $\mathcal{A}$, and the bound $D^2 - n + 1$. Assume $D \geq 5$.

Case 1: Suppose $(1,0) \in \mathcal{A}$. By Equation (3.10), it suffices to show that $\sigma(\mathcal{A}) \leq \frac{D^2-3D}{2} + 5 - |\mathcal{A}^{\langle 2 \rangle}|$. By Lemma 3.32, for all $(x,y) \in \mathbb{N}^2$ with $x+y \geq 2(D-2)D$ one has that $(x,y) \in s\mathcal{A}$ for all $s \geq \frac{x+y}{D}$. Moreover, by Proposition 3.48, for all $(x,y) \in \mathbb{N}^2$ with $(D-2)D \leq x+y \leq 2(D-2)D$ one has that $(x,y) \in 2(D-2)A$. Note that $2(D-2) \leq \frac{D^2-3D}{2} + 5$ since $D^2 - 7D + 18 = \left(D - \frac{7}{2}\right)^2 + \frac{23}{4} \geq 0$ for all D.

Case 1.1: If $\mathcal{A}^{\langle 2 \rangle} = \emptyset$, for all $(x,y) \in \langle \mathcal{A} \rangle$ with $x+y \leq (D-2)D$ one has that $(x,y) \in s\mathcal{A}$ for some $s \leq \frac{D^2-4}{3}$ by Lemma 3.49. Since we are assuming $D \geq 5$, then $2(D-2) \leq \frac{D^2-4}{3}$, and hence $\sigma(\mathcal{A}) \leq \frac{D^2-4}{3}$ by Proposition 3.39. Finally, note that $\frac{D^2-4}{3} \leq \frac{D^2-3D}{2} + 5$, which is true since $D^2 - 9D + 38 = \left(D - \frac{9}{2}\right)^2 + \frac{71}{4} \geq 0$. Therefore, we conclude $\sigma(\mathcal{A}) \leq \frac{D^2-3D}{2} + 5 \leq D^2 - |\mathcal{A}| + 2$ in this case.

Case 1.2: Assume $\mathcal{A}^{\langle 2 \rangle} \neq \emptyset$. By Lemma 3.50, for all $(x,y) \in \langle \mathcal{A} \rangle$ such that $x+y \leq (D-2)D$ one has that $(x,y) \in s\mathcal{A}$ for some $s \leq \frac{(D-2)D}{3} + \Sigma_0 - |\mathcal{A}^{\langle 2 \rangle}|$, for the number Σ_0 defined in that lemma. By Proposition 3.39 and the facts in Case 1, it suffices to show that $\frac{(D-2)D}{3} + \Sigma_0 - |\mathcal{A}^{\langle 2 \rangle}| \leq \frac{D^2-3D}{2} + 5 - |\mathcal{A}^{\langle 2 \rangle}|$, which is equivalent to showing

D	$n = \|\mathcal{A}\|-1$	$\max(\sigma(\mathcal{A}))$	$D^2 - n + 1$
3	6	2	4
	7	1	3
	8	1	2
4	6	4	11
	7	4	10
	8	3	9
	9	3	8
	10	3	7
	11	2	6
	12	2	5
	13	1	4

Table 3.1: Maximum value of $\sigma(\mathcal{A})$ for each $D \in \{3, 4\}$ and $n = |\mathcal{A}|-1$, with $\mathcal{A}$ as in Theorem 3.41. This table is part of the proof of Theorem 3.41.

that $\Sigma_0 \leq \frac{D^2-5D}{6} + 5$. For $D \geq 10$,

$$\Sigma_0 = D - 1 \leq \frac{D^2 - 5D}{6} + 5 \Leftrightarrow \left(D - \frac{11}{2}\right)^2 + \frac{23}{4} \geq 0\,,$$

and hence the desired inequality holds. For $5 \leq D < 10$, it is immediate to see that it is also true using the different definitions of Σ_0. Therefore, we conclude $\sigma(\mathcal{A}) \leq D^2 - |\mathcal{A}|+2$.

Case 2: Assume $(1, 0) \notin \mathcal{A}$ and let $m > 1$ be minimum such that $\mathcal{A}^{\langle m \rangle} \neq \emptyset$.

Case 2.1: Suppose $m \geq 3$. Note that the size of $\mathcal{A}$ satisfies

$$|\mathcal{A}| \leq 1 + (m + 1) + \cdots + (D + 1) = \frac{D(D + 3)}{2} - \frac{m(m + 1)}{2} + 2\,.$$

Hence, it suffices to show that $\sigma(\mathcal{A}) \leq \frac{D^2-3D}{2} + \frac{m^2+m}{2}$. By Proposition 3.48, for all $(x, y) \in \mathbb{N}^2$ with $(D+m-3)D \leq x+y \leq 2(D-2)D$ one has that $(x, y) \in 2(D-2)\mathcal{A}$. By Lemma 3.51, if $(x, y) \in \langle \mathcal{A} \rangle$ is such that $x+y \leq (D+m-3)D$, then $(x, y) \in s\mathcal{A}$ for some $s \leq \frac{(D+m-3)D}{m}$. Since $2(D - 2) \leq \frac{D^2-3D}{2}$, then by Proposition 3.39, it suffices to show that $\frac{(D+m-3)D}{m} \leq \frac{D^2-3D}{2} + \frac{m^2+m}{2}$. This is equivalent to showing that

$$\left(\frac{1}{2} - \frac{1}{m}\right) D^2 + \left(\frac{3}{m} - \frac{5}{2}\right) D + \frac{m^2 + m}{2} \geq 0\,.$$

Since we are assuming $m \geq 3$, the discriminant of this degree-2 polynomial in the variable D is

$$\left(\frac{3}{m}-\frac{5}{2}\right)^2-4\left(\frac{1}{2}-\frac{1}{m}\right)\frac{m^2+m}{2}\leq\frac{9}{4}-(m+1)(m-2)=\frac{17}{4}-m(m-1)<0\,,$$

for all $m \geq 3$, and hence the polynomial is nonnegative for all D. Thus, $\frac{(D+m-3)D}{m} \leq \frac{D^2-3D}{2}+\frac{m^2+m}{2}$, as we wanted to show.

Case 2.2: Assume $m = 2$, i.e., $\mathcal{A}^{\langle 2\rangle} \neq \emptyset$. By Equation (3.10), it suffices to show that $\sigma(\mathcal{A}) \leq \frac{D^2-3D}{2}+6-|\mathcal{A}^{\langle 2\rangle}|$. By Proposition 3.48, for all $(x,y) \in \mathbb{N}^2$ with $(D-1)D \leq x+y \leq 2(D-2)D$ one has that $(x,y) \in 2(D-2)\mathcal{A}$. By Lemma 3.52, for all $(x,y) \in \langle\mathcal{A}\rangle$ such that $x+y \leq (D-1)D$ one has that $(x,y) \in s\mathcal{A}$ for some $s \leq \frac{(D-1)D}{3}+\Sigma_0'-|\mathcal{A}^{\langle 2\rangle}|$, where Σ_0' is the number defined in that lemma. By Proposition 3.39, it suffices to show that $\frac{(D-1)D}{3}+\Sigma_0'-|\mathcal{A}^{\langle 2\rangle}| \leq \frac{D^2-3D}{2}+6-|\mathcal{A}^{\langle 2\rangle}|$, which is equivalent to $\Sigma_0' \leq \frac{D^2-7D}{6}+6$. For $D \geq 10$,

$$\Sigma_0'=\frac{D}{2}+1\leq\frac{D^2-7D}{6}+6\Leftrightarrow(D-5)^2+5\geq 0\,,$$

so the desired inequality holds. For $5 \leq D < 10$, it is immediate to see that it is also true by using that $\Sigma_0'=\frac{D+8}{3}$, and this concludes the proof. □

Lemma 3.49 (Case 1.1). *Assume $(1,0) \in \mathcal{A}$ and $\mathcal{A}^{\langle 2\rangle}=\emptyset$. If $(x,y) \in \langle\mathcal{A}\rangle$ is such that $x+y \leq (D-2)D$, then $(x,y) \in s\mathcal{A}$ for some $s \leq \frac{D^2-4}{3}$.*

Proof. Write (x,y) as in (3.9). Then $x+y \geq \alpha_{10}+3\sum_{i+j\geq 3}\alpha_{ij}$, so $\sum_{i+j\geq 3}\alpha_{ij} \leq \frac{x+y}{3}-\frac{\alpha_{10}}{3}$. Thus, the number of summands in (3.9) is

$$\sum_{i,j}\alpha_{ij}=\alpha_{10}+\sum_{i+j\geq 3}\alpha_{ij}\leq\frac{x+y}{3}+\frac{2}{3}\alpha_{10}\leq\frac{D(D-2)}{3}+\frac{2}{3}(D-2)=\frac{D^2-4}{3}\,,$$

so $(x,y) \in s\mathcal{A}$ for some $s \leq \frac{D^2-4}{3}$. □

Lemma 3.50 (Case 1.2). *Assume $(1,0) \in \mathcal{A}$, $\mathcal{A}^{\langle 2\rangle} \neq \emptyset$, and take $(x,y) \in \langle\mathcal{A}\rangle$. Then there exist $\alpha_{ij} \in \mathbb{N}$ such that $(x,y)=\sum_{i,j}\alpha_{ij}(i,j)$ and*

$$\Sigma:=\frac{2}{3}\alpha_{10}+\frac{1}{3}(\alpha_{20}+\alpha_{11}+\alpha_{02})+|\mathcal{A}^{\langle 2\rangle}|\leq\Sigma_0:=\begin{cases}\frac{D}{3}+\frac{10}{3} & \text{if } D<6\,,\\ \frac{5}{6}D+\frac{2}{3} & \text{if } 6\leq D<10\,,\\ D-1 & \text{if } D\geq 10\,.\end{cases}$$

Hence, if $x+y \leq (D-2)D$, then $(x,y) \in s\mathcal{A}$ for some $s \leq \frac{(D-2)D}{3}+\Sigma_0-|\mathcal{A}^{\langle 2\rangle}|$.

Proof. Observe that $(2,0)=2(1,0)$, $2(1,1)=(2,0)+(0,2)$, and $\lfloor \frac{D}{2} \rfloor (2,0) \in \{(D,0),(D-1,0)\}$ (it is $(D,0)$ when D is even and $(D-1,0)$ when D is odd), and the same holds for $(0,2)$. Using these relations and $\alpha_{ij} \le D-2$ when necessary, we can always assume that the coefficients α_{ij} in the writing of (x,y) satisfy the following conditions:

- If $|\mathcal{A}^{\langle 2\rangle}|=3$, we can assume $\alpha_{10}\le 1$, $\alpha_{11}\le 1$, and $\alpha_{20},\alpha_{02}\le \lfloor \frac{D}{2}\rfloor -1$. Thus, $\Sigma \le \frac{2}{3}+\frac{1}{3}\left[1+2\left(\frac{D}{2}-1\right)\right]+3=\frac{D}{3}+\frac{10}{3}$.
- If $\mathcal{A}^{\langle 2\rangle}=\{(2,0),(1,1)\}$, we can assume $\alpha_{10}\le 1$, and hence $\Sigma \le \frac{2}{3}+\frac{1}{3}\left(\frac{D}{2}-1+D-2\right)+2=\frac{D}{2}+\frac{5}{3}$.
- If $\mathcal{A}^{\langle 2\rangle}=\{(2,0),(0,2)\}$, we can assume $\alpha_{10}\le 1$, and hence $\Sigma \le \frac{2}{3}+\frac{1}{3}\cdot 2\left(\frac{D}{2}-1\right)+2=\frac{D}{3}+2$.
- If $\mathcal{A}^{\langle 2\rangle}=\{(1,1),(0,2)\}$, we can assume $\alpha_{11}\le 1$ (using the relation $2(1,1)=2(1,0)+(0,2)$), and hence $\Sigma \le \frac{2}{3}(D-2)+\frac{1}{3}\left(1+\frac{D}{2}-1\right)+2=\frac{5}{6}D+\frac{2}{3}$.
- If $|\mathcal{A}^{\langle 2\rangle}|=1$, then $\Sigma \le \frac{2}{3}(D-2)+\frac{1}{3}(D-2)+1=D-1$.

Therefore,

$$\Sigma \le \max\left\{\frac{D}{3}+\frac{10}{3},\frac{D}{2}+\frac{5}{3},\frac{5}{6}D+\frac{2}{3},D-1\right\}=\begin{cases}\frac{D}{3}+\frac{10}{3} & \text{if } D<6\,,\\ \frac{5}{6}D+\frac{2}{3} & \text{if } 6\le D<10\,,\\ D-1 & \text{if } D\ge 10\,.\end{cases}$$

Finally, if $x+y\le (D-2)D$ and we write (x,y) as in (3.9), then

$$x+y\ge \alpha_{10}+2\left(\alpha_{20}+\alpha_{11}+\alpha_{02}\right)+3\sum_{i+j\ge 3}\alpha_{ij}\,,$$

and from this expression we deduce that the number of summands in (3.9) is

$$\sum_{ij}\alpha_{ij}\le \frac{(D-2)D}{3}+\Sigma_0-|\mathcal{A}^{\langle 2\rangle}|\,,$$

and the lemma follows. □

Lemma 3.51 (Case 2.1)**.** *Assume $\mathcal{A}^{\langle 1\rangle}=\emptyset$ and let $m\ge 2$ be minimum such that $A^{\langle m\rangle}\ne\emptyset$. If $(x,y)\in\langle\mathcal{A}\rangle$ is such that $x+y\le (D+m-3)D$, then $(x,y)\in s\mathcal{A}$ for some $s\le \frac{(D+m-3)D}{m}$.*

Proof. Write (x,y) as in (3.9). Then $x+y\ge \left(\sum_{i,j}\alpha_{ij}\right)m$, and hence the number of summands in (3.9) is $\sum_{i,j}\alpha_{ij}\le \frac{x+y}{m}\le \frac{(D+m-3)D}{m}$. □

Lemma 3.52 (Case 2.2)**.** *Assume* $\mathcal{A}^{\langle 1\rangle} = \emptyset$*,* $\mathcal{A}^{\langle 2\rangle} \neq \emptyset$*, and take* $(x, y) \in \langle \mathcal{A} \rangle$*. Then there exist* $\alpha_{ij} \in \mathbb{N}$ *such that* $(x, y) = \sum_{i,j} \alpha_{ij}(i, j)$ *and*

$$\Sigma' := \frac{1}{3}(\alpha_{20} + \alpha_{11} + \alpha_{02}) + |\mathcal{A}^{\langle 2\rangle}| \leq \Sigma'_0 := \begin{cases} \frac{D}{3} + \frac{8}{3} & \textit{if} \quad D < 10\,, \\ \frac{D}{2} + 1 & \textit{if} \quad D \geq 10\,. \end{cases}$$

Hence, if $x + y \leq (D-1)D$*, then* $(x, y) \in s\mathcal{A}$ *for some* $s \leq \frac{(D-1)D}{3} + \Sigma'_0 - |\mathcal{A}^{\langle 2\rangle}|$*.*

Proof. Using the same relations as in the proof of Lemma 3.50, we can always assume that the coefficients α_{ij} in the writing of (x, y) satisfy the following conditions:

- If $|\mathcal{A}^{\langle 2\rangle}| = 3$, we can assume $\alpha_{11} \leq 1$, and $\alpha_{20}, \alpha_{02} \leq \lfloor \frac{D}{2} \rfloor - 1$. Thus, $\Sigma' \leq \frac{1}{3}\left[1 + 2\left(\frac{D}{2} - 1\right)\right] + 3 = \frac{D}{3} + \frac{8}{3}$.
- If $\mathcal{A}^{\langle 2\rangle} = \{(2, 0), (1, 1)\}$ or $\mathcal{A}^{\langle 2\rangle} = \{(1, 1), (0, 2)\}$, then $\Sigma' \leq \frac{1}{3}\left(\frac{D}{2} - 1 + D - 2\right) + 2 = \frac{D}{2} + 1$.
- If $\mathcal{A}^{\langle 2\rangle} = \{(2, 0), (0, 2)\}$, then $\Sigma' \leq \frac{1}{3} 2\left(\frac{D}{2} - 1\right) + 2 = \frac{D}{3} + \frac{4}{3}$.
- If $|\mathcal{A}^{\langle 2\rangle}| = 1$, then $\Sigma' \leq \frac{1}{3}(D - 2) + 1 = \frac{D}{3} + \frac{1}{3}$.

Therefore,

$$\Sigma' \leq \max\left\{\frac{D}{3} + \frac{8}{3}, \frac{D}{2} + 1\right\} = \begin{cases} \frac{D}{3} + \frac{8}{3} & \text{if} \quad D < 10\,, \\ \frac{D}{2} + 1 & \text{if} \quad D \geq 10\,. \end{cases}$$

To finish the proof, we can use the same argument as in the proof of Lemma 3.50 taking into account that in this case $\alpha_{10} = 0$ and $x + y \leq (D-1)D$. $\square$

Chapter 4

Regularity of simplicial projective toric varieties

"The Castelnuovo–Mumford regularity, or simply regularity, of an ideal is an important measure of how complicated the ideal is."
D. Eisenbud

In this chapter, we study the Castelnuovo-Mumford regularity of simplicial projective toric curves and surfaces. Moreover, we relate it to the sumsets theory in additive combinatorics, giving a proof of the Eisenbud-Goto Conjecture (Conj. 1.46) in some particular cases.

Let $\mathcal{A} = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$ be a finite set, $n, d \in \mathbb{Z}_{>0}$. When $d = 1$, assume that $\mathcal{A}$ is in normal form, i.e., $a_0 = 0 < a_1 < \cdots < a_n$ and $\gcd(a_1, \ldots, a_n) = 1$. Associated with $\mathcal{A}$, one has the set $\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \ldots, \underline{\mathbf{a}}_n\} \subset \mathbb{N}^2$, where $\underline{\mathbf{a}}_i = (D - a_i, a_i)$ for all $i = 0, \ldots, n$. Fix an infinite field $\Bbbk$ and consider the projective monomial curve determined by $\underline{\mathcal{A}}$, $\mathcal{C} = \mathcal{C}_{\underline{\mathcal{A}}} \subset \mathbb{P}^n_{\Bbbk}$. The coordinate ring of $\mathcal{C}$ is $\Bbbk[\mathcal{C}] = \Bbbk[x_0, \ldots, x_n]/I_{\underline{\mathcal{A}}}$, where $I_{\underline{\mathcal{A}}}$ is the toric ideal determined by $\underline{\mathcal{A}}$. In Section 4.1, we study the Castelnuovo-Mumford regularity of $\Bbbk[\mathcal{C}]$. The main result of Subsection 4.1.1 is Theorem 4.2, where we provide a combinatorial formula for the Castelnuovo-Mumford regularity of $\Bbbk[\mathcal{C}]$ in terms of the Apéry and the exceptional sets of the affine semigroup $\mathcal{S} = \langle \underline{\mathcal{A}} \rangle$. Moreover, in Theorem 4.9 ,we determine the step in a minimal graded free resolution (m.g.f.r.) of $\Bbbk[\mathcal{C}]$ in which the regularity is attained. In Subsection 4.1.2, we provide upper and lower bounds on the regularity of $\Bbbk[\mathcal{C}]$ in terms of the sumsets regularity of $\mathcal{A}$ (Theorem 4.13), and use this relation to give a combinatorial proof of the Eisenbud-Goto conjecture for projective monomial curves.

When $d \geq 2$, consider $D = \max\{|\mathbf{a}_i| = \sum_{j=1}^{d} a_{ij} : i = 0, \ldots, n\}$ and $\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \ldots, \underline{\mathbf{a}}_n\}$, where $\underline{\mathbf{a}}_i = (D - |\mathbf{a}_i|, \mathbf{a}_i) \in \mathbb{N}^{d+1}$ for all $i = 0, \ldots, n$. Fix an algebraically closed field $\Bbbk$ and consider the projective toric variety determined by $\underline{\mathcal{A}}$, $\mathcal{X} = \mathcal{X}_{\underline{\mathcal{A}}} \subset \mathbb{P}^n_{\Bbbk}$. We suppose that $\mathcal{X}$ is simplicial. The coordinate ring of $\mathcal{X}$ is $\Bbbk[\mathcal{X}] = \Bbbk[x_0, \ldots, x_n]/I_{\underline{\mathcal{A}}}$, where $I_{\underline{\mathcal{A}}}$ is the toric ideal determined by $\underline{\mathcal{A}}$. When $\mathcal{X}$ is smooth, one can assume that $\underline{\mathcal{A}}$ satisfies Theorem 1.72. The regularity of $\Bbbk[\mathcal{X}]$ has been already studied by Herzog and Hibi. In [48, Thm. 2.1], they proved that $\operatorname{reg}(\Bbbk[\mathcal{X}]) \leq d(D-2)$. Hence, the Eisenbud-Goto conjecture holds for simplicial and smooth projective toric varieties ([48, Cor. 2.2]), so the next step is to consider simplicial projective toric varieties with a single singular point. In Section 4.2, we study the Castelnuovo-Mumford regularity of simplicial projective monomial surfaces. In Subsection 4.2.1, the main result is Theorem 4.25, where we provide a combinatorial formula to compute the regularity of $\Bbbk[\mathcal{X}]$ in terms of some special subsets of the affine semigroup $\mathcal{S} = \langle \underline{\mathcal{A}} \rangle \subset \mathbb{N}^3$. This result holds for any simplicial projective toric surface. In Subsection 4.2.2, we focus on the surfaces with a single singular point. For this subclass, we give a relation between the Castelnuovo-Mumford regularity of $\Bbbk[\mathcal{X}]$ and the sumsets regularity of $\mathcal{A}$ in Theorem 4.27. This result, together with Theorem 3.41, provides a proof of the Eisenbud-Goto conjecture for the simplicial projective monomial surfaces with a single singular point whose degree is either maximal or minimal.

The results included in Section 4.1 are part of [39].

4.1 Projective monomial curves

Let $\mathcal{A} = \{a_0 = 0 < a_1 < \cdots < a_n = D\} \subset \mathbb{N}$ be a set in normal form, i.e., $\gcd(a_1, \ldots, a_n) = 1$. Consider the homogenization of $\mathcal{A}$, $\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \ldots, \underline{\mathbf{a}}_n\} \subset \mathbb{N}^2$, where $\underline{\mathbf{a}}_i = (D - a_i, a_i)$ for all $i = 0, \ldots, n$. Fix an infinite field $\Bbbk$ and consider the projective monomial curve determined by $\underline{\mathcal{A}}$, $\mathcal{C} = \mathcal{C}_{\underline{\mathcal{A}}} \subset \mathbb{P}^n_{\Bbbk}$. The coordinate ring of $\mathcal{C}$ is $\Bbbk[\mathcal{C}] = \Bbbk[x_0, \ldots, x_n]/I_{\underline{\mathcal{A}}}$, where $I_{\underline{\mathcal{A}}}$ is the toric ideal determined by $\underline{\mathcal{A}}$.

Consider the numerical semigroups $\mathcal{S}_1 = \langle a_1, \ldots, a_n \rangle$ and $\mathcal{S}_2 = \langle D - a_{n-1}, \ldots, D - a_1, D \rangle$, and the affine semigroup $\mathcal{S} = \langle \underline{\mathcal{A}} \rangle \subset \mathbb{N}^2$. Recall the definition of the Apéry and the exceptional sets of $\mathcal{S}$ (Definition 2.2):

- $\mathrm{AP}_{\mathcal{S}} = \{\mathbf{s} \in \mathcal{S} : \mathbf{s} - \underline{\mathbf{a}}_0 \notin \mathcal{S}, \mathbf{s} - \underline{\mathbf{a}}_n \notin \mathcal{S}\}$, and
- $E_{\mathcal{S}} = \{\mathbf{s} \in \mathcal{S} : \mathbf{s} - \underline{\mathbf{a}}_0 \in \mathcal{S}, \mathbf{s} - \underline{\mathbf{a}}_n \in \mathcal{S}, \mathbf{s} - \underline{\mathbf{a}}_0 - \underline{\mathbf{a}}_n \notin \mathcal{S}\}$.

In Subsection 4.1.1, we provide a combinatorial formula to compute $\operatorname{reg}(\Bbbk[\mathcal{C}])$ in terms of the elements in the Apéry and the exceptional sets of $\mathcal{S}$. In Subsection 4.1.2, we use this formula to relate $\operatorname{reg}(\Bbbk[\mathcal{C}])$ to the sumsets regularity of $\mathcal{A}$. This relation provides a nice combinatorial proof of the Eisenbud-Goto conjecture for projective monomial curves.

4.1.1 Formula for the regularity

To express $\operatorname{reg}(\Bbbk[\mathcal{C}])$ in terms of $\mathrm{AP}_{\mathcal{S}}$ and $E_{\mathcal{S}}$, we introduce the following notations. For all $s \in \mathbb{N}$, set $L_s := \{(x,y) \in \mathbb{N}^2 \mid x+y \leq sD\}$. If $F \subset \mathcal{S}$ is a finite set; for all $s \in \mathbb{N}$, define $F_s := F \cap L_s = F \cap s\mathcal{A}$, and $m(F) := \max\{s \in \mathbb{N} \mid F_s \neq \emptyset\}$, with the convention $m(F) = -\infty$ if $F = \emptyset$.

Remark 4.1. Since $\mathrm{AP}_{\mathcal{S}}$ and $E_{\mathcal{S}}$ are finite by Remark 3.18, one can consider the numbers $m(\mathrm{AP}_{\mathcal{S}}) \in \mathbb{N}$ and $m(E_{\mathcal{S}}) \in \mathbb{N} \cup \{-\infty\}$.

(1) One has that $m(E_{\mathcal{S}}) \leq \sigma$ and $m(\mathrm{AP}_{\mathcal{S}}) \leq \sigma + 1$, where $\sigma = \sigma(\mathcal{A})$ is the sumsets regularity of $\mathcal{A}$ (see Def. 3.5).

(2) Both $m(E_{\mathcal{S}})$ and $m(\mathrm{AP}_{\mathcal{S}})$ can be expressed in terms of the sumsets of $\mathcal{A}$ as follows:

- $m(\mathrm{AP}_{\mathcal{S}}) = \max(\{s \in \mathbb{N} : \exists \alpha \in s\mathcal{A}$, such that $\alpha \notin (s-1)\mathcal{A}$ and $\alpha - D \notin (s-1)\mathcal{A}\})$.
- $m(E_{\mathcal{S}}) = \max(\{s \in \mathbb{N} : \exists \alpha \in (s-1)\mathcal{A}$, such that $\alpha - D \in (s-1)\mathcal{A} \setminus (s-2)\mathcal{A}\})$, and

The following result gives a combinatorial way for computing the Castelnuovo-Mumford regularity of $\Bbbk[\mathcal{C}]$.

Theorem 4.2. *The Castelnuovo-Mumford regularity of the projective monomial curve $\mathcal{C}$ is*

$$\operatorname{reg}(\Bbbk[\mathcal{C}]) = \max\{m(\mathrm{AP}_{\mathcal{S}}), m(E_{\mathcal{S}}) - 1\}.$$

To prove this result, let us recall some known facts on the local cohomology modules of the coordinate ring of $\mathcal{C}$, $\Bbbk[\mathcal{C}]$. As observed in Section 1.3, $\Bbbk[\mathcal{C}] \cong \Bbbk[\mathcal{S}]$ as (standard) graded $\Bbbk[x_0,\ldots,x_n]$-modules. By Grothendieck's theorem, since $\dim(\Bbbk[\mathcal{S}]) = 1$ and $\operatorname{depth}(\Bbbk[\mathcal{S}]) \in \{1,2\}$, then for $\Bbbk[\mathcal{S}]$ there are at most two non-trivial local cohomology modules, $H^1_{\mathfrak{m}}(\Bbbk[\mathcal{S}])$ and $H^2_{\mathfrak{m}}(\Bbbk[\mathcal{S}])$, where $\mathfrak{m} = \langle x_0,\ldots,x_n\rangle$ is the maximal homogeneous ideal of $\Bbbk[x_0,\ldots,x_n]$. Furthermore, these two modules are completely characterized in terms of the semigroup $\mathcal{S}$.

Lemma 4.3 ([47, Lem. 2.2])**.** *Let $G \subset \mathbb{Z}^2$ be the group generated by $\mathcal{S}$ and $\mathcal{S}' = G \cap (\mathcal{S}_2 \times \mathcal{S}_1)$.*

(1) $H^1_{\mathfrak{m}}(\Bbbk[\mathcal{S}]) \cong \Bbbk[\mathcal{S}' \setminus \mathcal{S}]$, *and*

(2) $H^2_{\mathfrak{m}}(\Bbbk[\mathcal{S}]) \cong \Bbbk\left[G \cap ((\mathbb{Z} \setminus \mathcal{S}_2) \times (\mathbb{Z} \setminus \mathcal{S}_1))\right]$,

where the symbol $\cong$ *means that there exists an isomorphism of* $\mathbb{Z}$*-graded modules.*

When $\mathcal{C}$ is arithmetically Cohen-Macaulay, $\mathcal{S}' = \mathcal{S}$ by Proposition 2.4 (f), so $H^1_{\mathfrak{m}}(\Bbbk[\mathcal{S}]) = 0$ as we already know. By Theorem 1.45, one has that

$$\operatorname{reg}(\Bbbk[\mathcal{S}]) = \max\{\operatorname{end}\left(H^1_{\mathfrak{m}}(\Bbbk[\mathcal{S}])\right) + 1, \operatorname{end}\left(H^2_{\mathfrak{m}}(\Bbbk[\mathcal{S}])\right) + 2\}. \tag{4.1}$$

The proof of Theorem 4.2 will then be a consequence of the following two lemmas that relate the local cohomology modules $H^1_{\mathfrak{m}}(\Bbbk[\mathcal{S}])$ and $H^2_{\mathfrak{m}}(\Bbbk[\mathcal{S}])$ to the numbers $m(E_{\mathcal{S}})$ and $m(\mathrm{AP}_{\mathcal{S}})$. Note that the relation $m(E_{\mathcal{S}}) = \operatorname{end}(H^1_{\mathfrak{m}}(\Bbbk[\mathcal{S}])) + 2$ stated in Lemma 4.4 also holds when $\mathcal{C}$ is arithmetically Cohen-Macaulay, since both numbers are $-\infty$ in this case.

Lemma 4.4. *If* $\mathcal{S}' \neq \mathcal{S}$*, i.e., if* $\mathcal{C}$ *is not arithmetically Cohen-Macaulay, then*

$$\max\{s : E_{s+2} \neq \emptyset\} = \max\{s : (\mathcal{S}' \setminus \mathcal{S}) \cap L_s \neq \emptyset\}.$$

Therefore, $m(E_{\mathcal{S}}) = \operatorname{end}(H^1_{\mathfrak{m}}(\Bbbk[\mathcal{S}])) + 2$.

Proof. If $\mathcal{C}$ is not arithmetically Cohen-Macaulay, then the exceptional set $E_{\mathcal{S}} \neq \emptyset$ by Proposition 2.4 (e). Set $E'_{\mathcal{S}} := \{(x,y) \in \mathbb{N}^2 : (x,y) + \underline{\mathbf{a}}_0 + \underline{\mathbf{a}}_n \in E_{\mathcal{S}}\}$ and, for each $s \in \mathbb{N}$, $E'_s := E'_{\mathcal{S}} \cap L_s$. Note that $(x,y) \in E'_s$ if and only if $(x,y) + \underline{\mathbf{a}}_0 + \underline{\mathbf{a}}_n \in E_{s+2}$ so $\max\{s : E_{s+2} \neq \emptyset\} = \max\{s : E'_s \neq \emptyset\}$. Let us consider an element $(x,y) \in E'_{\mathcal{S}}$. It is clear that $(x,y) \in \mathcal{S}' \setminus \mathcal{S}$, since $(x,y) = (x+D,y) - (D,0) \in G$, where G is the group generated by $\mathcal{S}$. Therefore, $E'_{\mathcal{S}} \subset \mathcal{S}' \setminus \mathcal{S}$ and we get that $\max\{s : E'_s \neq \emptyset\} \leq \max\{s : (S' \setminus S) \cap L_s \neq \emptyset\}$.

Conversely, let $(x,y) \in (\mathcal{S}' \setminus \mathcal{S}) \cap L_s$ be an element such that s is maximum. Then, $(x,y) + \underline{\mathbf{a}}_0 \in \mathcal{S}$ and $(x,y) + \underline{\mathbf{a}}_n \in \mathcal{S}$, and hence, $(x,y) \in E'_s$. Therefore, $\max\{s : E'_s \neq \emptyset\} \geq \max\{s : (S' \setminus S) \cap L_s \neq \emptyset\}$ and the equality $\max\{s : E_{s+2} \neq \emptyset\} = \max\{s : (S' \setminus S) \cap L_s \neq \emptyset\}$ follows. By Lemma 4.3 (1), it implies that $m(E_{\mathcal{S}}) = \operatorname{end}(H^1_{\mathfrak{m}}(\Bbbk[\mathcal{S}])) + 2$. ☐

Observe that in the previous proof, we show that $E'_{\mathcal{S}} \subset \mathcal{S}' \setminus \mathcal{S}$. Equality, which would be a stronger result than the one stated in Lemma 4.4, is wrong in general. Using the example given in [47, Ex. 3.2], we show that those two sets may be different.

Example 4.5. For $\mathcal{A} = \{0,1,2,5,13,14,16,17\}$, the curve $\mathcal{C}$ is smooth. Thus, $\mathcal{S}_1 = \mathcal{S}_2 = \mathbb{N}$ and $G = \mathbb{Z}^2$, and hence, $\mathcal{S}' = G \cap \mathbb{N}^2 = \mathbb{N}^2$. Since $(9,8) \in \mathcal{S}' \setminus \mathcal{S}$ but $(9,8) \notin E'_{\mathcal{S}}$, because $(26,8) \notin \mathcal{S}$, one has that the inclusion $E'_{\mathcal{S}} \subset \mathcal{S}' \setminus \mathcal{S}$ is strict.

We now want to relate $m\,(\mathrm{AP}_{\mathcal{S}})$ to $\mathrm{end}\,(H^2_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}]))$. Let $(x,y)\in G\cap((\mathbb{Z}\setminus\mathcal{S}_2)\times(\mathbb{Z}\setminus\mathcal{S}_1))\cap L_s$ be an element with s maximal. Since $x\notin\mathcal{S}_2$ and $y\notin\mathcal{S}_1$, one has that $(x,y+D)\notin\mathcal{S}$ and $(x+D,y)\notin\mathcal{S}$. There are two possibilities, either $(x+D,y+D)\in\mathcal{S}$ or $(x+D,y+D)\notin\mathcal{S}$, and let us check that in both cases, the inequality (4.2) below holds. In the first case, note that $(x+D,y+D)\in\mathrm{AP}_{\mathcal{S}}\cap L_{s+2}$, so $\max\{s:\mathrm{AP}_{s+2}\neq\emptyset\}\geq\max\{s:G\cap((\mathbb{Z}\setminus\mathcal{S}_2)\times(\mathbb{Z}\setminus\mathcal{S}_1))\cap L_s\neq\emptyset\}$ and (4.2) follows from Lemma 4.3 (2). In the second case, using the notations in Proposition 2.4, there exists an index i, $0\leq i\leq D-1$, such that $x\equiv u_{D-i}\pmod D$ and $y\equiv v_i\pmod D$. Then, $u_{D-i}\geq x+D$ and $v_i\geq y+D$ and since $(x+D,y+D)\notin\mathcal{S}$, by Lemma 2.3, there exist natural numbers $x'\geq x+D$ and $y'\geq y+D$, being at least one of these two inequalities strict, such that $(x',y')\in\mathrm{AP}_{\mathcal{S}}$. Observe that $(x',y')\in L_{s'}$ for $s'\geq s+3$, so $\max\{s:\mathrm{AP}_{s+2}\neq\emptyset\}>\max\{s:G\cap((\mathbb{Z}\setminus\mathcal{S}_2)\times(\mathbb{Z}\setminus\mathcal{S}_1))\cap L_s\neq\emptyset\}$ in this case. In both cases, one has that

$$m\,(\mathrm{AP}_{\mathcal{S}})\geq\mathrm{end}\left(H^2_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}])\right)+2\,. \tag{4.2}$$

Adding an additional hypothesis, one gets equality in (4.2) as the following lemma shows.

Lemma 4.6. *If* $\mathrm{end}\,(H^2_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}]))+2>\mathrm{end}\,(H^1_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}]))+1=m\,(E_{\mathcal{S}})-1$, *then*

$$\max\{s:\mathrm{AP}_{s+2}\neq\emptyset\}=\max\{s:G\cap((\mathbb{Z}\setminus\mathcal{S}_2)\times(\mathbb{Z}\setminus\mathcal{S}_1))\cap L_s\neq\emptyset\}.$$

Therefore, in this case, one has that $m\,(\mathrm{AP}_{\mathcal{S}})=\mathrm{end}\,(H^2_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}]))+2$.

Proof. Let $(x,y)\in\mathrm{AP}_{s+2}$ be an element such that s is maximal and consider the element $(x-D,y-D)$. If $(x-D,y-D)\notin G\cap((\mathbb{Z}\setminus\mathcal{S}_2)\times(\mathbb{Z}\setminus\mathcal{S}_1))$, one can assume without loss of generality that $y-D\notin\mathcal{S}_1$. Then, there exists $x'\geq x+D$ such that $(x',y-D)\in\mathcal{S}$, so $(x',y)\in E_{s'}$ for some $s'\geq s+3$. Therefore, $\mathrm{end}\,(H^1_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}]))+1=m\,(E_{\mathcal{S}})-1\geq m\,(\mathrm{AP}_{\mathcal{S}})$ by Lemma 4.4, and using (4.2) we get that $\mathrm{end}\,(H^1_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}]))+1\geq\mathrm{end}\,(H^2_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}]))+2$, which is in contradiction with the hypothesis in the statement of the lemma. Thus, $(x-D,y-D)\in G\cap((\mathbb{Z}\setminus\mathcal{S}_2)\times(\mathbb{Z}\setminus\mathcal{S}_1))\cap L_s$, and hence, $\mathrm{end}\,(H^2_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}]))+2\geq m\,(\mathrm{AP}_{\mathcal{S}})$ by Lemma 4.3 (2). Using (4.2), we are done. □

Note that if one removes the hypothesis $\mathrm{end}\,(H^2_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}]))+2>\mathrm{end}\,(H^1_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}]))+1=m\,(E_{\mathcal{S}})-1$ in Lemma 4.6, the result may be wrong. To illustrate this fact, we use the example in [47, Ex. 3.2].

Example 4.7. For $\mathcal{A}=\{0,1,2,5,13,14,16,17\}$, as observed in Example 4.5, $\mathcal{S}_1=\mathcal{S}_2=\mathbb{N}$ and $G=\mathbb{Z}^2$. Therefore, $\mathrm{end}\,(H^2_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}]))=0$ by Lemma 4.3 (2), but $(43,8)\in\mathrm{AP}_3$ so $m\,(\mathrm{AP}_{\mathcal{S}})\neq\mathrm{end}\,(H^2_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}]))+2$.

Proof of Theorem 4.2. If $m\,(E_{\mathcal{S}}) - 1 \geq m\,(\mathrm{AP}_{\mathcal{S}})$, then by Lemma 4.4 one has that $m\,(E_{\mathcal{S}}) - 1 = \mathrm{end}\,(H^1_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}])) + 1$, and hence $\mathrm{end}\,(H^1_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}])) + 1 \geq \mathrm{end}\,(H^2_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}])) + 2$, because otherwise, by Lemma 4.6, one would have that $m\,(\mathrm{AP}_{\mathcal{S}}) > m\,(E_{\mathcal{S}}) - 1$, a contradiction. Thus, the equality $\mathrm{reg}(\Bbbk[\mathcal{C}]) = m\,(E_{\mathcal{S}}) - 1$ follows from Equation (4.1). Assume now that $m\,(\mathrm{AP}_{\mathcal{S}}) > m\,(E_{\mathcal{S}}) - 1$, and consider an element $(x, y) \in \mathrm{AP}_s$ with $s = m\,(\mathrm{AP}_{\mathcal{S}})$. Since $s > m\,(E_{\mathcal{S}}) - 1$, then $(x, y) - \underline{\mathbf{a}}_0 - \underline{\mathbf{a}}_n \in G \cap ((\mathbb{Z} \setminus \mathcal{S}_2) \times (\mathbb{Z} \setminus \mathcal{S}_1)) \cap L_{s-2}$ and hence

$$\mathrm{end}\left(H^2_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}])\right) + 2 \geq s = m\,(\mathrm{AP}_{\mathcal{S}}) > m\,(E_{\mathcal{S}}) - 1 = \mathrm{end}\left(H^1_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}])\right) + 1,$$

where the first inequality follows from Lemma 4.3 (2) and the last equality from Lemma 4.4. Therefore, $\mathrm{end}\,(H^2_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}])) + 2 > \mathrm{end}\,(H^1_{\mathfrak{m}}\,(\Bbbk[\mathcal{S}])) + 1$ and the equality $\mathrm{reg}(\Bbbk[\mathcal{C}]) = m\,(\mathrm{AP}_{\mathcal{S}})$ follows from Lemma 4.6 and Equation (4.1). □

Note that there exist curves such that the maximum in Theorem 4.2 is equal to $m\,(E_{\mathcal{S}}) - 1$ and not equal to $m\,(\mathrm{AP}_{\mathcal{S}})$, and vice versa. For instance, if $\mathcal{C}$ is arithmetically Cohen-Macaulay, then $m\,(\mathrm{AP}_{\mathcal{S}}) > m\,(E_{\mathcal{S}}) - 1 = -\infty$. However, there also exist non-arithmetically Cohen-Macaulay curves such that $m\,(\mathrm{AP}_{\mathcal{S}}) > m\,(E_{\mathcal{S}}) - 1$.

Example 4.8. (1) For $\mathcal{A} = \{0, 1, 3, 11, 13\}$, $m\,(E_{\mathcal{S}}) = 6$ and $m\,(\mathrm{AP}_{\mathcal{S}}) = 4$, so $\mathcal{C}$ is not arithmetically Cohen-Macaulay, and $\mathrm{reg}(\Bbbk[\mathcal{C}]) = 5 = m\,(E_{\mathcal{S}}) - 1 > m\,(\mathrm{AP}_{\mathcal{S}})$.

(2) [6, Ex. 4.3]. For $\mathcal{A} = \{0, 5, 9, 11, 20\}$, $m\,(E_{\mathcal{S}}) = 5$ and $m\,(\mathrm{AP}_{\mathcal{S}}) = 5$, so $\mathcal{C}$ is not arithmetically Cohen-Macaulay, and $\mathrm{reg}(\Bbbk[\mathcal{C}]) = 5 = m\,(\mathrm{AP}_{\mathcal{S}}) > m\,(E_{\mathcal{S}}) - 1$.

Recall that, as stated in (1.7), the regularity is always determined by the tail of a m.g.f.r.. Since in our case $1 \leq \mathrm{depth}(\Bbbk[\mathcal{C}]) \leq 2$, one has that the regularity is attained at one of the two last steps of a m.g.f.r. If $\Bbbk[\mathcal{C}]$ is Cohen-Macaulay, then the regularity is always attained at the last step. In the non Cohen-Macaulay case, our next result characterizes when the regularity is attained at the last step, in terms of the formula given in Theorem 4.2 and of the difference $\delta = \mathrm{reg}(\Bbbk[\mathcal{C}]) - \mathrm{r}(\Bbbk[\mathcal{C}])$, where $\mathrm{r}(\Bbbk[\mathcal{C}])$ is the regularity of the Hilbert polynomial of $\Bbbk[\mathcal{C}]$.

Theorem 4.9. *If $\mathcal{C}$ is not arithmetically Cohen-Macaulay, the following are equivalent:*

(a) The Castelnuovo-Mumford regularity of $\Bbbk[\mathcal{C}]$ is attained at the last step of a m.g.f.r.

(b) $\mathrm{reg}(\Bbbk[\mathcal{C}]) = m\,(E_{\mathcal{S}}) - 1$, *i.e.,* $m\,(E_{\mathcal{S}}) - 1 \geq m\,(\mathrm{AP}_{\mathcal{S}})$.

(c) $\mathrm{reg}(\Bbbk[\mathcal{C}]) = \mathrm{r}(\Bbbk[\mathcal{C}])$, *i.e.,* $\delta = 0$.

Proof. The equivalence (a) ⇔ (c) is a direct consequence of Theorem 1.51, as observed in Remark 1.52 (2). Therefore, we only have to prove (a) ⇔ (b). By Theorem 1.45, $\max\{j \mid \beta_{n+1-\text{depth}(M),j} \neq 0\} = \text{end}\left(H_{\mathfrak{m}}^{\text{depth}(M)}(M)\right) + n + 1$. If $\Bbbk[\mathcal{C}]$ is not Cohen-Macaulay, then by Theorem 4.2, its proof, and Lemma 4.4, one has that $\text{reg}(\Bbbk[\mathcal{C}]) = m(E_{\mathcal{S}})$ if and only if $\text{end}\left(H_{\mathfrak{m}}^1(\Bbbk[\mathcal{S}])\right) + 1 = m(E_{\mathcal{S}}) - 1 \geq \text{end}\left(H_{\mathfrak{m}}^2(\Bbbk[\mathcal{S}])\right) + 2$, i.e., if and only if the Castelnuovo-Mumford regularity is attained at the last step of a m.g.f.r. of $\Bbbk[\mathcal{C}]$, by (4.1) and the previous observation. This proves the equivalence between (a) and (b). □

Example 4.10. Different values of $\delta = \text{reg}(\Bbbk[\mathcal{C}]) - \text{r}(\Bbbk[\mathcal{C}])$ and different shapes for the Betti diagram of $\Bbbk[\mathcal{C}]$ are obtained in the following four examples of monomial curves in $\mathbb{P}_{\Bbbk}^4$.

(1) For $\mathcal{A} = \{0, 1, 3, 11, 13\}$, $\delta = 0$ and $\text{reg}(\Bbbk[\mathcal{C}])$ is attained at the last step of a m.g.f.r.

	0	1	2	3	4
0:	1	-	-	-	-
1:	-	1	-	-	-
2:	-	2	2	-	-
3:	-	2	2	-	-
4:	-	3	8	5	-
5:	-	-	2	4	2
total:	1	8	14	9	2

(2) For $\mathcal{A} = \{0, 2, 5, 6, 9\}$, $\delta = 1$ and $\text{reg}(\Bbbk[\mathcal{C}])$ is attained at the last step of a m.g.f.r.

	0	1	2	3
0:	1	-	-	-
1:	-	1	-	-
2:	-	7	12	5
total:	1	8	12	5

(3) For $\mathcal{A} = \{0, 6, 9, 13, 22\}$, $\delta = 1$ and $\operatorname{reg}(\Bbbk[\mathcal{C}])$ is not attained at the last step of a m.g.f.r.

	0	1	2	3	4
0:	1	-	-	-	-
1:	-	1	-	-	-
2:	-	1	-	-	-
3:	-	-	1	-	-
4:	-	5	9	5	1
5:	-	-	2	2	-
total:	1	7	12	7	1

(4) For $\mathcal{A} = \{0, 5, 9, 11, 20\}$, $\delta = 2$ and $\operatorname{reg}(\Bbbk[\mathcal{C}])$ is not attained at the last step of a m.g.f.r.

	0	1	2	3	4
0:	1	-	-	-	-
1:	-	1	-	-	-
2:	-	1	-	-	-
3:	-	1	1	-	-
4:	-	3	9	5	1
5:	-	-	-	1	-
total:	1	6	10	6	1

In the case of projective monomial curves $\mathcal{C} \subset \mathbb{P}^3_{\Bbbk}$, one can be more precise, since their codimension is 2.

Proposition 4.11. *Let $\mathcal{A} = \{a_0 = 0 < a_1 < a_2 < a_3 = D\} \subset \mathbb{N}$ be a set in normal form and consider the associated monomial curve $\mathcal{C} \subset \mathbb{P}^3_{\Bbbk}$.*

(1) The Castelnuovo-Mumford regularity is attained at the last step of a m.g.f.r. of $\Bbbk[\mathcal{C}]$.

(2) Setting $\delta := \operatorname{reg}(\Bbbk[\mathcal{C}]) - \operatorname{r}(\Bbbk[\mathcal{C}])$, one has that $0 \le \delta \le 1$. More precisely,

$$\delta = 0 \Leftrightarrow \Bbbk[\mathcal{C}] \text{ is not Cohen-Macaulay} \Leftrightarrow \operatorname{reg}(\Bbbk[\mathcal{C}]) = m\,(E_{\mathcal{S}}) - 1 \ge m\,(\mathrm{AP}_{\mathcal{S}})\,,$$
$$\delta = 1 \Leftrightarrow \Bbbk[\mathcal{C}] \text{ is Cohen-Macaulay} \Leftrightarrow \operatorname{reg}(\Bbbk[\mathcal{C}_A]) = m\,(\mathrm{AP}_{\mathcal{S}}) > m\,(E_{\mathcal{S}}) - 1.$$

Proof. Part (1) is a particular case of [7, Cor. 2.13]. By Theorem 1.51 and Remark 1.52 (2), this implies that either $\delta = 0$ if $\mathcal{C}$ is not arithmetically Cohen-

Macaulay, or $\delta = 1$ if $\mathcal{C}$ is arithmetically Cohen Macaulay. Part (2) then follows from Theorem 4.9. □

4.1.2 Relations with the sumsets regularity

The Castelnuovo-Mumford regularity of $\Bbbk[\mathcal{C}]$ can be upper and lower bounded in terms of $\sigma = \sigma(\mathcal{A})$, the sumsets regularity of $\mathcal{A}$. These bounds will be given in Theorem 4.13 where we distinguish two cases depending on the value of σ in Theorem 3.7. Let us first prove a lemma that will be needed in the proof. For $i = 1, 2$, set c_i the conductor of $\mathcal{S}_i$, and $F(\mathcal{S}_i)$ the Frobenius number of $\mathcal{S}_i$.

Lemma 4.12. *Set* $N := \lceil \frac{c_1+c_2}{D} \rceil$. *Then,* $\operatorname{reg}(\Bbbk[\mathcal{C}]) \geq \lceil \frac{N}{2} \rceil + 1$.

Proof. One has that $F(\mathcal{S}_1) + D \in \mathrm{Ap}_1$ and consider $x \in \mathrm{Ap}_2$, such that $F(\mathcal{S}_1) + D + x \equiv 0 \pmod{D}$. Note that $x \neq 0$. By Lemma 2.3, there are two options: either $(x, F(\mathcal{S}_1) + D) \in \mathrm{AP}_{\mathcal{S}}$, or there exists $x' \geq x$, such that $(x', F(\mathcal{S}_1) + D) \in \mathrm{AP}_{\mathcal{S}}$. In both cases, there exists $x \geq 1$, such that $(x, F(\mathcal{S}_1) + D) \in \mathrm{AP}_{\mathcal{S}}$ and, analogously, there exists $y \geq 1$, such that $(F(\mathcal{S}_2) + D, y) \in \mathrm{AP}_{\mathcal{S}}$. By Theorem 4.2,

$$\operatorname{reg}(\Bbbk[\mathcal{C}]) \geq \max\left\{\frac{F(\mathcal{S}_1)+D+x}{D}, \frac{F(\mathcal{S}_2)+D+y}{D}\right\} \geq \frac{1}{2}\frac{F(\mathcal{S}_1)+F(\mathcal{S}_2)+2}{D}+1$$
$$= \frac{c_1+c_2}{2D}+1.$$

Thus, $\operatorname{reg}(\Bbbk[\mathcal{C}]) \geq \lceil \frac{c_1+c_2}{2D} \rceil + 1 = \lceil \frac{N}{2} \rceil + 1$. □

Theorem 4.13. *We have the following bounds on the Castelnuovo-Mumford regularity of* $\Bbbk[\mathcal{C}]$*:*

(1) If $\sigma = \mathrm{r}(\Bbbk[\mathcal{C}]) \geq \lceil \frac{c_1+c_2}{D} \rceil$, *then* $\sigma \leq \operatorname{reg}(\Bbbk[\mathcal{C}]) \leq \sigma + 1$.

(2) If $\sigma = \lceil \frac{c_1+c_2}{D} \rceil > \mathrm{r}(\Bbbk[\mathcal{C}])$, *then* $\lceil \frac{\sigma}{2} \rceil + 1 \leq \operatorname{reg}(\Bbbk[\mathcal{C}]) \leq \sigma + 1$.

Proof. In both cases, the upper bound is a consequence of Theorem 4.2 and Remark 4.1 (1). If $\sigma = \mathrm{r}(\Bbbk[\mathcal{C}]) \geq \lceil \frac{c_1+c_2}{D} \rceil$, then we apply the known fact $\mathrm{r}(\Bbbk[\mathcal{C}]) \leq \operatorname{reg}(\Bbbk[\mathcal{C}])$ from Remark 1.52 (3), and in the other case, the lower bound is the one given in Lemma 4.12. □

Example 4.14. To illustrate that all the upper and lower bounds in Theorem 4.13 are sharp, the values of $\mathrm{r}(\Bbbk[\mathcal{C}])$, $\lceil \frac{c_1+c_2}{D} \rceil$, σ and $\operatorname{reg}(\Bbbk[\mathcal{C}])$ in four different examples are displayed in Table 4.1.

Table 4.1: Examples where the bounds in Theorem 4.13 are attainted.

$\mathcal{A}$	$\mathrm{r}(\Bbbk[\mathcal{C}])$	$\lceil \frac{c_1+c_2}{D} \rceil$	σ	$\mathrm{reg}(\Bbbk[\mathcal{C}])$
$\{0,1,3,11,13\}$	5	1	5	5
$\{0,1,3,5,6,12\}$	1	1	1	2
$\{0,4,5,9,16\}$	2	3	3	3
$\{0,5,9,11,20\}$	3	4	4	5

The following result is more precise than the one stated in Theorem 4.13 in a particular case. It gives, in this case, the precise relationship between the three regularities, in the sense of Castelnuovo-Mumford, of the Hilbert function, and of the sumsets.

Proposition 4.15. *If $\mathcal{C}$ is arithmetically Cohen-Macaulay and $(F(\mathcal{S}_2)+D, F(\mathcal{S}_1)+D) \in \mathrm{AP}_{\mathcal{S}}$, then*

$$\sigma = \left\lceil \frac{c_1+c_2}{D} \right\rceil, \ \mathrm{r}(\Bbbk[\mathcal{C}]) = \sigma, \ \textit{and} \ \mathrm{reg}(\Bbbk[\mathcal{C}]) = \sigma + 1\,.$$

Proof. Since $(F(\mathcal{S}_2)+D, F(\mathcal{S}_1)+D) \in \mathrm{AP}_{\mathcal{S}}$, then $(F(\mathcal{S}_2)+D, F(\mathcal{S}_1)+D) \in \mathrm{AP}_s$ for $s = m(\mathrm{AP}_{\mathcal{S}})$ and, as $\mathcal{C}$ is arithmetically Cohen-Macaulay, $\mathrm{reg}(\Bbbk[\mathcal{C}]) = m(\mathrm{AP}_{\mathcal{S}})$ by Theorem 4.2. Thus,

$$\mathrm{reg}(\Bbbk[\mathcal{C}]) = \frac{F(\mathcal{S}_1)+D+F(\mathcal{S}_2)+D}{D} = \frac{F(\mathcal{S}_1)+F(\mathcal{S}_2)}{D} + 2\,.$$

On the other hand,

$$\left\lceil \frac{c_1+c_2}{D} \right\rceil = \left\lceil \frac{F(\mathcal{S}_1)+F(\mathcal{S}_2)}{D} + \frac{2}{D} \right\rceil = \frac{F(\mathcal{S}_1)+F(\mathcal{S}_2)}{D} + 1\,,$$

so $\mathrm{reg}(\Bbbk[\mathcal{C}]) = \lceil \frac{c_1+c_2}{D} \rceil + 1$, and $\mathrm{r}(\Bbbk[\mathcal{C}]) = \lceil \frac{c_1+c_2}{D} \rceil$, since $\mathrm{reg}(\Bbbk[\mathcal{C}]) = \mathrm{r}(\Bbbk[\mathcal{C}]) + 1$ whenever $\mathcal{C}$ is arithmetically Cohen-Macaulay, by Remark 1.52 (2). Finally, $\sigma = \lceil \frac{c_1+c_2}{D} \rceil$ by Theorem 3.7. □

Remark 4.16. Note that, by Lemma 2.3, the condition $(F(\mathcal{S}_2)+D, F(\mathcal{S}_1)+D) \in \mathrm{AP}_{\mathcal{S}}$ is equivalent to $(F(\mathcal{S}_2)+D, F(\mathcal{S}_1)+D) \in \mathcal{S}$.

Corollary 4.17. *If $\mathcal{C}$ is arithmetically Gorenstein, then*

$$\sigma = \left\lceil \frac{c_1+c_2}{D} \right\rceil, \ \mathrm{r}(\Bbbk[\mathcal{C}]) = \sigma, \ \textit{and} \ \mathrm{reg}(\Bbbk[\mathcal{C}]) = \sigma + 1\,.$$

Proof. If $\mathcal{C}$ is arithmetically Gorenstein, then $(F(\mathcal{S}_2)+D, F(\mathcal{S}_1)+D) \in \mathrm{AP}_{\mathcal{S}}$ by the proof of Proposition 2.29, and the result follows from Proposition 4.15. □

In particular, by the proof of Theorem 2.32 and Corollary 4.17, it follows that the Castelnuovo-Mumford regularity of $\Bbbk[\mathcal{C}]$ is $\mathrm{reg}(\Bbbk[\mathcal{C}]) = 3$ for all the Gorenstein curves that we constructed in Example 2.31 and Theorem 2.32.

Example 4.18. For $\mathcal{A} = \{0,1,2,3,8\}$, $\Bbbk[\mathcal{C}]$ is Cohen-Macaulay, as shown in Example 2.5, and $(F(\mathcal{S}_2)+D, F(\mathcal{S}_1)+D) = (17,7) \in \mathrm{AP}_{\mathcal{S}}$. By Proposition 4.15, $\sigma = \mathrm{r}(\Bbbk[\mathcal{C}]) = \lceil \frac{c_1+c_2}{D} \rceil = 3$, and $\mathrm{reg}(\Bbbk[\mathcal{C}]) = \sigma + 1 = 4$.

Using the previous results, we can give a new proof for the bound obtained by J. Elias in [32] for arithmetically Cohen-Macaulay curves. First, recall a result of F. Lev that we will use in the proof.

Lemma 4.19 ([64, Thm. 1])**.** *Let $\mathcal{A} = \{a_0 = 0 < a_1 < \cdots < a_n = D\} \subset \mathbb{N}$ be a set in normal form. Then, for all $s \geq 2$ one has that*

$$|s\mathcal{A}| \geq |(s-1)\mathcal{A}| + \min(D, s(n-1)+1).$$

Proposition 4.20 ([32, Thm. 4.7])**.** *If $\mathcal{A} = \{a_0 = 0 < a_1 < \cdots < a_n = D\} \subset \mathbb{N}$ is a set in normal form, such that $\mathcal{C}$ is arithmetically Cohen-Macaulay, then*

$$\mathrm{reg}(\Bbbk[\mathcal{C}]) \leq \left\lceil \frac{D-1}{n-1} \right\rceil.$$

Proof. Set $s_0 := \lceil \frac{D-1}{n-1} \rceil$. By Corollary 3.21, the sequence $(|s\mathcal{A}| - |(s-1)\mathcal{A}|)_{s\in\mathbb{N}}$ is increasing and its limit is D. Indeed, as observed in the proof of this corollary, $|s\mathcal{A}| - |(s-1)\mathcal{A}| = \sum_{j=0}^{s} |\mathrm{AP}_j|$ for all $s \in \mathbb{N}$. On the other hand, $|\mathrm{AP}_{\mathcal{S}}| = D$ by Proposition 2.4 (d) and, by Lemma 4.19, $|s\mathcal{A}| - |(s-1)\mathcal{A}| \geq D$ if $s \geq s_0$. Therefore, $|\mathrm{AP}_s| = 0$ for all $s > s_0$, and hence, $\mathrm{reg}(\Bbbk[\mathcal{C}]) \leq s_0$, by Theorem 4.2. □

As a consequence of Theorem 4.13, one gets a sufficient condition for σ to be equal to $\lceil \frac{c_1+c_2}{D} \rceil$ in Theorem 3.7. The condition is expressed in terms of the difference between the Castelnuovo-Mumford regularity and the regularity of the Hilbert function of $\Bbbk[\mathcal{C}]$.

Corollary 4.21. *If $\delta = \mathrm{reg}(\Bbbk[\mathcal{C}]) - \mathrm{r}(\Bbbk[\mathcal{C}]) \geq 2$, then $\sigma = \lceil \frac{c_1+c_2}{D} \rceil > \mathrm{r}(\Bbbk[\mathcal{C}])$.*

Proof. If $\sigma = \mathrm{r}(\Bbbk[\mathcal{C}]) \geq \lceil \frac{c_1+c_2}{D} \rceil$, then $\sigma \leq \mathrm{reg}(\Bbbk[\mathcal{C}]) \leq \sigma + 1$ by Theorem 4.13, so $\delta \leq 1$. □

A combinatorial proof of the Gruson-Lazarsfeld-Peskine theorem (i.e., the Eisenbud-Goto conjecture)

Recall from Theorem 1.76 (1) that $\operatorname{reg}(\Bbbk[\mathcal{C}]) \leq D - n + 1$. Using the results of this section and Granville-Walker's bound on the sumsets regularity of $\mathcal{A}$ ($\sigma \leq s_0^{GW} = D - n + 1$) we can give an easy proof of the above bound. We distinguish three cases:

(1) If neither $\mathcal{A}$ nor $D - \mathcal{A}$ belongs to the two families listed in Theorem 3.6, then $\sigma \leq D - n$, and $\operatorname{reg}(\Bbbk[\mathcal{C}]) \leq D - n + 1$ follows from Theorem 4.13.

(2) If $\mathcal{A} = [0, D] \setminus \{a\}$ for some $a \in [2, D - 2]$, then $\sigma = 2$ and $\operatorname{reg}(\Bbbk[\mathcal{C}]) = 2$ as well by Theorem 4.2, and hence $\operatorname{reg}(\Bbbk[\mathcal{C}]) \leq D - n + 1$ holds for such a set $\mathcal{A}$, since in this case $n = D - 1$.

(3) If $\mathcal{A} = [0, 1] \sqcup [a + 1, D]$ for some $a \in [2, D - 2]$, then $s\mathcal{A} = [0, sD]$ for all $s \geq a$ and $a \notin (a - 1)\mathcal{A}$. Therefore, $\sigma = a$ and $\operatorname{reg}(\Bbbk[\mathcal{C}]) = a$ by Theorem 4.2, so $\operatorname{reg}(\Bbbk[\mathcal{C}]) \leq D - n + 1$ also follows from the bound $\sigma \leq s_0^{GW}$ in this case. One gets the same conclusion if $D - \mathcal{A} = [0, 1] \sqcup [a + 1, D]$ for some $a \in [2, D - 2]$.

4.2 Projective monomial surfaces

Consider $\mathcal{A} = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^2$ a finite set such that $|\mathbf{a}_i| = a_{i1} + a_{i2} \leq D$ for all $\mathbf{a}_i \in \mathcal{A}$. Denote $\underline{\mathcal{A}} = \{\underline{\mathbf{a}}_0, \ldots, \underline{\mathbf{a}}_n\}$, where $\underline{\mathbf{a}}_i = (a_{i0}, a_{i1}, a_{i2})$, $a_{i0} = D - a_{i1} - a_{i2}$, for all $i = 0, \ldots, n$.

Fix an infinite field $\Bbbk$ and consider $\mathcal{X} = \mathcal{X}_{\underline{\mathcal{A}}} \subset \mathbb{P}^n_{\Bbbk}$ the projective monomial surface determined by $\underline{\mathcal{A}}$. We assume that $\mathcal{X}$ is simplicial, i.e., $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2\} \subset \underline{\mathcal{A}}$, where $\mathbf{e}_i := D\boldsymbol{\epsilon}_i$ for $i = 0, 1, 2$ and $\{\boldsymbol{\epsilon}_0, \boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2\}$ is the canonical basis of $\mathbb{N}^3$. The coordinate ring of $\mathcal{X}$ is $\Bbbk[\mathcal{X}] = \Bbbk[x_0, \ldots, x_n]/I_{\underline{\mathcal{A}}}$, where $I_{\underline{\mathcal{A}}}$ is the toric ideal determined by $\underline{\mathcal{A}}$.

In Subsection 4.2.1, we provide a combinatorial formula to compute $\operatorname{reg}(\Bbbk[\mathcal{X}])$ in terms of the elements in the Apéry and the exceptional sets of $\mathcal{S}$. In Subsection 4.2.2, we use this formula to relate $\operatorname{reg}(\Bbbk[\mathcal{X}])$ with the sumsets regularity of $\mathcal{A}$. This relation provides a nice combinatorial proof of the Eisenbud-Goto conjecture for some of the projective monomial surfaces with a single singular point.

4.2.1 Formula for the regularity

Let $\mathcal{S} \subset \mathbb{N}^3$ be the affine semigroup generated by $\underline{\mathcal{A}}$, $\mathcal{S} = \langle \underline{\mathcal{A}} \rangle$. By hypothesis, $\mathcal{S}$ is simplicial, and the extremal rays of the cone $\operatorname{Pos}(\underline{\mathcal{A}})$ are $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2\}$. Recall that, in this case, the *Apéry set of* $\mathcal{S}$ is defined as $\operatorname{AP}_{\mathcal{S}} = \{\mathbf{s} \in \mathcal{S} \mid \mathbf{s} - \mathbf{e}_i \notin \mathcal{S}, \forall i = 0, 1, 2\}$, and it is finite by Remark 1.22 (2). We define now four special subsets of $\mathcal{S}$ that will be involved in the combinatorial formula for $\operatorname{reg}(\Bbbk[\mathcal{X}])$.

Definition 4.22. The *exceptional sets of $\mathcal{S}$* are:

- $E_{\mathcal{S}}^{3,1} = \{\mathbf{s} \in \mathcal{S} \mid \mathbf{s} - \mathbf{e}_j \in \mathcal{S}, \forall j;\ \mathbf{s} - (\mathbf{e}_{i_0} + \mathbf{e}_{i_1}) \in \mathcal{S}, \mathbf{s} - (\mathbf{e}_{i_0} + \mathbf{e}_{i_2}) \notin \mathcal{S}, \mathbf{s} - (\mathbf{e}_{i_1} + \mathbf{e}_{i_2}) \notin \mathcal{S},$ for a permutation (i_0, i_1, i_2) of $(0,1,2)\}$;
- $E_{\mathcal{S}}^{2,0} = \{\mathbf{s} \in \mathcal{S} \mid \mathbf{s} - \mathbf{e}_{i_0} \in \mathcal{S}, \mathbf{s} - \mathbf{e}_{i_1} \in \mathcal{S}, \mathbf{s} - \mathbf{e}_{i_2} \notin \mathcal{S};\ \mathbf{s} - (\mathbf{e}_{i_0} + \mathbf{e}_{i_1}) \notin \mathcal{S},$ for a permutation (i_0, i_1, i_2) of $(0,1,2)\}$;
- $E_{\mathcal{S}}^{3,0} = \{\mathbf{s} \in \mathcal{S} \mid \mathbf{s} - \mathbf{e}_i \in \mathcal{S}, \forall i;\ \mathbf{s} - (\mathbf{e}_i + \mathbf{e}_j) \notin \mathcal{S}, \forall i \neq j\}$;
- $E_{\mathcal{S}}^{3,3} = \{\mathbf{s} \in \mathcal{S} \mid \mathbf{s} - \mathbf{e}_i \in \mathcal{S}, \forall i;\ \mathbf{s} - (\mathbf{e}_i + \mathbf{e}_j) \in \mathcal{S}, \forall i \neq j;$ and $\mathbf{s} - (\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2) \notin \mathcal{S}\}$.

Figure 4.1 shows how elements in the Apéry and the exceptional sets of $\mathcal{S}$ look like.

In the notation $E_{\mathcal{S}}^{a,b}$ below, a is the number of indices i, $0 \leq i \leq 2$, such that $\mathbf{s} - \mathbf{e}_i \in \mathcal{S}$ and b is the number of pairs (i,j) of indices, $0 \leq i < j \leq 2$, such that $\mathbf{s} - \mathbf{e}_i - \mathbf{e}_j \in \mathcal{S}$ (according to this notation, the Apéry set defined before would be $E_{\mathcal{S}}^{0,0}$). Note that $0 \leq b \leq a \leq 3$. In Chapter 5 (more precisely, in Section 5.2), we will justify why the exceptional sets are finite.

Since $\underline{\mathcal{A}}$ is contained in the plane $\{(x,y,z) \in \mathbb{N}^3 : x + y + z = D\}$, for all $s \geq 0$, the s-fold sumset of $\underline{\mathcal{A}}$ is also contained in a plane,

$$s\underline{\mathcal{A}} \subset \{(x,y,z) \in \mathbb{N}^3 : x + y + z = sD\}\,.$$

For every $s \in \mathbb{N}$, set $H_s := \{(x,y,z) \in \mathbb{N}^3 : x + y + z = sD\}$, and for every subset F of $\mathcal{S}$, set $F_s := F \cap H_s$. Moreover, if $F \subset \mathcal{S}$ is a finite subset, we define the number $m(F)$ by

$$m(F) := \max\{s \in \mathbb{N} : F_s \neq \emptyset\}\,,$$

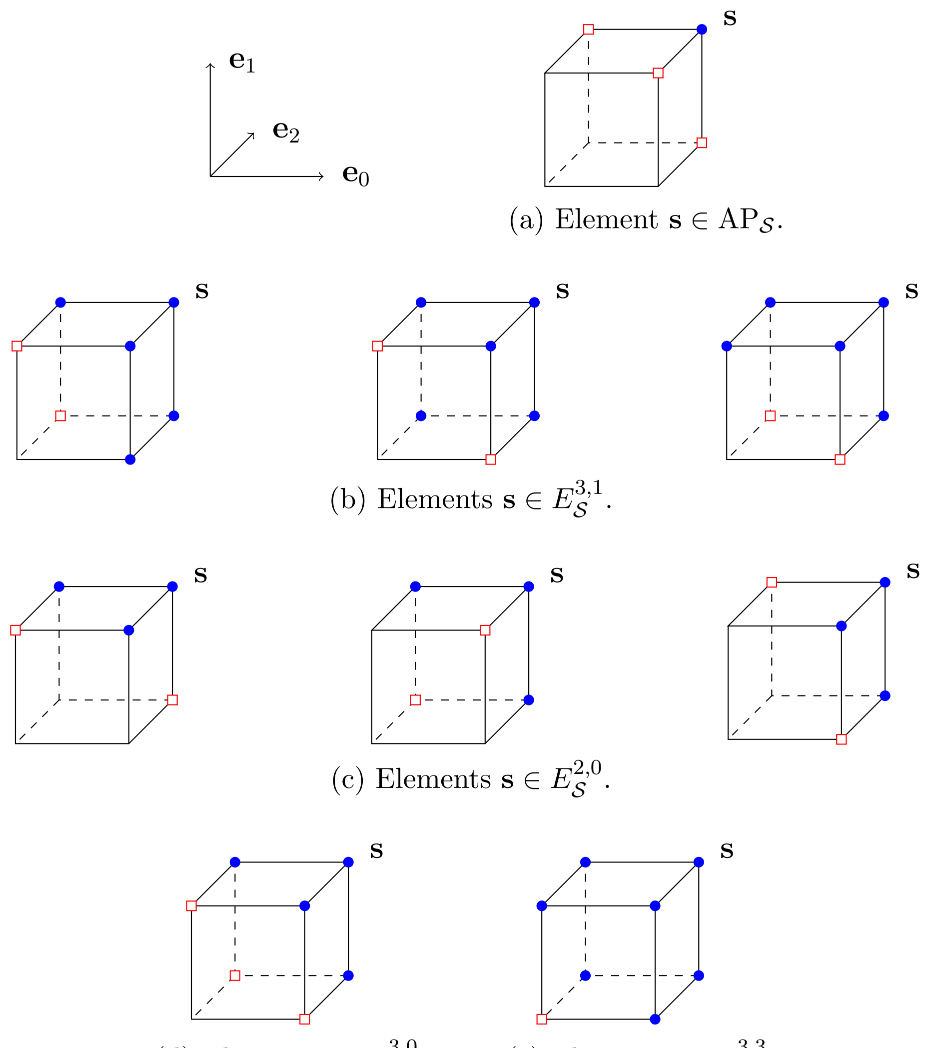
with the convention $m(F) = -\infty$ if $F = \emptyset$.

In the case of the Apéry set of $\mathcal{S}$, instead of writing $(\mathrm{AP}_{\mathcal{S}})_s$, we just write AP_s, and so we do with the exceptional sets. Using these notations, one can prove an analogous result to Proposition 3.19 for simplicial projective monomial surfaces.

Proposition 4.23. *For all $s \in \mathbb{N}$,*

$$|\mathrm{AP}_s| = (|s\underline{\mathcal{A}}| - 3|(s-1)\underline{\mathcal{A}}| + 3|(s-2)\underline{\mathcal{A}}| - |(s-3)\underline{\mathcal{A}}|) + |E_s^{3,1}| + |E_s^{2,0}| + 2|E_s^{3,0}| - |E_s^{3,3}|\,.$$

The proof of Proposition 4.23 will be a direct consequence of Theorem 5.15 and Proposition 5.9 (b). We will prove both results in Chapter 5.



(a) Element $\mathbf{s} \in \mathrm{AP}_{\mathcal{S}}$.

(b) Elements $\mathbf{s} \in E_{\mathcal{S}}^{3,1}$.

(c) Elements $\mathbf{s} \in E_{\mathcal{S}}^{2,0}$.

(d) Element $\mathbf{s} \in E_{\mathcal{S}}^{3,0}$.

(e) Element $\mathbf{s} \in E_{\mathcal{S}}^{3,3}$.

Figure 4.1: Points in $\mathrm{AP}_{\mathcal{S}}$ and the exceptional sets $E_{\mathcal{S}}^{3,1}$, $E_{\mathcal{S}}^{2,0}$, $E_{\mathcal{S}}^{3,0}$, and $E_{\mathcal{S}}^{3,3}$. Filled circles represent elements in $\mathcal{S}$, while empty squares represent elements outside $\mathcal{S}$.

Remark 4.24. Take $(x,y,z) \in \mathcal{S}$ and set $s = \frac{x+y+z}{D} \in \mathbb{N}$. One can characterize when (x,y,z) is in the Apéry set or in a exceptional set of $\mathcal{S}$ in terms of the element $(y,z) \in s\mathcal{A}$ and the sumsets $(s-1)\mathcal{A}$, $(s-2)\mathcal{A}$ and $(s-3)\mathcal{A}$. For instance, one has that

$$(x,y,z) \in \mathrm{AP}_{\mathcal{S}} \Leftrightarrow \begin{cases} (y,z),(y-D,z),(y,z-D) \notin (s-1)\mathcal{A}, \\ (y-D,z),(y,z-D),(y-D,z-D) \notin (s-2)\mathcal{A}, \text{ and} \\ (y-D,z-D) \notin (s-3)\mathcal{A}. \end{cases}$$

This is shown in Figure 4.2a. The analogous characterizations of the exceptional sets are shown in Figures 4.2b-4.2e. Note that there are two other variants of 4.2b and 4.2c corresponding to the possible permutations in the definition of $E_{\mathcal{S}}^{3,1}$ and $E_{\mathcal{S}}^{2,0}$.

The following result shows how to compute the Castelnuovo-Mumford regularity of $\Bbbk[\mathcal{X}]$ in terms of the Apéry and the exceptional sets of $\mathcal{S}$. This result will be a direct consequence of Theorem 5.15 from Chapter 5.

Theorem 4.25. *The Castelnuovo-Mumford regularity of the simplicial projective toric surface $\mathcal{X}$ is given by the formula*

$$\mathrm{reg}(\Bbbk[\mathcal{X}]) = \max\left\{m(\mathrm{AP}_{\mathcal{S}}), m(E_{\mathcal{S}}^{3,1})-1, m(E_{\mathcal{S}}^{2,0})-1, m(E_{\mathcal{S}}^{3,0})-1, m(E_{\mathcal{S}}^{3,3})-2\right\}.$$

The following example shows different sets $\mathcal{A} \subset \mathbb{N}^2$ for which the maximum in Theorem 4.25 is attained in the different terms that appear in the formula.

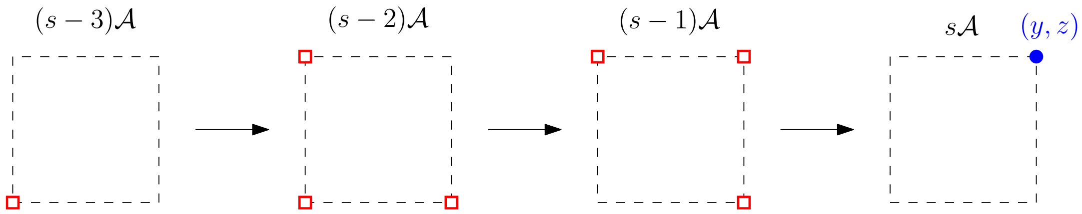
Example 4.26.
(1) For $\mathcal{A} = \{(0,0),(4,0),(0,4),(3,0)\}$, one has that $m(\mathrm{AP}_{\mathcal{S}}) = 4$ and $m(E_{\mathcal{S}}^{3,1}) = m(E_{\mathcal{S}}^{2,0}) = m(E_{\mathcal{S}}^{3,0}) = m(E_{\mathcal{S}}^{3,3}) = -\infty$. Hence, $\mathrm{reg}(\Bbbk[\mathcal{X}]) = m(\mathrm{AP}_{\mathcal{S}}) = 4$.

(2) For $\mathcal{A} = \{(0,0),(5,0),(0,5),(3,1),(0,2),(0,1),(4,0)\}$, one has that $m(\mathrm{AP}_{\mathcal{S}}) = 4$, $m(E_{\mathcal{S}}^{3,1}) = 6$, $m(E_{\mathcal{S}}^{2,0}) = 5$, and $m(E_{\mathcal{S}}^{3,0}) = m(E_{\mathcal{S}}^{3,3}) = -\infty$. Hence, $\mathrm{reg}(\Bbbk[\mathcal{X}]) = m(E_{\mathcal{S}}^{3,1}) - 1 = 5$.

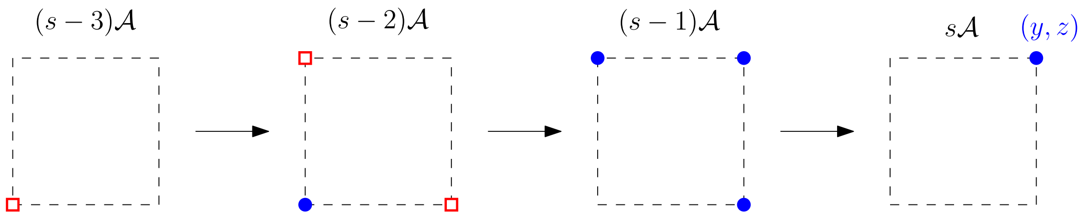
(3) For $\mathcal{A} = \{(0,0),(13,0),(0,13),(12,1),(10,3),(2,11)\}$, $m(\mathrm{AP}_{\mathcal{S}}) = 4$, $m(E_{\mathcal{S}}^{2,0}) = 6$, and $m(E_{\mathcal{S}}^{3,1}) = m(E_{\mathcal{S}}^{3,0}) = m(E_{\mathcal{S}}^{3,3}) = -\infty$. Hence, $\mathrm{reg}(\Bbbk[\mathcal{X}]) = m(E_{\mathcal{S}}^{2,0}) - 1 = 5$.

4.2.2 Surfaces with one singular point, sumsets, and the Eisenbud-Goto conjecture

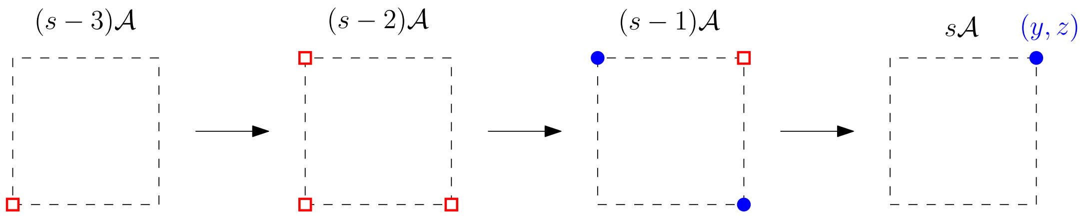
Suppose that $\underline{\mathcal{A}}$ contains $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, (0,D-1,1), (0,1,D-1), (e,D-e,0), (e,0,D-e)\}$, where $1 \leq e \leq D$ is a divisor of D that divides a_{i0} for all $i \in \{0,\ldots,n\}$, and if



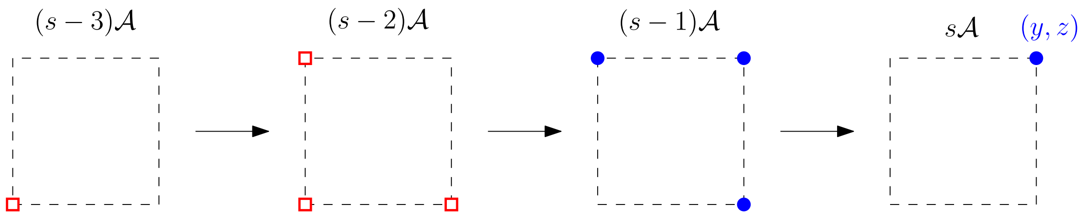
(a) An element $(y,z) \in s\mathcal{A}$ such that $(x,y,z) \in \mathrm{AP}_s$.



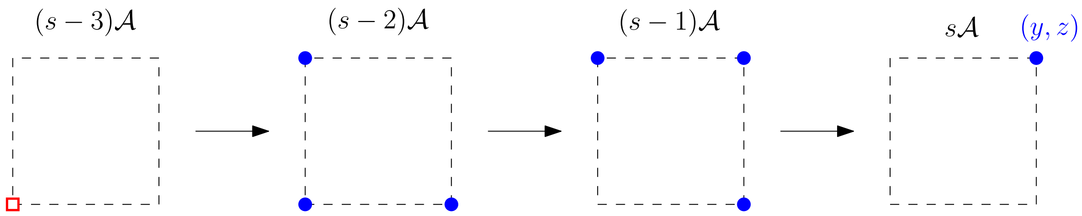
(b) An element $(y,z) \in s\mathcal{A}$ such that $(x,y,z) \in E_s^{3,1}$.



(c) An element $(y,z) \in s\mathcal{A}$ such that $(x,y,z) \in E_s^{2,0}$.



(d) An element $(y,z) \in s\mathcal{A}$ such that $(x,y,z) \in E_s^{3,0}$.



(e) An element $(y,z) \in s\mathcal{A}$ such that $(x,y,z) \in E_s^{3,3}$.

Figure 4.2: Characterization of the elements in $\mathrm{AP}_{\mathcal{S}}$ and the exceptional sets in terms of the sumsets of $\mathcal{A}$ (Remark 4.24).

$e = 1$ then either $(D-1, 1, 0) \notin \underline{\mathcal{A}}$ or $(D-1, 0, 1) \notin \underline{\mathcal{A}}$. By Theorem 3.35, if $\mathbb{k}$ is an algebraically closed field, the previous assumptions are equivalent to saying that $\mathcal{X}$ has a single singular point.

For all $s \in \mathbb{N}$, denote $\mathcal{T}_{s,e} := \{(x, y) \in \mathbb{N}_e^2 \mid x + y \leq sD\}$. The following theorem shows that the Castelnuovo-Mumford regularity of $\mathbb{k}[\mathcal{X}]$ is upper bounded by the sumsets regularity of $\mathcal{A}$.

Theorem 4.27. *Assume that $\mathcal{X} = \mathcal{X}_{\underline{\mathcal{A}}} \subset \mathbb{P}_{\mathbb{k}}^n$ is either smooth or has a single singular point. Then, the Castelnuovo-Mumford regularity of $\mathbb{k}[\mathcal{X}]$ and the sumsets regularity of $\mathcal{A}$ are related by the formula*

$$\operatorname{reg}(\mathbb{k}[\mathcal{X}]) \leq \sigma(\mathcal{A}) + 1 .$$

Proof. Set $\sigma = \sigma(\mathcal{A})$ the sumsets regularity of $\mathcal{A}$. By Theorem 4.25, we know that

$$\operatorname{reg}(\mathbb{k}[\mathcal{X}]) = \max\left\{m(\mathrm{AP}_{\mathcal{S}}), m(E_{\mathcal{S}}^{3,1}) - 1, m(E_{\mathcal{S}}^{2,0}) - 1, m(E_{\mathcal{S}}^{3,0}) - 1, m(E_{\mathcal{S}}^{3,3}) - 2\right\} .$$

We are going to show (i) $m(\mathrm{AP}_{\mathcal{S}}) \leq \sigma + 1$, (ii) $m(E_{\mathcal{S}}^{3,1}) \leq \sigma + 2$, (iii) $m(E_{\mathcal{S}}^{2,0}) \leq \sigma + 2$, (iv) $m(E_{\mathcal{S}}^{3,0}) \leq \sigma + 2$, and (v) $m(E_{\mathcal{S}}^{3,3}) \leq \sigma + 3$. Thus, we conclude $\operatorname{reg}(\mathbb{k}[\mathcal{X}]) \leq \sigma + 1$ by Theorem 4.25.

(i) Let $(x, y, z) \in \mathrm{AP}_{\mathcal{S}}$ and set $s = (x+y+z)/D$. Then (y, z) is as in Figure 4.2a. Assume by contradiction that $s > \sigma + 1$. Since $s - 2 \geq \sigma$ and $(y, z) \in s\mathcal{A} \setminus (s-1)\mathcal{A}$, then $(y, z) \in \mathcal{T}_{s,e} \setminus \mathcal{T}_{s-1,e}$ by Remark 3.37 (2). Thus, $(y-D, z), (y, z-D) \in \mathcal{T}_{s-1,e} \setminus \mathcal{T}_{s-2,e} \subset (s-1)\mathcal{A}$, a contradiction with $(x, y, z) \in \mathrm{AP}_{\mathcal{S}}$. Therefore, $s \leq \sigma + 1$ and hence $m(\mathrm{AP}_{\mathcal{S}}) \leq \sigma + 1$.

(ii) Let $(x, y, z) \in E_{\mathcal{S}}^{3,1}$ and set $s = (x + y + z)/D$. Let us prove that $s \geq \sigma + 2$. Suppose that $(x, y, z) - \mathbf{e}_0 - \mathbf{e}_1 \in \mathcal{S}$, i.e. the permutation in the definition of $E_{\mathcal{S}}^{3,1}$ is the identity. Then (y, z) is a as in Figure 4.2b. Assume by contradiction that $s > \sigma + 2$. Since $s - 3 \geq \sigma$ and $(y-D, z), (y, z-D) \in (s-1)\mathcal{A} \setminus (s-2)\mathcal{A}$, both elements are in $\mathcal{T}_{s-1,e} \setminus \mathcal{T}_{s-2,e}$ by Remark 3.37 (2). Thus $(y, z) \notin \mathcal{T}_{s-1,e}$, so $(y, z) \notin (s-1)\mathcal{A}$, a contradiction. The proof in the other two situations is analogous. Therefore, $m(E_{\mathcal{S}}^{3,1}) \leq \sigma + 2$.

(iii) Let $(x, y, z) \in E_{\mathcal{S}}^{2,0}$ and set $s = (x+y+z)/D$. Let us prove $s \geq \sigma+2$. Suppose $(x, y, z) - \mathbf{e}_2 \notin \mathcal{S}$, i.e. the permutation in the definition of $E_{\mathcal{S}}^{2,0}$ is the identity. Then (y, z) is as in Figure 4.2c. Assume by contradiction that $s > \sigma + 2$. Reasoning as in (ii), one has that $(y-D, z), (y, z-D) \in \mathcal{T}_{s-1,e} \setminus \mathcal{T}_{s-2,e}$. Thus, $(y-D, z-D) \in \mathcal{T}_{s-2,e} \setminus \mathcal{T}_{s-3,e} \subset (s-2)\mathcal{A}$, by Remark 3.37 (2), a contradiction. Hence, $s \leq \sigma + 2$. The proof in the other two situations is analogous, so we have proved that $m(E_{\mathcal{S}}^{2,0}) \leq \sigma + 2$.

(iv) Let $(x, y, z) \in E_{\mathcal{S}}^{3,0}$ and set $s = (x+y+z)/D$. Then (y, z) is as in Figure 4.2d. Let us prove $s \geq \sigma + 2$. Assume by contradiction that $s \geq \sigma + 2$. As in (ii), one has that $(y - D, z), (y, z - D) \in \mathcal{T}_{s-1,e} \setminus \mathcal{T}_{s-2,e}$, so $(y, z) \notin \mathcal{T}_{s-1,e}$. Hence, $(y, z) \notin (s-1)\mathcal{A}$, a contradiction. Therefore, $s \leq \sigma + 2$, and hence $m(E_{\mathcal{S}}^{3,0}) \leq \sigma + 2$.

(v) Let $(x, y, z) \in E_{\mathcal{S}}^{3,3}$ and set $s = (x+y+z)/D$. Then (y, z) is as in Figure 4.2e. Let us prove $s \geq \sigma + 3$. Assume by contradiction that $s > \sigma + 3$. Since $(y - D, z - D) \in (s-2)\mathcal{A} \setminus (s-3)\mathcal{A}$, then $(y - D, z - D) \notin \mathcal{T}_{s-3,e}$ by Remark 3.37 (2). Thus, $(y - D, z), (y, z - D) \notin \mathcal{T}_{s-2,e}$, a contradiction. Therefore, $s \leq \sigma + 3$, and hence $m(E_{\mathcal{S}}^{3,3}) \leq \sigma + 3$.

□

To finish this chapter, we explore the Eisenbud-Goto conjecture for simplicial projective monomial surfaces with a single singular point. In the following result, we compute their degree.

Proposition 4.28. *Suppose that* $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, (0, D-1, 1), (0, 1, D-1), (e, D-e, 0), (e, 0, D-e)\} \subset \underline{\mathcal{A}}$, *where* $1 \leq e \leq D$ *is a divisor of* D *that divides* a_{i0} *for all* $i = 0, \ldots, n$, *and let* $\mathcal{X} = \mathcal{X}_{\underline{\mathcal{A}}}$ *be the projective monomial surface determined by* $\underline{\mathcal{A}}$. *Then, the degree of* $\mathcal{X}$ *is* $\deg(\mathcal{X}) = \frac{D^2}{e}$.

Proof. Consider the matrix M of size $3 \times (n+1)$ whose columns are the elements of $\underline{\mathcal{A}}$. By Theorem 1.73, the degree of the toric variety $\mathcal{X}$ is $\deg(\mathcal{X}) = D^3/\theta_3$, where θ_3 is the g.c.d. of the 3×3 minors of the matrix M. Since the first row of M is a multiple of e and the sum of all its columns is D, then $D \cdot e$ divides θ_3. Moreover, $\begin{vmatrix} e & 0 & 0 \\ D-e & D & D-1 \\ 0 & 0 & 1 \end{vmatrix} = e \cdot D$, which shows $\theta_3 = D \cdot e$. Thus, $\deg(\mathcal{X}) = \frac{D^2}{e}$, by Theorem 1.73. □

Therefore, the Eisenbud-Goto conjecture (Conjecture 1.46) for projective monomial surfaces with a single singular point can be written as

$$\operatorname{reg}(\Bbbk[\mathcal{X}]) \leq \frac{D^2}{e} - n + 2, \tag{4.3}$$

where $n = |\mathcal{A}| - 1$.

Theorem 4.29. *Let* $\mathcal{X} \subset \mathbb{P}_{\Bbbk}^n$ *be a simplicial projective monomial surface whose degree is either minimal or maximal. Then,* $\Bbbk[\mathcal{X}]$ *satisfies the Eisenbud-Goto conjecture.*

Proof. By Theorem 3.35, there exists a set $\mathcal{A} = \{\mathbf{a}_0, \dots, \mathbf{a}_n\} \subset \mathbb{N}^2$, $\mathbf{a}_i = (a_{i1}, a_{i2})$, such that $a_{i1} + a_{i2} \leq D$, $\{(0,0), (D,0), (0,D)\} \subset \mathcal{A}$, for some $D > 0$, $1 \leq e \leq D$ a divisor of D such that $e \mid a_{i1} + a_{i2}$ for all i, and if $e = 1$, then either $(1,0) \notin \mathcal{A}$ or $(0,1) \notin \mathcal{A}$; with $\mathcal{X} = \mathcal{X}_{\underline{\mathcal{A}}}$. By Proposition 4.28, the degree of $\mathcal{X}$ is $\deg(\mathcal{X}) = \frac{D^2}{e}$. Note that it is maximal when $e = 1$, and it is minimal when $e = D$.

If $e = 1$, then the sumsets regularity of $\mathcal{A}$ satisfies $\sigma(\mathcal{A}) \leq D^2 - n + 1$, by Theorem 3.41. Thus, Equation (4.3) from Theorem 4.27.

If $e = D$, by Remark 1.79 (ii) one has that the rings $\mathbb{k}[\mathcal{X}]$ and $\mathbb{k}[x_1, \dots, x_n]/I_{\underline{\mathcal{A}'}}$ have the same minimal graded free resolution, where $I_{\underline{\mathcal{A}'}}$ is the defining ideal of a smooth projective monomial curve $\mathcal{C}'$. Since $\mathcal{C}'$ satisfies the Eisenbud-Goto conjecture, as we proved in Subsection 4.1.2, then $\mathbb{k}[\mathcal{C}]$ satisfies Equation (4.3). $\square$

Chapter 5

Effective computation of the short resolution

"Building a resolution consists of repeatedly solving systems of polynomial equations."
I. Peeva

Let $\Bbbk$ be an arbitrary field, $R = \Bbbk[x_1, \ldots, x_n]$ a polynomial ring over $\Bbbk$, and $I \subset R$ a ω-homogeneous ideal for some weight vector $\omega = (\omega_1, \ldots, \omega_n) \in (\mathbb{Z}_{>0})^n$, i.e., I is homogeneous for the grading on R induced by $\deg_\omega(x_i) = \omega_i$. We denote by $d \coloneqq \dim(R/I)$ the Krull dimension of R/I and assume that $A \coloneqq \Bbbk[x_{n-d+1}, \ldots, x_n]$ is a *Noether normalization* of R/I, that is, $A \hookrightarrow R/I$ is an integral ring extension. When this occurs, we will say that the variables are in *Noether position*. In this setting, R/I is a finitely generated graded A-module, so it has a finite minimal graded free resolution as A-module. This resolution has been referred to in the literature as the *short resolution* [75, 78] or *Noether resolution* [3] of R/I. We denote it by

$$\mathcal{F} : 0 \to \oplus_{v \in \mathcal{B}_p} A(-s_{p,v}) \xrightarrow{\psi_p} \ldots \xrightarrow{\psi_1} \oplus_{v \in \mathcal{B}_0} A(-s_{0,v}) \xrightarrow{\psi_0} R/I \to 0\,, \tag{5.1}$$

where its length $p = \operatorname{pd}_A(R/I)$ is the projective dimension of R/I as A-module, and for all $i \in \{0, \ldots, p\}$, $\mathcal{B}_i \subset R$ are finite sets of monomials, and $s_{i,v}$ are nonnegative integers.

The relation between the lengths of the short resolution of R/I and of its usual minimal graded free resolution as R-module is given by $\operatorname{pd}_R(R/I) = \operatorname{pd}_A(R/I) + n - d$. This follows from the Auslander-Buchsbaum formula and the fact that $\operatorname{depth}_A(R/I) = \operatorname{depth}_R(R/I)$; see, e.g. [13, Ex. 1.2.26(b)]. Hence, the short resolution is shorter than the usual minimal graded free resolution, and it contains valuable combinatorial, algebraic and geometric information about R/I. For example, since (5.1) is a graded free resolution of R/I, one gets that the (weighted) Hilbert series of R/I can be expressed as:

$$\mathrm{HS}_{R/I}(t) = \frac{\sum_{i=0}^{p} \sum_{v\in\mathcal{B}_i} (-1)^i \, t^{s_{i,v}}}{(1-t)^d},$$

and its numerator, $h(t) = \sum_{i=0}^{p} \sum_{v\in\mathcal{B}_i} (-1)^i \, t^{s_{i,v}}$, satisfies that $h(1) = \sum_{i=0}^{p} (-1)^i |\mathcal{B}_i|$ is $e(R/I)$, the multiplicity of R/I. Moreover, when I is homogeneous with respect to the standard grading, as a consequence of the Independence Theorem for local cohomology (see, e.g., [90, Sect. 1]), the Castelnuovo-Mumford regularity of R/I, $\mathrm{reg}(R/I)$, can be computed using the short resolution:

$$\mathrm{reg}(R/I) = \max\{s_{i,v} - i \mid 0 \le i \le p, v \in \mathcal{B}_i\}\,.$$

In [3], the authors describe how to compute short resolutions in some cases. The first step of the short resolution is given by [3, Prop. 1] that we recall in Proposition 5.1. This result provides the whole short resolution when R/I is Cohen-Macaulay. If R/I is not Cohen-Macaulay, the resolution has at least one more step. When $\dim(R/I) = 1$ and $\mathrm{depth}(R/I) = 0$, the second (and last) step of the short resolution is given in [3, Prop. 3]. Moreover, when $\dim(R/I) = 2$ and x_n is not a zero divisor on R/I, the whole short resolution is given in [3, Prop. 4]. In the first section, we study the short resolution in any dimension, and we also drop the assumption that x_n is a nonzero divisor on R/I. We will only assume that I is homogeneous for some grading $\omega \in (\mathbb{Z}_{>0})^n$, and that $A \hookrightarrow R/I$ is a Noether normalization. Note that this last assumption is not restrictive if I is homogeneous for the standard grading and $\Bbbk$ is infinite since linear changes of coordinates preserve homogeneity for the standard grading, and A is a Noether normalization of R/I after a generic linear change of coordinates; see [5, Lem. 4.1] for a Noether position test, and [6, App. A] for smaller changes of coordinates.

Our main results in the first section are Proposition 5.2 and Theorem 5.7. In Proposition 5.2, using the monomial generators of R/I as A-module given in [3, Prop. 1], we describe a generating set (that may not be minimal) of its module of syzygies, a submodule of a free A-module. This presentation of the A-module R/I by generators and relations allows to obtain its minimal graded free resolution by means of standard A-module computations. This gives the first way of contructing the short resolution of R/I (Algorithm 5.1). Another way to obtain the short resolution is as follows. In Theorem 5.7, we prove that the generating set given in Proposition 5.2 is, indeed, the reduced Gröbner basis of the syzygy submodule for a Schreyer-like monomial order, and hence we can build a graded free resolution (that does not need to be minimal) of R/I as A-module by an iterative application of Schreyer's Theorem (Theorem 1.29).

A case in which our results apply nicely is that of toric rings. Let $I \subset R$ be a simplicial toric ideal of height $n-d$ (i.e., $\dim(R/I) = d$). By Proposition 1.60, there exists a set $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^d$ such that $I = I_{\mathcal{A}}$ and $\{D\boldsymbol{\epsilon}_1, \ldots, D\boldsymbol{\epsilon}_d\} \subset \mathcal{A}$, where $\{\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_d\}$ is the canonical basis of $\mathbb{N}^d$ and $D \in \mathbb{Z}_{>0}$. Hence, we can always assume that the extremal rays of the rational cone spanned by $\mathcal{A}$ are $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$, with $\mathbf{e}_i \coloneqq D\boldsymbol{\epsilon}_i$, $i = 1, \ldots, d$. By [33, Prop. 1.1.12], one has that $A = \Bbbk[x_{n-d+1}, \ldots, x_n] \hookrightarrow R/I$ is a Noether normalization if and only if $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ are the last d elements of $\mathcal{A}$. Hence, without loss of generality, we will assume that $\mathbf{a}_{n-d+i} = \mathbf{e}_i$ for $i = 1, \ldots, d$. We focus on the simplicial toric rings of dimension 3.

In Section 5.2, we describe their short resolution and their Hilbert series and function in terms of the combinatorics of the associated semigroup translating some results of [78] and [75]. In the standard-graded homogeneous case, we provide formulas for the Castelnuovo-Mumford regularity of the toric ring. In Section 5.3, we devise an algorithm to compute the short resolution for 3-dimensional simplicial toric rings. This algorithm first constructs a non-minimal graded free resolution as A-module following Section 5.1 (Algorithm 5.2), and then minimalizes/prunes it to obtain the short one by applying Theorems 5.24 and 5.26 (Algorithm 5.3). The whole algorithm involves the computation of the reduced Gröbner bases of $I_{\mathcal{A}}$ and $I_{\mathcal{A}} + \langle x_{n-2} \rangle$, and the division of some monomials by those bases.

In Section 5.4, we provide an example of a simplicial semigroup whose toric ring has different projective dimensions, both as A-module and as R-module, depending on the characteristic of the field $\Bbbk$. Hence, both the usual and the short resolution depend on the characteristic of $\Bbbk$. To our knowledge, this is the first example in which this phenomenon is observed.

The results included in this chapter are part of the preprint [35], and the algorithms have been implemented in SageMath and are available in the GitHub repository [41].

5.1 Contruction of the short resolution via Gröbner bases

Let $\omega = (\omega_1, \ldots, \omega_n) \in (\mathbb{Z}_{>0})^n$ be a weight vector, $\Bbbk$ an arbitrary field and $R = \Bbbk[x_1, \ldots, x_n]$. Consider $I \subset R$ a ω-homogeneous ideal, i.e., a homogeneous ideal with respect to the grading induced by $\deg_\omega(x_i) = \omega_i$ for all $i \in \{1, \ldots, n\}$. Take

$d := \dim(R/I)$ and assume that $A = \Bbbk[x_{n-d+1},\ldots,x_n]$ is a Noether normalization of R/I. In this section we study the *short resolution* of R/I, i.e., the minimal graded free resolution of R/I as A-module:

$$\mathcal{F} : 0 \to \oplus_{v\in\mathcal{B}_p} A(-s_{p,v}) \xrightarrow{\psi_p} \ldots \xrightarrow{\psi_1} \oplus_{v\in\mathcal{B}_0} A(-s_{0,v}) \xrightarrow{\psi_0} R/I \to 0\,, \tag{5.2}$$

where $p = \mathrm{pd}_A(R/I)$, and for all $i \in \{0,\ldots,p\}$, $\mathcal{B}_i \subset R$ is a finite set and $s_{i,v}$ are nonnegative integers. In our description, the sets $\mathcal{B}_i$ will consist of monomials and $s_{i,v} = \deg_\omega(v)$ will be the ω-degree of the monomial $v \in \mathcal{B}_i$. Note that the sets $\mathcal{B}_i$ might not be unique, but their degrees are.

Consider the ω-graded reverse lexicographic order $>_\omega$ in R, i.e., the monomial order defined as follows: $\mathbf{x}^\alpha >_\omega \mathbf{x}^\beta$ if and only if

- $\deg_\omega(\mathbf{x}^\alpha) > \deg_\omega(\mathbf{x}^\beta)$, or
- $\deg_\omega(\mathbf{x}^\alpha) = \deg_\omega(\mathbf{x}^\beta)$ and the last nonzero entry of $\alpha - \beta \in \mathbb{Z}^n$ is negative.

For every polynomial $f \in R$, let $\mathrm{in}(f)$ denote the initial term of f with respect to $>_\omega$ (we include the coefficient in the initial term). Given an ideal $J \subset R$, $\mathrm{in}(J)$ denotes the initial ideal of J with respect to $>_\omega$, and $\mathcal{G}$ the reduced Gröbner basis of I with respect to $>_\omega$. Since I is ω-homogeneous, $\mathcal{G}$ consists of ω-homogeneous polynomials.

With these notations, the first step of the short resolution of R/I is given by the following result:

Proposition 5.1 ([3, Prop. 1])**.** *Let $\mathcal{B}_0 \subset R$ be the set of monomials that do not belong to* $\mathrm{in}(I) + \langle x_{n-d+1},\ldots,x_n\rangle$. *Then,*

$$\{u + I \mid u \in \mathcal{B}_0\}$$

is a minimal set of generators of R/I as A-module. The ω-graded A-module homomorphism $\psi_0 : \oplus_{v\in\mathcal{B}_0} A\,(-\deg_\omega(v)) \to R/I$ is defined by $\psi_0(\boldsymbol{\epsilon}_u) = u + I$, where $\{\boldsymbol{\epsilon}_u \mid u \in \mathcal{B}_0\}$ denotes the canonical basis of $\oplus_{v\in\mathcal{B}_0} A\,(-\deg_\omega(v))$, and hence the shifts at the first step of the short resolution (5.2) *are the ω-degrees of the elements $u \in \mathcal{B}_0$.*

This result provides the whole short resolution when R/I is a free A-module, i.e., when the projective dimension of R/I as A-module is 0, which is equivalent to R/I being Cohen-Macaulay. In Gröbner basis terms, this is also equivalent to the fact that variables $x_{n-d+1},\ldots,x_n$ do not divide any minimal generator of $\mathrm{in}(I)$; see, e.g., [5, Thm. 2.1] or [3, Prop. 2].

When R/I is not free, the resolution has at least one more step. In this case, we will describe the relations between the generators of R/I given in Proposition 5.1, i.e., provide a finite set of generators $\mathcal{H}$ of $\ker(\psi_0)$, which may not be minimal. This gives a presentation of R/I as A-module: R/I is isomorphic to the quotient of a free A-module by the submodule generated by $\mathcal{H}$, and the short resolution of R/I can then be obtained by standard A-module computations (see Section 1.2).

Let $\chi : R \to R$ be the evaluation morphism defined by $\chi(x_i) = x_i$ for $i \in \{1, \dots, n-d\}$ and $\chi(x_j) = 1$ for $j \in \{n-d+1, \dots, n\}$, and set $J := \chi(\mathrm{in}(I)).R$, the extension of the ideal $\mathrm{in}(I)$ by the ring homomorphism χ. Now, for every monomial $u \in \mathcal{B}_0 \cap J$, consider the ideal I_u defined by

$$I_u := (\mathrm{in}(I) : u) \cap \Bbbk[x_{n-d+1}, \dots, x_n] .$$

Since I_u is a monomial ideal, it has a unique minimal monomial generating set denoted by $G(I_u)$, and let $\mathcal{B}'_1$ be the following set of monomials:

$$\mathcal{B}'_1 = \{u \cdot M \mid u \in \mathcal{B}_0 \cap J, M \in G(I_u)\}. \tag{5.3}$$

Each monomial $\mathbf{x}^\alpha \in \mathcal{B}'_1$ can be written uniquely as $\mathbf{x}^\alpha = u \cdot M_\alpha$, where $u = \chi(\mathbf{x}^\alpha) \in \mathcal{B}_0 \cap J$ and $M_\alpha \in G(I_u)$. Let r_α be the remainder of the division of $\mathbf{x}^\alpha$ by $\mathcal{G}$, the reduced Gröbner basis of I with respect to $>_\omega$. Since every monomial in the expression of r_α does not belong to $\mathrm{in}(I)$, one can uniquely write $r_\alpha = \sum_{v \in \mathcal{B}_0} f_{\alpha,v} v$ with $f_{\alpha,v} \in A$. Using these notations, for all $\mathbf{x}^\alpha \in \mathcal{B}'_1$ set

$$\mathbf{h}_\alpha := M_\alpha \cdot \boldsymbol{\epsilon}_u - \sum_{v \in \mathcal{B}_0} f_{\alpha,v} \cdot \boldsymbol{\epsilon}_v \in \oplus_{v \in \mathcal{B}_0} A\left(-\deg_\omega(v)\right), \tag{5.4}$$

where $\{\boldsymbol{\epsilon}_v \mid v \in \mathcal{B}_0\}$ denotes the canonical basis of $\oplus_{v \in \mathcal{B}_0} A\left(-\deg_\omega(v)\right)$.

Since $\psi_0(\mathbf{h}_\alpha) = (\mathbf{x}^\alpha - r_\alpha) + I = 0$ for all $\mathbf{x}^\alpha \in \mathcal{B}'_1$, one has that $\langle \mathbf{h}_\alpha \mid \mathbf{x}^\alpha \in \mathcal{B}'_1 \rangle \subset \ker(\psi_0)$. The next result shows that this inclusion is indeed an equality.

Proposition 5.2. *The kernel of the A-module homomorphism ψ_0 is*

$$\ker(\psi_0) = \langle \mathbf{h}_\alpha \mid \mathbf{x}^\alpha \in \mathcal{B}'_1 \rangle .$$

Proof. Consider $\bar{\psi}_0 : \oplus_{v \in \mathcal{B}_0} A\left(-\deg_w(v)\right) \to R$ the A-module homomorphism defined by $\bar{\psi}_0(\boldsymbol{\epsilon}_v) = v$, for all $v \in \mathcal{B}_0$. Take $g \in \ker(\psi_0)$, and let us prove that $g \in \langle \mathbf{h}_\alpha \mid \mathbf{x}^\alpha \in \mathcal{B}'_1 \rangle$. We write $g = \sum_{v \in \mathcal{B}_0} g_v \boldsymbol{\epsilon}_v$ with $g_v \in A$ for all $v \in \mathcal{B}_0$. Since $g \in \ker(\psi_0)$, then $g' = \bar{\psi}_0(g) = \sum_{v \in \mathcal{B}_0} g_v \cdot v \in I$ and its initial term is $\mathrm{in}(g') = c \cdot w \cdot M_\gamma$ for some $c \in \Bbbk \setminus \{0\}$, $w \in \mathcal{B}_0$ and a monomial $M_\gamma \in A$. In fact, $w \in \mathcal{B}_0 \cap J$ and

$M_\gamma \in I_w = (\mathrm{in}(I) : w) \cap A$. Hence, there exists $\mathbf{x}^\alpha = M_\alpha w \in \mathcal{B}_1'$ such that M_α divides M_γ. Let us consider $g_1 = g - c\frac{M_\gamma}{M_\alpha}\mathbf{h}_\alpha \in \ker(\psi_0)$. If $g_1 = 0$, then $g \in \langle \mathbf{h}_\alpha \mid \mathbf{x}^\alpha \in \mathcal{B}_1' \rangle$. Otherwise, one has that $0 \neq \mathrm{in}\left(\bar{\psi}_0(g_1)\right) < \mathrm{in}\left(\bar{\psi}_0(g)\right)$ and we iterate this process. The result then follows by induction because $>_\omega$ is a well ordering. □

Proposition 5.2 provides a system of generators of $\ker(\psi_0)$. As a consequence, we get the next step of a non-necessarily minimal graded free resolution of R/I as A-module.

Corollary 5.3. *Consider the morphism of A-modules*

$$\psi_1' : \oplus_{\mathbf{x}^\alpha \in \mathcal{B}_1'} A\left(-\deg_\omega(\mathbf{x}^\alpha)\right) \to \oplus_{v \in \mathcal{B}_0} A\left(-\deg_\omega(v)\right)$$
$$\boldsymbol{\epsilon}_\alpha \mapsto \mathbf{h}_\alpha$$

where $\{\boldsymbol{\epsilon}_\alpha \mid \mathbf{x}^\alpha \in \mathcal{B}_1'\}$ is the canonical basis of $\oplus_{\mathbf{x}^\alpha \in \mathcal{B}_1'} A\left(-\deg_\omega(\mathbf{x}^\alpha)\right)$. Then, $\mathrm{Im}(\psi_1') = \mathrm{Ker}(\psi_0)$.

Since R/I and $\oplus_{v \in \mathcal{B}_0} A\left(-\deg_\omega(v)\right)/\ker(\psi_0)$ are isomorphic as graded A-modules, their minimal graded free resolutions coincide up to isomorphism. Thus, one can compute the short resolution by applying standard A-module computations to the submodule $\ker(\psi_0) = \langle \mathbf{h}_\alpha \mid \mathbf{x}^\alpha \in \mathcal{B}_1' \rangle \subset \oplus_{v \in \mathcal{B}_0} A(-\deg_\omega(v))$. The whole process to obtain the short resolution of R/I is shown in Algorithm 5.1. It has been implemented in the function `shortRes` of [41].

Example 5.4. Set $R = \mathbb{Q}[x_1, x_2, x_3, x_4, x_5]$, let $>$ be the degree reverse lexicographic order in R, and consider $I_\mathcal{C} \subset R$, the defining ideal of the projective monomial curve determined by $\underline{\mathcal{A}} = \{(1,6),(2,5),(6,1),(7,0),(0,7)\}$, i.e.,

$$I_{\underline{\mathcal{A}}} = \langle x_1 - t_1 t_2^6, x_2 - t_1^2 t_2^5, x_3 - t_1^6 t_2, x_4 - t_1^7, x_5 - t_2^7 \rangle \cap \mathbb{Q}[x_1, x_2, x_3, x_4, x_5].$$

Let L be the zero-dimensional ideal $L = \langle x_1^2 - x_2 x_3, x_2^3 - x_4 x_5^2, x_1 x_2, x_3^2, x_4^2 - x_5^2, x_2 x_5, x_5^4 \rangle$, and consider the ideal $I = I_{\underline{\mathcal{A}}} \cap L$. One has that I is homogeneous, $\dim(R/I) = \dim(R/I_{\underline{\mathcal{A}}}) = 2$, and variables are in Noether position, i.e., $A = \mathbb{Q}[x_4, x_5] \hookrightarrow R/I$ is a Noether normalization. Moreover, x_5 is a zero divisor on R/I because $f = x_2 x_3^2 - x_4^2 x_5 \notin I$ while $f x_5 \in I$, so we are not under the hypotheses of [3, Prop. 4]. By Proposition 5.1, a minimal system of generators of R/I as A-module is $\{u + I \mid u \in \mathcal{B}_0\}$ for

$$\mathcal{B}_0 = \{u_1 = x_3^4, u_2 = x_3^3, u_3 = x_2 x_3^2, u_4 = x_1 x_3^2, u_5 = x_3^2, u_6 = x_2 x_3, u_7 = x_1 x_3, u_8 = x_3, u_9 = x_2^3, u_{10} = x_1 x_2^2, u_{11} = x_2^2, u_{12} = x_1 x_2, u_{13} = x_2, u_{14} = x_1^2, u_{15} = x_1, u_{16} = 1\}.$$

Algorithm 5.1 Computation of the short resolution.

Input: $I \subset R$ a weighted homogeneous ideal with variables in Noether position
Output: Short resolution of R/I

$\mathcal{G} \leftarrow$ reduced Gröbner basis of I for $>_\omega$.
$\mathcal{B}_0 \leftarrow \Bbbk$-basis of $\mathrm{in}(I) + \langle x_{n-d+1}, \dots, x_n \rangle$ for $>_\omega$.
$J \leftarrow \chi(\mathrm{in}(I)).R$, where $\chi : R \to R$ is defined by $\chi(x_i) = x_i$ for $i \in \{1, \dots, n-d\}$, and $\chi(x_j) = 1$ for $j \in \{n-d+1, \dots, n\}$.
$I_u \leftarrow (\mathrm{in}(I) : u) \cap A$, $\forall u \in \mathcal{B}_0 \cap J$.
$G(I_u) \leftarrow$ minimal monomial generating set of I_u, $\forall u \in \mathcal{B}_0 \cap J$.
$\mathcal{B}_1' \leftarrow \{u \cdot M \mid u \in \mathcal{B}_0 \cap J, M \in G(I_u)\}$.
$r_\alpha \leftarrow$ remainder of $\mathbf{x}^\alpha$ by $\mathcal{G}$, $\forall \mathbf{x}^\alpha \in \mathcal{B}_1'$.
For all $\mathbf{x}^\alpha \in \mathcal{B}_1'$, write $\mathbf{x}^\alpha = M_\alpha u$ and $r_\alpha = \sum_{v \in \mathcal{B}_0} f_{\alpha,v} v$.
$\mathbf{h}_\alpha \leftarrow M_\alpha \boldsymbol{\epsilon}_u - \sum_{v \in \mathcal{B}_0} f_{\alpha,v} \boldsymbol{\epsilon}_v$, $\forall \mathbf{x}^\alpha \in \mathcal{B}_1'$.
$\ker(\psi_0) \leftarrow \langle \mathbf{h}_\alpha \mid \mathbf{x}^\alpha \in \mathcal{B}_1' \rangle$.
Compute the m.g.f.r. of $\ker(\psi_0)$.

If $\chi : R \to R$ is the ring homomorphism defined by $\chi(x_1) = x_1$, $\chi(x_2) = x_2$, $\chi(x_3) = x_3$, and $\chi(x_4) = \chi(x_5) = 1$, then $J = \chi(\mathrm{in}(I)).R = \langle x_3^5, x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3 \rangle$, and hence $\mathcal{B}_0 \cap J = \{u_3, u_4, u_6, u_7, u_9, u_{10}, u_{11}, u_{12}, u_{14}\}$, and $I_{u_3} = I_{u_4} = \langle x_4, x_5 \rangle$, $I_{u_6} = \langle x_4^2, x_5^2 \rangle$, $I_{u_7} = \langle x_5^4, x_4x_5^2, x_4^2 \rangle$, $I_{u_9} = \langle x_5^2, x_4 \rangle$, $I_{u_{10}} = \langle x_4 \rangle$, $I_{u_{11}} = \langle x_4^2 \rangle$, $I_{u_{12}} = \langle x_4^3 \rangle$, and $I_{u_{14}} = \langle x_4^2, x_5 \rangle$. Thus, the set $\mathcal{B}_1'$ defined in (5.3) is

$$\mathcal{B}_1' = \{x_2x_3^2x_4,\ x_2x_3^2x_5,\ x_1x_3^2x_4,\ x_1x_3^2x_5,\ x_2x_3x_4^2,\ x_2x_3x_5^2,\ x_1x_3x_5^4,\ x_1x_3x_4x_5^2,\ x_1x_3x_4^2, \\ x_2^3x_5^2,\ x_2^3x_4,\ x_1x_2^2x_4,\ x_2^2x_4^2,\ x_1x_2x_4^3,\ x_1^2x_4^2,\ x_1^2x_5\}.$$

Take the first element in $\mathcal{B}_1'$, $\mathbf{x}^\alpha = x_2x_3^2x_4 = x_4u_3$, and compute the remainder r_α of its division by the reduced Gröbner basis of I with respect to $>$, $r_\alpha = x_4^3x_5 = x_4^3x_5u_{16}$. The corresponding element as in (5.4) is $\mathbf{h}_\alpha = x_4\boldsymbol{\epsilon}_3 - x_4^3x_5\boldsymbol{\epsilon}_{16}$ where $\{\boldsymbol{\epsilon}_1, \dots, \boldsymbol{\epsilon}_{16}\}$ is the canonical basis of $\oplus_{i=1}^{16} A(-\deg(u_i))$. Doing the same for each monomial in $\mathcal{B}_1'$, one gets 16 elements that generate the submodule $\ker(\psi_0)$ of the free module $\oplus_{i=1}^{16} A(-\deg(u_i))$, and computing the minimal graded free resolution of this submodule, one gets the short resolution of R/I. Using the function `shortRes` of [41], one gets directly the Betti table of the short resolution:

```
            0     1     2
------------------------
    0:      1     -     -
    1:      3     -     -
    2:      6     1     -
    3:      5    11     2
    4:      1     2     2
    5:      -     -     1
------------------------
total:     16    14     5
```

Observe that in this example the set of 16 generators of $\ker(\psi_0)$ given by Proposition 5.2 is not minimal since the Betti table shows that $\ker(\psi_0)$ is minimally generated by 14 elements. We will come back to this example later in Example 5.8.

Interestingly, the system of generators provided in Proposition 5.2 is, in fact, a Gröbner basis for a monomial order in $\oplus_{v\in\mathcal{B}_0} A(-\deg_\omega(v))$ that we now introduce. This can be used to provide another method for computing a graded free resolution of R/I as A-module, applying Theorem 1.29 repeatedly.

Definition 5.5. Consider the monomial order $>_{\mathrm{SL}}$ in $\oplus_{v\in\mathcal{B}_0} A(-\deg_\omega(v))$ defined as follows: for all $M, M' \in A$ monomials and $u, v \in \mathcal{B}_0$,

$$M\boldsymbol{\epsilon}_u >_{\mathrm{SL}} M'\boldsymbol{\epsilon}_v \iff u \cdot M >_\omega v \cdot M' .$$

We call this monomial order the *Schreyer-like order* in $\oplus_{v\in\mathcal{B}_0} A\,(-\deg_\omega(v))$.

If $\bar{\psi}_0$ is the homomorphism of A-modules introduced in the proof of Proposition 5.2, $\bar{\psi}_0 : \oplus_{v\in\mathcal{B}_0} A\,(-\deg_\omega(v)) \to R$, $\boldsymbol{\epsilon}_v \mapsto v$, it is injective and maps monomials to monomials, and

$$M\boldsymbol{\epsilon}_u >_{\mathrm{SL}} M'\boldsymbol{\epsilon}_v \iff \bar{\psi}_0(M) >_\omega \bar{\psi}_0(M') .$$

This equivalent description of $>_{\mathrm{SL}}$ proves that it is a monomial order and justifies its name.

Remark 5.6. For each $\mathbf{x}^\alpha \in \mathcal{B}_1'$, the initial term of $\mathbf{h}_\alpha = M_\alpha \cdot \boldsymbol{\epsilon}_u - \sum_{v\in\mathcal{B}_0} f_{\alpha,v} \cdot \boldsymbol{\epsilon}_v$ for the Schreyer-like monomial order $>_{\mathrm{SL}}$ is $\mathrm{in}(\mathbf{h}_\alpha) = M_\alpha \cdot \boldsymbol{\epsilon}_u$.

Theorem 5.7. *The set $\mathcal{H} = \{\mathbf{h}_\alpha \mid \mathbf{x}^\alpha \in \mathcal{B}_1'\}$ is the reduced Gröbner basis of* $\ker(\psi_0)$ *for the Schreyer-like order $>_{\mathrm{SL}}$.*

Proof. By Proposition 5.2, $\ker(\psi_0) = \langle \mathcal{H} \rangle$. By Buchberger's criterion, $\mathcal{H}$ is a Gröbner basis if and only if, for all $\mathbf{h}_\alpha, \mathbf{h}_\beta \in \mathcal{H}$, the S-polynomial $S(\mathbf{h}_\alpha, \mathbf{h}_\beta)$ reduces to zero modulo $\mathcal{H}$. One has that $S(\mathbf{h}_\alpha, \mathbf{h}_\beta) = 0$ whenever $\mathrm{in}(\mathbf{h}_\alpha)$ and $\mathrm{in}(\mathbf{h}_\beta)$ are multiples of different elements in the canonical basis $\{\boldsymbol{\epsilon}_v \mid v \in \mathcal{B}_0\}$. Let $\mathbf{h}_\alpha, \mathbf{h}_\beta$ be two elements in $\mathcal{H}$ whose initial terms are multiples of the same element in the canonical basis. By Remark 5.6, there exist monomials $u \in \mathcal{B}_0$ and $M_\alpha, M_\beta \in A$, such that $\mathrm{in}(\mathbf{h}_\alpha) = M_\alpha \boldsymbol{\epsilon}_u$ and $\mathrm{in}(\mathbf{h}_\beta) = M_\beta \boldsymbol{\epsilon}_u$. Set $h'_\alpha = uM_\alpha - r_\alpha$ and $h'_\beta = uM_\beta - r_\beta$, where r_α and r_β are the remainder of the division of $\mathbf{x}^\alpha = uM_\alpha$ and $\mathbf{x}^\beta = uM_\beta$ by $\mathcal{G}$ (the reduced Gröbner basis of I for $>_\omega$), respectively. Let $M = \mathrm{lcm}(M_\alpha, M_\beta)$ be the least common multiple of M_α and M_β. Then, the S-polynomial of $\mathbf{h}_\alpha$ and $\mathbf{h}_\beta$ is

$$S_{\alpha,\beta} = S(\mathbf{h}_\alpha, \mathbf{h}_\beta) = \frac{M}{M_\alpha}\mathbf{h}_\alpha - \frac{M}{M_\beta}\mathbf{h}_\beta\,.$$

If $S(\mathbf{h}_\alpha, \mathbf{h}_\beta) = 0$, we are done. Otherwise, note that $\psi_0(S_{\alpha,\beta}) = 0$, so $\bar{\psi}_0(S_{\alpha,\beta}) \in I$, and hence in $\left(\bar{\psi}_0(S_{\alpha,\beta})\right) \in \mathrm{in}(I)$. Thus, there exist $c \in \mathbb{k}$, $w \in \mathcal{B}_0 \cap J$ and a monomial $M_\mu \in A$ such that $\mathrm{in}\left(\bar{\psi}_0(S_{\alpha,\beta})\right) = c \cdot w \cdot M_\mu$. Therefore, $M_\mu \in I_w$, and there exists a monomial $M_\gamma \in G(I_w)$ that divides M_μ. Let $\mathbf{h}_\gamma \in \mathcal{H}$ be the element whose initial term is $\mathrm{in}(\mathbf{h}_\gamma) = M_\gamma \boldsymbol{\epsilon}_w$. Consider $S'_{\alpha,\beta} = S_{\alpha,\beta} - c \cdot \frac{M_\mu}{M_\gamma}\mathbf{h}_\gamma$. If $S'_{\alpha,\beta} = 0$, we are done. Otherwise, one has that $\bar{\psi}_0(S'_{\alpha,\beta}) \in I$ and $0 \neq \mathrm{in}\left(\bar{\psi}_0(S'_{\alpha,\beta})\right) <_\omega \mathrm{in}\left(\bar{\psi}_0(S_{\alpha,\beta})\right)$. We can iterate this process and conclude that $S_{\alpha,\beta}$ reduces to zero modulo $\mathcal{H}$ by induction because $>_\omega$ is a well order. This shows that $\mathcal{H}$ is a Gröbner basis of $\ker(\psi_0)$ for $>_{\mathrm{SL}}$.

Moreover, since $\mathbf{x}^\alpha \nmid \mathbf{x}^\beta$ and $\mathbf{x}^\beta \nmid \mathbf{x}^\alpha$ for all $\mathbf{x}^\alpha \neq \mathbf{x}^\beta$ in $\mathcal{B}'_1$, $\mathcal{H}$ is minimal. Finally, for each $\mathbf{x}^\alpha \in \mathcal{B}'_1$, every monomial appearing in r_α (the remainder of the division of $\mathbf{x}^\alpha$ by $\mathcal{G}$), does not belong to $\mathrm{in}(I)$. Therefore, each monomial that appears in $\sum_{v \in \mathcal{B}_0} f_{\alpha,v} \cdot \boldsymbol{\epsilon}_v$ does not belong to $\langle \mathrm{in}(\mathbf{h}_\beta) \mid \mathbf{h}_\beta \in \mathcal{H} \rangle = \mathrm{in}\left(\ker(\psi_0)\right)$, and we are done. □

Since $\mathcal{H}$ is a Gröbner basis of $\ker(\psi_0)$, the reductions of the S-polynomials $S_{\alpha\beta}$ provide a generating set for the next syzygy module. This generating set is indeed a Gröbner basis by Schreyer's Theorem (Theorem 1.29). The order used here is the Schreyer order induced in $\oplus_{\mathbf{x}^\alpha \in \mathcal{B}'_1} A\left(-\deg_\omega(\mathbf{x}^\alpha)\right)$ by our Schreyer-like order in $\oplus_{v \in \mathcal{B}_0} A(-\deg_\omega(v))$. Applying repeatedly Schreyer's Theorem, we obtain the co-called Schreyer resolution that may not be minimal. Moreover, if we sort at each step the elements of the Gröbner basis as in Corollary 1.31, one variable disappears from the initial terms of the elements in the Gröbner basis at each step. Mimicking the proof of Hilbert's Syzygies Theorem that uses iteratedly Schreyer's Theorem, we obtain a ω-graded free resolution of R/I as A-module that may not be minimal but has at most d steps,

$$\mathcal{F}' : 0 \to \oplus_{v\in\mathcal{B}'_{p'}} A(-\deg_\omega(v)) \xrightarrow{\psi'_{p'}} \dots \xrightarrow{\psi'_1} \oplus_{v\in\mathcal{B}_0} A(-\deg_\omega(v)) \xrightarrow{\psi_0} R/I \to 0\,, \quad (5.5)$$

where $d \geq p' \geq p$ and $\mathcal{B}'_i \subset R$ is a set of monomials for all i. Minimalizing this resolution, a short resolution of R/I as in (5.2) is obtained with $\mathcal{B}_i \subset \mathcal{B}'_i$.

We now illustrate with an example how to build and minimalize Schreyer's resolution. We will see later in Section 5.3 how to explicitly obtain the short resolution (5.2) from Schereyer's resolution (5.5) when R/I is a simplicial toric ring of dimension 3.

Example 5.8. Consider the ideal $I \subset R = \mathbb{Q}[x_1, x_2, x_3, x_4, x_5]$ in Example 5.4. We have already determined $\mathcal{B}_0$ and $\mathcal{B}'_1$, and we now sort the elements in $\mathcal{B}'_1$ as follows:

$$\begin{aligned}\mathcal{B}'_1 = \{&v_1 = x_2x_3^2x_4, v_2 = x_2x_3^2x_5, v_3 = x_1x_3^2x_4, v_4 = x_1x_3^2x_5, v_5 = x_2x_3x_4^2, v_6 = x_2x_3x_5^2,\\ &v_7 = x_1x_3x_4^2, v_8 = x_1x_3x_4x_5^2, v_9 = x_1x_3x_5^4, v_{10} = x_2^3x_4, v_{11} = x_2^3x_5^2, v_{12} = x_1x_2^2x_4,\\ &v_{13} = x_2^2x_4^2, v_{14} = x_1x_2x_4^3, v_{15} = x_1^2x_4^2, v_{16} = x_1^2x_5\}\,.\end{aligned}$$

The 16 generators of $\ker(\psi_0)$ given by Proposition 5.2 are

$$\begin{array}{lll}\mathbf{h}_1 = x_4\boldsymbol{\epsilon}_3 - x_4^3x_5\boldsymbol{\epsilon}_{16}, & \mathbf{h}_2 = x_5\boldsymbol{\epsilon}_3 - x_4^2x_5^2\boldsymbol{\epsilon}_{16}, & \mathbf{h}_3 = x_4\boldsymbol{\epsilon}_4 - x_5\boldsymbol{\epsilon}_9 - x_4^2x_5\boldsymbol{\epsilon}_8 + x_5^3\boldsymbol{\epsilon}_8,\\ \mathbf{h}_4 = x_5\boldsymbol{\epsilon}_4 - x_4x_5^2\boldsymbol{\epsilon}_8, & \mathbf{h}_5 = x_4^2\boldsymbol{\epsilon}_6 - x_4^3\boldsymbol{\epsilon}_{15}, & \mathbf{h}_6 = x_5^2\boldsymbol{\epsilon}_6 - x_4x_5^2\boldsymbol{\epsilon}_{15},\\ \multicolumn{2}{l}{\mathbf{h}_7 = x_4^2\boldsymbol{\epsilon}_7 - x_4^3x_5\boldsymbol{\epsilon}_{16} - x_5^2\boldsymbol{\epsilon}_7 + x_4x_5^3\boldsymbol{\epsilon}_{16}} & \mathbf{h}_8 = x_4x_5^2\boldsymbol{\epsilon}_7 - x_4^2x_5^3\boldsymbol{\epsilon}_{16},\\ \mathbf{h}_9 = x_5^4\boldsymbol{\epsilon}_7 - x_4x_5^5\boldsymbol{\epsilon}_{16}, & \mathbf{h}_{10} = x_4\boldsymbol{\epsilon}_9 - x_4x_5^2\boldsymbol{\epsilon}_8, & \mathbf{h}_{11} = x_5^2\boldsymbol{\epsilon}_9 - x_5^4\boldsymbol{\epsilon}_8,\\ \mathbf{h}_{12} = x_4\boldsymbol{\epsilon}_{10} - x_5^2\boldsymbol{\epsilon}_5, & \mathbf{h}_{13} = x_4^2\boldsymbol{\epsilon}_{11} - x_5\boldsymbol{\epsilon}_2, & \mathbf{h}_{14} = x_4^3\boldsymbol{\epsilon}_{12} - x_5\boldsymbol{\epsilon}_1,\\ \mathbf{h}_{15} = x_4^2\boldsymbol{\epsilon}_{14} - x_4^2x_5\boldsymbol{\epsilon}_{13}, & \mathbf{h}_{16} = x_5\boldsymbol{\epsilon}_{14} - x_5^2\boldsymbol{\epsilon}_{13}\ , & \end{array}$$

where $\{\boldsymbol{\epsilon}_1, \dots, \boldsymbol{\epsilon}_{16}\}$ denotes the canonical basis of $\oplus_{i=1}^{16} A\,(-\deg(u_i))$.

By Theorem 5.7, $\mathcal{H} = \{\mathbf{h}_1, \dots, \mathbf{h}_{16}\}$ is the reduced Gröbner basis of $\ker(\psi_0)$ for our Schreyer-like order $>_{\mathrm{SL}}$. Moreover, in the above list, the first term of each element is its initial term, by Remark 5.6. Note that we have sorted $\mathbf{h}_1, \dots, \mathbf{h}_{16}$ (and, accordingly, $v_1, \dots, v_{16}$ in $\mathcal{B}'_1$) in such a way that, if for some $i < j$, the initial terms of $\mathbf{h}_i$ and $\mathbf{h}_j$ are multiples of the same element of the canonical basis, say $\mathrm{in}(\mathbf{h}_i) = M_i \cdot \boldsymbol{\epsilon}_u$ and $\mathrm{in}(\mathbf{h}_j) = M_j \cdot \boldsymbol{\epsilon}_u$ for some $u \in \mathcal{B}_0$ and two monomials M_i and M_j in $A = \mathbb{Q}[x_4, x_5]$, then $M_i > M_j$ for the lexicographic order $>$ with $x_4 > x_5$, i.e., if $M_i = x_4^{a_i}x_5^{b_i}$ and $M_j = x_4^{a_j}x_5^{b_j}$, $a_i > a_j$. This guarantees that x_4 will not appear in the leading terms of the generators of the next syzygy module obtained by applying Schreyer's Theorem, and hence we will be done.

The reductions of the S-polynomials $S(\mathbf{h}_i, \mathbf{h}_j)$ for all $1 \leq i < j \leq 16$, provide a Gröbner basis of the next syzygy module for the induced Schreyer order. Since the only S-polynomials that have to be computed and reduced by $\mathcal{H}$ are the $S(\mathbf{h}_i, \mathbf{h}_j)$ such that the initial terms of $\mathbf{h}_i$ and $\mathbf{h}_j$ are multiples of the same element in the canonical basis, one just has to focus on $S(\mathbf{h}_1, \mathbf{h}_2)$, $S(\mathbf{h}_3, \mathbf{h}_4)$, $S(\mathbf{h}_5, \mathbf{h}_6)$, $S(\mathbf{h}_7, \mathbf{h}_8)$, $S(\mathbf{h}_7, \mathbf{h}_9)$, $S(\mathbf{h}_8, \mathbf{h}_9)$, $S(\mathbf{h}_{10}, \mathbf{h}_{11})$ and $S(\mathbf{h}_{15}, \mathbf{h}_{16})$. Note that the leading term of the syzygy corresponding to the reduction of $S(\mathbf{h}_7, \mathbf{h}_9)$ is a multiple of the one coming from $S(\mathbf{h}_7, \mathbf{h}_8)$, and hence the syzygy coming from $S(\mathbf{h}_7, \mathbf{h}_9)$ will be discarded when the Gröbner basis is minimalized. Thus, we do not need to compute it, and by reducing the other seven S-polynomials, we get that the set of monomials $\mathcal{B}'_2$ is

$$\begin{aligned}\mathcal{B}'_2 = \{&w_1 = x_2x_3^2x_4x_5, w_2 = x_1x_3^2x_4x_5, w_3 = x_2x_3x_4^2x_5^2, w_4 = x_1x_3x_4^2x_5^2,\\ &w_5 = x_1x_3x_4x_5^4, w_6 = x_2^3x_4x_5^2, w_7 = x_1^2x_4^2x_5\}\,.\end{aligned}$$

Hence, a graded free resolution of R/I as A-module is

$$0 \to \oplus_{v\in\mathcal{B}'_2} A(-\deg(v)) \xrightarrow{\psi'_2} \oplus_{v\in\mathcal{B}'_1} A(-\deg(v)) \xrightarrow{\psi'_1} \oplus_{v\in\mathcal{B}_0} A(-\deg(v)) \xrightarrow{\psi_0} R/I \to 0\,,$$

where the matrix of ψ_0 is $\begin{pmatrix} u_1 + I & u_2 + I & \dots & u_{16} + I \end{pmatrix}$, the matrix of ψ'_1 is the square matrix $\begin{pmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \dots & \mathbf{h}_{16} \end{pmatrix}$, and the matrix of ψ'_2 is given by the reductions of the S-polynomials $S(\mathbf{h}_1, \mathbf{h}_2)$, $S(\mathbf{h}_3, \mathbf{h}_4)$, $S(\mathbf{h}_5, \mathbf{h}_6)$, $S(\mathbf{h}_7, \mathbf{h}_8)$, $S(\mathbf{h}_8, \mathbf{h}_9)$, $S(\mathbf{h}_{10}, \mathbf{h}_{11})$, and $S(\mathbf{h}_{15}, \mathbf{h}_{16})$. Since there are nonzero constants in the reduction of the second and fourth S-polynomials,

$$\begin{aligned}S(\mathbf{h}_3, \mathbf{h}_4) &= x_5\mathbf{h}_3 - x_4\mathbf{h}_4 = -x_5^2\boldsymbol{\epsilon}_9 + x_5^4\boldsymbol{\epsilon}_8 = -\mathbf{h}_{11}\,,\\ S(\mathbf{h}_7, \mathbf{h}_8) &= x_5^2\mathbf{h}_7 - x_4\mathbf{h}_8 = -x_5^4\boldsymbol{\epsilon}_7 + x_4x_5^5\boldsymbol{\epsilon}_{16} = -\mathbf{h}_9\,,\end{aligned}$$

the above resolution is not minimal. Making it minimal, we get the short resolution of R/I as in (5.2) for $\mathcal{B}_1 = \mathcal{B}'_1 \setminus \{v_9, v_{11}\}$ and $\mathcal{B}_2 = \mathcal{B}'_2 \setminus \{w_2, w_4\}$. Reordering, at each step, the generators (and hence the rows and columns of the matrices defining the morphisms), the short resolution of R/I shows as

$$\begin{aligned}0 \to A(-5)^2 \oplus A(-6)^2 \oplus A(-7) &\xrightarrow{\psi_2} A(-3) \oplus A(-4)^{11} \oplus A(-5)^2\\ &\xrightarrow{\psi_1} A \oplus A(-1)^3 \oplus A(-2)^6 \oplus A(-3)^5 \oplus A(-4) \xrightarrow{\psi_0} R/I \to 0\,,\end{aligned}$$

and the Betti table is the same as the one given in Example 5.4.

Since the Hilbert series of R/I can be determined using any graded free resolution of R/I, we can use the Schreyer resolution (5.5) to compute it, and we get

$$\mathrm{HS}_{R/I}(t) = \frac{\sum_{i=0}^{p'} \sum_{v\in\mathcal{B}'_i} (-1)^i\, t^{\deg_\omega(v)}}{(1-t)^d}\,, \tag{5.6}$$

by Equation (1.6). As $d = \dim(R/I)$, the numerator does not vanish at $t = 1$ and the expression of the Hilbert series cannot be simplified. As a consequence, one can compute the Hilbert-Samuel multiplicity of R/I (which is the degree of the projective algebraic variety defined by I whenever I is homogeneous) from the size of the sets $\mathcal{B}'_i$.

Proposition 5.9. *Denote by* $e(R/I)$ *the (Hilbert-Samuel) multiplicity of* R/I. *Then,*

(a) $e(R/I) = \sum_{i=0}^{p'} (-1)^i |\mathcal{B}'_i|$.

(b) For all $s \in \mathbb{N}$,

$$\sum_{k=0}^{d} (-1)^k \binom{d}{k} \mathrm{HF}_{R/I}(s-k) = \sum_{i=0}^{p'} (-1)^i |(\mathcal{B}'_i)_s| \,,$$

where $(\mathcal{B}'_i)_s := \{v \in \mathcal{B}'_i \mid \deg_\omega(v) = s\}$.

Proof. Evaluating the numerator of the Hilbert series (5.6) in $t = 1$, we obtain (a). For (b), note that

$$(1-t)^d \mathrm{HS}_{R/I}(t) = \sum_{i=0}^{p'} \sum_{v \in \mathcal{B}'_i} (-1)^i \, t^{\deg_\omega(v)}$$

and compare the coefficient of t^s in both sides of the equality. □

Although the Schreyer resolution (5.5) is not minimal in general, there are cases in which it is known to be: when R/I is Cohen-Macaulay ([3, Prop. 1]), when $\dim(R/I) = 1$ ([3, Prop. 3]), or when $\dim(R/I) = 2$ and x_n is not a zero divisor of R/I ([3, Prop. 4]). The following straightforward result provides another case in which it is minimal.

Proposition 5.10. *If, for all* $u \in \mathcal{B}_0 \cap J$, *the monomial ideal* $I_u = (\mathrm{in}(I) : u) \cap A$ *is principal, then*

$$0 \to \oplus_{v \in \mathcal{B}'_1} A(-\deg_\omega(v)) \xrightarrow{\psi'_1} \oplus_{v \in \mathcal{B}_0} A(-\deg_\omega(v)) \xrightarrow{\psi_0} R/I \to 0$$

is the short resolution of R/I, *i.e., it is the minimal graded free resolution of* R/I *as* A*-module. In particular,* $\mathrm{depth}(R/I) = d - 1$.

The condition in the previous result is not necessary and one can have that $\mathrm{depth}(R/I) = d - 1$ when I_u is not principal for some $u \in \mathcal{B}_0 \cap J$ as the following example shows.

Example 5.11. Consider $R = \mathbb{Q}[x_1,\ldots,x_7]$, $A = \mathbb{Q}[x_5,x_6,x_7]$, and R/I, the 3-dimensional simplicial toric ring determined by $\mathcal{A} = \{(1,3,5),(5,1,5),(3,5,3),(5,5,1),(2,0,0),(0,2,0),(0,0,2)\}$, i.e.,

$$I = \langle x_1 - t_1t_2^3t_3^5, x_2 - t_1^5t_2t_3^5, x_3 - t_1^3t_2^5t_3^3, x_4 - t_1^5t_2^5t_3, x_5 - t_1^2, x_6 - t_2^2, x_7 - t_3^2\rangle \cap \mathbb{Q}[x_1,\ldots,x_7]\,.$$

Variables are in Noether position, and I is ω-homogeneous for $\omega = (9,11,11,11,2,2,2)$. One can check, using for example [93], that $\mathcal{B}_0 = \{x_4,x_3,x_2,x_1,1\}$, $\mathcal{B}_0 \cap J = \{x_3,x_2,x_1\}$, $I_{x_3} = \langle x_5\rangle$, $I_{x_2} = \langle x_6^2\rangle$ and $I_{x_1} = \langle x_5^2, x_5x_6\rangle$. Hence, $\mathcal{B}_1' = \{x_3x_5, x_2x_6^2, x_1x_5^2, x_1x_5x_6\}$ and $\mathcal{B}_2' = \{x_1x_5^2x_6\}$. However, the ω-graded short resolution of R/I, which can be computed using the function `shortRes` of [41], is

$$0 \to A(-13)^3 \to A \oplus A(-9) \oplus A(-11)^3 \to R/I \to 0\,,$$

so $|\mathcal{B}_1| = 3$ and $|\mathcal{B}_2| = 0$. Therefore, $\mathrm{pd}_A(R/I) = 1$ and $\mathrm{depth}(R/I) = d-1$, although I_{x_1} is not principal.

5.2 Simplicial toric rings of dimension 3: a combinatorial description of the short resolution

Let $\mathcal{A} = \{\mathbf{a}_1,\ldots,\mathbf{a}_n\} \subset \mathbb{N}^d$ be a finite set of nonzero vectors, $\mathcal{S} = \langle \mathcal{A}\rangle \subset \mathbb{N}^d$ the affine semigroup generated by $\mathcal{A}$, and $I_{\mathcal{A}} \subset R := \Bbbk[x_1,\ldots,x_n]$ the toric ideal determined by $\mathcal{A}$. We suppose that the toric ideal $I_{\mathcal{A}}$ is simplicial and hence, by Proposition 1.60, we can assume without loss of generality that there exists $D \in \mathbb{Z}_{>0}$ such that $\mathbf{a}_{n-d+i} = D\boldsymbol{\epsilon}_i$ for all $i = 1,\ldots,d$ where $\{\boldsymbol{\epsilon}_1,\ldots,\boldsymbol{\epsilon}_d\}$ denotes the canonical basis of $\mathbb{N}^d$. Set $\mathbf{e}_i := D\boldsymbol{\epsilon}_i$ for all $i \in \{1,\ldots,d\}$ and denote by $\mathcal{E} = \{\mathbf{e}_1,\ldots,\mathbf{e}_d\}$ the set of extremal rays of the rational cone spanned by $\mathcal{A}$. Under these hypothesis, one has that $A := \Bbbk[x_{n-d+1},\ldots,x_n] \hookrightarrow R/I_{\mathcal{A}}$ is a Noether normalization.

The simplicial semigroup ring $\Bbbk[\mathcal{S}]$ is an $\mathcal{S}$-graded $\Bbbk$-algebra isomorphic to $R/I_{\mathcal{A}}$ (as $\mathcal{S}$-graded $\Bbbk[x_1,\ldots,x_n]$-modules). We study here the multigraded short resolution of $\Bbbk[\mathcal{S}]$ with respect to the multigrading $|x_i|_{\mathcal{S}} = \deg_{\mathcal{S}}(x_i) = \mathbf{a}_i \in \mathcal{S}$; namely,

$$\mathcal{F} : 0 \to \oplus_{\mathbf{s}\in\mathcal{S}_p} A(-\mathbf{s}) \xrightarrow{\psi_p} \ldots \xrightarrow{\psi_1} \oplus_{\mathbf{s}\in\mathcal{S}_0} A(-\mathbf{s}) \xrightarrow{\psi_0} \Bbbk[\mathcal{S}] \to 0\,, \qquad (5.7)$$

where $\mathcal{S}_i \subset \mathcal{S}$ is a multiset for all $i \in \{0,\ldots,p\}$. Note that this multigrading is a refinement of the grading given by the weight vector $\omega = (\omega_1,\ldots,\omega_n)$, where $\omega_i = |\mathbf{a}_i| = \sum_{j=1}^d a_{ij} \in \mathbb{Z}_{>0}$ for all $i \in \{1,\ldots,n\}$. Hence, $I_{\mathcal{A}}$ is ω-homogeneous and the results of Section 5.1 apply here. As in that section, we fix the ω-graded reverse lexicographic order $>_\omega$ in R.

Our goal here is to describe the resolution (5.7) in terms of the combinatorics of the semigroup $\mathcal{S}$ when $d = 3$, i.e., when the Krull dimension of $\Bbbk[\mathcal{S}]$ is 3. We start by recalling from [3] the first step of the resolution for any $d \geq 1$. We will later describe the multisets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$ that appear in the short resolution (5.7) of $\Bbbk[\mathcal{S}]$. This is a combinatorial transcription of results in [78] and [75] that will be useful in Section 5.3.

Definition 5.12. Let $\mathcal{S}$ be a simplicial semigroup and denote by $\mathcal{E} = \{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ the set of extremal rays of the rational cone spanned by $\mathcal{A}$. The Apéry set of $\mathcal{S}$ is

$$\mathrm{AP}_{\mathcal{S}} := \{\mathbf{s} \in \mathcal{S} : \mathbf{s} - \mathbf{e}_i \notin \mathcal{S} \text{ for all } i = 1, \dots, d\}.$$

Proposition 5.13 ([3, Prop. 5]). *The set $\mathcal{S}_0$ in the short resolution (5.7) is $\mathrm{AP}_{\mathcal{S}}$, the Apéry set of $\mathcal{S}$. The $\mathcal{S}$-graded A-module homomorphism $\psi_0 : \oplus_{\mathbf{s}\in\mathcal{S}_0} A(-\mathbf{s}) \to \Bbbk[\mathcal{S}]$ is defined by $\psi_0(\boldsymbol{\epsilon}_{\mathbf{s}}) = t^{\mathbf{s}}$, where $\{\boldsymbol{\epsilon}_{\mathbf{s}} \mid \mathbf{s} \in \mathcal{S}_0\}$ is the canonical basis of $\oplus_{\mathbf{s}\in\mathcal{S}_0} A(-\mathbf{s})$.*

To compute the multidegrees in the next steps of the resolution, we consider, for every $\mathbf{s} \in \mathcal{S}$, the abstract simplicial complex $T_{\mathbf{s}}$ defined by

$$T_{\mathbf{s}} := \left\{ \mathcal{F} \subset \mathcal{E} : \mathbf{s} - \sum_{\mathbf{e}\in\mathcal{F}} \mathbf{e} \in \mathcal{S} \right\}.$$

In [78, Prop. 2.1] and [75, Prop. 5.1], the authors prove that the number of syzygies of multidegree $\mathbf{s}$ at the $(i+1)$-th step of the minimal $\mathcal{S}$-graded resolution (5.7) is $\dim_{\Bbbk} \tilde{H}_i(T_{\mathbf{s}})$, where $\tilde{H}_i(-)$ denotes the i-th reduced homology $\Bbbk$-vector space of $T_{\mathbf{s}}$.

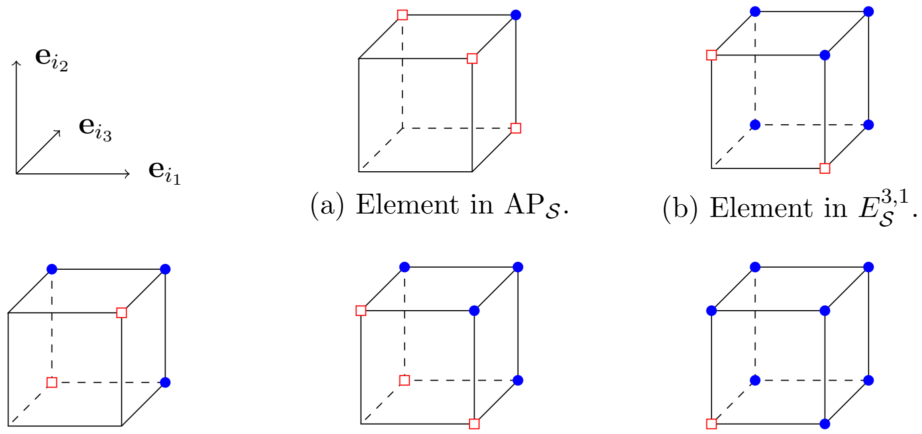
If $\mathbf{s} \in \mathcal{S}$ is such that $\mathbf{s} - \sum_{\mathbf{e}\in\mathcal{E}} \mathbf{e} \in \mathcal{S}$, then $T_{\mathbf{s}}$ is a simplex and $\dim_{\Bbbk} \tilde{H}_i(T_{\mathbf{s}}) = 0$ for all $i \in \mathbb{Z}$. Hence, such an element $\mathbf{s} \in \mathcal{S}$ does not belong to any of the multisets $\mathcal{S}_i$ in (5.7). We are thus interested in the elements $\mathbf{s} \in \mathcal{S}$ such that $\mathbf{s} - \sum_{\mathbf{e}\in\mathcal{E}} \mathbf{e} \notin \mathcal{S}$, which we will classify. We recall now the definition of the exceptional sets given in Section 4.2, adapting the notations to the setting we have now.

Definition 5.14. Let $\mathcal{S} \subset \mathbb{N}^3$ be a simplicial semigroup. We define the following subsets of $\mathcal{S}$, which we call the *exceptional sets of* $\mathcal{S}$:

- $E_{\mathcal{S}}^{3,1} = \{\mathbf{s} \in \mathcal{S} \mid \mathbf{s} - \mathbf{e}_j \in \mathcal{S}, \forall j;\ \mathbf{s} - (\mathbf{e}_{i_1} + \mathbf{e}_{i_2}) \in \mathcal{S}, \mathbf{s} - (\mathbf{e}_{i_1} + \mathbf{e}_{i_3}) \notin \mathcal{S}, \mathbf{s} - (\mathbf{e}_{i_2} + \mathbf{e}_{i_3}) \notin \mathcal{S},$ for a permutation (i_1, i_2, i_3) of $(1, 2, 3)\}$;
- $E_{\mathcal{S}}^{2,0} = \{\mathbf{s} \in \mathcal{S} \mid \mathbf{s} - \mathbf{e}_{i_1} \in \mathcal{S}, \mathbf{s} - \mathbf{e}_{i_2} \in \mathcal{S}, \mathbf{s} - \mathbf{e}_{i_3} \notin \mathcal{S};\ \mathbf{s} - (\mathbf{e}_{i_1} + \mathbf{e}_{i_2}) \notin \mathcal{S},$ for a permutation (i_1, i_2, i_3) of $(1, 2, 3)\}$;

- $E_{\mathcal{S}}^{3,0} = \{\mathbf{s} \in \mathcal{S} \mid \mathbf{s} - \mathbf{e}_i \in \mathcal{S}, \forall i;\ \mathbf{s} - (\mathbf{e}_i + \mathbf{e}_j) \notin \mathcal{S}, \forall i \neq j\}$;
- $E_{\mathcal{S}}^{3,3} = \{\mathbf{s} \in \mathcal{S} \mid \mathbf{s} - \mathbf{e}_i \in \mathcal{S}, \forall i;\ \mathbf{s} - (\mathbf{e}_i + \mathbf{e}_j) \in \mathcal{S}, \forall i \neq j;$ and $\mathbf{s} - (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \notin \mathcal{S}\}$.

Figure 5.1 shows how elements in the Apéry and the exceptional sets of $\mathcal{S}$ look like. In those figures, filled circles represent elements in $\mathcal{S}$, while empty squares represent elements outside $\mathcal{S}$.



(a) Element in $\mathrm{AP}_{\mathcal{S}}$. (b) Element in $E_{\mathcal{S}}^{3,1}$.

(c) Element in $E_{\mathcal{S}}^{2,0}$. (d) Element in $E_{\mathcal{S}}^{3,0}$. (e) Element in $E_{\mathcal{S}}^{3,3}$.

Figure 5.1: Points in $\mathrm{AP}_{\mathcal{S}}$ and the exceptional sets $E_{\mathcal{S}}^{3,1}$, $E_{\mathcal{S}}^{2,0}$, $E_{\mathcal{S}}^{3,0}$, and $E_{\mathcal{S}}^{3,3}$. Filled circles represent elements in $\mathcal{S}$, while empty squares represent elements outside $\mathcal{S}$.

Theorem 5.15. *If $\Bbbk[\mathcal{S}]$ is a simplicial semigroup ring of Krull dimension $d = 3$, the multisets $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$ that appear in the short resolution (5.7) are*

$$\mathcal{S}_0 = \mathrm{AP}_{\mathcal{S}}, \qquad \mathcal{S}_1 = E_{\mathcal{S}}^{3,1} \cup E_{\mathcal{S}}^{2,0} \cup E_{\mathcal{S}}^{3,0} \cup E_{\mathcal{S}}^{3,0}, \qquad \mathcal{S}_2 = E_{\mathcal{S}}^{3,3}.$$

Proof. We already know that $\mathcal{S}_0 = \mathrm{AP}_{\mathcal{S}}$. For any other $\mathbf{s} \in \mathcal{S}$, the simplicial complex $T_{\mathbf{s}}$ is one of those in Table 5.1, whose homologies are straightforward to compute. Then, the result follows from [78, Prop. 2.1] and [75, Prop. 5.1] □

Remark 5.16. As a consequence of Theorem 5.15, the sets $\mathrm{AP}_{\mathcal{S}}$, $E_{\mathcal{S}}^{3,1}$, $E_{\mathcal{S}}^{2,0}$, $E_{\mathcal{S}}^{3,0}$, and $E_{\mathcal{S}}^{3,3}$ are finite subsets of $\mathcal{S}$.

The Apéry and exceptional sets of $\mathcal{S}$ determine the multigraded Hilbert series of $\Bbbk[\mathcal{S}]$, as the following result shows.

$E_{\mathcal{S}}^{3,1}$	$E_{\mathcal{S}}^{2,0}$	$E_{\mathcal{S}}^{3,0}$	$E_{\mathcal{S}}^{3,3}$	Other configurations

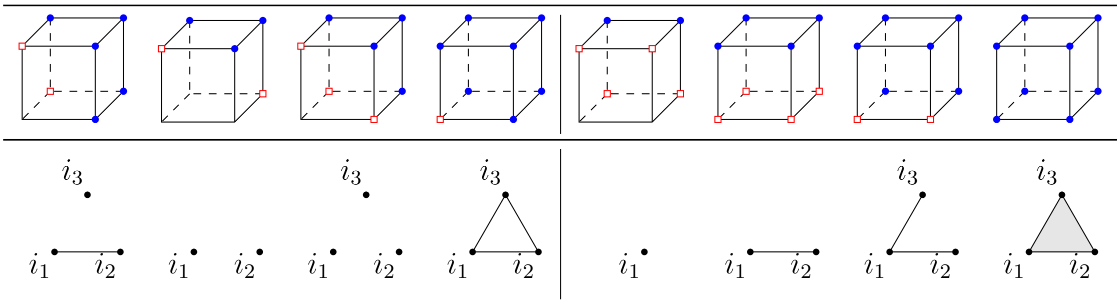


Table 5.1: Possible configurations of elements $\mathbf{s} \in \mathcal{S}$ and the associated simplicial complexes $T_{\mathbf{s}}$.

Corollary 5.17. *Let $\Bbbk[\mathcal{S}]$ be a simplicial semigroup ring of Krull dimension 3. The multigraded Hilbert series of $\Bbbk[\mathcal{S}]$ is:*

$$\mathrm{HS}_{\Bbbk[\mathcal{S}]}(\mathbf{t}) = \frac{\sum_{\mathbf{s}\in \mathrm{AP}_{\mathcal{S}}} \mathbf{t}^{\mathbf{s}} - \sum_{\mathbf{s}\in E_{\mathcal{S}}^{3,1}} \mathbf{t}^{\mathbf{s}} - \sum_{\mathbf{s}\in E_{\mathcal{S}}^{2,0}} \mathbf{t}^{\mathbf{s}} - 2\sum_{\mathbf{s}\in E_{\mathcal{S}}^{3,0}} \mathbf{t}^{\mathbf{s}} + \sum_{\mathbf{s}\in E_{\mathcal{S}}^{3,3}} \mathbf{t}^{\mathbf{s}}}{(1-t_1^{\omega_{n-2}})(1-t_2^{\omega_{n-1}})(1-t_3^{\omega_n})},$$

where $\omega_{n-2} = |\mathbf{e}_1|$, $\omega_{n-1} = |\mathbf{e}_2|$ and $\omega_n = |\mathbf{e}_3|$.

Proof. The multigraded Hilbert series of $\Bbbk[\mathcal{S}]$ is given by

$$\mathrm{HS}_{\Bbbk[\mathcal{S}]}(\mathbf{t}) = \sum_{\mathbf{s}=(s_1,s_2,s_3)\in\mathcal{S}} t_1^{s_1}t_2^{s_2}t_3^{s_3} = \frac{\sum_{\mathbf{s}\in\mathcal{S}_0}\mathbf{t}^{\mathbf{s}} - \sum_{\mathbf{s}\in\mathcal{S}_1}\mathbf{t}^{\mathbf{s}} + \sum_{\mathbf{s}\in\mathcal{S}_2}\mathbf{t}^{\mathbf{s}}}{(1-t_1^{\omega_{n-2}})(1-t_2^{\omega_{n-1}})(1-t_3^{\omega_n})},$$

and the result follows from Theorem 5.15. □

As already observed at the beginning of this section, the ideal $I_{\mathcal{A}}$ is ω-homogeneous for the weight vector $\omega = (\omega_1, \ldots, \omega_n)$, where $\omega_i = |\mathbf{a}_i|$ for $i \in \{1, \ldots, n\}$. Therefore, the short resolution of $\Bbbk[\mathcal{S}]$ with respect to this grading can be obtained from the multigraded one in a simple way as follows:

$$\mathcal{F} : 0 \to \oplus_{\mathbf{s}\in\mathcal{S}_2} A(-|\mathbf{s}|) \xrightarrow{\psi_2} \oplus_{\mathbf{s}\in\mathcal{S}_1} A(-|\mathbf{s}|) \xrightarrow{\psi_1} \oplus_{\mathbf{s}\in\mathcal{S}_0} A(-|\mathbf{s}|) \xrightarrow{\psi_0} \Bbbk[\mathcal{S}] \to 0\,.$$

Moreover, the weighted Hilbert series of $\Bbbk[\mathcal{S}]$ is obtained from the multigraded one by the transformation $t_1^{a_1}t_2^{a_2}t_3^{a_3} \mapsto t^{a_1+a_2+a_3}$.

When $I_{\mathcal{A}}$ is a (standard graded) homogeneous ideal, by Remark 1.71, without loss of generality we can assume that there exists $D \in \mathbb{Z}_{>0}$ such that $|\mathbf{a}_i| = D$ for all $i = 1, \ldots, n$. Thus, the short resolution of $\Bbbk[\mathcal{X}_\mathcal{A}]$ with respect to the standard grading is

$$\mathcal{F} : 0 \to \oplus_{\mathbf{s}\in\mathcal{S}_2} A(-|\mathbf{s}|/D) \xrightarrow{\psi_2} \oplus_{\mathbf{s}\in\mathcal{S}_1} A(-|\mathbf{s}|/D) \xrightarrow{\psi_1} \oplus_{\mathbf{s}\in\mathcal{S}_0} A(-|\mathbf{s}|/D) \xrightarrow{\psi_0} R/I_\mathcal{A} \to 0\,,$$

and hence, the Castelnuovo-Mumford regularity of $R/I_\mathcal{A}$ is

$$\operatorname{reg}(\Bbbk[\mathcal{X}_\mathcal{A}]) = \max\left(\left\{\frac{|\mathbf{s}|}{D} : \mathbf{s}\in\mathcal{S}_0\right\} \cup \left\{\frac{|\mathbf{s}|}{D} - 1 : \mathbf{s}\in\mathcal{S}_1\right\} \cup \left\{\frac{|\mathbf{s}|}{D} - 2 : \mathbf{s}\in\mathcal{S}_2\right\}\right) \tag{5.8}$$

and the Hilbert series of $\Bbbk[\mathcal{X}_\mathcal{A}]$ is obtained from the multigraded Hilbert series by applying the transformation $t_1^{a_1}t_2^{a_2}t_3^{a_3} \mapsto t^{(a_1+a_2+a_3)/D}$. Then, the formula for the regularity of a simplicial projective monomial surface in Theorem 4.25 follows from Equation (5.8) and Theorem 5.15. Moreover, Proposition 4.23 follows from Proposition 5.9 (b), Theorem 5.15, and the short resolution $\mathcal{F}$ above.

5.3 Pruning algorithm for simplicial toric rings of dimension 3

Consider now, as in Section 5.2, the toric ideal $I_\mathcal{A}$ defined by $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^3$, the generating set of a simplicial semigroup $\mathcal{S}$, and assume without loss of generality that the last three generators are the extremal rays of the rational cone spanned by $\mathcal{A}$. Setting $R := \Bbbk[x_1, \ldots, x_n]$ and $I := I_\mathcal{A}$, one has that R/I is a simplicial toric ring of dimension 3. Moreover, for $A := \Bbbk[x_{n-2}, x_{n-1}, x_n]$ and $\omega := (\omega_1, \ldots, \omega_n) \in \mathbb{N}^n$ with $\omega_i = |\mathbf{a}_i|$ for all i, $1 \le i \le n$, one has that I is ω-homogeneous and A is a Noether normalization of R/I, so the results in Section 5.1 apply. Our aim in this section is to build the Schreyer resolution and explicitly prune it in order to build directly the short resolution of R/I in this case.

Let $\mathcal{G}$ be the reduced Gröbner basis of I with respect to $>_\omega$, the ω-graded reverse lexicographic order. It is known that the elements in $\mathcal{G}$ are binomials. Take $\mathcal{B}_0$ the set of monomials not belonging to $\operatorname{in}(I) + \langle x_{n-2}, x_{n-1}, x_n\rangle$. Consider $\chi : R \to R$ the evaluation morphism defined by $\chi(x_i) = x_i$ for $i \in \{1, \ldots, n-3\}$ and $\chi(x_j) = 1$ for $j \in \{n-2, n-1, n\}$, and set J the extension of $\operatorname{in}(I)$ by χ. Now, for every $u \in \mathcal{B}_0 \cap J$, $G(I_u)$ denotes the minimal monomial generating set of

$$I_u := (\operatorname{in}(I) : u) \cap \Bbbk[x_{n-2}, x_{n-1}, x_n]\,.$$

Since the generators of $\operatorname{in}(I)$ do not involve the variable x_n because the ideal I is prime and $>_\omega$ is a reverse lexicographic order, every element in $G(I_u)$ is a monomial of the form $x_{n-2}^a x_{n-1}^b$ with $a, b \in \mathbb{N}$. Denote $\ell_u := |G(I_u)|$ and write $G(I_u) = \{M_{(u,1)}, \dots, M_{(u,\ell_u)}\}$, where the elements of $G(I_u)$ are sorted lexicographically, i.e., $M_{(u,1)} > \cdots > M_{(u,\ell_u)}$ with respect to the lexicographic order $x_n > x_{n-1} > x_{n-2}$. Now consider the set of monomials

$$\mathcal{B}_1' = \{uM_{(u,i)} \mid u \in \mathcal{B}_0 \cap J, 1 \le i \le \ell_u\}.$$

For each $\mathbf{x}^\alpha = uM_{(u,i)} \in \mathcal{B}_1'$, where $u \in \mathcal{B}_0 \cap J$ and $M_{(u,i)} = x_{n-2}^a x_{n-1}^b \in G(I_u)$, take r_α the remainder of the division of $\mathbf{x}^\alpha$ by $\mathcal{G}$. Since $\mathcal{G}$ consists of binomials and $M_{(u,i)} \in G(I_u)$, then $r_\alpha = x_{n-2}^{a'} x_{n-1}^{b'} x_n^{c'} v$ for some $a', b', c' \in \mathbb{N}$ such that $\gcd(M_{(u,i)}, x_{n-2}^{a'} x_{n-1}^{b'}) = 1$ and some $v \in \mathcal{B}_0$. By Theorem 5.7, the set

$$\mathcal{H} = \{\mathbf{h}_{(u,i)} := M_{(u,i)} \cdot \boldsymbol{\epsilon}_u - x_{n-2}^{a'} x_{n-1}^{b'} x_{n-1}^{c'} \cdot \boldsymbol{\epsilon}_v \mid u \in \mathcal{B}_0 \cap J, 1 \le i \le \ell_u\}$$

is the reduced Gröbner basis for the Schreyer-like monomial order $>_{\mathrm{SL}}$ in Definition 5.5, and $\mathrm{in}\big(\mathbf{h}_{(u,i)}\big) = M_{(u,i)} \cdot \boldsymbol{\epsilon}_u$ by Remark 5.6. Applying Schreyer's Theorem (Theorem 1.29), one gets that the syzygies of $\mathcal{H}$ are obtained by reducing the S-polynomials of all pairs of elements in $\mathcal{H}$ by $\mathcal{H}$. Note that only S-polynomials of the form $S(\mathbf{h}_{(u,i)}, \mathbf{h}_{(u,j)})$ with $u \in \mathcal{B}_0 \cap J$ and $1 \le i < j \le \ell_u$ must be considered and reduced since the other S-polynomials are zero. Furthermore, since the monomials $M_{(u,i)}$ only involve variables x_{n-2} and x_{n-1} and have been lexicographically sorted, $M_{(u,1)} > \ldots > M_{(u,\ell_u)}$, we only need to consider the reductions of the S-polynomials $S(\mathbf{h}_{(u,i)}, \mathbf{h}_{(u,i+1)})$ with $u \in \mathcal{B}_0 \cap J$ and $1 \le i < \ell_u$ since the other ones will be discarded when the resulting Gröbner basis of the syzygy module is made minimal. This implies that the initial terms of the resulting syzygies are pure powers of x_{n-2} located in different copies of A, and hence the module of syzygies of $\mathcal{H}$ obtained by applying Schreyer's Theorem is free. The Schreyer resolution of R/I has thus at most two steps, and it shows as follows:

$$0 \to \oplus_{v \in \mathcal{B}_2'} A(-\deg_\omega(v)) \xrightarrow{\psi_2'} \oplus_{v \in \mathcal{B}_1'} A(-\deg_\omega(v)) \xrightarrow{\psi_1'} \oplus_{v \in \mathcal{B}_0} A(-\deg_\omega(v)) \xrightarrow{\psi_0} R/I \to 0\,. \quad (5.9)$$

Algorithm 5.2 below takes advantage of the previous discussion and builds $\mathcal{B}_0$, $\mathcal{B}_1'$, and $\mathcal{B}_2'$, the sets of monomials in R involved in the above resolution. It is worth pointing out that this algorithm involves only a Gröbner basis computation and Gröbner-free manipulations with monomial ideals. It has been implemented in the function `schreyerResDim3` of [41].

As Examples 5.11 and 5.18 show, even when R/I is a 3-dimensional simplicial toric ring, the resolution (5.9) might not be minimal.

Algorithm 5.2 Computation of the sets $\mathcal{B}_i'$ for a simplicial toric ring of dim. 3.

Input: $I \subset R = \Bbbk[x_1, \ldots, x_n]$ a simplicial toric ideal of dimension 3 with variables in Noether position.
Output: The sets of monomials $\mathcal{B}_0, \mathcal{B}_1', \mathcal{B}_2' \subset R$ involved in the Schreyer resolution (5.9) of R/I as A-module, $A = \Bbbk[x_{n-2}, x_{n-1}, x_n]$.

$\mathcal{B}_0 \leftarrow$ monomial $\Bbbk$-basis of $R/\mathrm{in}(I) + \langle x_{n-2}, x_{n-1}, x_n \rangle$ for the degrevlex order $>_\omega$.
$J \leftarrow \chi\left(\mathrm{in}(I)\right).R$, where $\chi : R \to R$ is defined by $\chi(x_i) = x_i$ for $i \in \{1, \ldots, n-3\}$, and $\chi(x_{n-2}) = \chi(x_{n-1}) = \chi(x_n) = 1$.
$I_u \leftarrow (\mathrm{in}(I) : u) \cap A$, $\forall u \in \mathcal{B}_0 \cap J$.
$G(I_u) \leftarrow$ minimal generating set of I_u, $\forall u \in \mathcal{B}_0 \cap J$; $G(I_u) = \{M_{(u,1)}, \ldots, M_{(u,\ell_u)}\}$ ordered lexicographically with $x_n > x_{n-1} > x_{n-2}$.
$\mathcal{B}_1' \leftarrow \{u \cdot M_{(u,i)} \mid u \in \mathcal{B}_0 \cap J, 1 \le i \le \ell_u\}$.
$L_u \leftarrow \{\mathrm{lcm}(M_{(u,i)}, M_{(u,i+1)}) \mid 1 \le i < \ell_u\}$, $\forall u \in \mathcal{B}_0 \cap J$ such that $\ell_u \ge 2$.
$\mathcal{B}_2' \leftarrow \{u \cdot M \mid u \in \mathcal{B}_0 \cap J, \ell_u \ge 2,$ and $M \in L_u\}$.

Example 5.18. Set $R := \mathbb{Q}[x_1, \ldots, x_6]$, and let I be the toric ideal determined by $\mathcal{A} = \{(7,2,3), (1,8,3), (3,8,1), (12,0,0), (0,12,0), (0,0,12)\}$. One has that I is a homogeneous toric ideal and $A = \mathbb{Q}[x_4, x_5, x_6]$ is a Noether normalization of R/I, hence R/I is a 3-dimensional simplicial toric ring. Applying Algorithm 5.2 we obtain that $|\mathcal{B}_0| = 204$, $|\mathcal{B}_1'| = 174$ and $|\mathcal{B}_2'| = 42$. However, the Betti diagram of the short resolution, obtained by using the function `shortRes` of [41], is the following:

	0	1	2
0:	1	-	-
1:	3	-	-
2:	6	1	-
3:	10	3	-
4:	15	6	-
5:	21	10	-
6:	26	15	-
7:	29	20	-
8:	32	26	1
9:	29	26	2
10:	20	19	2
11:	9	9	1
12:	2	2	-
13:	1	1	-
total:	204	138	6

Our next aim is thus to minimalize Schreyer's resolution (5.9) using the results from Section 5.2. We will show how to obtain subsets $\mathcal{B}_1 \subset \mathcal{B}_1'$ and $\mathcal{B}_2 \subset \mathcal{B}_2'$, such that $\mathcal{B}_1$ and $\mathcal{B}_2$ provide the actual shifts that appear in the short resolution of R/I. We will refer to this process as *pruning* the resolution. Note that, by Proposition 5.9 (a),

$$e(R/I) = |\mathcal{B}_0| - |\mathcal{B}_1| + |\mathcal{B}_2| = |\mathcal{B}_0| - |\mathcal{B}_1'| + |\mathcal{B}_2'|$$

and, in particular, $|\mathcal{B}_1' \setminus \mathcal{B}_1| = |\mathcal{B}_2' \setminus \mathcal{B}_2|$.

In the process of pruning the resolution, we will use the following result several times.

Proposition 5.19. *Let $\mathcal{S} = \langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle \subset \mathbb{N}^d$ be an affine semigroup and $\mathbf{b}, \mathbf{c} \in \mathcal{S}$. Write $\mathbf{b} = \sum_{i=1}^n \beta_i \mathbf{a}_i$ and $\mathbf{c} = \sum_{i=1}^n \gamma_i \mathbf{a}_i$ with $\beta_i, \gamma_i \in \mathbb{N}$ and consider the monomials $\mathbf{x}^\beta := x_1^{\beta_1} \cdots x_n^{\beta_n}$ and $\mathbf{x}^\gamma := x_1^{\gamma_1} \cdots x_n^{\gamma_n} \in \Bbbk[x_1, \ldots, x_n]$. Then, $\mathbf{b} - \mathbf{c} \in \mathcal{S}$ if and only if $\mathbf{x}^\beta \in I_\mathcal{A} + \langle \mathbf{x}^\gamma \rangle$.*

Proof. We know that $R/I_\mathcal{A}$ and $\Bbbk[\mathcal{S}]$ are isomorphic as graded $\Bbbk$-algebras, and denote by $\tilde{\varphi}$ the corresponding graded isomorphism. Now, consider the ideal $\langle \mathbf{t}^\mathbf{c} \rangle$ of $\Bbbk[\mathcal{S}]$, and the canonical projection map $\pi : \Bbbk[\mathcal{S}] \to \Bbbk[\mathcal{S}]/\langle \mathbf{t}^\mathbf{c} \rangle$. Since $\tilde{\varphi}(\mathbf{x}^\gamma) = \mathbf{t}^\mathbf{c}$, we have that $\ker(\pi \circ \tilde{\varphi}) = (I_\mathcal{A} + \langle \mathbf{x}^\gamma \rangle)/I_\mathcal{A}$. Thus, by the third isomorphism theorem, there is a graded isomorphism of $\Bbbk$-algebras

$$\Psi : \Bbbk[\mathbf{x}]/(I_\mathcal{A} + \langle \mathbf{x}^\gamma \rangle) \longrightarrow \Bbbk[\mathcal{S}]/\langle \mathbf{t}^\mathbf{c} \rangle.$$

Moreover, $\Bbbk[\mathcal{S}]/\langle \mathbf{t}^\mathbf{c} \rangle$ has a unique monomial basis, which is $\{\mathbf{t}^\mathbf{d} \mid \mathbf{d} \in \mathcal{S} \text{ and } \mathbf{d} - \mathbf{c} \notin \mathcal{S}\}$. Finally, observe that the image of a monomial by Ψ is a monomial, and hence

$$\mathbf{x}^\beta \in I_\mathcal{A} + \langle \mathbf{x}^\gamma \rangle \Longleftrightarrow \Psi(\mathbf{x}^\beta) = 0 \Longleftrightarrow \mathbf{b} - \mathbf{c} \in \mathcal{S},$$

and we are done. □

To achieve our goal, consider the subset $C \subset \mathcal{B}_1'$ defined by $C = \{v \cdot x_{n-1}^b \in \mathcal{B}_1' \mid v \in \mathcal{B}_0 \text{ and } b \geq 2\}$. The following result shows that the elements in $\mathcal{B}_1' \setminus \mathcal{B}_1$ belong to C.

Lemma 5.20. $\mathcal{B}_1' \setminus \mathcal{B}_1 \subset C$.

Proof. Consider $\mathbf{x}^\alpha \in \mathcal{B}_1' \setminus \mathcal{B}_1$ and denote by $\mathbf{h}_\alpha$ the corresponding element of $\mathcal{H}$. Since $\mathbf{x}^\alpha \notin \mathcal{B}_1$, there exist $u \in \mathcal{B}_0 \cap J$ and $1 \leq i < \ell_u$ such that there appears a nonzero constant multiplying $\mathbf{h}_\alpha$ in the reduction of $S(\mathbf{h}_{(u,i)}, \mathbf{h}_{(u,i+1)})$ by $\mathcal{H}$. If

$\mathbf{h}_{(u,i)} = x_{n-2}^{a_1} x_{n-1}^{b_1} \boldsymbol{\epsilon}_u - x_n^{c_1} \boldsymbol{\epsilon}_v$ and $\mathbf{h}_{(u,i+1)} = x_{n-2}^{a_2} x_{n-1}^{b_2} \boldsymbol{\epsilon}_u - x_n^{c_2} \boldsymbol{\epsilon}_w$, for some $v, w \in \mathcal{B}_0$, $a_i, b_i, c_i \in \mathbb{N}$ $(i = 1, 2)$ with $c_1, c_2 \geq 1$, $a_1 < a_2$, and $b_1 > b_2$, then

$$S(\mathbf{h}_{(u,i)}, \mathbf{h}_{(u,i+1)}) = x_{n-2}^{a_2-a_1} \mathbf{h}_{(u,i)} - x_{n-1}^{b_1-b_2} \mathbf{h}_{(u,i+1)} = x_n \left(x_{n-1}^{b_1-b_2} x_n^{c_2-1} \boldsymbol{\epsilon}_w - x_{n-2}^{a_2-a_1} x_n^{c_1-1} \boldsymbol{\epsilon}_v \right) .$$

Hence, the reduction of $S(\mathbf{h}_{(u,i)}, \mathbf{h}_{(u,i+1)})$ by $\mathcal{H}$ does not involve nonzero constants.

Therefore, $\mathbf{h}_{(u,i)} = x_{n-2}^{a_1} \boldsymbol{\epsilon}_u - x_{n-1}^{b_1} \boldsymbol{\epsilon}_v$, and $\mathbf{h}_{(u,i+1)} = x_{n-2}^{a_2} x_{n-1}^{b_2} \boldsymbol{\epsilon}_u - x_n^{c} \boldsymbol{\epsilon}_w$, for some $v, w \in \mathcal{B}_0$ and $a_1, a_2, b_1, b_2, c \in \mathbb{N}$ with $a_1, b_1, b_2, c \geq 1$, $a_2 < a_1$ and $b_1 + b_2 = b \geq 2$. Hence,

$$S(\mathbf{h}_{(u,i)}, \mathbf{h}_{(u,i+1)}) = x_{n-1}^{b_2} \mathbf{h}_{(u,i)} - x_{n-2}^{a-a'} \mathbf{h}_{(u,i+1)} = -x_{n-1}^{b} \boldsymbol{\epsilon}_v + x_n \left(x_{n-2}^{a-a'} x_n^{c-1} \boldsymbol{\epsilon}_w \right) ,$$

and since there appears a nonzero constant in the reduction of $S(\mathbf{h}_{(u,i)}, \mathbf{h}_{(u,i+1)})$, one has that $x_{n-1}^b \in G(I_v)$, where $I_v = (\mathrm{in}(I) : v) \cap A$. Thus, $\mathbf{x}^\alpha = v x_{n-1}^b \in C$. □

Remark 5.21. As a direct consequence of the previous result, if $C = \emptyset$, then $\mathcal{B}_1' = \mathcal{B}_1$, $\mathcal{B}_2' = \mathcal{B}_2$, and hence the Schreyer resolution (5.9) is already minimal.

The inclusion $\mathcal{B}_1' \setminus \mathcal{B}_1 \subset C$ can be strict or not. In fact, if $C \neq \emptyset$, both cases $\mathcal{B}_1' = \mathcal{B}_1$ and $\mathcal{B}_1' \setminus \mathcal{B}_1 = C$ can happen, as the following examples show.

Example 5.22. In this example, computations are performed over the field $\mathbb{Q}$.

(1) Set $\mathcal{A} := \{(1,0,3),(3,0,1),(0,1,3),(3,1,0),(0,3,1),(1,3,0),(4,0,0),(0,4,0),(0,0,4)\}$, and let I be the toric ideal determined by $\mathcal{A}$. Applying Algorithm 5.2, one gets that $|\mathcal{B}_0| = 28$, $|\mathcal{B}_1'| = 18$, and $|\mathcal{B}_2'| = 6$. In this case, $\mathcal{B}_1 = \mathcal{B}_1'$ although $|C| = 3$ since the Betti diagram of the short resolution given by Algorithm 5.1 is

```
              0       1       2
------------------------
      0:      1       -       -
      1:      6       -       -
      2:     12       3       -
      3:      6       6       -
      4:      3       9       6
------------------------
total:       28      18       6
```

(2) If $\mathcal{A}$ is the set in Example 5.18, $|\mathcal{B}_1'| = 174$, $|\mathcal{B}_1| = 138$ and $|C| = 36$, so $\mathcal{B}_1' \setminus \mathcal{B}_1 = C$.

For each $\mathbf{x}^\beta \in C$, denote by r_β the remainder of $\mathbf{x}^\beta$ by the reduced Gröbner basis of I for the ω-graded reverse lexicographic order $>_\omega$ in R. Since x_{n-1} divides $\mathbf{x}^\beta$, then r_β is a multiple of x_n. Consider the partition $C = C_1 \sqcup C_2$, where

$$C_1 = \{\mathbf{x}^\beta \in C \mid r_\beta = w x_{n-2}^a x_n^c, \text{ for some } a, c \geq 1, w \in \mathcal{B}_0\}, \text{ and}$$
$$C_2 = \{\mathbf{x}^\beta \in C \mid r_\beta = w x_n^c, \text{ for some } c \geq 1, w \in \mathcal{B}_0\}.$$

We now show that one can decide whether a monomial $\mathbf{x}^\beta \in C$ is in $\mathcal{B}_1$ or not just by looking at its $\mathcal{S}$-degree. More precisely, it suffices to check if $|\mathbf{x}^\beta|_\mathcal{S} = \sum_{i=1}^n \beta_i \mathbf{a}_i$ appears as a shift in the first step of the short resolution, and this happens if and only if $|\mathbf{x}^\beta|_\mathcal{S} \in E_\mathcal{S}^{3,1} \cup E_\mathcal{S}^{2,0} \cup E_\mathcal{S}^{3,0}$ by Theorem 5.15. In Theorem 5.24, we characterize when the latter holds in terms of some monomials that may belong to the ideal $I_\mathcal{A} + \langle x_{n-2} \rangle$ or not. We will use the following easy lemma. As in Section 5.2, set $\mathbf{e}_i := D\boldsymbol{\epsilon}_i$ for all $i \in \{1, 2, 3\}$ and $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where $\{\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_3\}$ is the canonical basis of $\mathbb{N}^3$.

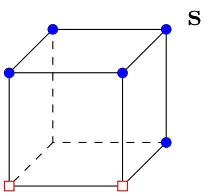
Lemma 5.23. *Let* $\mathbf{x}^\beta = v x_{n-1}^b \in C$ *and set* $\mathbf{s} = |\mathbf{x}^\beta|_\mathcal{S}$.

(1) If $\mathbf{x}^\beta \in C_1$, *then*

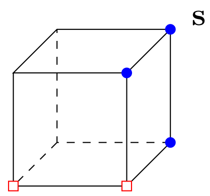
$$\mathbf{s} - \mathbf{e}_i \in \mathcal{S}, \forall i = 1, 2, 3;\ \mathbf{s} - (\mathbf{e}_1 + \mathbf{e}_3) \in \mathcal{S};\ \textit{and}\ \mathbf{s} - (\mathbf{e}_2 + \mathbf{e}_3) \notin \mathcal{S}.$$

(2) If $\mathbf{x}^\beta \in C_2$, *then*

$$\mathbf{s} - \mathbf{e}_2 \in \mathcal{S};\ \mathbf{s} - \mathbf{e}_3 \in \mathcal{S};\ \textit{and}\ \mathbf{s} - (\mathbf{e}_2 + \mathbf{e}_3) \notin \mathcal{S}.$$



(a) Situation in Lemma 5.23 (1).



(b) Situation in Lemma 5.23 (2).

Figure 5.2

Proof. Let us prove (1). If $\mathbf{x}^\beta = v x_{n-1}^b \in C_1$, there exist a monomial $w \in \mathcal{B}_0$ and natural numbers $a, c \geq 1$ such that $v x_{n-1}^b - w x_{n-2}^a x_n^c \in I_\mathcal{A}$. From this fact, it follows that $\mathbf{s} - \mathbf{e}_i \in \mathcal{S}$ for $i = 1, 2, 3$, and $\mathbf{s} - (\mathbf{e}_1 + \mathbf{e}_3) \in \mathcal{S}$. Suppose by contradiction that $\mathbf{s} - (\mathbf{e}_2 + \mathbf{e}_3) \in \mathcal{S}$. Then, there exists a monomial $M \in \Bbbk[x_1, \ldots, x_n]$ such that $v x_{n-1}^b - M x_{n-1} x_n \in I_\mathcal{A}$. Since $I_\mathcal{A}$ is prime, then $v x_{n-1}^{b-1} - M x_n \in I_\mathcal{A}$, and hence $v x_{n-1}^{b-1} \in \mathrm{in}(I_\mathcal{A})$, which contradicts with the minimality of $x_{n-1}^b \in G(I_v)$. The proof of (2) is analogous. □

Theorem 5.24. *Let* $\mathbf{x}^\beta = vx_{n-1}^b \in C$.

(1) If $\mathbf{x}^\beta \in C_1$, *then*

$$vx_{n-1}^b \in \mathcal{B}_1 \Longleftrightarrow vx_{n-1}^{b-1} \notin I_\mathcal{A} + \langle x_{n-2} \rangle \,.$$

(2) If $\mathbf{x}^\beta \in C_2$, *denote by* wx_n^c *the remainder of* $\mathbf{x}^\beta$ *by* $\mathcal{G}$. *Then,*

$$vx_{n-1}^b \in \mathcal{B}_1 \Longleftrightarrow vx_{n-1}^{b-1} \notin I_\mathcal{A} + \langle x_{n-2} \rangle \text{ or } wx_n^{c-1} \notin I_\mathcal{A} + \langle x_{n-2} \rangle \,.$$

Therefore,

$$\begin{aligned}\mathcal{B}_1 = (\mathcal{B}_1' \setminus C) &\cup \{vx_{n-1}^b \in C_1 \mid vx_{n-1}^{b-1} \notin I_\mathcal{A} + \langle x_{n-2} \rangle\} \\ &\cup \{vx_{n-1}^b \in C_2 \mid vx_{n-1}^{b-1} \notin I_\mathcal{A} + \langle x_{n-2} \rangle \text{ or } wx_n^{c-1} \notin I_\mathcal{A} + \langle x_{n-2} \rangle\} \,.\end{aligned}$$

Proof. By Theorem 5.15, we know that the multiset of $\mathcal{S}$-degrees appearing in the first step of the short resolution is

$$\mathcal{S}_1 = E_\mathcal{S}^{3,1} \cup E_\mathcal{S}^{2,0} \cup E_\mathcal{S}^{3,0} \cup E_\mathcal{S}^{3,0};$$

we observe that in $\mathcal{S}_1$ the elements of $E_\mathcal{S}^{3,1} \cup E_\mathcal{S}^{2,0}$ have multiplicity 1, and the elements of $E_\mathcal{S}^{3,0}$ have multiplicity two. We know that $\mathcal{S}_1$ is a (multi)subset of

$$\mathcal{S}_1' := \{|\mathbf{x}^\alpha|_\mathcal{S} \mid \mathbf{x}^\alpha \in \mathcal{B}_1' \setminus C\} \cup \{|\mathbf{x}^\alpha|_\mathcal{S} \mid \mathbf{x}^\alpha \in C\}.$$

Claim: Whenever $\mathbf{s} \in \mathcal{S}_1$, its multiplicities in $\mathcal{S}_1$ and in $\mathcal{S}_1'$ coincide.
Proof of the claim: By Lemma 5.20, we know that $\{|\mathbf{x}^\alpha|_\mathcal{S} \mid \mathbf{x}^\alpha \in \mathcal{B}_1' \setminus C\}$ is a (multi)subset of $\mathcal{S}_1$. Hence, to derive the claim it suffices to prove that:

(i) distinct elements of C have distinct $\mathcal{S}$-degrees, and

(ii) whenever an element of $\mathcal{B}_1' \setminus C$ and an element of C have the same $\mathcal{S}$-degree, then this $\mathcal{S}$-degree belongs to $E_\mathcal{S}^{3,0}$ and has multiplicity exactly two in $\mathcal{S}_1'$.

To prove (i), consider two elements in C with the same $\mathcal{S}$-degree, namely, $\mathbf{x}^\alpha = ux_{n-1}^b$ and $\mathbf{x}^\beta = u'x_{n-1}^{b'}$ and assume that $b \geq b'$. Then it follows that $f = ux_{n-1}^{b-b'} - u' \in I_\mathcal{A}$, so $f = 0$, and hence $u = u'$ and $b = b'$.

To prove (ii), consider $\mathbf{x}^\alpha \in \mathcal{B}_1' \setminus C$ and $\mathbf{x}^\beta \in C$ with $\mathbf{s} := |\mathbf{x}^\alpha|_\mathcal{S} = |\mathbf{x}^\beta|_\mathcal{S}$. We write $\mathbf{x}^\beta = ux_{n-1}^b$, $b \geq 2$, $\mathbf{x}^\alpha = u'x_{n-2}^{a'}x_{n-1}^{b'}$, $a' + b' \geq 1$. Suppose first that $\mathbf{x}^\beta \in C_1$, i.e., $r_\beta = vx_{n-2}^a x_n^c$ for some $a, c \in \mathbb{Z}_{>0}$. If $a' \geq 1$, then $u'x_{n-2}^{a'-1}x_{n-1}^{b'} - vx_{n-2}^{a-1}x_n^c \in I_\mathcal{A}$, so $u'x_{n-2}^{a'-1}x_{n-1}^{b'} \in \mathrm{in}(I_\mathcal{A})$, and hence $x_{n-2}^{a'-1}x_{n-1}^{b'} \in I_{u'}$, contradicting the minimality of $x_{n-2}^{a'}x_{n-1}^{b'} \in G(I_{u'})$. Therefore, $a' = 0$, so $|ux_{n-1}^b|_\mathcal{S} = |u'x_{n-1}^{b'}|_\mathcal{S}$, which implies that

$u = u'$ and $b = b'$, a contradiction. Hence, $\mathbf{x}^\beta \in C_2$, i.e., $r_\beta = vx_n^c$, for some $c \in \mathbb{Z}_{>0}$. Now, let us see that $\mathbf{s} \in E_{\mathcal{S}}^{3,0}$. If $b' \geq 1$, then $ux_{n-1}^{b-1} - u'x_{n-2}^{a'}x_{n-1}^{b'-1} \in I_{\mathcal{A}}$ is a nonzero binomial and neither ux_{n-1}^{b-1} nor $u'x_{n-2}^{a'}x_{n-1}^{b'-1}$ belongs to $\mathrm{in}(I_{\mathcal{A}})$, which is impossible. This proves that $b' = 0$. Since, $\mathbf{s} = |u'x_{n-2}^{a'}|_{\mathcal{S}} = |ux_{n-1}^b|_{\mathcal{S}} = |vx_n^c|_{\mathcal{S}} \in \mathcal{S}_1$, then either $\mathbf{s} \in E_{\mathcal{S}}^{3,1}$ or $\mathbf{s} \in E_{\mathcal{S}}^{3,0}$. Suppose $\mathbf{s} \in E_{\mathcal{S}}^{3,1}$. Then there exist $w \in \mathcal{B}_0$, $a'', b'', c'' \in \mathbb{N}$ with at least two of them nonzero, such that $\mathbf{s} = |wx_{n-2}^{a''}x_{n-1}^{b''}x_n^{c''}|_{\mathcal{S}}$. Combining this with $\mathbf{s} = |u'x_{n-2}^{a'}|_{\mathcal{S}}$ (if $a'' \neq 0$) or $\mathbf{s} = |ux_{n-1}^b|_{\mathcal{S}}$ (if $b'' \neq 0$), we get a contradiction. Hence, $\mathbf{s} \in E_{\mathcal{S}}^{3,0}$. Finally, let us see that there does not exist $\mathbf{x}^\gamma \in \mathcal{B}_1' \setminus C$, $\mathbf{x}^\gamma \neq \mathbf{x}^\alpha$, such that $|\mathbf{x}^\gamma|_{\mathcal{S}} = \mathbf{s}$. Let $\mathbf{x}^\gamma = u''x_{n-2}^{a''}x_{n-1}^{b''} \in \mathcal{B}_1' \setminus C$, $\mathbf{x}^\gamma \neq \mathbf{x}^\alpha$, such that $|\mathbf{x}^\gamma|_{\mathcal{S}} = \mathbf{s}$. Then, $a'', b'' \in \mathbb{Z}_{>0}$, or $a'' \in \mathbb{Z}_{>0}$ and $b'' = 0$, or $a'' = 0$ and $b'' = 1$. Proceeding as before each of these three cases leads to a contradiction. Therefore, the claim is proved.

As a consequence of the *Claim*, one has a criterion to detect if an element of $\mathcal{B}_1'$ belongs to $\mathcal{B}_1$ or not. More precisely, let $\mathbf{x}^\alpha \in \mathcal{B}_1'$, then:

$$\mathbf{x}^\alpha \in \mathcal{B}_1 \Longleftrightarrow |\mathbf{x}^\alpha|_{\mathcal{S}} \in \mathcal{S}_1 \,.$$

We now use this criterion to prove (1) and (2).

Let $\mathbf{x}^\beta = vx_{n-1}^b \in C_1$ and set $\mathbf{s} = |\mathbf{x}^\beta|_{\mathcal{S}}$. By Lemma 5.23 (1), one has that $\mathbf{x}^\beta \in \mathcal{B}_1$ if and only if $\mathbf{s} - (\mathbf{e}_1 + \mathbf{e}_2) \notin \mathcal{S}$. Then, by Proposition 5.19, one has that $\mathbf{s} - (\mathbf{e}_1 + \mathbf{e}_2) \in \mathcal{S}$ if and only if $vx_{n-1}^{b-1} \in I_{\mathcal{A}} + \langle x_{n-2} \rangle$.

Let $\mathbf{x}^\beta = vx_{n-1}^b \in C_2$ and set $\mathbf{s} = |\mathbf{x}^\beta|_{\mathcal{S}}$ and $r_\beta = wx_n^c$ the remainder of $\mathbf{x}^\beta$ by the reduced Gröbner basis of $I_{\mathcal{A}}$. By Lemma 5.23 (2), one has that $\mathbf{x}^\beta \in \mathcal{B}_1$ if and only if $\mathbf{s} - (\mathbf{e}_1 + \mathbf{e}_2) \notin \mathcal{S}$ or $\mathbf{s} - (\mathbf{e}_1 + \mathbf{e}_3) \notin \mathcal{S}$. Hence, the result follows again from Proposition 5.19.

The last claim in the theorem is a direct consequence of (1) and (2). □

In Theorem 5.24, we have obtained a test to decide algebraically if a monomial $\mathbf{x}^\beta \in C \subset \mathcal{B}_1'$ is in $\mathcal{B}_1$ or not, and hence we can obtain the set $\mathcal{B}_1$. To apply this criterion, one only has to test the membership of some monomials to the ideal $I_{\mathcal{A}} + \langle x_{n-2} \rangle$. Now, we do something similar to obtain the set $\mathcal{B}_2 \subset \mathcal{B}_2'$.

Lemma 5.25. *Let $\mathbf{x}^\alpha \in \mathcal{B}_2' \setminus \mathcal{B}_2$, and set $\mathbf{s} = |\mathbf{x}^\alpha|_{\mathcal{S}}$, then*

$$\mathbf{s} - \mathbf{e}_i \in \mathcal{S}, \forall i = 1,2,3;\ \mathbf{s} - (\mathbf{e}_1 + \mathbf{e}_2) \in \mathcal{S};\ \mathbf{s} - (\mathbf{e}_1 + \mathbf{e}_3) \in \mathcal{S};\ \textit{and}\ \mathbf{s} - (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \notin \mathcal{S}\,.$$

Proof. If a monomial $\mathbf{x}^\alpha \in \mathcal{B}_2'$ is not in $\mathcal{B}_2$, then it comes from a S-polynomial $S(\mathbf{h}, \mathbf{h}')$, $\mathbf{h}, \mathbf{h}' \in \mathcal{H}$, such that there appears a nonzero constant in the reduction of $S(\mathbf{h}, \mathbf{h}')$ by the Gröbner basis $\{\mathbf{h}_\alpha \mid \mathbf{x}^\alpha \in \mathcal{B}_1'\}$. The syzygies $\mathbf{h}, \mathbf{h}'$ have expressions $\mathbf{h} = x_{n-2}^a \boldsymbol{\epsilon}_u - x_{n-1}^b \boldsymbol{\epsilon}_v$, and $\mathbf{h}' = x_{n-2}^{a'} x_{n-1}^{b'} \boldsymbol{\epsilon}_u - x_n^c \boldsymbol{\epsilon}_w$, for some $u, v, w \in \mathcal{B}_0$, $a, b, b', c \geq 1$ and $a' \in \mathbb{N}$, with $a' < a$, as in the proof of Lemma 5.20. Therefore, $\mathbf{x}^\alpha = u \cdot \mathrm{lcm}(x_{n-2}^a, x_{n-2}^{a'} x_{n-1}^{b'}) = u \cdot x_{n-2}^a x_{n-1}^{b'}$, and we note that

$$|ux_{n-2}^a x_{n-1}^{b'}|_{\mathcal{S}} = |vx_{n-1}^{b+b'}|_{\mathcal{S}} = |wx_{n-2}^{a-a'} x_n^c|_{\mathcal{S}} \,.$$

From the previous equalities, we deduce that $\mathbf{s}-\mathbf{e}_i \in \mathcal{S}$ for all $i=1,2,3$, $\mathbf{s}-(\mathbf{e}_1+\mathbf{e}_2) \in \mathcal{S}$, and $\mathbf{s}-(\mathbf{e}_1+\mathbf{e}_3) \in \mathcal{S}$. Let us prove that $\mathbf{s}-(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3) \notin \mathcal{S}$. Assume by contradiction that $\mathbf{s}-(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3) \in \mathcal{S}$. Then, there exist a monomial $M \in \Bbbk[x_1,\dots,x_n]$, such that $ux_{n-2}^{a}x_{n-1}^{b'} - x_{n-2}x_{n-1}x_nM \in I_{\mathcal{A}}$, so $ux_{n-2}^{a-1}x_{n-1}^{b'-1} - x_nM \in I_{\mathcal{A}}$. Therefore, there exist natural numbers $a'' \leq a-1$ and $b'' \leq b'-1$ such that $x_{n-2}^{a''}x_{n-1}^{b''} \in G(I_u)$. Since $x_{n-2}^{a'}x_{n-1}^{b'} \in G(I_u)$ and $b'' < b'$, then $a'' > a'$. Hence, $\operatorname{lcm}\left(x_{n-2}^{a''}x_{n-1}^{b''}, x_{n-2}^{a'}x_{n-1}^{b'}\right) = x_{n-2}^{a''}x_{n-1}^{b'}$, which is a proper divisor of $x_{n-2}^{a}x_{n-1}^{b'}$, a contradiction with $\mathbf{x}^\alpha = ux_{n-2}^{a}x_{n-1}^{b'} \in \mathcal{B}_2'$. □

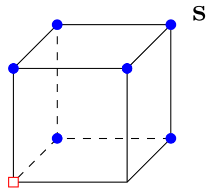


Figure 5.3: Situation in Lemma 5.25.

Theorem 5.26. *For all* $\mathbf{x}^\alpha = ux_{n-2}^{a}x_{n-1}^{b} \in \mathcal{B}_2'$,

$$\mathbf{x}^\alpha \in \mathcal{B}_2 \Longleftrightarrow ux_{n-2}^{a}x_{n-1}^{b-1} \in I_{\mathcal{A}} + \langle x_n \rangle .$$

Therefore,

$$\mathcal{B}_2 = \{ux_{n-2}^{a}x_{n-1}^{b} \in \mathcal{B}_2' \mid ux_{n-2}^{a}x_{n-1}^{b-1} \in I_{\mathcal{A}} + \langle x_n \rangle\} .$$

Proof. By Theorem 5.15, we know that the multiset of $\mathcal{S}$-degrees appearing in the second step of the short resolution is $\mathcal{S}_2 = E_{\mathcal{S}}^{3,3}$, and every element appears with mutiplicity 1. We know that $\mathcal{S}_2$ is a (multi)subset of

$$\mathcal{S}_2' := \{|\mathbf{x}^\alpha|_{\mathcal{S}} \mid \mathbf{x}^\alpha \in \mathcal{B}_2'\}.$$

Claim: Distinct elements of $\mathcal{B}_2'$ have distinct $\mathcal{S}$-degrees.
Proof of the claim. Take $\mathbf{x}^\alpha = ux_{n-2}^{a}x_{n-1}^{b} \in \mathcal{B}_2'$ and $\mathbf{x}^\beta = u'x_{n-2}^{a'}x_{n-1}^{b'} \in \mathcal{B}_2'$, $a,a',b,b' \in \mathbb{Z}_{>0}$, such that $|\mathbf{x}^\alpha|_{\mathcal{S}} = |\mathbf{x}^\beta|_{\mathcal{S}}$. Then, $ux_{n-2}^{a}x_{n-1}^{b} - u'x_{n-2}^{a'}x_{n-1}^{b'} \in I_{\mathcal{A}}$. If $u=u'$, then $a=a'$ and $b=b'$, and hence $\mathbf{x}^\alpha = \mathbf{x}^\beta$. Now suppose $u \neq u'$. In this case, $ux_{n-2}^{a}x_{n-1}^{b-1} - u'x_{n-2}^{a'}x_{n-1}^{b'-1} \in I_{\mathcal{A}}$. Assume without loss of generality that its initial term is $ux_{n-2}^{a}x_{n-1}^{b-1} \in \operatorname{in}(I_{\mathcal{A}})$, so $x_{n-2}^{a}x_{n-1}^{b-1} \in I_u$, contradicting the minimality of $x_{n-2}^{a}x_{n-1}^{b} \in G(I_u)$. Hence, the *Claim* follows.

As a consequence of the *Claim*, one has a criterion the detect if an element of $\mathcal{B}_2'$ belongs to $\mathcal{B}_2$ or not. More precisely, let $\mathbf{x}^\alpha \in \mathcal{B}_2'$, then:

$$\mathbf{x}^\alpha \in \mathcal{B}_2 \Longleftrightarrow |\mathbf{x}^\alpha|_{\mathcal{S}} \in \mathcal{S}_2 = E_{\mathcal{S}}^{3,3}$$

By Lemma 5.25, one has that for all $\mathbf{x}^\alpha \in \mathcal{B}_2'$,

$$|\mathbf{x}^\alpha|_{\mathcal{S}} \in E_{\mathcal{S}}^{3,3} \iff |\mathbf{x}^\alpha|_{\mathcal{S}} - (\mathbf{e}_2 + \mathbf{e}_3) \in \mathcal{S}\,.$$

Therefore, the result follows from Proposition 5.19. □

Algorithm 5.3 Pruning algorithm for a simplicial toric ring of dimension 3.

Input: $I \subset R = \Bbbk[x_1, \ldots, x_n]$ a simplicial toric ideal of dimension 3 with variables in Noether position
Output: The sets of monomials $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2 \subset R$ that appear in the short resolution of R/I.

1: $\mathcal{G} \leftarrow$ reduced Gröbner basis of I for the ω-graded reverse lexicographic order.
2: $\mathcal{B}_0, \mathcal{B}_1', \mathcal{B}_2' \leftarrow$ sets obtained in Algorithm 5.2.
3: $C \leftarrow \{v \cdot x_{n-1}^b \in \mathcal{B}_1' \mid v \in \mathcal{B}_0 \text{ and } b \geq 2\}$.
4: $r_\alpha \leftarrow$ remainder of $\mathbf{x}^\alpha$ by $\mathcal{G}$, $\forall \mathbf{x}^\alpha \in \mathcal{B}_1'$.
5: $C_1 \leftarrow \{\mathbf{x}^\alpha \in C \mid x_{n-2} \text{ divides } r_\alpha\}$.
6: $C_2 \leftarrow \{\mathbf{x}^\alpha \in C \mid x_{n-2} \text{ does not divide } r_\alpha\}$.
7: $\mathcal{B}_1 \leftarrow (\mathcal{B}_1' \setminus C) \cup \{\mathbf{x}^\alpha \in C_1 \mid \frac{\mathbf{x}^\alpha}{x_{n-1}} \notin I + \langle x_{n-2}\rangle\} \cup \{\mathbf{x}^\alpha \in C_2 \mid \frac{\mathbf{x}^\alpha}{x_{n-1}} \notin I + \langle x_{n-2}\rangle \text{ or } \frac{r_\alpha}{x_n} \notin I + \langle x_{n-2}\rangle\}$.
8: $\mathcal{B}_2 \leftarrow \{\mathbf{x}^\alpha \in \mathcal{B}_2' \mid \frac{\mathbf{x}^\alpha}{x_{n-1}} \in I + \langle x_n\rangle\}$.

Using Theorem 5.24 and Theorem 5.26, one can obtain the set $\mathcal{B}_2 \subset \mathcal{B}_2'$ in the short resolution. The whole pruning algorithm (for $\mathcal{B}_1'$ and $\mathcal{B}_2'$) is summarized in Algorithm 5.3. It is worth pointing out that this algorithm requires only the computation the Grobner basis $\mathcal{G}$ of I with respect to the ω-graded reverse lexicographic order, to compute the remainders of several monomials modulo $\mathcal{G}$, and to test membership of several monomials to the ideal $I + \langle x_{n-2}\rangle$. Algorithm 5.3 has been implemented in the function `pruningDim3` of [41].

The pruning algorithm presented in this section does not work if the ideal I is not toric. If I is not prime, it can happen that $\mathrm{pd}_A(R/I) = 3$, so the resolution has one more step and even Algorithm 5.2 fails. If I is prime but not binomial, Example 5.27 shows that Algorithm 5.3 can fail.

Example 5.27. Set $R := \mathbb{Q}[x_1, \ldots, x_7]$, and let $I \subset R$ be the ideal

$$I = \langle x_1 + t_1^2 t_2^2 - t_1^3 t_3, x_2 - t_1^3 t_2, x_3 - t_2^3 t_3, x_4 - t_2 t_3^3, x_5 - t_1^3, x_6 - t_2^3, x_7 - t_3^3\rangle \cap R\,.$$

The ideal I is prime, $\dim(R/I) = 3$, and the variables are in Noether position. However, I is not binomial, so it is not toric. Applying the results of Section 5.1, we obtain $|\mathcal{B}_0|= 28$, $|\mathcal{B}'_1|= 16$ and $|\mathcal{B}'_2|= 4$. Moreover, this resolution is minimal since its Betti diagram, obtained by using the function `shortRes` of [41], is

```
              0       1       2
------------------------
     0:       1       -       -
     1:       4       -       -
     2:       9       2       -
     3:      13      12       3
     4:       1       2       1
------------------------
total:       28      16       4
```

However, when applying Algorithm 5.3 to the sets $\mathcal{B}_0$, $\mathcal{B}'_1$ and $\mathcal{B}'_2$, one gets $\mathcal{B}_1 = \mathcal{B}'_1$ and $|\mathcal{B}'_2 \setminus \mathcal{B}_2|= 1$, so the algorithm fails in this case.

5.4 Dependence on the characteristic of $\Bbbk$

In this last section, we present an example of a simplicial toric ring $R/I_{\mathcal{A}}$ whose minimal graded free resolution depends on the characteristic of $\Bbbk$. Let $\mathcal{A} \subset \mathbb{N}^6$ be the set defined by the column vectors of the following matrix

$$\begin{pmatrix} 3&3&3&3&3&1&1&1&1&1&2&0&0&0&0&0\\ 3&3&1&1&1&3&3&3&1&1&0&2&0&0&0&0\\ 3&1&3&1&1&3&1&1&3&3&0&0&2&0&0&0\\ 1&1&3&3&1&1&3&3&3&1&0&0&0&2&0&0\\ 1&1&1&3&3&3&3&1&1&3&0&0&0&0&2&0\\ 1&3&1&1&3&1&1&3&3&3&0&0&0&0&0&2 \end{pmatrix},$$

and consider the toric ideal $I \subset R = \Bbbk[x_1, \ldots, x_{16}]$ determined by $\mathcal{A}$. Set $A = \Bbbk[x_{11}, \ldots, x_{16}]$. Then, $I_{\mathcal{A}}$ is ω-homogeneous for $\omega = (6, \ldots, 6, 1, \ldots, 1)$ and A is a Noether normalization of R/I.

We compute the short resolution of R/I when $\Bbbk$ is a field of characteristic 0 and when its characteristic is 2 using the function `shortRes` of [41]. This example shows that the short resolution of a simplicial toric ideal may depend on the characteristic of $\Bbbk$. Moreover the projective dimension as A-module is different for both characteristics, $\mathrm{pd}_A(R/I) = 2$ when $\mathrm{char}(\Bbbk) = 0$, while $\mathrm{pd}_A(R/I) = 3$ when $\mathrm{char}(\Bbbk) = 2$. Since $\mathrm{pd}_R(R/I) = \mathrm{pd}_A(R/I) + n - d = \mathrm{pd}_A(R/I) + 10$, the resolution of R/I as R-module also depends on the characteristic of $\Bbbk$. We could not compute the whole resolution but the second step already shows that the resolution depends on the characteristic of $\Bbbk$.

Betti diagram of the resolution of R/I as A-module:

char($\mathbb{k}$) = 0

	0	1	2
0:	1	-	-
1:	-	-	-
2:	-	-	-
3:	-	-	-
4:	-	-	-
5:	-	-	-
6:	10	15	6
total:	11	15	6

char($\mathbb{k}$) = 2

	0	1	2	3
0:	1	-	-	-
1:	-	-	-	-
2:	-	-	-	-
3:	-	-	-	-
4:	-	-	-	-
5:	-	-	-	-
6:	10	15	6	1
7:	-	-	1	-
total:	11	15	7	1

Beginning of the Betti diagram of the resolution as R-module:

char($\mathbb{k}$) = 0

	0	1	2
0:	1	-	-
1:	-	-	-
2:	-	-	-
3:	-	-	-
4:	-	-	-
5:	-	-	-
6:	-	15	6
7:	-	-	-
8:	-	-	-
9:	-	-	-
10:	-	-	-
11:	-	55	150
12:	-	-	-
13:	-	-	-
14:	-	-	-
15:	-	-	-
16:	-	-	330
total:	1	70	486

char($\mathbb{k}$) = 2

	0	1	2
0:	1	-	-
1:	-	-	-
2:	-	-	-
3:	-	-	-
4:	-	-	-
5:	-	-	-
6:	-	15	6
7:	-	-	1
8:	-	-	-
9:	-	-	-
10:	-	-	-
11:	-	55	150
12:	-	-	-
13:	-	-	-
14:	-	-	-
15:	-	-	-
16:	-	-	330
total:	1	70	487

Conclusions

In this thesis, we have addressed the study of several problems in the interface between commutative algebra and additive combinatorics, exploring the bridge established in the recent articles [18, 30, 32].

In Chapter 2, we have studied the equality of the Betti numbers of a projective monomial curve and one of its affine charts. In Theorem 2.12, we provide a sufficient condition in terms of the Apéry posets of the semigroups defined by the projective and the affine monomial curves.

In Chapter 3, we have studied the problem of determining the structure of the sumsets in additive combinatorics. To this end, we have applied some techniques from commutative algebra. We have given a complete understanding of the structure theorem for the sumsets of sets of integers, see Theorem 3.16. Furthermore, we have made the structure theorem in higher dimensions more explicit in some cases, providing upper bounds on the sumsets regularity that improve the ones in the literature; see Theorems 3.26 and 3.29 for the smooth case, and Theorems 3.35 and 3.41 for the case of surfaces with a single singular point.

In Chapter 4, we have provided combinatorial formulas for the Castelnuovo-Mumford regularity of projective monomial curves (Theorem 4.2) and simplicial projective monomial surfaces (Theorem 4.25). Moreover, we have established a relation between the Castelnuovo-Mumford regularity and the sumsets regularity in Theorems 4.13 and 4.27. This has allowed us to give a proof of the Eisenbud-Goto conjecture for projective monomial curves and simplicial projective monomial surfaces with a single singular point.

Finally, in Chapter 5, we have provided a Schreyer-like method to compute the short resolution of any weighted homogeneous ideal whenever the variables are in Noether position, which follows from Theorem 5.7 (Algorithm 5.1). Moreover, we have designed an algorithm for simplicial toric rings of dimension 3 that first constructs a non-minimal graded free resolution (Algorithm 5.2) and then minimalizes it to obtain the short resolution (Algorithm 5.3).

In view of the results of Chapters 3 and 4, we believe that these results can be extended to any simplicial projective monomial surface. We are therefore currently working on this extension to prove the Eisenbud-Goto conjecture for simplicial projective monomial surfaces.

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