A convergence analysis for the approximation to the solution of an age-structured population model with infinite lifespan

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Abstract

Considering the numerical approximation of the density distribution for an age-structured population model with unbounded lifespan on a compact interval $[0, T]$, we prove second order of convergence for a discretization that adaptively selects its truncated age-interval according to the exponential rate of decay with age of the solution of the model. It appears that the adaptive capacity of the length in the truncated age-interval of the discretization to the infinity lifespan is a very convenient approach for a long-time integration of the model to establish the asymptotic behavior of its dynamics numerically. The analysis of convergence uses an appropriate weighted maximum norm with exponential weights to cope with the unbounded age lifespan. We report experiments to exhibit numerically the theoretical results and the asymptotic behaviour of the dynamics for an age-structured squirrel population model introduced by Sulsky.

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1. Introduction

When the dynamics of a general population are modelled, there are biological reasons to distinguish the individuals in the population according to some relevant physiological characteristics. Thus, we say that the physiological variables structure the population. In human demography, for instance, the age of each individual is the physiological characteristic generally accounted for and, consequently, we say that the population is age-structured.

Age-structured population models were first considered by Sharpe and Lotka [15], and MacKendrick [13], and formulated in terms of a linear integro-partial differential equation with a nonlocal boundary condition, driving the evolution of the age-dependent density of individuals $u(a,t)$. These first age-structured population models were proposed under an infinite lifespan for individuals. Besides, we should mention age-structured population models in which the maximum age of individuals in the population is finite and fixed along the time evolution, as considered in [3, 11, 8]. Nonlinear models arise when the vital rates of the population are made depending on functionals of the density function as, for example, the total population size [10]

$$P(t) = \int_0^{+\infty} u(a,t)\,da, \quad t > 0. \tag{1.1}$$

The evolution of the population is modeled in terms of a hyperbolic partial differential equation

$$u_t + u_a = -\mu(a, I_\mu(t), t)\; u, \quad a > 0,\ t > 0, \tag{1.2}$$

and a nonlocal boundary condition, which represents the birth law of individuals in the population,

$$u(0,t) = \int_0^{\infty} \beta(a, I_\beta(t), t)\; u(a,t)\; da\ , \quad t > 0. \tag{1.3}$$

In (1.2), the nonnegative age-specific mortality rate function is given by $\mu(a, I_\mu(t), t) \geq 0$ and, in (1.3), $\beta(a, I_\beta(t), t)$ represents the nonnegative age-specific fertility rate function. The nonlinearity of the problem comes from the dependency of both functions on linear functionals of the age-dependent density function

$$I_\alpha(t) = \int_0^\infty \gamma_\alpha(a)\; u(a,t)\; da,\; t > 0,\; \alpha = \mu, \beta, \qquad (1.4)$$

to take into account the influence of the age-distribution of the population on the life history of individuals. For instance, $I_\mu(t)$ and $I_\beta(t)$ could be taken as the total population size (1.1), as it was first considered by Gurtin-MacCamy [10]. Finally, an initial condition

$$u(a,0) = u_0(a), \quad a \geq 0, \qquad (1.5)$$

defines the initial age-distribution of individuals in the population.

Nowadays, existence, uniqueness, and asymptotic behaviour of solutions to physiological structured problems are well established [17, 14, 9, 11]. For example, the existence of a unique solution up to time $T > 0$ (i.e. nonnegative function u on $\mathbb{R}^+ \times [0,T]$, with directional derivatives along the characteristics lines, and $u(\cdot,t) \in \mathcal{L}_1(\mathbb{R}^+)$, $I_\mu(t)$, $I_\beta(t)$ continuous for $t \in [0,T]$) to the nonlinear problem (1.2)-(1.5), with bounded autonomous functions $\gamma_\alpha(a)$, $a \geq 0$, $\alpha = \mu, \beta$, was attained with the following hypotheses [10]

- $u_0 \in \mathcal{L}_1(\mathbb{R}^+)$, piecewise continuous and nonnegative;
- $\mu, \beta \in \mathcal{C}(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+)$, and nonnegative; $\dfrac{\partial \mu}{\partial z}(a,z,t)$, $\dfrac{\partial \beta}{\partial z}(a,z,t)$ exists for all $a \geq 0$, $t \geq 0$, and $z \geq 0$;
- μ, β, $\dfrac{\partial \mu}{\partial z}$, $\dfrac{\partial \beta}{\partial z}$ are continuous and bounded functions with domain $\mathbb{R}^+$, for fixed values of age and time: $a, t \in \mathbb{R}^+$.

Moreover, again in [10], if $\bar{\beta} = \sup\{\beta(a,z,t), a \geq 0, z \geq 0, t \geq 0\} < \infty$, and $\underline{\mu} = \inf\{\mu(a,z,t), a \geq 0, z \geq 0, t \geq 0\} \geq 0$, and $u(a,t)$ is a solution in $\mathbb{R}^+ \times [0,T]$, then $u(a,t) \leq \bar{\beta}\, P(0)\, e^{(\bar{\beta}-\underline{\mu})\,t}\, e^{-\underline{\mu}\,a}$, when $a < t$, and $u(a,t) \leq e^{-\underline{\mu}\,a} \sup_{\tau\in[0,t]} (u_0(\tau))$, for $a \geq t$. In particular, this implies an exponential decay rate with age of the solution to (1.2)-(1.5).

From a numerical point of view, a general approach to discretize the problem (1.2)-(1.5) consists in the use of a truncation of the age-domain to a finite fixed interval $[0, A]$. This is justified when we assume that the age-dependent density function decays, fast enough, to zero when the age increases to infinity. This is the case, for example, when the initial condition u_0 has a compact suport and a finite time integration interval is assumed. Then, any of the methodologies considered in the past can be used: finite-difference schemes, characteristics methods, finite-element schemes, or discretizations of the identity

$$u(a+k, t+k) = u(a,t) \exp\left(- \int_0^k \mu(a+s, I_\mu(t+s), t+s)\, ds \right),\ a > 0,\ t > 0, \tag{1.6}$$

(see, for instance, [1] and the references therein). Nevertheless, this setting would not be useful enough if we were interested in a very long time integration in order to look for its equilibria and their asymptotic stability. Numerical methods that cope with the unbounded interval should be studied. A first approach was made in [4]. The authors introduced a change of independent age-variable and dependent density function that transforms the age-structured population problem in a population model structured by an artificial size variable whose domain is bounded. Thus, the original problem was analyzed through the properties of this model which was structured with the new artificial variable. A second order numerical method was proposed and analyzed to obtain an approximation to its solution by using a numerical subgrid of the so-called *natural grid* associated.

In this paper, we focus on a discretization in which the finite truncated age-interval $[0, A]$ is changing with the discretization parameter, according to the exponential rate of decay with age of the solution. In fact, the selection of A is given in terms of the last point in the natural grid provided by the artificial structural variable. Therefore, the only effect of the natural grid in the discretization is by means of the definition of the quadrature rules to approximate the nonlocal terms in (1.2)-(1.5) with the unbounded age-domain. In this way, the analysis of the new proposed numerical scheme avoids the difficulties of the unbounded age-interval.

The paper is organized as follows. In the next section, we propose a second order numerical method, described in detail. We carry out its convergence analysis in section 3. It is the highest order method analyzed for this setting, therefore we require the data functions and the solution of the problem to satisfy some technical restrictive smoothness conditions. We conclude with a section devoted to numerical results, with an experiment that confirms the theoretical order of convergence, and another one that describes the evolution of a squirrel population.

2. Numerical approximation

In this section, a numerical method is developed to compute the solution to model (1.2)-(1.5) at a final integration time $T \in \mathbb{R}$. With this aim, time and age variables will be discretized to introduce a grid where the solution to the model will be approximated. Thus, we consider $N \in \mathbb{N}$, which provides us the time-discretization step, $k = T/N$, and the discrete time levels which are described as $t^n = k\,n$, $n = 0, 1, \ldots, N$. In the case of the age variable, the lifespan is unbounded and, initially, we will introduce the age discretization defined by $a_j = j\,k$, $j = 0, 1, \ldots$ Besides, the solution u of (1.2)-(1.5) satisfies the identity

$$u(a_j + k, t^n + k) = u(a_j, t^n) \exp\left(- \int_0^k \mu(a_j + s, I_\mu(t^n + s), t^n + s)\, ds \right), \tag{2.1}$$

as the result of the integration of the problem along its characteristic lines, and the boundary condition

$$u(0, t^n) = \int_0^\infty \beta(a, I_\beta(t^n), t^n)\, u(a, t^n)\, da. \tag{2.2}$$

Then, after replacing the integral terms with appropriate quadrature rules, we obtain the following explicit scheme for advancing the solution from time t^n to time t^{n+1}, $0 \leq n \leq N-1$,

$$U_{j+1}^{n+1,*} = U_j^n \exp(-k\,\mu_j^n), \quad j \geq 0, \tag{2.3}$$

$$U_0^{n+1,*} = \mathcal{Q}_k\left(\boldsymbol{\beta}^{n+1}(\mathbf{U}^{n+1,*}) \cdot \mathbf{U}^{n+1,*}\right), \tag{2.4}$$

$$U_{j+1}^{n+1} = U_j^n \exp\left(-\frac{k}{2}\left(\mu_j^n + \mu_{j+1}^{n+1,*}\right)\right), \quad j \geq 0, \tag{2.5}$$

$$U_0^{n+1} = \mathcal{Q}_k\left(\boldsymbol{\beta}^{n+1}(\mathbf{U}^{n+1}) \cdot \mathbf{U}^{n+1}\right), \tag{2.6}$$

where

$$\boldsymbol{\beta}^n(\mathbf{U}^n) = \left(\beta_0^n(\mathbf{U}^n), \beta_1^n(\mathbf{U}^n), \ldots\right), \; \beta_j^n(\mathbf{U}^n) = \beta(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\beta \cdot \mathbf{U}^n), t^n),$$

$\mu_j^n = \mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{U}^n), t^n)$, $0 \leq n \leq N$,

$$\boldsymbol{\beta}^{n+1}(\mathbf{U}^{n+1,*}) = \left(\beta_0^{n+1}(\mathbf{U}^{n+1,*}), \beta_1^{n+1}(\mathbf{U}^{n+1,*}), \ldots\right),$$

$\beta_j^{n+1}(\mathbf{U}^{n+1,*})=\beta(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\beta \cdot \mathbf{U}^{n+1,*}), t^{n+1})$, $\mu_{j+1}^{n+1,*}=\mu(a_{j+1}, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{U}^{n+1,*}), t^{n+1})$, $0 \leq n \leq N-1$, and $\boldsymbol{\gamma}_s$, $s = \beta, \mu$, represent vectors with components $(\gamma_s)_j = \gamma_s(a_j)$, $j \geq 0$. The vectorial products $\boldsymbol{\beta}^n(\mathbf{U}^n) \cdot \mathbf{U}^n$, $\boldsymbol{\beta}^{n+1}(\mathbf{U}^{n+1,*}) \cdot \mathbf{U}^{n+1,*}$, $\boldsymbol{\gamma}_\beta \cdot \mathbf{U}^n$, $\boldsymbol{\gamma}_\mu \cdot \mathbf{U}^n$, $\boldsymbol{\gamma}_\beta \cdot \mathbf{U}^{n+1,*}$ and $\boldsymbol{\gamma}_\mu \cdot \mathbf{U}^{n+1,*}$are considered componentwise.

We have not completely determined the numerical method yet, because we should cope with the difficulties arising from the approximation of the integrals over the unbounded age interval. Assuming that the solution u decays exponentially fast enough to zero when age tends to infinity, an usual numerical procedure is to consider first a truncation of the age interval before proceeding with the time and age discretization. In this case, the numerical contribution of the individuals of the population after an *a priori* maximum age A is neglected and the numerical scheme (2.3)-(2.6) is well defined for approximations U_j^n, $0 \leq j \leq \tilde{J}$, with $\tilde{J} = \lfloor A/k \rfloor$. Alternatively, we propose a discretization in which the truncation of the age interval changes with the discretization parameter and depends on how we fix the last grid point of the age discretization.

Although the integrals we want to approximate

$$\int_0^\infty \beta(a, I_\beta(t), t)\, u(a,t)\, da, \quad \int_0^\infty \gamma_s(a)\, u(a,t)\, da, \quad s = \mu, \beta, \qquad (2.7)$$

are given in terms of an age variable, we propose a change of the integration variable that transforms the unbounded age interval in a bounded computational interval. In general, we assume for the sake of simplicity that the new integration variable is related with the age by means of $a = \alpha(x)$, with $x \in [0, 1)$, and $\alpha(x)$ a monotone increasing and unbounded function. Then the uniform age grid $a_j, j = 0, 1, \ldots,$ is transformed into a non uniform grid for the new variable x that evolves with time as given by $x' = g(x)$, $x \in [0, 1)$, where $g(x) = 1/\alpha'(x)$. We emphasize that the transformed grid $\{x_j\}_{j \geq 0}$ is a natural grid in the sense that the grid point x_{j+1} at time t^{n+1} is always in the solution curve of $x' = g(x)$ that goes through (x_j, t^n). Although this kind of grid has been used in the numerical approximation along the characteristics for size-structured population models [2, 5, 6, 7], its role in the numerical scheme we propose is only a convenient tool for defining the quadrature nodes in approximating (2.7). In particular, we propose the following specific expression for the change of variable

$$\alpha(x) = -\frac{1}{K_\alpha} \log(1 - x), \quad x \in [0, 1),$$

where K_α is an appropriate positive constant. As we will see below, the choice of the constant K_α will be made in terms of the exponential rate of decay of u and its derivatives. This change of variable transforms the integrals in (2.7) as

$$\int_0^1 \beta(\alpha(x), I_\beta(t), t)\, \frac{u(\alpha(x), t)}{K_\alpha\,(1-x)}\, dx, \quad \int_0^1 \gamma_s(\alpha(x))\, \frac{u(\alpha(x), t)}{K_\alpha\,(1-x)}\, dx, \quad s = \mu, \beta,$$

to be approximated with a quadrature rule based on quadrature nodes given by $x_j = 1 - \exp(-K_\alpha\, k\, j)$, $j = 1, 2, \ldots$ With this explicit expression for the quadrature nodes, we close the selection of the grid nodes in the algorithm (2.3)-(2.6) by choosing the one corresponding to the first x_J of the x-grid that satisfies the following inequality

$$1 - x_J = \exp(-K_\alpha\, k\, J) < K_1\, k, \tag{2.8}$$

as the last node in the age-grid, with K_1 a fixed constant that does not depend on the discretization parameter. We can assume, from the beginning, that k is small enough to satisfy $J > N$. We note that, in terms of the age-grid, we are just truncating the age-interval in the discretization using a maximum age given by a_J that increases as the discretization parameter tends to zero.

Now we define the quadrature rule $\mathcal{Q}_k$ that approaches $\displaystyle\int_0^\infty f(a)\, da$, in terms of the quadrature grid given by the nodes x_j, $j = 0, \ldots, J$, as

$$\begin{aligned}\int_0^1 \frac{f(\alpha(x))}{K_\alpha\,(1-x)}\, dx \approx \mathcal{Q}_k(\mathbf{f}) &= x_1\, \frac{f_1}{K_\alpha\,(1-x_1)} \\ &+ \sum_{j=1}^{J-1} \frac{x_{j+1} - x_j}{2} \left(\frac{f_j}{K_\alpha\,(1-x_j)} + \frac{f_{j+1}}{K_\alpha\,(1-x_{j+1})} \right) + \frac{f_J}{K_\alpha},\end{aligned} \tag{2.9}$$

note that $f_j = f(\alpha(x_j)) = f(a_j)$, $j = 0, \ldots, J$. We can rewrite this quadrature rule by using the description of x_j, $j = 0, \ldots, J$,

$$\begin{aligned}\mathcal{Q}_k(\mathbf{f}) \;=\; & \frac{1}{K_\alpha} \left\{ \left(e^{K_\alpha\, k} \left(1 - \frac{1}{2}\, e^{-K_\alpha\, 2k} \right) - \frac{1}{2} \right) f_1 \right. \\ & \left. + \frac{1}{2} \sum_{j=2}^{J-1} e^{K_\alpha\, k} \left(1 - e^{-K_\alpha\, 2k} \right) f_j + \frac{1}{2} \left(1 + e^{K_\alpha\, k} \right) f_J \right\}.\end{aligned} \tag{2.10}$$

Now the numerical method is fully described.

In the following section, the convergence analysis of this numerical method is developed. First, we show that the quadrature rule is second order of accuracy. Thus, we introduce the regularity requierements for the solution and the data functions involved that we will need in our consistency and stability analysis. Let $T, \sigma \in \mathbb{R}^+$; we define

$$\mathcal{A}_\sigma = \{f \in \mathcal{C}^2([0,\infty)); \exists\, a_0(f) > 0,\ |f^{(n)}(a)| \leq C\, e^{-\sigma\, a},\ a \geq a_0(f), n = 0, 1, 2\}$$

and

$$\begin{aligned} \mathcal{A}^0_\sigma &= \{f \in \mathcal{C}^2([0,\infty)); f \text{ bounded} \\ &\quad \text{and} \, \exists\, a_0(f) > 0,\ |f^{(n)}(a)| \leq C\, e^{-\sigma\, a},\ a \geq a_0(f), n = 1, 2\}. \end{aligned}$$

Throughout the paper, we assume the following hypotheses,

(H1) • $u \in \mathcal{C}^2([0,\infty) \times [0,T])$, is nonnegative and $u(\cdot,t) \in \mathcal{A}_\sigma$, $t > 0$,

(H2) • $\beta \in \mathcal{C}^2([0,\infty) \times D_\beta \times [0,T])$, is nonnegative and $\beta(\cdot,z,t) \in \mathcal{A}^0_\sigma$, $(z,t) \in D_\beta \times [0,T]$, and D_β is a compact neighbourghood of

$$\left\{ \int_0^\infty \gamma_\beta(a)\, u(a,t)\, da, \quad 0 \leq t \leq T \right\},$$

(H3) • $\mu \in \mathcal{C}^2([0,\infty) \times D_\mu \times [0,T])$, is nonnegative and $\mu(\cdot,z,t) \in \mathcal{A}^0_\sigma$, $(z,t) \in D_\mu \times [0,T]$, and D_μ is a compact neighbourghood of

$$\left\{ \int_0^\infty \gamma_\mu(a)\, u(a,t)\, da, \quad 0 \leq t \leq T \right\},$$

(H4) • $\gamma_\mu, \gamma_\beta \in \mathcal{A}^0_\sigma$, are nonnegative.

These assumptions would be enough to assure the convergence of our numerical approximations. In particular, in the following sections and for $f \in \mathcal{A}_\sigma$, we use a negative exponential bound on the values of f,

$$|f(a)| \leq C\, e^{-\sigma\, a}, \quad a \in [0, +\infty), \tag{2.11}$$

with an appropriate constant C, depending on f. Thus, under the assumptions (H1), at $t = 0$, (H2)-(H4) the problem (1.2)-(1.5) has a unique nonnegative solution, which is global in time (they are more restrictive hypotheses, about the regularity of the solution, than the ones assumed in [10]). Furthermore, it could be proved that $u \in \mathcal{C}^2(\mathbb{R}^+ \times [0,T])$ with an additional second order compatibility condition. And, hypotheses (H1), at $t = 0$, (H2)-(H4) ensure $u(\cdot, t) \in \mathcal{L}^1(\mathbb{R})$. However, we would note that the aim of the paper is the proposal of a new numerical method to approach an age-structured population model with an unbounded domain and its corresponding convergence analysis. Therefore, we are not involved in the theoretical study of existence, uniqueness, regularity and other properties of the solution to the problem.

Thus, we first show that under the problem hypotheses the quadrature rule is second order of accuracy.

Proposition 1. *Let be* $f \in \mathcal{A}_\sigma$, $3\,K_\alpha < \sigma$, $a_j = j\,k$, $0 \le j \le J$, $\mathbf{f} = (f(a_0), f(a_1), \ldots, f(a_J))$, *then*

$$\int_0^\infty f(a)\,da - \mathcal{Q}_k(\mathbf{f}) = \mathcal{O}(k^2), \tag{2.12}$$

for k *sufficiently small.*

Proof. First, we should remember the change on the integration variable given by $a = \alpha(x) = -\dfrac{1}{K_\alpha}\,\log(1-x)$. Thus,

$$\begin{aligned}
&\int_0^\infty f(a)\,da - \mathcal{Q}_k(\mathbf{f}) = \left(\int_0^{x_1} \frac{f\,(\alpha(x))}{K_\alpha\,(1-x)}\,dx - x_1\,\frac{f(\alpha(x_1))}{K_\alpha\,e^{-K_\alpha\,k}}\right) \\
&+ \sum_{j=1}^{J-1}\left(\int_{x_j}^{x_{j+1}} \frac{f\,(\alpha(x))}{K_\alpha\,(1-x)}\,dx - \frac{x_{j+1}-x_j}{2}\left(\frac{f(\alpha(x_j))}{K_\alpha\,e^{-K_\alpha\,k\,j}} + \frac{f(\alpha(x_{j+1}))}{K_\alpha\,e^{-K_\alpha\,k\,(j+1)}}\right)\right) \\
&+ \left(\int_{x_J}^{1} \frac{f\,(\alpha(x))}{K_\alpha\,(1-x)}\,dx - \frac{f(\alpha(x_J))}{K_\alpha}\right).
\end{aligned}$$

The rectangular and trapezoidal quadrature errors in the above expression can be bounded by $\mathcal{O}(k^2)$ and $\mathcal{O}(k^3)$ terms, respectively, by assuming that $h(x) := \dfrac{f(\alpha(x))}{K_\alpha\,(1-x)}$ has an $\mathcal{C}^2$ extension in $[0,1]$. The hypotheses $f \in \mathcal{A}_\sigma$ and $3\,K_\alpha < \sigma$ are sufficient for it to be true [4, Lemma 1]. Thus,

$$\begin{aligned}
&\left|\int_0^\infty f(a)\,da - \mathcal{Q}_k(\mathbf{f})\right| \\
&= \mathcal{O}(k^2) + \sum_{j=1}^{J-1}(x_{j+1}-x_j)\,\mathcal{O}(k^2) + \mathcal{O}(k^2) = \mathcal{O}(k^2), \quad (k \to 0). \blacksquare
\end{aligned}$$

3. Convergence analysis

In this section we analyze the convergence of the numerical method presented in section 2 by means of the study of its consistency and stability properties. It will be analyzed following the discretization framework developed by López-Marcos and Sanz-Serna [12]. Therefore, we introduce the following notation.

We assume that the discretization parameter k takes values in the set $H = \{k > 0 : k = T/N, N \in \mathbb{N}\}$, and for each $k \in H$, and J as in (2.8), we define the vector spaces $\mathcal{X}_k = (\mathbb{R}^{J+1})^{N+1}$ and $\mathcal{Y}_k = (\mathbb{R}^{J+1}) \times \mathbb{R}^N \times (\mathbb{R}^J)^N$. If $\mathbf{V} = (V_0, V_1, \ldots, V_J) \in \mathbb{R}^{J+1}$, and $\|\mathbf{V}\|_{\infty,J+1} = \max_{0\le j\le J} \{e^{K_\alpha\, k\, j}\, |V_j|\}$, we define the following norm on $\mathcal{X}_k$, if $(\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N) \in \mathcal{X}_k$

$$\|(\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N)\|_{\mathcal{X}_k} = \max_{0\le n\le N} \|\mathbf{V}^n\|_{\infty,J+1}.$$

On the other hand, If $\mathbf{Z} \in \mathbb{R}^N$, $\mathbf{W} \in \mathbb{R}^J$, $\|\mathbf{Z}\|_{\infty,N} = \max_{1\le n\le N} |Z^n|$, $\|\mathbf{W}\|_{\infty,J} = \max_{1\le j\le J} \{e^{K_\alpha\, k\, j}\, |W_j|\}$, then for $(\mathbf{Z}^0, \mathbf{Z}_0, \mathbf{Z}_*^1, \ldots, \mathbf{Z}_*^N) \in \mathcal{Y}_k$, we define

$$\|(\mathbf{Z}^0, \mathbf{Z}_0, \mathbf{Z}_*^1, \ldots, \mathbf{Z}_*^N)\|_{\mathcal{Y}_k} = \|\mathbf{Z}^0\|_{\infty,J+1} + \|\mathbf{Z}_0\|_{\infty,N} + \sum_{n=1}^{N} k\, \|\mathbf{Z}_*^n\|_{\infty,J}.$$

We also define the following seminorm based on the definition of the corresponding quadrature rule,

$$\begin{aligned} \|\mathbf{V}\|_{1,J} &= \frac{1}{K_\alpha} \left\{ \left(e^{K_\alpha\, k} \left(1 - \frac{1}{2}\, e^{-2\, K_\alpha\, k}\right) - \frac{1}{2}\right) |V_1| \right. \\ &\quad \left. + \frac{1}{2} \sum_{j=2}^{J-1} e^{K_\alpha\, k} \left(1 - e^{-2\, K_\alpha\, k}\right) |V_j| + \frac{1}{2} \left(1 + e^{K_\alpha\, k}\right) |V_J| \right\}, \end{aligned}$$

with $\mathbf{V} = (V_0, V_1, \ldots, V_J)$. This seminorm plays a crucial role in the stability study. Finally, we employ the usual notation in the maximum norm $\|\mathbf{V}\|_\infty = \max_{0\le j\le J} \{|V_j|\}$, with $\mathbf{V} = (V_0, V_1, \ldots, V_J) \in \mathbb{R}^{J+1}$.

Now, for each $k \in H$, we introduce the grid restriction of the solution, let be $\mathbf{u}_k = (\mathbf{u}^0, \mathbf{u}^1, \ldots, \mathbf{u}^N) \in \mathcal{X}_k$, $\mathbf{u}^n = (u_0^n, u_1^n, \ldots, u_J^n) \in \mathbb{R}^{J+1}$, $u_j^n = u(a_j, t^n)$, $0 \le j \le J$, $0 \le n \le N$, where u is the solution of (1.2)-(1.5). Let $r > 0$, we denote by $\mathcal{B}(\mathbf{u}_k, r) \subset \mathcal{X}_k$, the open ball with center $\mathbf{u}_k$ and radius r.

Let R be a fixed positive constant, we introduce the mapping $\mathbf{\Phi}_k : \mathcal{B}(\mathbf{u}_k, R\,k) \subset \mathcal{X}_k \to \mathcal{Y}_k$ defined by the equations

$$\mathbf{\Phi}_k(\mathbf{W}^0, \mathbf{W}^1, \ldots, \mathbf{W}^N) = (\mathbf{Z}^0, \mathbf{Z}_0, \mathbf{Z}^1_*, \ldots, \mathbf{Z}^N_*), \tag{3.1}$$

$$\begin{aligned}
\mathbf{Z}^0 &= \mathbf{W}^0 - \mathbf{U}^0 && (3.2)\\
Z^{n+1}_{j+1} &= \frac{1}{k}\Bigg(W^{n+1}_{j+1} - W^n_j \exp\Bigg(-\frac{k}{2}\,\left(\mu\left(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{W}^n), t^n\right)\right.\\
&\qquad \left.+\mu\left(a_{j+1}, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{W}^{n+1,*}), t^{n+1}\right)\right)\Bigg)\Bigg), && (3.3)\\
Z^{n+1}_0 &= W^{n+1}_0 - \mathcal{Q}_k(\boldsymbol{\beta}^{n+1}(\mathbf{U}^{n+1}) \cdot \mathbf{W}^{n+1}), && (3.4)
\end{aligned}$$

where $\mathbf{W}^{n+1,*} = \left(W^{n+1,*}_0, W^{n+1,*}_1, \ldots, W^{n+1,*}_J\right)$ with

$$W^{n+1,*}_{j+1} = W^n_j \exp\left(-k\,\mu\left(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{W}^n), t^n\right)\right), \quad 0 \le j \le J-1,$$

and

$$W^{n+1,*}_0 = \mathcal{Q}_k(\boldsymbol{\beta}^{n+1}(\mathbf{W}^{n+1,*}) \cdot \mathbf{W}^{n+1,*}),$$

$0 \le n \le N-1$. We emphasize that $W^{n+1,*}_0$ is an approximation to $u(0, t^{n+1})$ that is not used in the numerical scheme. We also introduce the vector $\mathbf{u}^{n+1,*} = \left(u^{n+1,*}_0, u^{n+1,*}_1, \ldots, u^{n+1,*}_J\right)$ given by the equations

$$u^{n+1,*}_{j+1} = u^n_j \exp\left(-k\,\mu\left(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{u}^n), t^n\right)\right), \quad 0 \le j \le J-1,$$

and

$$u^{n+1,*}_0 = \mathcal{Q}_k(\boldsymbol{\beta}^{n+1}(\mathbf{u}^{n+1,*}) \cdot \mathbf{u}^{n+1,*}),$$

$0 \le n \le N-1$.

We can study the properties of the numerical scheme (2.3)-(2.6) through this mapping due to the following main property: $\mathbf{U}_k = (\mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^N) \in \mathcal{X}_k$, is a solution of the scheme (2.3)-(2.6) if and only if

$$\mathbf{\Phi}_k(\mathbf{U}_k) = \mathbf{0}. \tag{3.5}$$

Now, we study the convergence of our scheme.

The following result shows that the operator (3.1) is well defined.

Proposition 2. *Assuming hypotheses (H1)-(H4), with $3\,K_\alpha < \sigma$, on the functions data and the solution to (1.2)-(1.5). If $(\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N) \in \mathcal{B}(\mathbf{u}_k, R_k)$, with $R_k = o(1)$, then, for k sufficiently small,*

$$\mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^n) \in D_\mu, \quad \mathcal{Q}_k(\boldsymbol{\gamma}_\beta \cdot \mathbf{V}^n) \in D_\beta, \tag{3.6}$$

$0 \le n \le N$, and

$$\mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^{n,*}) \in D_\mu, \quad \mathcal{Q}_k(\boldsymbol{\gamma}_\beta \cdot \mathbf{V}^{n,*}) \in D_\beta, \tag{3.7}$$

$1 \le n \le N$.

Proof The use of the hypotheses (H1) and (H4), the convergence properties of the quadrature rule $\mathcal{Q}_k$ allows us to arrive at

$$\begin{aligned} |I_s(t^n) - \mathcal{Q}_k(\boldsymbol{\gamma}_s \cdot \mathbf{V}^n)| &\le |I_s(t^n) - \mathcal{Q}_k(\boldsymbol{\gamma}_s \cdot \mathbf{u}^n)| + |\mathcal{Q}_k\left(\boldsymbol{\gamma}_s \cdot (\mathbf{u}^n - \mathbf{V}^n)\right)| \\ &\le \mathcal{O}(k^2) + \frac{1}{K_\alpha}\, \|\boldsymbol{\gamma}_s\|_\infty\, R_k, \end{aligned} \tag{3.8}$$

$s = \beta, \mu$, $0 \le n \le N-1$, and (3.6) is done.

With respect to (3.7), we use the definition (2.5), the Mean Value Theorem, Proposition 1 and regularity hypotheses (H1) and (H3), using the bound (2.11), to obtain

$$\begin{aligned} \left|u_{j+1}^{n+1} - V_{j+1}^{n+1,*}\right| &= \Big| u_j^n \, \exp\left(-\int_0^k \mu(a_j + s, I_\mu(t^n + s), t^n + s)\, ds\right) \\ &\qquad - V_j^n \, \exp\left(-k\, \mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^n), t^n)\right) \Big| \\ &\le u_j^n \left| \exp\left(-\int_0^k \mu(a_j + s, I_\mu(t^n + s), t^n + s)\, ds\right) - \exp\left(-k\, \mu(a_j, I_\mu(t^n), t^n)\right) \right| \\ &\quad + u_j^n \left| \exp\left(-k\, \mu(a_j, I_\mu(t^n), t^n)\right) - \exp\left(-k\, \mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{u}^n), t^n)\right) \right| \\ &\quad + u_j^n \left| \exp\left(-k\, \mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{u}^n), t^n)\right) - \exp\left(-k\, \mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^n), t^n)\right) \right| \\ &\quad + \left| u_j^n - V_j^n \right| \, \exp\left(-k\, \mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^n), t^n)\right) \\ &\le C\, e^{-\sigma\, a_j}\, \mathcal{O}(k^2) + C\, e^{-\sigma\, a_j}\, k\, \mathcal{O}(k^2) \\ &\quad + C\, e^{-\sigma\, a_j}\, k\, \left| \mathcal{Q}_k\left(\boldsymbol{\gamma}_\mu \cdot (\mathbf{u}^n - \mathbf{V}^n)\right) \right| + R_k\, e^{-K_\alpha\, a_j}, \end{aligned}$$

$0 \le j \le J-1$, $0 \le n \le N-1$, where, with the fact that $K_\alpha < \sigma$, we arrive at

$$\left|u_{j+1}^{n+1} - V_{j+1}^{n+1,*}\right| \le R_k^*\, e^{-K_\alpha\, a_j}, \tag{3.9}$$

with $R_k^* = o(1)$, $0 \le j \le J-1$, $0 \le n \le N-1$. And, again, we employ hypotheses (H1) and (H4) and inequality (3.9) to achieve

$$\begin{aligned} & \left| I_s(t^{n+1}) - \mathcal{Q}_k(\boldsymbol{\gamma}_s \cdot \mathbf{V}^{n+1,*}) \right| \\ \le \; & \left| I_s(t^{n+1}) - \mathcal{Q}_k(\boldsymbol{\gamma}_s \cdot \mathbf{u}^{n+1}) \right| + \left| \mathcal{Q}_k \left(\boldsymbol{\gamma}_s \cdot \left(\mathbf{u}^{n+1} - \mathbf{V}^{n+1,*} \right) \right) \right| \\ \le \; & \mathcal{O}(k^2) + \frac{1}{K_\alpha} \left\| \boldsymbol{\gamma}_s \right\|_\infty R_k^*, \end{aligned}$$

$s = \beta, \mu$, $0 \le n \le N-1$, that finishes with the desired result (3.7). ∎

3.1. Consistency

We define the *local discretization error* as

$$\mathbf{l}_k = \boldsymbol{\Phi}_k(\mathbf{u}_k) \in \mathcal{Y}_k, \tag{3.10}$$

and we say that the discretization (3.1) is *consistent* if, as $k \to 0$,

$$\lim_{k \to 0} \|\boldsymbol{\Phi}_k(\mathbf{u}_k)\|_{\mathcal{Y}_k} = \lim_{k \to 0} \|\mathbf{l}_k\|_{\mathcal{Y}_k} = 0. \tag{3.11}$$

The next theorem establishes the consistency of the numerical scheme (2.3)-(2.6).

Theorem 1. *Assuming hypotheses (H1)-(H4) on the functions data and the solution to (1.2)-(1.5) and $3\,K_\alpha < \sigma$, as $k \to 0$, the local discretization error satisfies*

$$\|\boldsymbol{\Phi}_k(\mathbf{u}_k)\|_{\mathcal{Y}_k} = \|\mathbf{u}^0 - \mathbf{U}^0\|_{\infty, J+1} + O(k^2). \tag{3.12}$$

Proof Before we demonstrate the main result of the theorem, we need to obtain some complementary inequalities. On the one hand, we obtain the consistency of the auxiliary terms; therefore, we describe the following difference,

$$u_{j+1}^{n+1} - u_{j+1}^{n+1,*} \; = \; u_{j+1}^{n+1} - u_j^n \, \exp\left(-k \, \mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{u}^n), t^n) \right),$$

$0 \le j \le J-1$, $\quad 0 \le n \le N-1$. The bound is achieved through the Mean Value Theorem applied to the exponential function and the error properties of $\mathcal{Q}_k$ and the rectangular quadrature rule, due to the smoothness and boundness properties of the functions u, μ y γ_μ, hypotheses (H1)-(H4), and bound (2.11),

$$
\begin{aligned}
|u_{j+1}^{n+1} - u_{j+1}^{n+1,*}| \le |u_j^n| & \left\{ \exp\left(-\int_0^k \mu(a_j + s, I_\mu(t^n+s), t^n+s)\, ds \right) \right. \\
& \left. - \exp\left(-k\,\mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{u}^n), t^n) \right) \right\} \\
\le |u_j^n| & \left\{ \exp\left(-\int_0^k \mu(a_j + s, I_\mu(t^n+s), t^n+s) ds \right) - \exp\left(-k\,\mu(a_j, I_\mu(t^n), t^n) \right) \right. \\
& \left. + \exp\left(-k\,\mu(a_j, I_\mu(t^n), t^n) \right) - \exp\left(-k\,\mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{u}^n), t^n) \right) \right\} \\
\le C\, e^{-\sigma\, a_j}\, k^2, & \quad (k \to 0), \qquad (3.13)
\end{aligned}
$$

$0 \le j \le J-1$, $\;0 \le n \le N-1$. We also obtain the following inequality from (3.13) (and considering that $K_\alpha < \sigma$)

$$
\left| \mathcal{Q}_k \left(\boldsymbol{\gamma}_s \cdot \left(\mathbf{u}^{n+1} - \mathbf{u}^{n+1,*} \right) \right) \right| \le \|\boldsymbol{\gamma}_s\|_\infty \left| \mathcal{Q}_k \left(\mathbf{u}^{n+1} - \mathbf{u}^{n+1,*} \right) \right| \le C\, k^2, \quad (3.14)
$$

$0 \le n \le N-1$.

On the other hand, we have the following inequality from the error properties of $\mathcal{Q}_k$, because of hypotheses (H1)-(H4), and inequality (3.14)

$$
\begin{aligned}
& \left| I_s(t^{n+1}) - \mathcal{Q}_k(\boldsymbol{\gamma}_s \cdot \mathbf{u}^{n+1,*}) \right| \\
& \quad \le \left| I_s(t^{n+1}) - \mathcal{Q}_k(\boldsymbol{\gamma}_s \cdot \mathbf{u}^{n+1}) \right| + \left| \mathcal{Q}_k \left(\boldsymbol{\gamma}_s \cdot \left(\mathbf{u}^{n+1} - \mathbf{u}^{n+1,*} \right) \right) \right| \\
& \quad \le C\, k^2, \qquad (3.15)
\end{aligned}
$$

$s = \beta, \mu$, $0 \le n \le N-1$.

In the following, we obtain the main result of the theorem. We denote $\boldsymbol{\Phi}_k(\mathbf{u}_k) = (\mathbf{L}^0, \mathbf{L}_0, \mathbf{L}^1, \ldots, \mathbf{L}^N)$, thus

$$
\begin{aligned}
L_{j+1}^{n+1} = \frac{1}{k} \Bigg(u_{j+1}^{n+1} - u_j^n \exp\Bigg(-\frac{k}{2} \big(& \mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{u}^n), t^n) \\
& + \mu(a_{j+1}, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{u}^{n+1,*}), t^{n+1}) \big) \Bigg) \Bigg),
\end{aligned}
$$

$0 \le j \le J-1$, $\;0 \le n \le N-1$. We applied the Mean Value Theorem to the exponential and mortality functions. The error properties of $\mathcal{Q}_k$ and the trapezoidal quadrature rule, hypotheses (H1)-(H4) and inequality (3.15) with $s=\mu$, allow us to arrive at

$$\begin{aligned}
|L_{j+1}^{n+1}| \leq \frac{1}{k}\,|u_j^n|\, \Bigg\{ \Bigg| & \exp\left(-\int_0^k \mu(a_j+s, I_\mu(t^n+s), t^n+s)\,ds\right) \\
& - \exp\left(-\frac{k}{2}\left(\mu(a_j, I_\mu(t^n), t^n) + \mu(a_{j+1}, I_\mu(t^{n+1}), t^{n+1})\right)\right)\Bigg| \\
& + \Bigg|\exp\left(-\frac{k}{2}\left(\mu(a_j, I_\mu(t^n), t^n) + \mu(a_{j+1}, I_\mu(t^{n+1}), t^{n+1})\right)\right) \\
& - \exp\left(-\frac{k}{2}\left(\mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu\cdot\mathbf{u}^n), t^n) + \mu(a_{j+1}, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu\cdot\mathbf{u}^{n+1,*}), t^{n+1})\right)\right)\Bigg|\Bigg\} \\
\leq\ & C\,|u_j^n|\,\left\{k^2 + \left|I_\mu(t^n) - \mathcal{Q}_k(\boldsymbol{\gamma}_\mu\cdot\mathbf{u}^n)\right| + \left|I_\mu(t^{n+1}) - \mathcal{Q}_k(\boldsymbol{\gamma}_\mu\cdot\mathbf{u}^{n+1,*})\right|\right\} \\
\leq\ & C\,e^{-\sigma\,a_j}\,k^2, \qquad (3.16)
\end{aligned}$$

$0 \leq j \leq J-1, \quad 0 \leq n \leq N-1$.

The last step consists in the treatment of the boundary term by means of

$$L_0^n = u_0^n - \mathcal{Q}_k(\boldsymbol{\beta}^n(\mathbf{u}^n)\cdot\mathbf{u}^n),$$

$1 \leq n \leq N$. This term can be bounded with the use of the convergence properties of the quadrature rule $\mathcal{Q}_k$, the Mean Value Theorem and the hypotheses (H1)-(H4) and the inequality (3.15) with $s=\beta$,

$$\begin{aligned}
|L_0^n| &= \left|\int_0^\infty \beta(a, I_\beta(t^n), t^n)\,u(a,t^n)\,da - \mathcal{Q}_k(\boldsymbol{\beta}^n(\mathbf{u}^n)\cdot\mathbf{u}^n)\right| \\
&\leq \left|\int_0^\infty \beta(a, I_\beta(t^n), t^n)\,u(a,t^n)\,da - \mathcal{Q}_k(\boldsymbol{\beta}_I^n\cdot\mathbf{u}^n)\right| + \left|\mathcal{Q}_k\left((\boldsymbol{\beta}_I^n - \boldsymbol{\beta}^n(\mathbf{u}^n))\cdot\mathbf{u}^n\right)\right| \\
&\leq C\,k^2 + C\,\left\|\boldsymbol{\beta}_I^n - \boldsymbol{\beta}^n(\mathbf{u}^n)\right\|_\infty\,\left|\mathcal{Q}_k(\mathbf{u}^n)\right| \\
&\leq C\,k^2 + C\,\left|I_\beta(t^n) - \mathcal{Q}_k(\boldsymbol{\gamma}_\beta\cdot\mathbf{u}^n)\right|\,\left|\mathcal{Q}_k(\mathbf{u}^n)\right|
\end{aligned}$$

$1 \leq n \leq N$, where we have introduced the following auxiliar notation $(\boldsymbol{\beta}_I^n)_j = \beta(a_j, I_\beta(t^n), t^n)$, $0 \leq j \leq J$. Finally, the bound $|u_j^n| \leq C\,e^{-\sigma\,a_j}$, $0 \leq j \leq J$, given by (2.11), within the quadrature term $|\mathcal{Q}_k(\mathbf{u}^n)|$ and the convergence properties of the quadrature rule $\mathcal{Q}_k$ allow us to arrive at, for k enough small,

$$|L_0^n| \leq C\,k^2, \qquad (3.17)$$

$1 \leq n \leq N$. Thus, the combination of (3.16) and (3.17) provides (3.12). ∎

3.2. Stability

Another notion that plays an important role in the analysis of our numerical method is the *stability*. For each $k \in H$, let R_k be a positive real number or $+\infty$ *(the stability threshold)*. We say that the discretization (3.1) is *stable* for $\mathbf{u}_k$ restricted to the thresholds R_k, if there exist two positive constants k_0 and S *(the stability constant)* such that, for any k in H with $k \leq k_0$ the open ball $\mathcal{B}(\mathbf{u}_k, R_k)$ is contained in the domain of $\mathbf{\Phi}_k$ and for all $\mathbf{V}_k$, $\mathbf{W}_k \in \mathcal{B}(\mathbf{u}_k, R_k)$

$$\|\mathbf{V}_k - \mathbf{W}_k\|_{\mathcal{X}_k} \leq S \, \|\mathbf{\Phi}_k(\mathbf{V}_k) - \mathbf{\Phi}_k(\mathbf{W}_k)\|_{\mathcal{Y}_k}.$$

Theorem 2. *Assuming the hypotheses of Theorem 1, the discretization (3.1)-(3.5) is stable for $\mathbf{u}_k$ with stability threshold $R_k = R\,k$, $(k \to 0)$.*

Proof Let $(\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N)$, $(\mathbf{W}^0, \mathbf{W}^1, \ldots, \mathbf{W}^N)$ be in the ball $\mathcal{B}(\mathbf{u}_k, R_k)$ of the space $\mathcal{X}_k$. We set

$$\mathbf{E}^n = \mathbf{V}^n - \mathbf{W}^n \in \mathbb{R}^{J+1}, \quad 0 \leq n \leq N;$$

$$\mathbf{\Phi}_k(\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N) = (\mathbf{Z}^0, \mathbf{Z}_0, \mathbf{Z}^1, \ldots, \mathbf{Z}^N),$$
$$\mathbf{\Phi}_k(\mathbf{W}^0, \mathbf{W}^1, \ldots, \mathbf{W}^N) = (\mathbf{S}^0, \mathbf{S}_0, \mathbf{S}^1, \ldots, \mathbf{S}^N).$$

By (3.3), we can write

$$\begin{aligned}
E_{j+1}^{n+1} &= V_j^n \, \exp\left(-\frac{k}{2}\left(\mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^n), t^n) + \mu(a_{j+1}, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^{n+1,*}), t^{n+1})\right)\right) \\
&\quad - W_j^n \, \exp\left(-\frac{k}{2}\left(\mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{W}^n), t^n) + \mu(a_{j+1}, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{W}^{n+1,*}), t^{n+1})\right)\right) \\
&\quad\quad + k\,(Z_{j+1}^{n+1} - S_{j+1}^{n+1}) \\
&= \; E_j^n \, \exp\left(-\frac{k}{2}\left(\mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^n), t^n) + \mu(a_{j+1}, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^{n+1,*}), t^{n+1})\right)\right) \\
&\quad + W_j^n \left(\exp\left(-\frac{k}{2}\left(\mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^n), t^n) + \mu(a_{j+1}, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^{n+1,*}), t^{n+1})\right)\right)\right. \\
&\quad \left. - \exp\left(-\frac{k}{2}\left(\mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{W}^n), t^n) + \mu(a_{j+1}, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{W}^{n+1,*}), t^{n+1})\right)\right)\right) \\
&\quad\quad + k\,(Z_{j+1}^{n+1} - S_{j+1}^{n+1}),
\end{aligned} \tag{3.18}$$

$0 \le j \le J-1, \quad 0 \le n \le N-1$. Due to hypothesis (H3), we derive

$$\exp\left(-\frac{k}{2}\left(\ \mu(x_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^n), t^n) + \mu(x_{j+1}, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^{n+1,*}), t^{n+1})\right)\right) \le 1, \tag{3.19}$$

$0 \le j \le J-1, \quad 0 \le n \le N-1$. Now, we obtain the following inequality with the use of hypotheses (H3) and (H4),

$$\begin{aligned}
&\Bigg|\exp\left(-\frac{k}{2}\left(\mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^n), t^n) + \mu(a_{j+1}, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^{n+1,*}), t^{n+1})\right)\right) \\
&\quad - \exp\left(-\frac{k}{2}\left(\ \mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{W}^n), t^n) + \mu(a_{j+1}, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{W}^{n+1,*}), t^{n+1})\right)\right)\Bigg| \\
&\quad \le\ k\ \big(\big|\mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^n), t^n) - \mu(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{W}^n), t^n)\big| \\
&\qquad + \big|\mu(a_{j+1}, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^{n+1,*}), t^{n+1}) - \mu(a_{j+1}, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{W}^{n+1,*}), t^{n+1})\big|\big) \\
&\quad \le\ C\,k\ \left(\big|\mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot (\mathbf{V}^n - \mathbf{W}^n))\big| + \big|\mathcal{Q}_k\left(\boldsymbol{\gamma}_\mu \cdot (\mathbf{V}^{n+1,*} - \mathbf{W}^{n+1,*})\right)\big|\right) \\
&\quad \le\ C\,k\,\|\boldsymbol{\gamma}_\mu\|_\infty\ \left(\left|\mathcal{Q}_k\left(\mathbf{V}^n - \mathbf{W}^n\right)\right| + \left|\mathcal{Q}_k\left(\mathbf{V}^{n+1,*} - \mathbf{W}^{n+1,*}\right)\right|\right) \\
&\quad \le\ C\,k\ \left(\|\mathbf{V}^n - \mathbf{W}^n\|_{1,J} + \left\|\mathbf{V}^{n+1,*} - \mathbf{W}^{n+1,*}\right\|_{1,J}\right),
\end{aligned} \tag{3.20}$$

$0 \le j \le J-1, \quad 0 \le n \le N-1$. Now, formula in (2.3) allows us to write

$$\begin{aligned}
V_{j+1}^{n+1,*} - W_{j+1}^{n+1,*} &= V_j^n\ \exp\left(-k\,\mu\left(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^n), t^n\right)\right) \\
&\quad - W_j^n\ \exp\left(-k\,\mu\left(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{W}^n), t^n\right)\right) \\
&= E_j^n\ \exp\left(-k\,\mu\left(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^n), t^n\right)\right) \\
&\quad + W_j^n\ \left(\exp\left(-k\,\mu\left(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^n), t^n\right)\right)\right. \\
&\qquad \left. - \exp\left(-k\,\mu\left(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{W}^n), t^n\right)\right)\right),
\end{aligned} \tag{3.21}$$

$0 \le j \le J-1, \quad 0 \le n \le N-1$. Again, due to (H3), we derive,

$$\exp\left(-k\,\mu^*\left(a_j, \mathcal{Q}_k(\boldsymbol{\gamma}_\mu \cdot \mathbf{V}^n), t^n\right)\right) \le 1, \tag{3.22}$$

$0 \le j \le J-1, \quad 0 \le n \le N-1$. We use similar arguments as in (3.20), and the fact that $|W_j^n| \le C\,e^{-K_\alpha\,a_j}$, $0 \le j \le J$, for k small enough, to arrive at

$$|V_{j+1}^{n+1,*} - W_{j+1}^{n+1,*}| \le |E_j^n| + C\,k\,e^{-K_\alpha\,a_j}\,\|\mathbf{E}^n\|_{1,J}\,, \tag{3.23}$$

$0 \le j \le J-1, \quad 0 \le n \le N-1$. Finally, we obtain

$$\left\|\mathbf{V}^{n+1,*} - \mathbf{W}^{n+1,*}\right\|_{1,J} \le (1 + C\,k)\ \|\mathbf{E}^n\|_{1,J}\,, \tag{3.24}$$

$0 \leq n \leq N-1$. Thus, combining (3.18)-(3.24), we have,

$$|E_{j+1}^{n+1}| \leq |E_j^n| + C\,k\,e^{-K_\alpha\,a_j}\,\|\mathbf{E}^n\|_{1,J} + k\,|Z_{j+1}^{n+1} - S_{j+1}^{n+1}|, \tag{3.25}$$

$0 \leq j \leq J-1$, $\quad 0 \leq n \leq N-1$. The property $|W_j^n| \leq C\,e^{-K_\alpha\,a_j}$, $0 \leq j \leq J$, also implies

$$\begin{aligned}
\|\mathbf{W}^n\|_{1,J} = \frac{1}{K_\alpha}\,&\left\{\left(e^{K_\alpha\,k}\left(1-\frac{1}{2}\,e^{-K_\alpha\,2\,k}\right)-\frac{1}{2}\right)|W_1^n|\right.\\
&\left.+\frac{1}{2}\sum_{j=2}^{J-1} e^{K_\alpha\,k}\left(1-e^{-K_\alpha\,2\,k}\right)|W_j^n| + \frac{1}{2}\left(1+e^{K_\alpha\,k}\right)|W_J^n|\right\}\\
= \quad \frac{1}{K_\alpha}\,&\left\{\frac{1}{2}\left(1-e^{-K_\alpha\,k}\right)e^{K_\alpha\,k}\,|W_1^n|\right.\\
&+\frac{1}{2}\sum_{j=1}^{J-1}\left(e^{-K_\alpha\,k\,(j-1)} - e^{-K_\alpha\,k\,(j+1)}\right)e^{K_\alpha\,k\,j}\,|W_j^n|\\
&\left.+\frac{1}{2}\,e^{-K_\alpha\,k\,J}\left(1+e^{K_\alpha\,k}\right)e^{K_\alpha\,k\,J}\,|W_J^n|\right\} \leq \frac{C}{K_\alpha},
\end{aligned} \tag{3.26}$$

where, in the last inequality, the cancellation of terms is due to the telescopic sum of exponentials after bounding $|W_j^n| \leq C\,e^{-K_\alpha\,a_j}$, $0 \leq j \leq J$.

Now, if $n \leq j \leq J$, by a recursive argument

$$|E_j^n| \leq |E_{j-n}^0| + C\,k\sum_{m=0}^{n-1} e^{-K_\alpha\,a_{j-n+m}}\,\|\mathbf{E}^m\|_{1,J} + k\sum_{m=1}^{n}|Z_{j-n+m}^m - S_{j-n+m}^m|, \tag{3.27}$$

and, when $N \geq n > j$,

$$|E_j^n| \leq |E_0^{n-j}| + C\,k\sum_{m=n-j}^{n-1} e^{-K_\alpha\,a_{j-n+m}}\,\|\mathbf{E}^m\|_{1,J} + k\sum_{m=n-j+1}^{n}|Z_{j-n+m}^m - S_{j-n+m}^m|. \tag{3.28}$$

Furthermore, equation (3.4), hypotheses (H2), (H4), and property (3.26), allow us to achieve

$$
\begin{aligned}
&|E_0^n| \le |\mathcal{Q}_k(\boldsymbol{\beta}^n(\mathbf{V}^n)\cdot\mathbf{V}^n) - \mathcal{Q}_k(\boldsymbol{\beta}^n(\mathbf{W}^n)\cdot\mathbf{W}^n)| + |Z_0^n - S_0^n| \\
&\le C\ |\mathcal{Q}_k(\boldsymbol{\beta}^n(\mathbf{V}^n)\cdot\mathbf{E}^n)| + C\ |\mathcal{Q}_k\left((\boldsymbol{\beta}^n\left(\mathbf{V}^n\right) - \boldsymbol{\beta}^n(\mathbf{W}^n))\cdot\mathbf{W}^n\right)| + |Z_0^n - S_0^n| \\
&\le C\,\|\boldsymbol{\beta}^n(\mathbf{V}^n)\|_\infty\,\|\mathbf{E}^n\|_{1,J} + C\,\|(\boldsymbol{\beta}^n(\mathbf{V}^n) - \boldsymbol{\beta}^n(\mathbf{W}^n))\|_\infty\ \|\mathbf{W}^n\|_{1,J} + |Z_0^n - S_0^n| \\
&\le C\ \left(\|\mathbf{E}^n\|_{1,J} + \left|\mathcal{Q}_k\left(\boldsymbol{\gamma}_\beta\cdot\mathbf{E}^n\right)\right|\right) + |Z_0^n - S_0^n| \\
&\le C\ \|\mathbf{E}^n\|_{1,J} + |Z_0^n - S_0^n|, \qquad (3.29)
\end{aligned}
$$

$1 \le n \le N$. Next, we deal with a bound for the seminorm $\|\mathbf{E}^n\|_{1,J}$, $1 \le n \le N$. Since $1 \le n \le J$, and inequalities (3.27)-(3.28) gives different bounds for $|E_j^n|$ depending on $j < n$ or $j \ge n$, we write down $\|\mathbf{E}^n\|_{1,J}$ as

$$
\begin{aligned}
\|\mathbf{E}^n\|_{1,J} = \frac{1}{K_\alpha}\ \Bigg\{&\left(e^{K_\alpha\,k}\ \left(1 - \frac{1}{2}\,e^{-K_\alpha\,2\,k}\right) - \frac{1}{2}\right)\ |E_1^n| \\
&+ \frac{1}{2}\sum_{j=2}^{n-1} e^{K_\alpha\,k}\left(1 - e^{-K_\alpha\,2\,k}\right)\ |E_j^n| \\
&+ \frac{1}{2}\sum_{j=n}^{J-1} e^{K_\alpha\,k}\left(1 - e^{-K_\alpha\,2\,k}\right)\ |E_j^n| + \frac{1}{2}\ \left(1 + e^{K_\alpha\,k}\right)\ |E_J^n|\Bigg\}. \qquad (3.30)
\end{aligned}
$$

Therefore, if we substitute (3.27)-(3.28) into (3.30), we obtain

$$
\begin{aligned}
\|\mathbf{E}^n\|_{1,J} \leq \frac{1}{K_\alpha} \Bigg\{ & \left(e^{K_\alpha\, k} \left(1 - \frac{1}{2}\, e^{-K_\alpha\, 2\, k} \right) - \frac{1}{2} \right) \left(|E_0^{n-1}| + C\, k\, \left\|\mathbf{E}^{n-1}\right\|_{1,J} \right. \\
& \qquad\qquad \left. + k\, |Z_1^n - S_1^n| \right) \\
& + \frac{1}{2} \sum_{j=2}^{n-1} e^{K_\alpha\, k} \left(1 - e^{-K_\alpha\, 2\, k}\right) \left(|E_0^{n-j}| + C\, k \sum_{m=n-j}^{n-1} e^{-K_\alpha\, a_{j-n+m}} \|\mathbf{E}^m\|_{1,J} \right. \\
& \qquad\qquad \left. + k \sum_{m=n-j+1}^{n} |Z_{j-n+m}^m - S_{j-n+m}^m| \right) \\
& + \frac{1}{2} \sum_{j=n}^{J-1} e^{K_\alpha\, k} \left(1 - e^{-K_\alpha\, 2\, k}\right) \left(|E_{j-n}^0| + C\, k \sum_{m=0}^{n-1} e^{-K_\alpha\, a_{j-n+m}} \|\mathbf{E}^m\|_{1,J} \right. \\
& \qquad\qquad \left. + k \sum_{m=1}^{n} |Z_{j-n+m}^m - S_{j-n+m}^m| \right) \\
& + \frac{1}{2} \left(1 + e^{K_\alpha\, k}\right) \left(|E_{J-n}^0| + C\, k \sum_{m=0}^{n-1} e^{-K_\alpha\, a_{J-n+m}} \|\mathbf{E}^m\|_{1,J} \right. \\
& \qquad\qquad \left. + k \sum_{m=1}^{n} |Z_{J-n+m}^m - S_{J-n+m}^m| \right) \Bigg\},
\end{aligned}
$$

$1 \leq n \leq J$. Thus,

$$
\begin{aligned}
\|\mathbf{E}^n\|_{1,J} \leq \frac{1}{K_\alpha} & \Bigg\{ \frac{1}{2}\,\left(e^{K_\alpha\,k}-1\right)\,|E_0^{n-1}| + \frac{1}{2}\sum_{j=1}^{n-1} e^{K_\alpha\,k}\left(1-e^{-K_\alpha\,2\,k}\right)\,|E_0^{n-j}| \\
& + \frac{1}{2}\sum_{j=n}^{J-1} e^{K_\alpha\,k}\left(1-e^{-K_\alpha\,2\,k}\right)\,|E_{j-n}^0| + \frac{1}{2}\,\left(1+e^{K_\alpha\,k}\right)\,|E_{J-n}^0| \\
& + C\,k\,\frac{1}{2}\,\left(e^{K_\alpha\,k}-1\right)\,\left\|\mathbf{E}^{n-1}\right\|_{1,J} \\
& + C\,k\,\frac{1}{2}\sum_{j=1}^{n-1} e^{K_\alpha\,k}\left(1-e^{-K_\alpha\,2\,k}\right)\,\sum_{m=n-j}^{n-1} e^{-K_\alpha\,a_{j-n+m}}\,\|\mathbf{E}^m\|_{1,J} \\
& + C\,k\,\frac{1}{2}\sum_{j=n}^{J-1} e^{K_\alpha\,k}\left(1-e^{-K_\alpha\,2\,k}\right)\,\sum_{m=0}^{n-1} e^{-K_\alpha\,a_{j-n+m}}\,\|\mathbf{E}^m\|_{1,J} \\
& + Ck\frac{1}{2}\left(1+e^{K_\alpha\,k}\right)\,\sum_{m=0}^{n-1} e^{-K_\alpha\,a_{J-n+m}}\,\|\mathbf{E}^m\|_{1,J} + k\,\frac{1}{2}\left(e^{K_\alpha\,k}-1\right)|Z_1^n - S_1^n| \\
& + k\,\frac{1}{2}\sum_{j=1}^{n-1} e^{K_\alpha\,k}\left(1-e^{-K_\alpha\,2\,k}\right)\,\sum_{m=n-j+1}^{n} |Z_{j-n+m}^m - S_{j-n+m}^m| \\
& + k\,\frac{1}{2}\sum_{j=n}^{J-1} e^{K_\alpha\,k}\left(1-e^{-K_\alpha\,2\,k}\right)\,\sum_{m=1}^{n} |Z_{j-n+m}^m - S_{j-n+m}^m| \\
& + k\,\frac{1}{2}\,\left(1+e^{K_\alpha\,k}\right)\,\sum_{m=1}^{n} |Z_{J-n+m}^m - S_{J-n+m}^m| \Bigg\}, \\
= \;\; & \frac{1}{K_\alpha}\,\left\{(I)+(II)+(III)+(IV)\right\}, \qquad (3.31)
\end{aligned}
$$

$1 \leq n \leq J$, where we have grouped the different terms in the right hand-side of the inequality for $\|\mathbf{E}^n\|_{1,J}$ (3.31), as indicated above to make the following bounds clearer.

With respect to the first term, the different elements involving components on the boundary, $|E_0^m|$, $1 \leq m \leq n-1$, are bounded using the inequality (3.29),

$$
\begin{aligned}
(I) = & \frac{1}{2}\,\left(e^{K_\alpha\,k}-1\right)\,|E_0^{n-1}| + \frac{1}{2}\sum_{j=1}^{n-1} e^{K_\alpha\,k}\left(1-e^{-K_\alpha\,2\,k}\right)\,|E_0^{n-j}| \\
\leq & \;\frac{1}{2}\,\left(e^{K_\alpha\,k}-1\right)\,\left(|Z_0^{n-1}-S_0^{n-1}| + C\,\left\|\mathbf{E}^{n-1}\right\|_{1,J}\right) \\
& + \frac{1}{2}\sum_{j=1}^{n-1} e^{K_\alpha\,k}\left(1-e^{-K_\alpha\,2\,k}\right)\,\left(|Z_0^{n-j}-S_0^{n-j}| + C\,\left\|\mathbf{E}^{n-j}\right\|_{1,J}\right) \\
\leq & \;\frac{1}{2}\,\left(e^{K_\alpha\,k}-1\right)\,|Z_0^{n-1}-S_0^{n-1}| + \frac{1}{2}\sum_{j=1}^{n-1} e^{K_\alpha\,k}\left(1-e^{-K_\alpha\,2\,k}\right)\,|Z_0^{n-j}-S_0^{n-j}| \\
& + C\,\frac{1}{2}\,\left(e^{K_\alpha\,k}-1\right)\,\left\|\mathbf{E}^{n-1}\right\|_{1,J} + C\,\frac{1}{2}\sum_{m=1}^{n-1} e^{K_\alpha\,k}\left(1-e^{-K_\alpha\,2\,k}\right)\,\|\mathbf{E}^{m}\|_{1,J} \\
\leq & \;C\,k\sum_{m=1}^{n-1}|Z_0^m-S_0^m| + C\,k\sum_{m=1}^{n-1}\|\mathbf{E}^m\|_{1,J} \\
\leq & \;C\,\|\mathbf{Z}_0-\mathbf{S}_0\|_{\infty,N} + C\,k\sum_{m=0}^{n-1}\|\mathbf{E}^m\|_{1,J}\,,
\end{aligned}
\tag{3.32}
$$

$1 \leq n \leq J$. In the second part, we bound the terms related with the initial condition,

$$
\begin{aligned}
(II) \;=\; & \frac{1}{2}\sum_{j=n}^{J-1} e^{K_\alpha\,k}\left(1-e^{-K_\alpha\,2\,k}\right)\,|E_{j-n}^0| + \frac{1}{2}\,\left(1+e^{K_\alpha\,k}\right)\,|E_{J-n}^0| \\
=\; & \frac{1}{2}e^{K_\alpha\,k}\left(1-e^{-K_\alpha\,2\,k}\right)\sum_{j=0}^{J-n-1} e^{-K_\alpha\,k\,j}\,e^{K_\alpha\,k\,j}\,|E_j^0| \\
& + \frac{1}{2}\,e^{-K_\alpha\,k\,(J-n)}\,\left(1+e^{K_\alpha\,k}\right)\,e^{K_\alpha\,k\,(J-n)}\,|E_{J-n}^0| \\
\leq\; & \frac{1}{2}e^{K_\alpha\,k}\left(1-e^{-K_\alpha\,2\,k}\right)\frac{1-e^{-K_\alpha\,k\,(J-n)}}{1-e^{-K_\alpha\,k}}\,\left\|\mathbf{E}^0\right\|_{\infty,J+1} \\
& + \frac{1}{2}\,e^{-K_\alpha\,k\,(J-n)}\,\left(1+e^{K_\alpha\,k}\right)\,\left\|\mathbf{E}^0\right\|_{\infty,J+1} \\
=\; & \frac{1}{2}\,\left(e^{K_\alpha\,k}+1\right)\left\|\mathbf{E}^0\right\|_{\infty,J+1} \\
\leq\; & C\,\left\|\mathbf{E}^0\right\|_{\infty,J+1}\,,
\end{aligned}
\tag{3.33}
$$

$1 \leq n \leq J$. Now, we treat the term corresponding to the sums of error norms,

$$\begin{aligned}
(III) = {} & C\,k\,\frac{1}{2}\,\left(e^{K_\alpha\,k}-1\right)\,\left\|\mathbf{E}^{n-1}\right\|_{1,J} \\
& + C\,k\,\frac{1}{2}e^{K_\alpha\,k}\left(1-e^{-K_\alpha\,2\,k}\right)\left(\sum_{j=1}^{n-1}\sum_{m=n-j}^{n-1} e^{-K_\alpha\,a_{j-n+m}}\,\|\mathbf{E}^m\|_{1,J}\right. \\
& \qquad \left. + \sum_{j=n}^{J-1}\sum_{m=0}^{n-1} e^{-K_\alpha\,a_{j-n+m}}\,\|\mathbf{E}^m\|_{1,J}\right) \\
& + C\,k\,\frac{1}{2}\,\left(1+e^{K_\alpha\,k}\right)\,\sum_{m=0}^{n-1} e^{-K_\alpha\,a_{J-n+m}}\,\|\mathbf{E}^m\|_{1,J}\,. \qquad (3.34)
\end{aligned}$$

We deal first with the double sums,

$$\begin{aligned}
& \sum_{j=1}^{n-1}\sum_{m=n-j}^{n-1} e^{-K_\alpha\,a_{j-n+m}}\,\|\mathbf{E}^m\|_{1,J} + \sum_{j=n}^{J-1}\sum_{m=0}^{n-1} e^{-K_\alpha\,a_{j-n+m}}\,\|\mathbf{E}^m\|_{1,J} \\
= {} & \sum_{m=1}^{n-1}\|\mathbf{E}^m\|_{1,J}\sum_{j=n-m}^{n-1} e^{-K_\alpha\,k\,(j-n+m)} + \sum_{m=0}^{n-1}\|\mathbf{E}^m\|_{1,J}\sum_{j=n}^{J-1} e^{-K_\alpha\,k\,(j-n+m)} \\
= {} & \sum_{m=1}^{n-1}\|\mathbf{E}^m\|_{1,J}\,\frac{1-e^{-K_\alpha\,k\,m}}{1-e^{-K_\alpha\,k}} + \sum_{m=0}^{n-1}\|\mathbf{E}^m\|_{1,J}\,e^{-K_\alpha\,k\,m}\,\frac{1-e^{-K_\alpha\,k\,(J-n)}}{1-e^{-K_\alpha\,k}} \\
= {} & \frac{1}{1-e^{-K_\alpha\,k}}\left(\sum_{m=1}^{n-1}\|\mathbf{E}^m\|_{1,J}\,(1-e^{-K_\alpha\,k\,m})\right. \\
& \left. + \sum_{m=0}^{n-1}\|\mathbf{E}^m\|_{1,J}\,e^{-K_\alpha\,k\,m}\,(1-e^{-K_\alpha\,k\,(J-n)})\right) \\
= {} & \frac{1}{1-e^{-K_\alpha\,k}}\,\sum_{m=0}^{n-1}\|\mathbf{E}^m\|_{1,J}\,(1-e^{-K_\alpha\,k\,(J-n+m)}). \qquad (3.35)
\end{aligned}$$

Then, we substitute (3.35) in (3.34) to obtain

$$
\begin{aligned}
(III) = C\,k\,\frac{1}{2}\,\left(e^{K_\alpha\,k}-1\right)\,\left\|\mathbf{E}^{n-1}\right\|_{1,J} \\
+\,C\,k\,\frac{1}{2}\left(1+e^{K_\alpha\,k}\right)\sum_{m=0}^{n-1}\|\mathbf{E}^m\|_{1,J}\,(1-e^{-K_\alpha\,k\,(J-n+m)}) \\
+\,C\,k\,\frac{1}{2}\,\left(1+e^{K_\alpha\,k}\right)\,\sum_{m=0}^{n-1}e^{-K_\alpha\,k\,(J-n+m)}\,\|\mathbf{E}^m\|_{1,J} \\
= \;\; C\,k\,\frac{1}{2}\,\left(e^{K_\alpha\,k}-1\right)\,\left\|\mathbf{E}^{n-1}\right\|_{1,J}+C\,k\,\frac{1}{2}\left(1+e^{K_\alpha\,k}\right)\sum_{m=0}^{n-1}\|\mathbf{E}^m\|_{1,J} \\
\leq \;\; C\,k\,\sum_{m=0}^{n-1}\|\mathbf{E}^m\|_{1,J}\,,
\end{aligned}
\tag{3.36}
$$

$1 \leq n \leq J$. And, finally, the sums related with the residual

$$
\begin{aligned}
(IV) = k\,\frac{1}{2}\,\left(e^{K_\alpha\,k}-1\right)\,|Z_1^n-S_1^n| \\
+\,k\,\frac{1}{2}\sum_{j=1}^{n-1}e^{K_\alpha\,k}\left(1-e^{-K_\alpha\,2\,k}\right)\,\sum_{m=n-j+1}^{n}|Z_{j-n+m}^m-S_{j-n+m}^m| \\
+\,k\,\frac{1}{2}\sum_{j=n}^{J-1}e^{K_\alpha\,k}\left(1-e^{-K_\alpha\,2\,k}\right)\,\sum_{m=1}^{n}|Z_{j-n+m}^m-S_{j-n+m}^m| \\
+\,k\,\frac{1}{2}\,\left(1+e^{K_\alpha\,k}\right)\,\sum_{m=1}^{n}|Z_{J-n+m}^m-S_{J-n+m}^m|.
\end{aligned}
\tag{3.37}
$$

Again, we treat first the double sums, which we bound as

$$
\begin{aligned}
&\sum_{j=1}^{n-1}\sum_{m=n-j+1}^{n}|Z^m_{j-n+m}-S^m_{j-n+m}|+\sum_{j=n}^{J-1}\sum_{m=1}^{n}|Z^m_{j-n+m}-S^m_{j-n+m}|\\
&\quad\le \sum_{j=1}^{n-1}\sum_{m=n-j+1}^{n} e^{-K_\alpha\, k\,(j-n+m)}\|\mathbf{Z}^m_*-\mathbf{S}^m_*\|_{\infty,J}\\
&\qquad+\sum_{j=n}^{J-1}\sum_{m=1}^{n} e^{-K_\alpha\, k\,(j-n+m)}\,\|\mathbf{Z}^m_*-\mathbf{S}^m_*\|_{\infty,J}\\
&\quad= \sum_{m=2}^{n}\|\mathbf{Z}^m_*-\mathbf{S}^m_*\|_{\infty,J}\sum_{j=n-m+1}^{n-1} e^{-K_\alpha\, k\,(j-n+m)}\\
&\qquad+\sum_{m=1}^{n}\|\mathbf{Z}^m_*-\mathbf{S}^m_*\|_{\infty,J}\sum_{j=n}^{J-1} e^{-K_\alpha\, k\,(j-n+m)}\\
&\quad= \sum_{m=2}^{n}\|\mathbf{Z}^m_*-\mathbf{S}^m_*\|_{\infty,J}e^{-K_\alpha\, k}\frac{1-e^{-K_\alpha\, k\,(m-1)}}{1-e^{-K_\alpha\, k}}\\
&\qquad\sum_{m=1}^{n}\|\mathbf{Z}^m_*-\mathbf{S}^m_*\|_{\infty,J}e^{-K_\alpha\, k\, m}\frac{1-e^{-K_\alpha\, k\,(J-n)}}{1-e^{-K_\alpha\, k}}\\
&\quad= \sum_{m=1}^{n}\|\mathbf{Z}^m_*-\mathbf{S}^m_*\|_{\infty,J}e^{-K_\alpha\, k}\frac{1-e^{-K_\alpha\, k\,(J-n+m-1)}}{1-e^{-K_\alpha\, k}},
\end{aligned}
\tag{3.38}
$$

and, therefore, we substitute (3.38) in (3.37) to arrive at

$$
\begin{aligned}
(IV)&\le k\,\frac{1}{2}\,\left(1-e^{-K_\alpha\, k}\right)\,\|\mathbf{Z}^n_*-\mathbf{S}^n_*\|_{\infty,J}\\
&\quad+k\,\frac{1}{2}\left(1+e^{-K_\alpha\, k}\right)\sum_{m=1}^{n}\|\mathbf{Z}^m_*-\mathbf{S}^m_*\|_{\infty,J}\left(1-e^{-K_\alpha\, k\,(J-n+m-1)}\right)\\
&\quad+k\,\frac{1}{2}\,\left(1+e^{K_\alpha\, k}\right)\,\sum_{m=1}^{n} e^{-K_\alpha\, k\,(J-n+m)}\|\mathbf{Z}^m_*-\mathbf{S}^m_*\|_{\infty,J}\\
&= k\,\frac{1}{2}\,\left(1-e^{-K_\alpha\, k}\right)\,\|\mathbf{Z}^n_*-\mathbf{S}^n_*\|_{\infty,J}+k\,\frac{1}{2}\left(1+e^{-K_\alpha\, k}\right)\sum_{m=1}^{n}\|\mathbf{Z}^m_*-\mathbf{S}^m_*\|_{\infty,J}\\
&\le C\,\sum_{m=1}^{n} k\,\|\mathbf{Z}^m_*-\mathbf{S}^m_*\|_{\infty,J},
\end{aligned}
\tag{3.39}
$$

$1 \leq n \leq J$. Then, we combine (3.31) with (3.32), (3.33), (3.36), (3.39) to arrive at,

$$\|\mathbf{E}^n\|_{1,J} \;\leq\; C\,\left\|\mathbf{E}^0\right\|_{\infty,J+1} + C\,k\sum_{m=0}^{n-1}\|\mathbf{E}^m\|_{1,J} + C\,\|\mathbf{Z}_0-\mathbf{S}_0\|_{\infty,N} + C\sum_{m=1}^{n} k\,\|\mathbf{Z}_*^m-\mathbf{S}_*^m\|_{\infty,J}, \tag{3.40}$$

$1 \leq n \leq J$. Finally, we use the Discrete Gronwall Lemma to obtain

$$\|\mathbf{E}^n\|_{1,J} \leq C\left\{\|\mathbf{E}^0\|_{\infty,J+1} + \|\mathbf{Z}_0-\mathbf{S}_0\|_{\infty,N} + \sum_{m=1}^{n} k\,\|\mathbf{Z}_*^m-\mathbf{S}_*^m\|_{\infty,J}\right\}. \tag{3.41}$$

Thus, from (3.29), the following inequality follows with respect to the boundary terms,

$$|E_0^n| \;\leq\; C\left\{\|\mathbf{E}^0\|_{\infty,J+1} + \|\mathbf{Z}_0-\mathbf{S}_0\|_{\infty,N} + \sum_{m=1}^{n} k\,\|\mathbf{Z}_*^m-\mathbf{S}_*^m\|_{\infty,J}\right\}, \tag{3.42}$$

$1 \leq n \leq N$. Now, on the one hand, if $n \leq N$, $n \leq j \leq J$, then by means of (3.27) and (3.41), we arrive at

$$\begin{aligned} e^{K_\alpha\,k\,j}\,|E_j^n| &\leq e^{K_\alpha\,k\,j}\,|E_{j-n}^0| + C\,k\sum_{m=0}^{n-1} e^{-K_\alpha\,k\,(m-n)}\,\|\mathbf{E}^m\|_{1,J} \\ &\quad + k\,e^{K_\alpha\,k\,j}\sum_{m=1}^{n}|Z_{j-n+m}^m - S_{j-n+m}^m| \\ &\leq C\left\{\|\mathbf{E}^0\|_{\infty,J+1} + \|\mathbf{Z}_0-\mathbf{S}_0\|_{\infty,N} + \sum_{m=1}^{n} k\,\|\mathbf{Z}_*^m-\mathbf{S}_*^m\|_{\infty,J}\right\}. \end{aligned} \tag{3.43}$$

On the other hand, when $N \geq n > j \geq 0$, taking into account (3.28), (3.42) and (3.41) we obtain

$$
\begin{aligned}
e^{K_\alpha k j}\,|E_j^n| &\le e^{K_\alpha k j}\,|E_0^{n-j}| + C\,k \sum_{m=n-j}^{n-1} e^{-K_\alpha k (m-n)}\,\|\mathbf{E}^m\|_{1,J} \\
&\quad + k\,e^{K_\alpha k j} \sum_{m=n-j+1}^{n} |Z_{j-n+m}^m - S_{j-n+m}^m| \\
&\le C \left\{ \|\mathbf{E}^0\|_{\infty,J+1} + \|\mathbf{Z}_0 - \mathbf{S}_0\|_{\infty,N} + \sum_{m=1}^{n} k\,\|\mathbf{Z}_*^m - \mathbf{S}_*^m\|_{\infty,J} \right\}. \quad (3.44)
\end{aligned}
$$

Thus by (3.42)-(3.44), we achieve

$$
\|(\mathbf{E}^0, \dots, \mathbf{E}^N)\|_{\mathcal{X}_h} \le C\,\|\mathbf{V}^0 - \mathbf{W}^0, \mathbf{Z}_0 - \mathbf{S}_0, \mathbf{Z}_*^1 - \mathbf{S}_*^1, \dots, \mathbf{Z}_*^N - \mathbf{S}_*^N\|_{\mathcal{Y}_h}. \blacksquare
$$

3.3. Existence and Convergence

We say that the discretization (3.1) is *convergent* if there exists $k_0 > 0$ such that for each k in H with $k \le k_0$, (3.5) has a solution $\mathbf{U}_k$ for which,

$$
\lim_{k\to 0} \|\mathbf{u}_k - \mathbf{U}_k\|_{\mathcal{X}_k} = 0.
$$

We define the *global discretization error* as

$$
\mathbf{e}_k = \mathbf{u}_k - \mathbf{U}_k \in \mathcal{X}_k.
$$

To derive the existence and convergence of numerical solutions of (2.3)-(2.6), we shall use a result of the general discretization framework introduced by López-Marcos and Sanz-Serna [12].

Theorem 3. *Assume that (3.1) is consistent and stable with thresholds R_k. If $\mathbf{\Phi}_k$ is continuous in $\mathcal{B}(\mathbf{u}_k, R_k)$ and $\|\mathbf{l}_k\|_{\mathcal{Y}_k} = o(R_k)$ as $k \to 0$, then:*

i) *for k sufficiently small, the discrete equations (3.5) possess a unique solution in $\mathcal{B}(\mathbf{u}_k, R_k)$,*
ii) *as $k \to 0$, the solutions converge and $\|\mathbf{e}_k\|_{\mathcal{X}_k} = O(\|\mathbf{l}_k\|_{\mathcal{Y}_k})$.*

Now the existence and convergence are immediately obtained by means of Theorem 1 (consistency), Theorem 2 (stability), and Theorem 3. We emphasize that this theorem establishes the existence of a unique solution of the nonlinear system of equations for the approximation derived through the discretization of the problem. The analysis can be extended even if the quadrature rule (2.10) were closed at $a_0 = 0$, once we establish the consistency and stability properties of the new numerical scheme.

Theorem 4. *Under the hypotheses of Theorem 2, let the numerical initial condition* $\mathbf{U}^0$ *be such that*

$$\|\mathbf{U}^0 - \mathbf{u}^0\|_{\infty,J+1} = o(R_k),$$

as $k \to 0$. *Then, for* k *sufficiently small, there exists a unique solution*

$$(\mathbf{U}^0, \mathbf{U}^1, \dots, \mathbf{U}^N)$$

in the ball $\mathcal{B}(\mathbf{u}_k, R_k)$ *of* $\mathcal{X}_k$, *of equations (2.3)-(2.6) and*

$$\max_{0\le n\le N} \|\mathbf{U}^n - \mathbf{u}^n\|_{\infty,J+1} = \mathcal{O}(\|\mathbf{U}^0 - \mathbf{u}^0\|_{\infty,J+1} + k^2).$$

Note that, in particular, if $\mathbf{U}^0$ is taken as the grid restriction $\mathbf{u}^0$ of the initial condition (1.5), then our scheme is second order accurate.

4. Numerical results

We have carried out numerical experiments with the scheme (2.3)-(2.6) in a theoretical test problem which presents meaningful nonlinearities. Let $c \in \mathbb{R}^+$; we choose the age-dependent mortality rate as $\mu(a, z, t) = c^2\, z$, the age-specific birth rate as

$$\beta(a, z, t) = \frac{4\, c^3\, z\, a\, e^{-c\, a}}{(1 + c\, z)^2} \frac{(2 + e^{-c\, t})^2}{1 + e^{-c\, t}},$$

and the weight functions $\gamma_\mu(a) = \gamma_\beta(a) = 1$. Thus, the solution to (1.2)-(1.5) is

$$u(a, t) = \frac{e^{-c\, a}}{1 + e^{-c\, t}}.$$

The numerical integration of this problem is carried out in the time interval $[0, 10]$. Since the exact solution is known, we can show the accuracy of our numerical method through its global error and the numerical order of convergence quantitatively.

In this test problem, parameter c provides the ratio of the exponential decay of the solution. We perform all the experimentation with the choice $c = 0.01$, a small value that is considered to retard the rate of decline to zero of the age-specific density function as much as possible.

We try different values for parameters K_α and K_1 to grasp the effect of these parameters in the numerical order of convergence of the approximations. Both parameters K_α and K_1 determine together the natural grid in the interval $[0, 1]$, and, correspondingly, the extension of the uniform grid on the age variable. It is straightforward to derive from formula in (2.8) that the number of nodes in the natural grid, J, should satisfy

$$J > -\frac{\log(K_1\, k)}{K_\alpha\, k}.$$

Parameter K_α is chosen in terms of the exponential rate decay of the solution and for fixed K_α, as K_1 decreases, the length of the truncated age interval in the discretization increases to infinity. Alternatively, keeping K_1 constant, we also get an increasing length for the truncated age-interval in the discretization as we decrease K_α.

The existence of an analytical formula for the solution to model (1.2)-(1.5) allows us to both compare with the numerical solution and compute the error caused by the numerical approximation. Then, once the value of each parameter K_1, K_α, and the time-discretization parameter k are fixed, and the corresponding numerical solution $\mathbf{U}_k = (\mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^N)$ is computed, we can obtain the global error, with the formula,

$$e_k = \max_{0 \le n \le N} \|\mathbf{u}^n - \mathbf{U}^n\|_{\infty, J+1}. \tag{4.1}$$

We can also obtain the numerical order of convergence $order_{2\,k}$ by means of the following well-known formula

$$order_{2\,k} = \frac{\log(e_{2\,k}/e_k)}{\log(2)}.$$

Note that, errors are measured in a maximum norm in which nodal values are weighted with an exponential increasing factor in age of rate K_α. The size of K_α is naturally chosen to balance the assumed exponential decay of the solution and the corresponding numerical approximations.

We have made a wide numerical experimentation with different values for all the parameters. Some of the experimental results are shown on Tables 1-3. Each table represents a different value of the parameter K_α ($K_\alpha = 0.01$, $K_\alpha = 0.005$, and $K_\alpha = 0.02$, respectively). In Tables 1-2, each column and each row of the corresponding table represents a computation with different

k	K_1=1e-1	K_1=1e-2	K_1=1e-3	K_1=1e-4	K_1=1e-5
5e-2	1.3179e-04	2.2027e-06	2.8174e-07	2.5628e-07	2.5596e-07
2.5e-2	3.7710e-05	5.9816e-07	7.0940e-08	6.4104e-08	6.4021e-08
	1.81	1.88	1.99	2.00	2.00
1.25e-2	1.0616e-05	1.6146e-07	1.7858e-08	1.6031e-08	1.6009e-08
	1.83	1.89	1.99	2.00	2.00
6.25e-3	2.9509e-06	4.3339e-08	4.4947e-09	4.0085e-09	4.0027e-09
	1.85	1.90	1.99	2.00	2.00
3.125e-3	8.1205e-07	1.1578e-08	1.1312e-09	1.0023e-09	1.0007e-09
	1.86	1.90	1.99	2.00	2.00
1.5625e-3	2.2157e-07	3.0801e-09	2.8465e-10	2.5059e-10	2.5020e-10
	1.87	1.91	1.99	2.00	2.00
7.8125e-4	6.0035e-08	8.1644e-10	7.1635e-11	6.2663e-11	6.2556e-11
	1.88	1.92	1.99	2.00	2.00
3.9063e-4	1.6169e-08	2.1573e-10	1.8053e-11	1.5696e-11	1.5670e-11
	1.90	1.92	1.99	2.00	2.00

Table 1: Theoretical Experiment. Errors and numerical order with parameters $c = 0.01$, $K_\alpha = 0.01$.

values of the parameter K_1 ($K_1 = 1e-1$, $K_1 = 1e-2$, $K_1 = 1e-3$, $K_1 = 1e-4$, and $K_1 = 1e-5$), and the discretization parameter k ($k = 5e-2$, $k = 2.5e-2$, $k = 1.25e-2$, $k = 6.25e-3$, $k = 3.125e-3$, $k = 1.5625e-3$, $k = 7.8125e-4$ and $k = 3.90625e-4$), respectively. The upper number of each entry in columns two to six of the table represents the global error, e_k, and the lower quantity is the experimental order of convergence, $order_{2k}$.

The results on Tables 1-2 show numerically the expected theoretical order of convergence. This is the case in which the exponential rate of decay of the solution is greater than the exponentially increasing rate of the weights in the maximum norm (4.1). We emphasize that the second order of convergence is obtained under a weaker restriction on K_α than the one fixed in the convergence analysis. Lower and higher values of parameters K_1 and K_α have been considered but the results are not reported because they confirm the second order of convergence shown on Tables 1-2. We also observe that the effect of K_1 on the error is diminished with lower values of K_α.

The values on Table 3 illustrate how the convergence fails when $K_\alpha > c$. We only show results corresponding to $K_1 = 1e-5$, with $c = 0.01$ and $K_\alpha = 0.02$, because the use of other values of K_1 provides the same bad behaviour. The method is effective even when the parameter K_α underestimates c, the theoretical rate of decay of the solution.

k	K_1=1e-1	K_1=1e-2	K_1=1e-3	K_1=1e-4	K_1=1e-5
5e-2	3.7530e-06	2.6439e-07	2.3060e-07	2.3026e-07	2.3025e-07
2.5e-2	9.3524e-07	6.6131e-08	5.7682e-08	5.7598e-08	5.7597e-08
	2.00	2.00	2.00	2.00	2.00
1.25e-2	2.3360e-07	1.6537e-08	1.4425e-08	1.4404e-08	1.4403e-08
	2.00	2.00	2.00	2.00	2.00
6.25e-3	5.8384e-08	4.1348e-09	3.6067e-09	3.6014e-09	3.6014e-09
	2.00	2.00	2.00	2.00	2.00
3.125e-3	1.4596e-08	1.0338e-09	9.0174e-10	9.0042e-10	9.0041e-10
	2.00	2.00	2.00	2.00	2.00
1.5625e-3	3.6489e-09	2.5845e-10	2.2545e-10	2.2511e-10	2.2511e-10
	2.00	2.00	2.00	2.00	2.00
7.8125e-4	9.1223e-10	6.4629e-11	5.6374e-11	5.6293e-11	5.6293e-11
	2.00	2.00	2.00	2.00	2.00
3.9063e-4	2.2808e-10	1.6194e-11	1.4136e-11	1.4114e-11	1.4113e-11
	2.00	2.00	2.00	2.00	2.00

Table 2: Theoretical Experiment. Errors and numerical order with parameters $c = 0.01$, $K_\alpha = 0.005$

As a motivational test of the numerical approach considered with a more biological insight, we propose the following numerical experiment introduced by Sulsky [16]. The model describes the dynamics of a gray squirrel population (*Sciurus carolinensis*) with a continuous age-structure. Fecundity as a function of age and total population, $P(t)$, is given by

$$\beta(a, P) = b(a)\, B(P), \quad b(a) = \frac{1}{0.438 + 19.56\, e^{-4.04\, a}}, \quad B(P) = \frac{75}{25 + P}. \tag{4.2}$$

The constants in the function b are chosen to assure $b(0) = 0.05$, $b(1) = 1.28$, and $\lim_{a\to\infty} b(a) = 2.28$ that match with field data experiment. On the

k	e_k
5e-2	1.3314468e-02
2.5e-2	1.3320279e-02
1.25e-2	1.3322284e-02
6.25e-3	1.3322941e-02
3.125e-3	1.3323129e-02
1.5625e-3	1.3323160e-02
7.8125e-4	1.3323143e-02
3.90625e-4	1.3323117e-02

Table 3: Theoretical Experiment. Errors with parameters $c = 0.01$, $K_\alpha = 0.02$, $K_1 = 1e-5$.

other hand the density dependence, which appears as a multiplicative factor, concerns unavailable data; it is supposed to be a nonincreasing function with $\lim_{P\to\infty} B(P) = 0$.

Mortality is described as,

$$\mu(a,P) = d(a)\,D(P), \quad d(a) = 2.75\,e^{-2.5\,a} + 0.275\,e^{0.15\,(a-2)}, \quad D(P) = \frac{1.5\,P}{25+P}. \tag{4.3}$$

In this case, mortality is due to a term which increases slowly and corresponds with the main mortality for advanced ages, and another one that is higher but whose influence disappears soon. With respect to the density dependence, it is a nondecreasing bounded function. As a final data, the initial condition is taken as

$$u_0(a) = 5\,\chi_{[0,5]}(a). \tag{4.4}$$

Due to the difficulty to obtain an analytical solution of this particular model, the use of numerical methods is necessary to get it. Besides, a principal interest of this model is to identify, if any, the asymptotic age-structure equilibrium of the population. Then, our approach consists in doing long-time numerical integrations with the numerical scheme analyzed. We have employed as parameter values $K_\alpha = 0.1$, $K_1 = 0.1$, the discretization parameter as $k = 1.5625e-03$, and the final integration time is $T = 100$, which is enough to capture the stability of the equilibria in the model.

If the dependence on age of the vital functions is avoided, it is simple to obtain the asymptotic behaviour of the model, because we can describe a time-evolution model for the total population

$$P'(t) = P(t)\,(\beta(P(t)) - \mu(P(t))),$$

whose stationary equilibria correspond to $P^* = 0$, which is unstable, and to the solution $P^* = 50$ of equation $\mu(P^*) = \beta(P^*)$, which is stable in the case we are dealing with, as we can see on the left plot of Figure 1. However, when the vital functions depend on the age, we need the help of an effective numerical method to describe the model, not only the approximation to the time evolution of the age-dependent density function, but also the age-dependent structure of the asymptotically stable stationary equilibrium. The theoretical study of equilibria led us to the following characteristic equation for the nontrivial equilibrium solution of the model, $u^*(a)$ (and P^*),

$$1 = \int_0^\infty \beta(a, P^*)\, \exp\left(-\int_0^a \mu(s, P^*)\, ds\right) da. \tag{4.5}$$

When our numerical method is applied to the age-dependent problem in a

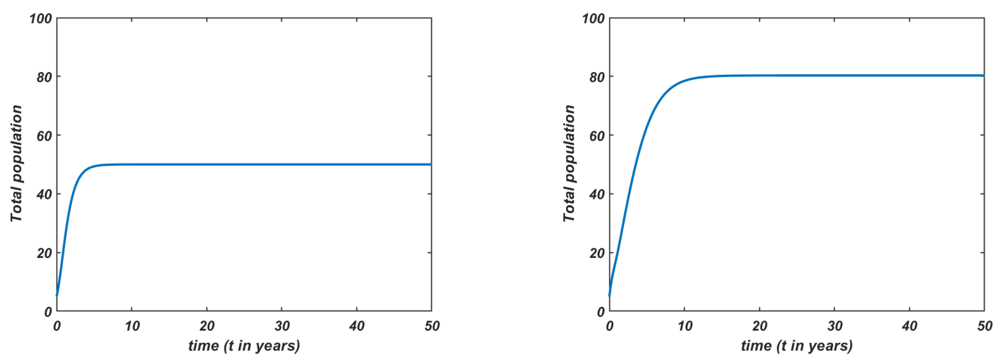

Figure 1: Evolution of the total population with time, $K_\alpha = 0.1$, $K_1 = 0.1$, $k = 1.5625e-03$, and $T = 100$. Plot in the left hand side: age-independent case, $P^* = 50$. Plot in the right hand side: age-dependent case, $P^* = 80.3172290$.

long-time integration, a steady state also appears. We perform a numerical simulation using the same parameter values ($K_\alpha = 0.1$, $K_1 = 0.1$, the discretization parameter as $k = 1.5625e-03$, and the final integration time as $T = 100$). This allows us, on one hand, to establish the existence of a nontrivial stable age-dependent steady state, and, on the other hand, to compute an approximation to the total population of such age-structured

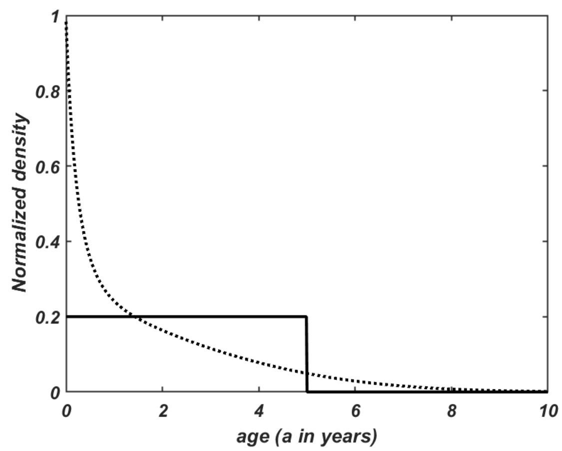

Figure 2: Initial density (solid line) and equilibrium density (dotted line). Age-dependent case.

distribution, $P^* = 80.3172290$, as it can be observed on the right plot of Figure 1. In Figure 2, the normalized density of the distribution at equilibrium is compared with the normalized initial density to describe the evolution of the population age-structure. Equation (4.5) can be solved with standard algorithms for finding zeros of nonlinear equations, which results in $P^* = 80.3171998$, close to our predicted value and far from the numerical prediction of [16], $P^* = 50.5$. A crude estimation of the exponential rate of decay of the age-dependent density profile is given by $\sigma = 4.827$. Values of K_α that underestimate this rate are equally effective in the simulations to obtain the right asymptotic steady state.

5. Conclusions

In age-structured population models, it is common to consider an unbounded age-interval for the lifespan. This is the case of the pioneering works of Sharpe-Lotka [15], McKendrick [13] and Gurtin-MacCamy [10], among other authors. Numerical discretization of this kind of problems should cope with the unbounded age-interval through some strategy for truncating the age-domain.

In this paper, we focus on a discretization in which the finite truncated age-interval is adaptively increasing in length, as the discretization parameter decreases, according to the exponential rate of decay with age of the solution. The long-time integration of these models in order to determine, if any, the stationary asymptotic state and the asymptotic behaviour of the solutions makes this approach very convenient.

Furthermore, we prove second order of convergence for a numerical scheme that adaptively selects the truncated age-interval of the discretization in terms of the last point in the *natural grid* provided by an artificial structural variable. Convergence analysis is made straightforward through a weighted maximum norm (with exponential increasing weights) on the nodal values of the approximation errors to the age density function of the population. We avoid any reformulation of the problem as an artificial size-structured population model on a bounded domain and the corresponding discretization, as reported in [4]. Numerical experiments with an appropriate test problem confirm the second-order of convergence even when we relax some of the hypothesis assumed by the convergence analysis. We also point out the robustness of the method against an underestimation of the exponential rate of decay of the solution given by σ.

Lastly, we report the effectiveness of this methodology to approximate the stationary equilibrium state for a model proposed in [16] for the dynamics of an age-dependent squirrel population. With the technique introduced, we improve some of the approximations to the asymptotic steady-state equilibrium reported in [16], and we support them analytically.

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